Development of high-order well-balanced schemes for geophysical flows

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Thursday, September 29th, 2016

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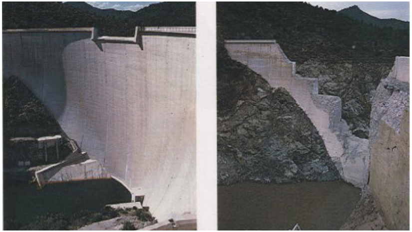
Several kinds of destructive geophysical flows



Dam failure (Malpasset, France, 1959)

Tsunami (Tōhoku, Japan, 2011)

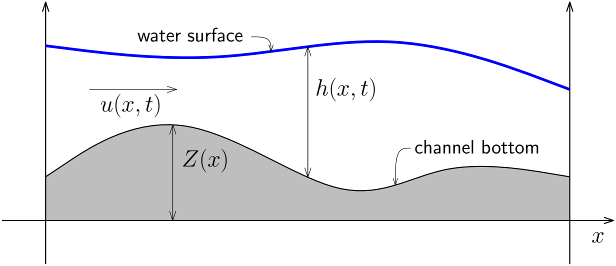
Flood (La Faute sur Mer, France, 2010)

Mudslide (Madeira, Portugal, 2010)

The shallow-water equations and their source terms

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \dfrac{1}{2} g h^2 \right) = -gh \partial_x Z - \dfrac{kq|q|}{h^\eta} \end{cases} \text{(with } q = hu\text{)}$$

We can rewrite the equations as $\partial_t W + \partial_x F(W) = S(W)$, with $W = \begin{pmatrix} h \\ q \end{pmatrix}$.



- $\eta = 7/3$ and g is the gravitational constant
- $k \geq 0$ is the so-called Manning coefficient: a higher k leads to a stronger Manning friction

Steady state solutions

Definition: Steady state solutions

W is a steady state solution iff $\partial_t W = 0$, i.e. $\partial_x F(W) = S(W)$.

Taking $\partial_t W = 0$ in the shallow-water equations leads to

$$\begin{cases} \partial_x q = 0 \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq|q|}{h^\eta}. \end{cases}$$

The steady state solutions are therefore given by

$$\begin{cases} q = \text{cst} = q_0 \\ \partial_x \left(\dfrac{q_0^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq_0|q_0|}{h^\eta}. \end{cases}$$

Steady states for the friction source term

Assume a flat bottom ($Z = \mathrm{cst}$): the steady states are given by

$$\partial_x \left(\frac{q_0^2}{h} + \frac{1}{2} g h^2 \right) = -\frac{k q_0 |q_0|}{h^\eta}.$$

Assuming smooth steady state solutions and integrating this relation between some $x_0 \in \mathbb{R}$ and $x \in \mathbb{R}$ yields (with $h = h(x)$ and $h_0 = h(x_0)$):

$$-\frac{q_0^2}{\eta - 1}\left(h^{\eta-1} - h_0^{\eta-1}\right) + \frac{g}{\eta + 2}\left(h^{\eta+2} - h_0^{\eta+2}\right) + k q_0 |q_0| (x - x_0) = 0.$$

next step: study of the above nonlinear equation, denoted by $\chi(h; x, h_0, x_0, q_0) = 0$

Steady states for the friction source term

1. We show that $\dfrac{\partial \chi}{\partial h}(h; x, h_0, x_0, q_0) < 0$ if and only if $h < h_c$, where

$$h_c = \left(\frac{q_0^2}{g}\right)^{1/3}.$$

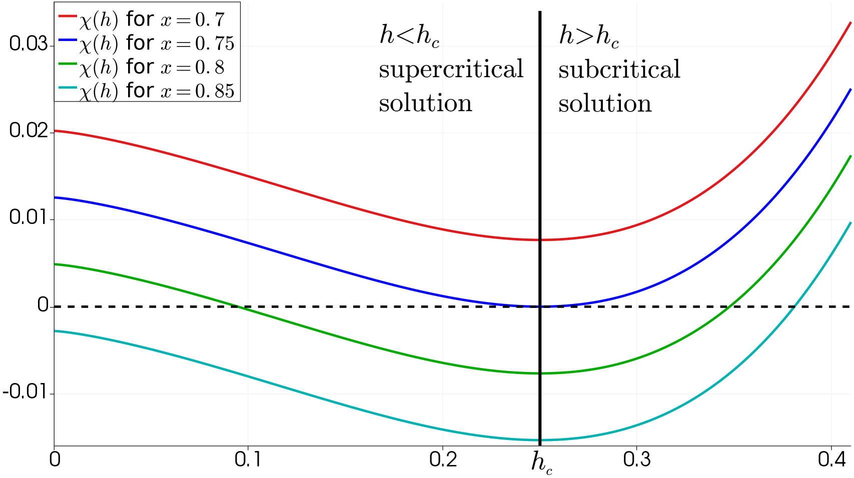
As a consequence, $\chi(h)$ is:
- decreasing for $h < h_c$;
- increasing for $h > h_c$.

2. In the context of a steady state solution, the Froude number is defined, using the sound speed $c = \sqrt{gh}$, by:

$$\mathrm{Fr}(h) = \frac{u}{c} = \frac{q_0^2}{\sqrt{gh^3}}.$$

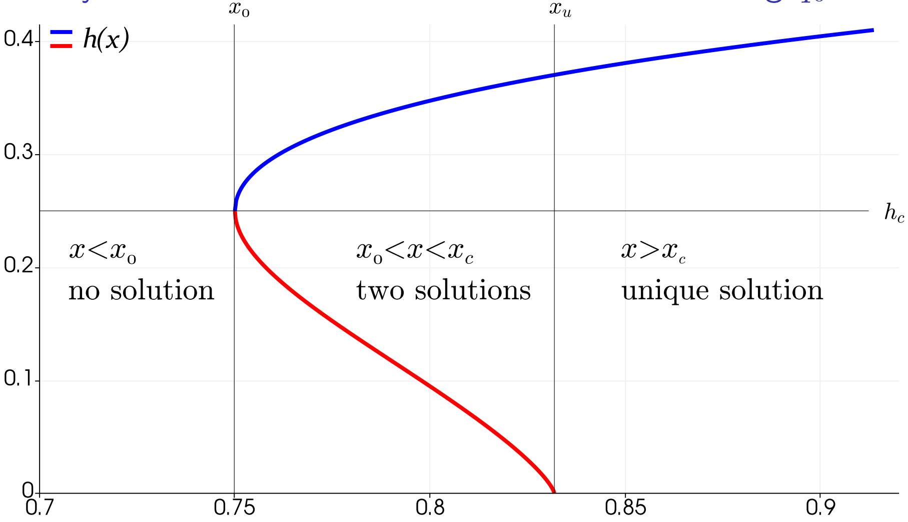
Therefore, $\mathrm{Fr}(h_c) = 1$ and the steady state is:
- supercritical if $h < h_c$;
- subcritical if $h > h_c$.

Steady states for the friction source term, assuming $q_0 < 0$



Sketches of $\chi(h; x)$ for $h \in [0, 0.41]$, for different values of x, and for $h_c = 0.25$. We are interested in the solutions of $\chi(h) = 0$.

Steady states for the friction source term, assuming $q_0 < 0$



blue curve: subcritical solution; red curve: supercritical

Objectives

1. Derive a scheme that:
   - is well-balanced for the shallow-water equations with friction and/or topography, i.e.:
     - preservation of all steady states with $k = 0$ and $Z \neq \mathrm{cst}$,
     - preservation of all steady states with $k \neq 0$ and $Z = \mathrm{cst}$,
     - preservation of steady states with $k \neq 0$ and $Z \neq \mathrm{cst}$;
   - preserves the non-negativity of the water height;
   - is able to deal with wet/dry transitions, where the friction source term is stiff.
2. Provide two-dimensional and high-order extensions of this scheme, while keeping the above properties.

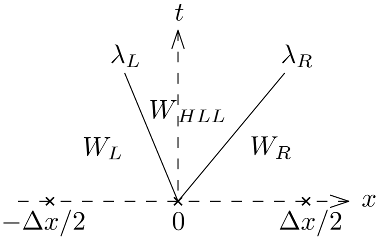
Non-exhaustive state of the art

Well-balanced schemes for the shallow-water equations

- introduction of the well-balance property:
  Bermudez-Vazquez (1994), *Greenberg-LeRoux* (1996)
- preservation of the lake at rest:
  Audusse et al. (2004), *Berthon-Foucher* (2012), *Audusse* et al. (2015)
- 1D fully well-balanced schemes:
  Gosse (2000), *Castro* et al. (2007), *Fjordholm* et al. (2011), *Xing* et al. (2011), *Berthon-Chalons* (2016)
- 1D high-order schemes that preserve steady states:
  Castro et al. (2006), *Castro Díaz* et al. (2013)
- 2D schemes preserving the lake at rest on unstructured meshes:
  Duran et al. (2013), *Clain-Figueiredo* (2014)
- for the friction source term:
  Liang-Marche (2009), *Chertock* et al. (2015)

The HLL scheme

To approximate solutions of $\partial_t W + \partial_x F(W) = 0$, the HLL scheme (Harten, Lax, van Leer (1983)) may be chosen; it uses the approximate Riemann solver $\widetilde{W}$, displayed on the right.



The consistency condition (as per Harten and Lax) holds if:

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(\Delta t, x; W_L, W_R)dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x; W_L, W_R)dx,$$

which gives $W_{HLL} = \dfrac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \dfrac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L} = \begin{pmatrix} h_{HLL} \\ q_{HLL} \end{pmatrix}$.

Note that, if $h_L > 0$ and $h_R > 0$, then $h_{HLL} > 0$ for $|\lambda_L|$ and $|\lambda_R|$ large enough.

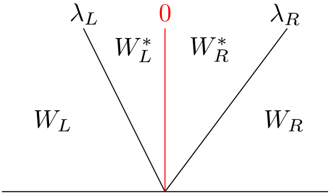
Modification of the HLL scheme

With $Y(t,x) = x$, we rewrite the shallow-water equations with a generic source term S as follows:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{1}{2} g h^2 \right) - S \partial_x Y = 0, \\ \partial_t Y = 0. \end{cases}$$

The equation $\partial_t Y = 0$ induces a stationary wave associated to the source term; we also note that q is a Riemann invariant for this wave.

To approximate solutions of $\partial_t W + \partial_x F(W) = S(W)$, we thus use the approximate Riemann solver displayed on the right (assuming $\lambda_L < 0 < \lambda_R$).



Modification of the HLL scheme

We now wish to apply the Harten-Lax consistency condition to $\partial_t W + \partial_x F(W) = S(W)$. Recall this condition:

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(\Delta t, x; W_L, W_R)dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x; W_L, W_R)dx,$$

first step: compute $\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(\Delta t, x)dx$ (straightforward)

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(\Delta t, x)dx = \frac{W_L + W_R}{2} - \lambda_R \frac{\Delta t}{\Delta x}(W_R - W_R^*) + \lambda_L \frac{\Delta t}{\Delta x}(W_L - W_L^*)$$

second step: compute $\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx$

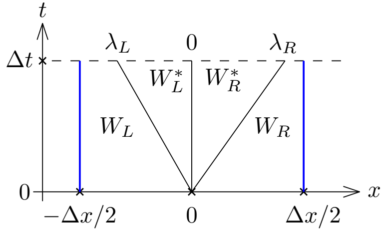
Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) \qquad\qquad)\,dx\,dt = 0$$

Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) \qquad\qquad) \, dx \, dt = 0$$

$$0 = \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx - \int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(0, x)dx\right) + \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, -\frac{\Delta x}{2}\right)dt - \int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, \frac{\Delta x}{2}\right)dt\right)$$



Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) \qquad\qquad)\, dx\, dt = 0$$

$$0 = \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx - \int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(0, x)dx\right) +$$
$$\frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, -\frac{\Delta x}{2}\right)dt - \int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, \frac{\Delta x}{2}\right)dt\right)$$

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx = \frac{W_L + W_R}{2} - \frac{\Delta t}{\Delta x}(F(W_R) - F(W_L))$$

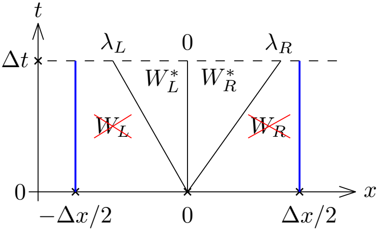
Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) - S(W_{\mathcal{R}}))\, dx\, dt = 0$$

Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}\left(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) - S(W_{\mathcal{R}})\right)\,dx\,dt = 0$$

$$\begin{aligned}0 =& \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx - \int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(0, x)dx\right) + \\ & \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, -\frac{\Delta x}{2}\right)dt - \int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, \frac{\Delta x}{2}\right)dt\right) - \\ & \frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2} S(W_{\mathcal{R}})(t,x)\,dx\,dt\end{aligned}$$



Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) - S(W_{\mathcal{R}}))\,dx\,dt = 0$$

$$\begin{aligned} 0 = & \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx - \int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(0, x)dx\right) + \\ & \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, -\frac{\Delta x}{2}\right)dt - \int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, \frac{\Delta x}{2}\right)dt\right) - \\ & \frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2} S(W_{\mathcal{R}})(t,x)\,dx\,dt \end{aligned}$$

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx \simeq \frac{W_L + W_R}{2} - \frac{\Delta t}{\Delta x}(F(W_R) - F(W_L)) + \overline{S}\Delta t$$

Modification of the HLL scheme

We have 4 unknowns to determine: $W_L^* = \begin{pmatrix} h_L^* \\ q_L^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q_R^* \end{pmatrix}$.

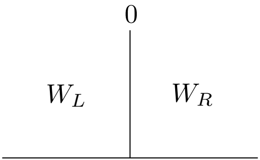
- q is a 0-Riemann invariant $\rightsquigarrow$ we take $q_L^* = q_R^* = q^*$ (relation 1)
- Harten-Lax consistency gives us the following two relations:
  - $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$ (relation 2)
  - $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$ (relation 3)
- **next step**: obtain a fourth relation

Obtaining an additional relation

Assume that W_L and W_R define a steady state, i.e. that they satisfy the following discrete version of the steady relation $\partial_x F(W) = S(W)$ (where $[X] = X_R - X_L$):

$$\frac{1}{\Delta x}\left(q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right]\right) = \overline{S}.$$

For the steady state to be preserved, it is sufficient to have $h_L^* = h_L$, $h_R^* = h_R$ and $q^* = q_0$.



Assuming a steady state, we show that $q^* = q_0$, as follows:

$$q^* = q_{HLL} + \frac{\overline{S}\Delta x}{\lambda_R - \lambda_L} = q_0 - \frac{1}{\lambda_R - \lambda_L}\left(q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] - \overline{S}\Delta x\right) = q_0.$$

Obtaining an additional relation

In order to determine an addition relation, we consider the discrete steady relation, satisfied when W_L and W_R define a steady state:

$$q_0^2 \left(\frac{1}{h_R} - \frac{1}{h_L} \right) + \frac{g}{2} \left((h_R)^2 - (h_L)^2 \right) = \overline{S} \Delta x.$$

To ensure that $h_L^* = h_L$ and $h_R^* = h_R$, we impose that h_L^* and h_R^* satisfy the above relation, as follows:

$$q_0^2 \left(\frac{1}{h_R^*} - \frac{1}{h_L^*} \right) + \frac{g}{2} \left((h_R^*)^2 - (h_L^*)^2 \right) = \overline{S} \Delta x.$$

Determination of h_L^* and h_R^*

The intermediate water heights satisfy the following relation:

$$-q_0^2\left(\frac{h_R^* - h_L^*}{h_L^* h_R^*}\right) + \frac{g}{2}(h_L^* + h_R^*)(h_R^* - h_L^*) = \overline{S}\Delta x.$$

Recall that q^* is known and is equal to q_0 for a steady state.
Instead of the above relation, we choose the following linearization:

$$\frac{-(q^*)^2}{h_L h_R}(h_R^* - h_L^*) + \frac{g}{2}(h_L + h_R)(h_R^* - h_L^*) = \overline{S}\Delta x,$$

which can be rewritten as follows:

$$\underbrace{\left(\frac{-(q^*)^2}{h_L h_R} + \frac{g}{2}(h_L + h_R)\right)}_{\alpha}(h_R^* - h_L^*) = \overline{S}\Delta x.$$

Determination of h_L^* and h_R^*

With the consistency relation between h_L^* and h_R^*, the intermediate water heights satisfy the following linear system:

$$\begin{cases} \alpha(h_R^* - h_L^*) = \overline{S}\Delta x, \\ \lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}. \end{cases}$$

Using both relations linking h_L^* and h_R^*, we obtain

$$\begin{cases} h_L^* = h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \\ h_R^* = h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \end{cases}$$

where $\alpha = \left(\dfrac{-(q^*)^2}{h_L h_R} + \dfrac{g}{2}(h_L + h_R) \right)$ with $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$.

Correction to ensure non-negative h_L^* and h_R^*

However, these expressions of h_L^* and h_R^* do not guarantee that the intermediate heights are non-negative: instead, we use the following cutoff (see Audusse, Chalons, Ung (2014)):

$$\begin{cases} h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \bar{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \bar{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right). \end{cases}$$

Note that this cutoff does not interfere with:

- the consistency condition $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$;
- the well-balance property, since it is not activated when W_L and W_R define a steady state.

Summary

The two-state approximate Riemann solver with intermediate states $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$ given by

$$\begin{cases} q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}, \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right), \end{cases}$$

is consistent, non-negativity-preserving and well-balanced.

next step: determination of $\overline{S}$ according to the source term definition (topography and/or friction).

The topography source term

We now consider $S(W) = S^t(W) = -gh\partial_x Z$:
the smooth steady states are governed by

$$\left.\begin{aligned} &\partial_x\left(\frac{q_0^2}{h}\right) + \frac{g}{2}\partial_x(h^2) = -gh\partial_x Z, \\ &\frac{q_0^2}{2}\partial_x\left(\frac{1}{h^2}\right) + g\partial_x(h+Z) = 0, \end{aligned}\right\} \xrightarrow[\text{discretization}]{} \left\{\begin{aligned} &q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}[h^2] = \overline{S}^t\Delta x, \\ &\frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h+Z] = 0. \end{aligned}\right.$$

We can exhibit an expression of q_0^2 and thus obtain

$$\overline{S}^t = -g\frac{2h_L h_R}{h_L + h_R}\frac{[Z]}{\Delta x} + \frac{g}{2\Delta x}\frac{[h]^3}{h_L + h_R}.$$

However, when $Z_L = Z_R$, we have $\overline{S}^t \neq \mathcal{O}(\Delta x)$, i.e. a loss of consistency with S^t (see for instance Berthon, Chalons (2016)).

The topography source term

Instead, we set, for some constant $C > 0$,

$$\begin{cases} \overline{S}^t = -g \dfrac{2h_L h_R}{h_L + h_R} \dfrac{[Z]}{\Delta x} + \dfrac{g}{2\Delta x} \dfrac{[h]_c^3}{h_L + h_R}, \\ [h]_c = \begin{cases} h_R - h_L & \text{if } |h_R - h_L| \leq C\Delta x, \\ \operatorname{sgn}(h_R - h_L)\, C\Delta x & \text{otherwise.} \end{cases} \end{cases}$$

Theorem: Well-balance for the topography source term

If W_L and W_R define a smooth steady state, i.e. if they satisfy

$$\frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h + Z] = 0,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$ and the approximate Riemann solver is well-balanced.

The friction source term

We consider, in this case, $S(W) = S^f(W) = -kq|q|h^{-\eta}$.

The average of S^f we choose is $\overline{S}^f = -k\bar{q}|\bar{q}|\overline{h^{-\eta}}$, with

- $\bar{q}$ the harmonic mean of q_L and q_R (note that $\bar{q} = q_0$ at the equilibrium);
- $\overline{h^{-\eta}}$ a well-chosen discretization of $h^{-\eta}$, depending on h_L and h_R, and ensuring the well-balance property.

We determine $\overline{h^{-\eta}}$ using the same technique (with $\mu_0 = \operatorname{sgn}(q_0)$):

$$\left.\begin{aligned} &\partial_x\left(\frac{q_0^2}{h}\right) + \frac{g}{2}\partial_x\left(h^2\right) = -kq_0|q_0|h^{-\eta}, \\ &q_0^2\frac{\partial_x h^{\eta-1}}{\eta-1} - g\frac{\partial_x h^{\eta+2}}{\eta+2} = kq_0|q_0|, \end{aligned}\right\} \xrightarrow[\text{discretization}]{} \left\{\begin{aligned} &q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = -k\mu_0 q_0^2\overline{h^{-\eta}}\Delta x, \\ &q_0^2\frac{\left[h^{\eta-1}\right]}{\eta-1} - g\frac{\left[h^{\eta+2}\right]}{\eta+2} = k\mu_0 q_0^2\Delta x. \end{aligned}\right.$$

The friction source term

The expression for q_0^2 we obtained is now used to get:

$$\overline{h^{-\eta}} = \frac{[h^2]}{2} \frac{\eta + 2}{[h^{\eta+2}]} - \frac{\mu_0}{k \Delta x} \left(\left[\frac{1}{h} \right] + \frac{[h^2]}{2} \frac{[h^{\eta-1}]}{\eta - 1} \frac{\eta + 2}{[h^{\eta+2}]} \right),$$

which gives $\overline{S}^f = -k \bar{q} |\bar{q}| \overline{h^{-\eta}}$ ($\overline{h^{-\eta}}$ is consistent with $h^{-\eta}$).

Theorem: Well-balance for the friction source term

If W_L and W_R define a smooth steady state, i.e. verify

$$q_0^2 \frac{[h^{\eta-1}]}{\eta - 1} + g \frac{[h^{\eta+2}]}{\eta + 2} = -k q_0 |q_0| \Delta x,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$ and the approximate Riemann solver is well-balanced.

Friction and topography source terms

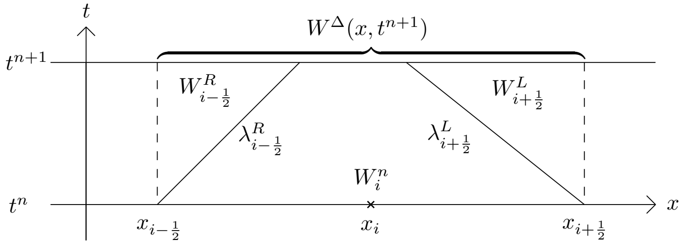
With both source terms, the scheme preserves the following discretization of the steady relation $\partial_x F(W) = S(W)$:

$$q_0^2 \left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = \overline{S}^t \Delta x + \overline{S}^f \Delta x.$$

The intermediate states are therefore given by:

$$\begin{cases} q^* = q_{HLL} + \dfrac{(\overline{S}^t + \overline{S}^f)\Delta x}{\lambda_R - \lambda_L}; \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R(\overline{S}^t + \overline{S}^f)\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right); \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L(\overline{S}^t + \overline{S}^f)\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right). \end{cases}$$

The full Godunov-type scheme



We define $W_i^{n+1} = \dfrac{1}{\Delta x} \displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W^\Delta(x, t^{n+1}) dx$: then

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^L_{i+\frac{1}{2}}\left(W^L_{i+\frac{1}{2}} - W_i^n\right) - \lambda^R_{i-\frac{1}{2}}\left(W^R_{i-\frac{1}{2}} - W_i^n\right)\right],$$

which can be rewritten, after straightforward computations,

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}}\right) + \Delta t \left(\begin{pmatrix} 0 \\ \dfrac{(\mathcal{S}^t)^n_{i-\frac{1}{2}} + (\mathcal{S}^t)^n_{i+\frac{1}{2}}}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{(\mathcal{S}^f)^n_{i-\frac{1}{2}} + (\mathcal{S}^f)^n_{i+\frac{1}{2}}}{2} \end{pmatrix} \right).$$

Summary

We have presented a scheme that:

- is consistent with the shallow-water equations with friction and topography;
- is well-balanced for friction and topography steady states;
- preserves the non-negativity of the water height;
- is not able to correctly approximate wet/dry interfaces due to the stiffness of the friction: we require a semi-implicitation of the friction source term.

next step: introduction of this semi-implicitation

Semi-implicit finite volume scheme

We use a splitting method with an explicit treatment of the flux and the topography and an implicit treatment of the friction.

1. explicitly solve $\partial_t W + \partial_x F(W) = S^t(W)$ to get

$$W_i^{n+\frac{1}{2}} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+\frac{1}{2}}^n - \mathcal{F}_{i-\frac{1}{2}}^n\right) + \Delta t \begin{pmatrix} 0 \\ \frac{1}{2}\left((\mathcal{S}^t)_{i-\frac{1}{2}}^n + (\mathcal{S}^t)_{i+\frac{1}{2}}^n\right) \end{pmatrix}$$

2. implicitly solve $\partial_t W = S^f(W)$ to get

$$\begin{cases} h_i^{n+1} = h_i^{n+\frac{1}{2}} \\ \text{IVP: } \begin{cases} \partial_t q = -kq|q|(h_i^{n+1})^{-\eta} \\ q(x_i, t^n) = q_i^{n+\frac{1}{2}} \end{cases} \rightsquigarrow q_i^{n+1} \end{cases}$$

Semi-implicit finite volume scheme

Solving the IVP yields:

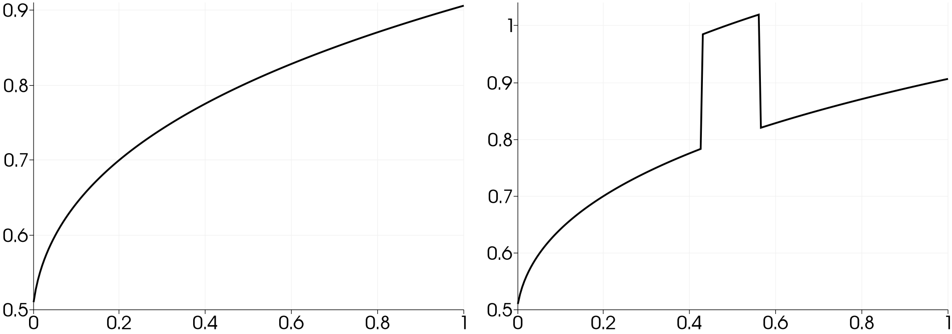
$$q_i^{n+1} = \frac{(h_i^{n+1})^\eta q_i^{n+\frac{1}{2}}}{(h_i^{n+1})^\eta + k\,\Delta t\left|q_i^{n+\frac{1}{2}}\right|}.$$

We use the following approximation of $(h_i^{n+1})^\eta$, which provides us with an expression of q_i^{n+1} that is equal to q_0 at the equilibrium:

$$(\overline{h^\eta})_i^{n+1} = \frac{2\mu_i^{n+\frac{1}{2}}\mu_i^n}{\left(\overline{h^{-\eta}}\right)_{i-\frac{1}{2}}^{n+1} + \left(\overline{h^{-\eta}}\right)_{i+\frac{1}{2}}^{n+1}} + k\,\Delta t\,\mu_i^{n+\frac{1}{2}} q_i^n.$$

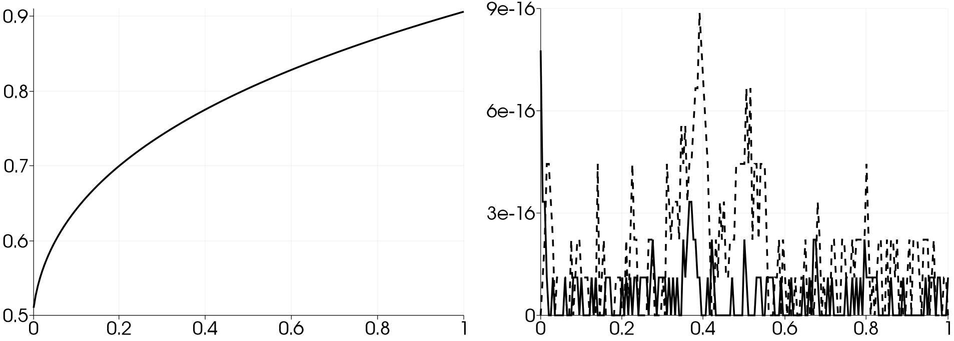
- semi-implicit treatment of the friction source term
  $\rightsquigarrow$ scheme able to model wet/dry transitions
- scheme still well-balanced and non-negativity-preserving

Verification of the well-balance: friction



left panel: water height for the subcritical steady state solution
right panel: water height for the perturbed steady state solution

Verification of the well-balance: friction

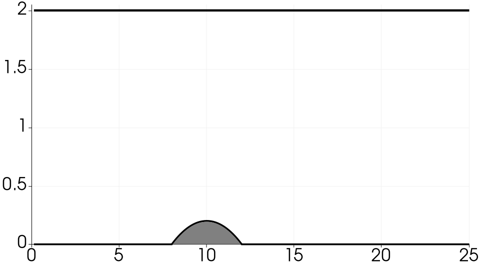


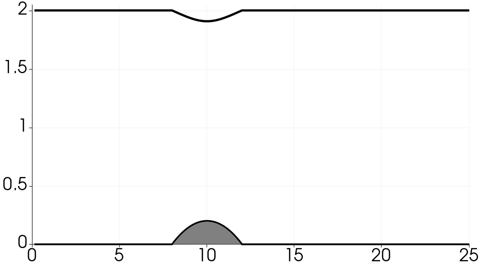
left panel: convergence to the unperturbed steady state

right panel: errors to the steady state (solid: h, dashed: q)

Verification of the well-balance: topography

subcritical flow test case (see Goutal, Maurel (1997))





left panel: initial free surface at rest; water is injected from the left boundary

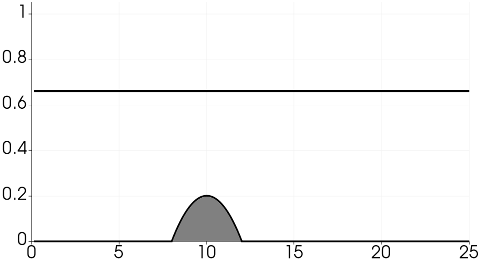
right panel: free surface for the steady state solution, after a transient state

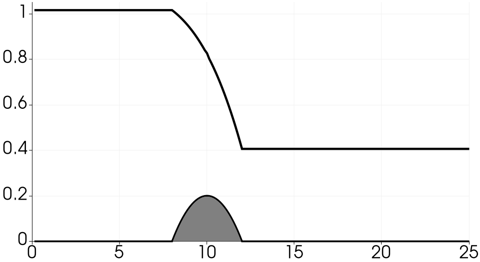
$$\mathcal{E} = \frac{u^2}{2} + g(h+Z)$$

	L^1	L^2	L^∞
errors on q	6.65e-14	6.99e-14	8.26e-14
errors on $\mathcal{E}$	1.18e-13	1.25e-13	1.53e-13

Verification of the well-balance: topography

transcritical flow test case (see Goutal, Maurel (1997))





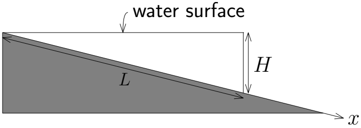
left panel: initial free surface at rest; water is injected from the left boundary

right panel: free surface for the steady state solution, after a transient state

$$\mathcal{E} = \frac{u^2}{2} + g(h + Z)$$

	L^1	L^2	L^∞
errors on q	1.47e-14	1.58e-14	2.04e-14
errors on $\mathcal{E}$	1.67e-14	2.13e-14	4.26e-14

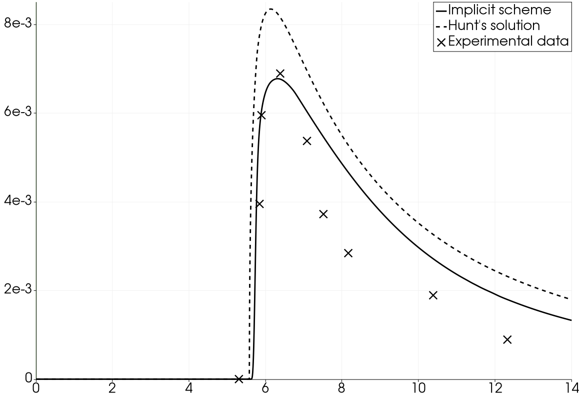
Dry dam-break: Hunt's asymptotic solution



↑

initial condition for the dry dam-break on a sloping channel

water height with respect to the time at a fixed position →



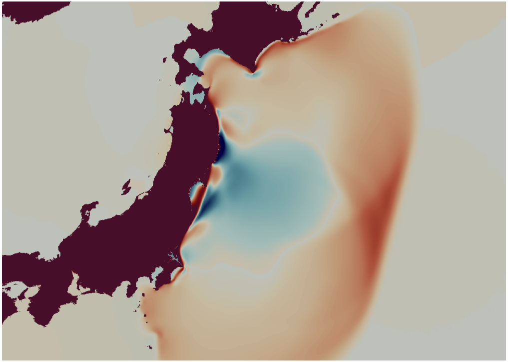
See Hunt (1984) for the experimental points and the solution, valid far enough away from the initial dam.

Two-dimensional extension

2D shallow-water model: $\partial_t W + \boldsymbol{\nabla} \cdot \boldsymbol{F}(W) = \boldsymbol{S}^t(W) + \boldsymbol{S}^f(W)$

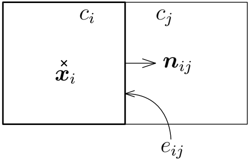
$$\begin{cases} \partial_t h + \boldsymbol{\nabla} \cdot \boldsymbol{q} = 0 \\ \partial_t \boldsymbol{q} + \boldsymbol{\nabla} \cdot \left(\dfrac{\boldsymbol{q} \otimes \boldsymbol{q}}{h} + \dfrac{1}{2} g h^2 \mathbb{I}_2 \right) = -gh \boldsymbol{\nabla} Z - \dfrac{k\boldsymbol{q}\|\boldsymbol{q}\|}{h^\eta} \end{cases}$$

to the right: simulation of the 2011 Japan tsunami



Two-dimensional extension

space discretization: Cartesian mesh



With $\mathcal{F}_{ij}^n = \mathcal{F}(W_i^n, W_j^n; \boldsymbol{n}_{ij})$, the scheme reads:

$$W_i^{n+\frac{1}{2}} = W_i^n - \Delta t \sum_{j \in \nu_i} \frac{|e_{ij}|}{|c_i|} \mathcal{F}_{ij}^n + \frac{\Delta t}{2} \sum_{j \in \nu_i} (\boldsymbol{\mathcal{S}}^t)_{ij}^n.$$

W_i^{n+1} is obtained from $W_i^{n+\frac{1}{2}}$ with a splitting strategy:

$$\begin{cases} \partial_t h = 0 \\ \partial_t \boldsymbol{q} = -k\, \boldsymbol{q} \|\boldsymbol{q}\| h^{-\eta} \end{cases} \rightsquigarrow \begin{cases} h_i^{n+1} = h_i^{n+\frac{1}{2}} \\ \boldsymbol{q}_i^{n+1} = \dfrac{(\overline{h^\eta})_i^{n+1} \boldsymbol{q}_i^{n+\frac{1}{2}}}{(\overline{h^\eta})_i^{n+1} + k\,\Delta t \left\| \boldsymbol{q}_i^{n+\frac{1}{2}} \right\|} \end{cases}$$

Two-dimensional extension

The 2D scheme is:

- non-negativity-preserving for the water height:
  $\forall i \in \mathbb{Z}, h_i^n \geq 0 \Longrightarrow \forall i \in \mathbb{Z}, h_i^{n+1} \geq 0$;
- able to deal with wet/dry transitions thanks to the semi-implicitation with the splitting method;
- well-balanced by direction for the shallow-water equations with friction and/or topography, i.e.:
  - it preserves all steady states at rest,
  - it preserves friction and/or topography steady states in the x-direction and the y-direction,
  - it does not preserve the fully 2D steady states.

next step: high-order extension of this 2D scheme

High-order extension: the polynomial reconstruction

polynomial reconstruction (see Diot, Clain, Loubère (2012)):

$$\widehat{W}_i^n(x) = W_i^n + \sum_{|k|=1}^{d} \alpha_i^k \Big[(x - x_i)^k - M_i^k \Big]$$

- The polynomial coefficients α_i^k are chosen to minimize the least squares error between the reconstruction and W_j^n, for all j in the stencil S_i^d.
- We have $M_i^k = \dfrac{1}{|c_i|} \displaystyle\int_{c_i} (x - x_i)^k dx$ such that

  the conservation property is verified: $\dfrac{1}{|c_i|} \displaystyle\int_{c_i} \widehat{W}_i^n(x) dx = W_i^n$.

High-order extension: the scheme

High-order space accuracy

$$W_i^{n+1} = W_i^n - \Delta t \sum_{j\in\nu_i} \frac{|e_{ij}|}{|c_i|} \sum_{r=0}^{R} \xi_r \mathcal{F}_{ij,r}^n + \Delta t \sum_{q=0}^{Q} \eta_q \Big((\mathcal{S}^t)_{i,q}^n + (\mathcal{S}^f)_{i,q}^n \Big)$$

- $\mathcal{F}_{ij,r}^n = \mathcal{F}(\widehat{W}_i^n(\sigma_r), \widehat{W}_j^n(\sigma_r); \boldsymbol{n}_{ij})$
- $(\mathcal{S}^t)_{i,q}^n = S^t(\widehat{W}_i^n(x_q))$ and $(\mathcal{S}^f)_{i,q}^n = S^f(\widehat{W}_i^n(x_q))$

We have set:

- $(\xi_r, \sigma_r)_r$, a quadrature rule on the edge e_{ij};
- $(\eta_q, x_q)_q$, a quadrature rule on the cell c_i.

The high-order time accuracy is achieved by the use of SSPRK methods (see Gottlieb, Shu (1998)).

MOOD method

High-order schemes induce oscillations: we use the MOOD method to get rid of the oscillations and to restore the non-negativity preservation (see Clain, Diot, Loubère (2011)).

MOOD loop

1. compute a candidate solution W^c with the high-order scheme
2. determine whether W^c is admissible, i.e.
   - if h^c is non-negative (PAD criterion)
   - if W^c does not present spurious oscillations (DMP and u2 criteria)
3. where necessary, decrease the degree of the reconstruction
4. compute a new candidate solution

Well-balance recovery (1D)

reconstruction procedure $\rightsquigarrow$ scheme no longer well-balanced

Well-balance recovery

We suggest a convex combination between the high-order scheme W_{HO} and the well-balanced scheme W_{WB}:

$$W_i^{n+1} = \theta_i^n (W_{HO})_i^{n+1} + (1 - \theta_i^n)(W_{WB})_i^{n+1},$$

with θ_i^n the parameter of the convex combination, such that:

- if $\theta_i^n = 0$, then the well-balanced scheme is used;
- if $\theta_i^n = 1$, then the high-order scheme is used.

next step: derive a suitable expression for θ_i^n

Choice of θ_i^n

Steady state detector

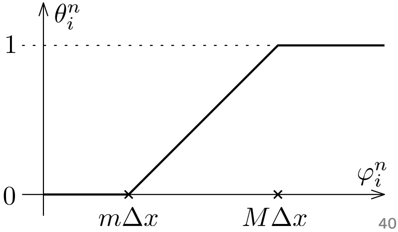
steady state solution: $\begin{cases} q_L = q_R = q_0, \\ \dfrac{q_0^2}{h_R} - \dfrac{q_0^2}{h_L} + \dfrac{g}{2}(h_R^2 - h_L^2) = (\overline{S}^t + \overline{S}^f)\Delta x \end{cases}$

steady state detector: $\varphi_i^n = \left\| \begin{pmatrix} q_i^n - q_{i-1}^n \\ [\mathcal{E}]_{i-\frac{1}{2}}^n \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} q_{i+1}^n - q_i^n \\ [\mathcal{E}]_{i+\frac{1}{2}}^n \end{pmatrix} \right\|_2$

$\varphi_i^n = 0$ if there is a steady state between W_{i-1}^n, W_i^n and W_{i+1}^n

$\rightsquigarrow$ in this case, we take $\theta_i^n = 0$

$\rightsquigarrow$ otherwise, we take $0 < \theta_i^n \leq 1$

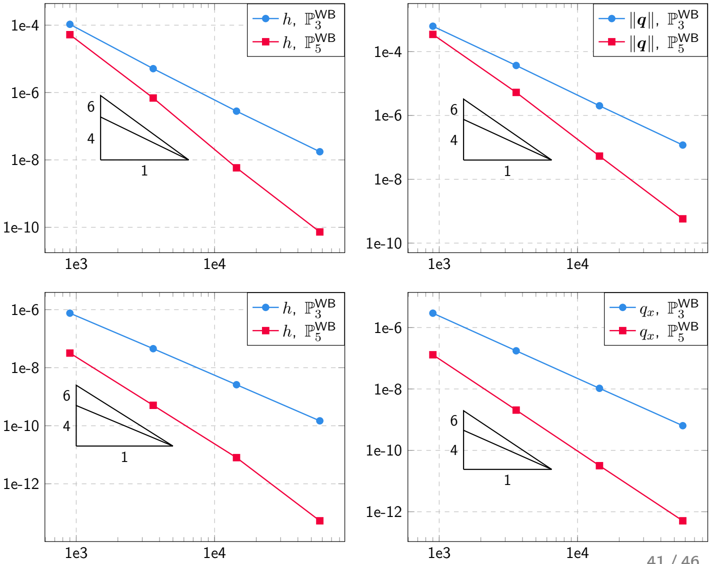


Order of accuracy assessment

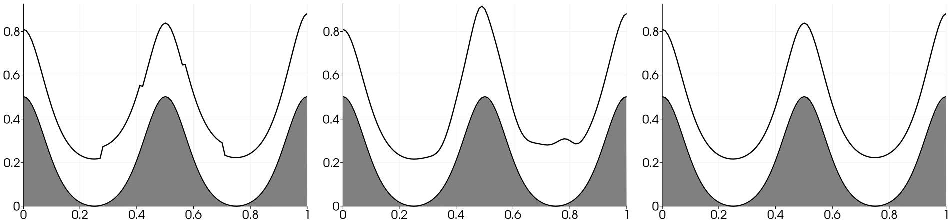
L^2 errors with respect to the number of cells

top graphs: 2D steady solution with topography

bottom graphs: 2D steady solution with friction and topography

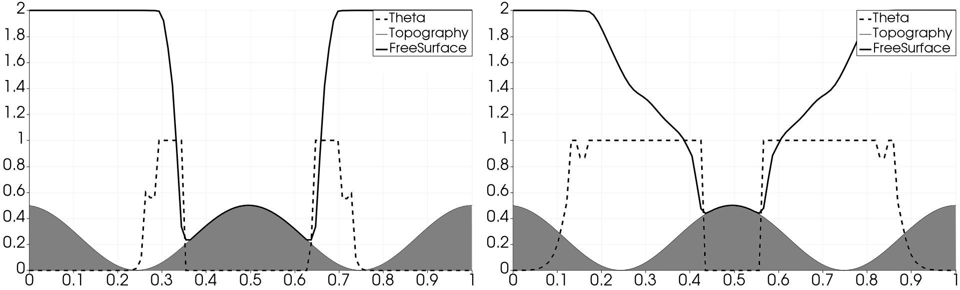


Perturbed pseudo-1D steady state



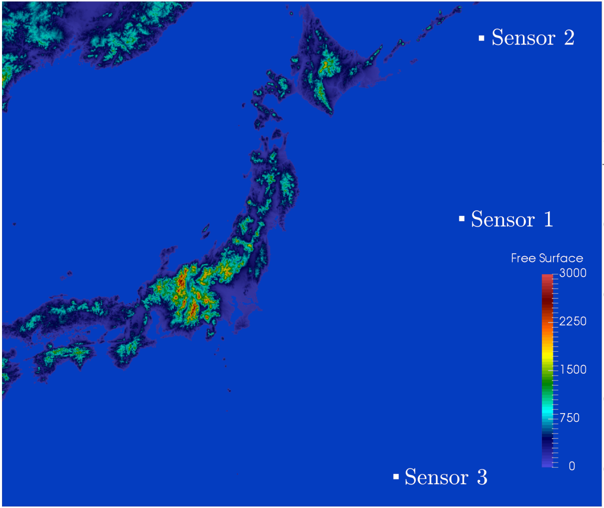
	h			$\|\|\boldsymbol{q}\|\|$		
	L^1	L^2	L^∞	L^1	L^2	L^∞
$\mathbb{P}_0$	1.22e-15	1.71e-15	6.27e-15	2.34e-15	3.02e-15	9.10e-15
$\mathbb{P}_5$	5.01e-05	1.47e-04	1.16e-03	2.32e-04	2.63e-04	1.18e-03
$\mathbb{P}_5^{\mathsf{WB}}$	8.50e-14	1.05e-13	3.35e-13	2.82e-13	3.37e-13	6.76e-13

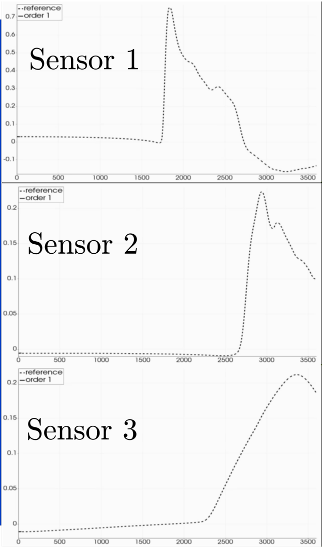
Double dry dam-break on a sinusoidal bottom



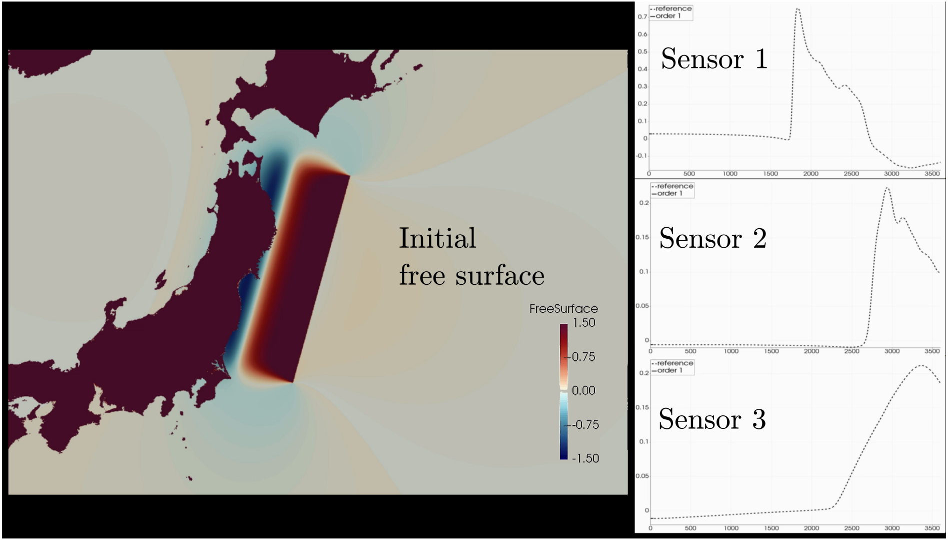
- near the edges, steady state at rest ⇝ well-balanced scheme
- away from the edges, far from steady state ⇝ high-order scheme
- center, dry area ⇝ well-balanced scheme

Simulation of the 2011 Tōhoku tsunami

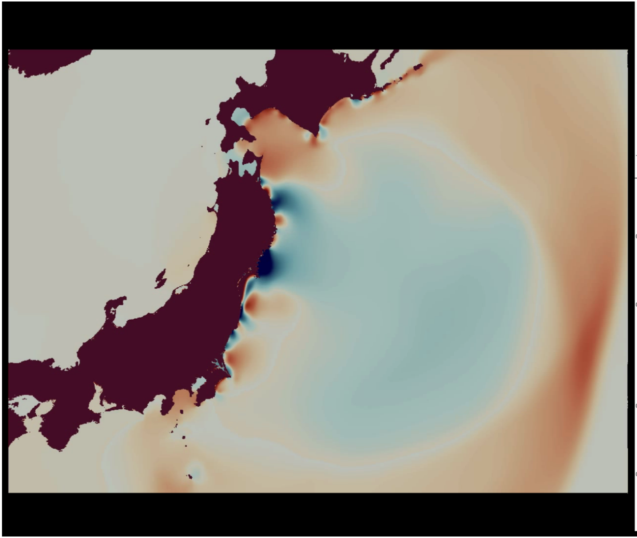


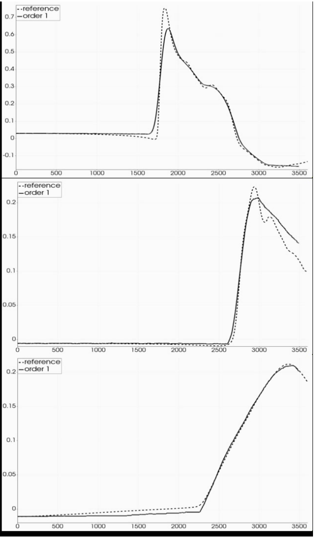


Simulation of the 2011 Tōhoku tsunami



Simulation of the 2011 Tōhoku tsunami





Conclusion

1D scheme:

- well-balanced for the shallow-water equations with friction and topography
- non-negativity-preserving for the water height
- provides a suitable approximation of interfaces between wet and dry areas
- able to be applied to other source terms or combinations of source terms

2D scheme:

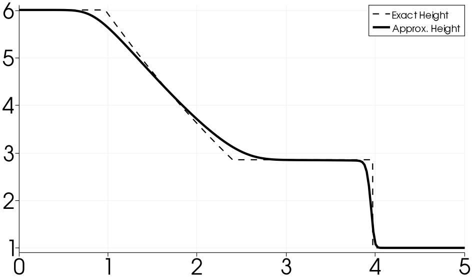
- well-balanced by direction
- non-negativity-preserving, handles wet/dry transitions
- high-order accurate in space and time

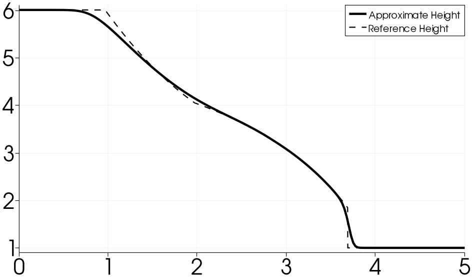
Perspectives

- application to other source terms:
  - Coriolis force source term
  - breadth variation source term
- stability of the scheme:
  - values of C, λ_L and λ_R to ensure the entropy preservation
  - entropy criterion in the MOOD method
- high-order accuracy:
  - rigorous proof of the order of the convex combination
  - reconstruction based on the moving steady states

Thank you for your attention!

Riemann problems between two wet areas

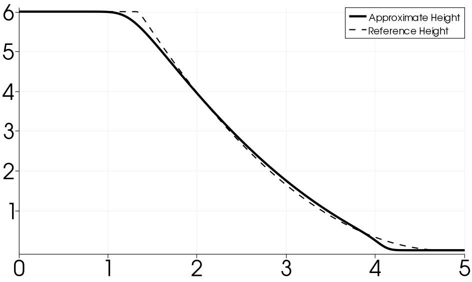


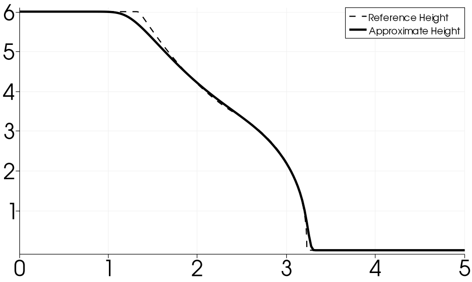


left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.2s

Riemann problems with a wet/dry transition

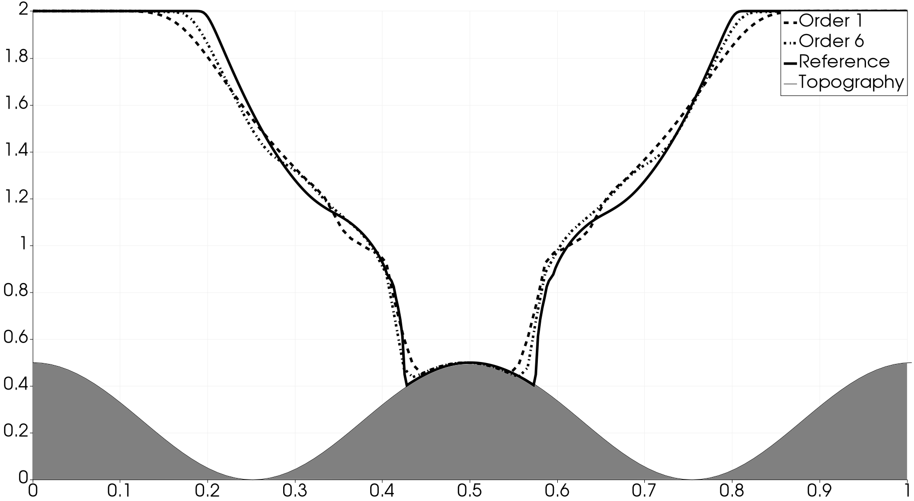




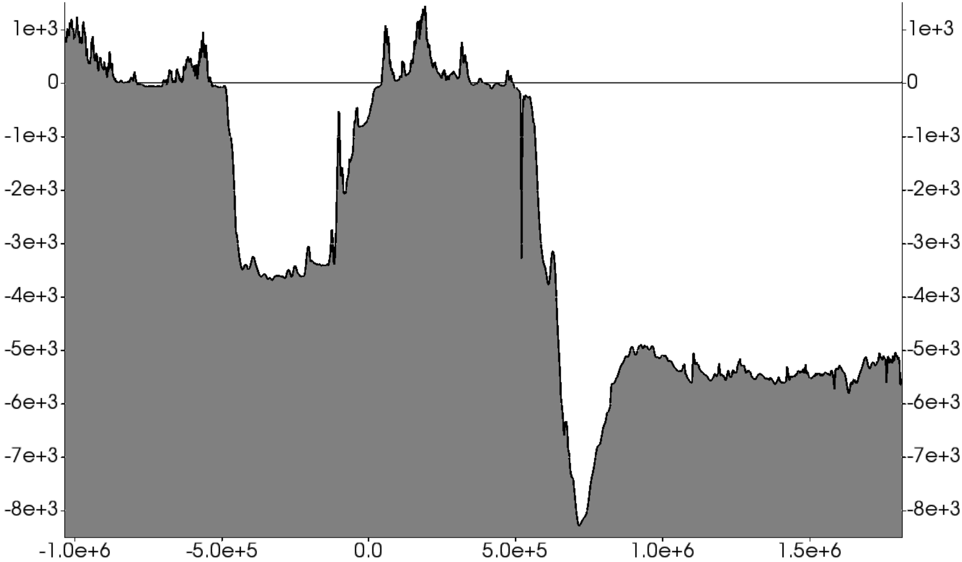
left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.15s

Double dry dam-break on a sinusoidal bottom



Japan tsunami: 1D slice



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