Consistent section-averaged shallow water equations with bottom friction

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Motivation: 2D/1D coupling for estuary simulation

Gironde estuary: satellite picture

Gironde estuary: 2D mesh

Existing approaches

Regarding the shape of the river bed, as of now,

- the derivation of 1D models is well-understood [1,2] in the ideal case of a $\sqcup$-shaped channel;
- for more complex shapes, the water surface of uniform stationary flows is recovered [3,4] using a empiric terms or data assimilation;
- fully 2D models are used but they are computationally costly.

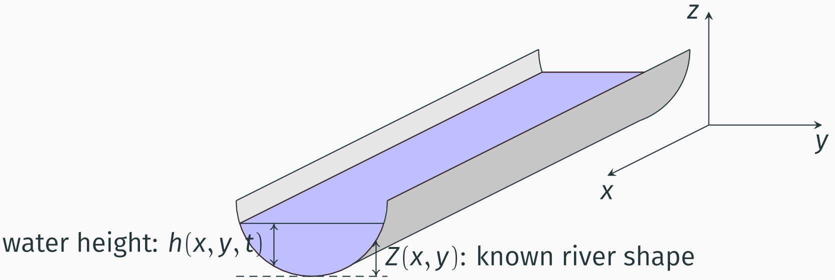
[1]see Bresch and Noble, 2007, in the context of laminar flows
[2]see Richard, Rambaud and Vila, 2017, in the context of turbulent flows
[3]see Decoene, Bonaventura, Miglio and Saleri, 2009
[4]see Marin and Monnier, 2009

Specifications of the 1D model

The goal of this work is to develop a new model, based on the shallow water equations, that is:

- generic enough to not require empiric friction coefficients;
- consistent with the 2D shallow water equations in the asymptotic regime corresponding to an estuary or a river;
- hyperbolic;
- easily implementable (collaboration with the SHOM for flood simulations, ocean model forcing, ...);
- able to handle the meanders of the river.

The non-conservative 2D shallow water system



$$\begin{cases} h_t + \nabla \cdot (h\mathbf{u}) = 0 \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g \nabla h = g \left(-\nabla Z - \dfrac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\, h^p} \right) \end{cases}$$

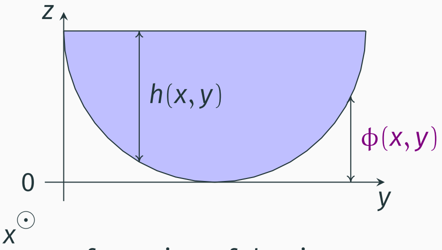
- $\mathbf{u} = (u, v)$ is the water velocity
- g is the gravity constant
- $C_h(x, y)$ is the (known) Chézy friction coefficient
- $p = 4/3$ is the friction law exponent

Introduction of reference scales: the topography

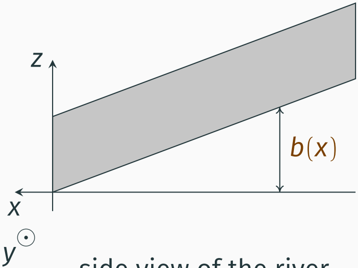
Regarding the geometry, we assume that $Z(x, y) = b(x) + \phi(x, y)$, where:

- $b(x)$ represents the main longitudinal topography, driving the flow from upstream to downstream;
- $\phi(x, y)$ represents small longitudinal and transverse variations.

Thus, $h + \phi$ represents the altitude of the water surface.

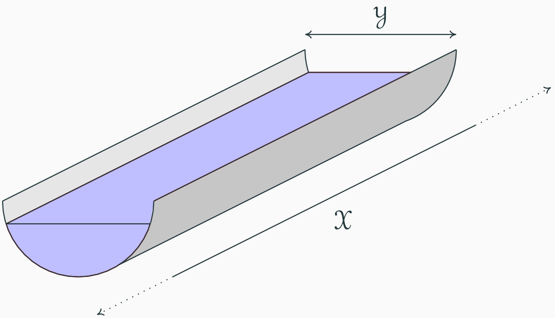


front view of the river



side view of the river

Introduction of reference scales: the coordinates



	dimensional quantity	reference scale	non-dimensional quantity
longitudinal coordinates	$x \in (0\mathrm{m}, 60000\mathrm{m})$	$\mathcal{X} = 2000\mathrm{m}$	$\bar{x} = \frac{x}{\mathcal{X}} \in (0, 30)$
transverse coordinates	$y \in (-100\mathrm{m}, 100\mathrm{m})$	$\mathcal{Y} = 100\mathrm{m}$	$\bar{y} = \frac{y}{\mathcal{Y}} \in (-1, 1)$

Non-dimensional form of the 2D shallow water system

We introduce the following non-dimensional numbers to emphasize the different scales of the flow:

- F^2, the reference Froude number (ratio material/acoustic velocity),
- δ, the shallow water parameter (ratio height/reference length),
- R_u, the quasi-1D parameter (ratio transverse/longitudinal velocity),
- I_0 and J_0, the reference topography and friction slopes.

Finally, the non-dimensional form of the 2D shallow water system is:

$$
\begin{cases}
\bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\
\bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \dfrac{1}{F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{x}} = \dfrac{1}{\delta F^2}\left(-J_0 \dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2 \bar{v}^2}}{\bar{C}^2 \bar{h}^p} - I_0 \bar{b}_{\bar{x}}\right), \\
\bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \dfrac{1}{R_u^2 F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{y}} = -\dfrac{J_0}{\delta F^2} \dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2 \bar{v}^2}}{\bar{C}^2 \bar{h}^p}.
\end{cases}
$$

Asymptotic expansions setup

In the regime under consideration, we have

- $\varepsilon := \dfrac{\delta F^2}{J_0} \ll 1$ (in practice, $F^2 \ll 1$, $\delta \ll 1$, $J_0 \ll 1$ and $J_0 \sim \delta$),
- $R_u \ll 1$ (quasi-unidimensional setting), and $R_u = \mathcal{O}(\varepsilon)$.

Highlighting the dominant terms in the system, we get:

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ u_t + uu_x + vu_y + \dfrac{1}{\varepsilon}\dfrac{\delta}{J_0}(h+\phi)_x = \dfrac{1}{\varepsilon}\left(-\dfrac{u\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} - \dfrac{I_0}{J_0} b_x\right), \\ v_t + uv_x + vv_y + \dfrac{1}{\varepsilon^3}\dfrac{\delta}{J_0}(h+\phi)_y = -\dfrac{1}{\varepsilon}\dfrac{v\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p}. \end{cases}$$

Goal: Perform asymptotic expansions in this regime, to better understand the weak dependency of the solution in y.

We consider the third equation:

$$v_t + uv_x + vv_y + \frac{1}{\varepsilon^3}\frac{\delta}{J_0}(h+\phi)_y = -\frac{1}{\varepsilon}\frac{v\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p},$$

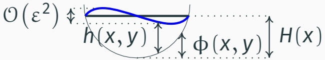
which we rewrite as follows to highlight the dominant term:

$$\frac{\delta}{J_0}(h+\phi)_y = \varepsilon^2\frac{v\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} + \varepsilon^3(v_t + uv_x + vv_y).$$

Neglecting the $\mathcal{O}(\varepsilon^2)$ terms, we get

$$\frac{\delta}{J_0}(h+\phi)_y = \mathcal{O}(\varepsilon^2),$$

and there exists $H = H(x,t)$ such that



$$H(x,t) = h(x,y,t) + \phi(x,y) + \mathcal{O}(\varepsilon^2).$$

$\rightsquigarrow$ the free surface $h+\phi$ is almost flat in the y-direction, up to $\mathcal{O}(\varepsilon^2)$

Longitudinal velocity expansion

Highlighting the dominant terms, the second equation reads:

$$u_t + uu_x + vu_y + \frac{1}{\varepsilon}\frac{\delta}{J_0}(h+\phi)_x = \frac{1}{\varepsilon}\left(-\frac{u\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} - \frac{I_0}{J_0} b_x\right).$$

To perform the asymptotic expansion of u with respect to ε, we write

$$u(x,y,t) = u_{2D}^{(0)}(x,y,t) + \mathcal{O}(\varepsilon).$$

Since $h + \phi = H + \mathcal{O}\left(\varepsilon^2\right)$, straightforward computations yield:

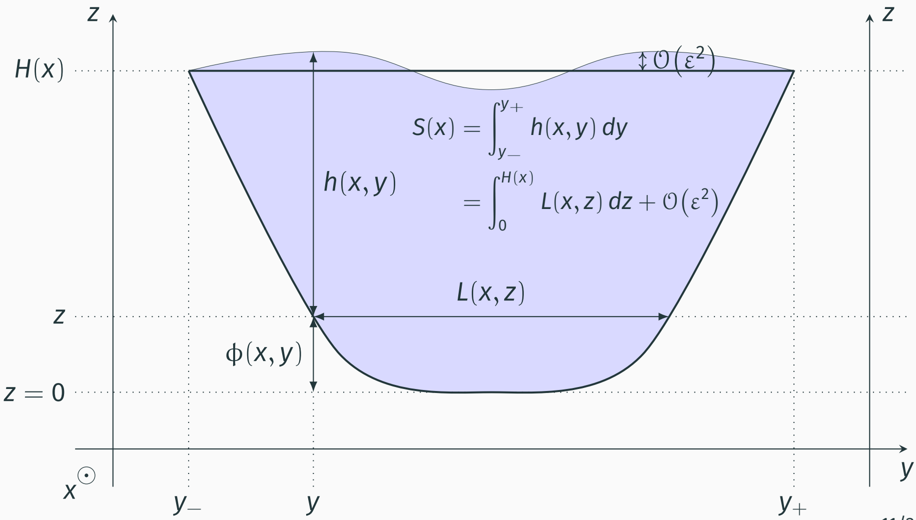
$$u_{2D}^{(0)} = C\frac{\Lambda}{\sqrt{|\Lambda|}}(H-\phi)^{p/2},$$

where we have defined the corrected slope $\Lambda(x,t) = -\frac{I_0}{J_0}b_x - \frac{\delta}{J_0}H_x$.

Next step: Build a 1D model consistent with these expansions.

The river cross-section

To obtain a 1D model, we start by averaging the 2D equations: below, we display the cross-section of the river, with respect to x.



Averaging the 2D system over the river width

1. The original mass conservation equation reads:

$$h_t + (hu)_x + (hv)_y = 0.$$

Therefore, since $v(y_-) = v(y_+) = 0$, we get:

$$\int_{y_-}^{y_+} h_t \, dy + \int_{y_-}^{y_+} (hu)_x \, dy = 0 \quad \Longrightarrow \quad S_t + Q_x = 0,$$

where the averaged discharge Q is given by $Q = \displaystyle\int_{y_-}^{y_+} hu \, dy$.

2. Arguing the mass conservation and integrating the second equation (times h) between y_- and y_+ yields:

$$Q_t + \left(\int_{y_-}^{y_+} hu^2 \, dy \right)_x = \frac{1}{\varepsilon} \int_{y_-}^{y_+} h \left(-\frac{I_0}{J_0} b_x - \frac{\delta}{J_0} (h + \phi)_x \right) dy$$

$$- \frac{1}{\varepsilon} \int_{y_-}^{y_+} \frac{u\sqrt{u^2 + \varepsilon^2 v^2}}{C^2 h^{p-1}} \, dy.$$

Averaging the 2D system

Finally, the averaged system reads as follows, up to $\mathcal{O}(\varepsilon^2)$:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\int_{y_-}^{y_+} hu^2 \, dy \right)_x = \frac{1}{\varepsilon} \left(\Lambda S - \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}} \, dy \right) + \mathcal{O}(\varepsilon). \end{cases}$$

Next step: From the averaged system, build a truly 1D model that is zeroth-order accurate (up to $\mathcal{O}(\varepsilon)$).

That is to say, the new model needs to ensure $Q = Q_{2D}^{(0)} + \mathcal{O}(\varepsilon)$, where

$$\begin{aligned} Q_{2D}^{(0)} &= \int_{y_-}^{y_+} hu_{2D}^{(0)} \, dy \\ &= \sqrt{|\Lambda|} \, \mathrm{sgn}(\Lambda) \int_{y_-}^{y_+} C \, (H - \phi)^{1+p/2} \, dy. \end{aligned}$$

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3. Transverse averaging

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5. Numerical treatment of real data

6. Numerical validation of the model on an academic test case

7. Conclusion and perspectives

Setting up the model

The integrated discharge equation, highlighting the dominant terms and multiplying by ε, is

$$\Lambda S - \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}}\, dy = \varepsilon \left(Q_t + \left(\int_{y_-}^{y_+} h u^2\, dy \right)_x \right) + \mathcal{O}(\varepsilon^2).$$

At the zeroth order, i.e. up to $\mathcal{O}(\varepsilon)$, the right-hand side of this equation is neglected, and we get:

$$\Lambda S - \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}}\, dy = \mathcal{O}(\varepsilon).$$

We cannot directly use this equation in a 1D model, since it contains the unknown u, which depends on y.

Instead, we approximate the integral, up to $\mathcal{O}(\varepsilon)$, with a new 1D friction term.

The friction model

First, we choose this 1D friction term as a usual hydraulic engineering model. Thus, we impose the following formula:

$$\frac{Q|Q|}{C_{1D}^2 S} = \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}} \, dy + \mathcal{O}(\varepsilon).$$

It contains a 1D friction coefficient[5] C_{1D}, to be determined.

According to the discharge equation, we get, up to $\mathcal{O}(\varepsilon)$:

$$\frac{Q|Q|}{C_{1D}^2 S} = \Lambda S + \mathcal{O}(\varepsilon) \quad \Longrightarrow \quad C_{1D}^2 = \frac{Q|Q|}{\Lambda S^2} + \mathcal{O}(\varepsilon).$$

Second, we impose $Q = Q_{2D}^{(0)} + \mathcal{O}(\varepsilon)$, to get the following expression of the friction coefficient:

$$C_{1D}^2 = \frac{Q_{2D}^{(0)} \left| Q_{2D}^{(0)} \right|}{\Lambda S^2} = \frac{1}{S^2} \left(\int_{y_-}^{y_+} C \, (H - \phi)^{1+p/2} \, dy \right)^2.$$

[5]The coefficient C_{1D}^2 usually contains the hydraulic radius, the Chézy coefficient, …

The final system

With the new friction model, the discharge equation reads

$$\Lambda S - \frac{Q|Q|}{C_{1D}^2 S} = \varepsilon\left(Q_t + \left(\int_{y_-}^{y_+} hu^2\, dy\right)_x\right) + \mathcal{O}(\varepsilon).$$

We choose to approximate the integral in the flux to describe the advection of the discharge:

$$\varepsilon \int_{y_-}^{y_+} hu^2\, dy = \varepsilon \frac{\left(\int_{y_-}^{y_+} hu\, dy\right)^2}{\int_{y_-}^{y_+} h\, dy} + \mathcal{O}(\varepsilon) = \varepsilon\, \frac{Q^2}{S} + \mathcal{O}(\varepsilon).$$

The resulting discharge equation is

$$S\Big(\Lambda - \underbrace{\frac{Q|Q|}{C_{1D}^2 S^2}}_{\mathcal{J}}\Big) = \varepsilon\left(Q_t + \left(\frac{Q^2}{S}\right)_x\right) + \mathcal{O}(\varepsilon).$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J}). \end{cases}$$

Let us double check that this model is sufficient to recover the zeroth-order expansion of Q.

With $Q = Q^{(0)}_{\text{model}} + \mathcal{O}(\varepsilon)$, we get, at the zeroth order:

$$\Lambda = \mathcal{J} + \mathcal{O}(\varepsilon) \implies \Lambda = \overbrace{\Lambda \frac{Q|Q|}{Q^{(0)}_{2D}\left|Q^{(0)}_{2D}\right|}}^{\mathcal{J}} + \mathcal{O}(\varepsilon) = \Lambda \frac{Q^{(0)}_{\text{model}}\left|Q^{(0)}_{\text{model}}\right|}{Q^{(0)}_{2D}\left|Q^{(0)}_{2D}\right|} + \mathcal{O}(\varepsilon)$$

$$\implies Q^{(0)}_{\text{model}} = Q^{(0)}_{2D} + \mathcal{O}(\varepsilon).$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = \dfrac{1}{\varepsilon} S \left(-\dfrac{I_0}{J_0} b_x - \dfrac{\delta}{J_0} H_x - \mathcal{J} \right). \end{cases}$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = \dfrac{1}{\varepsilon} S \Big(\underbrace{-\dfrac{I_0}{J_0} b_x}_{\mathcal{J}} - \dfrac{\delta}{J_0} H_x - \mathcal{J} \Big). \end{cases}$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x + \dfrac{SH_x}{F^2} = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}). \end{cases}$$

This form is quite similar to that of the the usual models.
All the complexity lies within the friction model $\mathcal{J}$ and in the expression of the friction coefficient C_{1D}.

⇝ We have derived a zeroth-order model governed by a hyperbolic system of balance laws.
⇝ We also enhance this approach to derive a first-order model, based on the energy equation.

Next step: Numerical validation of these models on real data.

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5. **Numerical treatment of real data**

6. Numerical validation of the model on an academic test case

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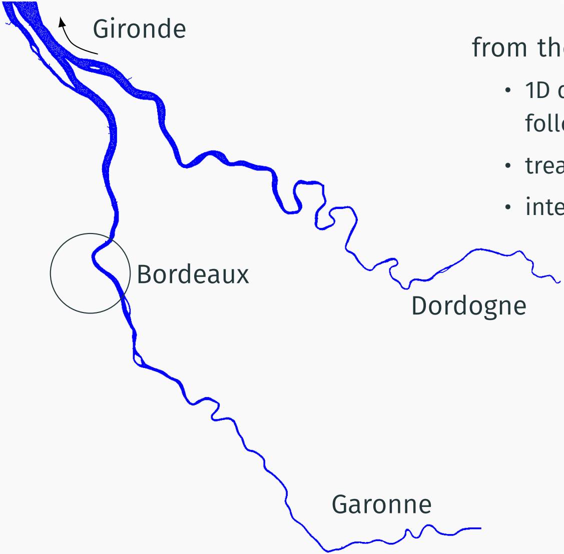
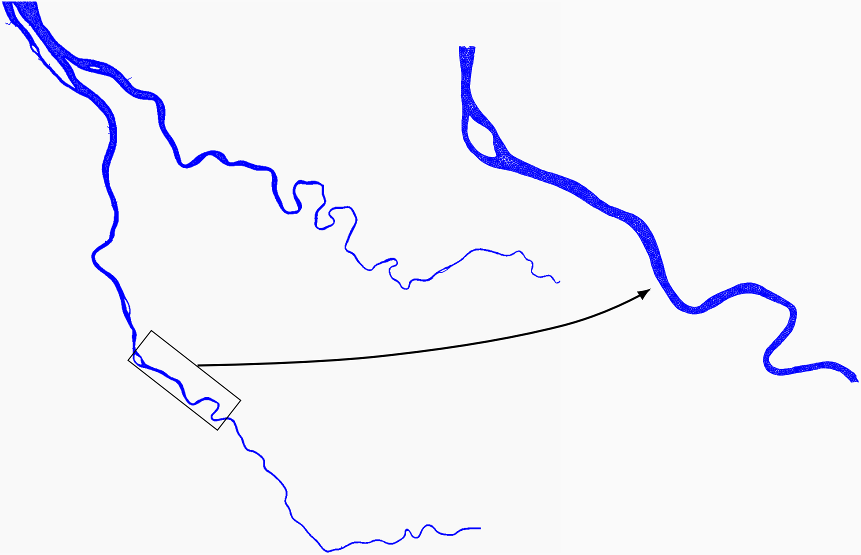
Extracting 1D quantities from a 2D mesh



from the 2D mesh:

- 1D description of the rivers following their meanders
- treatment of the confluence
- interaction with the tide

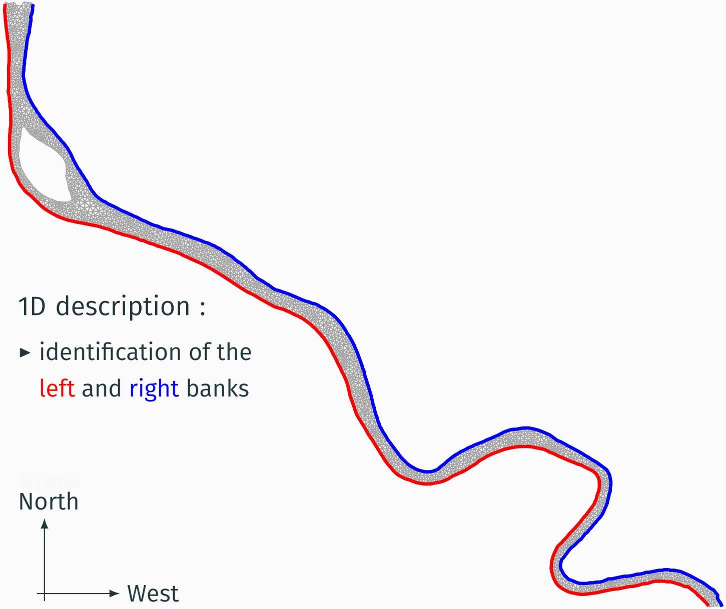
Extracting 1D quantities from a 2D mesh



Extracting 1D quantities from a 2D mesh

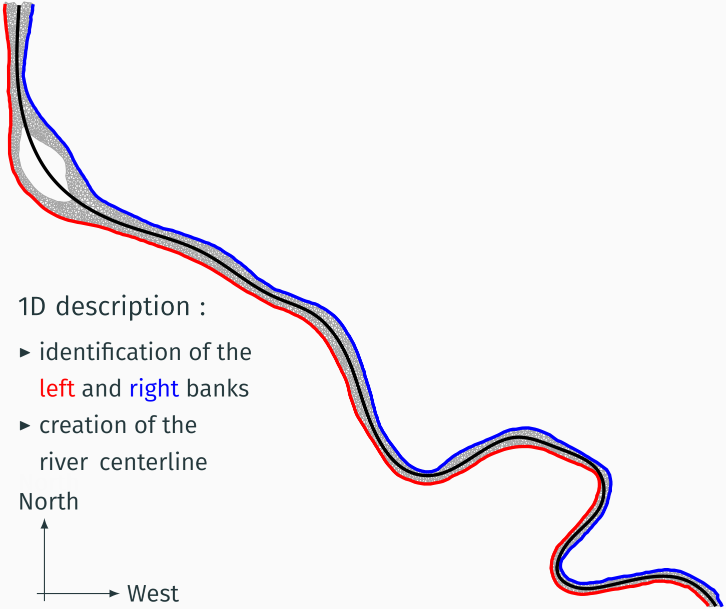
1D description :

Extracting 1D quantities from a 2D mesh



1D description :

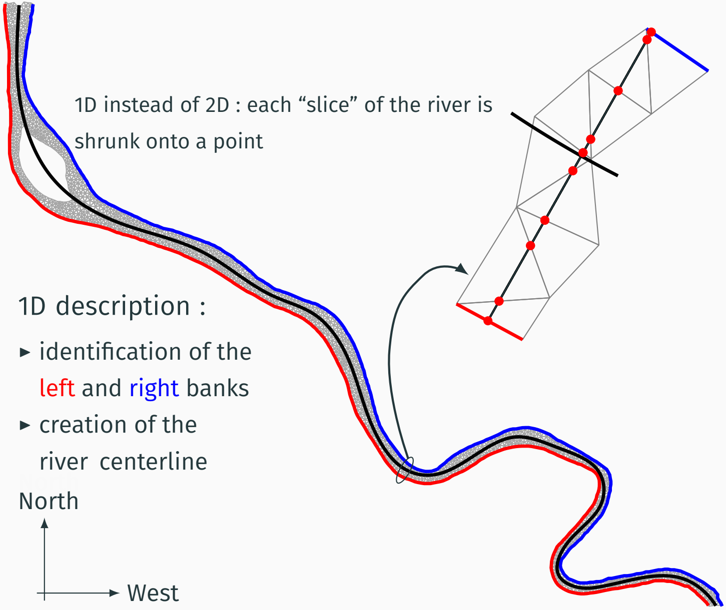
- identification of the left and right banks



1D description :

- identification of the left and right banks
- creation of the river centerline

Extracting 1D quantities from a 2D mesh

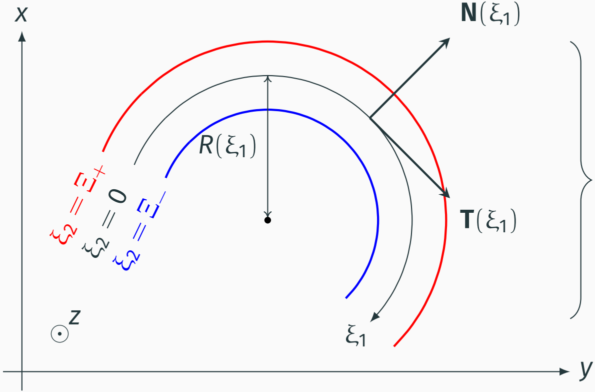


1D instead of 2D : each “slice” of the river is shrunk onto a point

1D description :

- identification of the left and right banks
- creation of the river centerline

Rewriting the 2D system in local coordinates



meander to the left:

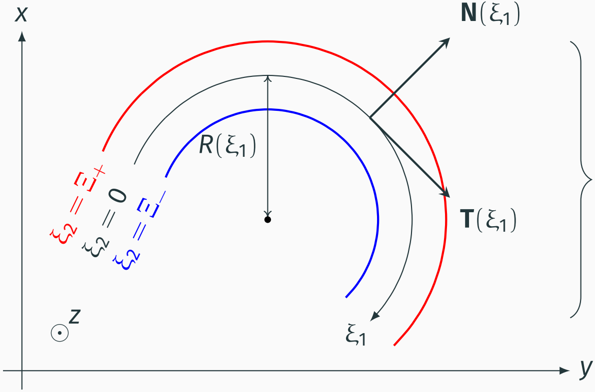
$\sigma(\xi_1) = -1$

Jacobian determinant:

$|\mathcal{F}| = 1 - \xi_2 \dfrac{\sigma(\xi_1)}{R(\xi_1)}$

$$
\begin{cases}
h_t + (hu)_x + (hv)_y = 0 \\
u_t + uu_x + vu_y + g(h+Z)_x = -\dfrac{gu\sqrt{u^2+v^2}}{C_h^2\, h^p} \\
v_t + uv_x + vv_y + g(h+Z)_y = -\dfrac{gv\sqrt{u^2+v^2}}{C_h^2\, h^p}
\end{cases}
$$

Rewriting the 2D system in local coordinates



meander to the left:

$\sigma(\xi_1) = -1$

Jacobian determinant:

$|\mathcal{F}| = 1 - \xi_2 \dfrac{\sigma(\xi_1)}{R(\xi_1)}$

$$
\begin{cases}
(|\mathcal{F}|h)_t + (|\mathcal{F}|hu)_{\xi_1} + (|\mathcal{F}|hv)_{\xi_2} = 0 \\
u_t + uu_{\xi_1} + vu_{\xi_2} + \dfrac{g}{|\mathcal{F}|^2}(h+Z)_{\xi_1} + \dfrac{\xi_2 R'}{|\mathcal{F}|R}\dfrac{u^2}{R} - \dfrac{2\sigma uv}{|\mathcal{F}|R} = -\dfrac{gu\sqrt{|\mathcal{F}|^2u^2+v^2}}{C_h^2\,h^p} \\
v_t + uv_{\xi_1} + vv_{\xi_2} + g(h+Z)_{\xi_2} + \sigma|\mathcal{F}|\dfrac{u^2}{R} = -\dfrac{gv\sqrt{|\mathcal{F}|^2u^2+v^2}}{C_h^2\,h^p}
\end{cases}
$$

Numerical schemes

To handle the stiff relaxation source term, we introduce an implicit splitting procedure.

The zeroth-order model is made of a non-stiff part and a stiff part:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x + \dfrac{1}{\varepsilon}\dfrac{\delta}{J_0} S H_x = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}). \end{cases}$$

First, we consider the non-stiff part:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = 0, \end{cases}$$

which we discretize using an upwind finite difference scheme.

Numerical schemes

Second, we consider the stiff part:

$$\begin{cases} S_t = 0, \\ Q_t + \dfrac{1}{\varepsilon}\dfrac{\delta}{J_0} S H_x = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}). \end{cases}$$

Since $S_t = 0$, we are left with the following ODE on Q:

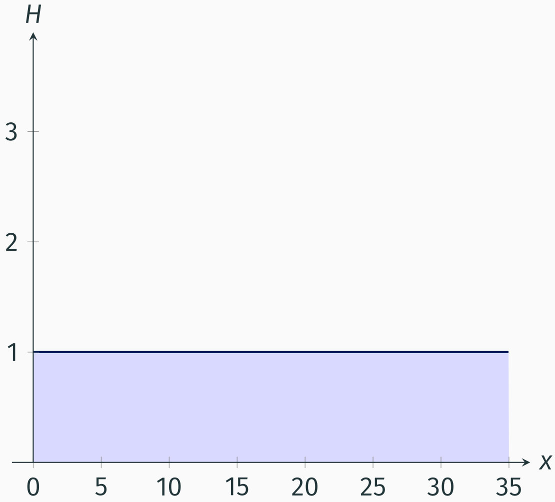
$$Q_t = \frac{1}{\varepsilon} S \Lambda \left(1 - \frac{Q^2}{\left(Q_{2D}^{(0)}\right)^2} \right),$$

which we can solve exactly, to get

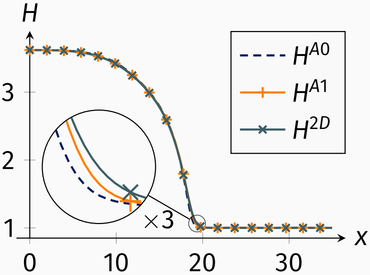
$$Q(t) = Q_{2D}^{(0)} \frac{\tanh\left(\dfrac{1}{\varepsilon} \dfrac{S|\Lambda|}{|Q_{2D}^{(0)}|} t \right) + \dfrac{Q(0)}{Q_{2D}^{(0)}}}{1 + \tanh\left(\dfrac{1}{\varepsilon} \dfrac{S|\Lambda|}{|Q_{2D}^{(0)}|} t \right) \dfrac{Q(0)}{Q_{2D}^{(0)}}} \xrightarrow[\varepsilon \to 0]{} Q_{2D}^{(0)}.$$

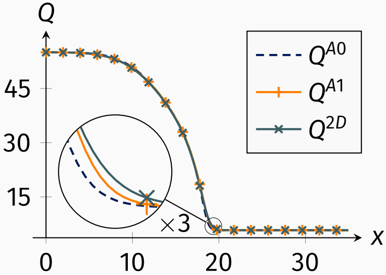
Unsteady flood flow

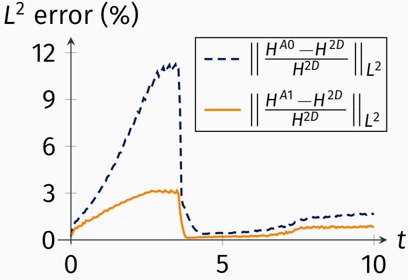
We consider a 5-year flood for the a simplified Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.

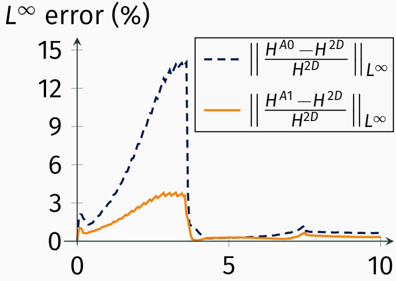


Unsteady flood flow (2D: ref. sol., A0: 0th-order, A1: 1st-order)









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Conclusion

We have developed a new 1D model, based on the 2D shallow water equations, that is:

- consistent, up to first-order, with the 2D model in the asymptotic regime corresponding to a river flow:
  - the zeroth-order is obtained with a new explicit friction term,
  - the first-order relies on new equations describing the evolution of the energy;
- hyperbolic;
- easily implementable and numerically validated.

The preprint related to these results is available on HAL:

V. Michel-Dansac, P. Noble et J.-P. Vila, **Consistent section-averaged shallow water equations with bottom friction**, 2018.
https://hal.archives-ouvertes.fr/hal-01962186

Work in progress and perspectives

Work related to the implementation and scientific computation (collaboration in progress with the SHOM):

- adapt an explicit low Froude method to improve the scheme [6]
- compare the 1D results to the ones given by a fully 2D code, in real test cases (Garonne, Lèze, Gironde, Amazon, ...)
- couple the 1D and 2D equations in the context of the Gironde estuary

Work related to the model:

- adapt this methodology to treat confluences
- model sedimentation with a time-dependent topography

[6] see Couderc, Duran and Vila, 2017

Thank you for your attention!

First-order model

The first-order model is:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x + \left(1 - \dfrac{S\Psi_{2D}^{(0)}}{\left(Q_{2D}^{(0)}\right)^2}\right)\dfrac{SH_x}{F^2} = \dfrac{1}{\varepsilon} S \left(\mathcal{I} - \mathcal{J} - \dfrac{S\Psi_{2D}^{(0)}}{\left(Q_{2D}^{(0)}\right)^2}(\mathcal{I} - \mathcal{J}_\Psi)\right), \\ \left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x + \dfrac{QH_x}{F^2} = \dfrac{1}{\varepsilon} Q(\mathcal{I} - \mathcal{J}), \\ \left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = \dfrac{1}{\varepsilon} Q \dfrac{S\Pi_{2D}^{(0)}}{\left(Q_{2D}^{(0)}\right)^2}(\mathcal{J}_\Psi - \mathcal{J}_\Pi). \end{cases}$$

It ensures the correct asymptotic regime, that is to say

$$Q = Q_{2D}^{(0)} + \varepsilon Q_{2D}^{(1)} + \mathcal{O}\left(\varepsilon^2\right).$$

In addition, it is hyperbolic and linearly stable.

Non-dimensional form of the 2D shallow water system

To emphasize the different scales of the flow, we perform a non-dimensionalization of the 2D system.

We introduce the following dimensionalization scales and related non-dimensional quantities (which are denoted with a bar, like $\bar{x}$):

$$h := \mathcal{H}\bar{h}, \quad u := \mathcal{U}\bar{u}, \quad v := \mathcal{V}\bar{v}, \quad x := \mathcal{X}\bar{x}, \quad y := \mathcal{Y}\bar{y}, \quad t := \mathcal{T}\bar{t}, \quad \mathcal{T} := \frac{\mathcal{X}}{\mathcal{U}}.$$

The mass conservation equation

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

then becomes

$$\frac{\mathcal{H}}{\mathcal{T}}\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\mathcal{H}\mathcal{U}}{\mathcal{X}}\frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{\mathcal{H}\mathcal{V}}{\mathcal{Y}}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0.$$

Non-dimensional form of the 2D shallow water system

The non-dimensional conservation equation is

$$\frac{\mathcal{H}}{\mathcal{T}}\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\mathcal{H}\mathcal{U}}{\mathcal{X}}\frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{\mathcal{H}\mathcal{V}}{\mathcal{Y}}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0, \quad \text{i.e.} \quad \frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{\mathcal{V}}{\mathcal{U}}\frac{\mathcal{X}}{\mathcal{Y}}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0.$$

We set $R_u := \mathcal{V}/\mathcal{U}$ and $R_x := \mathcal{Y}/\mathcal{X}$, to get

$$\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{R_u}{R_x}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0.$$

We have

- $\mathcal{V} \ll \mathcal{U}$ (quasi-unidimensional flow) $\implies R_u \ll 1$,
- $\mathcal{Y} \ll \mathcal{X}$ (quasi-unidimensional geometry) $\implies R_x \ll 1$.

We assume $R_u = R_x$ to keep the mass conservation equation unchanged from the dimensional case.

Non-dimensional form of the 2D shallow water system

Regarding the geometry, we assume that $Z(x, y) = b(x) + \phi(x, y)$, where:

- $b(x)$ represents the main longitudinal topography, driving the flow from upstream to downstream;
- $\phi(x, y)$ represents small longitudinal and transverse variations.

The related non-dimensional quantities are

$$b = \mathcal{B}\bar{b}\left(\frac{x}{\mathcal{X}}\right) \quad \text{and} \quad \phi = \mathcal{H}\bar{\phi}\left(\frac{x}{\mathcal{X}}, \frac{y}{\mathcal{Y}}\right).$$

The non-dimensional topography gradient then reads:

$$\boldsymbol{\nabla} Z = \begin{pmatrix} \dfrac{\mathcal{B}}{\mathcal{X}} \dfrac{\partial \bar{b}}{\partial \bar{x}}(\bar{x}) + \dfrac{\mathcal{H}}{\mathcal{X}} \dfrac{\partial \bar{\phi}}{\partial \bar{x}}(\bar{x}, \bar{y}) \\ \dfrac{\mathcal{H}}{\mathcal{Y}} \dfrac{\partial \bar{\phi}}{\partial \bar{y}}(\bar{x}, \bar{y}) \end{pmatrix}.$$

Non-dimensional form of the 2D shallow water system

Regarding the friction, we take $C_h = \mathcal{C}\,\bar{C}(\bar{x}, \bar{y})$.

The non-dimensional friction source term then reads:

$$\frac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\,h^p} = \begin{pmatrix} \dfrac{\mathcal{U}}{\mathcal{C}\mathcal{H}^p} \cdot \dfrac{\bar{u}\sqrt{\mathcal{U}^2\bar{u}^2 + \mathcal{V}^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} \\ \dfrac{\mathcal{V}}{\mathcal{C}\mathcal{H}^p} \cdot \dfrac{\bar{v}\sqrt{\mathcal{U}^2\bar{u}^2 + \mathcal{V}^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} \end{pmatrix} = \begin{pmatrix} \dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p} \cdot \dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} \\ \dfrac{\mathcal{V}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p} \cdot \dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} \end{pmatrix}.$$

Non-dimensional form of the 2D shallow water system

We are finally able to write the non-dimensional form of the 2D shallow water system: from the dimensional system

$$\begin{cases} h_t + \nabla \cdot (h\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g\nabla h = g\left(-\nabla Z - \dfrac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\, h^p}\right), \end{cases}$$

we get the following non-dimensional form:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \dfrac{\mathcal{U}^2}{\mathcal{X}}\bar{u}_{\bar{t}} + \dfrac{\mathcal{U}^2}{\mathcal{X}}\bar{u}\bar{u}_{\bar{x}} + \dfrac{\mathcal{U}\mathcal{V}}{\mathcal{Y}}\bar{v}\bar{u}_{\bar{y}} + \dfrac{g\mathcal{H}}{\mathcal{X}}\left(\bar{h} + \bar{\Phi}\right)_{\bar{x}} = g\left(-\dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - \dfrac{\mathcal{B}}{\mathcal{X}}\bar{b}_{\bar{x}}\right), \\ \dfrac{\mathcal{V}\mathcal{U}}{\mathcal{X}}\bar{v}_{\bar{t}} + \dfrac{\mathcal{V}\mathcal{U}}{\mathcal{X}}\bar{u}\bar{v}_{\bar{x}} + \dfrac{\mathcal{V}^2}{\mathcal{Y}}\bar{v}\bar{v}_{\bar{y}} + \dfrac{g\mathcal{H}}{\mathcal{Y}}\left(\bar{h} + \bar{\Phi}\right)_{\bar{y}} = g\left(-\dfrac{\mathcal{V}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p}\right). \end{cases}$$

Non-dimensional form of the 2D shallow water system

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we get the following non-dimensional form:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \dfrac{g\mathcal{H}}{\mathcal{U}^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{x}} = \dfrac{g\mathcal{X}}{\mathcal{U}^2}\left(-\dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - \dfrac{\mathcal{B}}{\mathcal{X}}\bar{b}_{\bar{x}}\right), \\ \bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \dfrac{g\mathcal{H}\mathcal{X}}{\mathcal{V}\mathcal{U}\mathcal{Y}}\left(\bar{h} + \bar{\phi}\right)_{\bar{y}} = \dfrac{g\mathcal{X}}{\mathcal{U}^2}\left(-\dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p}\right). \end{cases}$$

Non-dimensional form of the 2D shallow water system

We introduce:

- $F^2 = \dfrac{\mathcal{U}^2}{g\mathcal{H}}$ the reference Froude number,
- $\delta = \dfrac{\mathcal{H}}{\mathcal{X}}$ the shallow water parameter,
- $I_0 = \dfrac{\mathcal{B}}{\mathcal{X}}$ and $J_0 = \dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}$ the topography and friction slopes.

With $\dfrac{g\mathcal{X}}{\mathcal{U}^2} = \dfrac{g\mathcal{H}}{\mathcal{U}^2}\dfrac{\mathcal{X}}{\mathcal{H}} = \dfrac{1}{\delta F^2}$ and $\dfrac{g\mathcal{H}\mathcal{X}}{\mathcal{V}\mathcal{U}\mathcal{Y}} = \dfrac{g\mathcal{H}}{\mathcal{U}^2}\dfrac{\mathcal{U}}{\mathcal{V}}\dfrac{\mathcal{X}}{\mathcal{Y}} = \dfrac{1}{R_u^2 F^2}$, we finally get:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \dfrac{1}{F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{x}} = \dfrac{1}{\delta F^2}\left(-J_0\dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - I_0\bar{b}_{\bar{x}}\right), \\ \bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \dfrac{1}{R_u^2 F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{y}} = -\dfrac{J_0}{\delta F^2}\dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p}. \end{cases}$$