A well-balanced scheme for the shallow-water equations with topography and bottom friction

C. Berthon[1], S. Clain[2], F. Foucher[1,3], V. Michel-Dansac[1]

[1]Laboratoire de Mathématiques Jean Leray, Université de Nantes
[2]Centre of Mathematics, Minho University
[3]École Centrale de Nantes

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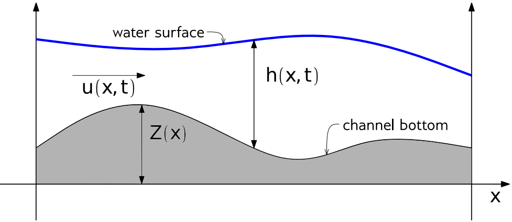
The shallow-water equations and their source terms

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq|q|}{h^\eta} \end{cases} \text{(with } q = hu\text{)}$$

note we can rewrite the equations as $\partial_t W + \partial_x F(W) = S(W)$



- $\eta = 7/3$ and g is the gravitational constant
- $k \geq 0$ is the so-called Manning coefficient: a higher k leads to a stronger bottom friction

Steady states

Definition: Steady states

W is a steady state iff $\partial_t W = 0$, i.e. $\partial_x F(W) = S(W)$

taking $\partial_t W = 0$ in the shallow-water equations leads to

$$\begin{cases} \partial_x q = 0 \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq|q|}{h^\eta} \end{cases}$$

the steady states are therefore given by

$$\begin{cases} q = \mathrm{cst} = q_0 \\ \partial_x \left(\dfrac{q_0^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq_0|q_0|}{h^\eta} \end{cases}$$

Smooth steady states for the friction source term

assume a flat bottom ($Z = \text{cst}$): the steady states are given by

$$\partial_x \left(\frac{q_0^2}{h} + \frac{1}{2} g h^2 \right) = - \frac{k q_0 |q_0|}{h^\eta}$$

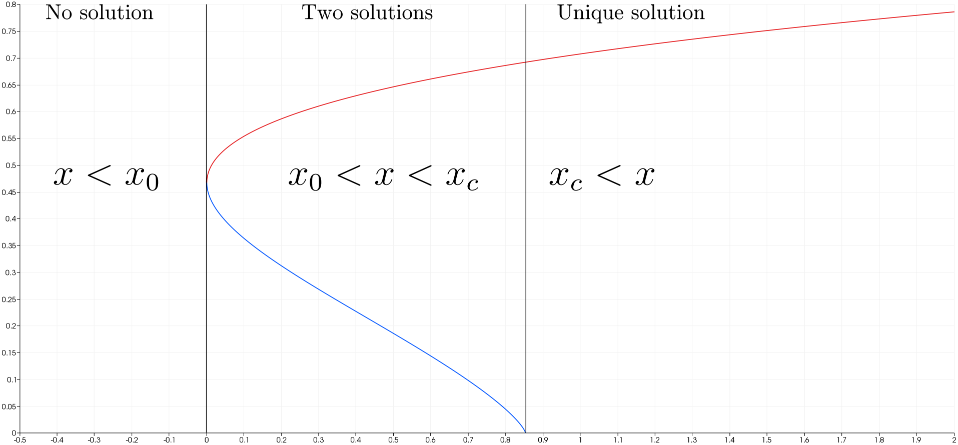
assuming smooth steady states and integrating this steady equation between some $x_0 \in \mathbb{R}$ and $x \in \mathbb{R}$ yields the algebraic relation (with $h = h(x)$ and $h_0 = h(x_0)$):

$$-\frac{q_0^2}{\eta - 1} \left(h^{\eta-1} - h_0^{\eta-1} \right) + \frac{g}{\eta + 2} \left(h^{\eta+2} - h_0^{\eta+2} \right) + k q_0 |q_0| \, (x - x_0) = 0$$

but: $\rightsquigarrow$ no global solution $h(x)$ for all $x \in \mathbb{R}$

$\rightsquigarrow$ for fixed x, we have 0, 1 or 2 solutions

Smooth steady states for the friction source term



- zones and variations: analytical study with $q_0 < 0$
- solution shape: Newton's method

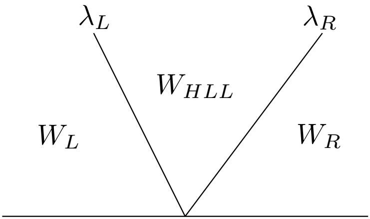
Objectives

derive a scheme that:

- is well-balanced for the shallow-water equations with friction and/or topography, i.e.:
  - preservation of all steady states with $k = 0$ and $Z \neq \text{cst}$
  - preservation of all steady states with $k \neq 0$ and $Z = \text{cst}$
  - preservation of steady states with $k \neq 0$ and $Z \neq \text{cst}$
- preserves the positivity of the water height
- is able to deal with wet/dry transitions

The HLL scheme

to approximate solutions to $\partial_t W + \partial_x F(W) = 0$, we use the HLL scheme (Harten, Lax, van Leer (1983)), which uses the following approximate Riemann solver:



the consistency condition (as per Harten and Lax) holds if:

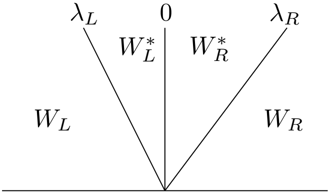
$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(x,\Delta t)dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx$$

which gives $W_{HLL} = \dfrac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \dfrac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L} = \begin{pmatrix} h_{HLL} \\ q_{HLL} \end{pmatrix}$

note that $h_{HLL} > 0$ for $|\lambda_L|$ and $|\lambda_R|$ large enough

Modification of the HLL scheme

to approximate solutions to $\partial_t W + \partial_x F(W) = S(W)$, we use the following approximate Riemann solver (assuming $\lambda_L < 0 < \lambda_R$):



$\rightsquigarrow$ 3 unknowns to determine: $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$;

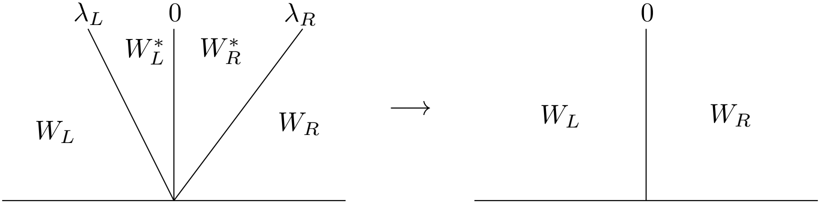
Harten-Lax consistency gives us

- $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$
- $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$ (with $\overline{S} = \overline{S}\,(W_L, W_R)$ approximating the mean of $S(W)$, to be determined)

Determination of h_L^* and h_R^*

assume that W_L and W_R define a steady state, i.e. verify

$$q_0^2 \left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = \overline{S}\Delta x$$



for the steady state to be preserved, we need

- $W_L^* = W_L$ and $W_R^* = W_R$, i.e.
- $h_L^* = h_L$, $h_R^* = h_R$ and $q^* = q_0$

as soon as W_L and W_R define a steady state

Determination of h_L^* and h_R^*

two unknowns $\rightsquigarrow$ we need two equations

- we have $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$
- we choose $\alpha \left(h_R^* - h_L^* \right) = \overline{S} \Delta x$

where $\alpha = \dfrac{-\bar{q}^2}{h_L h_R} + \dfrac{g}{2} \left(h_L + h_R \right)$, with $\bar{q}$ to be determined

$\rightsquigarrow$ using both relations, we obtain

$$\begin{cases} h_L^* = h_{HLL} - \dfrac{\lambda_R \overline{S} \Delta x}{\alpha \left(\lambda_R - \lambda_L \right)} \\ h_R^* = h_{HLL} - \dfrac{\lambda_L \overline{S} \Delta x}{\alpha \left(\lambda_R - \lambda_L \right)} \end{cases}$$

Correction to ensure positive h_L^* and h_R^*

however, these expressions of h_L^* and h_R^* do not guarantee their positivity: instead, we use (see Audusse, Chalons, Ung (2014))

$$\begin{cases} h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right) \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right) \end{cases}$$

note that this cutoff does not interfere with:

- the consistency condition $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$
- the well-balance property, since it does not activate when W_L and W_R define a steady state

Summary

using a two-state approximate Riemann solver with intermediate states $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$ given by

$$\begin{cases} q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L} \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right) \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right) \end{cases}$$

yields a scheme that is consistent, positivity-preserving and well-balanced; we now need to find $\overline{S}$ and $\bar{q}$ in agreement with the source term definition

The topography source term

we now consider $S(W) = -gh\partial_x Z$: the discrete equilibrium is

$$q_0^2 \left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = \overline{S}\Delta x$$

$$\frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g\left[h + Z\right] = 0$$

we can have an expression of q_0^2 and thus obtain

$$\overline{S}\Delta x = -2g\left[Z\right]\frac{h_L h_R}{h_L + h_R} + \frac{g}{2}\frac{\left[h\right]^3}{h_L + h_R}$$

but when $Z_L = Z_R$, we have $\overline{S} \neq \mathcal{O}(\Delta x) \rightsquigarrow$ loss of consistency with S (see for instance Berthon, Chalons (2015))

The topography source term

instead, we set, for some constant C,

$$\begin{cases} \overline{S}\Delta x = -2g\,[Z]\,\dfrac{h_L h_R}{h_L + h_R} + \dfrac{g}{2}\dfrac{[h]_c^3}{h_L + h_R} \\ [h]_c = \begin{cases} h_R - h_L & \text{if } |h_R - h_L| \leq C\,\Delta x \\ \operatorname{sgn}(h_R - h_L)\,C\,\Delta x & \text{otherwise.} \end{cases} \end{cases}$$

Theorem: Well-balancedness for the topography source term

If W_L and W_R define a steady state, i.e. verify

$$\frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g\,[h + Z] = 0,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$.

this result holds for any $\bar{q}$: we choose $\bar{q} = q^*$

The friction source term

we consider, in this case, $S(W) = -kq|q|h^{-\eta}$

the average of S we choose is $\overline{S} = -k\tilde{q}|\tilde{q}|\overline{h^{-\eta}}$, with

- $\tilde{q}$ the harmonic mean of q_L and q_R (note that $\tilde{q} = q_0$ at the equilibrium), and
- $\overline{h^{-\eta}}$ a well-chosen discretization of $h^{-\eta}$, depending on h_L and h_R, and ensuring the well-balancedness

we determine $\overline{h^{-\eta}}$ using the same technique (with $\delta_0 = \operatorname{sgn}(q_0)$):

$$q_0^2 \left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = -k\delta_0 q_0^2 \overline{h^{-\eta}} \Delta x$$

$$-q_0^2 \frac{\left[h^{\eta-1}\right]}{\eta-1} + g\frac{\left[h^{\eta+2}\right]}{\eta+2} = -k\delta_0 q_0^2 \Delta x$$

The friction source term

the expression for q_0^2 we obtained can be used to get:

$$\overline{h^{-\eta}} = \frac{-\delta_0}{k\Delta x}\left(\left[\frac{1}{h}\right] + \frac{[h^2]}{2}\frac{\eta+2}{[h^{\eta+2}]}\left(\frac{[h^{\eta-1}]}{\eta-1} - k\delta_0\Delta x\right)\right),$$

which gives $\overline{S} = -k\tilde{q}|\tilde{q}|\overline{h^{-\eta}}$ (we have $\overline{h^{-\eta}}$ consistent with $h^{-\eta}$)

Theorem: Well-balancedness for the friction source term

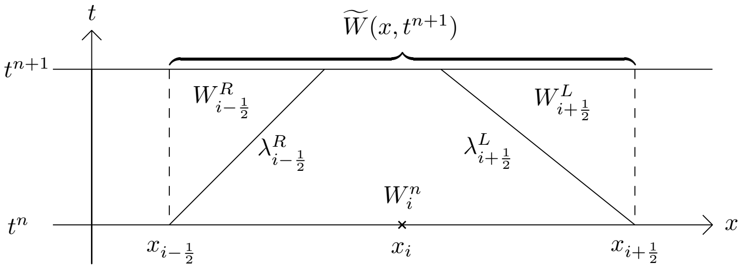
If W_L and W_R define a steady state, i.e. verify

$$q_0^2\frac{[h^{\eta-1}]}{\eta-1} + g\frac{[h^{\eta+2}]}{\eta+2} = -kq_0|q_0|\Delta x,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$.

this result holds for any $\bar{q}$: we choose $\bar{q} = \tilde{q}$

The full Godunov-type scheme



define $W_i^{n+1} = \dfrac{1}{\Delta x} \displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{W}(x, t^{n+1}) dx$: then

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} \left[\lambda^L_{i+\frac{1}{2}} \left(W^L_{i+\frac{1}{2}} - W_i^n \right) - \lambda^R_{i-\frac{1}{2}} \left(W^R_{i-\frac{1}{2}} - W_i^n \right) \right],$$

which can be rewritten, after straightforward computations,

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}} \right) + \Delta t \left(\begin{pmatrix} 0 \\ (\mathcal{S}_{\text{topo}})_i^n \end{pmatrix} + \begin{pmatrix} 0 \\ (\mathcal{S}_{\text{fric}})_i^n \end{pmatrix} \right)$$

Summary

we have presented a scheme that:

- is consistent with the shallow-water equations with friction and topography
- is well-balanced for friction and topography steady states
- preserves the positivity of the water height
- is not able to correctly model wet/dry interfaces: we need a semi-implicitation of the friction source term

$\rightsquigarrow$ how to introduce this semi-implicitation?

Semi-implicit finite volume scheme

we use a splitting method: explicit treatment of the flux and implicit treatment of the source terms

1. $\partial_t W + \partial_x F(W) = 0 \rightsquigarrow W_i^{n+1,=} = W_i^n - \dfrac{\Delta t}{\Delta x}\left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}}\right)$
   (explicit)
2. $\partial_t W = S_{\text{topo}}(W) \rightsquigarrow \begin{cases} h_i^{n+1,-} = h_i^{n+1,=} \\ q_i^{n+1,-} = q_i^{n+1,=} + \Delta t \left(\mathcal{S}_{\text{topo}}\right)_i^{n+1,-} \end{cases}$
   (implicit)
3. $\partial_t W = S_{\text{fric}}(W) \rightsquigarrow \begin{cases} h_i^{n+1} = h_i^{n+1,-} \\ \text{IVP: } \begin{cases} \partial_t q = -kq|q|(h_i^{n+1})^{-\eta} \\ q(x_i, t^n) = q_i^{n+1,-} \end{cases} \rightsquigarrow q_i^{n+1} \end{cases}$
   (exact)

Semi-implicit finite volume scheme

solving the IVP yields:

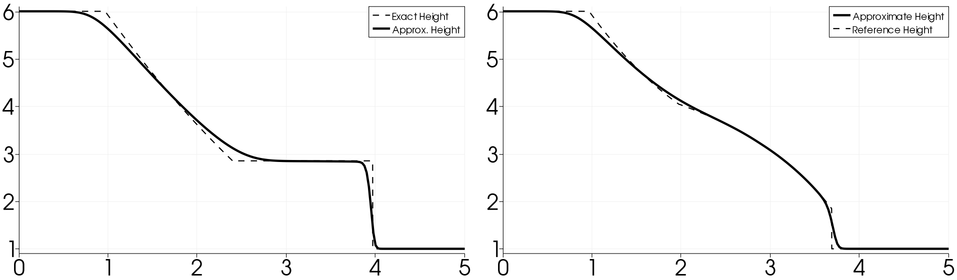
$$q_i^{n+1} = \frac{(h_i^{n+1})^\eta q_i^{n+1,-}}{(h_i^{n+1})^\eta + k\,\Delta t\,|q_i^{n+1,-}|}$$

we use the following approximation of $(h_i^{n+1})^\eta$: this provides us with an expression of q_i^{n+1} that is equal to q_0 at the equilibrium

$$\overline{h^\eta} = 2\frac{\delta_i^{n+1,-}}{\delta_i^n}\frac{1}{\left(\overline{h^{-\eta}}\right)_{i-\frac{1}{2}} + \left(\overline{h^{-\eta}}\right)_{i+\frac{1}{2}}} + k\,\Delta t\,\delta_i^{n+1,-} q_i^n$$

- semi-implicit treatment of the friction source term $\rightsquigarrow$ scheme able to model wet/dry transitions
- scheme still well-balanced and positivity-preserving

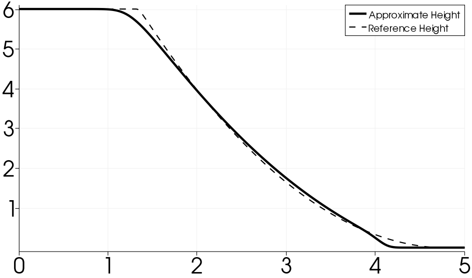
Riemann problems between two wet areas

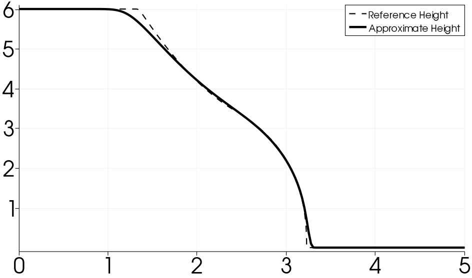


left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.2s

Riemann problems with a wet/dry transition



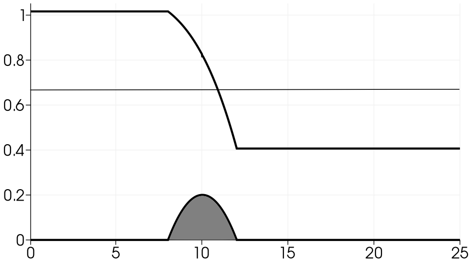


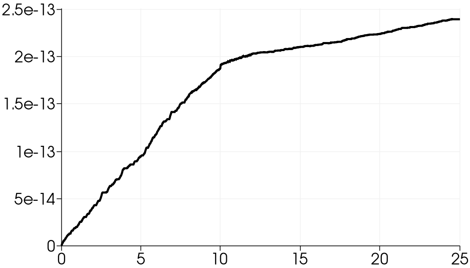
left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.15s

Verification of the well-balancedness: topography

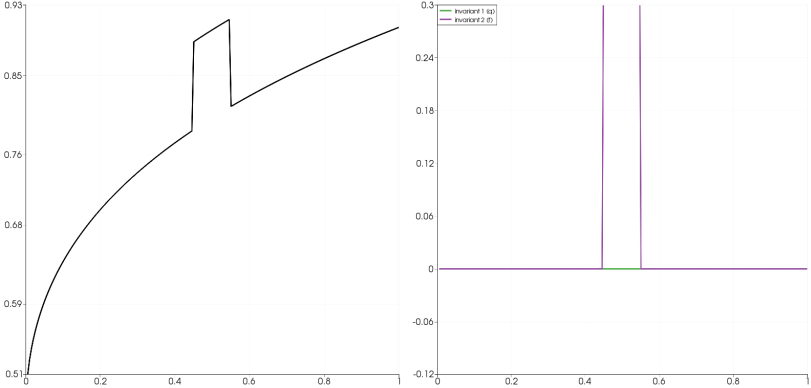
we show the so-called transcritical test case, introduced by Goutal and Maurel (1997): here, we assume $k = 0$





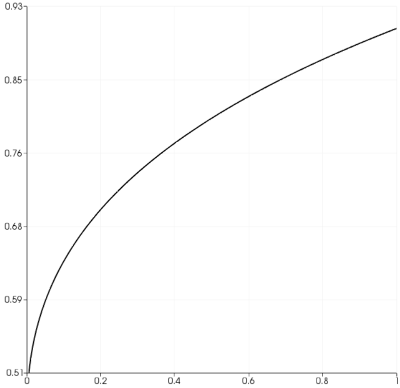
left: initial free surface and free surface for the steady state solution, obtained after a transient state
right: error on the discharge

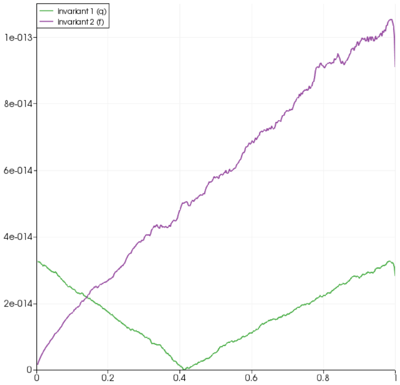
Verification of the well-balancedness: friction



small perturbation of a steady state solution
left: water height; right: errors to the equilibrium

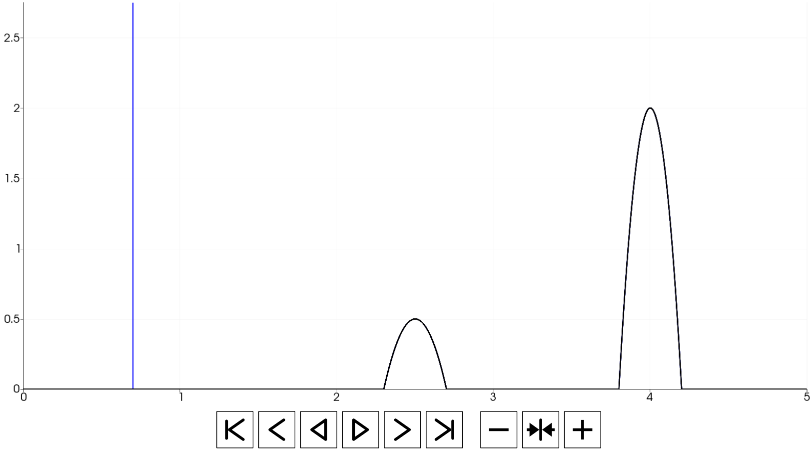
Verification of the well-balancedness: friction





small perturbation of a steady state solution
left: water height; right: errors to the equilibrium

A more complex test case, with topography



$k = 0.1$, $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, final time 3.5s

Conclusion

- well-balancedness for the shallow-water equations with friction and topography
- preservation of the water height positivity
- accurate approximation of wet/dry interfaces

Perspectives

- high-order extension, using MOOD-like techniques to recover the well-balancedness of the scheme
- 2D extension

Thank you for your attention!