A high-order well-balanced scheme for the shallow-water equations with topography and Manning friction

C. Berthon[1], S. Clain[2], F. Foucher[1,3], R. Loubère[4], V. Michel-Dansac[5]

[1]Laboratoire de Mathématiques Jean Leray, Université de Nantes
[2]Centre of Mathematics, Minho University
[3]École Centrale de Nantes
[4]CNRS et Institut de Mathématiques de Bordeaux
[5]Institut de Mathématiques de Toulouse et INSA Toulouse

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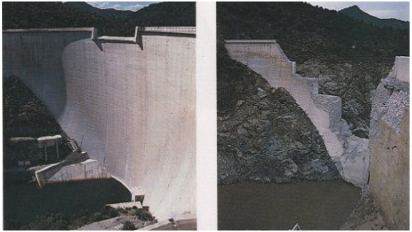
Several kinds of destructive geophysical flows



Dam failure (Malpasset, France, 1959)

Tsunami (Tōhoku, Japan, 2011)

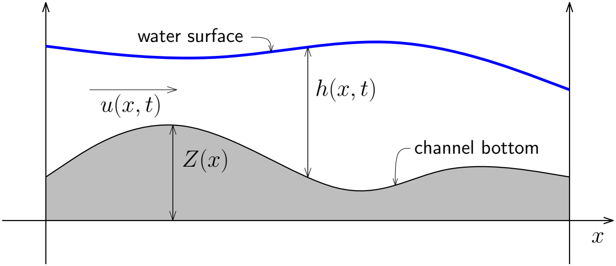
Flood (La Faute sur Mer, France, 2010)

Mudslide (Madeira, Portugal, 2010)

The shallow-water equations and their source terms

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq|q|}{h^{7/3}} \end{cases} \text{(with } q = hu\text{)}$$

We can rewrite the equations as $\partial_t W + \partial_x F(W) = S(W)$, with $W = \begin{pmatrix} h \\ q \end{pmatrix}$.



- $Z(x)$ is the known topography
- k is the Manning coefficient
- g is the gravitational constant
- we label the water discharge $q := hu$

Steady state solutions

Definition: Steady state solutions

W is a steady state solution iff $\partial_t W = 0$, i.e. $\partial_x F(W) = S(W)$.

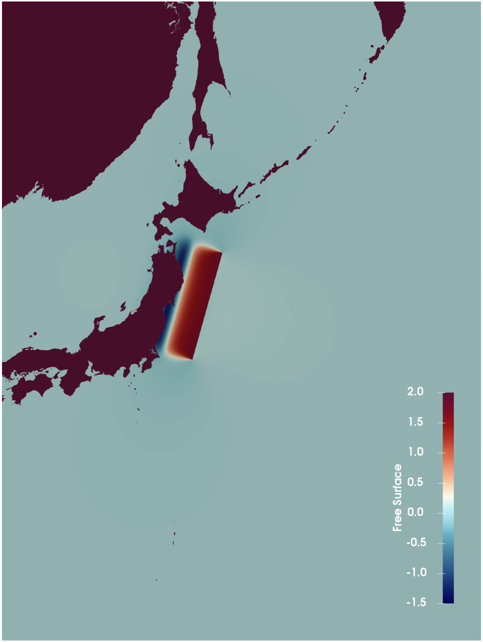
Taking $\partial_t W = 0$ in the shallow-water equations leads to

$$\begin{cases} \partial_x q = 0 \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq|q|}{h^{7/3}}. \end{cases}$$

The steady state solutions are therefore given by

$$\begin{cases} q = \mathrm{cst} = q_0 \\ \partial_x \left(\dfrac{q_0^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq_0|q_0|}{h^{7/3}}. \end{cases}$$

A real-life simulation: the 2011 Tōhoku tsunami.
The water is close to a steady state at rest far from the tsunami.
This steady state is not preserved by a non-well-balanced scheme!



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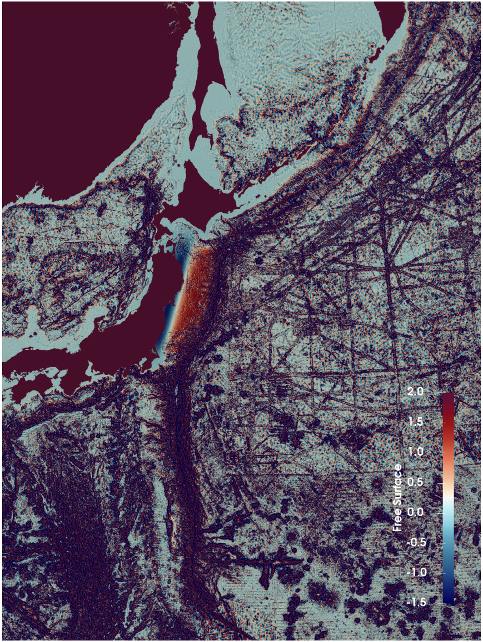


Objectives

Our goal is to derive a numerical method for the shallow-water model with topography and Manning friction that exactly preserves its stationary solutions on every mesh.

To that end, we seek a numerical scheme that:

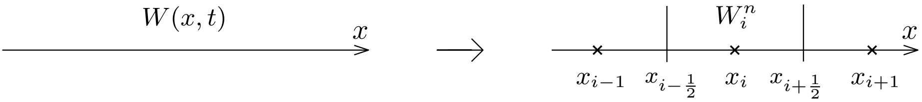
1. is well-balanced for the shallow-water equations with topography and friction, i.e. it exactly preserves and captures the steady states without having to solve the governing nonlinear differential equation;
2. preserves the non-negativity of the water height;
3. ensures a discrete entropy inequality;
4. can be easily extended for other source terms of the shallow-water equations (e.g. breadth).

Setting: finite volume schemes

Objective: Approximate the solution $W(x,t)$ of the system $\partial_t W + \partial_x F(W) = S(W)$, with suitable initial and boundary conditions.

We partition the space domain in *cells*, of volume Δx and of evenly spaced centers x_i, and we define:

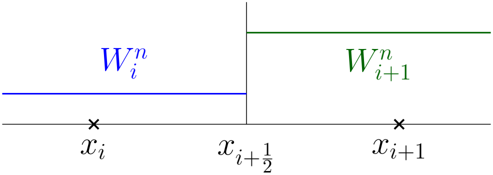
- $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$, the boundaries of the cell i;
- W_i^n, an approximation of $W(x,t)$, constant in the cell i and at time t^n, which is defined as $W_i^n = \dfrac{1}{\Delta x}\displaystyle\int_{\Delta x/2}^{\Delta x/2} W(x,t^n)dx$.



Godunov-type scheme (approximate Riemann solver)

As a consequence, at time t^n, we have a succession of Riemann problems (Cauchy problems with discontinuous initial data) at the interfaces between cells:

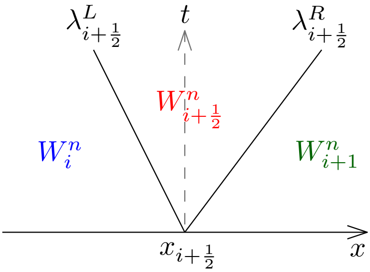
$$\begin{cases} \partial_t W + \partial_x F(W) = S(W) \\ W(x,t^n) = \begin{cases} W_i^n \text{ if } x < x_{i+\frac{1}{2}} \\ W_{i+1}^n \text{ if } x > x_{i+\frac{1}{2}} \end{cases} \end{cases}$$



For $S(W) \neq 0$, the exact solution to these Riemann problems is unknown or costly to compute $\rightsquigarrow$ we require an approximation.

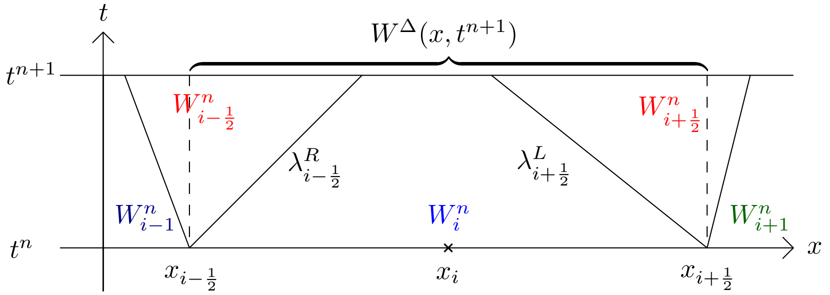
Godunov-type scheme (approximate Riemann solver)

We choose to use an approximate Riemann solver, as follows.



- $W^n_{i+\frac{1}{2}}$ is an approximation of the interaction between W^n_i and W^n_{i+1} (i.e. of the solution to the Riemann problem), possibly made of several constant states separated by discontinuities.
- $\lambda^L_{i+\frac{1}{2}}$ and $\lambda^R_{i+\frac{1}{2}}$ are approximations of the largest wave speeds of the system.

Godunov-type scheme (approximate Riemann solver)



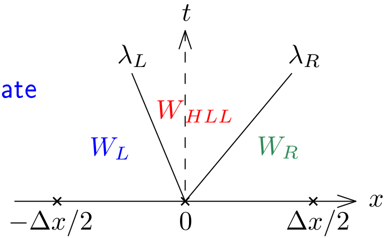
We define the time update as follows:

$$W_i^{n+1} := \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W^{\Delta}(x, t^{n+1}) dx.$$

Since $W^n_{i-\frac{1}{2}}$ and $W^n_{i+\frac{1}{2}}$ are made of constant states, the above integral is easy to compute.

The HLL approximate Riemann solver

To approximate solutions of $\partial_t W + \partial_x F(W) = 0$, the HLL approximate Riemann solver (Harten, Lax, van Leer (1983)) may be chosen; it is denoted by W^Δ and displayed on the right.



The consistency condition (as per Harten and Lax) holds if:

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W^\Delta(\Delta t, x; W_L, W_R)dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x; W_L, W_R)dx,$$

which gives $W_{HLL} = \dfrac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \dfrac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L} = \begin{pmatrix} h_{HLL} \\ q_{HLL} \end{pmatrix}$.

Note that, if $h_L > 0$ and $h_R > 0$, then $h_{HLL} > 0$ for $|\lambda_L|$ and $|\lambda_R|$ large enough.

Modification of the HLL approximate Riemann solver

The shallow-water equations with the topography and friction source terms read as follows:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) + g h \partial_x Z + \dfrac{k q |q|}{h^{7/3}} = 0. \end{cases}$$

Modification of the HLL approximate Riemann solver

With $Y(t,x) := x$, we can add the equations $\partial_t Z = 0$ and $\partial_t Y = 0$, which correspond to the fixed geometry of the problem:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) + g h \partial_x Z + \dfrac{k q |q|}{h^{7/3}} \partial_x Y = 0, \\ \partial_t Y = 0, \\ \partial_t Z = 0. \end{cases}$$

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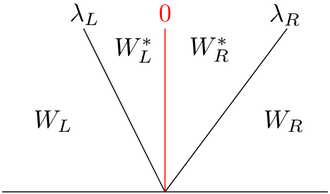
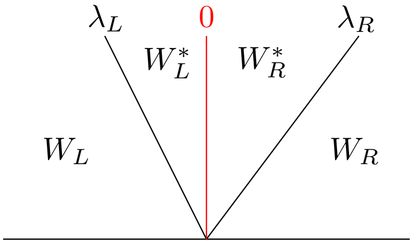
The equations $\partial_t Y = 0$ and $\partial_t Z = 0$ induce stationary waves associated to the source term (of which q is a Riemann invariant).

To approximate solutions of $\partial_t W + \partial_x F(W) = S(W)$, we thus use the approximate Riemann solver displayed on the right (assuming $\lambda_L < 0 < \lambda_R$).



Modification of the HLL approximate Riemann solver

We have 4 unknowns to determine: $W_L^* = \begin{pmatrix} h_L^* \\ q_L^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q_R^* \end{pmatrix}$.



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    where $\overline{S} \simeq \dfrac{1}{\Delta x}\dfrac{1}{\Delta t}\displaystyle\int_{-\Delta x/2}^{\Delta x/2}\int_0^{\Delta t} S(W_{\mathcal{R}}(x,t))\,dt\,dx$.

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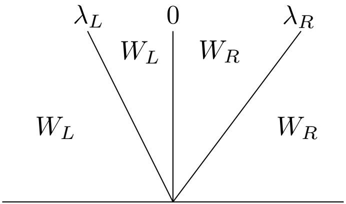
- **next step**: obtain a fourth relation

Obtaining an additional relation

Assume that W_L and W_R define a steady state, i.e. that they satisfy the following discrete version of the steady relation $\partial_x F(W) = S(W)$ (where $[X] = X_R - X_L$):

$$\frac{1}{\Delta x}\left(q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right]\right) = \overline{S}.$$

For the steady state to be preserved, it is sufficient to have $h_L^* = h_L$, $h_R^* = h_R$ and $q^* = q_0$.



Assuming a steady state, we show that $q^* = q_0$, as follows:

$$q^* = q_{HLL} + \frac{\overline{S}\Delta x}{\lambda_R - \lambda_L} = q_0 - \frac{1}{\lambda_R - \lambda_L}\left(q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] - \overline{S}\Delta x\right) = q_0.$$

Obtaining an additional relation

In order to determine an additional relation, we consider the discrete steady relation, satisfied when W_L and W_R define a steady state:

$$q_0^2 \left(\frac{1}{h_R} - \frac{1}{h_L} \right) + \frac{g}{2}\left((h_R)^2 - (h_L)^2\right) = \overline{S}\Delta x.$$

To ensure that $h_L^* = h_L$ and $h_R^* = h_R$, we impose that h_L^* and h_R^* satisfy the above relation, as follows:

$$q_0^2 \left(\frac{1}{h_R^*} - \frac{1}{h_L^*} \right) + \frac{g}{2}\left((h_R^*)^2 - (h_L^*)^2\right) = \overline{S}\Delta x.$$

Determination of h_L^* and h_R^*

The intermediate water heights satisfy the following relation:

$$-q_0^2\left(\frac{h_R^*-h_L^*}{h_L^*h_R^*}\right)+\frac{g}{2}(h_L^*+h_R^*)(h_R^*-h_L^*)=\overline{S}\Delta x.$$

Recall that q^* is known and is equal to q_0 for a steady state.
Instead of the above relation, we choose the following linearization:

$$\frac{-(q^*)^2}{h_Lh_R}(h_R^*-h_L^*)+\frac{g}{2}(h_L+h_R)(h_R^*-h_L^*)=\overline{S}\Delta x,$$

which can be rewritten as follows:

$$\underbrace{\left(\frac{-(q^*)^2}{h_Lh_R}+\frac{g}{2}(h_L+h_R)\right)}_{\alpha}(h_R^*-h_L^*)=\overline{S}\Delta x.$$

Determination of h_L^* and h_R^*

With the consistency relation between h_L^* and h_R^*, the intermediate water heights satisfy the following linear system:

$$\begin{cases} \alpha(h_R^* - h_L^*) = \overline{S}\Delta x, \\ \lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}. \end{cases}$$

Using both relations linking h_L^* and h_R^*, we obtain

$$\begin{cases} h_L^* = h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \\ h_R^* = h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \end{cases}$$

where $\alpha = \left(\dfrac{-(q^*)^2}{h_L h_R} + \dfrac{g}{2}(h_L + h_R) \right)$ with $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$.

Correction to ensure non-negative h_L^* and h_R^*

However, these expressions of h_L^* and h_R^* do not guarantee that the intermediate heights are non-negative: instead, we use the following cutoff (see Audusse, Chalons, Ung (2015)):

$$\begin{cases} h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right). \end{cases}$$

Note that this cutoff does not interfere with:

- the consistency condition $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$;
- the well-balance property, since it is not activated when W_L and W_R define a steady state.

Summary

The two-state approximate Riemann solver with intermediate states $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$ given by

$$\begin{cases} q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}, \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right), \end{cases}$$

is consistent, non-negativity-preserving, entropy preserving and well-balanced.

next step: determination of $\overline{S}$ according to the source term definition (topography or friction).

The topography source term

We now consider $S(W) = S^t(W) = -gh\partial_x Z$:
the smooth steady states are governed by

$$\left.\begin{aligned} \partial_x\left(\frac{q_0^2}{h}\right) + \frac{g}{2}\partial_x\left(h^2\right) &= -gh\partial_x Z, \\ \frac{q_0^2}{2}\partial_x\left(\frac{1}{h^2}\right) + g\partial_x(h+Z) &= 0, \end{aligned}\right\} \xrightarrow[\text{discretization}]{} \left\{\begin{aligned} q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] &= \overline{S}^t\Delta x, \\ \frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h+Z] &= 0. \end{aligned}\right.$$

We can exhibit an expression of q_0^2 and thus obtain

$$\overline{S}^t = -g\frac{2h_L h_R}{h_L + h_R}\frac{[Z]}{\Delta x} + \frac{g}{2\Delta x}\frac{[h]^3}{h_L + h_R}.$$

However, when $Z_L = Z_R$, we have $\overline{S}^t \neq \mathcal{O}(\Delta x)$, i.e. a loss of consistency with S^t (see for instance Berthon, Chalons (2016)).

The topography source term

Instead, we set, for some constant $C > 0$,

$$\begin{cases} \overline{S}^t = -g \dfrac{2h_L h_R}{h_L + h_R} \dfrac{[Z]}{\Delta x} + \dfrac{g}{2\Delta x} \dfrac{[h]_c^3}{h_L + h_R}, \\ [h]_c = \begin{cases} h_R - h_L & \text{if } |h_R - h_L| \leq C\Delta x, \\ \operatorname{sgn}(h_R - h_L)\, C\Delta x & \text{otherwise.} \end{cases} \end{cases}$$

Theorem: Well-balance for the topography source term

If W_L and W_R define a smooth steady state, i.e. if they satisfy

$$\frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h+Z] = 0,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$ and the approximate Riemann solver is well-balanced. By construction, the Godunov-type scheme using this approximate Riemann solver is consistent, well-balanced, non-negativity-preserving and entropy preserving.

The friction source term

We consider, in this case, $S(W) = S^f(W) = -kq|q|h^{-\eta}$, where we have set $\eta = 7/3$.

The average of S^f we choose is $\overline{S}^f = -k\bar{q}|\bar{q}|\overline{h^{-\eta}}$, with

- $\bar{q}$ the harmonic mean of q_L and q_R (note that $\bar{q} = q_0$ at the equilibrium);
- $\overline{h^{-\eta}}$ a well-chosen discretization of $h^{-\eta}$, depending on h_L and h_R, and ensuring the well-balance property.

We determine $\overline{h^{-\eta}}$ using the same technique (with $\mu_0 = \operatorname{sgn}(q_0)$):

$$\left.\begin{aligned} &\partial_x\left(\frac{q_0^2}{h}\right) + \frac{g}{2}\partial_x\left(h^2\right) = -kq_0|q_0|h^{-\eta}, \\ &q_0^2\frac{\partial_x h^{\eta-1}}{\eta-1} - g\frac{\partial_x h^{\eta+2}}{\eta+2} = kq_0|q_0|, \end{aligned}\right\} \xrightarrow[\text{discretization}]{} \left\{\begin{aligned} &q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = -k\mu_0 q_0^2\overline{h^{-\eta}}\Delta x, \\ &q_0^2\frac{\left[h^{\eta-1}\right]}{\eta-1} - g\frac{\left[h^{\eta+2}\right]}{\eta+2} = k\mu_0 q_0^2\Delta x. \end{aligned}\right.$$

The friction source term

The expression for q_0^2 we obtained is now used to get:

$$\overline{h^{-\eta}} = \frac{[h^2]}{2} \frac{\eta+2}{[h^{\eta+2}]} - \frac{\mu_0}{k\Delta x}\left(\left[\frac{1}{h}\right] + \frac{[h^2]}{2} \frac{[h^{\eta-1}]}{\eta-1} \frac{\eta+2}{[h^{\eta+2}]}\right),$$

which gives $\overline{S}^f = -k\bar{q}|\bar{q}|\overline{h^{-\eta}}$ ($\overline{h^{-\eta}}$ is consistent with $h^{-\eta}$ if a cutoff is applied to the second term of $\overline{h^{-\eta}}$).

Theorem: Well-balance for the friction source term

If W_L and W_R define a smooth steady state, i.e. verify

$$q_0^2 \frac{[h^{\eta-1}]}{\eta-1} + g\frac{[h^{\eta+2}]}{\eta+2} = -kq_0|q_0|\Delta x,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$ and the approximate Riemann solver is well-balanced. By construction, the Godunov-type scheme using this approximate Riemann solver is consistent, well-balanced, non-negativity-preserving and entropy preserving.

Friction and topography source terms

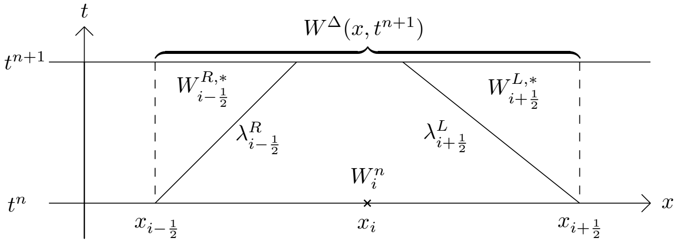
With both source terms, the scheme preserves the following discretization of the steady relation $\partial_x F(W) = S(W)$:

$$q_0^2 \left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = \overline{S}^t \Delta x + \overline{S}^f \Delta x.$$

The intermediate states are therefore given by:

$$\begin{cases} q^* = q_{HLL} + \dfrac{(\overline{S}^t + \overline{S}^f)\Delta x}{\lambda_R - \lambda_L}; \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R(\overline{S}^t + \overline{S}^f)\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL} \right); \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L(\overline{S}^t + \overline{S}^f)\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL} \right). \end{cases}$$

The full Godunov-type scheme



We recall $W_i^{n+1} = \dfrac{1}{\Delta x} \displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W^\Delta(x, t^{n+1}) dx$: then

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} \left[\lambda^L_{i+\frac{1}{2}} \left(W^{L,*}_{i+\frac{1}{2}} - W_i^n \right) - \lambda^R_{i-\frac{1}{2}} \left(W^{R,*}_{i-\frac{1}{2}} - W_i^n \right) \right],$$

which can be rewritten, after straightforward computations,

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}} \right) + \Delta t \left(\begin{pmatrix} 0 \\ \dfrac{(\mathcal{S}^t)^n_{i-\frac{1}{2}} + (\mathcal{S}^t)^n_{i+\frac{1}{2}}}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{(\mathcal{S}^f)^n_{i-\frac{1}{2}} + (\mathcal{S}^f)^n_{i+\frac{1}{2}}}{2} \end{pmatrix} \right).$$

Summary

We have presented a scheme that:

- is consistent with the shallow-water equations with friction and topography;
- is well-balanced for friction and topography steady states;
- preserves the non-negativity of the water height;
- ensures a discrete entropy inequality;
- is not able to correctly approximate wet/dry interfaces due to the stiffness of the friction $kq|q|h^{-7/3}$: the friction term should be treated implicitly.

next step: introduction of this semi-implicit scheme

Semi-implicit finite volume scheme

We use a splitting method with an explicit treatment of the flux and the topography and an implicit treatment of the friction.

1. explicitly solve $\partial_t W + \partial_x F(W) = S^t(W)$ as follows:

$$W_i^{n+\frac{1}{2}} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+\frac{1}{2}}^n - \mathcal{F}_{i-\frac{1}{2}}^n\right) + \Delta t \begin{pmatrix} 0 \\ \frac{1}{2}\left((\mathcal{S}^t)_{i-\frac{1}{2}}^n + (\mathcal{S}^t)_{i+\frac{1}{2}}^n\right) \end{pmatrix}$$

2. implicitly solve $\partial_t W = S^f(W)$ as follows:

$$\begin{cases} h_i^{n+1} = h_i^{n+\frac{1}{2}} \\ \text{IVP: } \begin{cases} \partial_t q = -kq|q|(h_i^{n+1})^{-\eta} \\ q(x_i, t^n) = q_i^{n+\frac{1}{2}} \end{cases} \rightsquigarrow q_i^{n+1} \end{cases}$$

Semi-implicit finite volume scheme

Solving the IVP yields:

$$q_i^{n+1} = \frac{(h_i^{n+1})^\eta q_i^{n+\frac{1}{2}}}{(h_i^{n+1})^\eta + k\,\Delta t\left|q_i^{n+\frac{1}{2}}\right|}.$$

We use the following approximation of $(h_i^{n+1})^\eta$, which provides us with an expression of q_i^{n+1} that is equal to q_0 at the equilibrium:

$$(\overline{h^\eta})_i^{n+1} = \frac{2\mu_i^{n+\frac{1}{2}}\mu_i^n}{\left(\overline{h^{-\eta}}\right)_{i-\frac{1}{2}}^{n+1} + \left(\overline{h^{-\eta}}\right)_{i+\frac{1}{2}}^{n+1}} + k\,\Delta t\,\mu_i^{n+\frac{1}{2}} q_i^n.$$

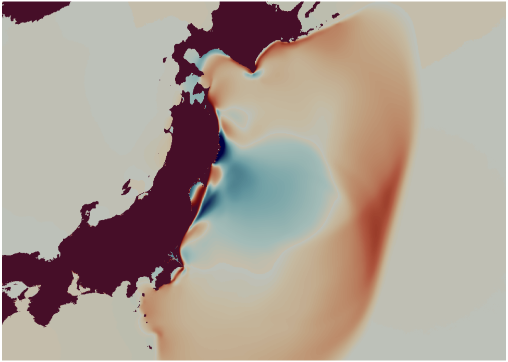
- semi-implicit treatment of the friction source term
  $\rightsquigarrow$ scheme able to model wet/dry transitions
- scheme still well-balanced and non-negativity-preserving

Two-dimensional extension

2D shallow-water model: $\partial_t W + \boldsymbol{\nabla} \cdot \boldsymbol{F}(W) = \boldsymbol{S}^t(W) + \boldsymbol{S}^f(W)$

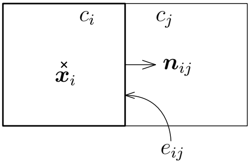
$$\begin{cases} \partial_t h + \boldsymbol{\nabla} \cdot \boldsymbol{q} = 0 \\ \partial_t \boldsymbol{q} + \boldsymbol{\nabla} \cdot \left(\dfrac{\boldsymbol{q} \otimes \boldsymbol{q}}{h} + \dfrac{1}{2} g h^2 \mathbb{I}_2 \right) = -gh \boldsymbol{\nabla} Z - \dfrac{k \boldsymbol{q} \|\boldsymbol{q}\|}{h^\eta} \end{cases}$$

to the right: simulation of the 2011 Japan tsunami



Two-dimensional extension

space discretization: Cartesian mesh



With $\mathcal{F}_{ij}^n = \mathcal{F}(W_i^n, W_j^n; \boldsymbol{n}_{ij})$ and ν_i the neighbors of c_i, the scheme reads:

$$W_i^{n+\frac{1}{2}} = W_i^n - \Delta t \sum_{j \in \nu_i} \frac{|e_{ij}|}{|c_i|} \mathcal{F}_{ij}^n + \frac{\Delta t}{2} \sum_{j \in \nu_i} (\boldsymbol{\mathcal{S}}^t)_{ij}^n.$$

W_i^{n+1} is obtained from $W_i^{n+\frac{1}{2}}$ with a splitting strategy:

$$\begin{cases} \partial_t h = 0 \\ \partial_t \boldsymbol{q} = -k\,\boldsymbol{q}\|\boldsymbol{q}\|h^{-\eta} \end{cases} \rightsquigarrow \begin{cases} h_i^{n+1} = h_i^{n+\frac{1}{2}} \\[2ex] \boldsymbol{q}_i^{n+1} = \dfrac{(\overline{h^\eta})_i^{n+1} \boldsymbol{q}_i^{n+\frac{1}{2}}}{(\overline{h^\eta})_i^{n+1} + k\,\Delta t \left\|\boldsymbol{q}_i^{n+\frac{1}{2}}\right\|} \end{cases}$$

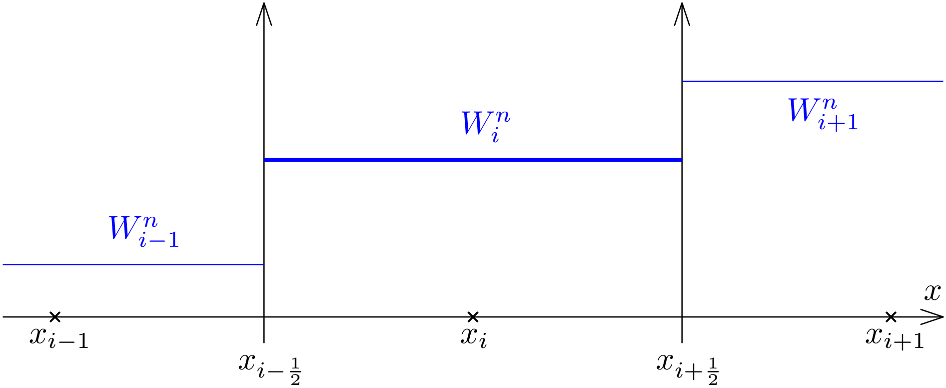
Two-dimensional extension

The 2D scheme is:

- non-negativity-preserving for the water height:
  $\forall i \in \mathbb{Z},\ h_i^n \geq 0 \implies \forall i \in \mathbb{Z},\ h_i^{n+1} \geq 0;$
- able to deal with wet/dry transitions thanks to the semi-implicitation with the splitting method;
- well-balanced by direction for the shallow-water equations with friction and/or topography, i.e.:
  - it preserves all steady states at rest,
  - it preserves friction and/or topography steady states in the x-direction and the y-direction,
  - it does not preserve the fully 2D steady states.

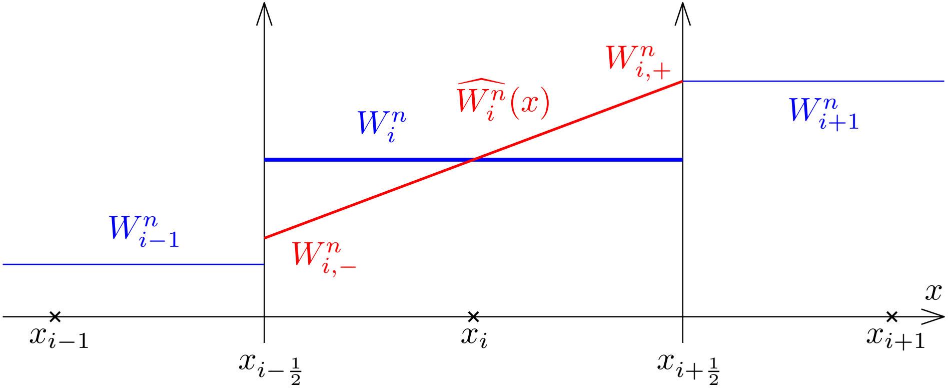
next step: high-order extension of this 2D scheme

High-order extension: the basics, in 1D



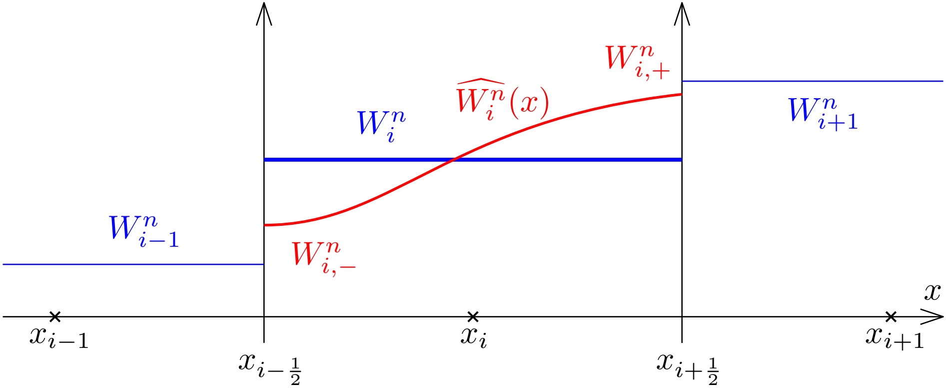
$W_i^n \in \mathbb{P}_0$: constant (order 1 scheme)

High-order extension: the basics, in 1D



$\widehat{W_i^n} \in \mathbb{P}_1$: linear (order 2 scheme)

High-order extension: the basics, in 1D



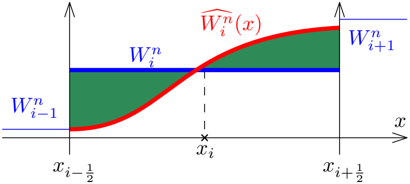
$\widehat{W_i^n} \in \mathbb{P}_d$: polynomial (order $d+1$ scheme)

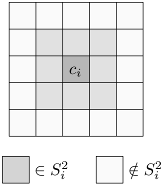
High-order extension: the polynomial reconstruction

polynomial reconstruction (see Diot, Clain, Loubère (2012)):

$$\widehat{W}_i^n(x) = W_i^n + \sum_{|k|=1}^{d} \alpha_i^k \left[(x - x_i)^k - M_i^k \right]$$

- We have $M_i^k = \dfrac{1}{|c_i|} \displaystyle\int_{c_i} (x - x_i)^k dx$ such that the conservation property is verified: $\dfrac{1}{|c_i|} \displaystyle\int_{c_i} \widehat{W}_i^n(x) dx = W_i^n$.





- The polynomial coefficients α_i^k are chosen to minimize the least squares error between the reconstruction and W_j^n, for all j in the stencil S_i^d.

High-order extension: the scheme

High-order space accuracy

$$W_i^{n+1} = W_i^n - \Delta t \sum_{j\in\nu_i} \frac{|e_{ij}|}{|c_i|} \sum_{r=0}^{R} \xi_r \mathcal{F}_{ij,r}^n + \Delta t \sum_{q=0}^{Q} \eta_q \Big((\mathcal{S}^t)_{i,q}^n + (\mathcal{S}^f)_{i,q}^n \Big)$$

- $\mathcal{F}_{ij,r}^n = \mathcal{F}(\widehat{W}_i^n(\sigma_r), \widehat{W}_j^n(\sigma_r); \boldsymbol{n}_{ij})$
- $(\mathcal{S}^t)_{i,q}^n = S^t(\widehat{W}_i^n(x_q))$ and $(\mathcal{S}^f)_{i,q}^n = S^f(\widehat{W}_i^n(x_q))$

We have set:

- $(\xi_r, \sigma_r)_r$, a quadrature rule on the edge e_{ij};
- $(\eta_q, x_q)_q$, a quadrature rule on the cell c_i.

The high-order time accuracy is achieved by the use of SSPRK methods (see Gottlieb, Shu (1998)).

Well-balance recovery (1D): a convex combination

reconstruction procedure $\rightsquigarrow$ the scheme no longer preserves steady states

Well-balance recovery

We suggest a convex combination between the high-order scheme W_{HO} and the well-balanced scheme W_{WB}:

$$W_i^{n+1} = \theta_i^n (W_{HO})_i^{n+1} + (1 - \theta_i^n)(W_{WB})_i^{n+1},$$

with θ_i^n the parameter of the convex combination, such that:

- if $\theta_i^n = 0$, then the well-balanced scheme is used;
- if $\theta_i^n = 1$, then the high-order scheme is used.

next step: derive a suitable expression for θ_i^n

Well-balance recovery (1D): a steady state detector

Steady state detector

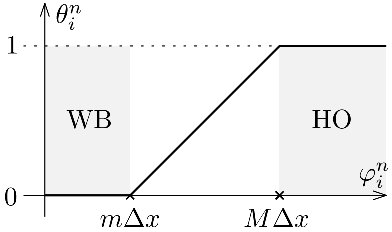
steady state solution: $\begin{cases} q_L = q_R = q_0, \\ \mathcal{E} := \dfrac{q_0^2}{h_R} - \dfrac{q_0^2}{h_L} + \dfrac{g}{2}(h_R^2 - h_L^2) - (\overline{S}^t + \overline{S}^f)\Delta x = 0 \end{cases}$

steady state detector: $\varphi_i^n = \left\| \begin{pmatrix} q_i^n - q_{i-1}^n \\ [\mathcal{E}]_{i-\frac{1}{2}}^n \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} q_{i+1}^n - q_i^n \\ [\mathcal{E}]_{i+\frac{1}{2}}^n \end{pmatrix} \right\|_2$

$\varphi_i^n = 0$ if there is a steady state between W_{i-1}^n, W_i^n and W_{i+1}^n

⇝ in this case, we take $\theta_i^n = 0$

⇝ otherwise, we take $0 < \theta_i^n \leq 1$



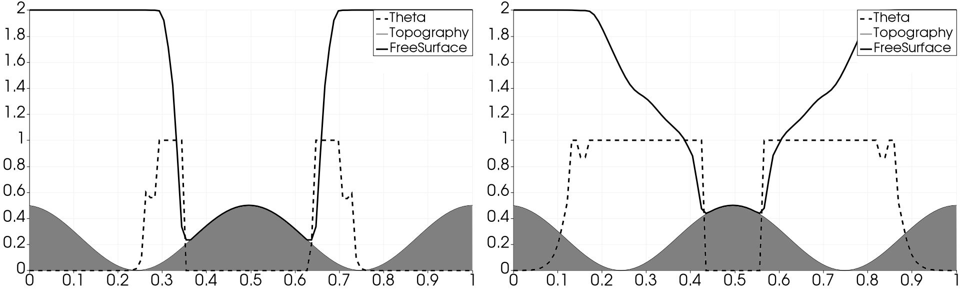
MOOD method

High-order schemes induce oscillations: we use the MOOD method to get rid of the oscillations and to restore the non-negativity preservation (see Clain, Diot, Loubère (2011)).

MOOD loop

1. compute a candidate solution W^c with the high-order scheme
2. determine whether W^c is admissible, i.e.
   - if h^c is non-negative (PAD criterion)
   - if W^c does not present spurious oscillations (DMP and u2 criteria)
3. where necessary, decrease the degree of the reconstruction
4. compute a new candidate solution

Pseudo-1D double dry dam-break on a sinusoidal bottom



The $\mathbb{P}_5^{\mathrm{WB}}$ scheme is used in the whole domain:

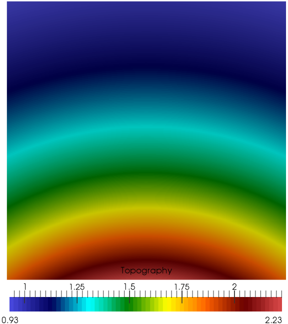
- near the boundaries, steady state at rest $\rightsquigarrow$ well-balanced scheme;
- away from the boundaries, far from steady state $\rightsquigarrow$ high-order scheme;
- center, dry area $\rightsquigarrow$ well-balanced scheme.

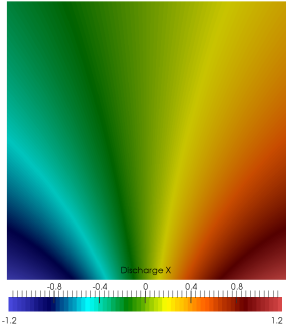
Order of accuracy assessment

To assess the order of accuracy, we take the following exact steady solution of the 2D shallow-water system, where $\boldsymbol{r} = {}^t(x, y)$:

$$h = 1 \ ; \ \boldsymbol{q} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} \ ; \ Z = \frac{2k\|\boldsymbol{r}\| - 1}{2g\|\boldsymbol{r}\|^2}.$$

With $k = 10$, this solution is depicted below on the space domain $[-0.3, 0.3] \times [0.4, 1]$.



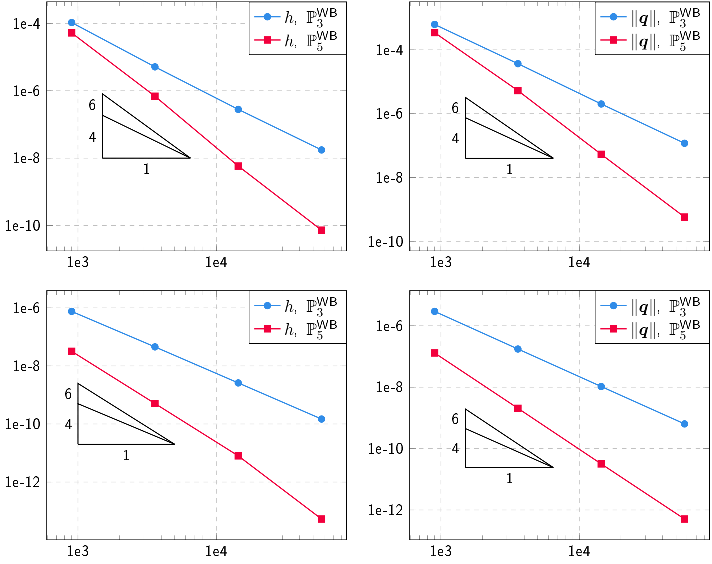


Order of accuracy assessment

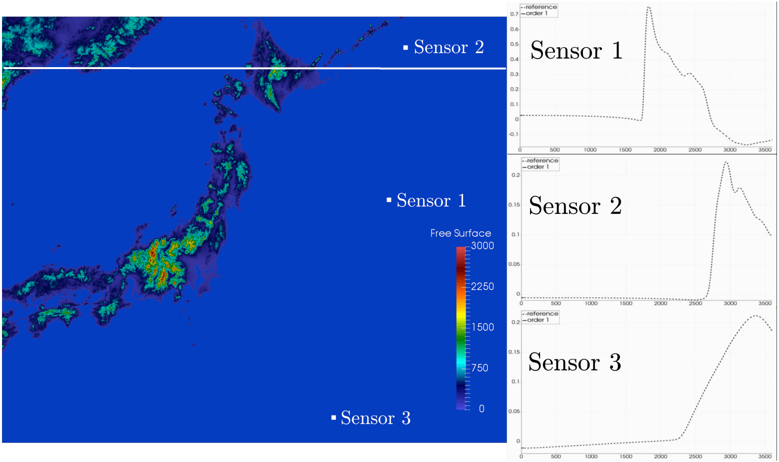
L^2 errors with respect to the number of cells

top graphs: 2D steady solution with topography

bottom graphs: 2D steady solution with friction and topography

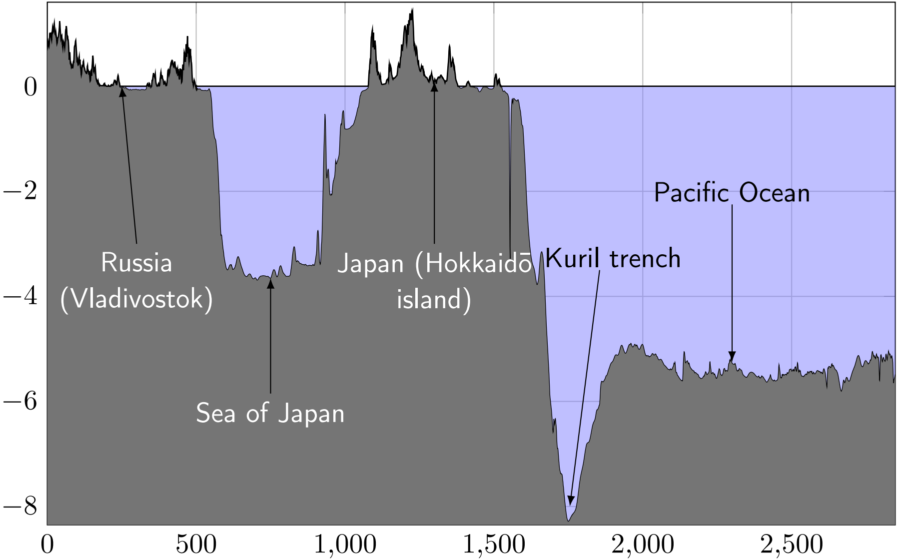


2011 Tōhoku tsunami



Tsunami simulation on a Cartesian mesh: 13 million cells, Fortran code parallelized with OpenMP, run on 48 cores.

2011 Tōhoku tsunami

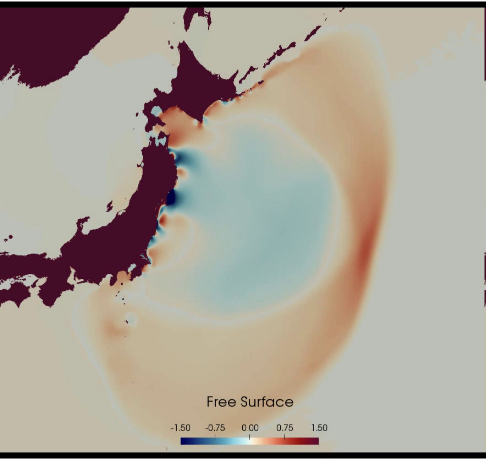


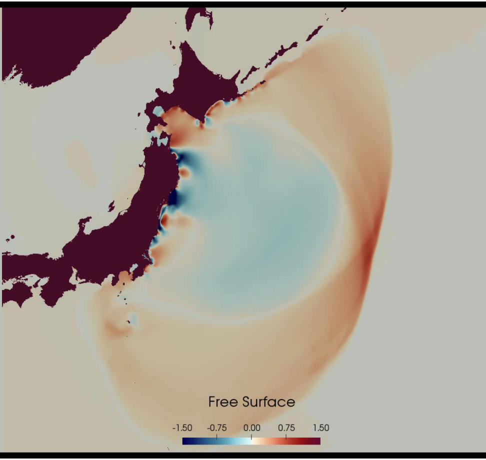
1D slice of the topography (unit: kilometers).

2011 Tōhoku tsunami

2011 Tōhoku tsunami

physical time of the simulation: 1 hour





first-order scheme
CPU time: ~ 1.1 hour

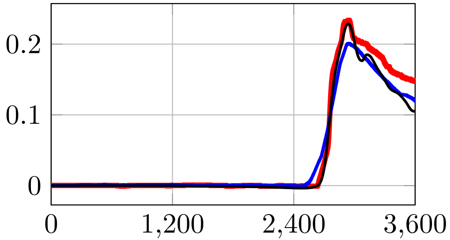
second-order scheme
CPU time: ~ 2.7 hours

2011 Tōhoku tsunami

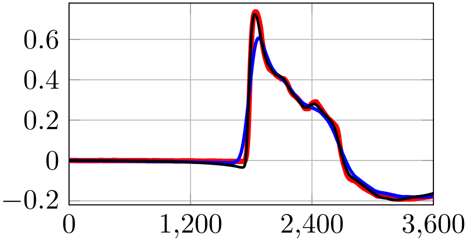
Water depth at the sensors:

- #1: 5700 m;
- #2: 6100 m;
- #3: 4400 m.

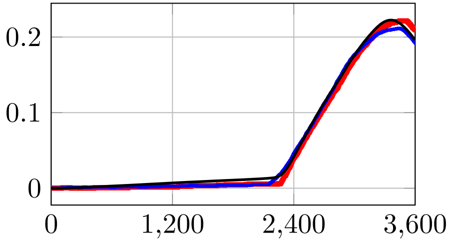
Graphs of the time variation of the water height (in meters).
data in black, order 1 in blue, order 2 in red



Sensor #2



Sensor #1



Sensor #3

Conclusion

- We have presented a well-balanced, non-negativity-preserving and entropy preserving numerical scheme for the shallow-water equations with topography and Manning friction, able to be applied to other source terms or combinations of source terms.
- We have also displayed results from the 2D high-order extension of this numerical method, coded in Fortran and parallelized with OpenMP.

This work has been published:

V. M.-D., C. Berthon, S. Clain and F. Foucher.
"A well-balanced scheme for the shallow-water equations with topography".
Comput. Math. Appl. 72(3):568–593, 2016.

V. M.-D., C. Berthon, S. Clain and F. Foucher.
"A well-balanced scheme for the shallow-water equations with topography or Manning friction". *J. Comput. Phys.* 335:115–154, 2017.

C. Berthon, R. Loubère, and V. M.-D.
"A second-order well-balanced scheme for the shallow-water equations with topography". Accepted in *Springer Proc. Math. Stat.*, 2017.

C. Berthon and V. M.-D.
"A simple fully well-balanced and entropy preserving scheme for the shallow-water equations". Submitted.

Perspectives

Work in progress

- high-order simulation of the 2011 Tōhoku tsunami
- application to other source terms:
  - Coriolis force source term
  - breadth variation source term

Long-term perspectives

- ensure the entropy preservation for the high-order scheme (use of an e-MOOD method)
- simulation of rogue waves

Thank you for your attention!

The discrete entropy inequality

The following non-conservative entropy inequality is satisfied by the shallow-water system:

$$\partial_t \eta(W) + \partial_x G(W) \leq \frac{q}{h} S(W); \; \eta(W) = \frac{q^2}{2h} + \frac{gh^2}{2}; \; G(W) = \frac{q}{h}\left(\frac{q^2}{2h} + gh^2\right).$$

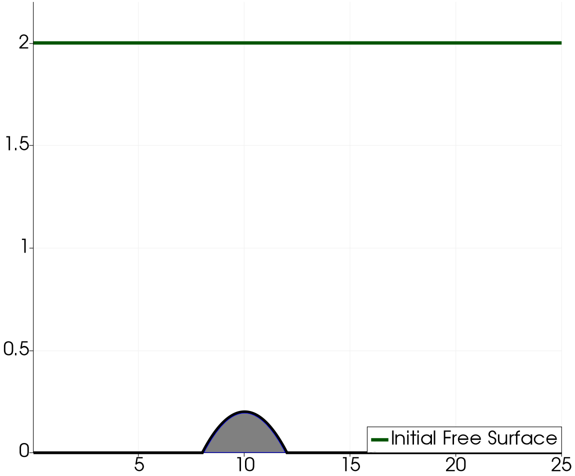
At the discrete level, we show that:

$$\lambda_R(\eta_R^* - \eta_R) - \lambda_L(\eta_L^* - \eta_L) + (G_R - G_L) \leq \frac{q_{HLL}}{h_{HLL}} \overline{S} \Delta x + \mathcal{O}(\Delta x^2).$$

main ingredients:

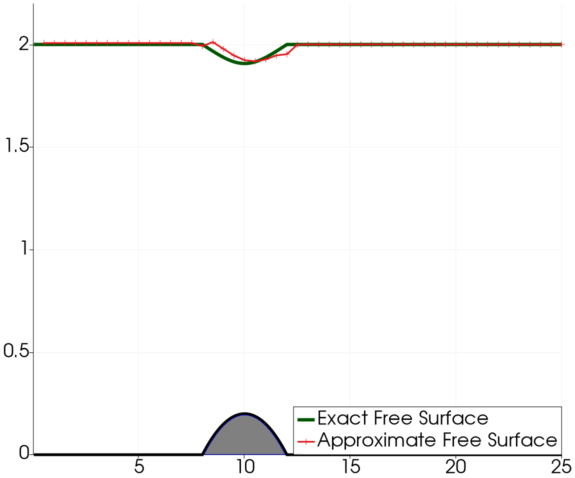
- $h_L^* = h_{HLL} - \overline{S} \Delta x \dfrac{\lambda_R}{\alpha(\lambda_R - \lambda_L)}$
  (and similar expressions for h_R^* and q^*)
- $(\lambda_R - \lambda_L)\eta_{HLL} \leq \lambda_R \eta_R - \lambda_L \eta_L - (G_R - G_L)$
  from Harten, Lax, van Leer (1983)

Verification of the well-balance: topography



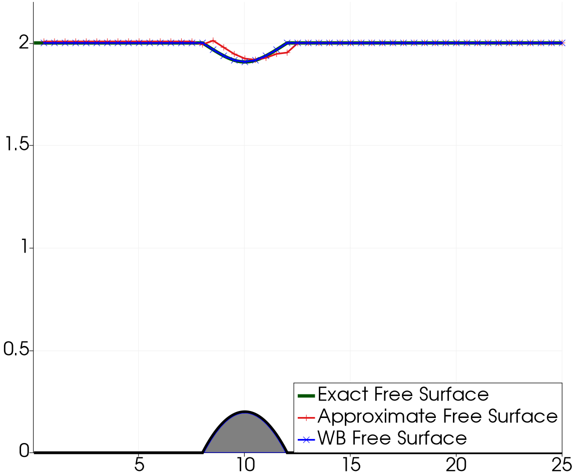
The initial condition is at rest; water is injected through the left boundary.

Verification of the well-balance: topography



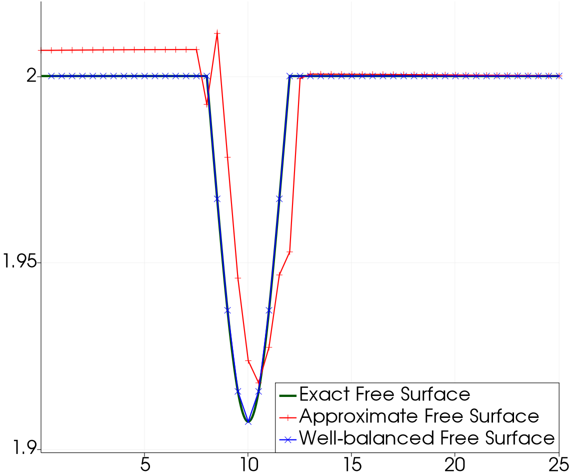
The non-well-balanced HLL scheme converges towards a numerical steady state which does not correspond to the physical one.

Verification of the well-balance: topography



The non-well-balanced HLL scheme yields a numerical steady state which does not correspond to the physical one. The well-balanced scheme exactly yields the physical steady state.

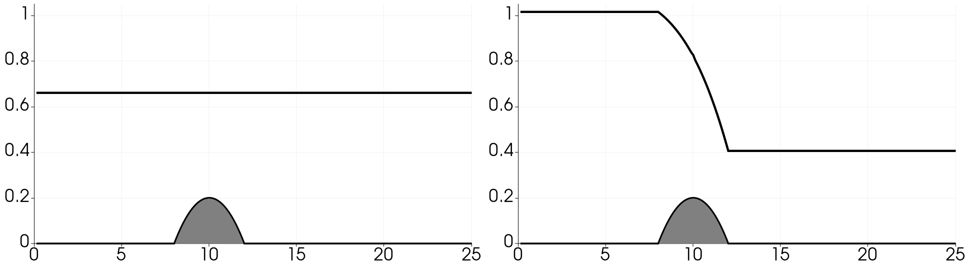
Verification of the well-balance: topography



The non-well-balanced HLL scheme yields a numerical steady state which does not correspond to the physical one. The well-balanced scheme exactly yields the physical steady state.

Verification of the well-balance: topography

transcritical flow test case (see Goutal, Maurel (1997))



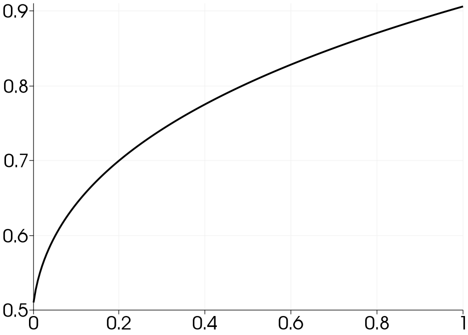
left panel: initial free surface at rest; water is injected from the left boundary

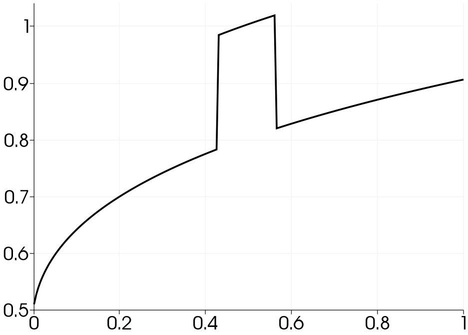
right panel: free surface for the steady state solution, after a transient state

$$\Phi = \frac{u^2}{2} + g(h+Z)$$

	L^1	L^2	L^∞
errors on q	1.47e-14	1.58e-14	2.04e-14
errors on Φ	1.67e-14	2.13e-14	4.26e-14

Verification of the well-balance: friction

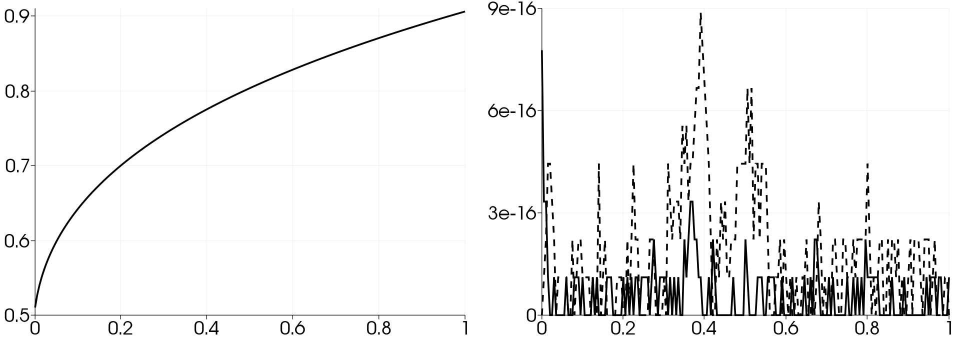




left panel: water height for the subcritical steady state solution

right panel: water height for the perturbed steady state solution

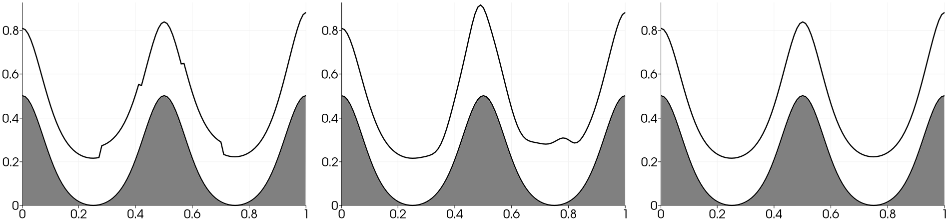
Verification of the well-balance: friction



left panel: convergence to the unperturbed steady state

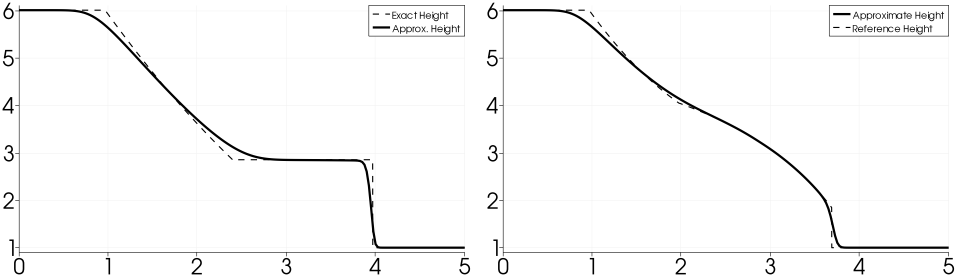
right panel: errors to the steady state (solid: h, dashed: q)

Perturbed pseudo-1D friction and topography steady state



	h			$\|\boldsymbol{q}\|$		
	L^1	L^2	L^∞	L^1	L^2	L^∞
$\mathbb{P}_0$	1.22e-15	1.71e-15	6.27e-15	2.34e-15	3.02e-15	9.10e-15
$\mathbb{P}_5$	5.01e-05	1.47e-04	1.16e-03	2.32e-04	2.63e-04	1.18e-03
$\mathbb{P}_5^{\mathsf{WB}}$	8.50e-14	1.05e-13	3.35e-13	2.82e-13	3.37e-13	6.76e-13

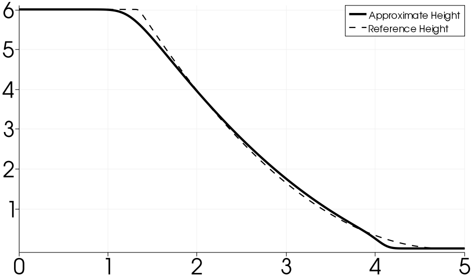
Riemann problems between two wet areas

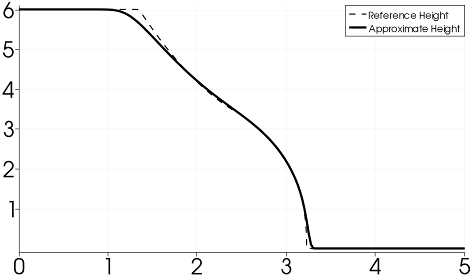


left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.2s

Riemann problems with a wet/dry transition





left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.15s

Double dry dam-break on a sinusoidal bottom