Consistent section-averaged shallow water equations with bottom friction

Victor Michel-Dansac[†], Pascal Noble[†], Jean-Paul Vila[†]

Thursday, March 21st, 2019

Journées Jeunes ÉDPistes, Rennes

[†]Institut de Mathématiques de Toulouse et INSA Toulouse

Motivation: 2D/1D coupling for estuary simulation

Gironde estuary: satellite picture

Gironde estuary: 2D mesh

Existing approaches

Regarding the shape of the river bed, as of now,

- the derivation of 1D models is well-understood [1,2] in the ideal case of a ⊔-shaped channel;
- for more complex shapes, the water surface of uniform stationary flows is recovered [3,4] using a empiric terms or data assimilation;
- fully 2D models are used but they are computationally costly.

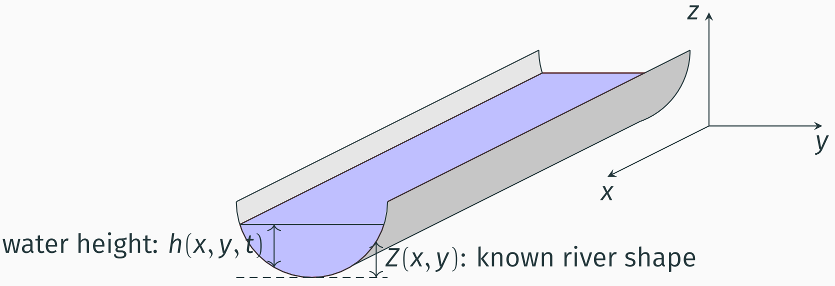
[1] see Bresch and Noble, 2007, in the context of laminar flows
[2] see Richard, Rambaud and Vila, 2017, in the context of turbulent flows
[3] see Decoene, Bonaventura, Miglio and Saleri, 2009
[4] see Marin and Monnier, 2009

Specifications of the 1D model

The goal of this work is to develop a new model, based on the shallow water equations, that is:

- generic enough to not require empiric friction coefficients;
- consistent with the 2D shallow water in the asymptotic regime corresponding to an estuary or a river;
- hyperbolic;
- easily implementable (collaboration with the SHOM for flood simulations, ocean model forcing, ...).

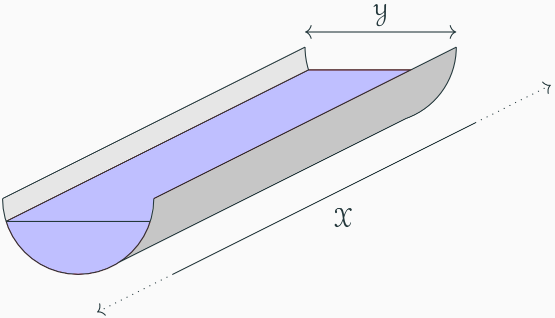
The non-conservative 2D shallow water system



$$
\begin{cases}
h_t + \boldsymbol{\nabla} \cdot (h\mathbf{u}) = 0 \\
\mathbf{u}_t + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} + g\boldsymbol{\nabla} h = g\left(-\boldsymbol{\nabla} Z - \dfrac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\, h^p}\right)
\end{cases}
$$

- $\mathbf{u} = (u, v)$ is the water velocity
- g is the gravity constant
- $C_h(x, y)$ is the (known) Chézy friction coefficient
- $p = 4/3$ is the friction exponent

Introduction of reference scales: the coordinates



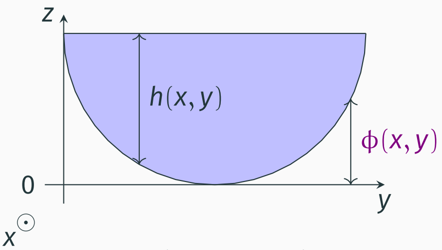
	dimensional quantity	reference scale	non-dimensional quantity
longitudinal coordinates	$x \in (0\mathrm{m}, 60000\mathrm{m})$	$\mathcal{X} = 2000\mathrm{m}$	$\bar{x} = \dfrac{x}{\mathcal{X}} \in (0, 30)$
transverse coordinates	$y \in (-25\mathrm{m}, 25\mathrm{m})$	$\mathcal{Y} = 50\mathrm{m}$	$\bar{y} = \dfrac{y}{\mathcal{Y}} \in (-0.5, 0.5)$

Introduction of reference scales: the topography

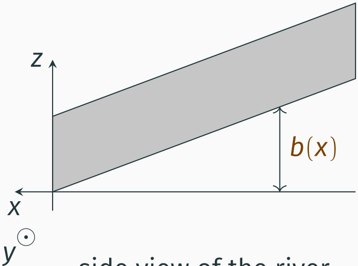
Regarding the geometry, we assume that $Z(x, y) = b(x) + \phi(x, y)$, where:

- $b(x)$ represents the main longitudinal topography, driving the flow from upstream to downstream;
- $\phi(x, y)$ represents small longitudinal and transverse variations.

Thus, $h + \phi$ represents the altitude of the water surface.



front view of the river



side view of the river

Non-dimensional form of the 2D shallow water system

We introduce the following non-dimensional numbers to emphasize the different scales of the flow:

- F^2, the reference Froude number (ratio material/acoustic velocity),
- δ, the shallow water parameter (ratio height/reference length),
- R_u, the quasi-1D parameter (ratio transverse/longitudinal velocity),
- I_0 and J_0, the reference topography and friction slopes.

Finally, the non-dimensional form of the 2D shallow water system is:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \dfrac{1}{F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{x}} = \dfrac{1}{\delta F^2}\left(-J_0 \dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2 \bar{v}^2}}{\bar{C}^2 \bar{h}^p} - I_0 \bar{b}_{\bar{x}}\right), \\ \bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \dfrac{1}{R_u^2 F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{y}} = -\dfrac{J_0}{\delta F^2} \dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2 \bar{v}^2}}{\bar{C}^2 \bar{h}^p}. \end{cases}$$

Asymptotic expansions setup

In the regime under consideration, we have

- $\varepsilon := \dfrac{\delta F^2}{J_0} \ll 1$ (in practice, $F^2 \ll 1$, $\delta \ll 1$, $J_0 \ll 1$ and $J_0 \sim \delta$),
- $R_u \ll 1$ (quasi-unidimensional setting), and $R_u = \mathcal{O}(\varepsilon)$.

Highlighting the dominant terms in the system, we get:

$$
\begin{cases}
h_t + (hu)_x + (hv)_y = 0, \\
u_t + uu_x + vu_y + \dfrac{1}{\varepsilon}\dfrac{\delta}{J_0}(h+\phi)_x = \dfrac{1}{\varepsilon}\left(-\dfrac{u\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} - \dfrac{I_0}{J_0} b_x\right), \\
v_t + uv_x + vv_y + \dfrac{1}{\varepsilon^3}\dfrac{\delta}{J_0}(h+\phi)_y = -\dfrac{1}{\varepsilon}\dfrac{v\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p}.
\end{cases}
$$

Goal: Perform asymptotic expansions in this regime, to better understand the weak dependency of the solution in y.

Free surface expansion

We consider the third equation:

$$v_t + uv_x + vv_y + \frac{1}{\varepsilon^3}\frac{\delta}{J_0}(h+\phi)_y = -\frac{1}{\varepsilon}\frac{v\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p},$$

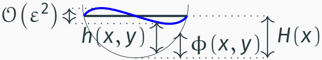
which we rewrite as follows to highlight the dominant term:

$$\frac{\delta}{J_0}(h+\phi)_y = \varepsilon^2 \frac{v\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} + \varepsilon^3 (v_t + uv_x + vv_y).$$

Neglecting the $\mathcal{O}(\varepsilon^2)$ terms, we get

$$\frac{\delta}{J_0}(h+\phi)_y = \mathcal{O}(\varepsilon^2),$$

and there exists $H = H(x,t)$ such that



$$H(x,t) = h(x,y,t) + \phi(x,y) + \mathcal{O}(\varepsilon^2).$$

$\rightsquigarrow$ the free surface is almost flat in the y-direction, up to $\mathcal{O}(\varepsilon^2)$

Longitudinal velocity expansion

Highlighting the dominant terms, the second equation (times ε) reads:

$$u_t + uu_x + vu_y + \frac{1}{\varepsilon}\frac{\delta}{J_0}(h+\phi)_x = \frac{1}{\varepsilon}\left(-\frac{u\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} - \frac{I_0}{J_0} b_x\right).$$

To perform the asymptotic expansion of u with respect to ε, we write

$$u(x,y,t) = u_{2D}^{(0)}(x,y,t) + \mathcal{O}(\varepsilon).$$

Since $h + \phi = H + \mathcal{O}\left(\varepsilon^2\right)$, straightforward computations yield:

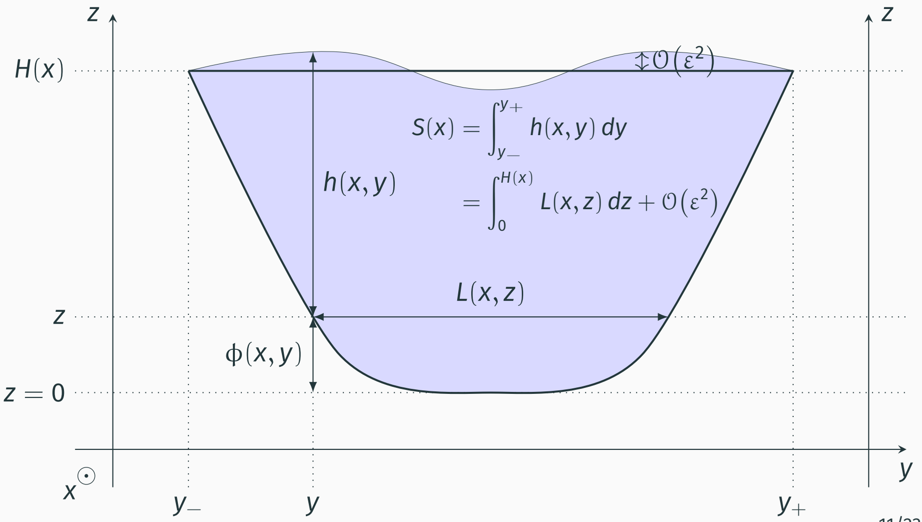
$$u_{2D}^{(0)} = C\frac{\Lambda}{\sqrt{|\Lambda|}}(H-\phi)^{p/2},$$

where we have defined the corrected slope $\Lambda(x,t) = -\frac{I_0}{J_0}b_x - \frac{\delta}{J_0}H_x$.

Next step: Build a 1D model consistent with these expansions.

The river cross-section

To obtain a 1D model, we start by averaging the 2D equations: below, we display the cross-section of the river, with respect to x.



Averaging the 2D system over the river width

1. The original mass conservation equation reads:

$$h_t + (hu)_x + (hv)_y = 0.$$

Therefore, since $v(y_-) = v(y_+) = 0$, we get:

$$\int_{y_-}^{y_+} h_t \, dy + \int_{y_-}^{y_+} (hu)_x \, dy = 0 \quad \Longrightarrow \quad S_t + Q_x = 0,$$

where the averaged discharge Q is given by $Q = \displaystyle\int_{y_-}^{y_+} hu \, dy$.

2. Arguing the mass conservation and integrating the second equation between y_- and y_+ yields:

$$Q_t + \left(\int_{y_-}^{y_+} hu^2 \, dy \right)_x = \frac{1}{\varepsilon} \int_{y_-}^{y_+} h \left(-\frac{I_0}{J_0} b_x - \frac{\delta}{J_0} (h + \phi)_x \right) dy$$

$$- \frac{1}{\varepsilon} \int_{y_-}^{y_+} \frac{u\sqrt{u^2 + \varepsilon^2 v^2}}{C^2 h^{p-1}} \, dy.$$

Averaging the 2D system

Finally, the averaged system reads as follows, up to $\mathcal{O}(\varepsilon^2)$:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\displaystyle\int_{y_-}^{y_+} hu^2 \, dy \right)_x = \dfrac{1}{\varepsilon} \left(\Lambda S - \displaystyle\int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}} \, dy \right) + \mathcal{O}(\varepsilon). \end{cases}$$

Next step: From the averaged system, build a truly 1D model that is zeroth-order accurate (up to $\mathcal{O}(\varepsilon)$).

That is to say, the new model needs to ensure $Q = Q_{2D}^{(0)} + \mathcal{O}(\varepsilon)$, where

$$\begin{aligned} Q_{2D}^{(0)} &= \int_{y_-}^{y_+} h u_{2D}^{(0)} \, dy \\ &= \sqrt{|\Lambda|} \operatorname{sgn}(\Lambda) \int_{y_-}^{y_+} C \, (H - \phi)^{1+p/2} \, dy. \end{aligned}$$

Setting up the model

The integrated discharge equation, highlighting the dominant terms and multiplying by ε, is

$$\Lambda S - \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}}\, dy = \varepsilon\left(Q_t + \left(\int_{y_-}^{y_+} hu^2\, dy\right)_x\right) + \mathcal{O}\left(\varepsilon^2\right).$$

At the zeroth order, i.e. up to $\mathcal{O}(\varepsilon)$, the right-hand side of this equation is neglected, and we get:

$$\Lambda S - \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}}\, dy = \mathcal{O}(\varepsilon).$$

We cannot directly use this equation in a 1D model, since it contains the unknown u, which depends on y.

Instead, we approximate the integral, up to $\mathcal{O}(\varepsilon)$, with a new 1D friction term.

The friction model

First, we choose this 1D friction term as a usual hydraulic engineering model. Thus, we impose the following formula:

$$\frac{Q|Q|}{C_{1D}^2 S} = \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}}\, dy + \mathcal{O}(\varepsilon).$$

It contains a 1D friction coefficient[5] C_{1D}, to be determined.

According to the discharge equation, we get, up to $\mathcal{O}(\varepsilon)$:

$$\frac{Q|Q|}{C_{1D}^2 S} = \Lambda S + \mathcal{O}(\varepsilon) \quad \Longrightarrow \quad C_{1D}^2 = \frac{Q|Q|}{\Lambda S^2} + \mathcal{O}(\varepsilon).$$

Second, we impose $Q = Q_{2D}^{(0)} + \mathcal{O}(\varepsilon)$, to get the following expression of the friction coefficient:

$$C_{1D}^2 = \frac{Q_{2D}^{(0)} \left| Q_{2D}^{(0)} \right|}{\Lambda S^2}.$$

[5] The coefficient C_{1D}^2 usually contains the hydraulic radius, the Chézy coefficient, ...

The final system

With the new friction model, the discharge equation reads

$$\Lambda S - \frac{Q|Q|}{C_{1D}^2 S} = \varepsilon\left(Q_t + \left(\int_{y_-}^{y_+} hu^2\,dy\right)_x\right) + \mathcal{O}(\varepsilon).$$

We choose to approximate the integral in the flux to describe the advection of the discharge:

$$\varepsilon \int_{y_-}^{y_+} hu^2\,dy = \varepsilon \frac{\left(\displaystyle\int_{y_-}^{y_+} hu\,dy\right)^2}{\displaystyle\int_{y_-}^{y_+} h\,dy} + \mathcal{O}(\varepsilon) = \varepsilon \frac{Q^2}{S} + \mathcal{O}(\varepsilon).$$

The resulting discharge equation is

$$\varepsilon\left(Q_t + \left(\frac{Q^2}{S}\right)_x\right) = S\left(\Lambda - \underbrace{\frac{Q|Q|}{C_{1D}^2 S^2}}_{\mathcal{J}}\right) + \mathcal{O}(\varepsilon).$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J}). \end{cases}$$

Let us double check that this model is sufficient to recover the zeroth-order expansion of Q.

With $Q = Q^{(0)}_{\text{model}} + \mathcal{O}(\varepsilon)$, we get, at the zeroth order:

$$\Lambda = \mathcal{J} + \mathcal{O}(\varepsilon) \implies \Lambda = \overbrace{\Lambda \frac{Q|Q|}{Q^{(0)}_{2D}\left|Q^{(0)}_{2D}\right|}}^{\mathcal{J}} + \mathcal{O}(\varepsilon) = \Lambda \frac{Q^{(0)}_{\text{model}}\left|Q^{(0)}_{\text{model}}\right|}{Q^{(0)}_{2D}\left|Q^{(0)}_{2D}\right|} + \mathcal{O}(\varepsilon)$$

$$\implies Q^{(0)}_{\text{model}} = Q^{(0)}_{2D} + \mathcal{O}(\varepsilon).$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = \dfrac{1}{\varepsilon} S\left(-\dfrac{I_0}{J_0} b_x - \dfrac{\delta}{J_0} H_x - \mathcal{J}\right). \end{cases}$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = \dfrac{1}{\varepsilon} S \Bigg(\underbrace{-\dfrac{I_0}{J_0} b_x}_{\mathcal{J}} - \dfrac{\delta}{J_0} H_x - \mathcal{J} \Bigg). \end{cases}$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x + \dfrac{SH_x}{F^2} = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}). \end{cases}$$

This form is quite similar to that of the the usual models.
All the complexity lies within the friction model $\mathcal{J}$ and in the expression of the friction coefficient C_{1D}.

$\rightsquigarrow$ We have derived a zeroth-order model governed by a hyperbolic system of balance laws.
$\rightsquigarrow$ We also enhance this approach to derive a first-order model, based on the energy equation.

Next step: Numerical validation of these models.

Numerical schemes

To handle the stiff relaxation source term, we introduce an implicit splitting procedure.

The zeroth-order model is made of a non-stiff part and a stiff part:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x + \dfrac{1}{\varepsilon}\dfrac{\delta}{J_0} S H_x = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}). \end{cases}$$

First, we consider the non-stiff part:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = 0, \end{cases}$$

which we discretize using an upwind finite difference scheme.

Numerical schemes

Second, we consider the stiff part:

$$\begin{cases} S_t = 0, \\ Q_t + \dfrac{1}{\varepsilon}\dfrac{\delta}{J_0} S H_x = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}). \end{cases}$$

Since $S_t = 0$, we are left with the following ODE on Q:

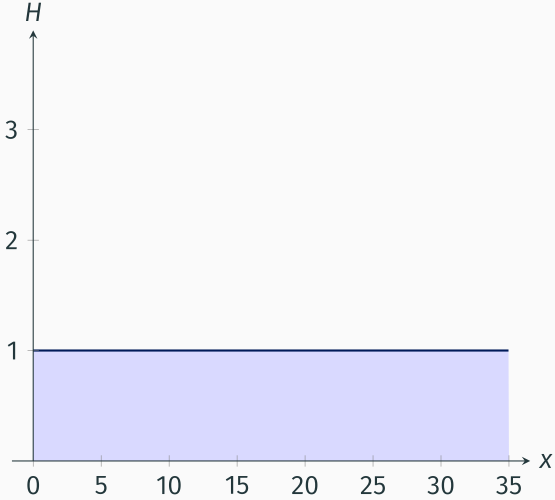
$$Q_t = \frac{1}{\varepsilon} S \Lambda \left(1 - \frac{Q^2}{\left(Q_{2D}^{(0)}\right)^2} \right),$$

which we can solve exactly, to get

$$Q(t) = Q_{2D}^{(0)} \frac{\tanh\left(\dfrac{1}{\varepsilon}\dfrac{S|\Lambda|}{|Q_{2D}^{(0)}|} t\right) + \dfrac{Q(0)}{Q_{2D}^{(0)}}}{1 + \tanh\left(\dfrac{1}{\varepsilon}\dfrac{S|\Lambda|}{|Q_{2D}^{(0)}|} t\right) \dfrac{Q(0)}{Q_{2D}^{(0)}}} \xrightarrow[\varepsilon \to 0]{} Q_{2D}^{(0)}.$$

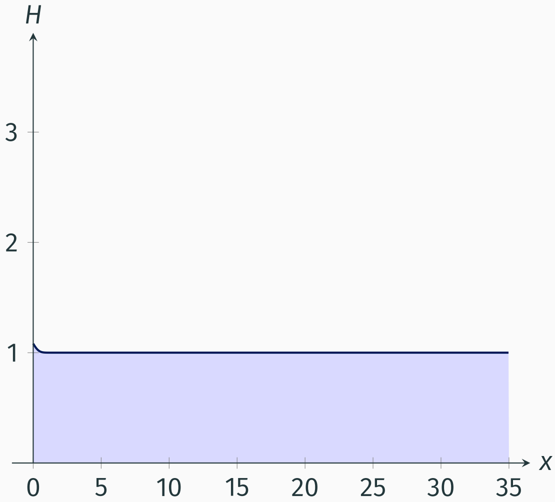
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



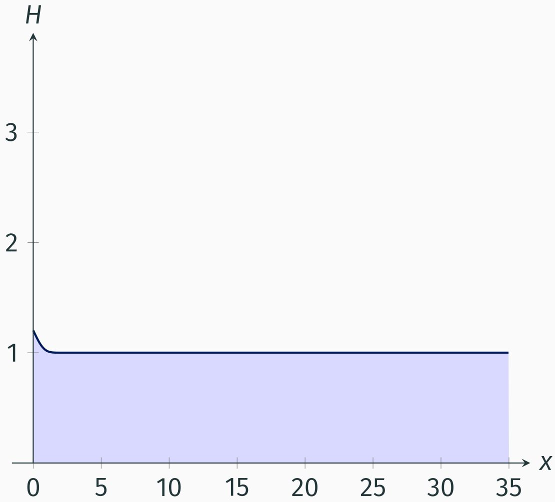
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



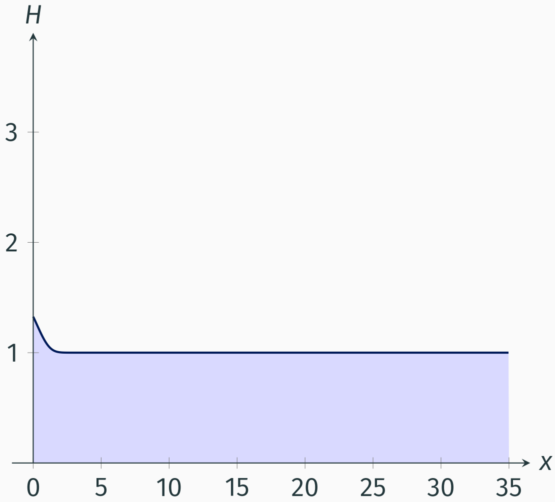
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



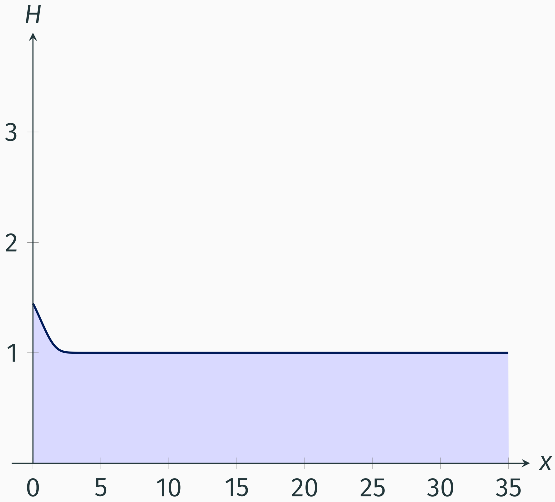
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



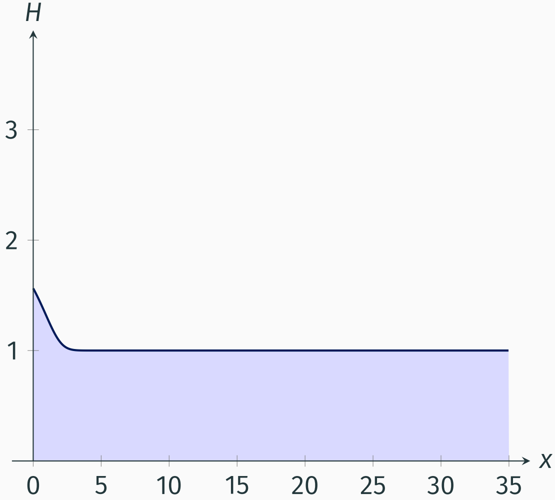
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



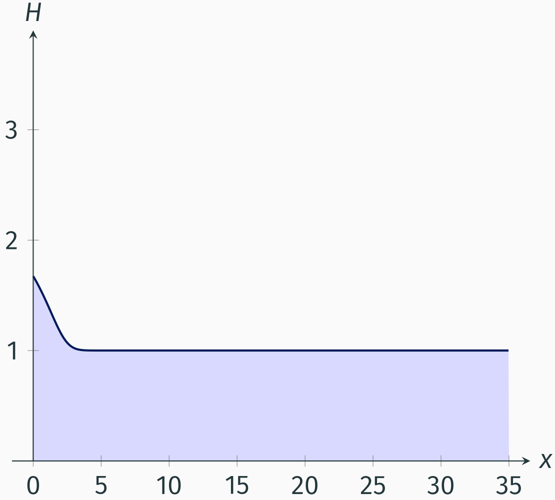
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



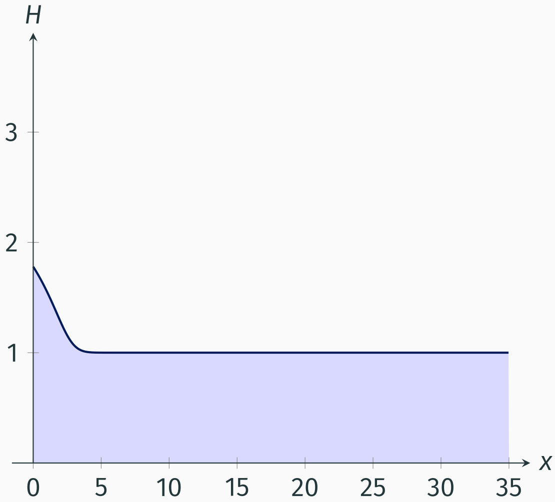
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



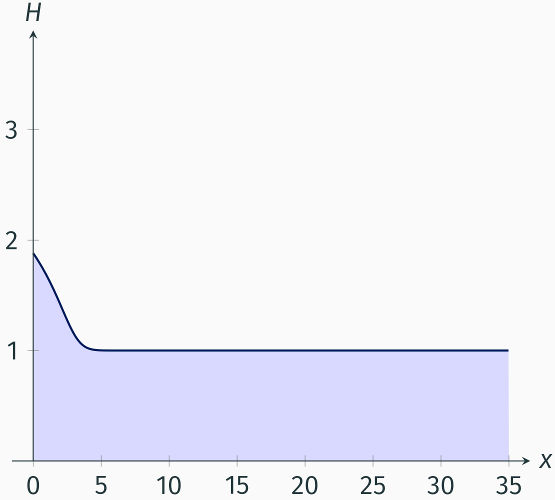
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



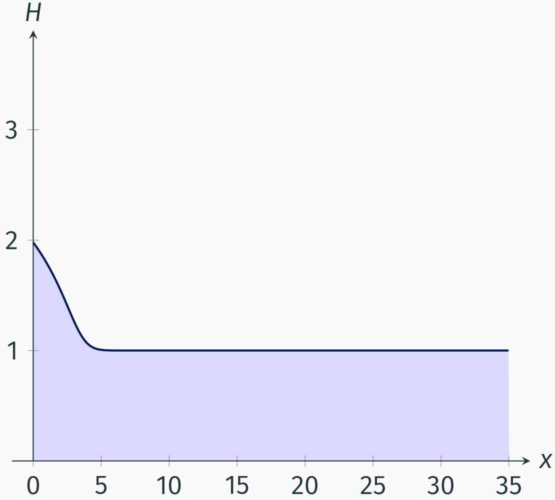
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



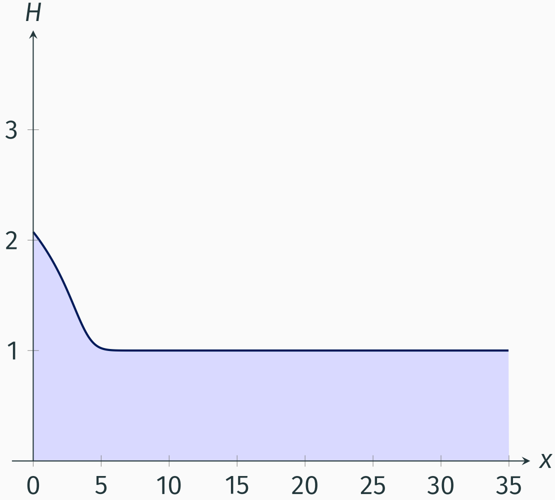
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



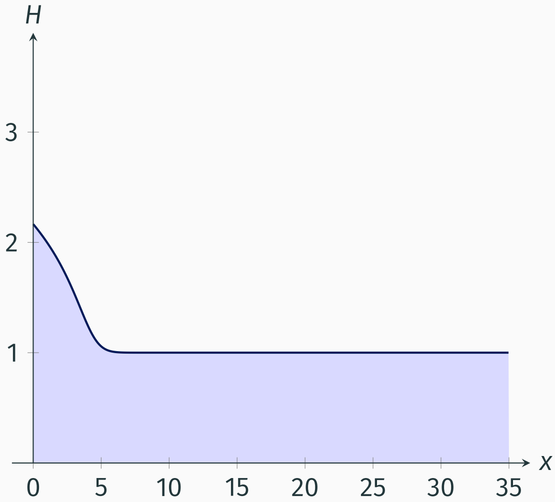
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



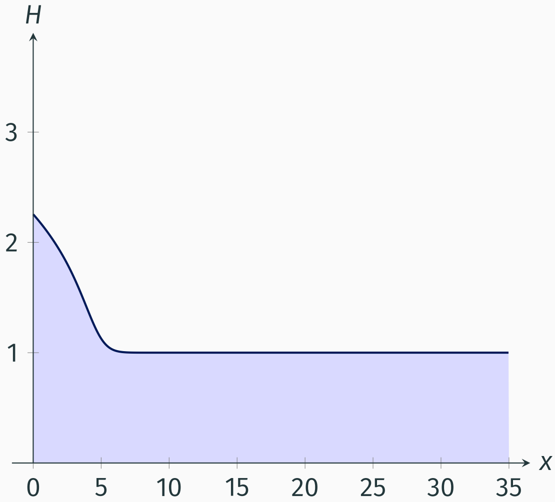
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



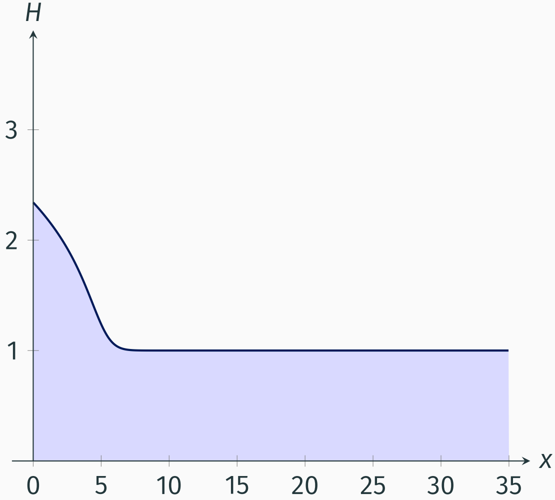
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



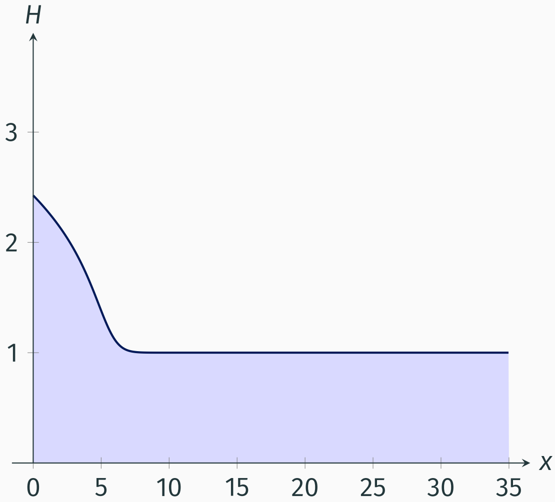
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



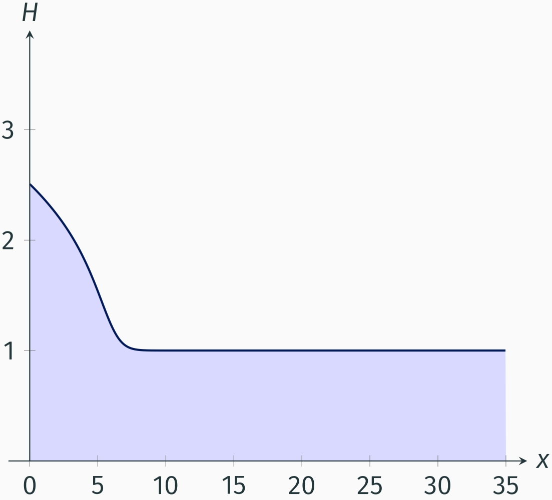
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



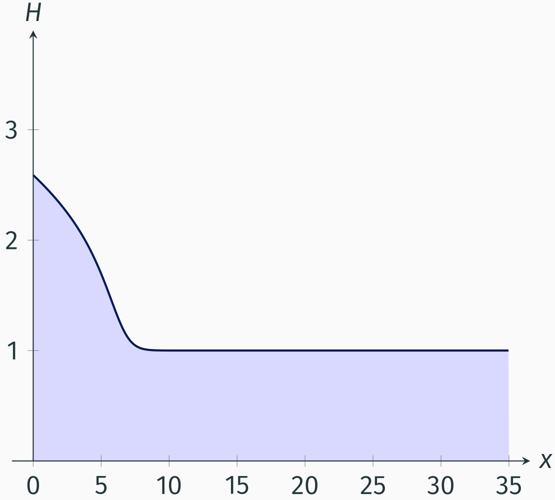
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



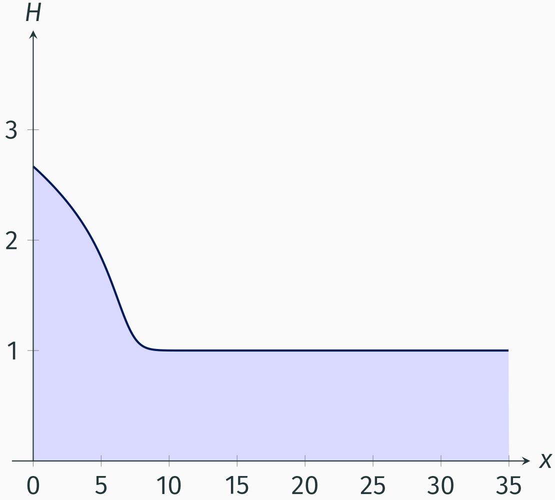
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



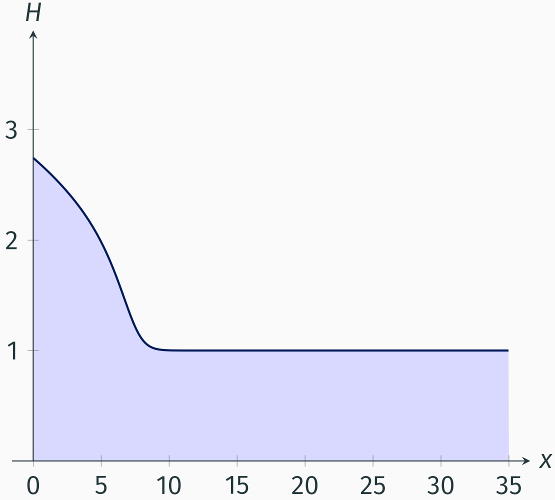
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



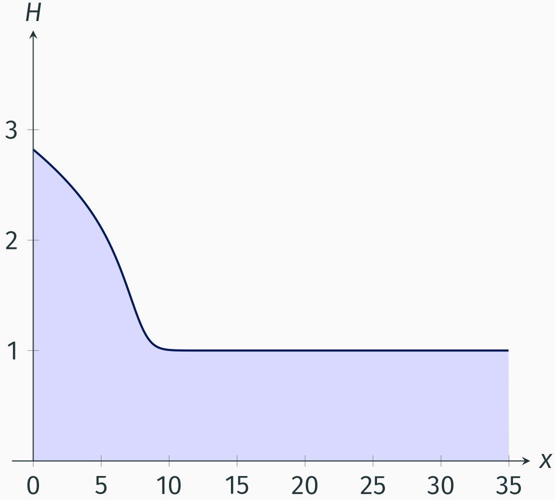
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



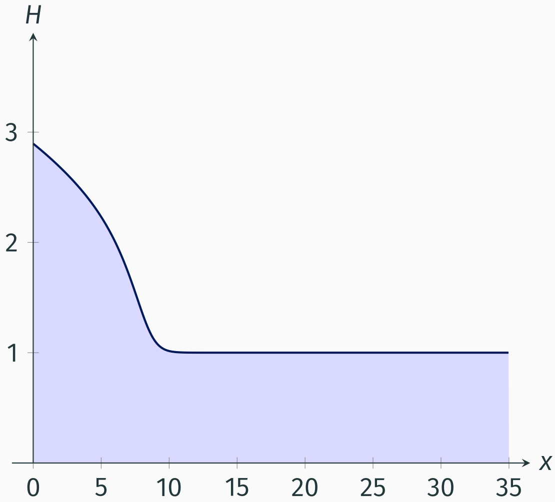
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



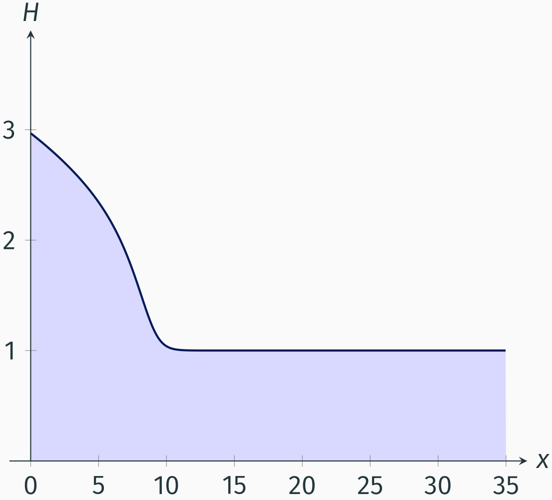
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



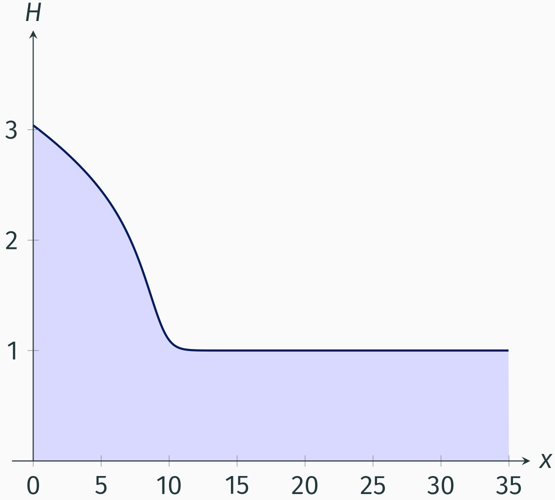
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



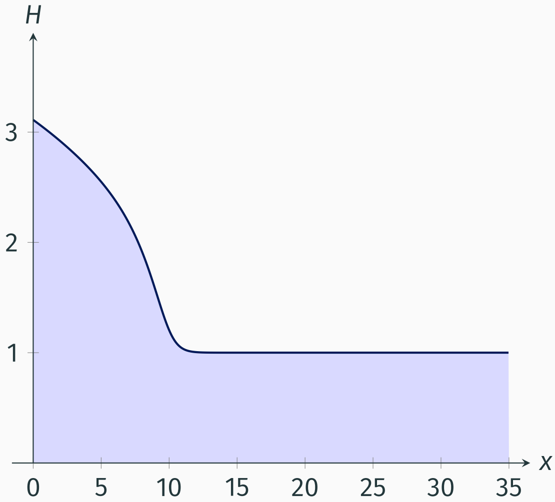
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



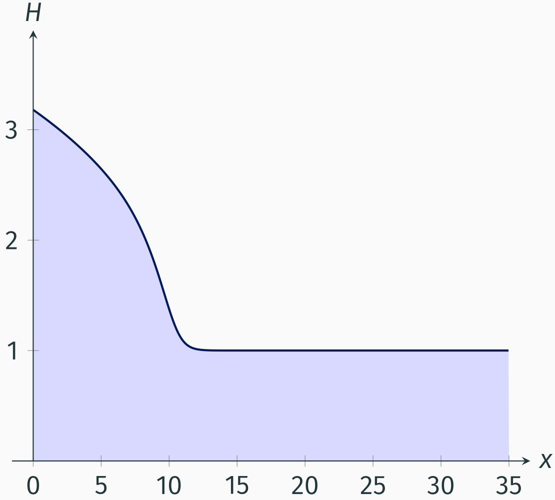
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



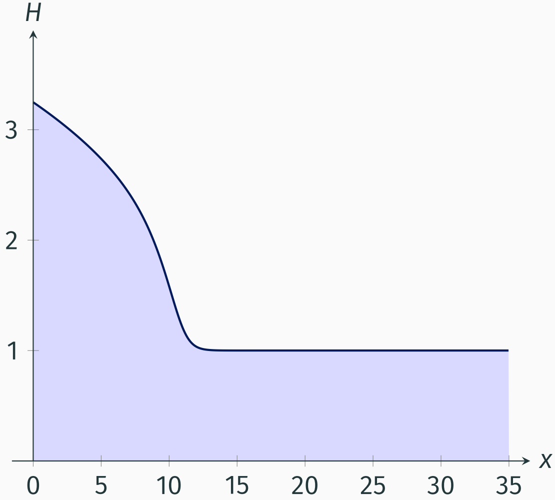
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



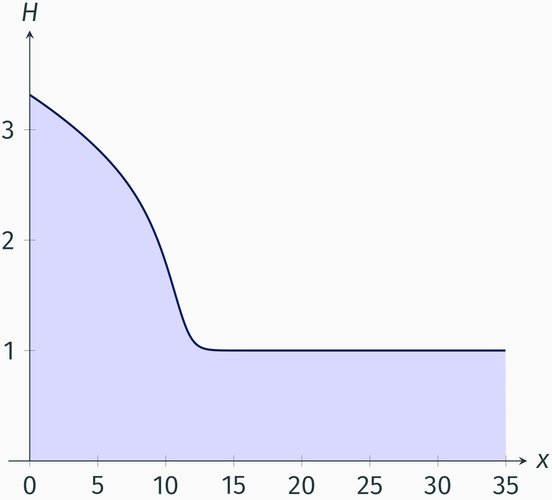
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



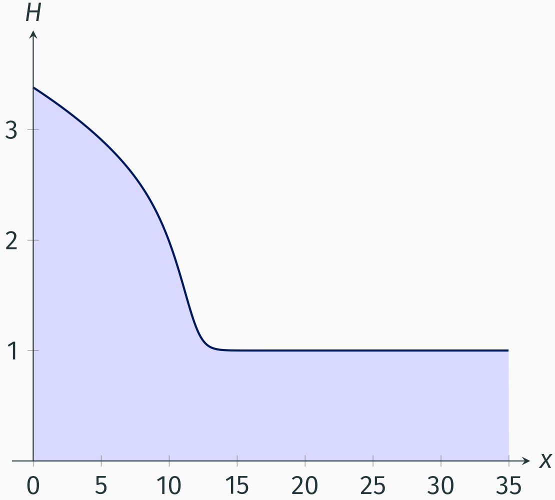
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



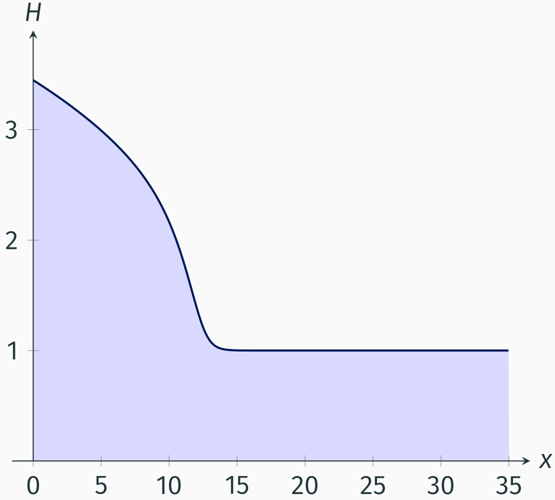
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



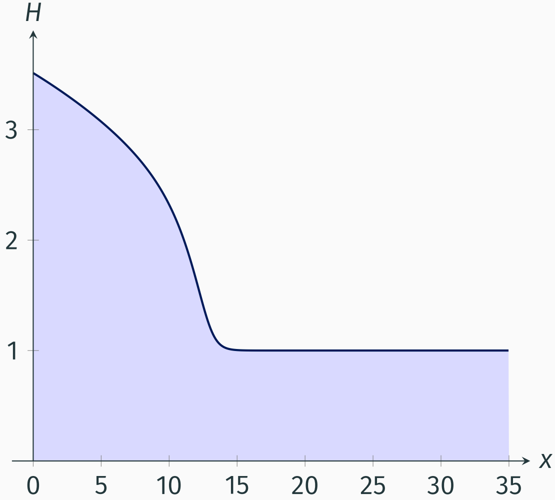
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



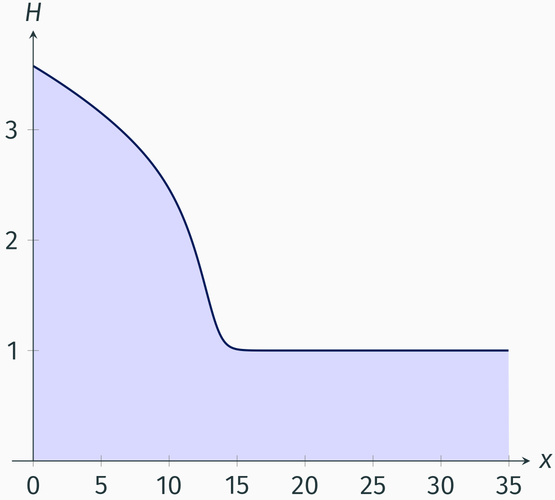
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



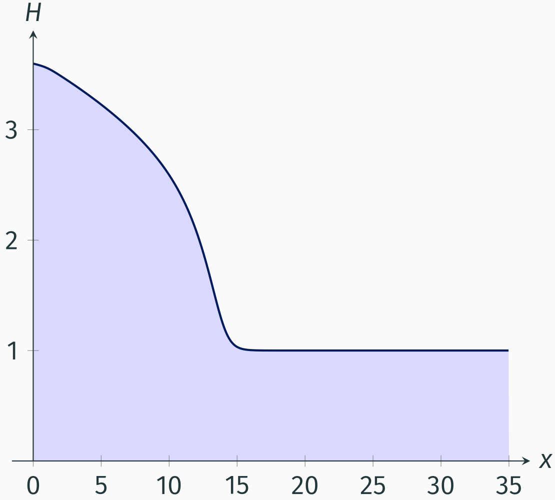
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



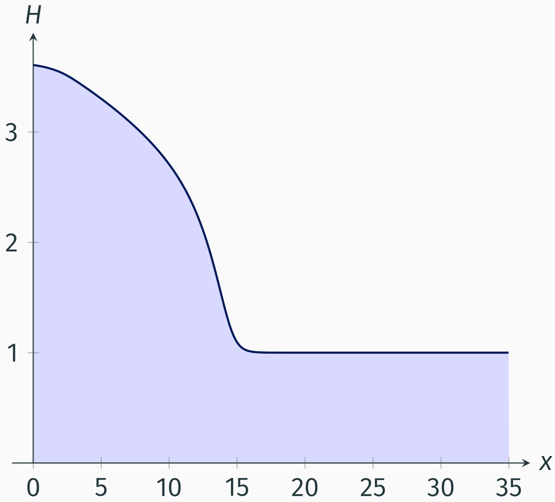
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



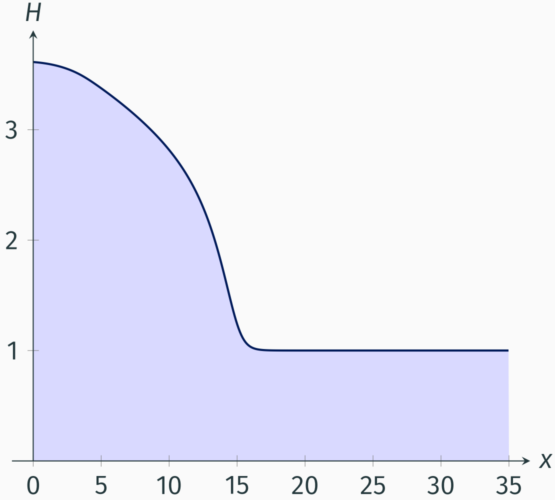
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



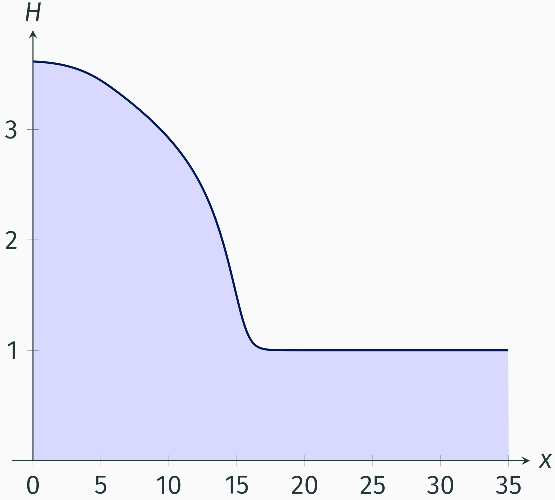
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



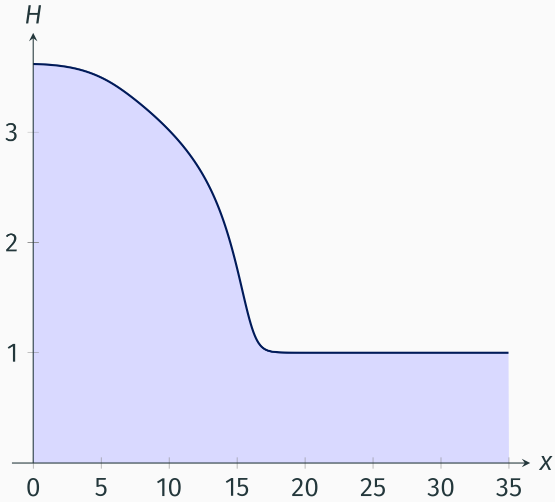
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



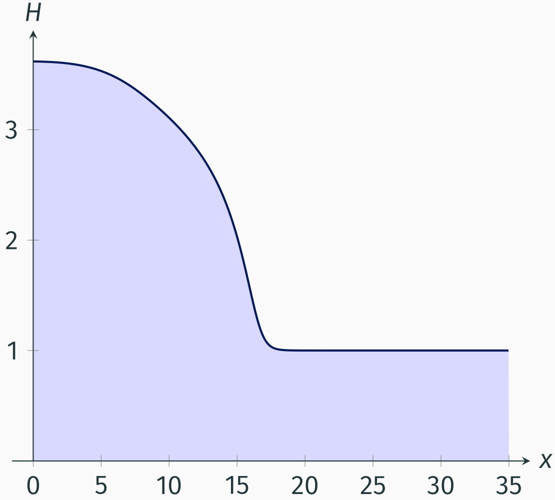
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



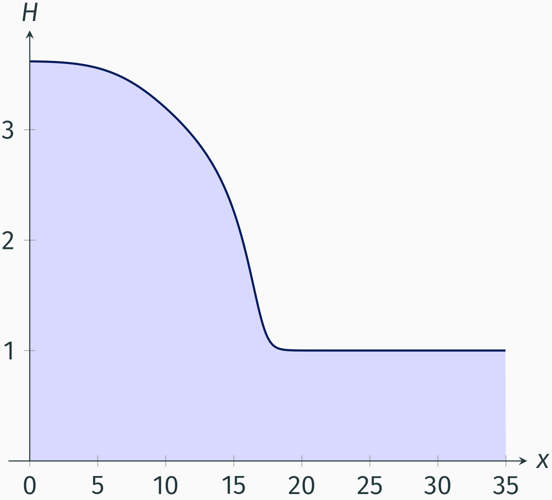
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



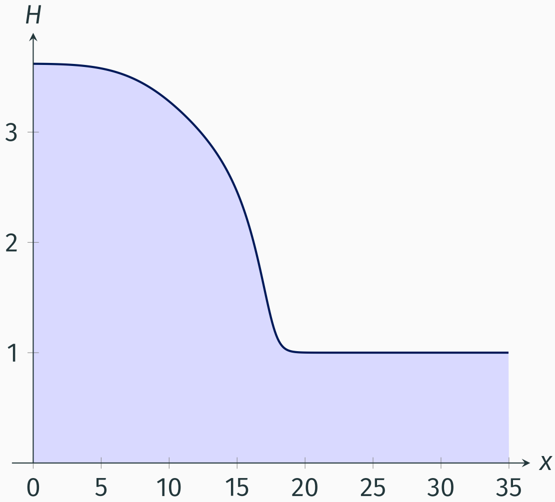
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



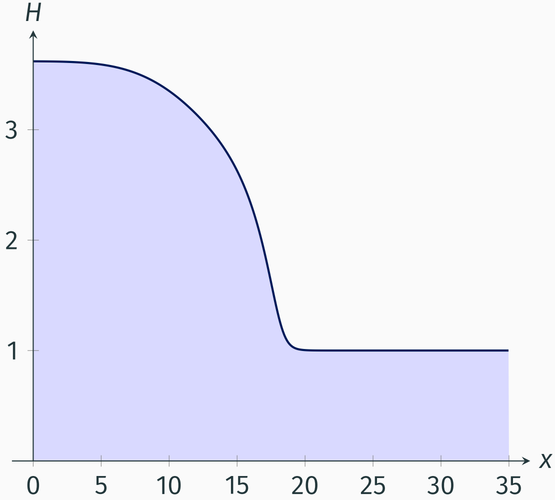
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



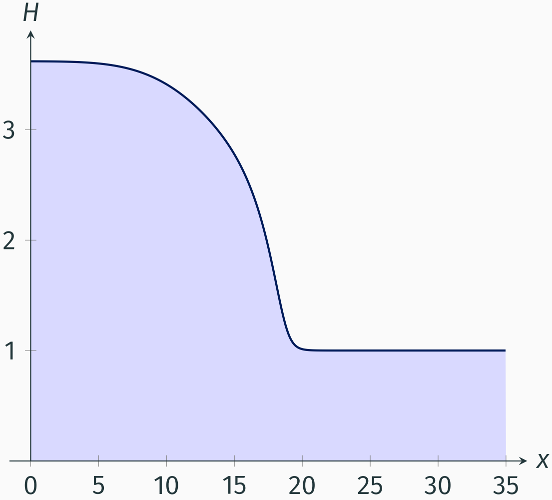
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



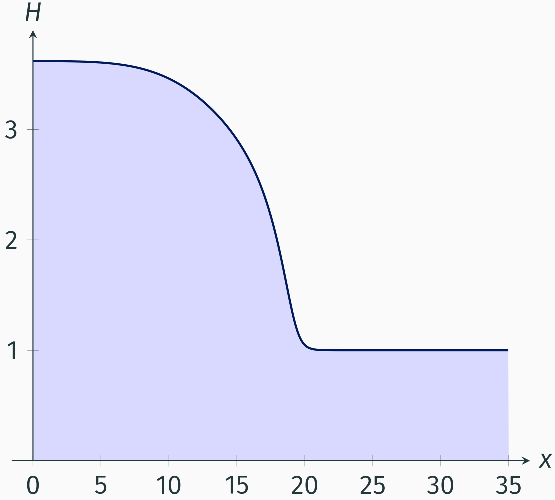
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



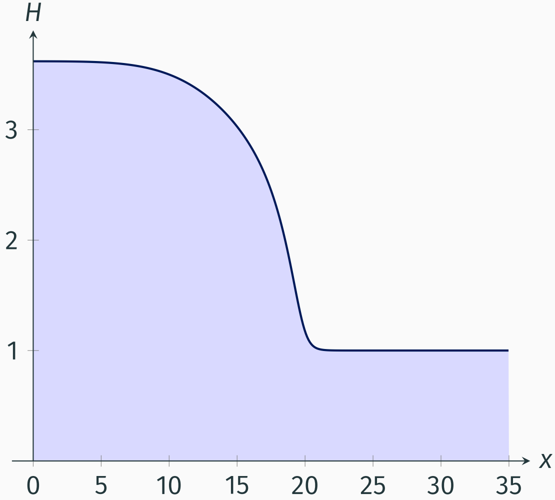
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



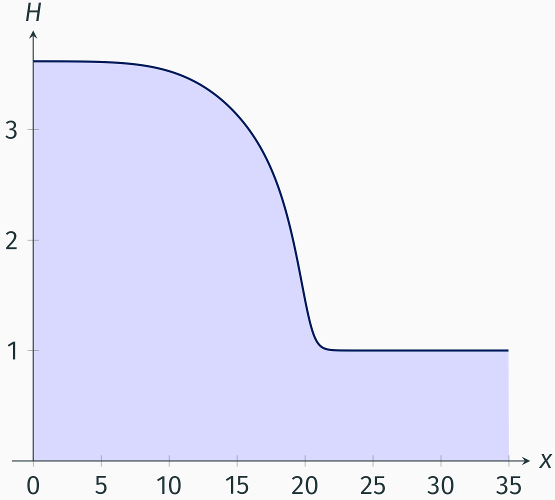
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



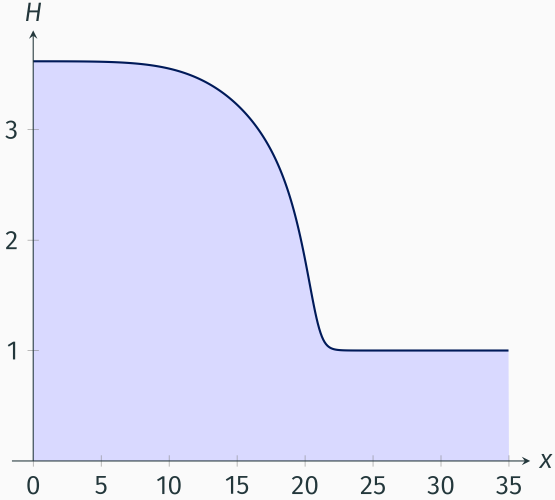
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



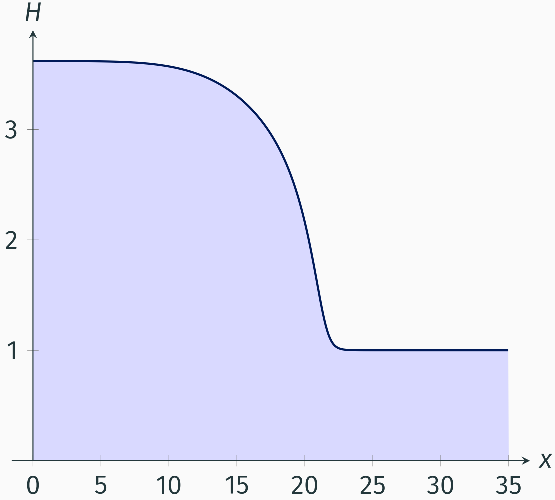
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



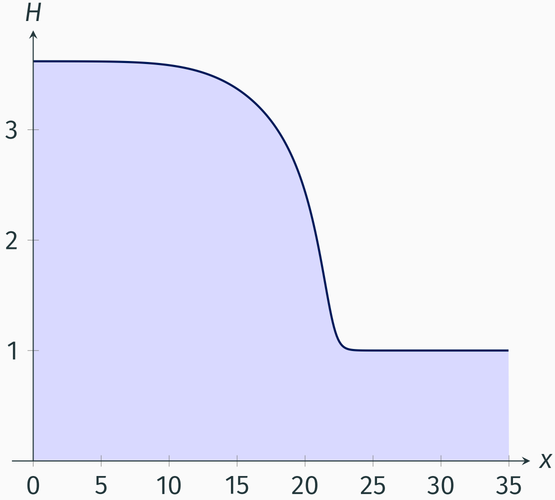
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



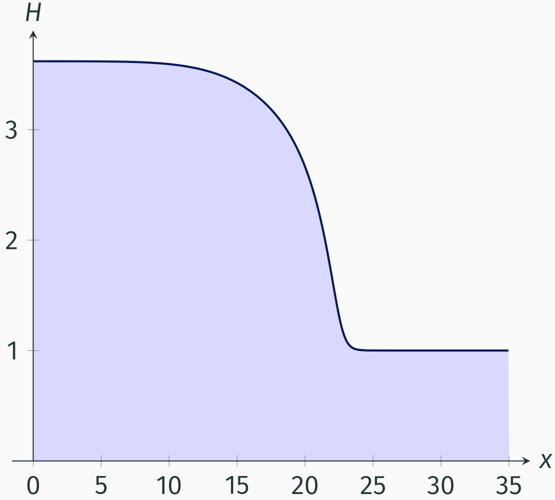
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



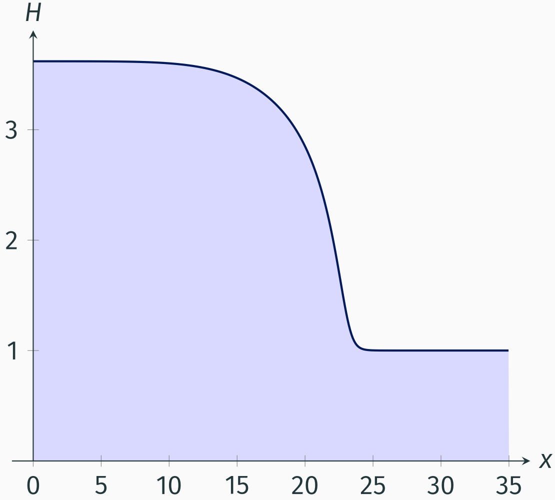
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



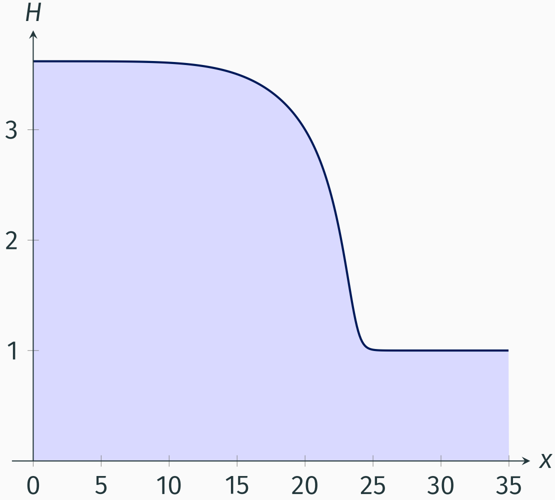
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



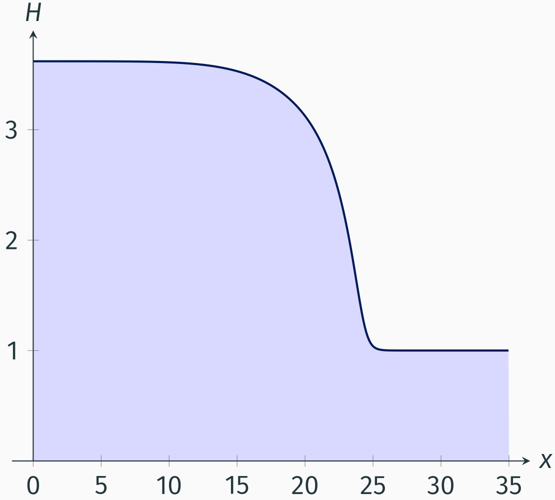
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



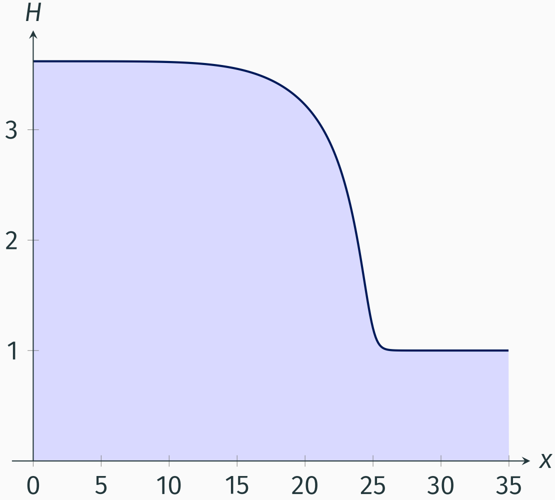
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



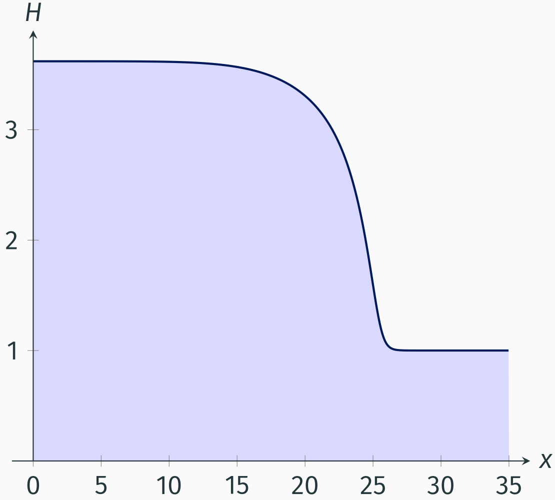
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



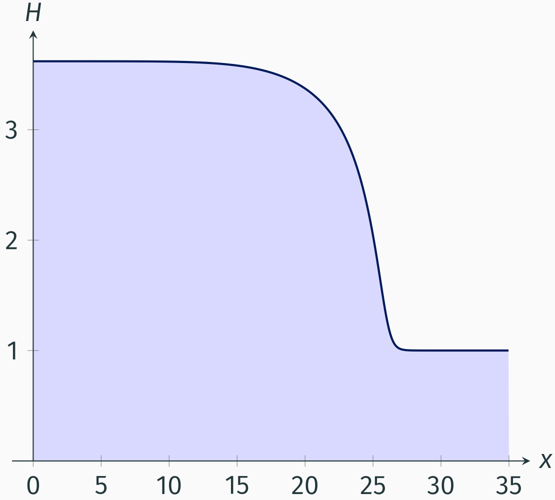
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



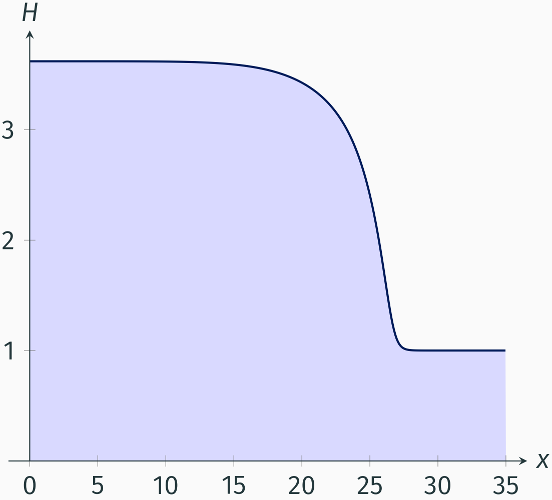
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



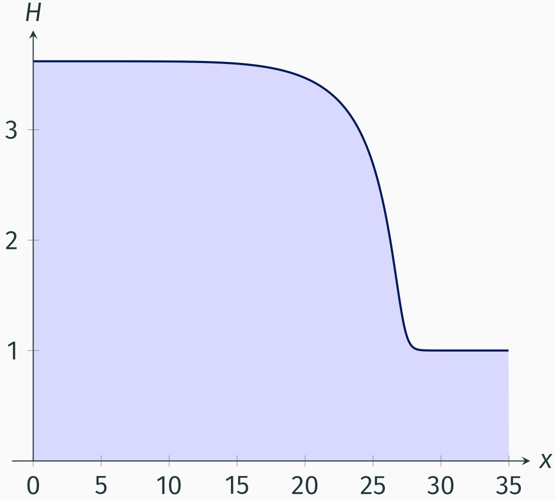
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



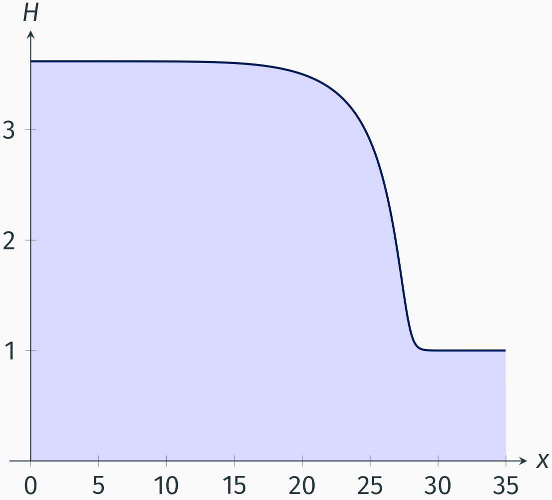
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



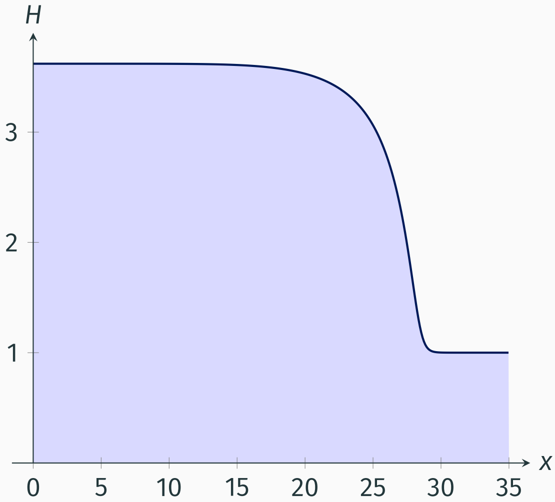
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



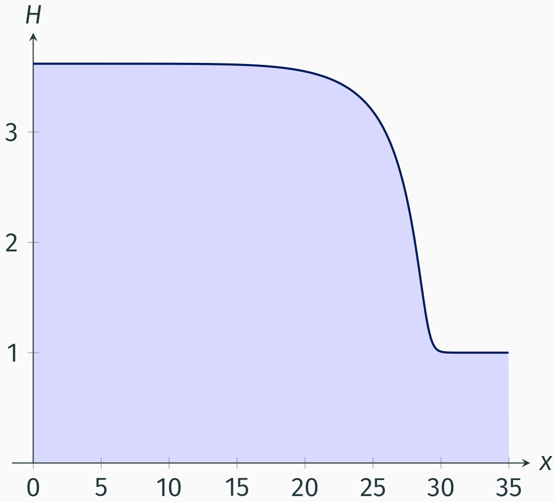
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



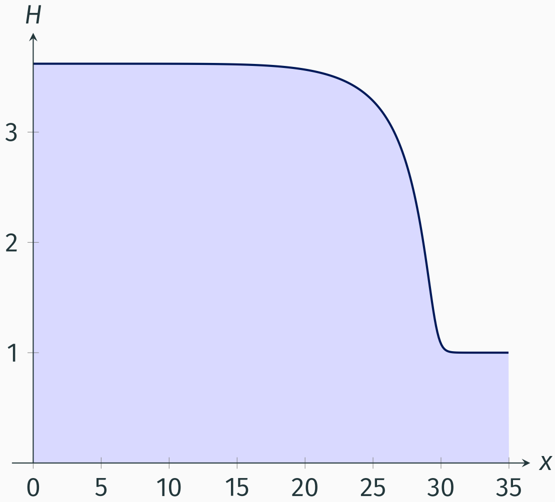
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



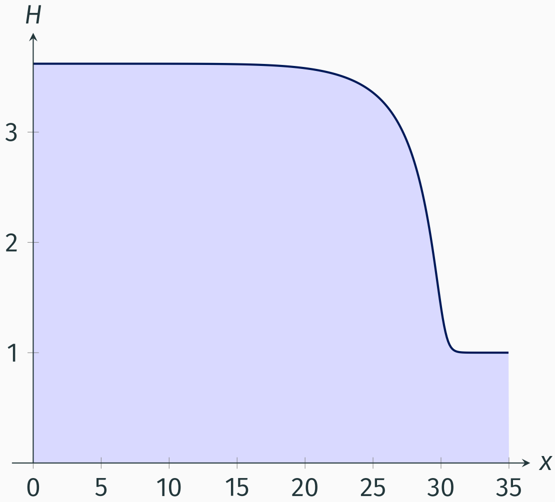
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



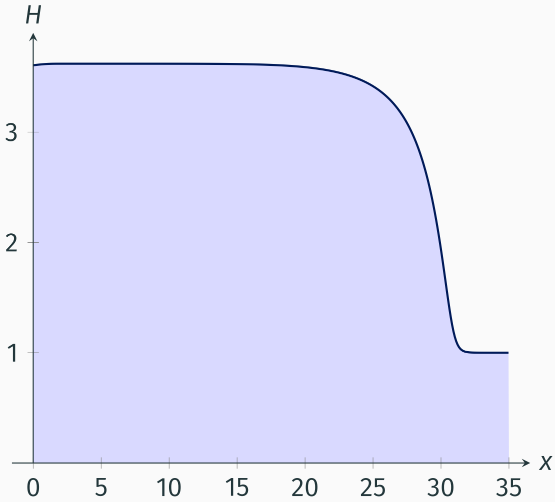
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



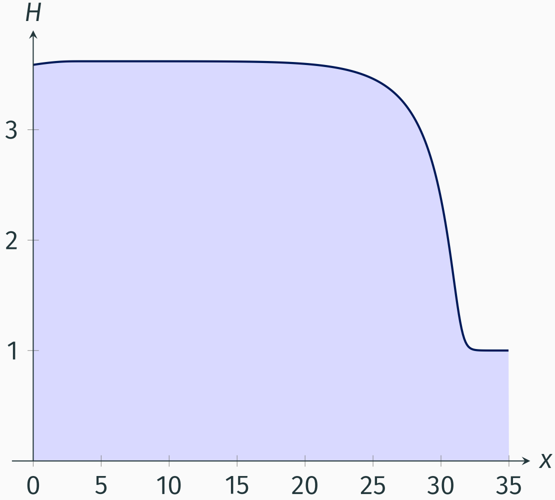
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



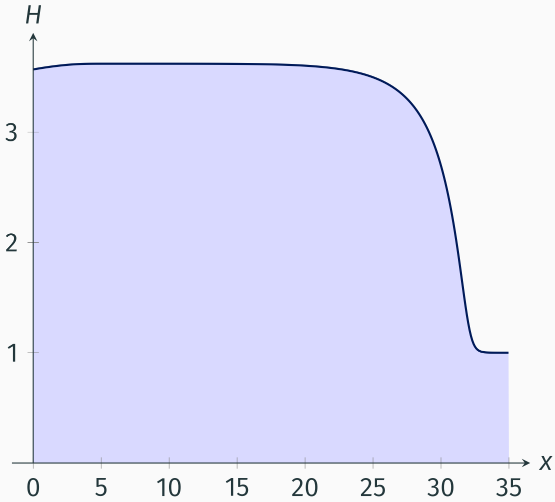
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



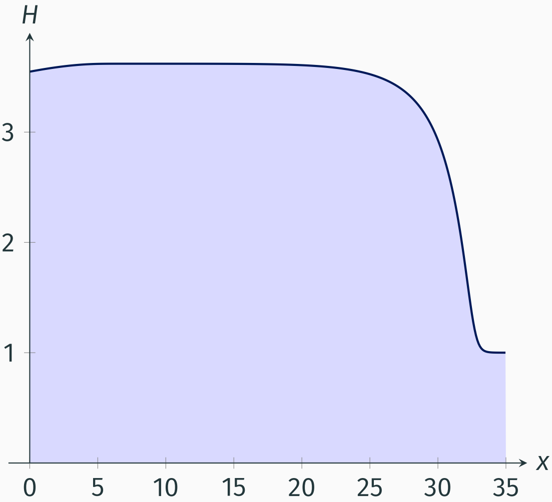
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

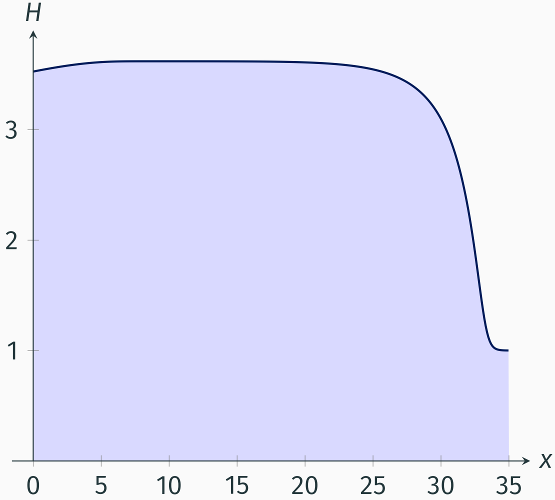


Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

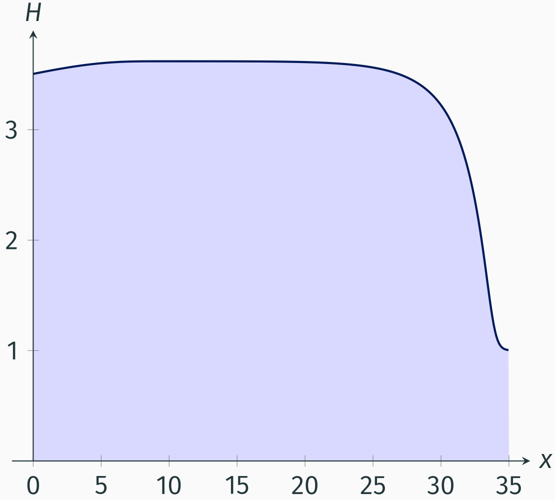


We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

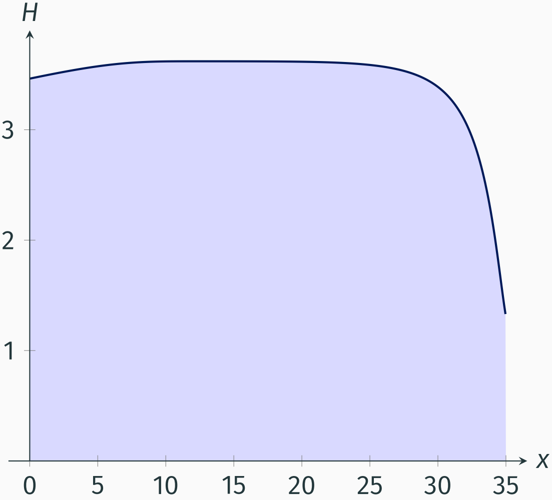


Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

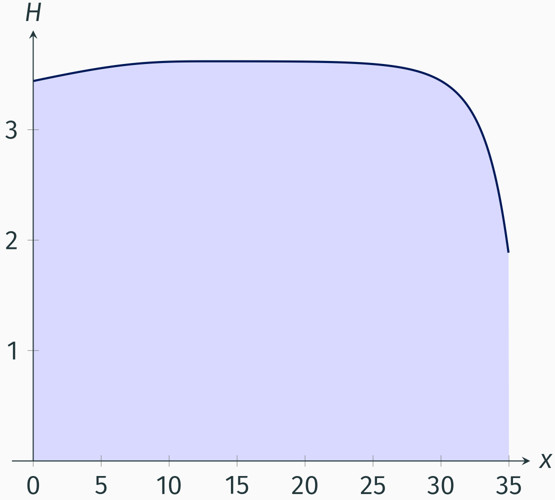
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



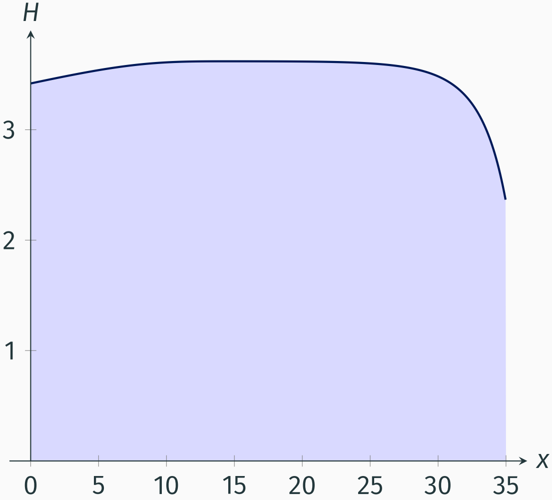
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



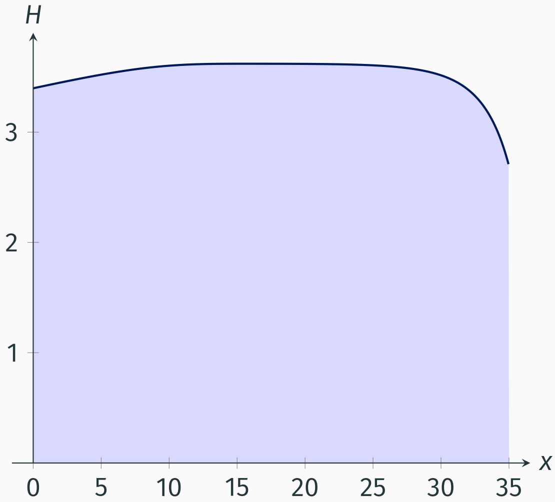
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



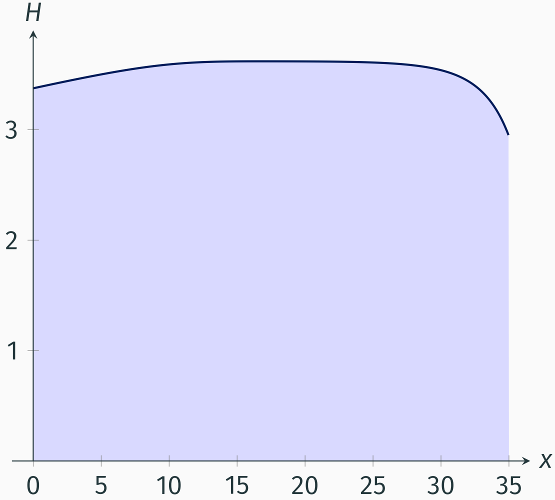
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



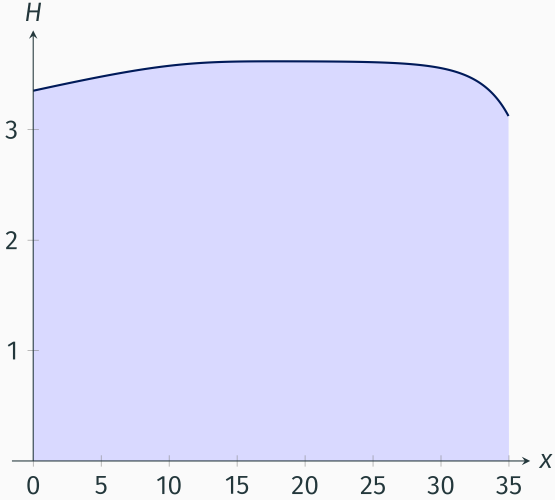
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



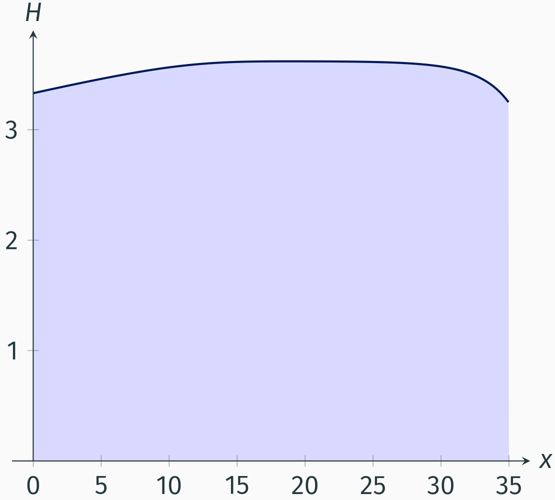
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



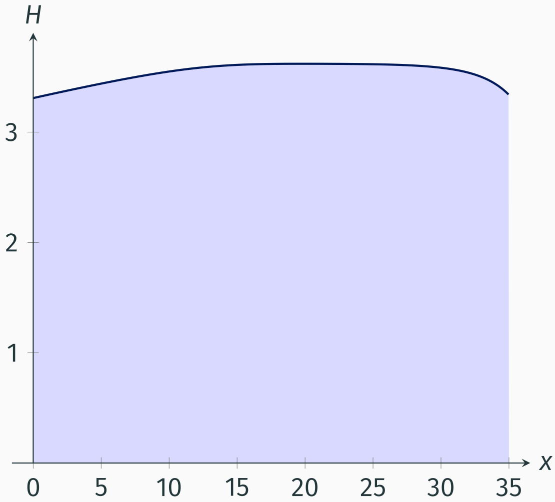
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



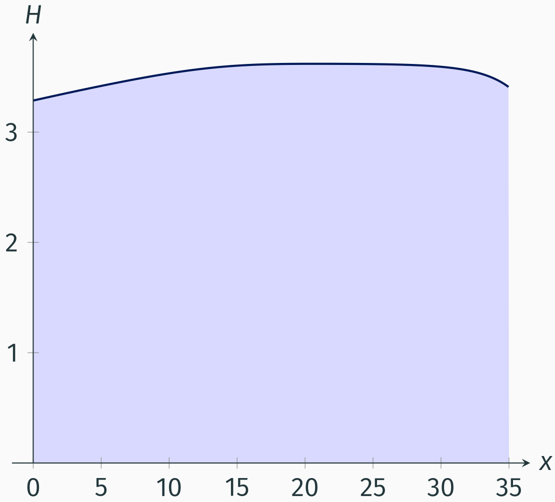
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



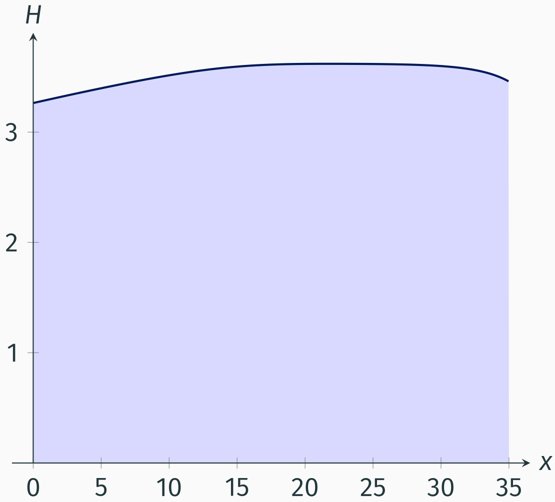
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



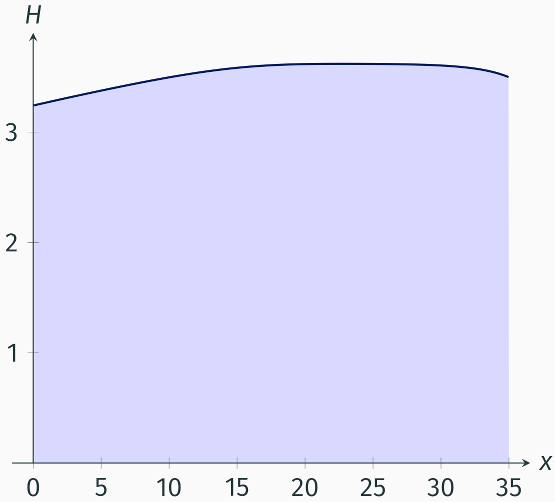
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



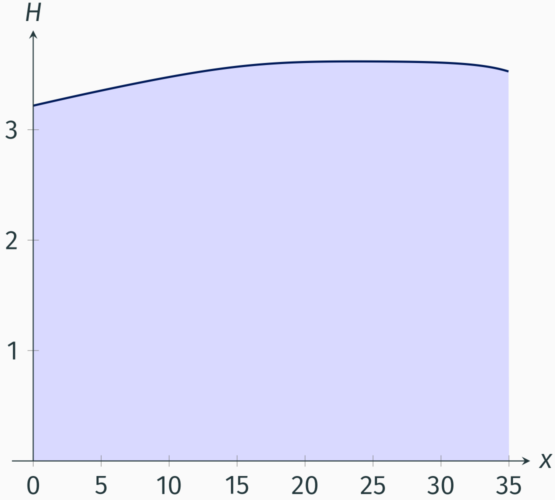
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



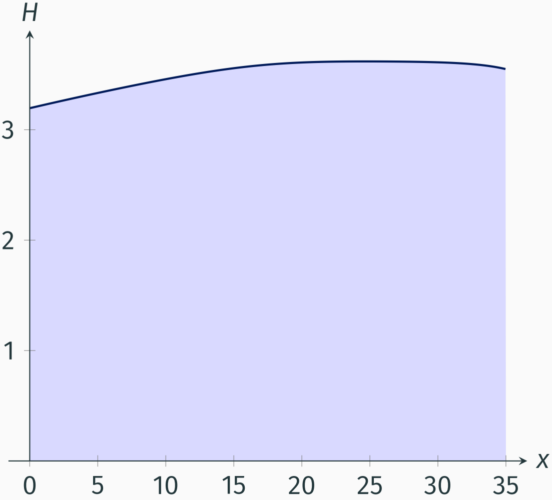
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



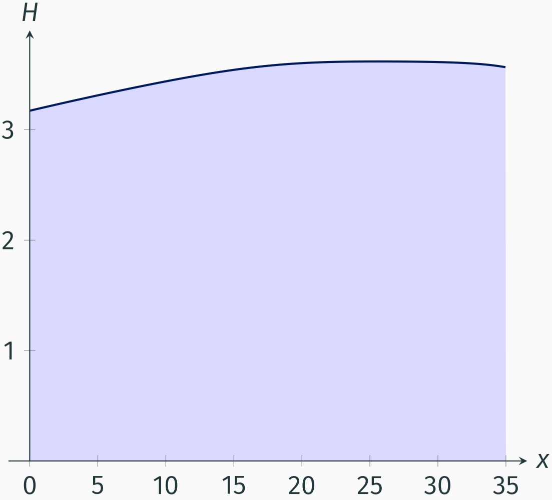
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



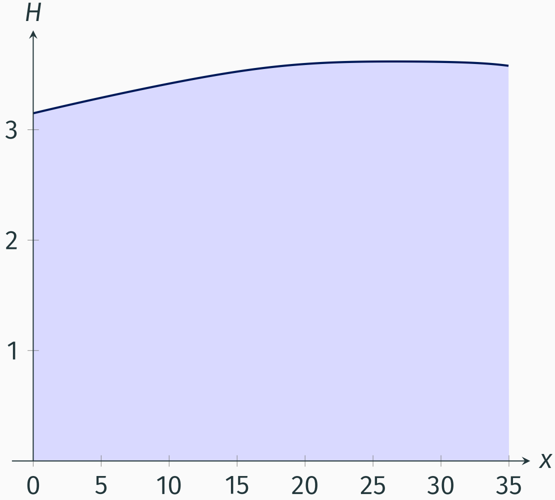
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



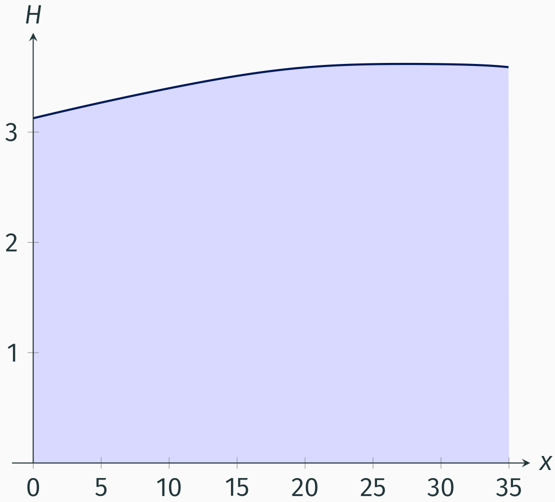
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



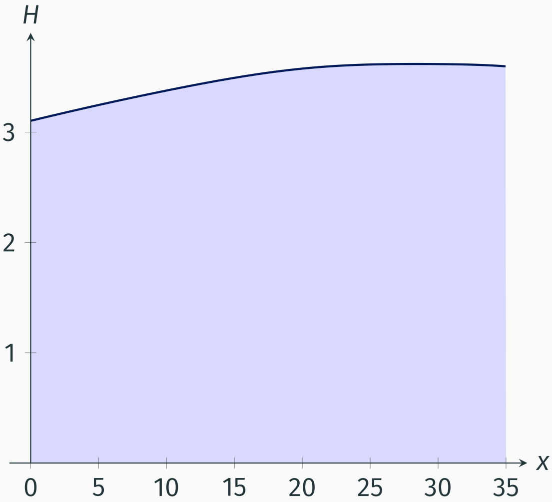
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



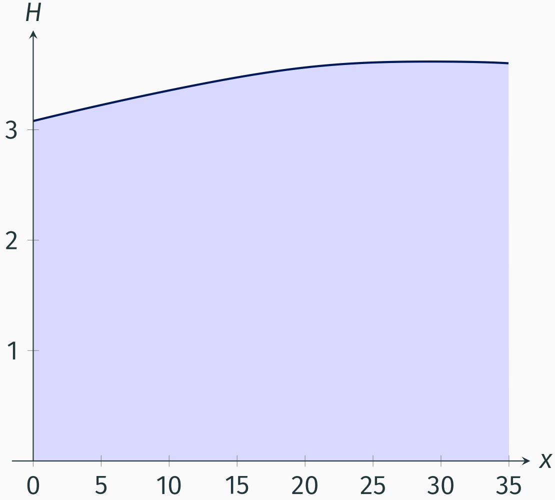
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



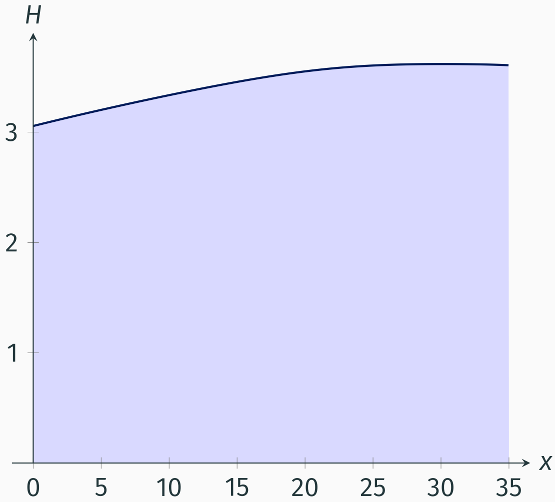
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



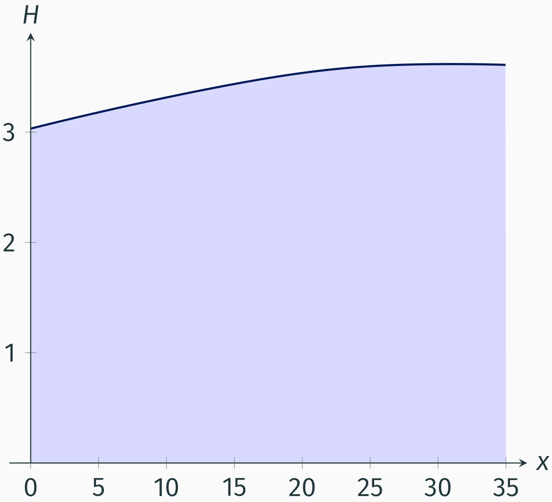
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



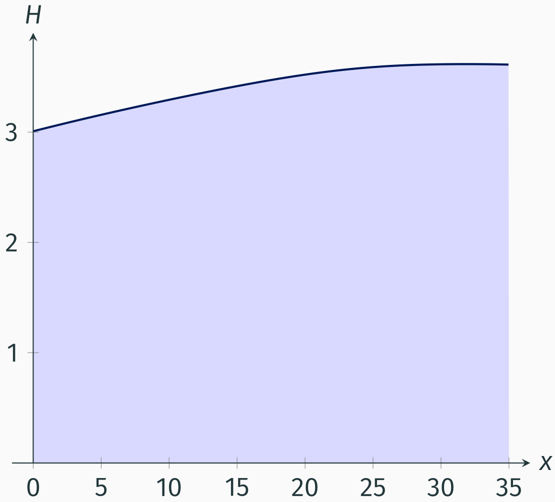
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



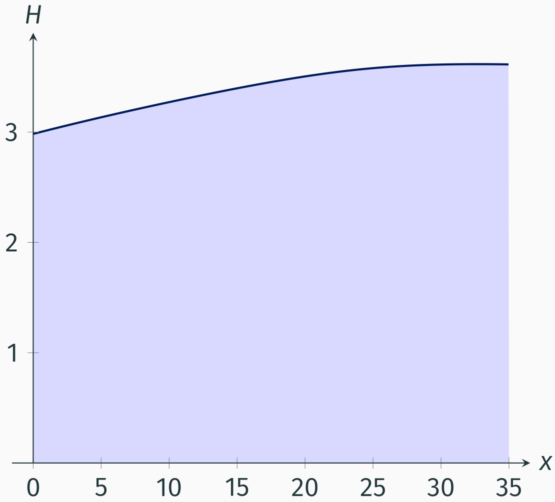
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

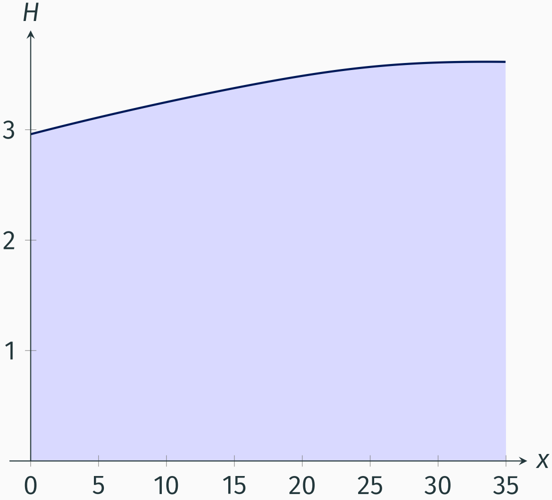


Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

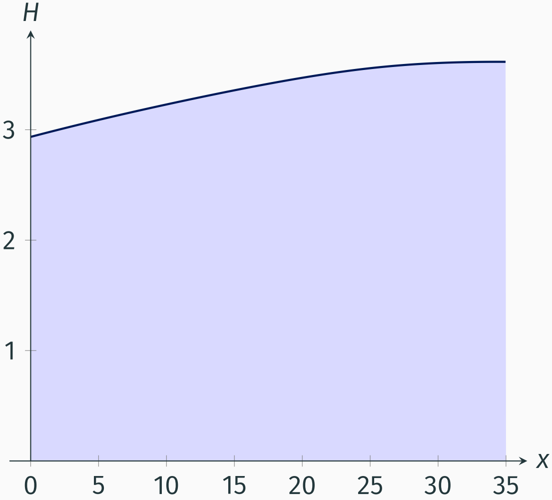


We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



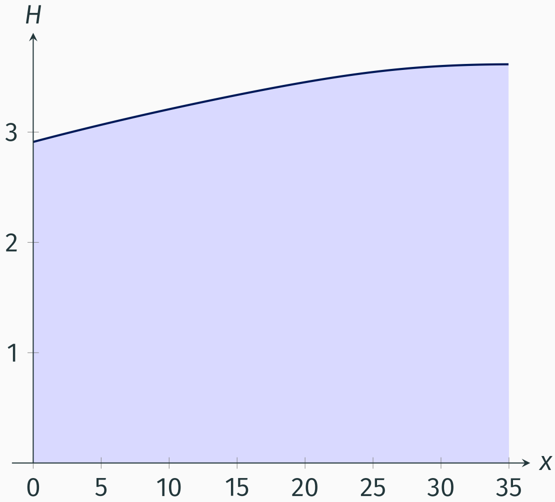
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



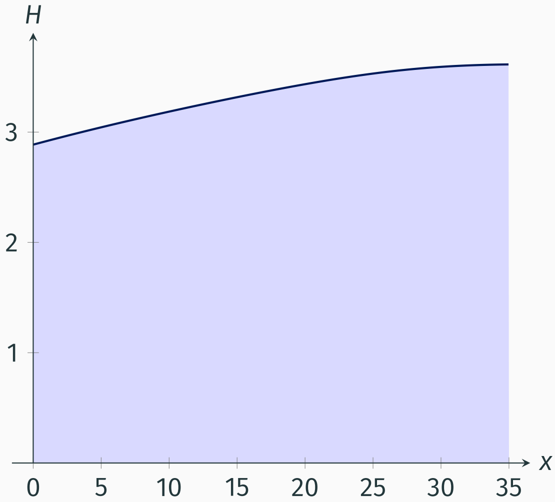
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



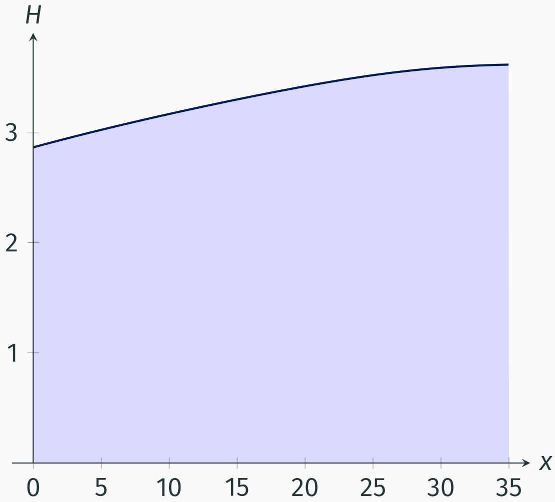
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



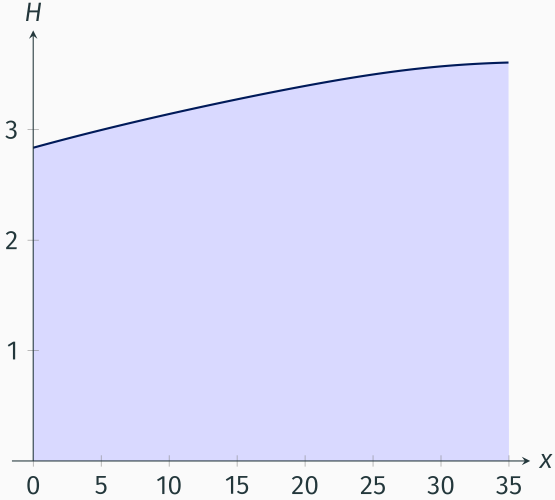
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



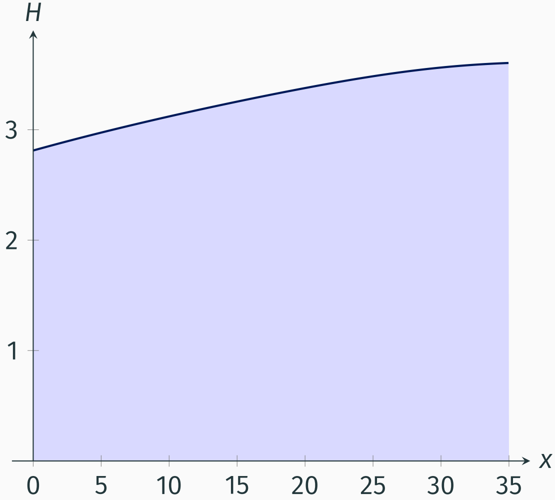
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



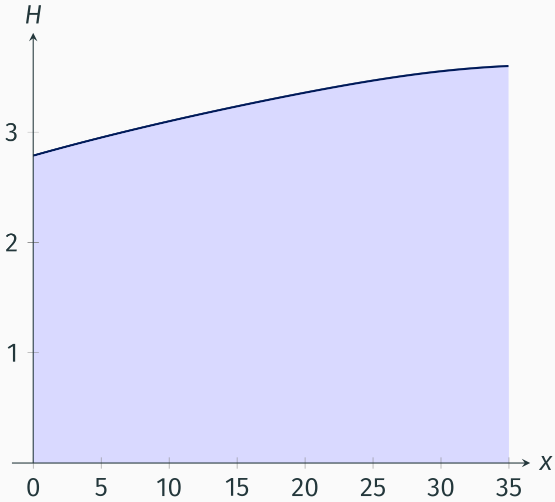
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



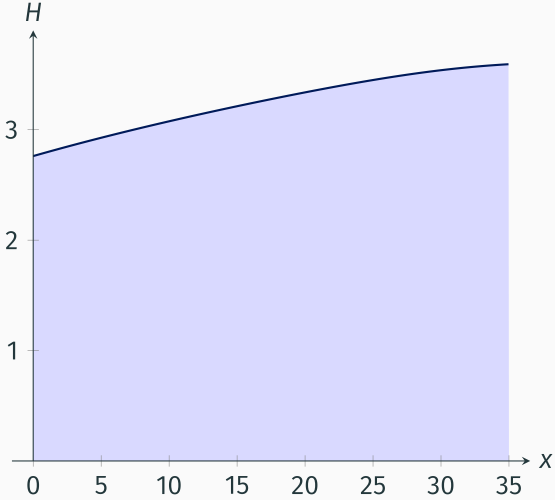
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



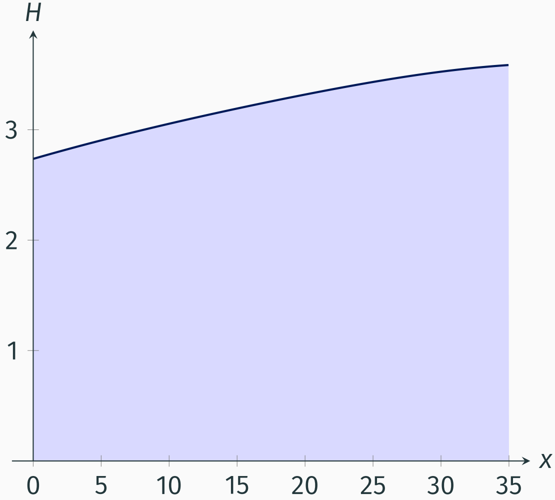
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



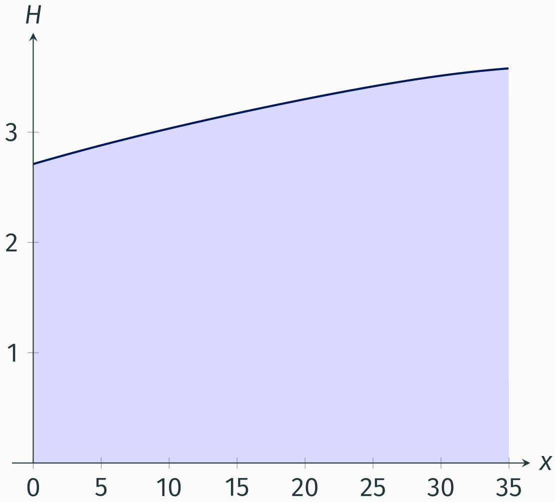
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



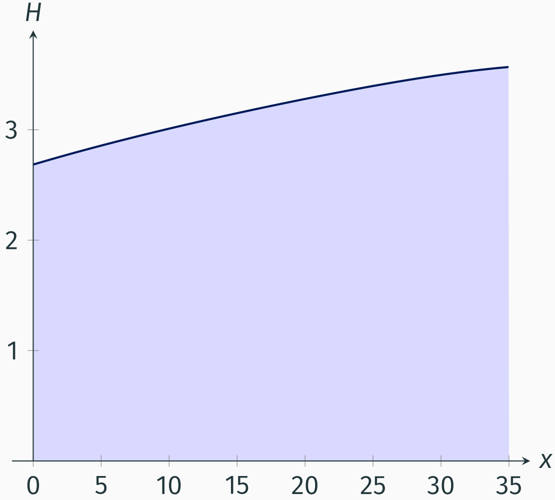
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



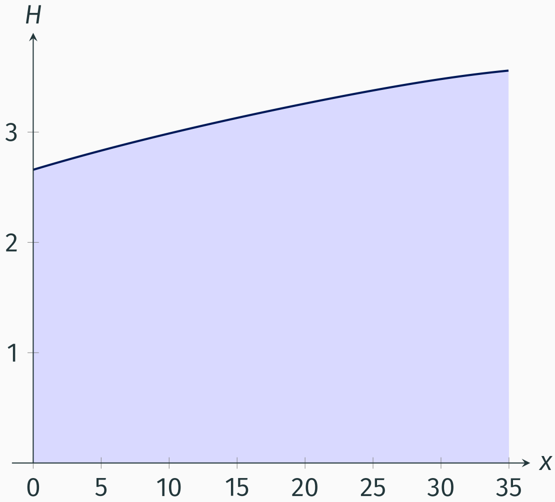
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



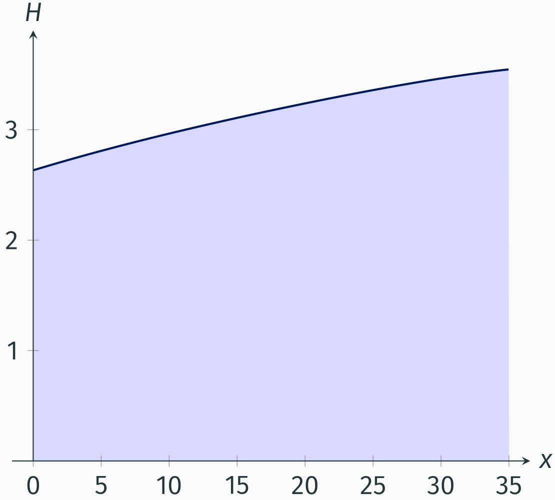
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



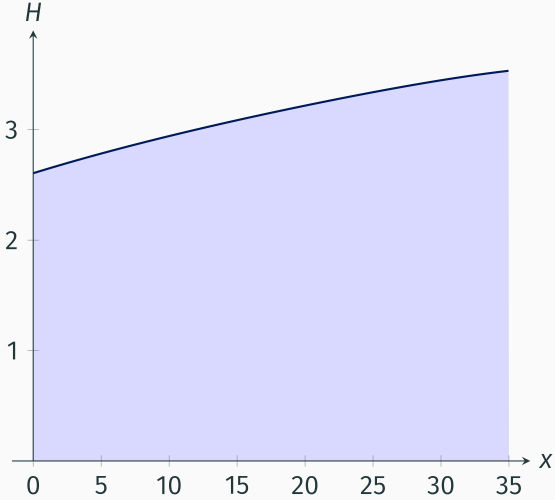
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



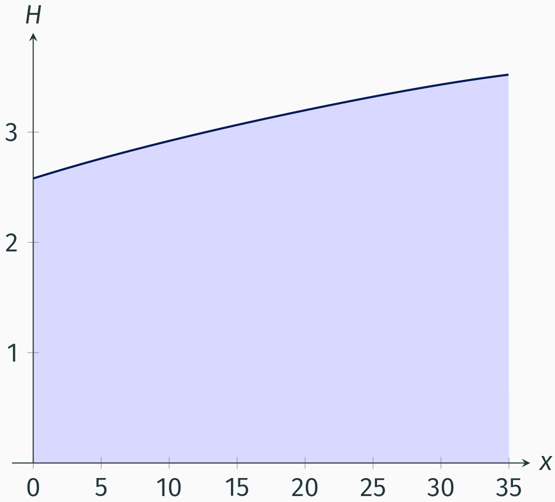
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



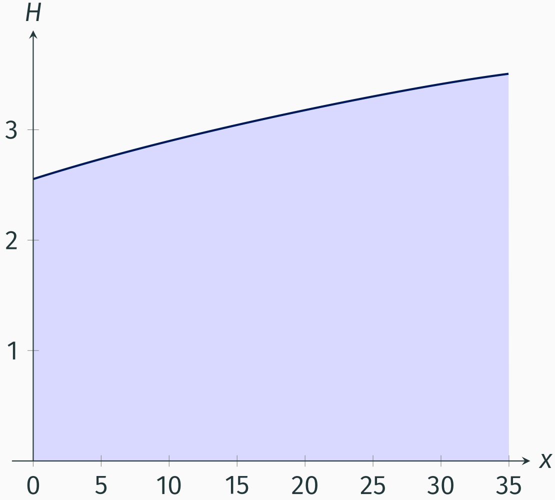
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



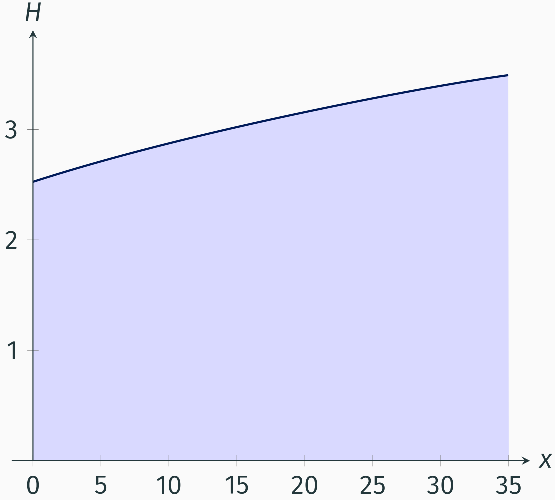
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



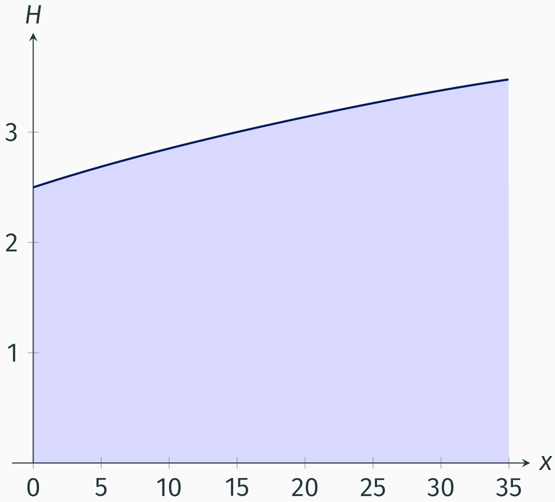
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



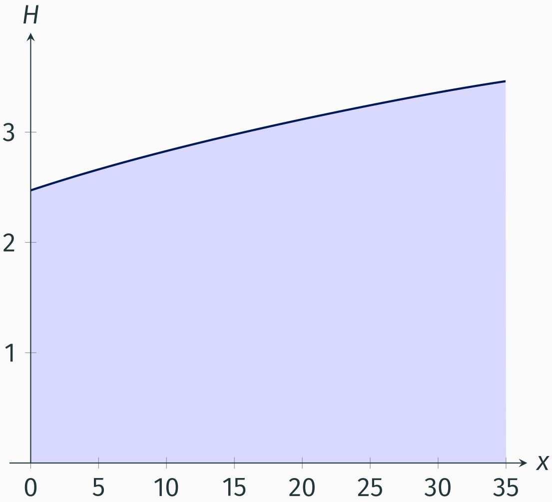
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



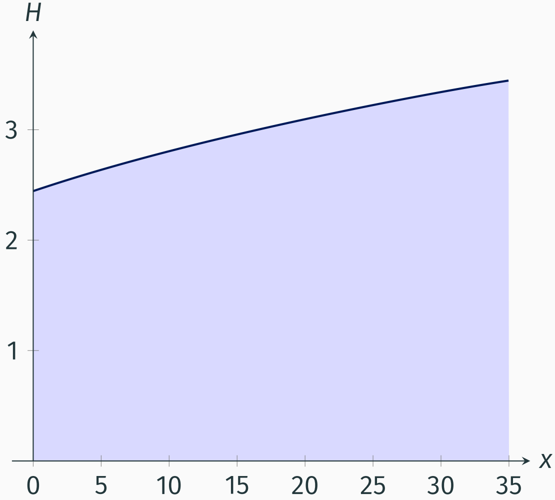
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



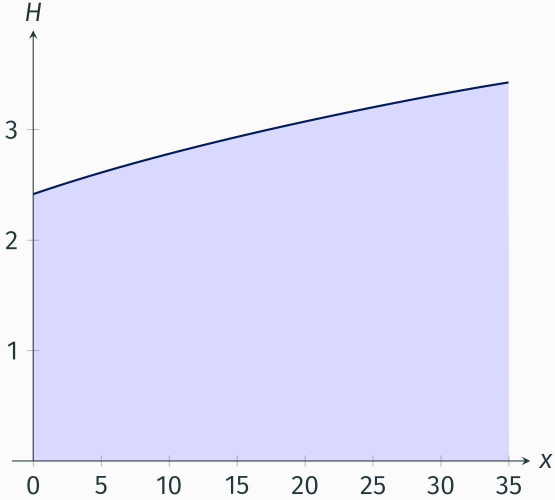
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



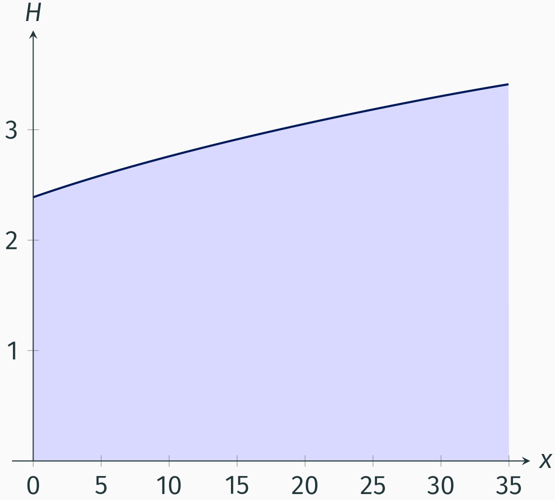
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



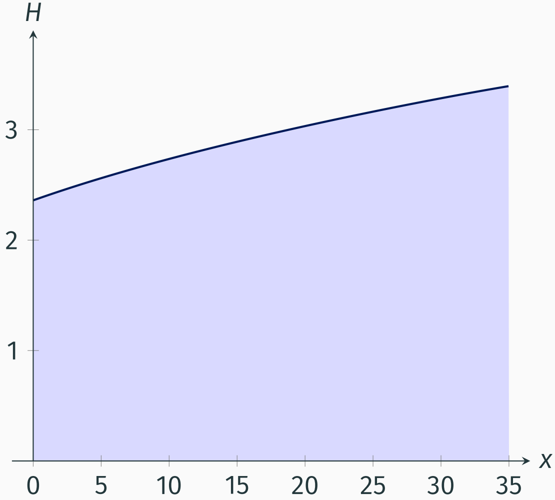
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



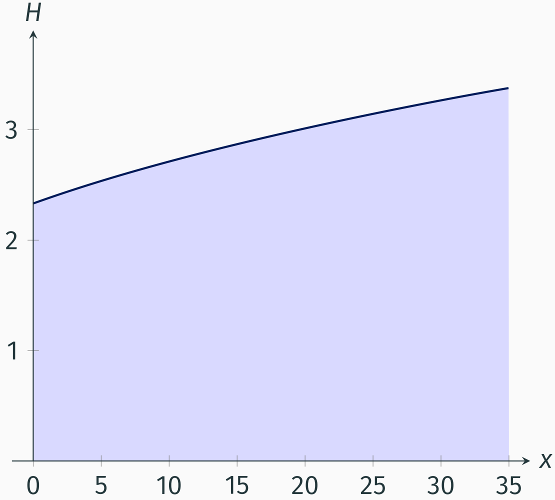
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



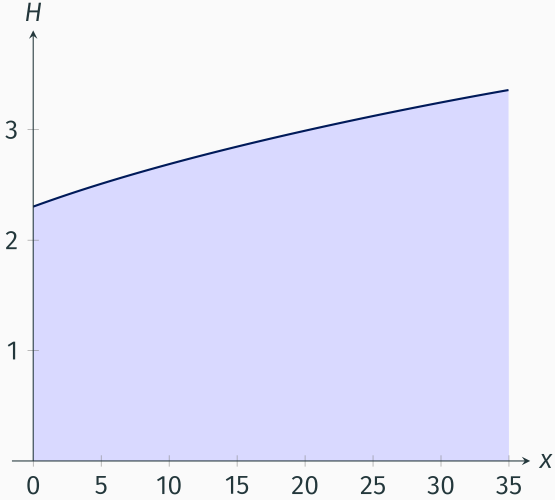
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



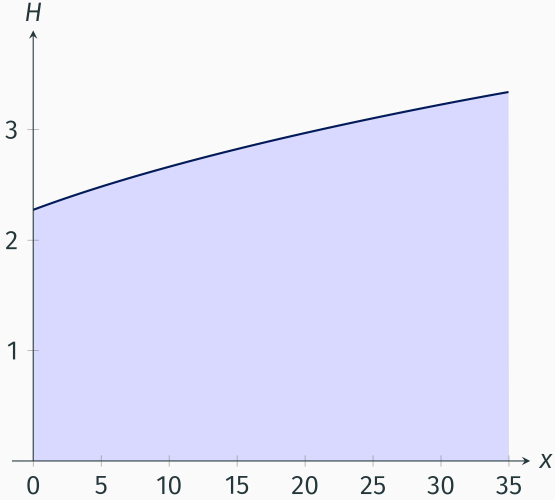
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



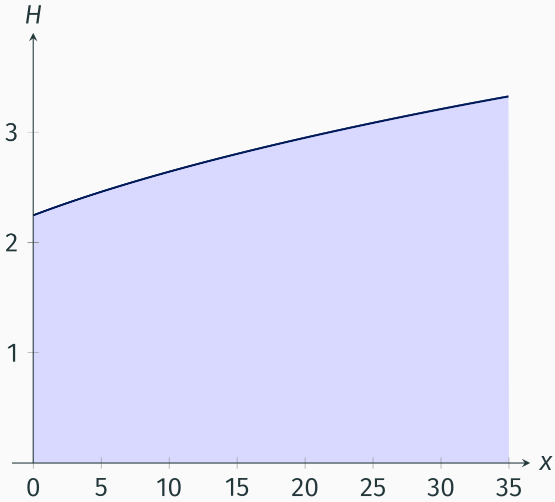
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



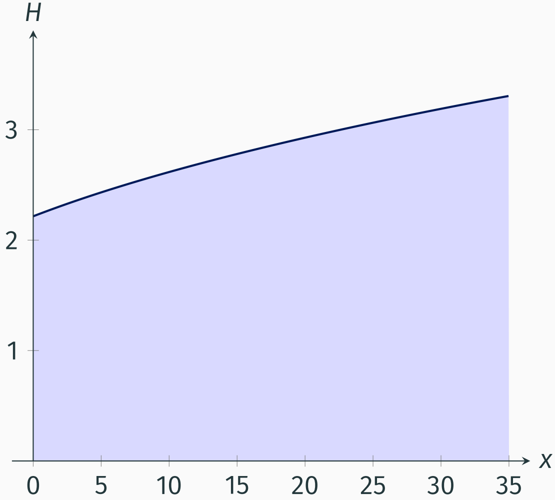
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



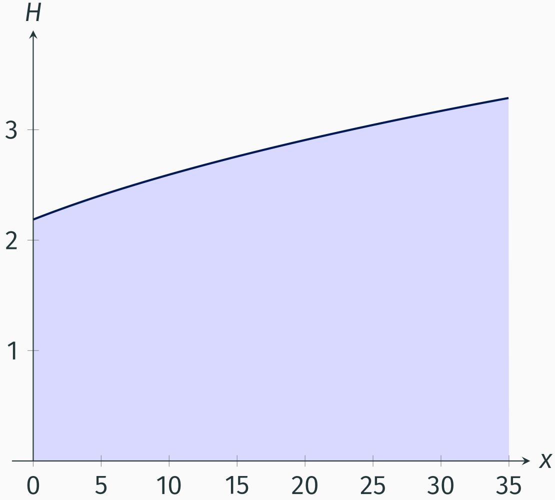
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



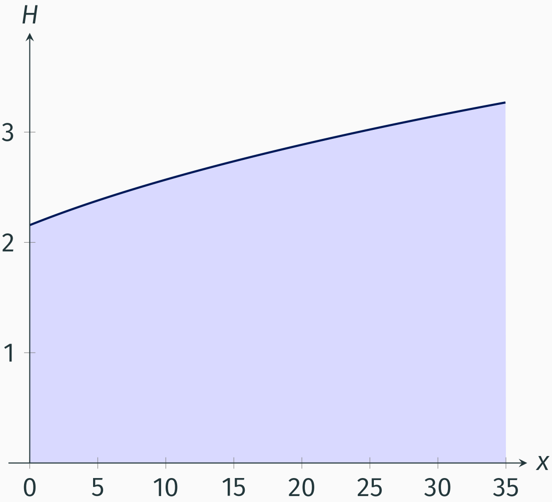
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



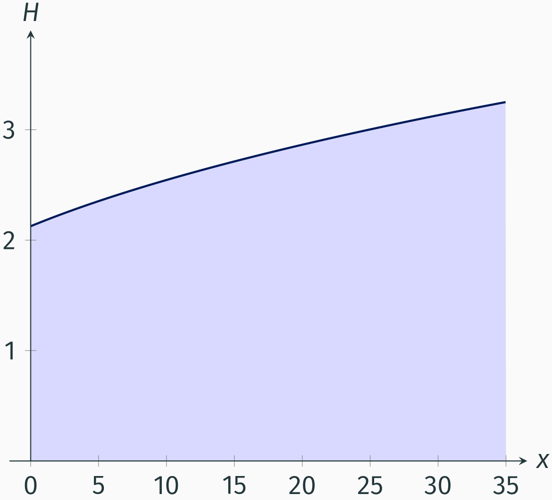
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



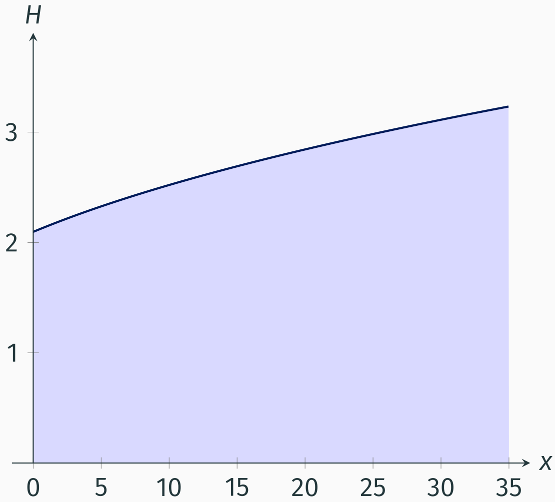
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



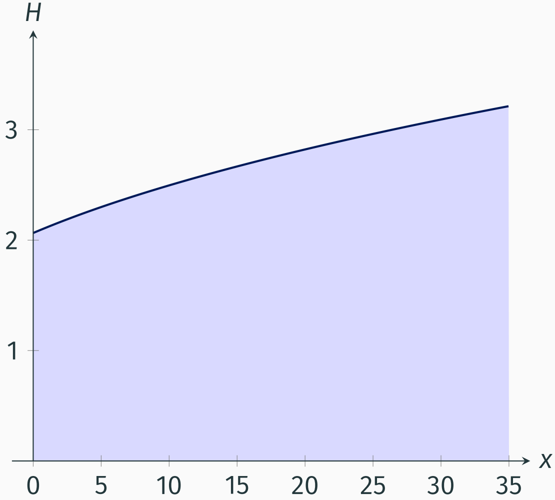
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



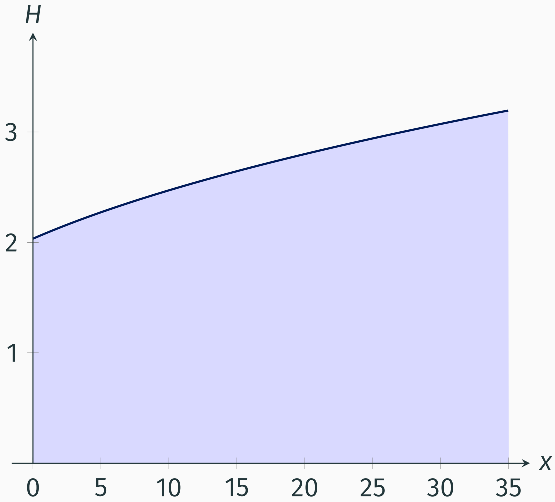
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



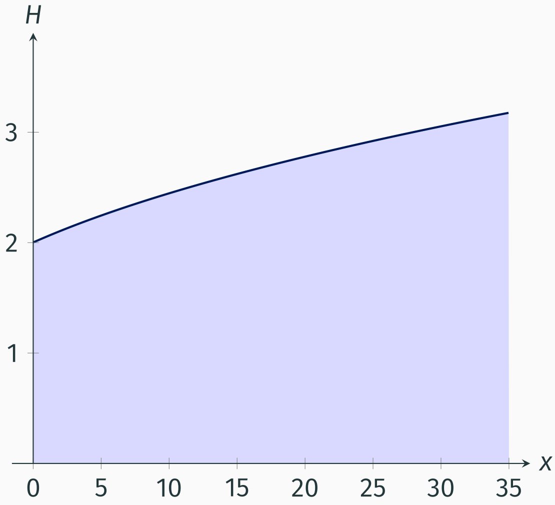
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

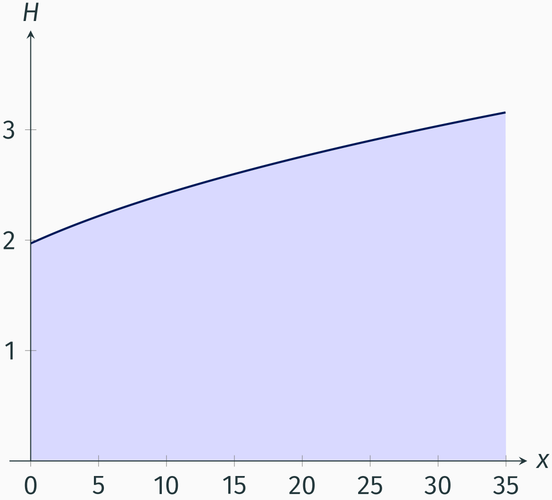


Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

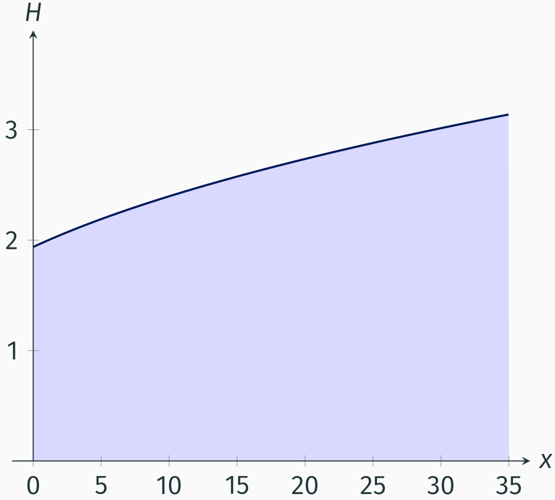


We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



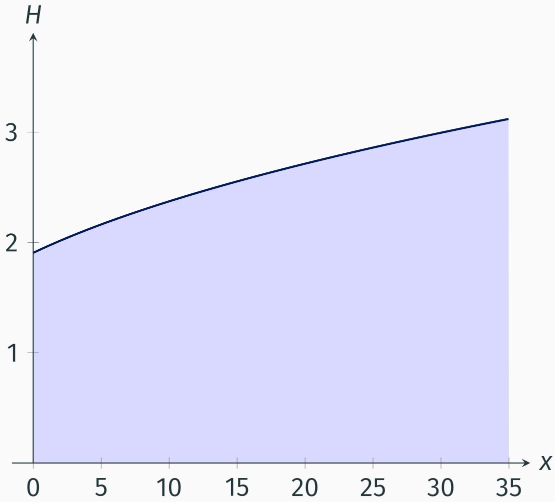
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



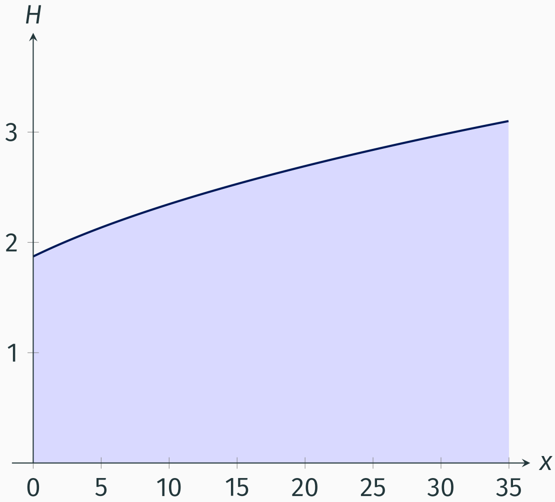
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



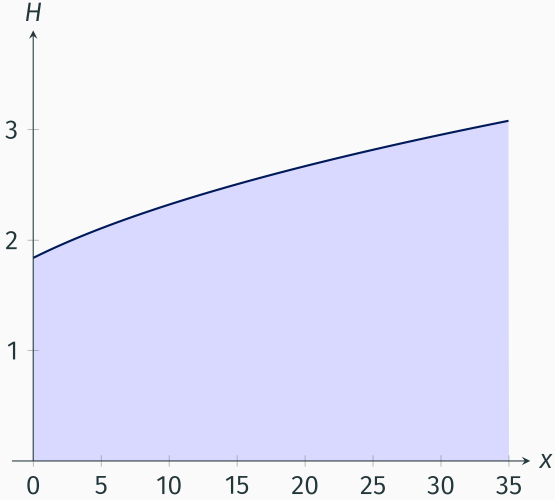
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



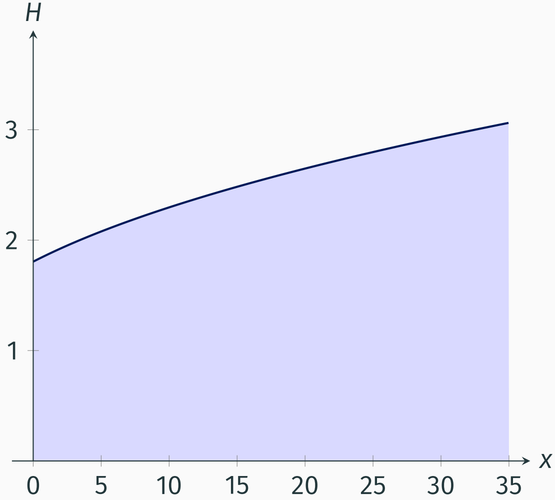
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



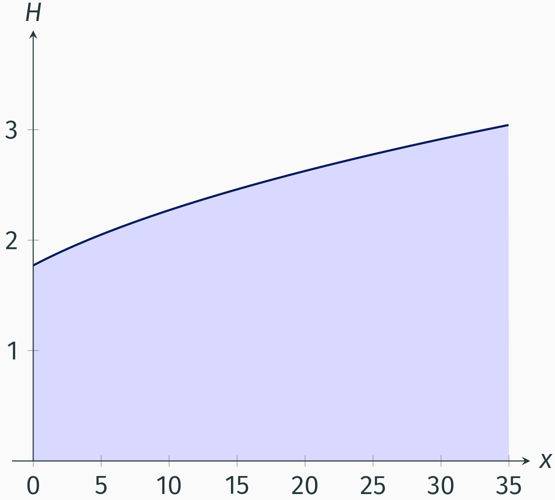
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



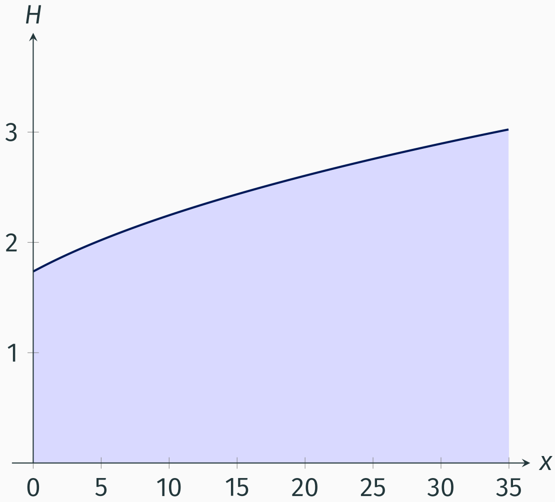
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



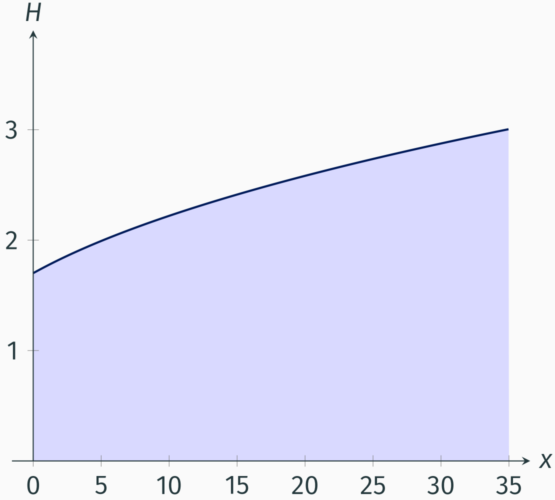
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



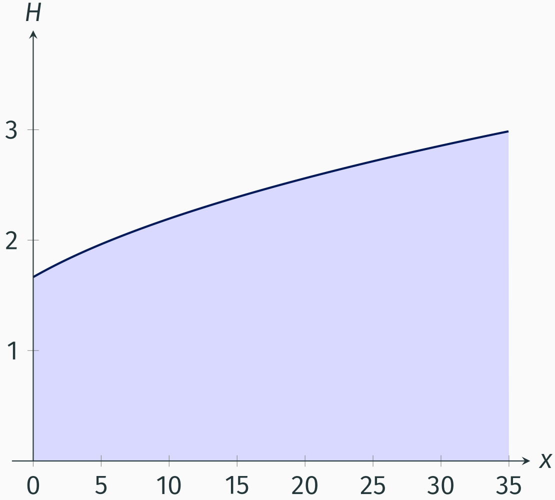
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



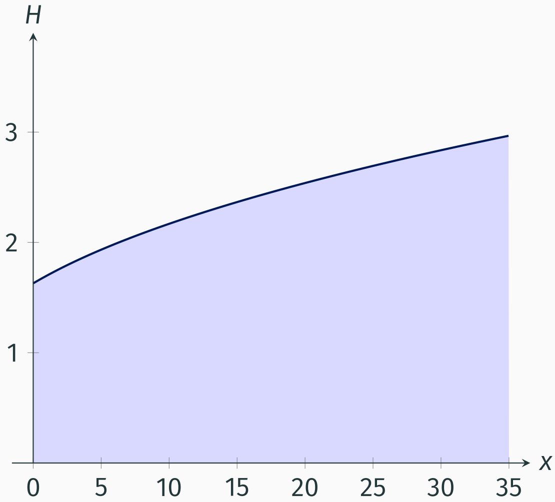
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



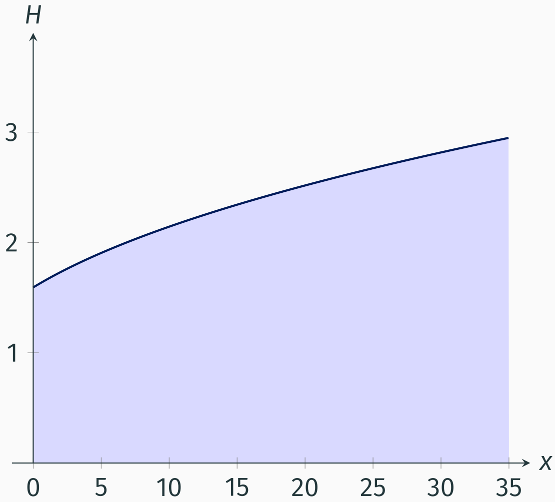
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



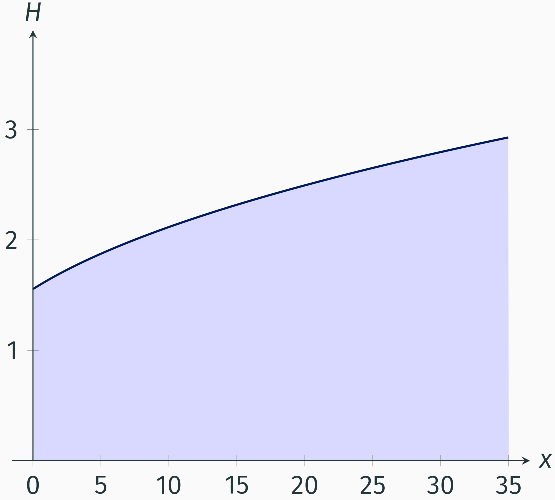
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



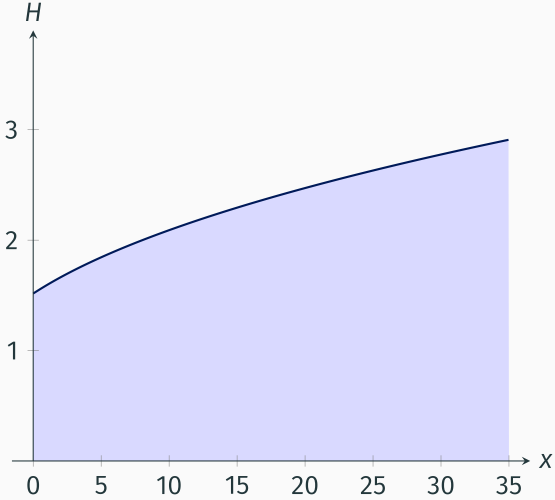
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



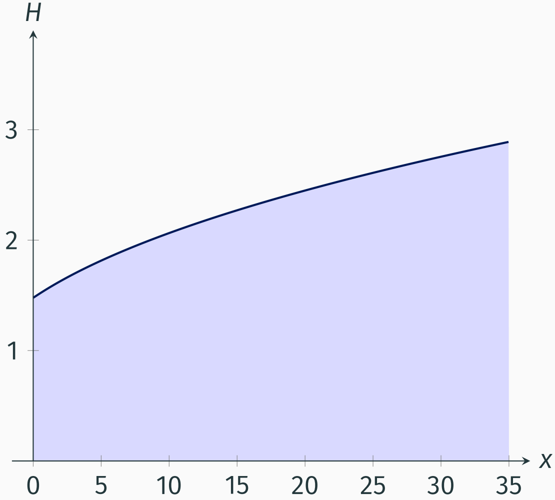
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



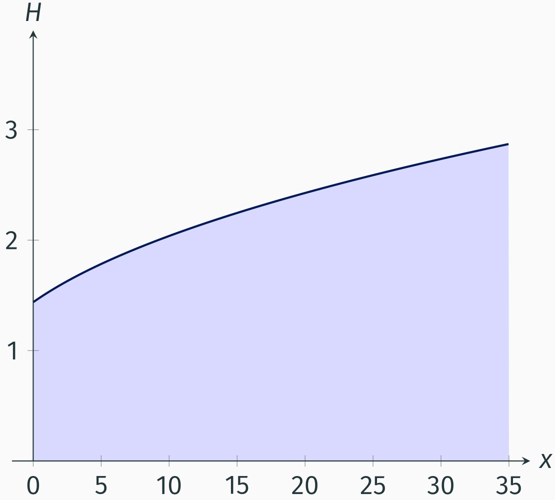
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



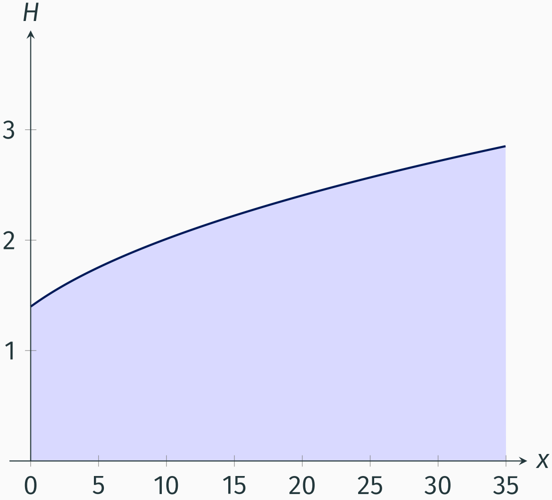
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



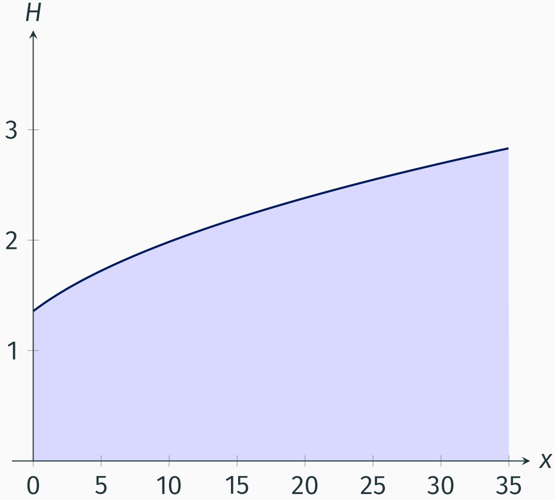
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



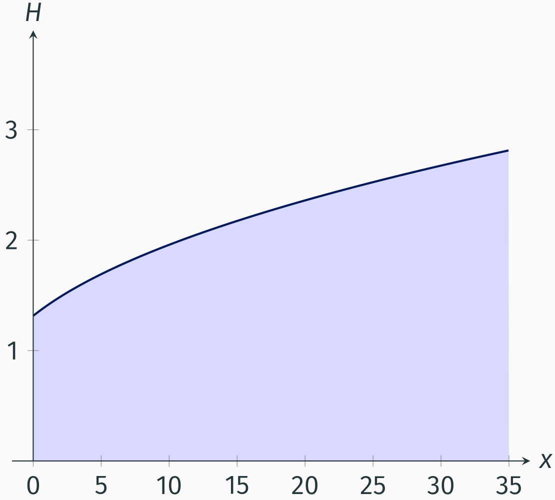
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



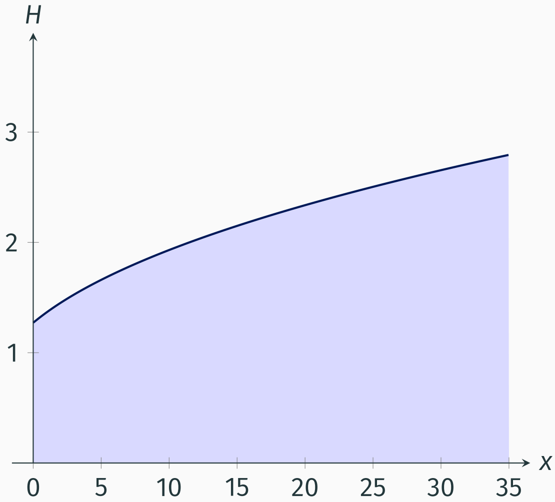
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



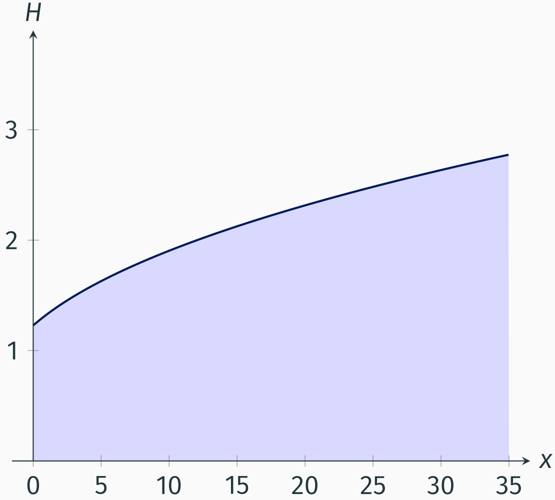
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



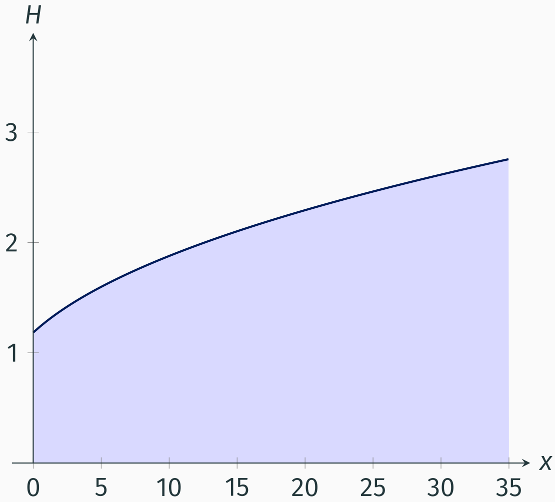
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



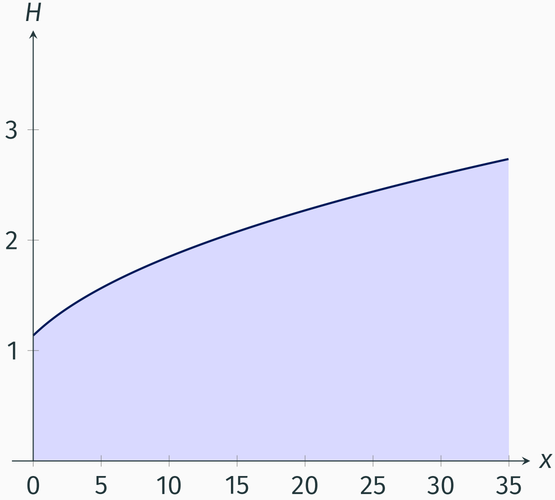
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



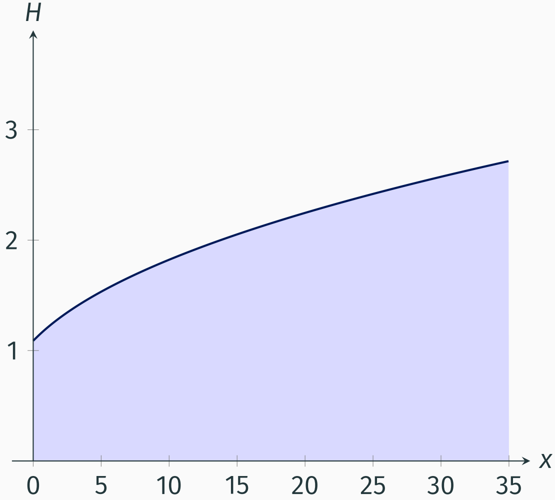
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



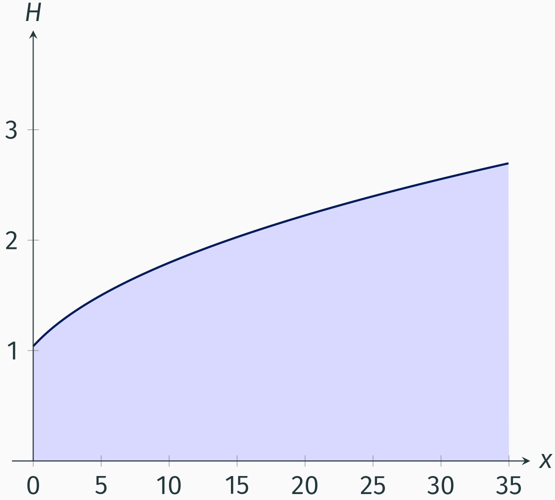
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



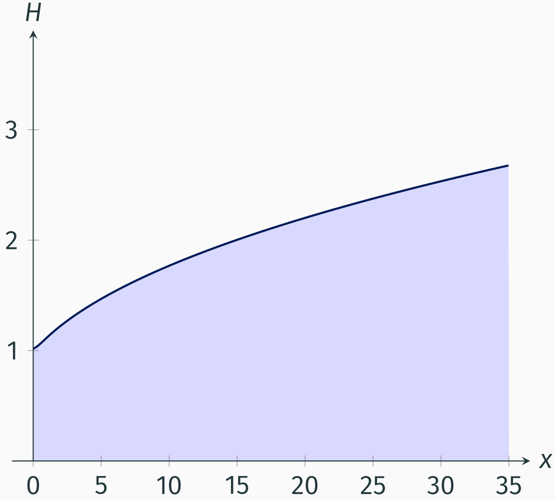
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



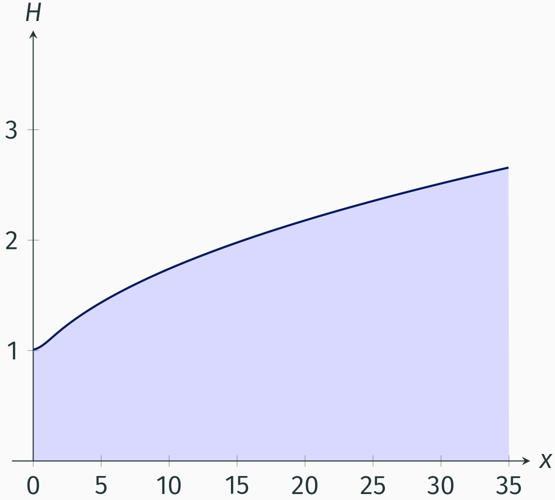
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



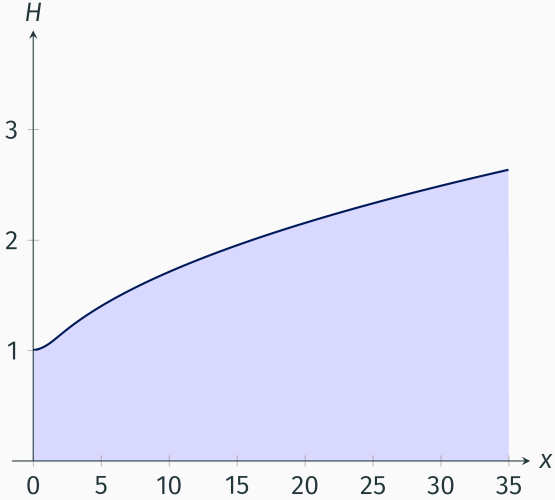
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



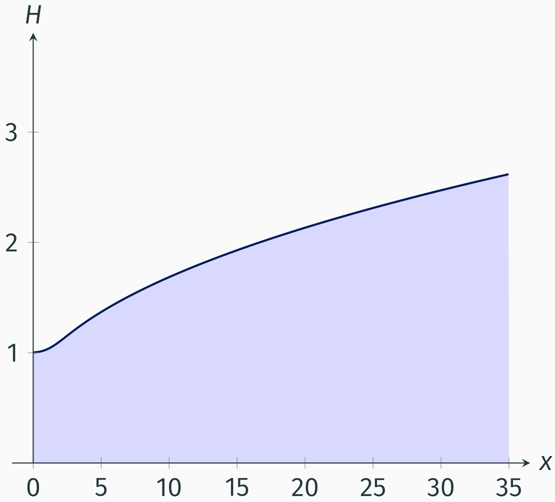
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



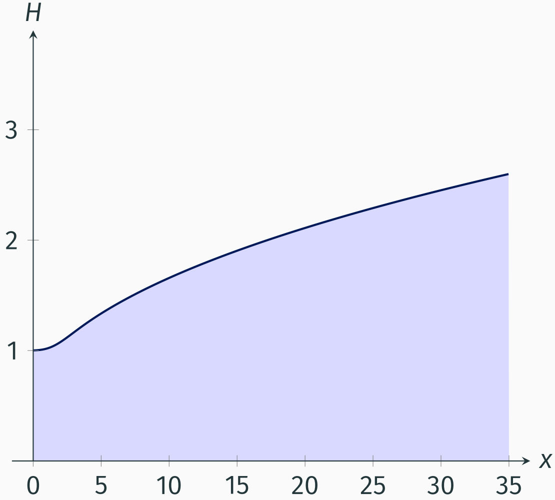
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



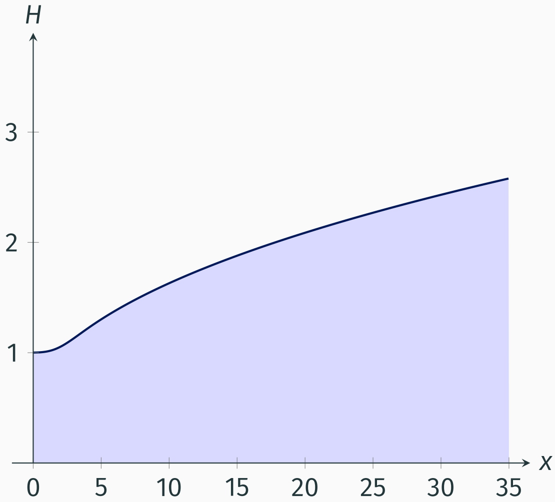
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



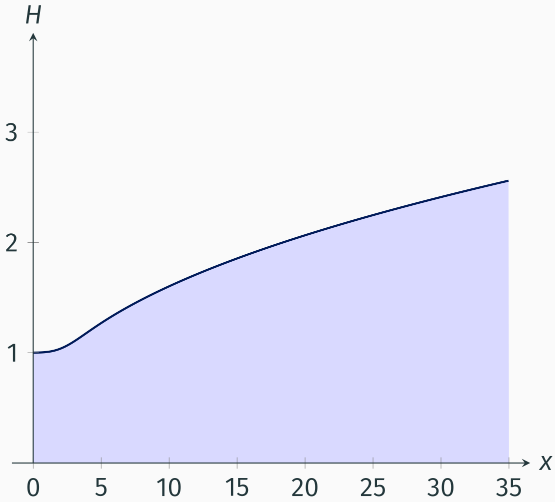
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



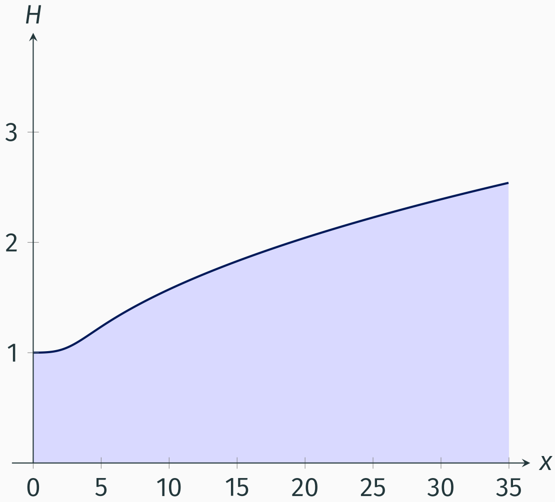
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



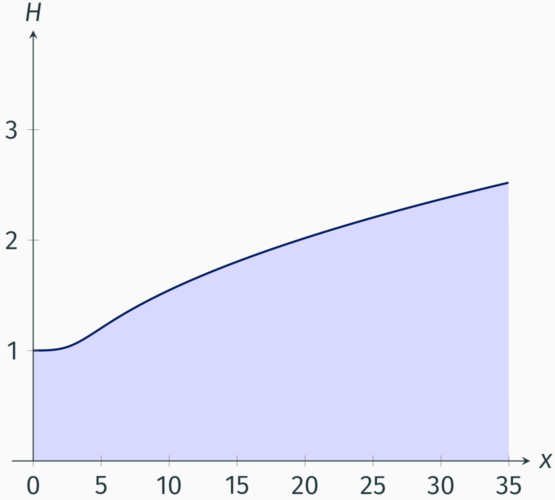
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



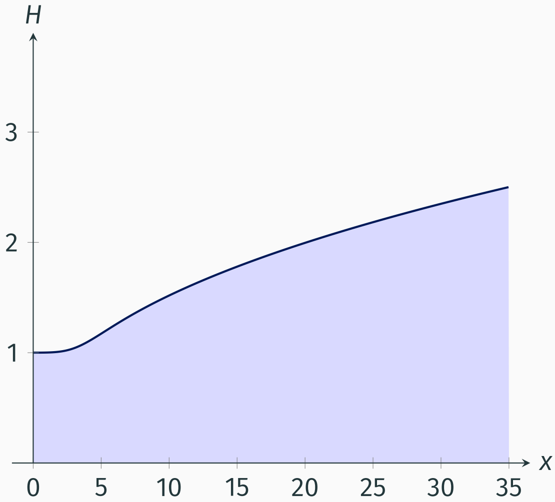
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



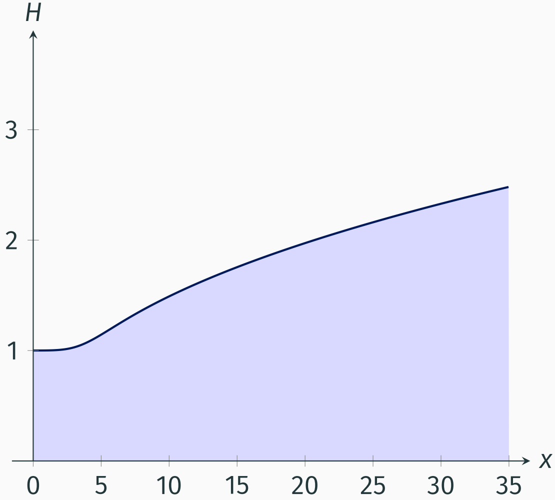
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



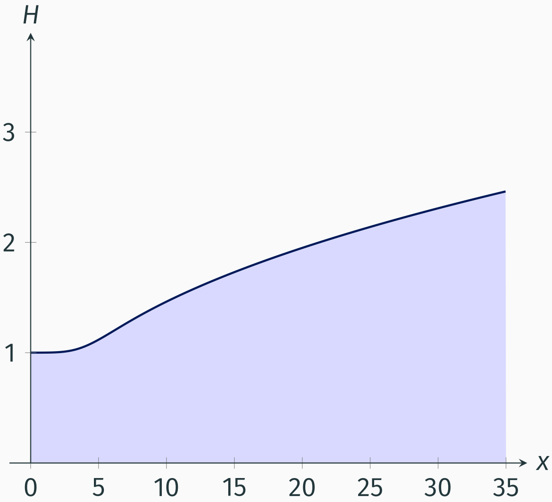
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



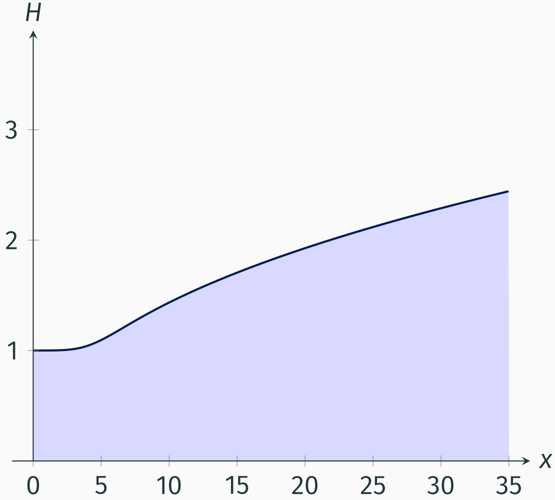
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



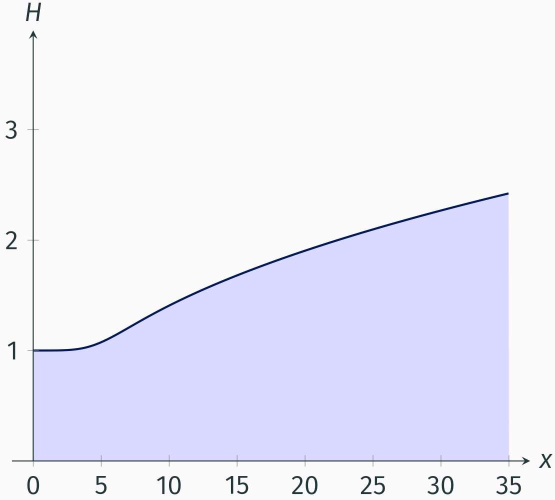
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



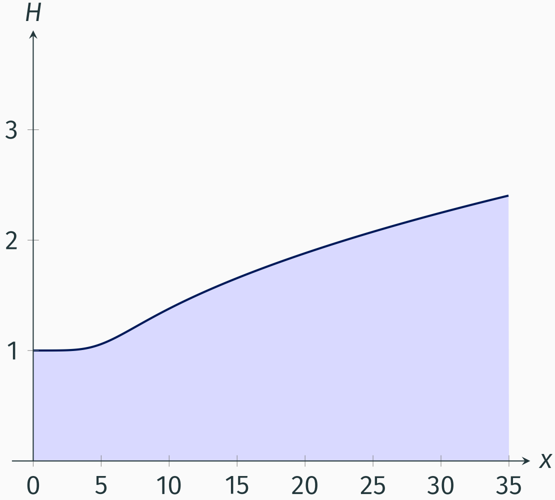
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



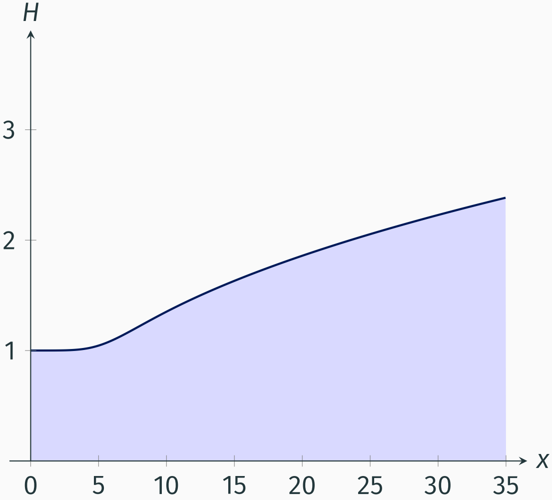
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



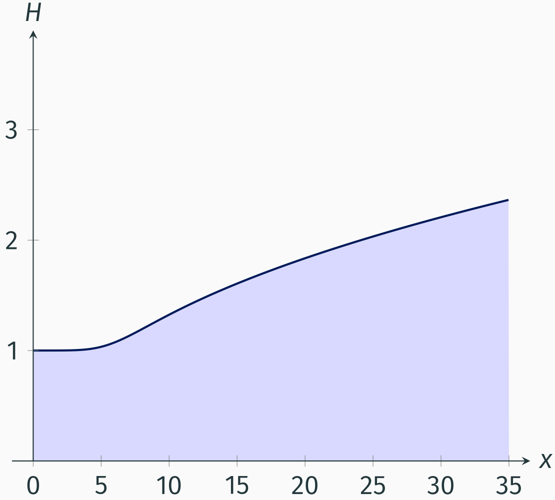
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



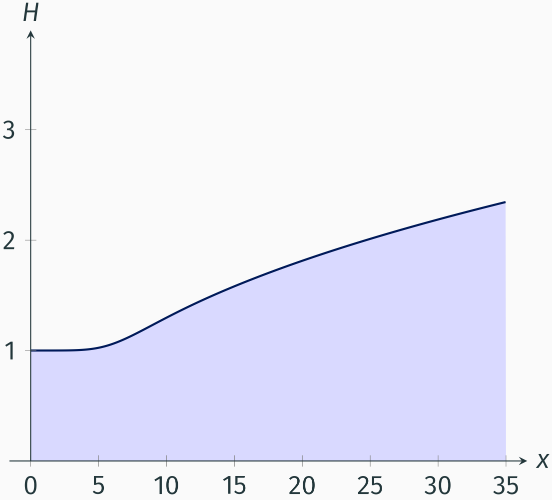
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



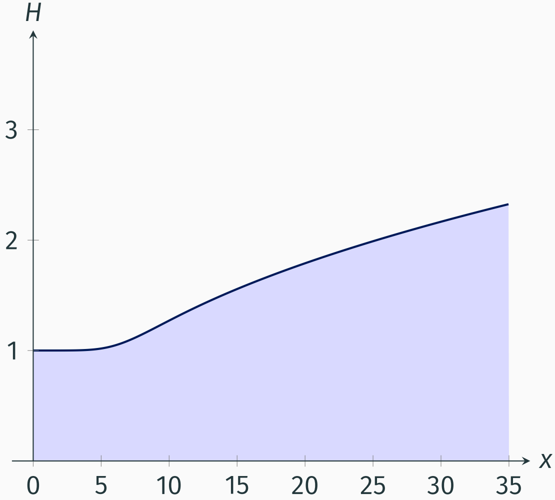
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



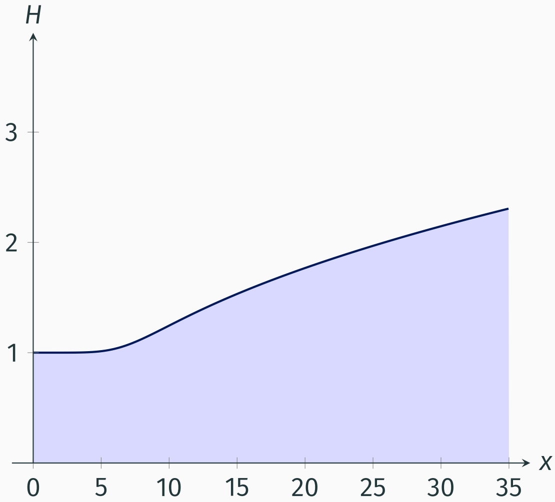
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



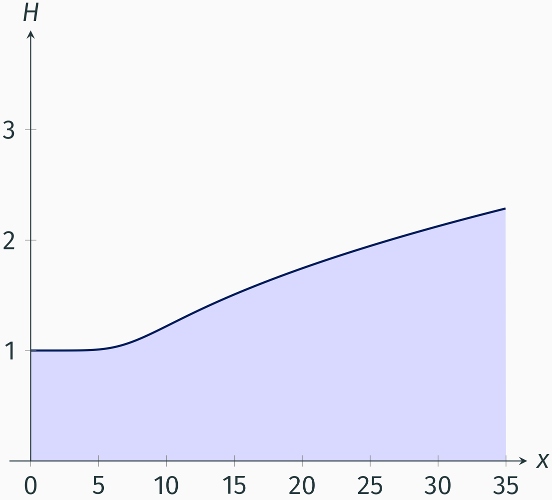
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



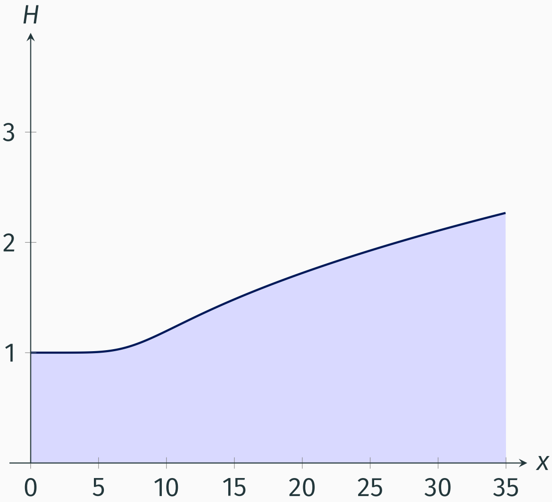
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



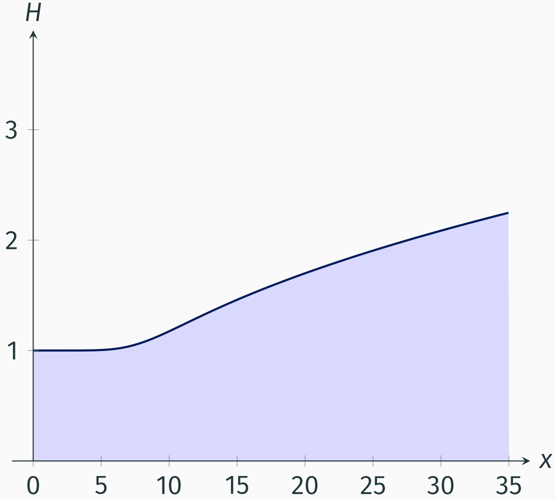
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



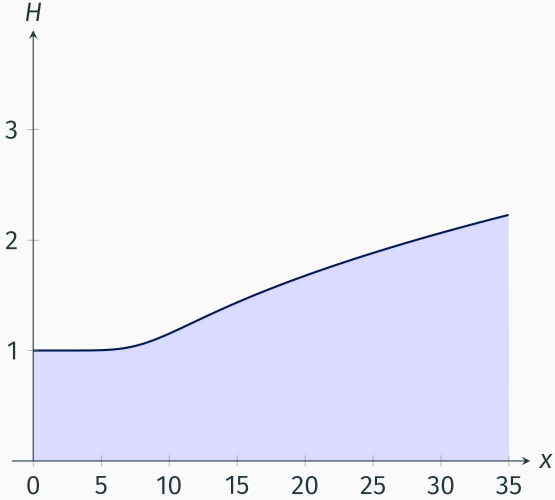
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



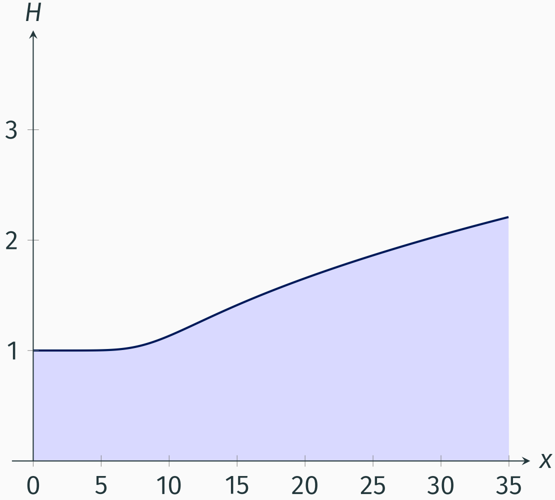
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



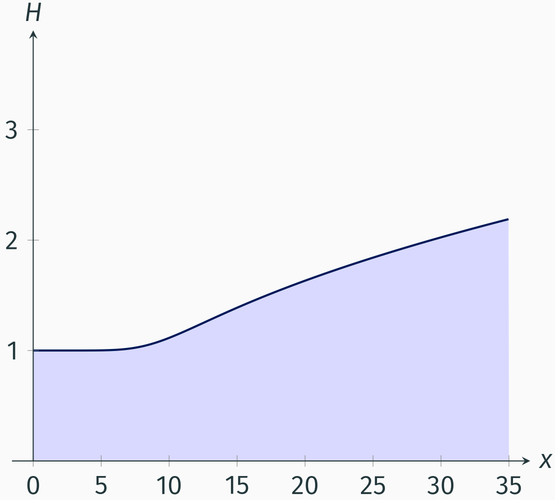
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



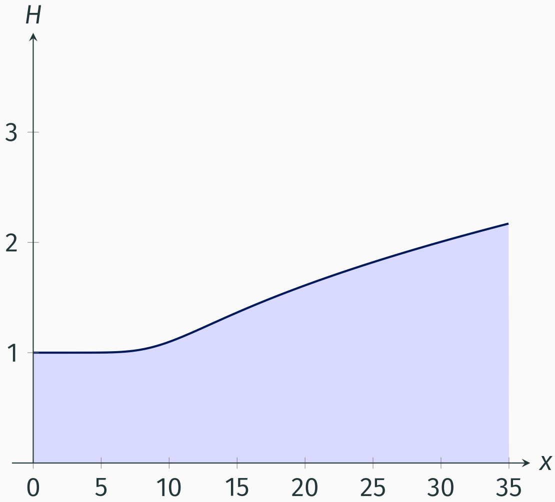
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

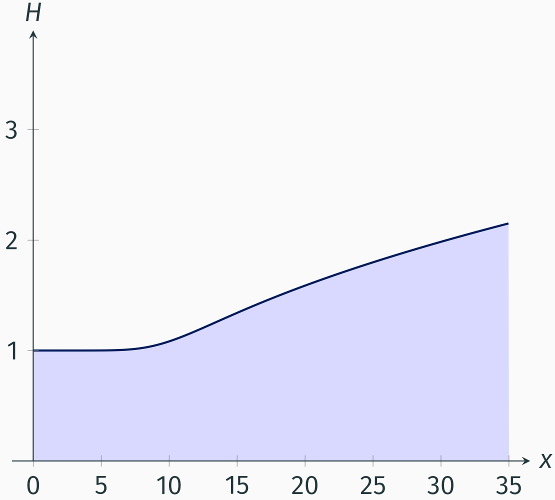


Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

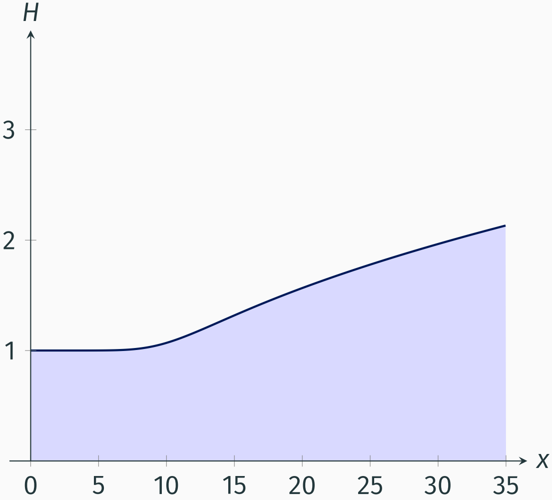


We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



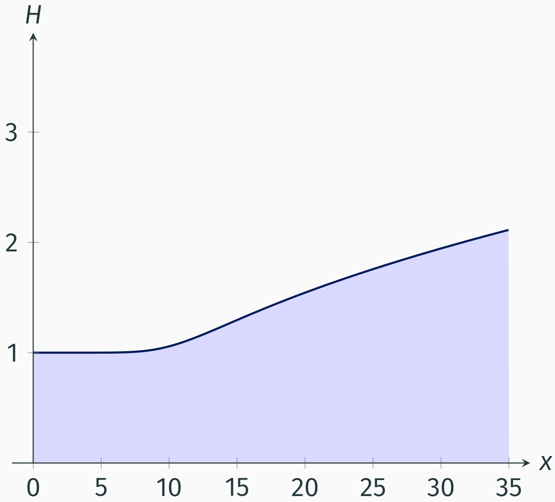
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



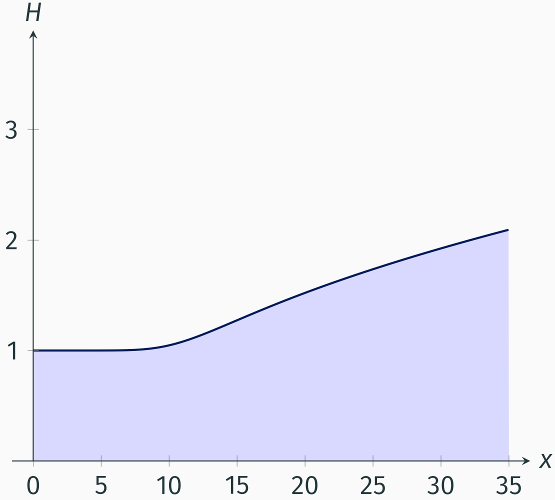
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



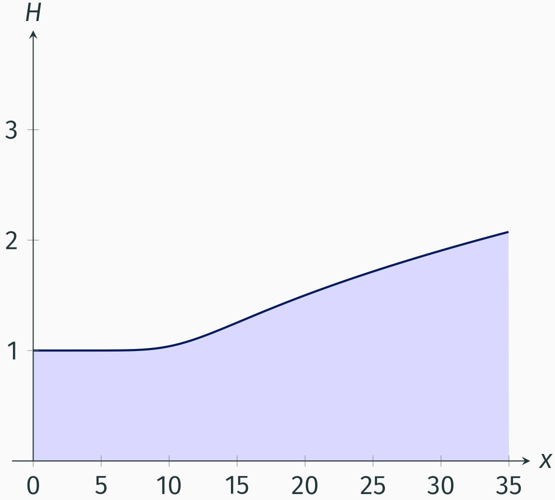
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



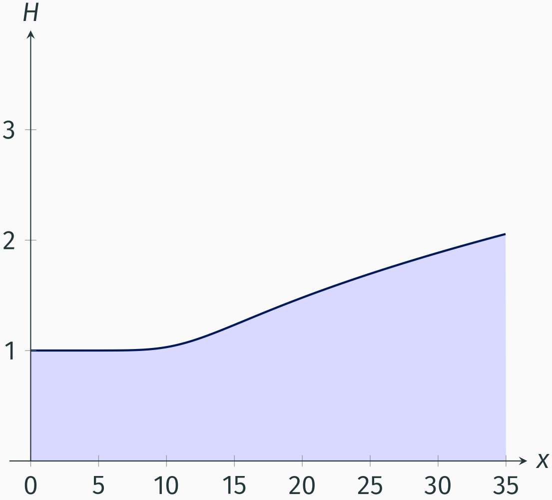
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



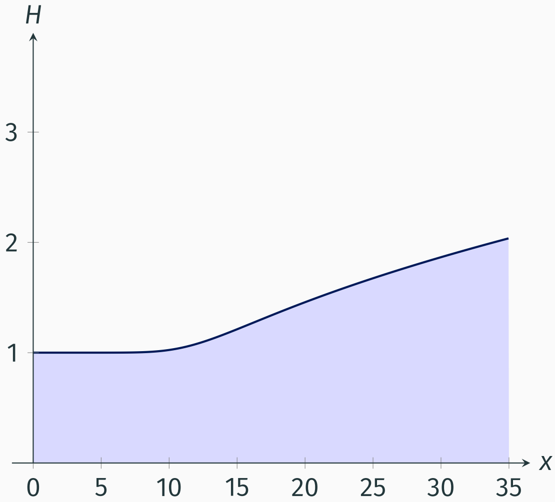
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



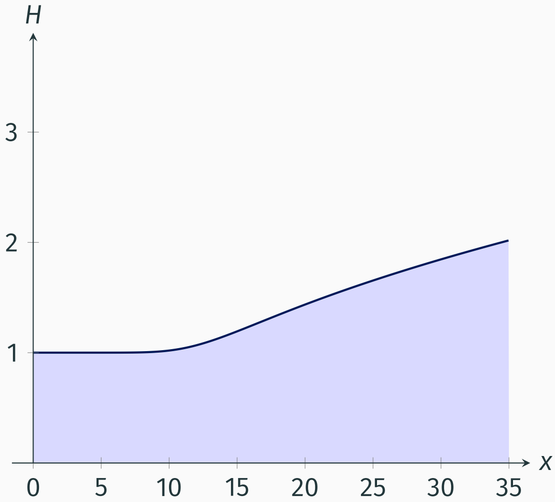
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



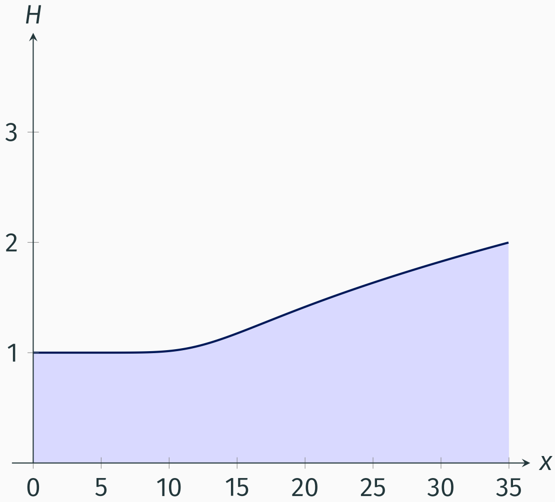
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



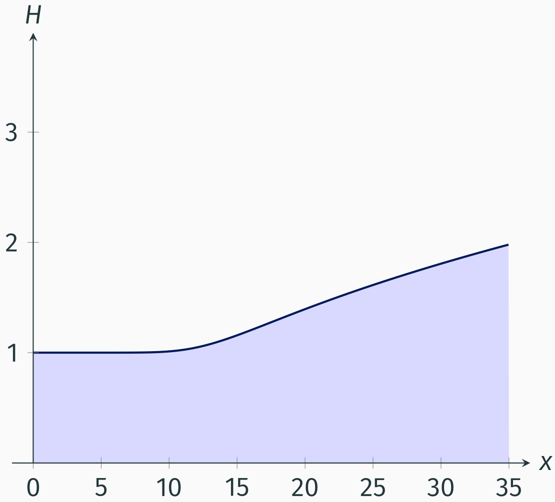
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



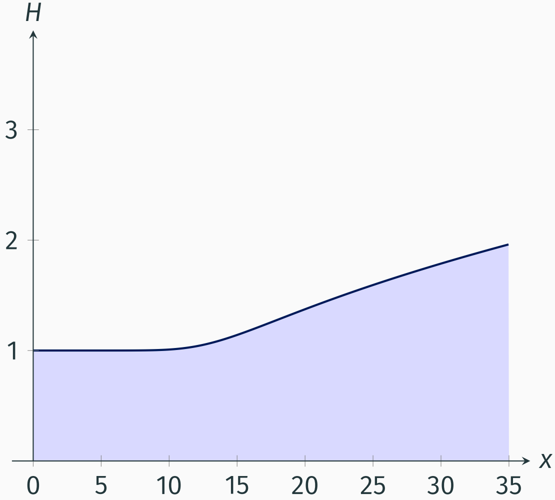
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



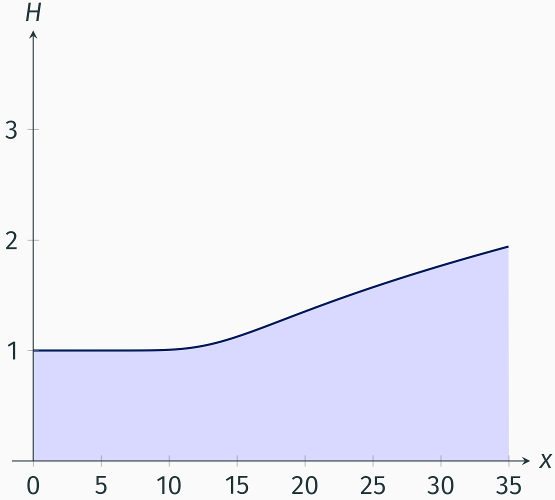
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



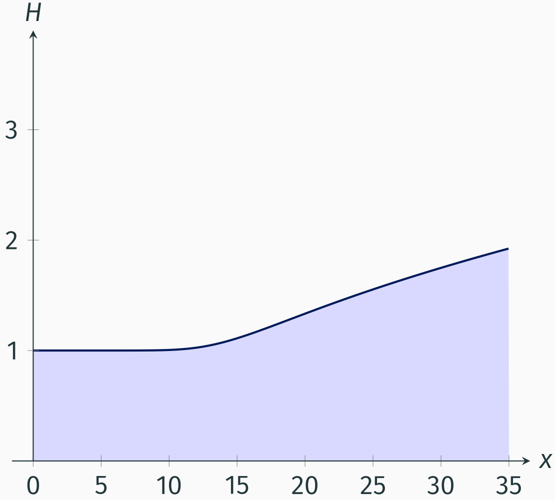
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



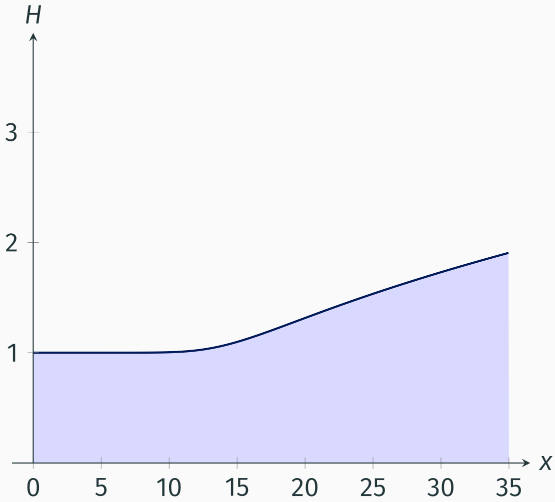
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



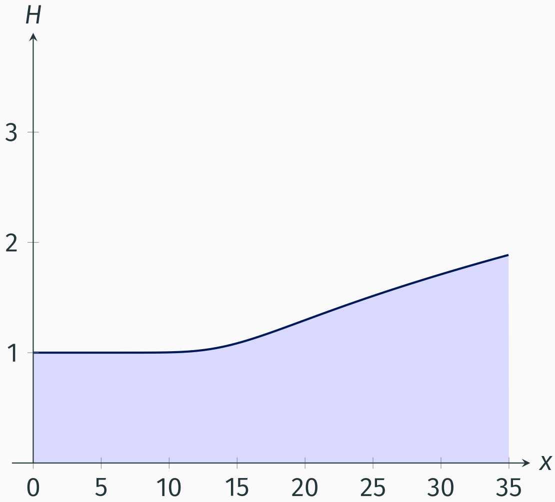
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



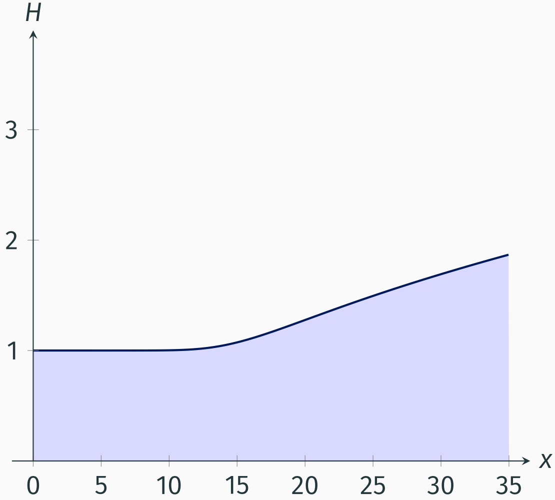
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



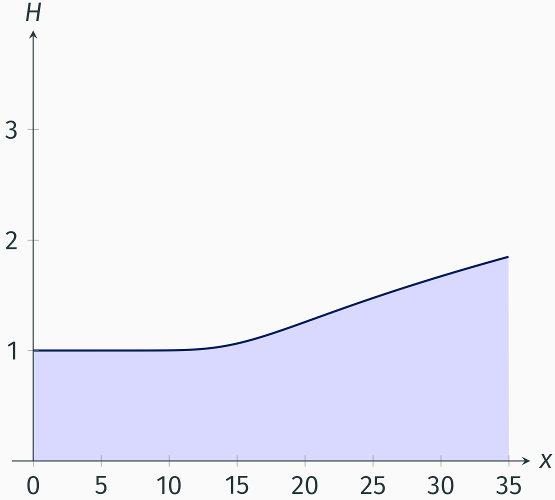
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



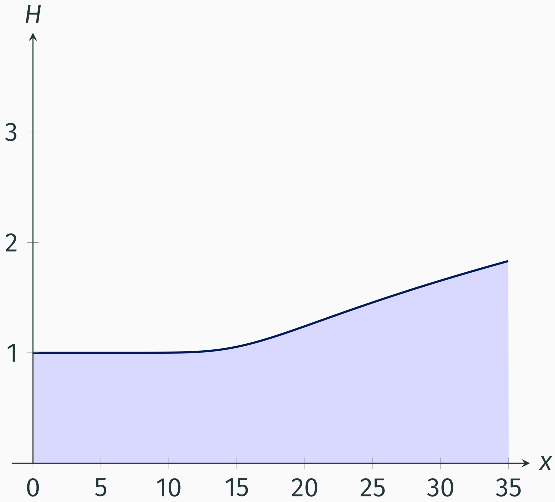
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



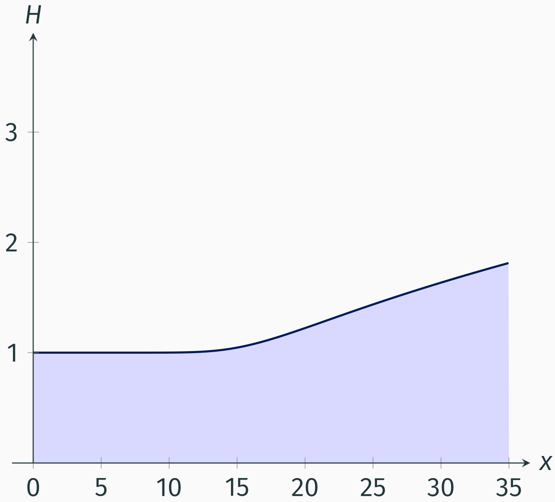
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



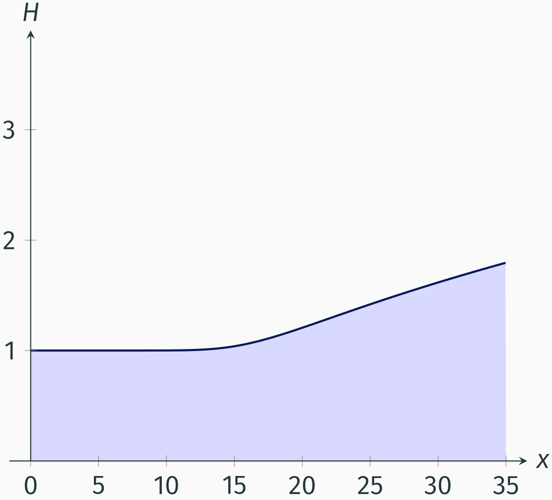
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



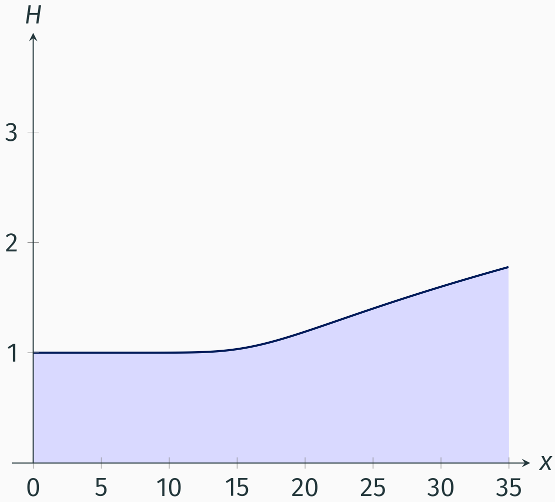
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



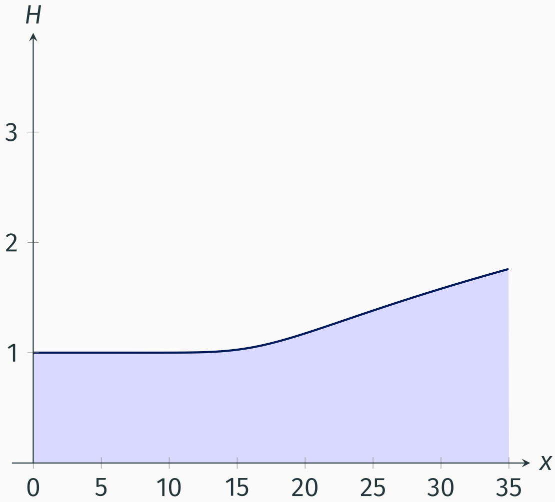
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



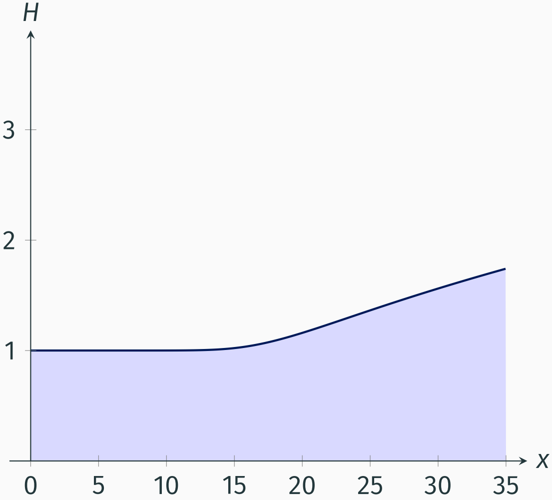
Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

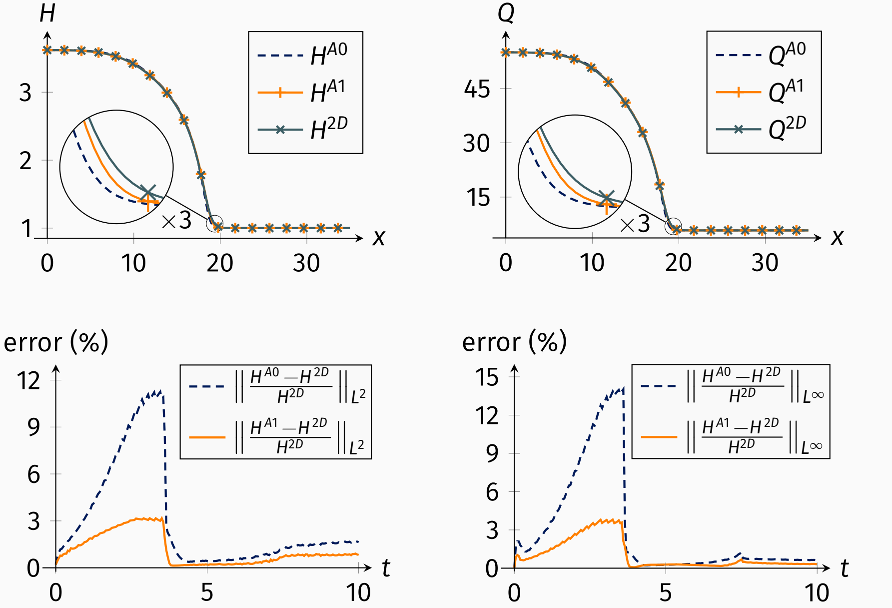


Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



Unsteady flood flow (2D: ref. sol., A0: 0th-order, A1: 1st-order)



1. Governing equations
2. Asymptotic expansions
3. Transverse averaging
4. A zeroth-order model
5. Numerical validation of the model
6. Conclusion and perspectives

Conclusion

We have developed a new 1D model, based on the 2D shallow water equations, that is:

- consistent, up to first-order, with the 2D model in the asymptotic regime corresponding to a river flow:
  - the zeroth-order is obtained with a new explicit friction term,
  - the first-order relies on new equations describing the evolution of the energy;
- hyperbolic;
- easily implementable and numerically validated.

The preprint related to these results is available on HAL:

V. Michel-Dansac, P. Noble et J.-P. Vila, **Consistent section-averaged shallow water equations with bottom friction**, 2018.
https://hal.archives-ouvertes.fr/hal-01962186

Perspectives

Work related to the model:

- improve the treatment of the river meanders by going to the first-order instead of the zeroth-order
- adapt this methodology to treat confluences
- consider a time-dependent topography to model the effects of sedimentation

Work related to the implementation and scientific computation:

- compare the 1D results to the ones given by a fully 2D code, in real test cases (Garonne, Lèze, Gironde, Amazon, ...)
- couple the 1D and 2D equations in the context of the Gironde estuary (collaboration in progress with the SHOM)

Thank you for your attention!

First-order model

The first-order model is:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x + \left(1 - \dfrac{S\Psi_{2D}^{(0)}}{\left(Q_{2D}^{(0)}\right)^2}\right)\dfrac{SH_x}{F^2} = \dfrac{1}{\varepsilon} S\left(\mathcal{I} - \mathcal{J} - \dfrac{S\Psi_{2D}^{(0)}}{\left(Q_{2D}^{(0)}\right)^2}(\mathcal{I} - \mathcal{J}_\Psi)\right), \\ \left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x + \dfrac{QH_x}{F^2} = \dfrac{1}{\varepsilon} Q(\mathcal{I} - \mathcal{J}), \\ \left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = \dfrac{1}{\varepsilon} Q\dfrac{S\Pi_{2D}^{(0)}}{\left(Q_{2D}^{(0)}\right)^2}(\mathcal{J}_\Psi - \mathcal{J}_\Pi). \end{cases}$$

It ensures the correct asymptotic regime, that is to say

$$Q = Q_{2D}^{(0)} + \varepsilon Q_{2D}^{(1)} + \mathcal{O}\left(\varepsilon^2\right).$$

In addition, it is hyperbolic and linearly stable.

Non-dimensional form of the 2D shallow water system

To emphasize the different scales of the flow, we perform a non-dimensionalization of the 2D system.

We introduce the following dimensionalization scales and related non-dimensional quantities (which are denoted with a bar, like $\bar{x}$):

$$h := \mathcal{H}\bar{h}, \quad u := \mathcal{U}\bar{u}, \quad v := \mathcal{V}\bar{v}, \quad x := \mathcal{X}\bar{x}, \quad y := \mathcal{Y}\bar{y}, \quad t := \mathcal{T}\bar{t}, \quad \mathcal{T} := \frac{\mathcal{X}}{\mathcal{U}}.$$

The mass conservation equation

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

then becomes

$$\frac{\mathcal{H}}{\mathcal{T}}\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\mathcal{H}\mathcal{U}}{\mathcal{X}}\frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{\mathcal{H}\mathcal{V}}{\mathcal{Y}}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0.$$

Non-dimensional form of the 2D shallow water system

The non-dimensional conservation equation is

$$\frac{\mathcal{H}}{\mathcal{T}}\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\mathcal{H}\mathcal{U}}{\mathcal{X}}\frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{\mathcal{H}\mathcal{V}}{\mathcal{Y}}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0, \quad \text{i.e.} \qquad \frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{\mathcal{V}}{\mathcal{U}}\frac{\mathcal{X}}{\mathcal{Y}}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0.$$

We set $R_u := \mathcal{V}/\mathcal{U}$ and $R_x := \mathcal{Y}/\mathcal{X}$, to get

$$\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{R_u}{R_x}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0.$$

We have

- $\mathcal{V} \ll \mathcal{U}$ (quasi-unidimensional flow) $\implies R_u \ll 1$,
- $\mathcal{Y} \ll \mathcal{X}$ (quasi-unidimensional geometry) $\implies R_x \ll 1$.

We assume $R_u = R_x$ to keep the mass conservation equation unchanged from the dimensional case.

Non-dimensional form of the 2D shallow water system

Regarding the geometry, we assume that $Z(x,y) = b(x) + \phi(x,y)$, where:

- $b(x)$ represents the main longitudinal topography, driving the flow from upstream to downstream;
- $\phi(x,y)$ represents small longitudinal and transverse variations.

The related non-dimensional quantities are

$$b = \mathcal{B}\bar{b}\left(\frac{x}{\mathcal{X}}\right) \quad \text{and} \quad \phi = \mathcal{H}\bar{\phi}\left(\frac{x}{\mathcal{X}}, \frac{y}{\mathcal{Y}}\right).$$

The non-dimensional topography gradient then reads:

$$\boldsymbol{\nabla} Z = \begin{pmatrix} \dfrac{\mathcal{B}}{\mathcal{X}} \dfrac{\partial \bar{b}}{\partial \bar{x}}(\bar{x}) + \dfrac{\mathcal{H}}{\mathcal{X}} \dfrac{\partial \bar{\phi}}{\partial \bar{x}}(\bar{x},\bar{y}) \\ \dfrac{\mathcal{H}}{\mathcal{Y}} \dfrac{\partial \bar{\phi}}{\partial \bar{y}}(\bar{x},\bar{y}) \end{pmatrix}.$$

Non-dimensional form of the 2D shallow water system

Regarding the friction, we take $C_h = \mathcal{C}\,\overline{C}(\bar{x}, \bar{y})$.

The non-dimensional friction source term then reads:

$$\frac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\,h^p} = \begin{pmatrix} \dfrac{\mathcal{U}}{\mathcal{C}\mathcal{H}^p} \cdot \dfrac{\bar{u}\sqrt{\mathcal{U}^2\bar{u}^2 + \mathcal{V}^2\bar{v}^2}}{\overline{C}^2\,\overline{h}^p} \\ \dfrac{\mathcal{V}}{\mathcal{C}\mathcal{H}^p} \cdot \dfrac{\bar{v}\sqrt{\mathcal{U}^2\bar{u}^2 + \mathcal{V}^2\bar{v}^2}}{\overline{C}^2\,\overline{h}^p} \end{pmatrix} = \begin{pmatrix} \dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p} \cdot \dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\overline{C}^2\,\overline{h}^p} \\ \dfrac{\mathcal{V}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p} \cdot \dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\overline{C}^2\,\overline{h}^p} \end{pmatrix}.$$

Non-dimensional form of the 2D shallow water system

We are finally able to write the non-dimensional form of the 2D shallow water system: from the dimensional system

$$\begin{cases} h_t + \nabla \cdot (h\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g \nabla h = g\left(-\nabla Z - \dfrac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\, h^p} \right), \end{cases}$$

we get the following non-dimensional form:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \dfrac{\mathcal{U}^2}{\mathcal{X}}\bar{u}_{\bar{t}} + \dfrac{\mathcal{U}^2}{\mathcal{X}}\bar{u}\bar{u}_{\bar{x}} + \dfrac{\mathcal{U}\mathcal{V}}{\mathcal{Y}}\bar{v}\bar{u}_{\bar{y}} + \dfrac{g\mathcal{H}}{\mathcal{X}}\left(\bar{h} + \bar{\Phi}\right)_{\bar{x}} = g\left(-\dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p} \dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - \dfrac{\mathcal{B}}{\mathcal{X}}\bar{b}_{\bar{x}} \right), \\ \dfrac{\mathcal{V}\mathcal{U}}{\mathcal{X}}\bar{v}_{\bar{t}} + \dfrac{\mathcal{V}\mathcal{U}}{\mathcal{X}}\bar{u}\bar{v}_{\bar{x}} + \dfrac{\mathcal{V}^2}{\mathcal{Y}}\bar{v}\bar{v}_{\bar{y}} + \dfrac{g\mathcal{H}}{\mathcal{Y}}\left(\bar{h} + \bar{\Phi}\right)_{\bar{y}} = g\left(-\dfrac{\mathcal{V}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p} \dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} \right). \end{cases}$$

Non-dimensional form of the 2D shallow water system

We are finally able to write the non-dimensional form of the 2D shallow water system: from the dimensional system

$$\begin{cases} h_t + \nabla \cdot (h\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g \nabla h = g\left(-\nabla Z - \dfrac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\, h^p}\right), \end{cases}$$

we get the following non-dimensional form:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \dfrac{g\mathcal{H}}{\mathcal{U}^2}\left(\bar{h} + \bar{\Phi}\right)_{\bar{x}} = \dfrac{g\mathcal{X}}{\mathcal{U}^2}\left(-\dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - \dfrac{\mathcal{B}}{\mathcal{X}}\bar{b}_{\bar{x}}\right), \\ \bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \dfrac{g\mathcal{H}\mathcal{X}}{\mathcal{V}\mathcal{U}\mathcal{Y}}\left(\bar{h} + \bar{\Phi}\right)_{\bar{y}} = \dfrac{g\mathcal{X}}{\mathcal{U}^2}\left(-\dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p}\right). \end{cases}$$

Non-dimensional form of the 2D shallow water system

We introduce:

- $F^2 = \dfrac{\mathcal{U}^2}{g\mathcal{H}}$ the reference Froude number,
- $\delta = \dfrac{\mathcal{H}}{\mathcal{X}}$ the shallow water parameter,
- $I_0 = \dfrac{\mathcal{B}}{\mathcal{X}}$ and $J_0 = \dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}$ the topography and friction slopes.

With $\dfrac{g\mathcal{X}}{\mathcal{U}^2} = \dfrac{g\mathcal{H}}{\mathcal{U}^2}\dfrac{\mathcal{X}}{\mathcal{H}} = \dfrac{1}{\delta F^2}$ and $\dfrac{g\mathcal{H}\mathcal{X}}{\mathcal{V}\mathcal{U}\mathcal{Y}} = \dfrac{g\mathcal{H}}{\mathcal{U}^2}\dfrac{\mathcal{U}}{\mathcal{V}}\dfrac{\mathcal{X}}{\mathcal{Y}} = \dfrac{1}{R_u^2 F^2}$, we finally get:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \dfrac{1}{F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{x}} = \dfrac{1}{\delta F^2}\left(-J_0 \dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - I_0\bar{b}_{\bar{x}}\right), \\ \bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \dfrac{1}{R_u^2F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{y}} = -\dfrac{J_0}{\delta F^2}\dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p}. \end{cases}$$