Consistent section-averaged shallow water equations with bottom friction

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Motivation: 2D/1D coupling for estuary simulation

Loire estuary

Thames estuary

Gironde estuary

Objectives and existing approaches

We wish to take into account:

1. the shape of the river bed,
2. the meanders of the river.

This presentation focuses on the shape of the river bed:

- the derivation of 1D models is well-understood [1] [2] in the ideal case of a ⊔-shaped channel;
- for more complex shapes, uniform stationary flows are recovered [3] [4] using a complex friction term and an additional term in the discharge flux;
- fully 2D models could be used but they are computationally costly.

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[1] see Bresch and Noble, 2007, in the context of laminar flows
[2] see Richard, Rambaud and Vila, 2017, in the context of turbulent flows
[3] see Decoene, Bonaventura, Miglio and Saleri, 2009
[4] see Marin and Monnier, 2009

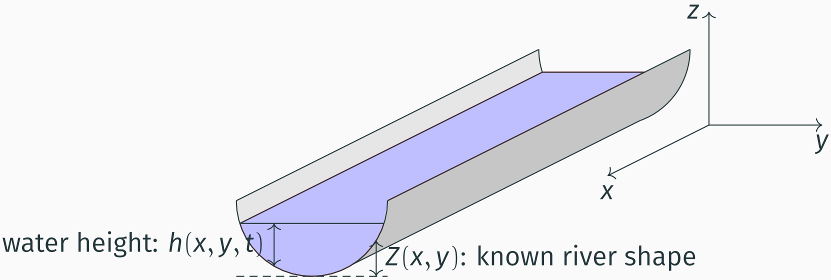
Specifications of the 1D model

The goal of this work is to develop a new model, based on the shallow water equations, that is:

- generic enough to not require empiric friction coefficients;
- consistent with the 2D shallow water in the asymptotic regime corresponding to an estuary or a river;
- hyperbolic and linearly stable;
- easily implementable (collaboration with the SHOM for flood simulations, ocean model forcing, ...).

Numerical experiments, on steady and unsteady flows, will help validate this approach.

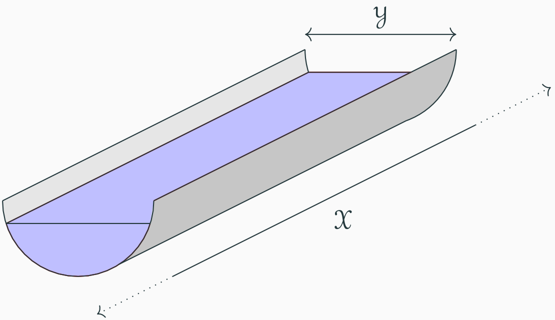
The non-conservative 2D shallow water system



$$\begin{cases} h_t + \boldsymbol{\nabla} \cdot (h\mathbf{u}) = 0 \\ \mathbf{u}_t + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} + g\boldsymbol{\nabla} h = g\left(-\boldsymbol{\nabla} Z - \dfrac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\, h^p}\right) \end{cases}$$

- $\mathbf{u} = (u, v)$ is the water velocity
- g is the gravity constant
- $C_h(x, y)$ is the (known) Chézy friction coefficient
- $p = 4/3$ is the friction exponent

Introduction of reference scales: the coordinates

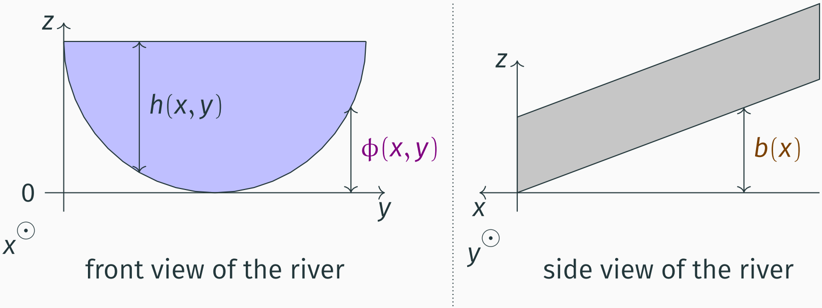


	dimensional quantity	reference scale	non-dimensional quantity
longitudinal coordinates	$x \in (0\mathrm{m}, 60000\mathrm{m})$	$\mathcal{X} = 2000\mathrm{m}$	$\bar{x} = \dfrac{x}{\mathcal{X}} \in (0, 30)$
transverse coordinates	$y \in (-25\mathrm{m}, 25\mathrm{m})$	$\mathcal{Y} = 50\mathrm{m}$	$\bar{y} = \dfrac{y}{\mathcal{Y}} \in (-0.5, 0.5)$

Introduction of reference scales: the topography

Regarding the geometry, we assume that $Z(x, y) = b(x) + \phi(x, y)$, where:

- $b(x)$ represents the main longitudinal topography, driving the flow from upstream to downstream;
- $\phi(x, y)$ represents small longitudinal and transverse variations.



Non-dimensional form of the 2D shallow water system

We introduce the following non-dimensional numbers to emphasize the different scales of the flow:

- F^2, the reference Froude number (ratio material/acoustic velocity),
- δ, the shallow water parameter (ratio height/reference length),
- R_u, the quasi-1D parameter (ratio transverse/longitudinal velocity),
- I_0 and J_0, the reference topography and friction slopes.

Finally, the non-dimensional form of the 2D shallow water system is:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \dfrac{1}{F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{x}} = \dfrac{1}{\delta F^2}\left(-J_0 \dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - I_0 \bar{b}_{\bar{x}}\right), \\ \bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \dfrac{1}{R_u^2 F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{y}} = -\dfrac{J_0}{\delta F^2}\dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p}. \end{cases}$$

Asymptotic expansions setup

In the regime under consideration, we have

- $\varepsilon := \dfrac{\delta F^2}{J_0} \ll 1$ (in practice, $F^2 \ll 1$, $\delta \ll 1$, $J_0 \ll 1$ and $J_0 \sim \delta$),
- $R_u \ll 1$ (quasi-unidimensional setting), and $R_u = \mathcal{O}(\varepsilon)$.

Highlighting the dominant terms in the system, we get:

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ u_t + uu_x + vu_y + \dfrac{1}{\varepsilon}\dfrac{\delta}{J_0}(h+\phi)_x = \dfrac{1}{\varepsilon}\left(-\dfrac{u\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} - \dfrac{I_0}{J_0} b_x\right), \\ v_t + uv_x + vv_y + \dfrac{1}{\varepsilon^3}\dfrac{\delta}{J_0}(h+\phi)_y = -\dfrac{1}{\varepsilon}\dfrac{v\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p}. \end{cases}$$

Goal: Perform asymptotic expansions in this regime, to better understand the weak dependency of the solution in y.

Free surface expansion

We consider the third equation:

$$v_t + uv_x + vv_y + \frac{1}{\varepsilon^3}\frac{\delta}{J_0}(h+\phi)_y = -\frac{1}{\varepsilon}\frac{v\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p},$$

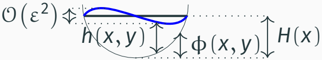
which we rewrite as follows to highlight the dominant term:

$$\frac{\delta}{J_0}(h+\phi)_y = \varepsilon^2\frac{v\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} + \varepsilon^3(v_t + uv_x + vv_y).$$

Neglecting the $\mathcal{O}(\varepsilon^2)$ terms, we get

$$\frac{\delta}{J_0}(h+\phi)_y = \mathcal{O}(\varepsilon^2),$$

and there exists $H = H(x,t)$ such that



$$H(x,t) = h(x,y,t) + \phi(x,y) + \mathcal{O}(\varepsilon^2).$$

$\rightsquigarrow$ the free surface is almost flat in the y-direction, up to $\mathcal{O}(\varepsilon^2)$

Longitudinal velocity expansion

We now consider the second equation. Highlighting the dominant terms, it reads:

$$u_t + uu_x + vu_y + \frac{1}{\varepsilon}\frac{\delta}{J_0}(h+\phi)_x = \frac{1}{\varepsilon}\left(-\frac{u\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} - \frac{I_0}{J_0} b_x\right).$$

Rearranging the terms, we get:

$$\begin{gathered}\frac{\delta}{J_0}(h+\phi)_x + \frac{I_0}{J_0} b_x + \frac{u\sqrt{u^2+\varepsilon^2 v^2}}{C^2 h^p} = \varepsilon(u_t + uu_x + vu_y),\\ \frac{\delta}{J_0}(h+\phi)_x + \frac{I_0}{J_0} b_x + \frac{u|u|}{C^2 h^p} = \mathcal{O}(\varepsilon).\end{gathered}$$

To perform the asymptotic expansion of u with respect to ε, we write

$$u(x,y,t) = u_{2D}^{(0)}(x,y,t) + \varepsilon u_{2D}^{(1)}(x,y,t) + \mathcal{O}\left(\varepsilon^2\right).$$

Longitudinal velocity expansion

Plugging $u_{2D}^{(0)}$, the second equation becomes, up to $\mathcal{O}(\varepsilon)$:

$$\frac{\delta}{J_0}(h+\phi)_x + \frac{I_0}{J_0}b_x + \frac{u_{2D}^{(0)}\left|u_{2D}^{(0)}\right|}{C^2 h^p} = \mathcal{O}(\varepsilon).$$

Since $h+\phi = H + \mathcal{O}\left(\varepsilon^2\right)$, we obtain

$$\frac{\delta}{J_0}H_x + \frac{I_0}{J_0}b_x + \frac{u_{2D}^{(0)}\left|u_{2D}^{(0)}\right|}{C^2 (H-\phi)^p} = \mathcal{O}(\varepsilon).$$

Straightforward computations yield:

$$u_{2D}^{(0)}(x,y,t) = \frac{\Lambda(x,t)}{\sqrt{|\Lambda(x,t)|}}\left(C(x,y)\right)\left(H(x,t)-\phi(x,y)\right)^{p/2},$$

where we have defined the corrected slope $\Lambda(x,t) = -\frac{I_0}{J_0}b_x - \frac{\delta}{J_0}H_x$.

Longitudinal velocity expansion

At the first order, the second equation becomes

$$\frac{u_{2D}^{(0)}\left|u_{2D}^{(0)}\right| - \left(u_{2D}^{(0)} + \varepsilon u_{2D}^{(1)}\right)\left|u_{2D}^{(0)} + \varepsilon u_{2D}^{(1)}\right|}{C^2(H-\phi)^p} = \varepsilon(u_t + uu_x + vu_y) + \mathcal{O}\left(\varepsilon^2\right),$$

which yields the following expression for $u_{2D}^{(1)}$:

$$u_{2D}^{(1)} = -\frac{u_{2D}^{(0)}}{2\Lambda}\left(\left(u_{2D}^{(0)}\right)_t + u_{2D}^{(0)}\left(u_{2D}^{(0)}\right)_x + v_{2D}^{(0)}\left(u_{2D}^{(0)}\right)_y\right).$$

Summary: At this level, we have obtained the asymptotic expansions of the free surface and the longitudinal velocity.

The goal is now to build a 1D model consistent with these asymptotic expansions.

1. Governing equations

2. Asymptotic expansions

3. Transverse averaging

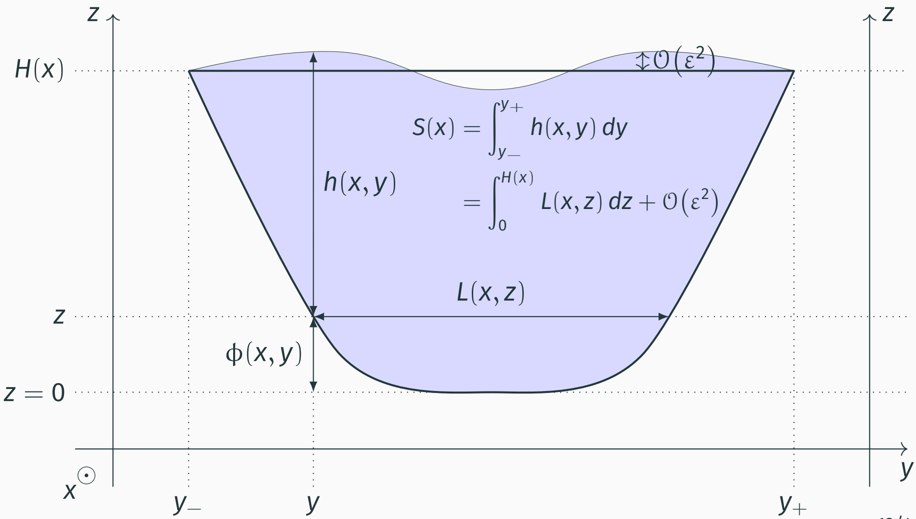
4. A zeroth-order model

5. A first-order model

6. Numerical validation of the model

7. Conclusion and perspectives

To obtain a 1D model, we start by averaging the 2D equations: below, we display the cross-section of the river, with respect to x.



Notations and asymptotic expansions

Let us introduce the following 1D notations:

$$S(x,t) = \int_{y_-}^{y_+} h(x,y,t)\, dy \quad \text{and} \quad Q(x,t) = \int_{y_-}^{y_+} h(x,y,t)\, u(x,y,t)\, dy.$$

We compute the asymptotic expansions of $Q = Q_{2D}^{(0)} + \varepsilon Q_{2D}^{(1)} + \mathcal{O}\left(\varepsilon^2\right)$:

- $Q_{2D}^{(0)} = \int_{y_-}^{y_+} h u_{2D}^{(0)}\, dy = \int_{y_-}^{y_+} \sqrt{|\Lambda|}\, \mathrm{sgn}(\Lambda)\, C\, h^{1+p/2}\, dy;$
- $Q_{2D}^{(1)} = \int_{y_-}^{y_+} h u_{2D}^{(1)}\, dy = \dfrac{-1}{4\Lambda}\left[\left(\int_{y_-}^{y_+} h\left(u_{2D}^{(0)}\right)^2 dy\right)_t + \left(\int_{y_-}^{y_+} h\left(u_{2D}^{(0)}\right)^3 dy\right)_x\right].$

We now integrate the 2D equations over the width of the river, to naturally introduce equations on S and Q.

Averaging the 2D system

1. The original mass conservation equation reads:

$$h_t + (hu)_x + (hv)_y = 0.$$

Therefore, since $v(y_-) = v(y_+) = 0$, we get:

$$\int_{y_-}^{y_+} h_t \, dy + \int_{y_-}^{y_+} (hu)_x \, dy = 0 \quad \Longrightarrow \quad S_t + Q_x = 0.$$

2. Now, we consider the second equation (which we multiply by h):

$$hu_t + huu_x + hvu_y + \frac{1}{\varepsilon}\frac{\delta}{J_0} h(h+\phi)_x = \frac{1}{\varepsilon} h \left(-\frac{u\sqrt{u^2 + R_u^2 v^2}}{C^2 h^p} - \frac{I_0}{J_0} b_x \right).$$

Arguing the mass conservation and integrating between y_- and y_+ yields:

$$Q_t + \left(\int_{y_-}^{y_+} hu^2 \, dy \right)_x = \frac{1}{\varepsilon} \int_{y_-}^{y_+} h \left(-\frac{I_0}{J_0} b_x - \frac{\delta}{J_0} (h+\phi)_x \right) dy$$

$$- \frac{1}{\varepsilon} \int_{y_-}^{y_+} \frac{u\sqrt{u^2 + R_u^2 v^2}}{C^2 h^{p-1}} \, dy.$$

Averaging the 2D system

We can simplify, in the current setting, the topography and friction source term within the discharge equation; recall $h + \phi = H + \mathcal{O}(\varepsilon^2)$.

$$Q_t + \left(\int_{y_-}^{y_+} hu^2 \, dy \right)_x = \frac{1}{\varepsilon} \int_{y_-}^{y_+} h \left(-\frac{I_0}{J_0} b_x - \frac{\delta}{J_0} (h + \phi)_x \right) dy$$

$$- \frac{1}{\varepsilon} \int_{y_-}^{y_+} \frac{u \sqrt{u^2 + R_u^2 v^2}}{C^2 h^{p-1}} \, dy$$

Averaging the 2D system

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$$Q_t + \left(\int_{y_-}^{y_+} h u^2 \, dy \right)_x = \frac{1}{\varepsilon} \int_{y_-}^{y_+} h \overbrace{\left(-\frac{I_0}{J_0} b_x - \frac{\delta}{J_0} (h + \phi)_x \right)}^{\Lambda(x,t) + \mathcal{O}\left(\varepsilon^2\right)} dy$$

$$- \frac{1}{\varepsilon} \int_{y_-}^{y_+} \underbrace{u \sqrt{u^2 + R_u^2 v^2}}_{u|u| + \mathcal{O}\left(\varepsilon^2\right)} \; \frac{1}{C^2 h^{p-1}} \, dy$$

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$$Q_t + \left(\int_{y_-}^{y_+} h u^2 \, dy \right)_x = \frac{1}{\varepsilon} \; \Lambda \int_{y_-}^{y_+} h \, dy$$
$$- \frac{1}{\varepsilon} \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}} \, dy + \mathcal{O}(\varepsilon)$$

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We can simplify, in the current setting, the topography and friction source term within the discharge equation; recall $h + \phi = H + \mathcal{O}(\varepsilon^2)$.

$$Q_t + \left(\int_{y_-}^{y_+} hu^2 \, dy \right)_x = \frac{1}{\varepsilon} \, \Lambda S$$

$$- \frac{1}{\varepsilon} \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}} \, dy + \mathcal{O}(\varepsilon)$$

Averaging the 2D system

Finally, the averaged system reads as follows, up to $\mathcal{O}(\varepsilon^2)$:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\int_{y_-}^{y_+} hu^2 \, dy \right)_x = \dfrac{1}{\varepsilon} \left(\Lambda S - \int_{y_-}^{y_+} \dfrac{u|u|}{C^2 h^{p-1}} \, dy \right) + \mathcal{O}(\varepsilon). \end{cases}$$

Next step: Build a truly 1D model, either zeroth-order accurate (up to $\mathcal{O}(\varepsilon)$) or first-order accurate (up to $\mathcal{O}(\varepsilon^2)$), from the averaged system.

That is to say:

- for the zeroth-order model, we need $Q = Q_{2D}^{(0)} + \mathcal{O}(\varepsilon)$;
- for the first-order model, we need $Q = Q_{2D}^{(0)} + \varepsilon Q_{2D}^{(1)} + \mathcal{O}(\varepsilon^2)$.

Setting up the model

The integrated discharge equation, highlighting the dominant terms and multiplying by ε, is

$$\Lambda S - \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}}\, dy = \varepsilon\left(Q_t + \left(\int_{y_-}^{y_+} hu^2\, dy\right)_x\right) + \mathcal{O}\left(\varepsilon^2\right).$$

At the zeroth order, i.e. up to $\mathcal{O}(\varepsilon)$, the right-hand side of this equation is neglected, and we get:

$$\Lambda S - \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}}\, dy = \mathcal{O}(\varepsilon).$$

We cannot directly use this equation in a 1D model, since it contains the unknown u, which depends on y.

Instead, we approximate the integral, up to $\mathcal{O}(\varepsilon)$, with a new 1D friction term.

The friction model

First, we choose this 1D friction term as a usual hydraulic engineering model. Thus, we impose the following formula:

$$\frac{Q|Q|}{C_{1D}^2 S} = \int_{y_-}^{y_+} \frac{u|u|}{C^2 h^{p-1}} \, dy + \mathcal{O}(\varepsilon).$$

It contains a 1D friction coefficient[5] C_{1D}, to be determined.

According to the discharge equation, we get, up to $\mathcal{O}(\varepsilon)$:

$$\frac{Q|Q|}{C_{1D}^2 S} = \Lambda S + \mathcal{O}(\varepsilon) \quad \Longrightarrow \quad C_{1D}^2 = \frac{Q|Q|}{\Lambda S^2} + \mathcal{O}(\varepsilon).$$

Second, we impose $Q = Q_{2D}^{(0)} + \mathcal{O}(\varepsilon)$, to get the following expression of the friction coefficient:

$$C_{1D}^2 = \frac{Q_{2D}^{(0)} \left| Q_{2D}^{(0)} \right|}{\Lambda S^2}.$$

[5] The coefficient C_{1D}^2 usually contains the hydraulic radius, the Chézy coefficient, ...

The final system

With the new friction model, the discharge equation reads

$$\Lambda S - \frac{Q|Q|}{C_{1D}^2 S} = \varepsilon\left(Q_t + \left(\int_{y_-}^{y_+} hu^2\,dy\right)_x\right) + \mathcal{O}(\varepsilon).$$

We choose to approximate the integral in the flux to describe the advection of the discharge:

$$\varepsilon\int_{y_-}^{y_+} hu^2\,dy = \varepsilon\frac{\left(\int_{y_-}^{y_+} hu\,dy\right)^2}{\int_{y_-}^{y_+} h\,dy} + \mathcal{O}(\varepsilon) = \varepsilon\frac{Q^2}{S} + \mathcal{O}(\varepsilon).$$

The resulting discharge equation, divided by ε, is

$$Q_t + \left(\frac{Q^2}{S}\right)_x = \frac{1}{\varepsilon}S\left(\Lambda - \underbrace{\frac{Q|Q|}{C_{1D}^2 S^2}}_{\mathcal{J}}\right) + \mathcal{O}(1).$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J}). \end{cases}$$

Let us double check that this model indeed recovers the zeroth-order expansion of Q.

Since $Q = Q^{(0)} + \mathcal{O}(\varepsilon)$, we get, at the zeroth order:

$$\Lambda = \mathcal{J} + \mathcal{O}(\varepsilon) \implies \Lambda = \overbrace{\Lambda \frac{Q|Q|}{Q_{2D}^{(0)}\left|Q_{2D}^{(0)}\right|}}^{\mathcal{J}} + \mathcal{O}(\varepsilon) = \Lambda \frac{Q^{(0)}\left|Q^{(0)}\right|}{Q_{2D}^{(0)}\left|Q_{2D}^{(0)}\right|} + \mathcal{O}(\varepsilon)$$

$$\implies Q^{(0)} = Q_{2D}^{(0)} + \mathcal{O}(\varepsilon).$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = \dfrac{1}{\varepsilon} S \left(-\dfrac{I_0}{J_0} b_x - \dfrac{\delta}{J_0} H_x - \mathcal{J} \right). \end{cases}$$

The final system

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$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = \dfrac{1}{\varepsilon} S \left(\underbrace{-\dfrac{I_0}{J_0} b_x}_{\mathcal{J}} - \dfrac{\delta}{J_0} H_x - \mathcal{J} \right). \end{cases}$$

The final system

Finally, the zeroth-order accurate 1D system reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x + \dfrac{SH_x}{F^2} = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}). \end{cases}$$

This is quite similar to the usual models: all the complexity lies within the friction model $\mathcal{J}$, and the expression of the friction coefficient C_{1D}.

$\rightsquigarrow$ We have derived a zeroth-order model.

What about a first-order one?

1. Governing equations

2. Asymptotic expansions

3. Transverse averaging

4. A zeroth-order model

5. A first-order model

6. Numerical validation of the model

7. Conclusion and perspectives

Derivation of a consistent energy equation

The procedure developed in the previous section introduces an error in $\mathcal{O}(1)$ on the discharge equation. To get a first-order accurate model, we need to lower this error to $\mathcal{O}(\varepsilon)$.

However, we already have a nice structure (hyperbolicity, ...): we keep this discharge equation and we focus on the the energy equation[6].

With $E_{2D} = \frac{h}{2}\|\mathbf{u}\|^2 + \frac{1}{2}gh^2$, the 2D energy equation is:

$$(E_{2D})_t + \boldsymbol{\nabla}\cdot\left(\mathbf{u}\left(E_{2D} + \frac{1}{2}gh^2\right)\right) = gh\left(-\mathbf{u}\cdot\boldsymbol{\nabla}Z - \frac{\|\mathbf{u}\|^3}{C_h^2 h^p}\right).$$

[6] see Luchini and Charru, 2010, in the context of thin film flows

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However, we already have a nice structure (hyperbolicity, ...): we keep this discharge equation and we focus on the the energy equation[6].

With $E_{2D} = \frac{1}{2}hu^2 + \frac{h^2}{2F^2} + \mathcal{O}\left(\varepsilon^2\right)$, the integrated equation is, up to $\mathcal{O}\left(\varepsilon^2\right)$:

$$\left(\int_{y_-}^{y_+} \frac{1}{2}hu^2\,dy\right)_t + \left(\int_{y_-}^{y_+} \frac{1}{2}hu^3\,dy\right)_x = \frac{1}{\varepsilon}\left(\Lambda Q - \int_{y_-}^{y_+} \frac{u^2|u|}{C^2h^{p-1}}\,dy\right) + \mathcal{O}(\varepsilon).$$

We build a 1D equation consistent, up to $\mathcal{O}\left(\varepsilon^2\right)$, with the energy equation: the first step is the introduction of a new 1D source term.

[6] see Luchini and Charru, 2010, in the context of thin film flows

Derivation of a consistent energy equation

The asymptotic expansion of Q is $Q = Q^{(0)} + \varepsilon Q^{(1)} + \mathcal{O}\left(\varepsilon^2\right)$.

At the moment, we have a zeroth order model:

- with the source term $\dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J})$ on the discharge equation,
- that recovers $Q = Q_{2D}^{(0)} + \mathcal{O}(\varepsilon)$.

To obtain a first order model, we build an energy equation:

- whose source term is, by analogy, $\dfrac{1}{\varepsilon} Q(\Lambda - \mathcal{J})$,
- that recovers $Q = Q_{2D}^{(0)} + \varepsilon Q_{2D}^{(1)} + \mathcal{O}\left(\varepsilon^2\right)$.

We show that the resulting equation deviates with $\mathcal{O}\left(\varepsilon^2\right)$ from the integrated 2D energy equation.

Derivation of a consistent energy equation

We impose $Q = Q_{2D}^{(0)} + \varepsilon Q_{2D}^{(1)} + \mathcal{O}\left(\varepsilon^2\right)$.

The friction model $\mathcal{J}$ then satisfies:

$$\mathcal{J} = \Lambda \frac{Q|Q|}{Q_{2D}^{(0)}|Q_{2D}^{(0)}|} = \Lambda\left(1 + 2\varepsilon \frac{Q_{2D}^{(1)}}{Q_{2D}^{(0)}}\right) + \mathcal{O}\left(\varepsilon^2\right).$$

Therefore, the source term of the energy equation is:

$$\begin{aligned} \frac{1}{\varepsilon} Q(\Lambda - \mathcal{J}) &= -2\Lambda Q_{2D}^{(1)} \frac{Q}{Q_{2D}^{(0)}} + \mathcal{O}(\varepsilon) = -2\Lambda Q_{2D}^{(1)} + \mathcal{O}(\varepsilon) \\ &= -2\Lambda \cdot \underbrace{\frac{-1}{4\Lambda}\left[\left(\int_{y_-}^{y_+} h\left(u_{2D}^{(0)}\right)^2 dy\right)_t + \left(\int_{y_-}^{y_+} h\left(u_{2D}^{(0)}\right)^3 dy\right)_x\right]}_{Q_{2D}^{(1)}} + \mathcal{O}(\varepsilon). \end{aligned}$$

Derivation of a consistent energy equation

The equation we have just derived reads

$$\left(\frac{1}{2}\int_{y_-}^{y_+} hu^2\,dy\right)_t + \left(\frac{1}{2}\int_{y_-}^{y_+} hu^3\,dy\right)_x = \frac{1}{\varepsilon}Q(\Lambda - \mathcal{J}) + \mathcal{O}(\varepsilon).$$

Compare this new equation to the integrated 2D energy equation:

$$\left(\int_{y_-}^{y_+} \frac{1}{2}hu^2\,dy\right)_t + \left(\int_{y_-}^{y_+} \frac{1}{2}hu^3\,dy\right)_x = \frac{1}{\varepsilon}\left(\Lambda Q - \int_{y_-}^{y_+} \frac{u^2|u|}{C^2 h^{p-1}}\,dy\right) + \mathcal{O}(\varepsilon).$$

Therefore, the new equation:

- is consistent with the integrated energy equation, up to $\mathcal{O}\left(\varepsilon^2\right)$;
- is based on the expression of $Q_{2D}^{(1)}$, and ensures its recovery.

The system in conservative form

We have thus obtained the following system:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\displaystyle\int_{y_-}^{y_+} hu^2 \, dy \right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J}) + \mathcal{O}(1), \\ \left(\dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^2 \, dy \right)_t + \left(\dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^3 \, dy \right)_x = \dfrac{1}{\varepsilon} Q(\Lambda - \mathcal{J}) + \mathcal{O}(\varepsilon). \end{cases}$$

The system in conservative form

We have thus obtained the following system:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\displaystyle\int_{y_-}^{y_+} hu^2 \, dy \right)_x + \dfrac{SH_x}{F^2} = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}) + \mathcal{O}(1), \\ \left(\dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^2 \, dy \right)_t + \left(\dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^3 \, dy \right)_x + \dfrac{QH_x}{F^2} = \dfrac{1}{\varepsilon} Q(\mathcal{I} - \mathcal{J}) + \mathcal{O}(\varepsilon). \end{cases}$$

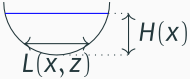
The system in conservative form

We have thus obtained the following conservative system:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(P^{hy} + \displaystyle\int_{y_-}^{y_+} hu^2\,dy \right)_x = P^{lat} + \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}) + \mathcal{O}(1), \\ \left(\mathcal{E} + \dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^2\,dy \right)_t + \left(\dfrac{Q}{S}\left(\mathcal{E} + P^{hy}\right) + \dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^3\,dy \right)_x = \dfrac{1}{\varepsilon} Q(\mathcal{I} - \mathcal{J}) + \mathcal{O}(\varepsilon), \end{cases}$$

where we have defined

- $L(x,z)$ such that $S = \displaystyle\int_0^H L(x,z)\,dz$,



- $P^{hy} = \dfrac{1}{F^2} \displaystyle\int_0^H (H - z) L\,dz, \quad P^{lat} = \dfrac{1}{F^2} \displaystyle\int_0^H (H - z) L_x\,dz, \quad \mathcal{E} = \dfrac{1}{F^2} \displaystyle\int_0^H zL\,dz.$

Introduction of a pressure and an energy

The goal is now to rewrite the homogeneous part of the model under an Euler-like formulation with pressure P_e and energy E_e, as follows:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(P^{hy} + \displaystyle\int_{y_-}^{y_+} hu^2\,dy \right)_x = 0, \\ \left(\mathcal{E} + \dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^2\,dy \right)_t + \left(\dfrac{Q}{S} \left(\mathcal{E} + P^{hy} \right) + \dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^3\,dy \right)_x = 0, \end{cases} \leftrightsquigarrow \begin{cases} S_t + (SU)_x = 0, \\ (SU)_t + (SU^2 + P_e)_x = 0, \\ (E_e)_t + (U(E_e + P_e))_x = 0. \end{cases}$$

By analogy, the Euler variables satisfy:

- U such that $U = \dfrac{Q}{S}$, as expected;
- P_e such that $P_e = P^{hy} + \displaystyle\int_{y_-}^{y_+} hu^2\,dy - SU^2$;
- E_e such that $E_e = \mathcal{E} + \dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^2\,dy$.

Introduction of auxiliary variables

To take care of the integrals, we introduce a new variable, the enstrophy[7] Ψ, defined as the variance of the velocity, by

$$\Psi = \int_{y_-}^{y_+} h(u-U)^2\,dy = \int_{y_-}^{y_+} hu^2\,dy - SU^2.$$

We therefore define a pressure and an energy in the 1D model by:

$$P = P^{hy} + \Psi \quad \text{and} \quad E = \mathcal{E} + \frac{1}{2}SU^2 + \frac{1}{2}\Psi.$$

We also introduce the potential Π, defined by

$$\Pi = \frac{1}{U}\int_{y_-}^{y_+} hu^3\,dy - SU^2 = \int_{y_-}^{y_+} h(u-U)^2\left(2+\frac{u}{U}\right)dy.$$

In practice, we cannot directly compute Ψ and Π since we do not know u. The final 1D model will have to address this issue.

[7] see Richard and Gavrilyuk, 2012

Introduction of auxiliary variables

With the integrals, the homogeneous model reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(P^{hy} + \int_{y_-}^{y_+} hu^2 \, dy \right)_x = 0, \\ \left(\mathcal{E} + \frac{1}{2} \int_{y_-}^{y_+} hu^2 \, dy \right)_t + \left(\frac{Q}{S} \left(\mathcal{E} + P^{hy} \right) + \frac{1}{2} \int_{y_-}^{y_+} hu^3 \, dy \right)_x = 0. \end{cases}$$

Introduction of auxiliary variables

With the energy and the pressure, the homogeneous model reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + P \right)_x = 0, \\ E_t + \left(\dfrac{Q}{S} \left(\mathcal{E} + P^{hy} \right) + \dfrac{1}{2} \displaystyle\int_{y_-}^{y_+} hu^3 \, dy \right)_x = 0. \end{cases}$$

Introduction of auxiliary variables

With the energy and the pressure, the homogeneous model reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + P \right)_x = 0, \\ E_t + \left(\dfrac{Q}{S}(E + P) \right)_x + \left(\dfrac{1}{2} \dfrac{Q}{S}(\Pi - 3\Psi) \right)_x = 0. \end{cases}$$

Introduction of auxiliary variables

With the energy and the pressure, the homogeneous model reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + P\right)_x = 0, \\ \underbrace{E_t + \left(\dfrac{Q}{S}(E+P)\right)_x}_{\text{Euler-like}} + \underbrace{\left(\dfrac{1}{2}\dfrac{Q}{S}(\Pi - 3\Psi)\right)_x}_{\text{non-Euler-like}} = 0. \end{cases}$$

$\rightsquigarrow$ How to handle the non-Euler-like part?

We introduce a new variable, the internal energy $e = e(S, \Psi, \Pi)$, which satisfies the equation

$$e_t + \left(\frac{Q}{S}e\right)_x - \left(\frac{1}{2}\frac{Q}{S}(\Pi - 3\Psi)\right)_x = 0.$$

Introduction of auxiliary variables

With the internal energy, the homogeneous model reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + P\right)_x = 0, \\ E_t + \left(\dfrac{Q}{S}(E+P)\right)_x + \left(\dfrac{1}{2}\dfrac{Q}{S}(\Pi - 3\Psi)\right)_x = 0, \\ e_t + \left(\dfrac{Q}{S}e\right)_x - \left(\dfrac{1}{2}\dfrac{Q}{S}(\Pi - 3\Psi)\right)_x = 0. \end{cases}$$

Introduction of auxiliary variables

With the internal energy, the homogeneous model reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + P\right)_x = 0, \\ (E+e)_t + \left(\dfrac{Q}{S}(E+e+P)\right)_x = 0, \\ e_t + \left(\dfrac{Q}{S}\left(e - \dfrac{1}{2}(\Pi - 3\Psi)\right)\right)_x = 0. \end{cases}$$

We get an Euler-like model, with energy $E + e$. How to make it hyperbolic?

We take $e = \dfrac{1}{2}(\Pi - 3\Psi)$: the wave velocities then are

$$0,\ U,\ U \pm \sqrt{\frac{S}{F^2 L(H)} + \frac{\Pi}{S}}, \quad \text{as opposed to } U \pm \sqrt{\frac{S}{F^2 L(H)}} \text{ for classical SW.}$$

Final homogeneous model

The final homogeneous model reads:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + P\right)_x = 0, \\ (E+e)_t + \left(\dfrac{Q}{S}(E+e+P)\right)_x = 0, \\ e_t = 0, \end{cases}$$

with the pressure and the total energy satisfying:

- $P(S,U,E) = 2\left(E - \dfrac{1}{2}SU^2\right) + \dfrac{1}{F^2}\displaystyle\int_0^H (H-3z)\,L\,dz;$
- $E + e = \mathcal{E} + \dfrac{1}{2}SU^2 + \dfrac{1}{2}\displaystyle\int_{y_-}^{y_+} h(u-U)^2\,dy + \dfrac{1}{2U}\int_{y_-}^{y_+} h(u-U)^3\,dy.$

Model with source terms: zeroth-order

Let us write the non-homogeneous model with Ψ and Π:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J}), \\ \left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x = \dfrac{1}{\varepsilon} Q(\Lambda - \mathcal{J}), \\ \left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = 0. \end{cases}$$

We check the zeroth-order expansion: we write $Q = Q^{(0)} + \mathcal{O}(\varepsilon)$, to get

$$\begin{aligned} \Lambda = \mathcal{J} + \mathcal{O}(\varepsilon) \implies \Lambda &= \Lambda \frac{Q|Q|}{Q_{2D}^{(0)}|Q_{2D}^{(0)}|} + \mathcal{O}(\varepsilon) \\ \implies Q^{(0)} &= Q_{2D}^{(0)} + \mathcal{O}(\varepsilon), \end{aligned}$$

and the zeroth-order expansion of Q is indeed still recovered.

What about the first-order expansion of Q?

Model with source terms: first-order

We study the first-order expansion of the energy equation: with $Q = Q_{2D}^{(0)} + \varepsilon Q^{(1)} + \mathcal{O}\left(\varepsilon^2\right)$, $\Psi = \Psi^{(0)} + \mathcal{O}(\varepsilon)$ and $\Pi = \Pi^{(0)} + \mathcal{O}(\varepsilon)$, we get

$$-2\Lambda Q^{(1)} = \left(\frac{1}{2}\frac{\left(Q_{2D}^{(0)}\right)^2}{S} + \frac{1}{2}\Psi^{(0)}\right)_t + \left(\frac{Q_{2D}^{(0)}}{S}\left(\frac{1}{2}\frac{\left(Q_{2D}^{(0)}\right)^2}{S} + \frac{1}{2}\Pi^{(0)}\right)\right)_x .$$

Moreover, straighforward computations show that

$$-2\Lambda Q_{2D}^{(1)} = \left(\frac{1}{2}\frac{\left(Q_{2D}^{(0)}\right)^2}{S} + \frac{1}{2}\Psi_{2D}^{(0)}\right)_t + \left(\frac{Q_{2D}^{(0)}}{S}\left(\frac{1}{2}\frac{\left(Q_{2D}^{(0)}\right)^2}{S} + \frac{1}{2}\Pi_{2D}^{(0)}\right)\right)_x .$$

To ensure $Q^{(1)} = Q_{2D}^{(1)}$, it is sufficient for the enstrophy and potential to satisfy $\Psi^{(0)} = \Psi_{2D}^{(0)} + \mathcal{O}(\varepsilon)$ and $\Pi^{(0)} = \Pi_{2D}^{(0)} + \mathcal{O}(\varepsilon)$.

Model with source terms: first-order

To recover the zeroth-order expansions of Ψ and Π, we introduce two new relaxation source terms in the system. These source terms have to ensure that $\Psi = \Psi_{2D}^{(0)} + \mathcal{O}(\varepsilon)$ and $\Pi = \Pi_{2D}^{(0)} + \mathcal{O}(\varepsilon)$.

We choose the following forms, where K_1 and K_2 are to be determined:

$$\frac{K_1}{\varepsilon} S\Lambda\left(1 - \frac{\Psi}{\Psi_{2D}^{(0)}}\right) \qquad \text{and} \qquad \frac{K_2}{\varepsilon} Q\Lambda\left(\frac{\Psi}{\Psi_{2D}^{(0)}} - \frac{\Pi}{\Pi_{2D}^{(0)}}\right).$$

At the zeroth-order, we indeed get $\Psi = \Psi_{2D}^{(0)} + \mathcal{O}(\varepsilon)$ and $\Pi = \Pi_{2D}^{(0)} + \mathcal{O}(\varepsilon)$.

We introduce the condensed notations

$$\mathcal{J}_\Psi = \Lambda \frac{\Psi}{\Psi_{2D}^{(0)}} \quad \text{and} \quad \mathcal{J}_\Pi = \Lambda \frac{\Pi}{\Pi_{2D}^{(0)}} \quad \left(\text{similarly to } \mathcal{J} = \Lambda \frac{Q|Q|}{Q_{2D}^{(0)}|Q_{2D}^{(0)}|}\right).$$

Model with source terms: first-order

We introduce these relaxation source terms in the system[8], to get:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J} + K_1(\Lambda - \mathcal{J}_\Psi)), \\ \left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x = \dfrac{1}{\varepsilon} Q(\Lambda - \mathcal{J}), \\ \left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = \dfrac{1}{\varepsilon} Q K_2(\mathcal{J}_\Psi - \mathcal{J}_\Pi). \end{cases}$$

[8] In order to ensure the recovery of $Q_{2D}^{(1)}$, we do not modify the energy equation.

Model with source terms: first-order

We introduce these relaxation source terms in the system[8], to get:

$$
\begin{cases}
S_t + Q_x = 0, \\
Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J} + K_1(\Lambda - \mathcal{J}_\Psi)), \\
\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x = \dfrac{1}{\varepsilon} Q(\Lambda - \mathcal{J}), \\
\left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = \dfrac{1}{\varepsilon} Q K_2(\mathcal{J}_\Psi - \mathcal{J}_\Pi).
\end{cases}
$$

Recall that, at the zeroth-order, the energy equation yields:

$$
\Lambda = \mathcal{J} + \mathcal{O}(\varepsilon) \quad \Longrightarrow \quad Q = Q_{2D}^{(0)} + \mathcal{O}(\varepsilon).
$$

[8] In order to ensure the recovery of $Q_{2D}^{(1)}$, we do not modify the energy equation.

Model with source terms: first-order

We introduce these relaxation source terms in the system[8], to get:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J} + K_1(\Lambda - \mathcal{J}_\Psi)), \\ \left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x = \dfrac{1}{\varepsilon} Q(\Lambda - \mathcal{J}), \\ \left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = \dfrac{1}{\varepsilon} Q K_2(\mathcal{J}_\Psi - \mathcal{J}_\Pi). \end{cases}$$

At the zeroth-order, from the discharge equation, we get:

$$\Lambda - \mathcal{J} + K_1(\Lambda - \mathcal{J}_\Psi) = \mathcal{O}(\varepsilon) \quad \Longrightarrow \quad \Lambda = \mathcal{J}_\Psi + \mathcal{O}(\varepsilon) \quad \Longrightarrow \quad \Psi = \Psi_{2D}^{(0)} + \mathcal{O}(\varepsilon).$$

[8] In order to ensure the recovery of $Q_{2D}^{(1)}$, we do not modify the energy equation.

Model with source terms: first-order

We introduce these relaxation source terms in the system[8], to get:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J} + K_1(\Lambda - \mathcal{J}_\Psi)), \\ \left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x = \dfrac{1}{\varepsilon} Q(\Lambda - \mathcal{J}), \\ \left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = \dfrac{1}{\varepsilon} Q K_2(\mathcal{J}_\Psi - \mathcal{J}_\Pi). \end{cases}$$

At the zeroth-order, from the internal energy equation, we get:

$$\mathcal{J}_\Pi = \mathcal{J}_\Psi + \mathcal{O}(\varepsilon) \quad \Longrightarrow \quad \mathcal{J}_\Pi = \Lambda + \mathcal{O}(\varepsilon) \quad \Longrightarrow \quad \Pi = \Pi_{2D}^{(0)} + \mathcal{O}(\varepsilon).$$

[8] In order to ensure the recovery of $Q_{2D}^{(1)}$, we do not modify the energy equation.

Model with source terms: first-order

We introduce these relaxation source terms in the system[8], to get:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x = \dfrac{1}{\varepsilon} S(\Lambda - \mathcal{J} + K_1(\Lambda - \mathcal{J}_\Psi)), \\ \left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x = \dfrac{1}{\varepsilon} Q(\Lambda - \mathcal{J}), \\ \left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = \dfrac{1}{\varepsilon} Q K_2(\mathcal{J}_\Psi - \mathcal{J}_\Pi). \end{cases}$$

At the first-order, from the energy equation, we get, up to $\mathcal{O}(\varepsilon)$:

$$\left(\frac{1}{2}\frac{(Q_{2D}^{(0)})^2}{S} + \frac{1}{2}\Psi_{2D}^{(0)}\right)_t + \left(\frac{Q_{2D}^{(0)}}{S}\left(\frac{1}{2}\frac{(Q_{2D}^{(0)})^2}{S} + \frac{1}{2}\Pi_{2D}^{(0)}\right)\right)_x = -2\Lambda Q^{(1)} \implies Q^{(1)} = Q_{2D}^{(1)}.$$

[8] In order to ensure the recovery of $Q_{2D}^{(1)}$, we do not modify the energy equation.

Summary

At this level, the model is, with no differential terms in the source:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x + (1+K_1)\dfrac{SH_x}{F^2} = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J} + K_1(\mathcal{I} - \mathcal{J}_\Psi)), \\ \left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x + \dfrac{QH_x}{F^2} = \dfrac{1}{\varepsilon} Q(\mathcal{I} - \mathcal{J}), \\ \left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = \dfrac{1}{\varepsilon} QK_2(\mathcal{J}_\Psi - \mathcal{J}_\Pi). \end{cases}$$

It ensures the correct asymptotic regime, that is to say

$$Q = Q_{2D}^{(0)} + \varepsilon Q_{2D}^{(1)} + \mathcal{O}\left(\varepsilon^2\right).$$

The quantities K_1 and K_2 still need to be determined: can we ensure the hyperbolicity and the linear stability of the model?

Mathematical properties

After straightforward but tedious computations, we show that, for small enough F^2 and ε:

- a necessary condition for the hyperbolicity is $1 + K_1 > 0$;
- taking the values

$$K_1 = -\frac{S\Psi^{(0)}}{\left(Q^{(0)}\right)^2} \quad \text{and} \quad K_2 = \frac{S\Pi^{(0)}}{\left(Q^{(0)}\right)^2}$$

  ensures the linear stability of the system, under the condition

$$\left(\frac{Q^{(0)}(S)}{S}\right)^2 < \frac{4S}{L(H)} + \frac{\Pi^{(0)}(S)}{S}.$$

  Note that, in this case, $1 + K_1 > 0$ in the usual applications.

Summary

The final model is:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x + \left(1 - \dfrac{S\Psi^{(0)}}{\left(Q^{(0)}\right)^2}\right)\dfrac{SH_x}{F^2} = \dfrac{1}{\varepsilon}S\left(\mathcal{I} - \mathcal{J} - \dfrac{S\Psi^{(0)}}{\left(Q^{(0)}\right)^2}(\mathcal{I} - \mathcal{J}_\Psi)\right), \\ \left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x + \dfrac{QH_x}{F^2} = \dfrac{1}{\varepsilon}Q(\mathcal{I} - \mathcal{J}), \\ \left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = \dfrac{1}{\varepsilon}Q\dfrac{S\Pi^{(0)}}{\left(Q^{(0)}\right)^2}(\mathcal{J}_\Psi - \mathcal{J}_\Pi). \end{cases}$$

It ensures the correct asymptotic regime, that is to say

$$Q = Q_{2D}^{(0)} + \varepsilon Q_{2D}^{(1)} + \mathcal{O}\left(\varepsilon^2\right).$$

In addition, it is hyperbolic and linearly stable.

Next step: numerical validation of this model

Numerical schemes

To handle the stiff relaxation source term, we introduce an implicit splitting procedure.

We present this procedure on the zeroth-order model for clarity:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x + \dfrac{1}{\varepsilon}\dfrac{\delta}{J_0} S H_x = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}). \end{cases}$$

First, we consider the non-stiff part:

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\dfrac{Q^2}{S}\right)_x = 0, \end{cases}$$

which we discretize using an upwind finite difference scheme.

Numerical schemes

Second, we consider the stiff part:

$$\begin{cases} S_t = 0, \\ Q_t + \dfrac{1}{\varepsilon}\dfrac{\delta}{J_0} S H_x = \dfrac{1}{\varepsilon} S(\mathcal{I} - \mathcal{J}). \end{cases}$$

Since $S_t = 0$, we are left with the following ODE on Q:

$$Q_t = \frac{1}{\varepsilon} S \Lambda \left(1 - \frac{Q^2}{\left(Q_{2D}^{(0)}\right)^2} \right),$$

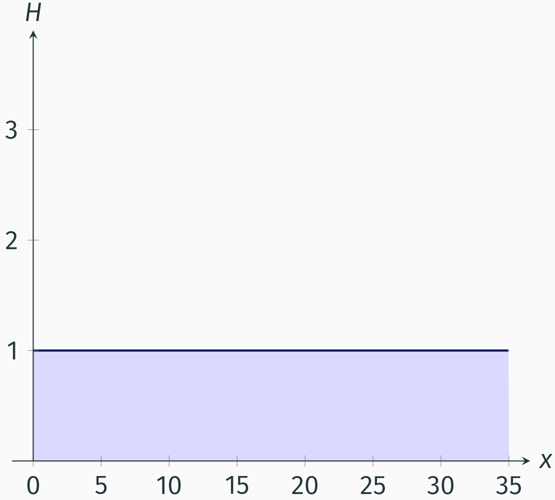
which we can solve exactly, to get

$$Q(t) = Q_{2D}^{(0)} \frac{\tanh\left(\dfrac{1}{\varepsilon} \dfrac{S|\Lambda|}{|Q_{2D}^{(0)}|} t \right) + \dfrac{Q(0)}{Q_{2D}^{(0)}}}{1 + \tanh\left(\dfrac{1}{\varepsilon} \dfrac{S|\Lambda|}{|Q_{2D}^{(0)}|} t \right) \dfrac{Q(0)}{Q_{2D}^{(0)}}}.$$

The same procedure is applied to the first-order model.

Unsteady flood flow

We consider a 5-year flood for the Garonne river upstream of Toulouse; we take $F = 0.09$ and $\varepsilon \simeq 0.175$.



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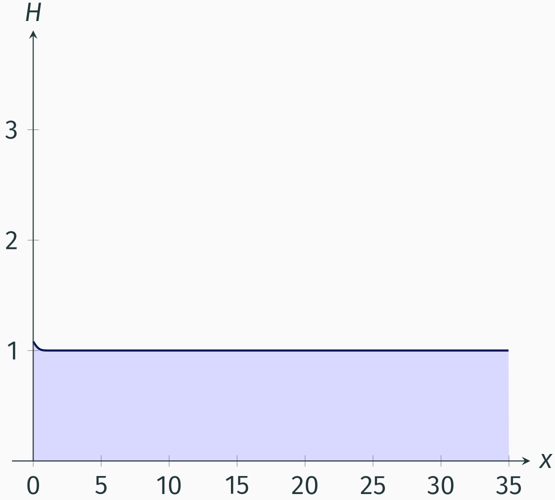

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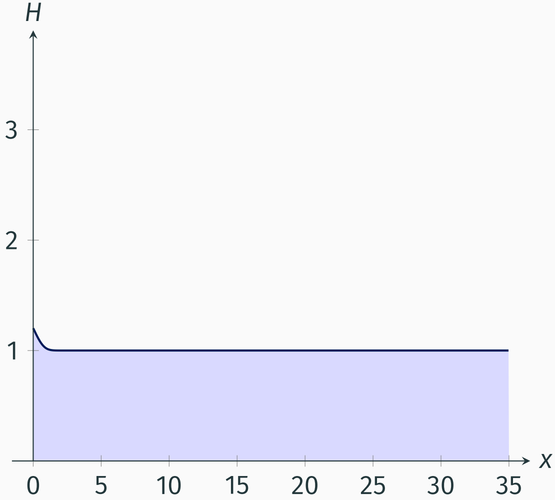

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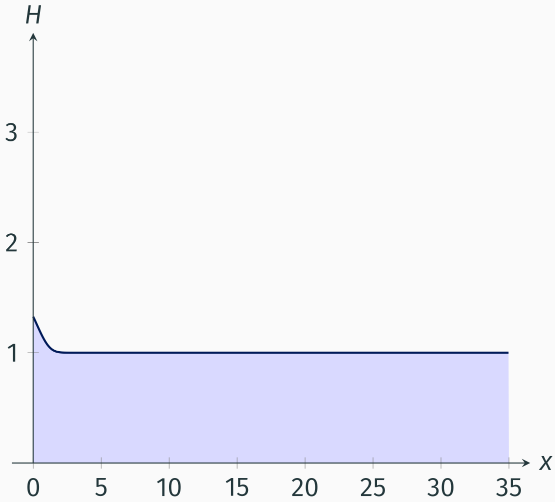

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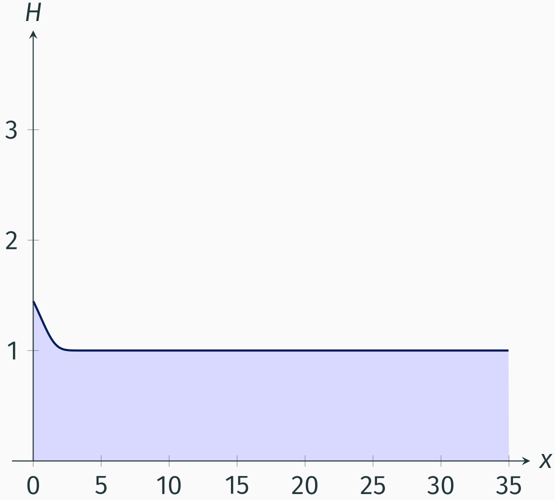

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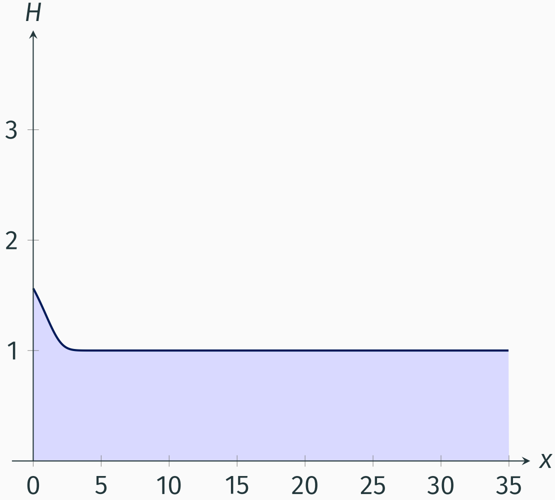

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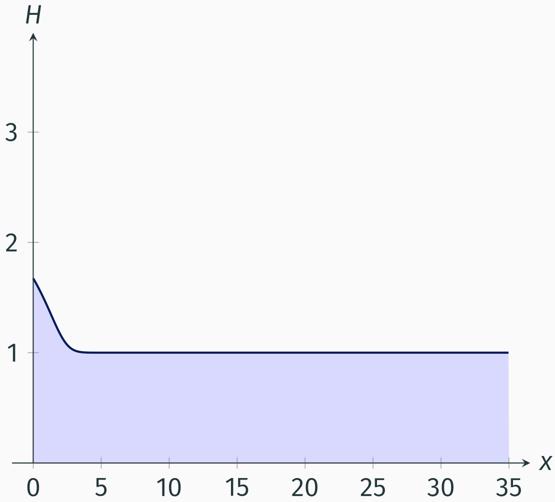

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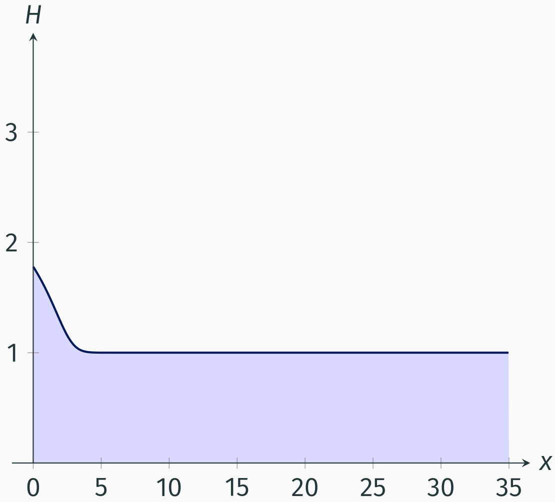

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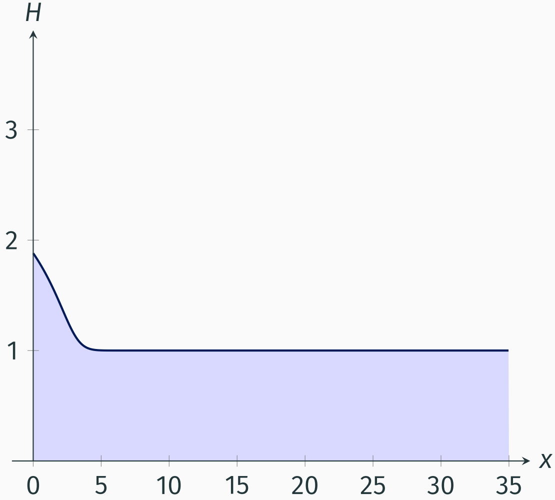

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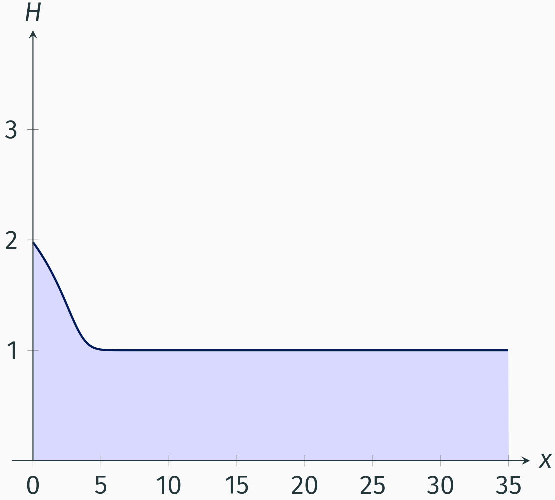

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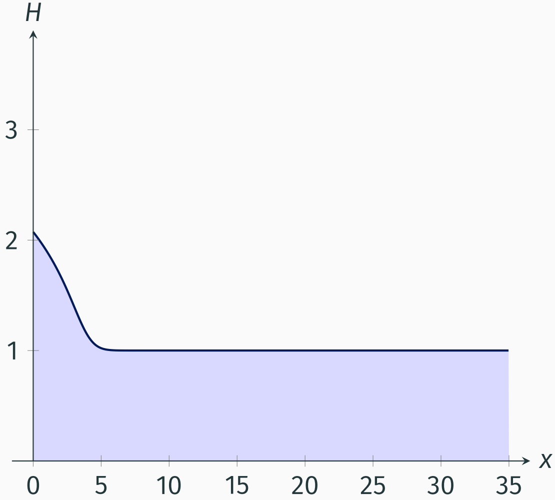

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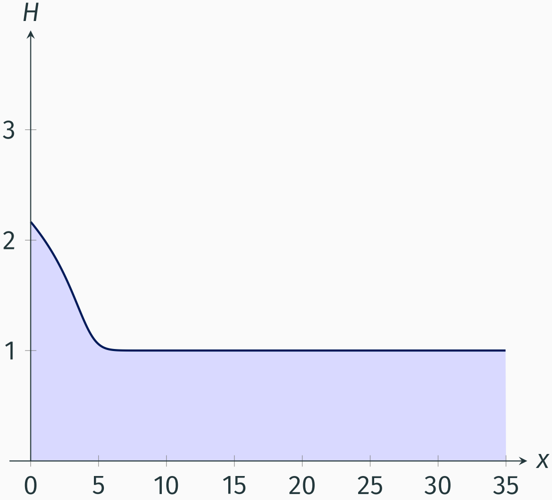

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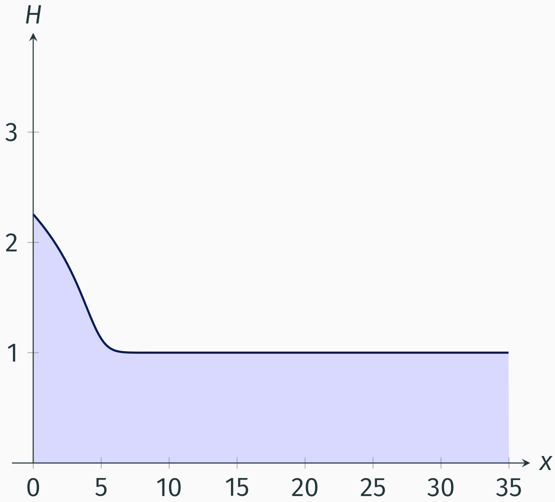

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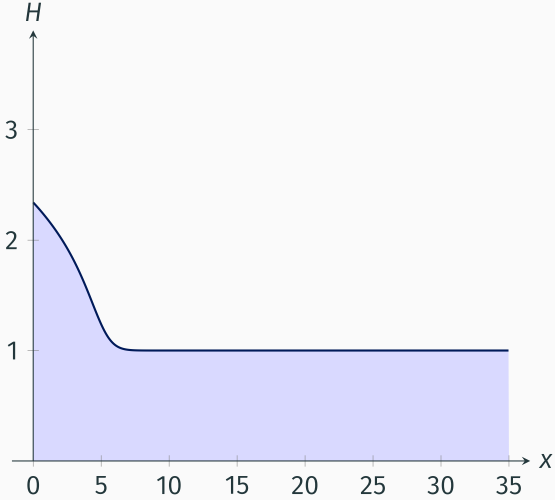

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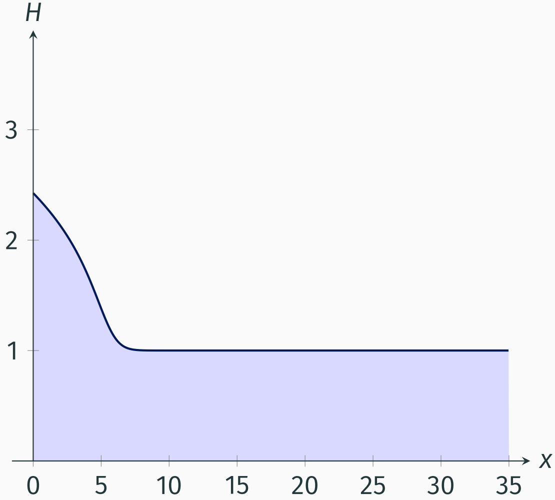

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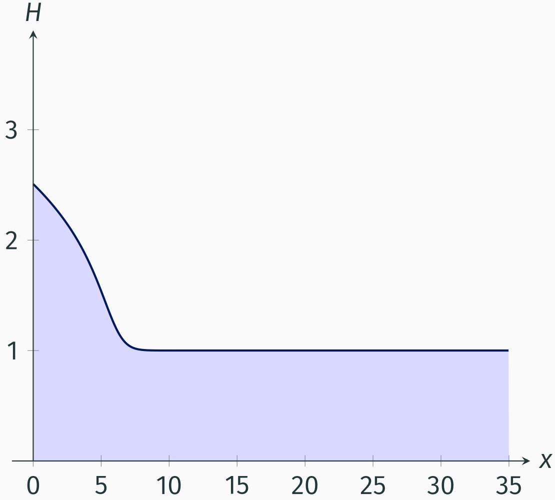

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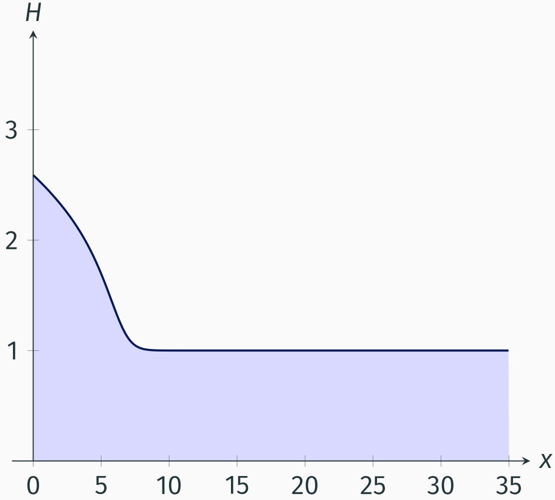

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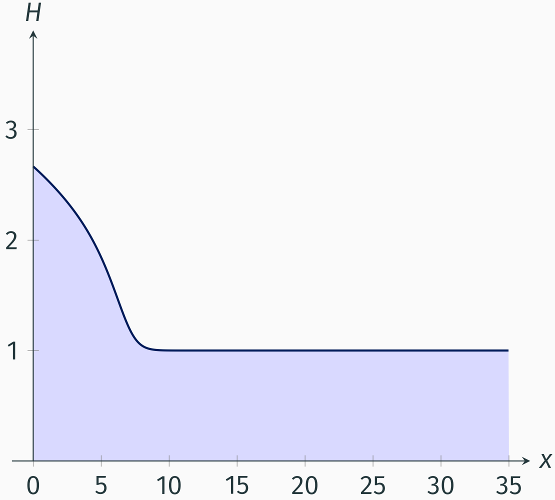

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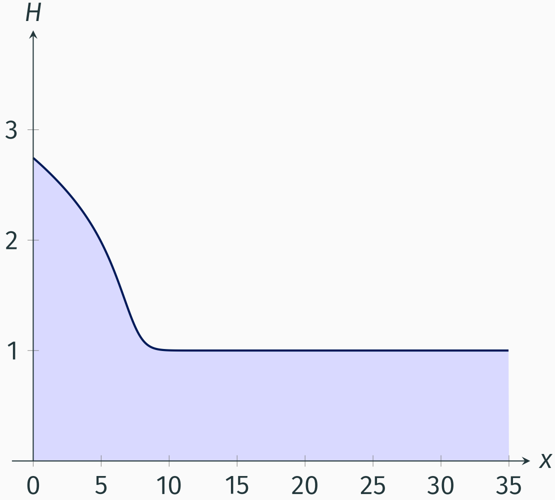

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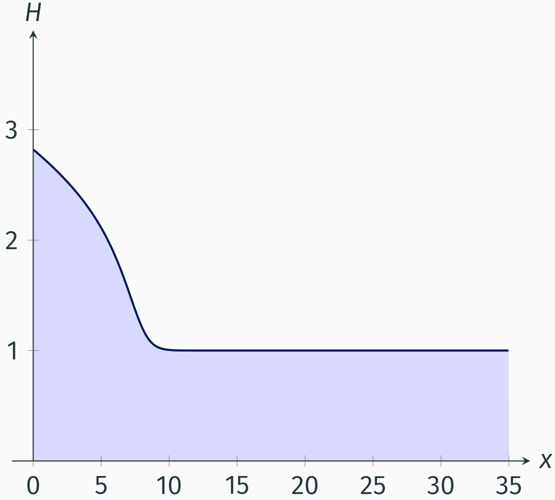

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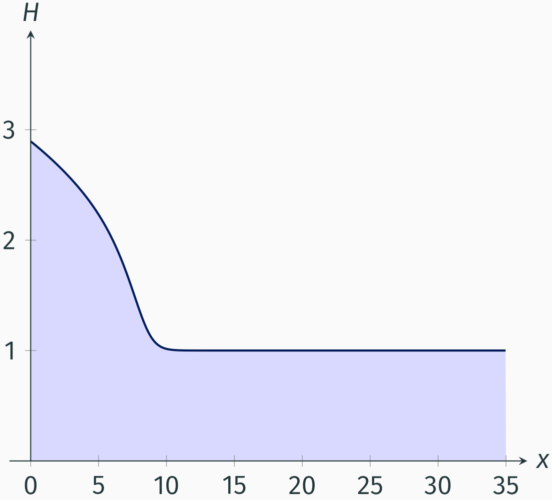

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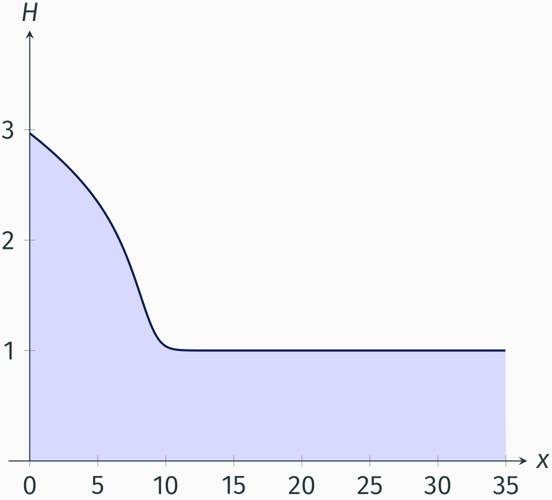

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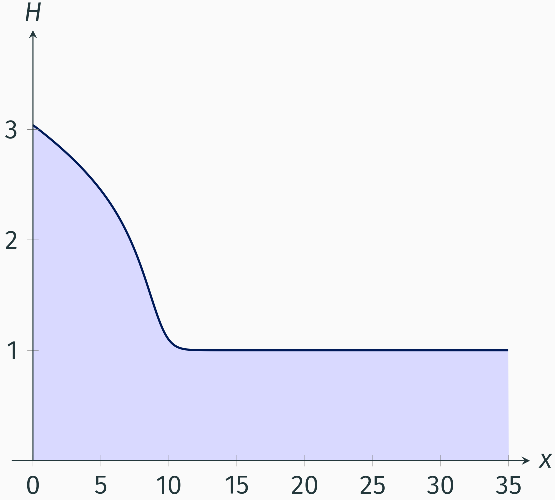

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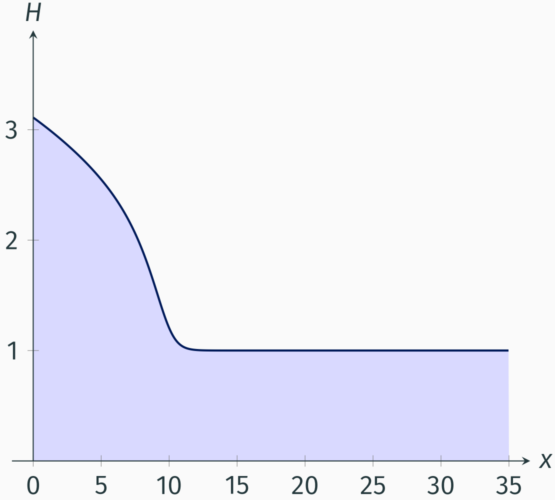

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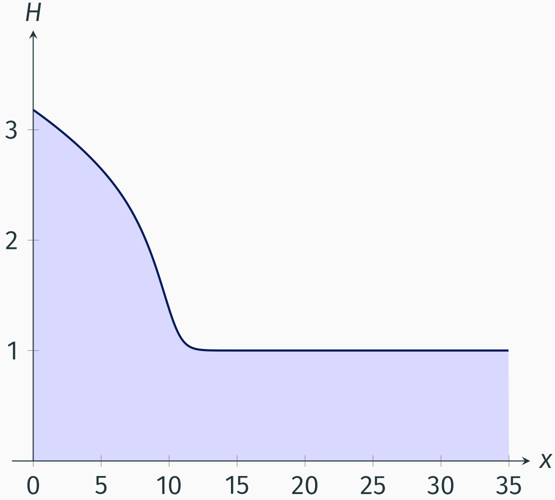

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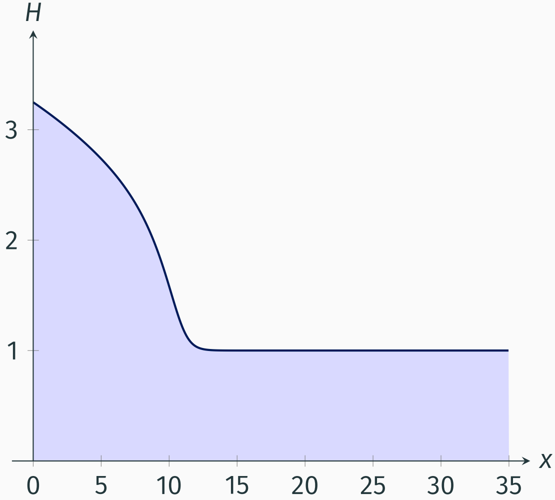

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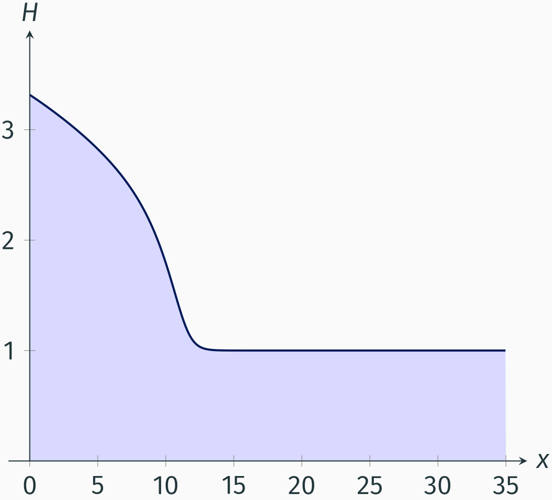

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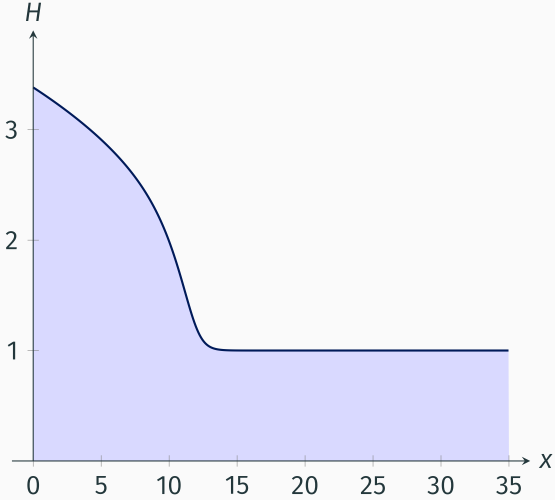

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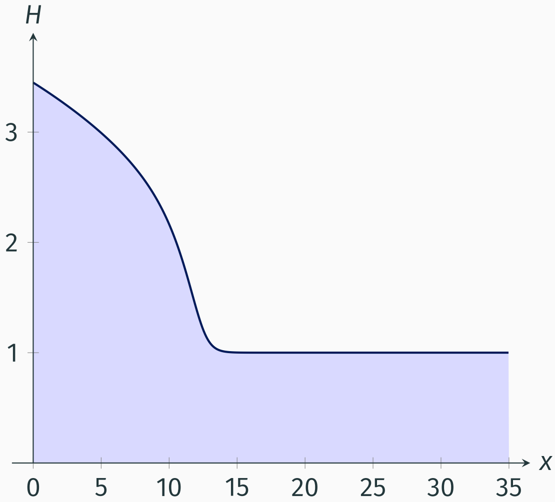

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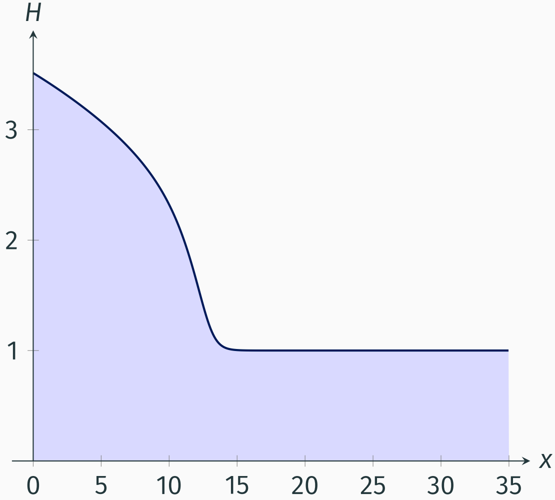

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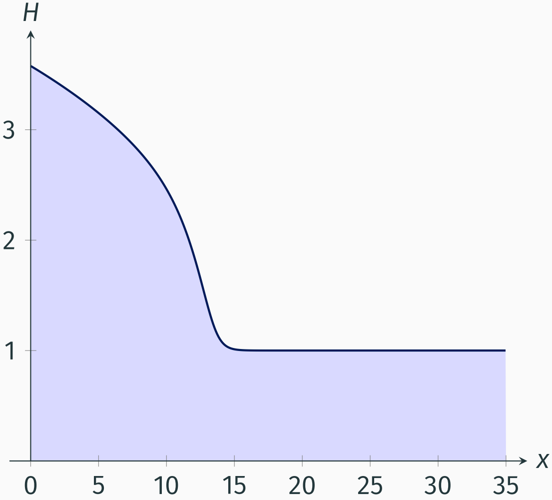

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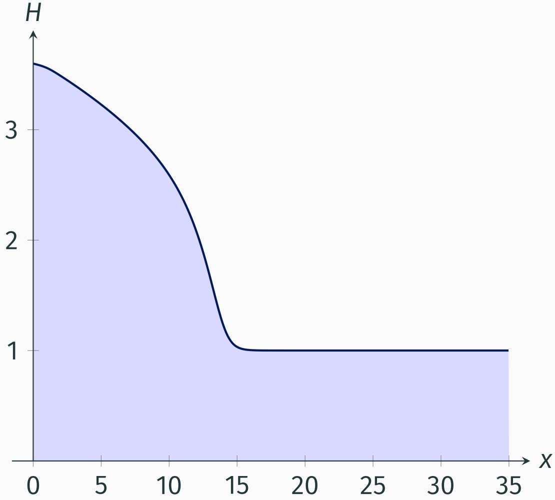

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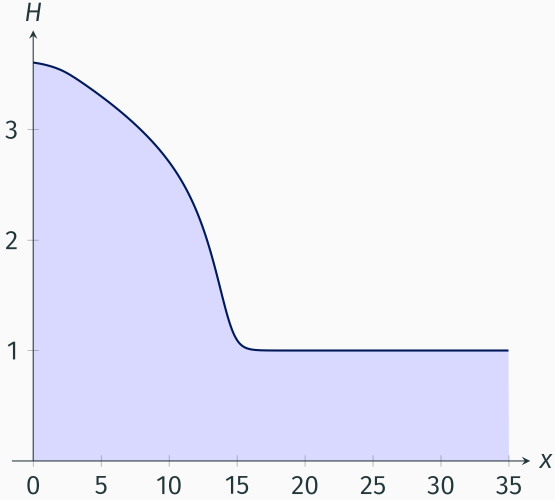

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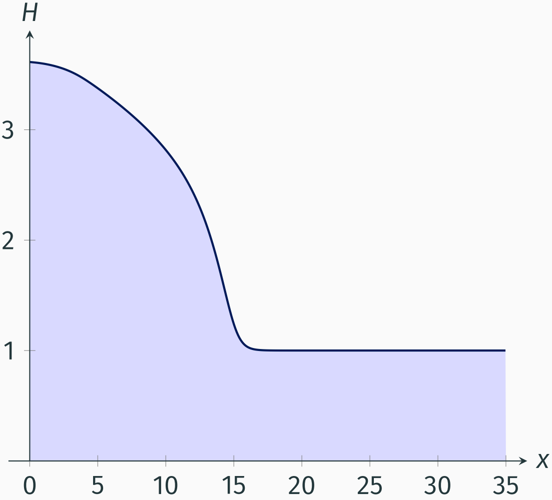

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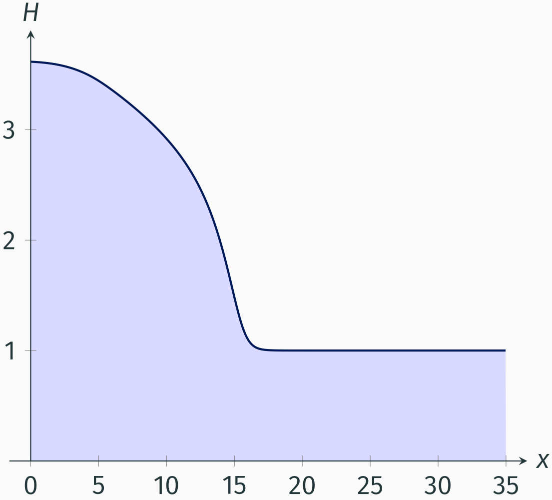

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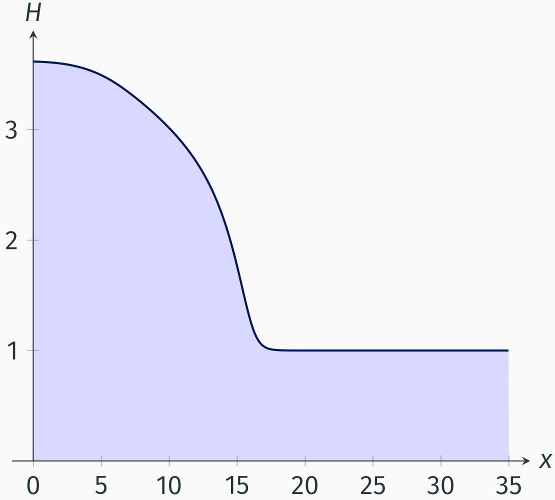

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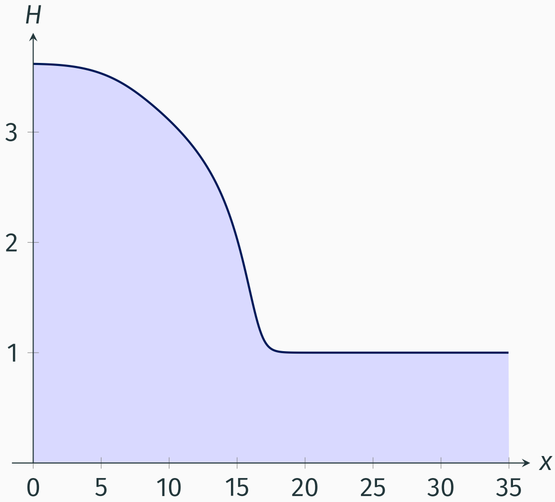

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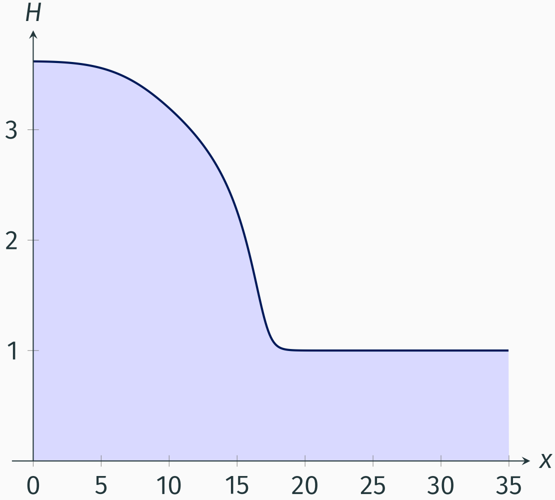

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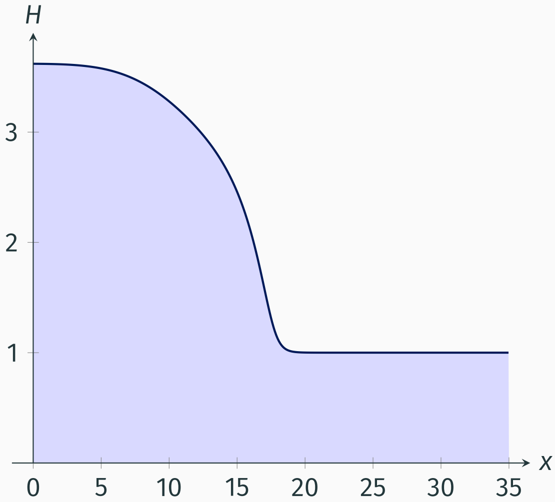

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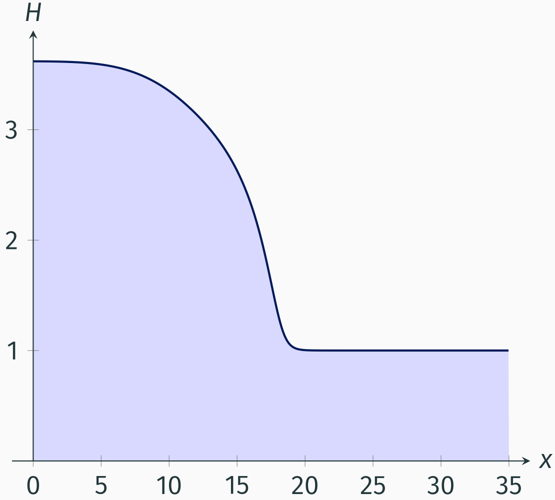

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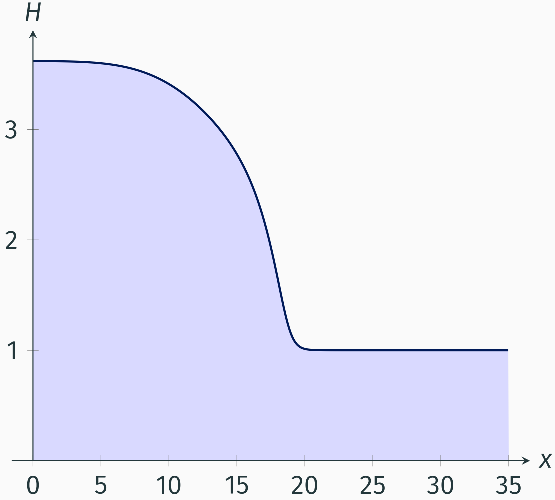

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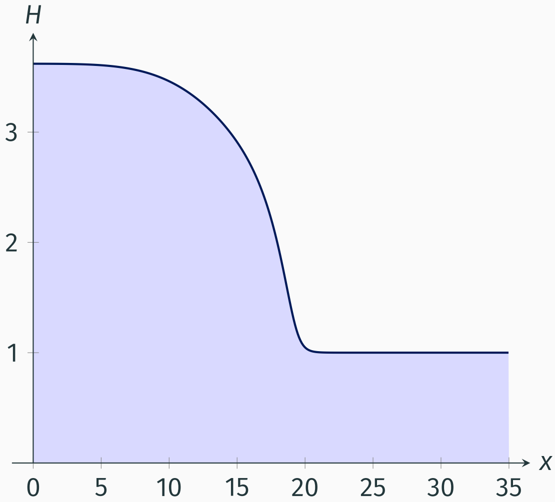

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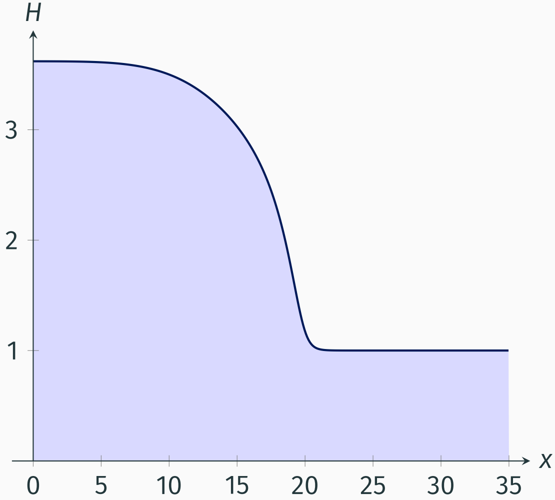

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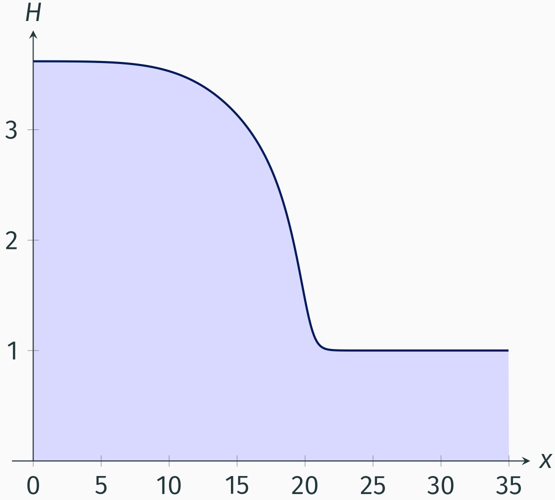

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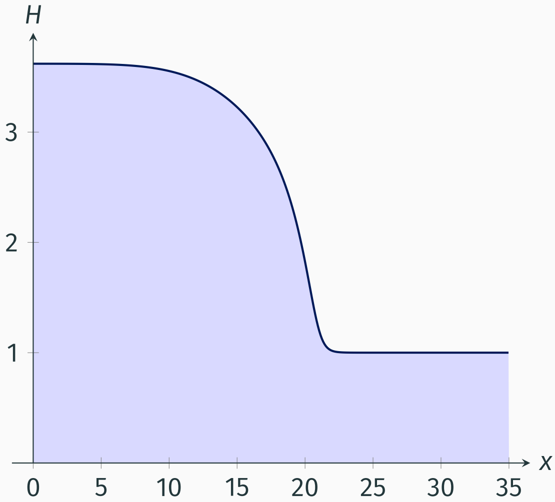

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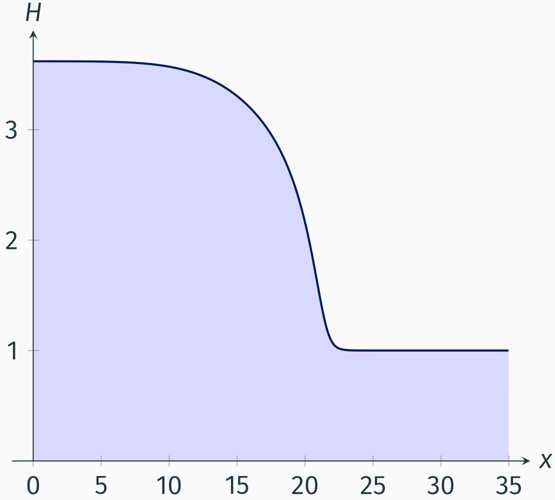

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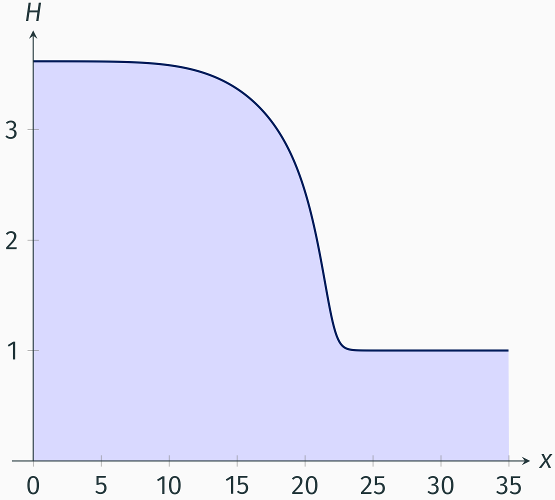

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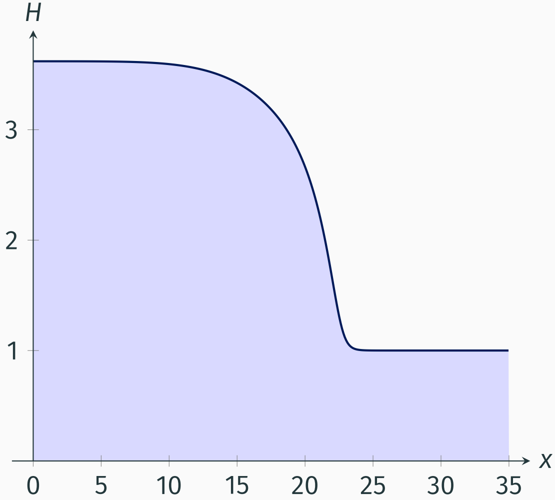

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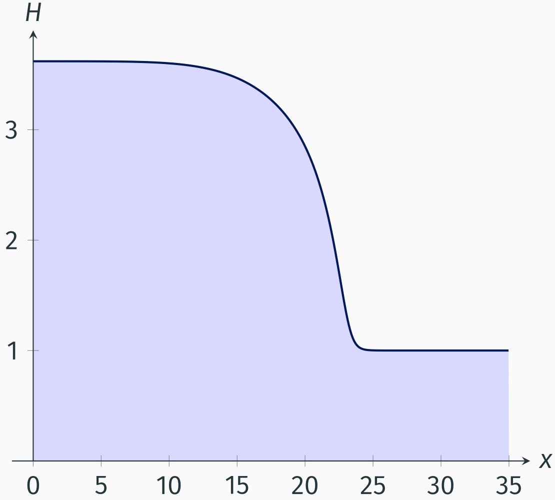

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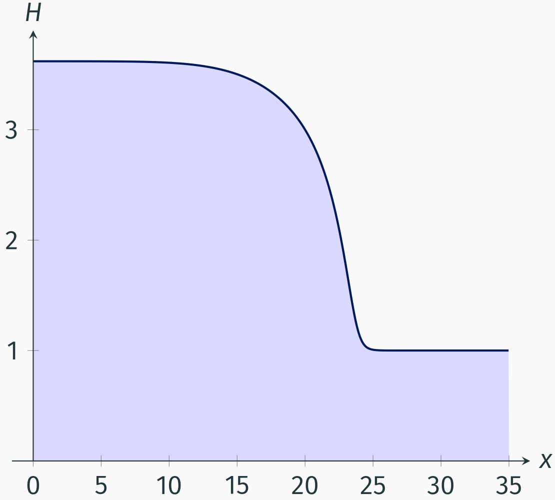

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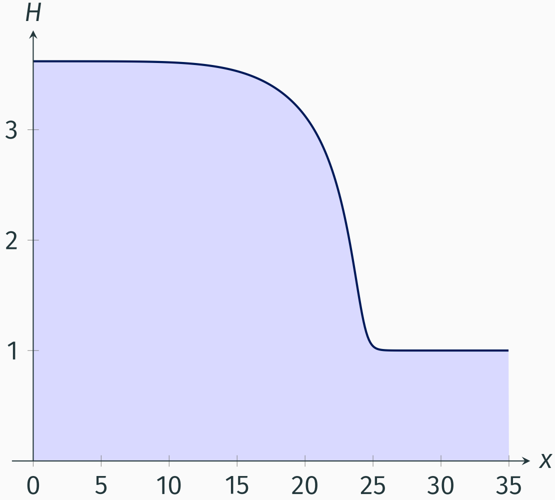

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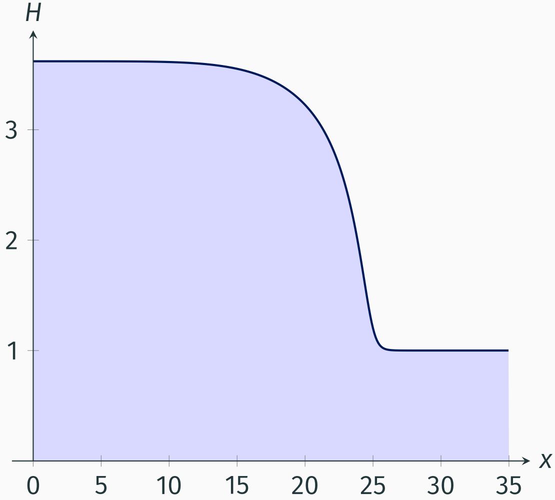

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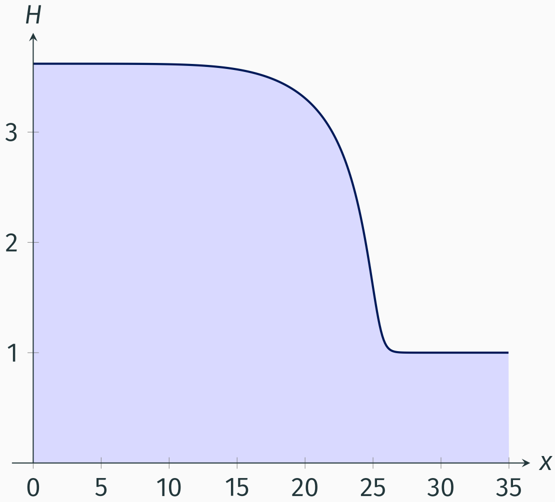

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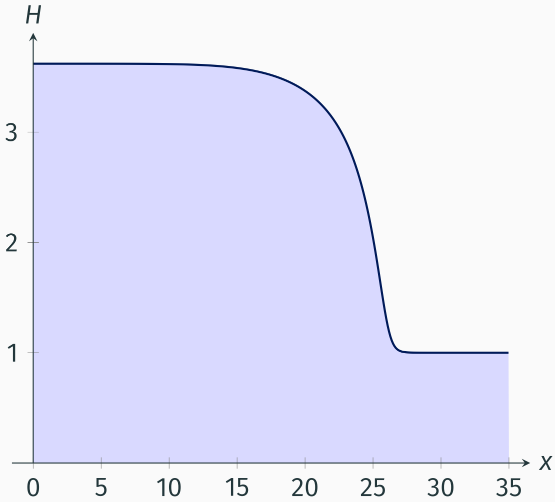

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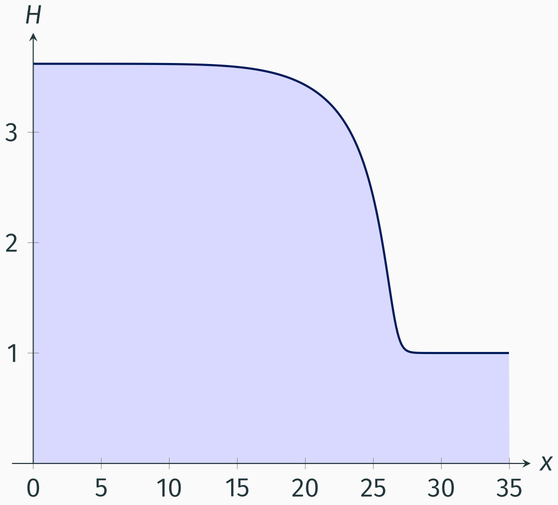

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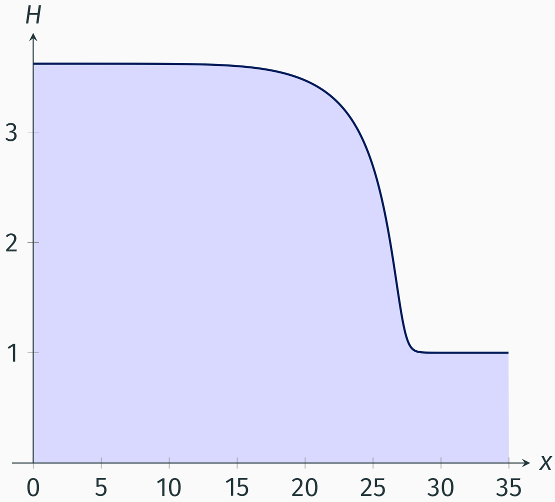

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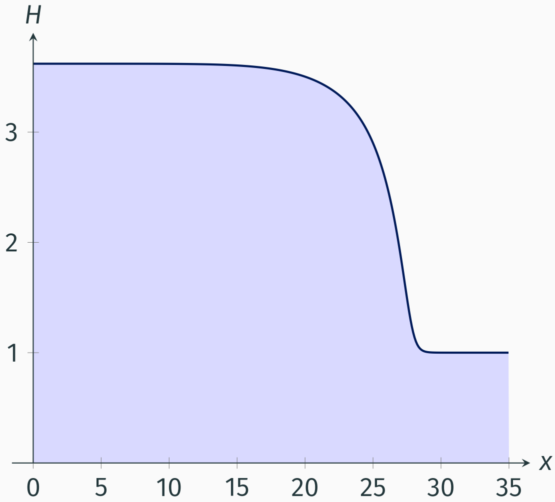

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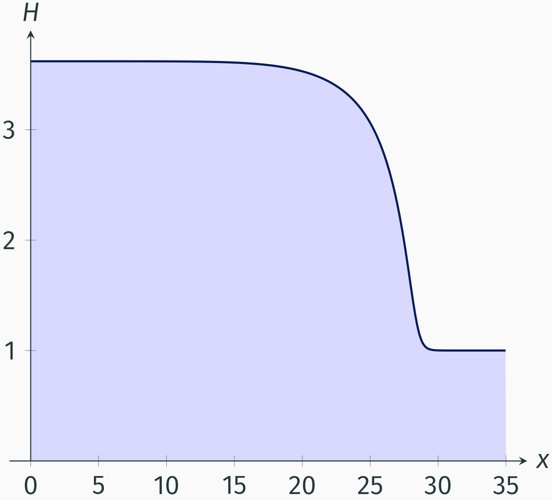

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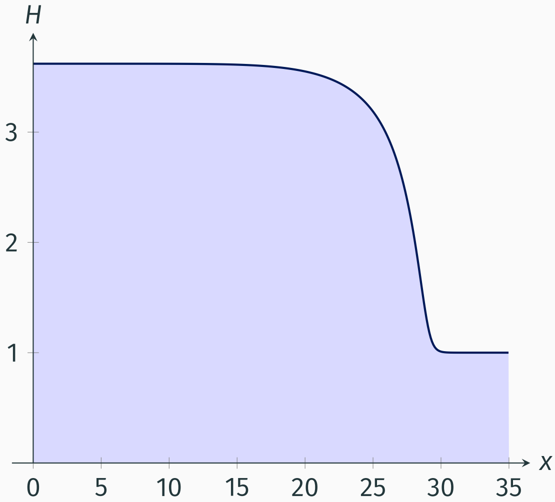

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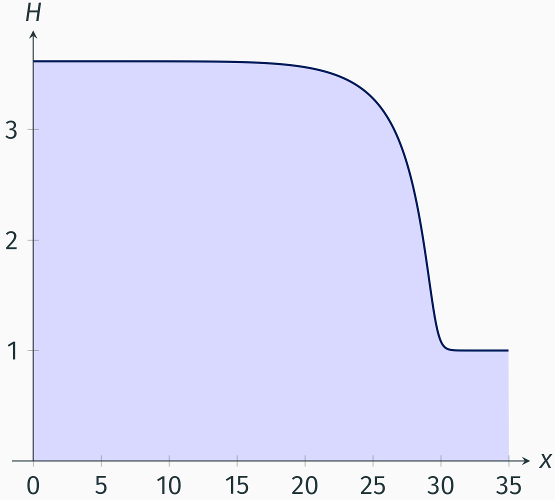

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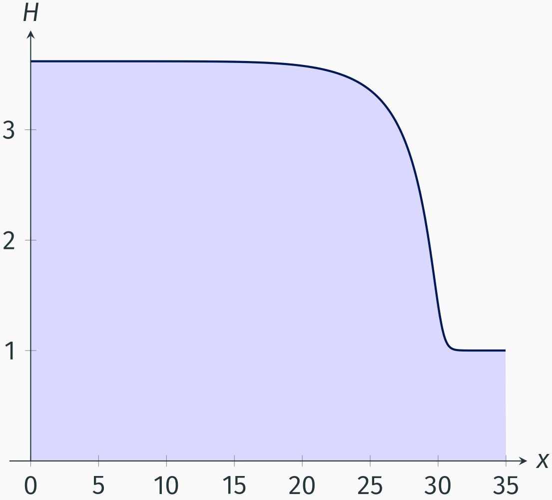

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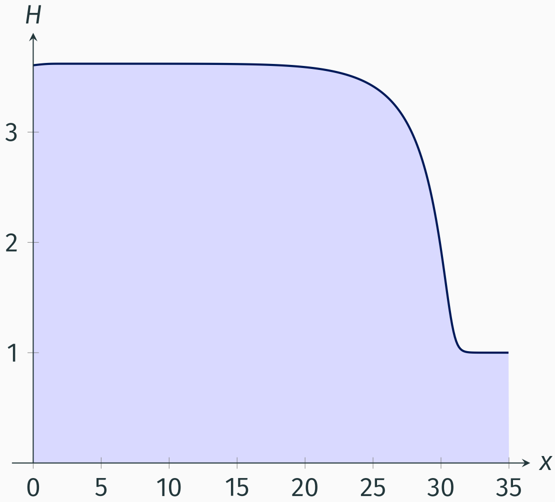

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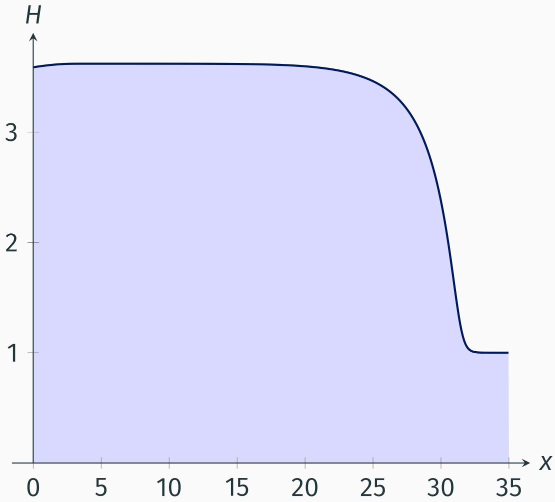

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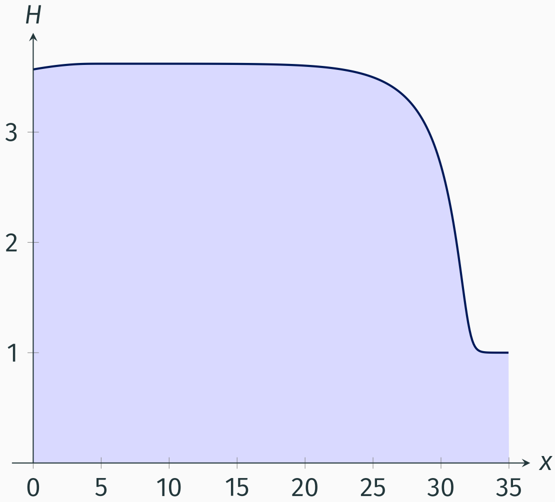

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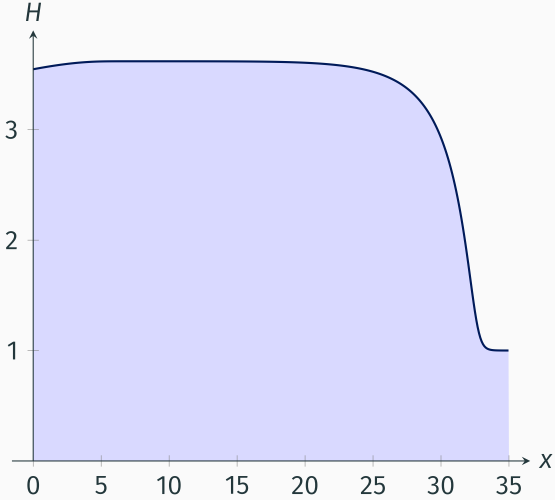

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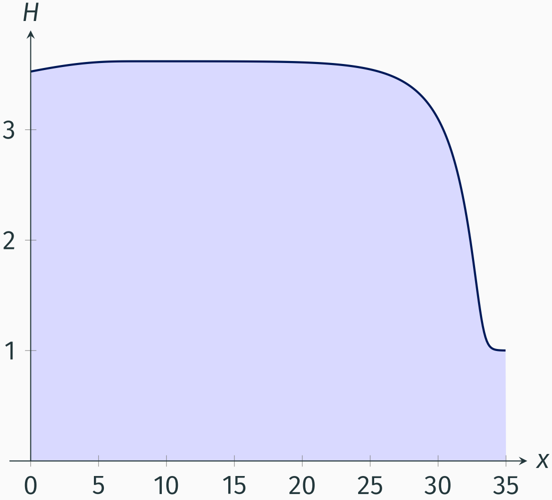

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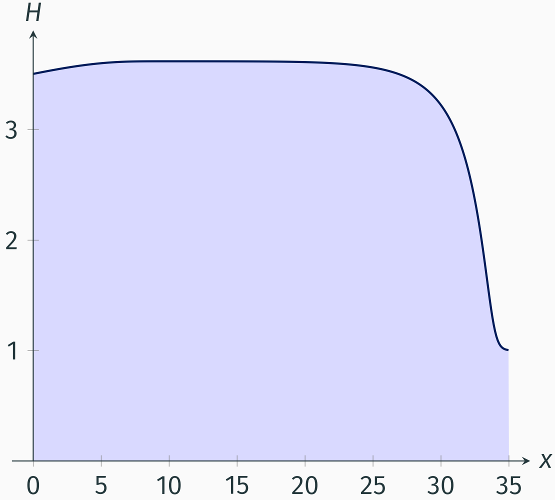

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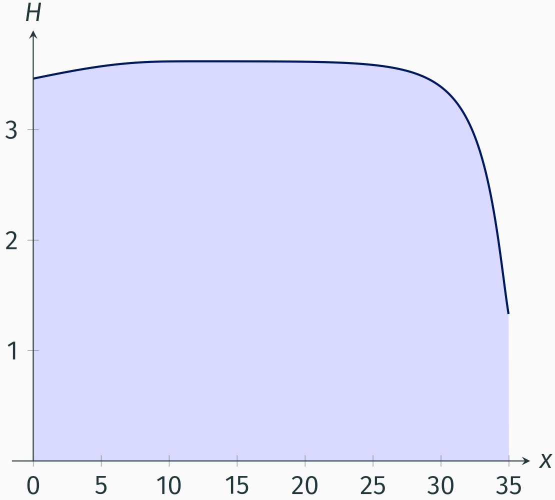

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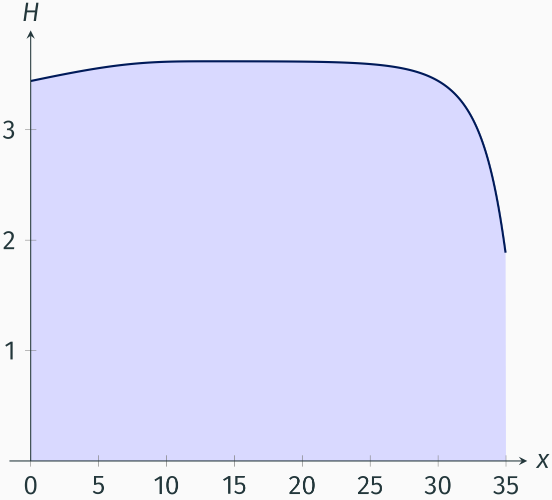

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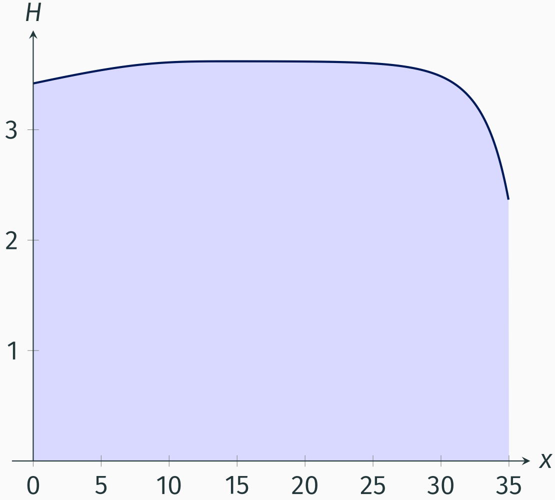

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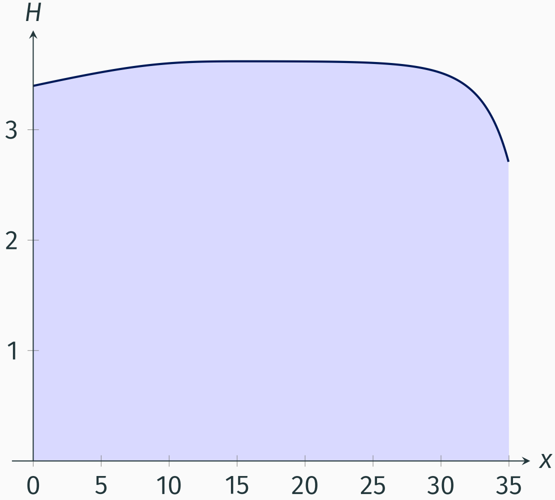

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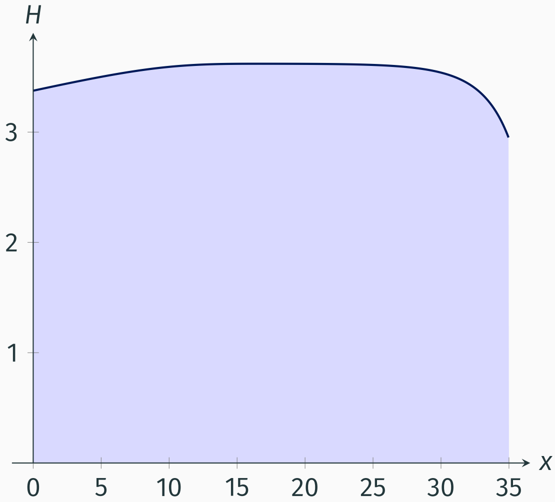

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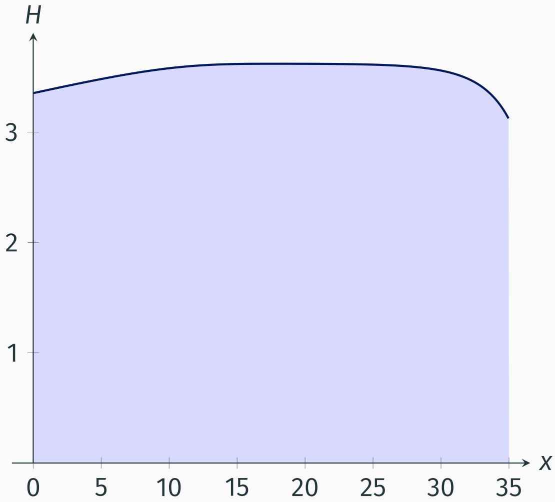

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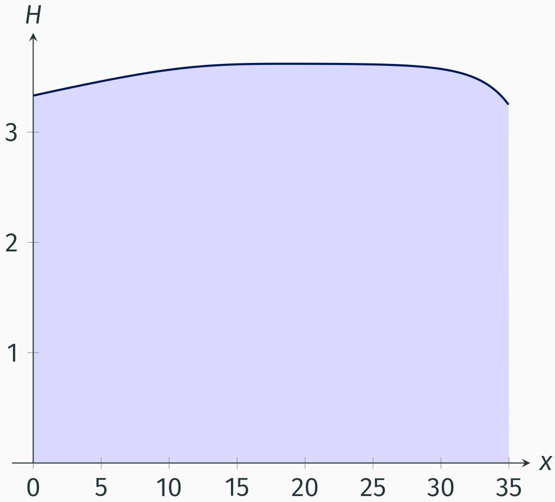

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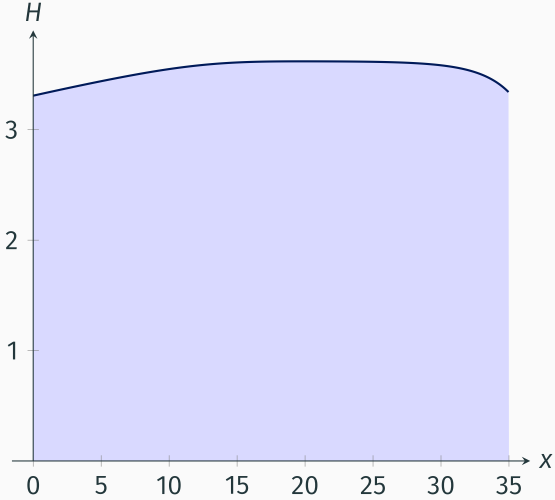

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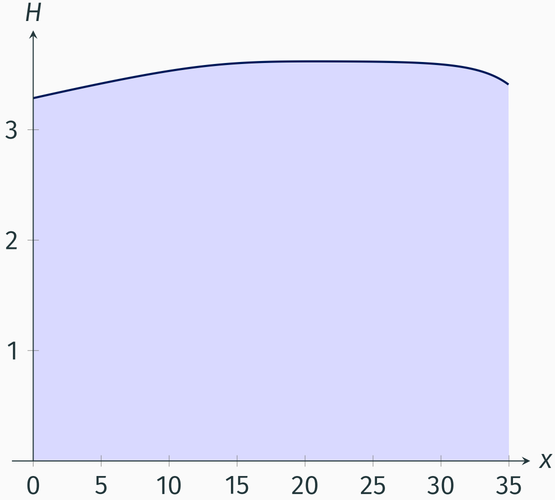

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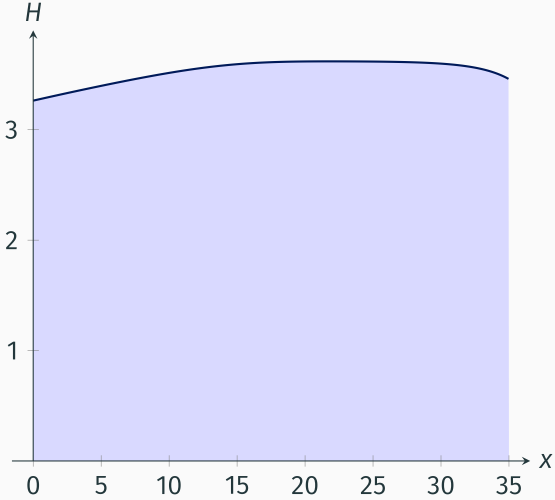

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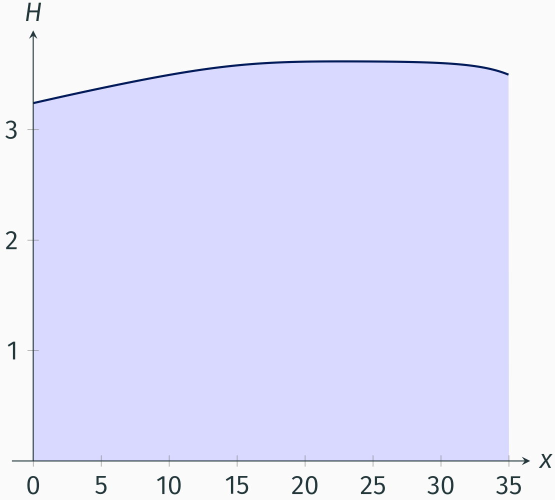

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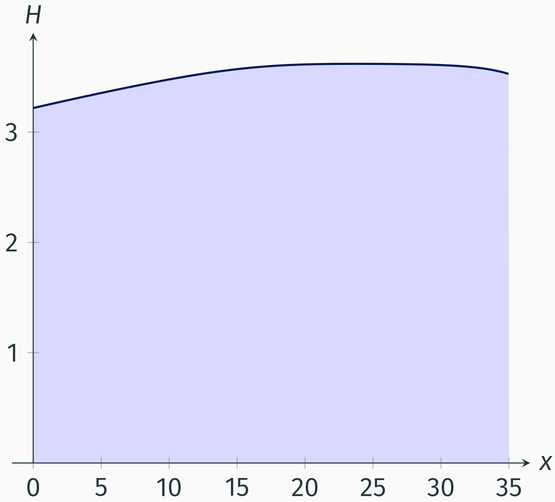

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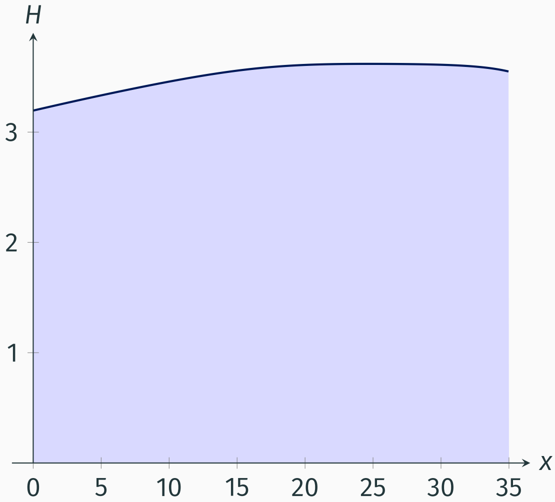

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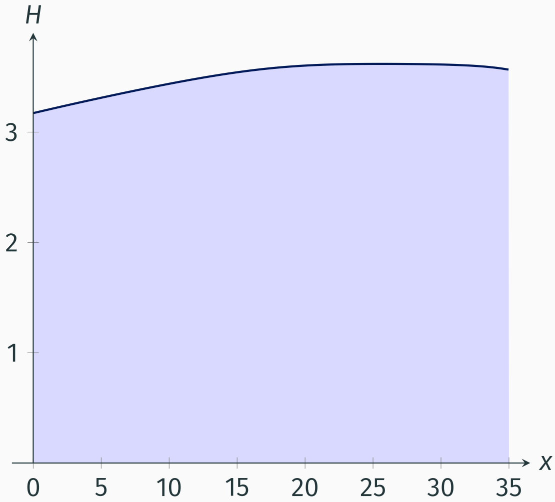

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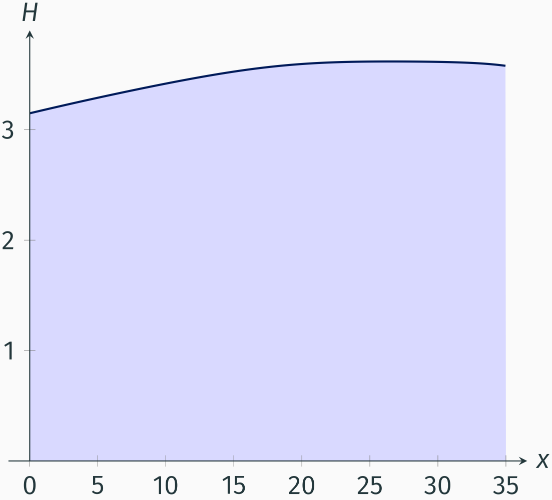

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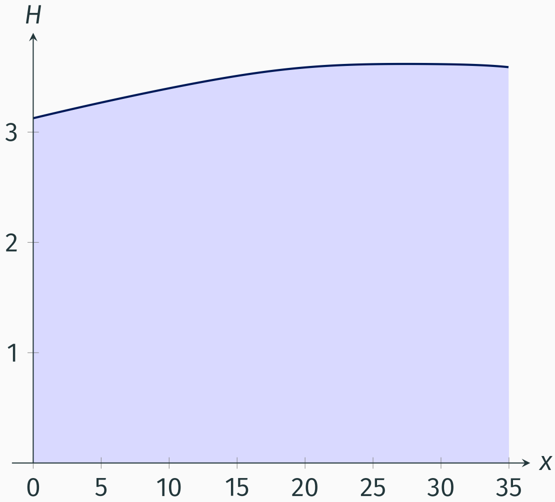

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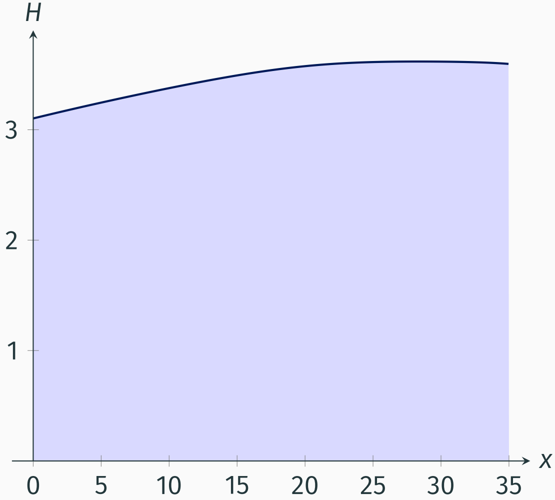

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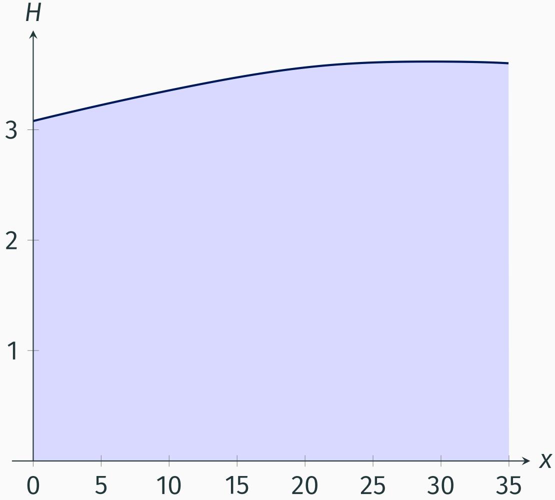

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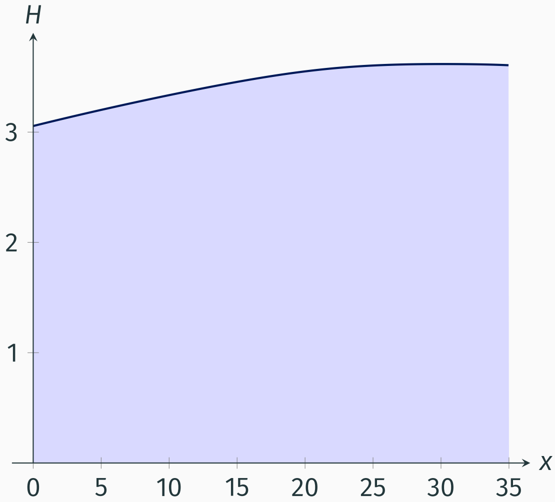

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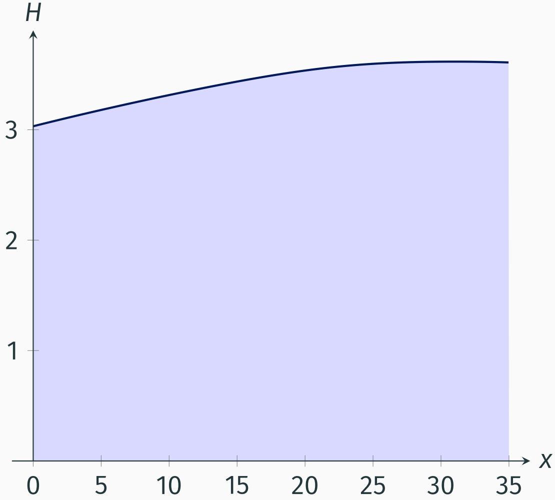

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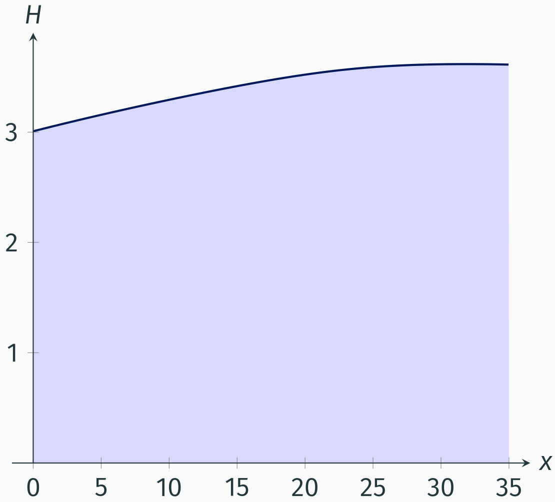

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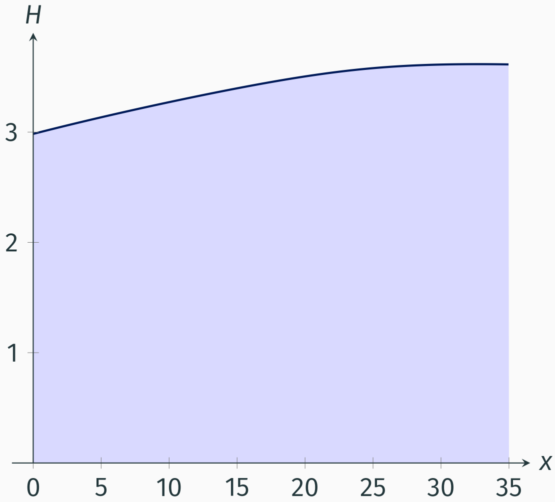

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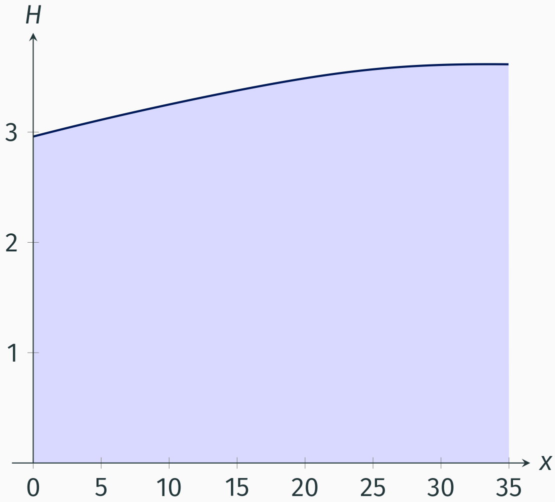

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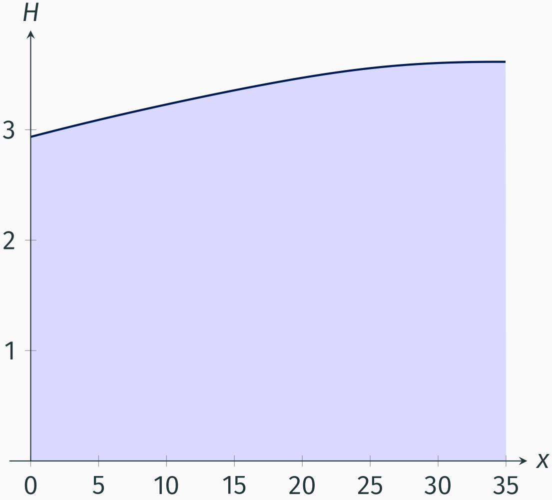

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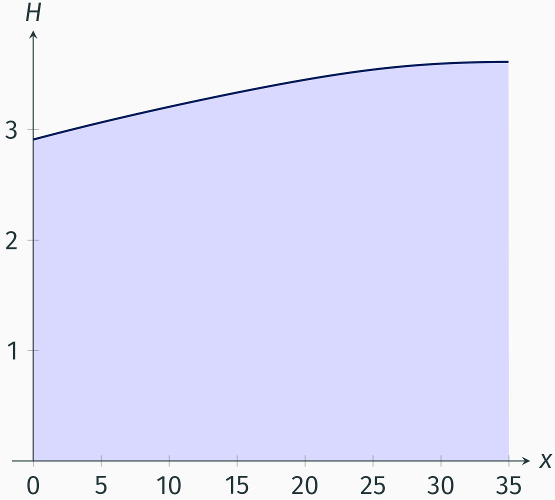

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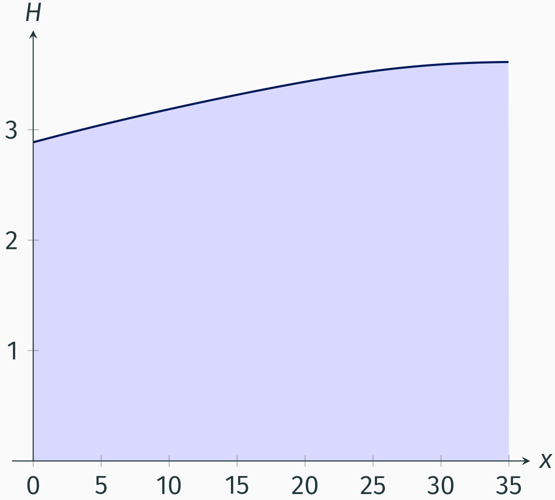

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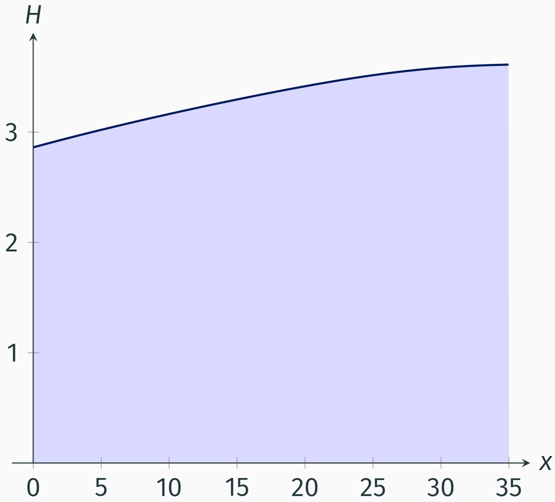

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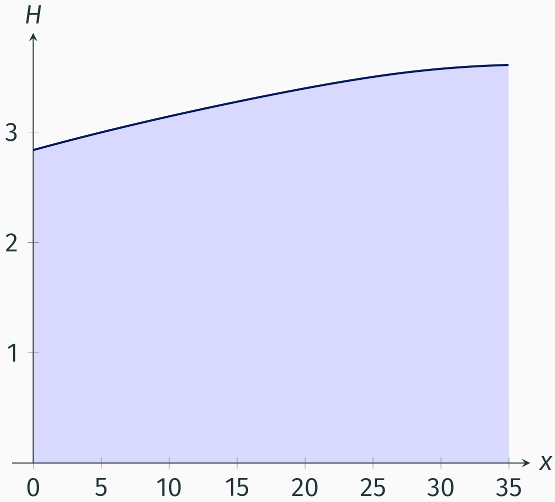

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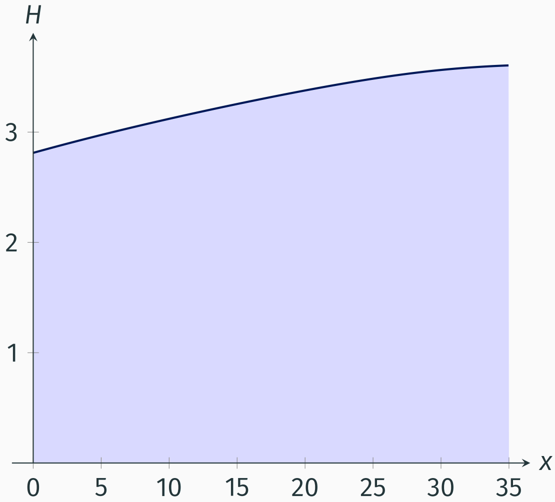

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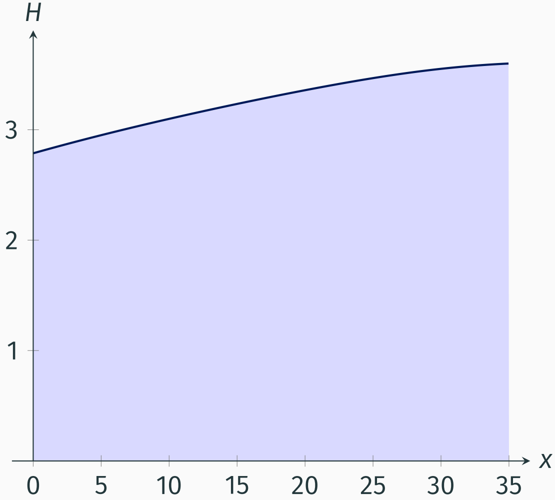

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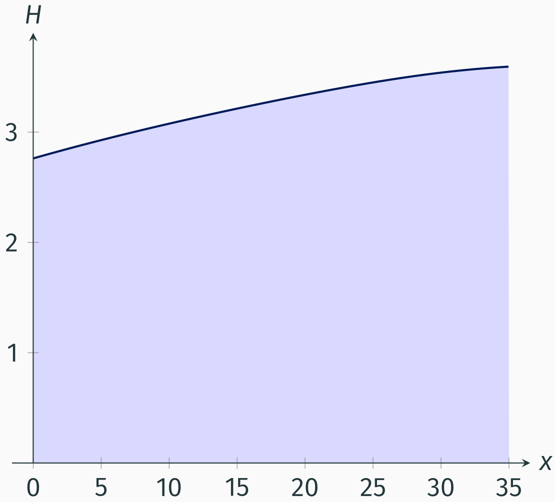

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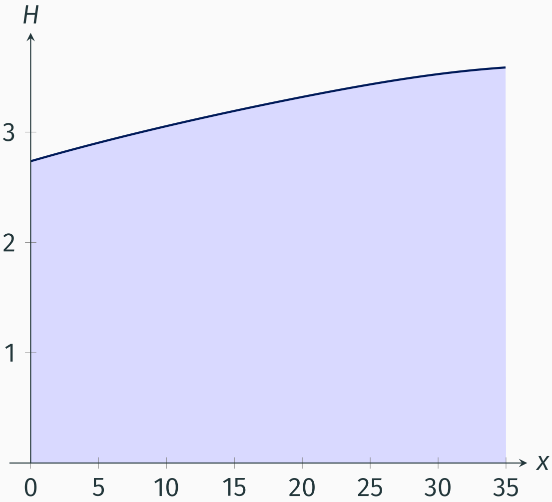

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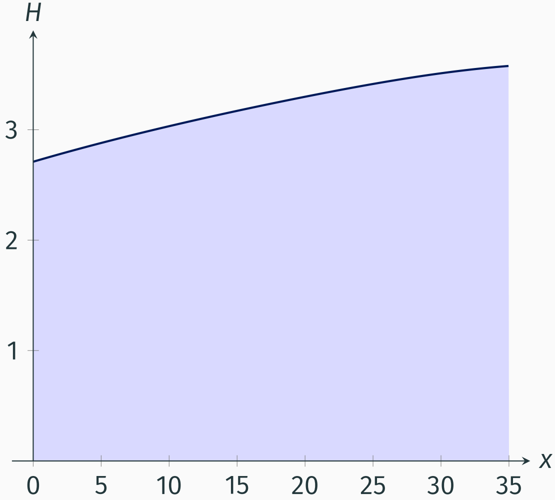

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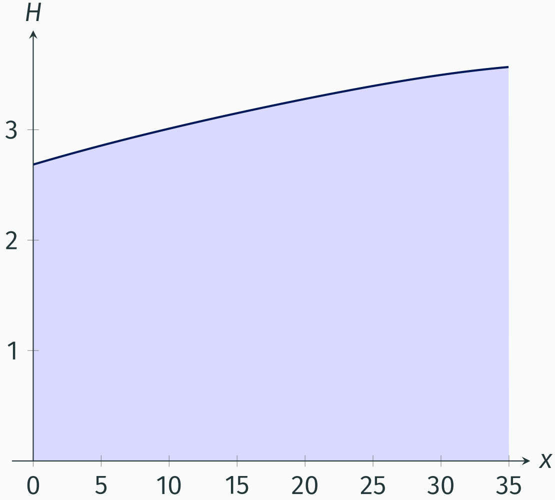

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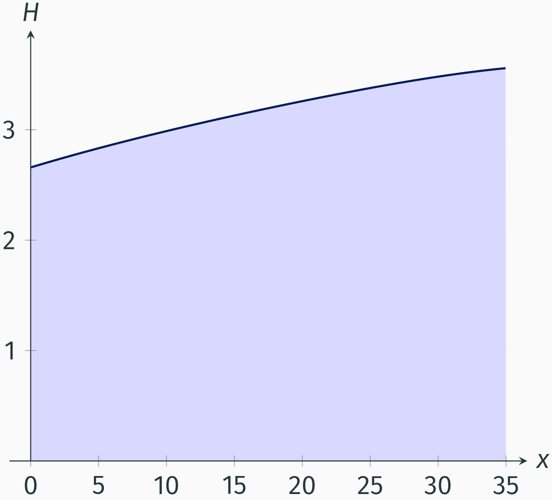

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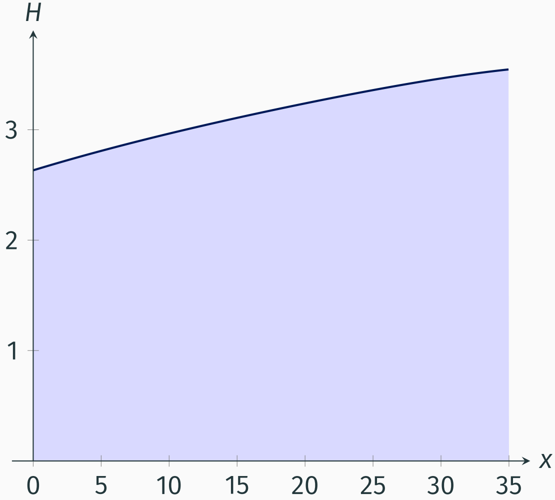

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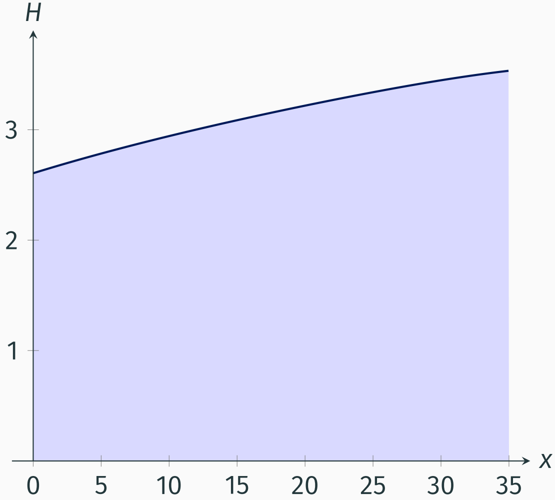

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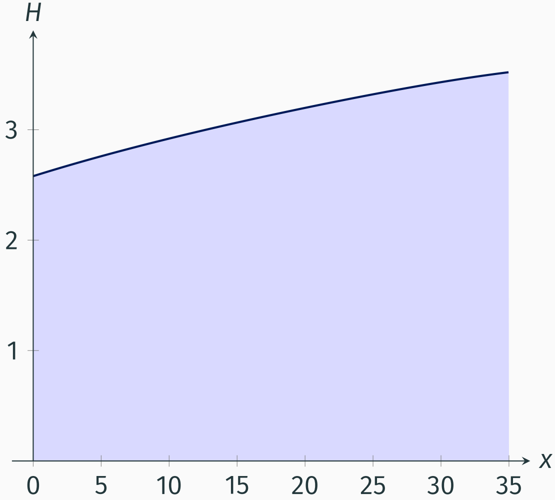

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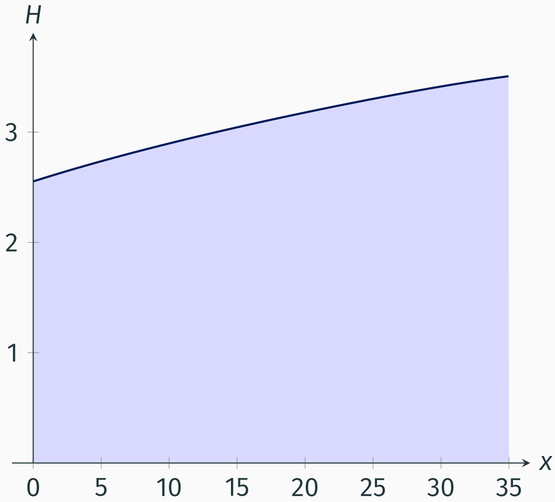

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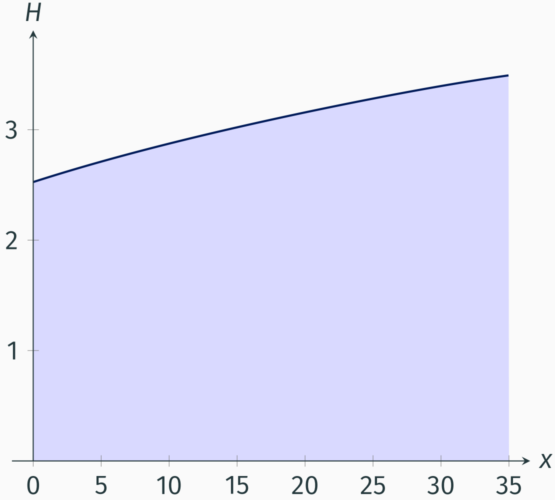

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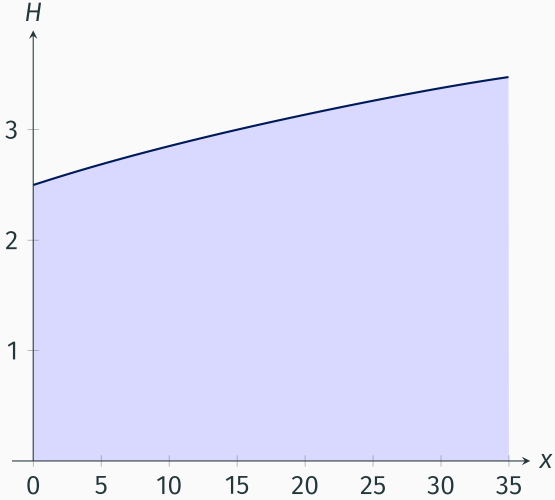

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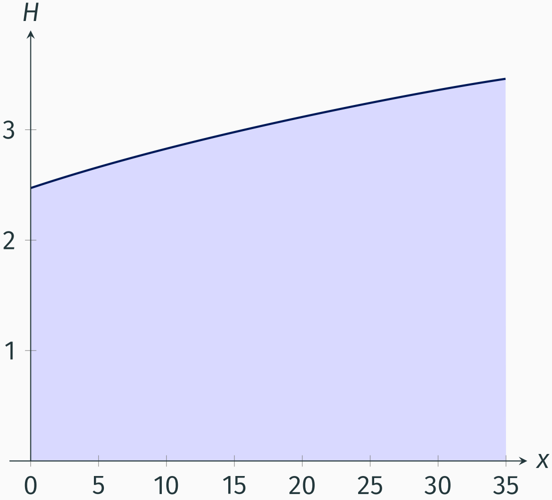

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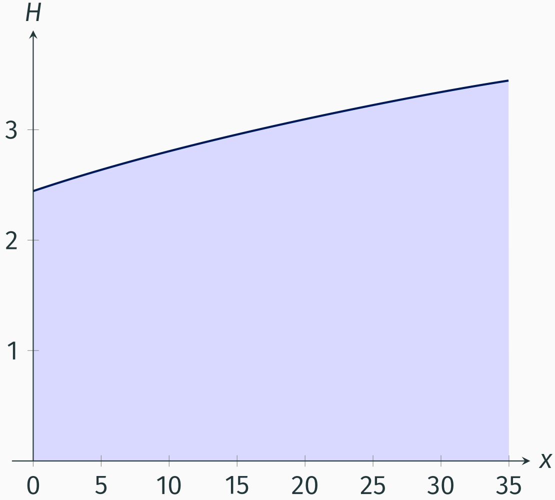

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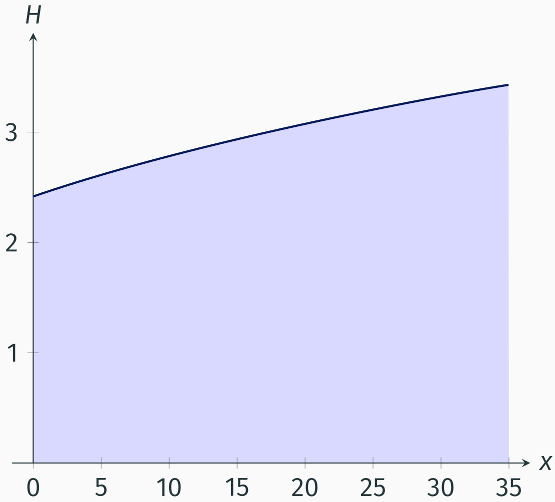

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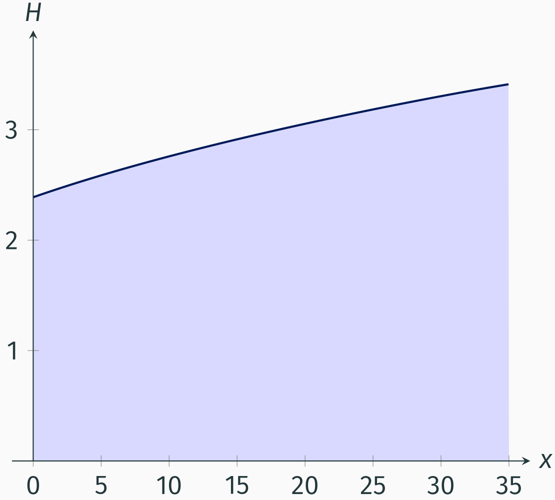

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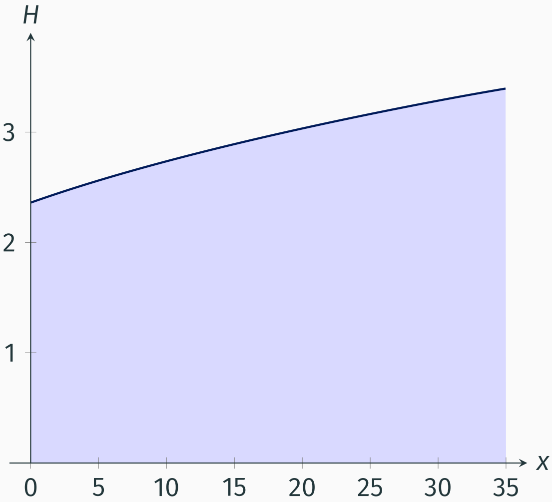

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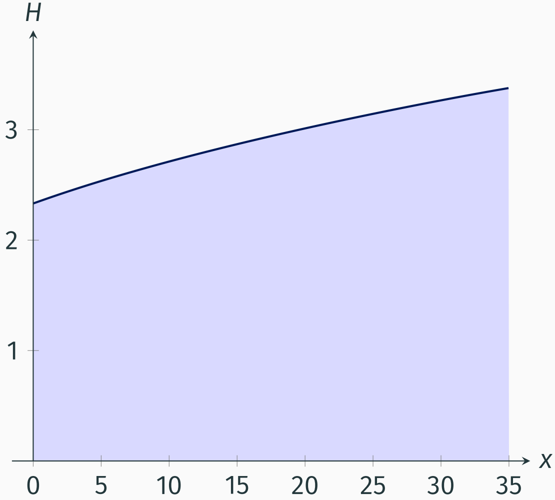

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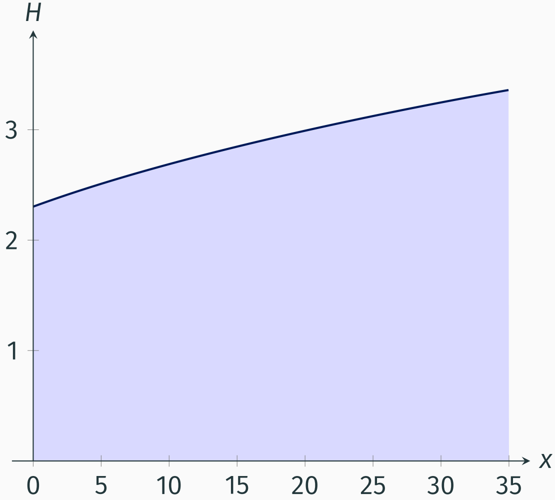

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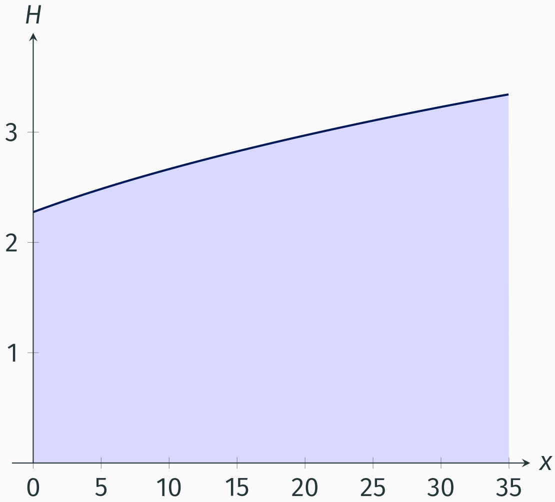

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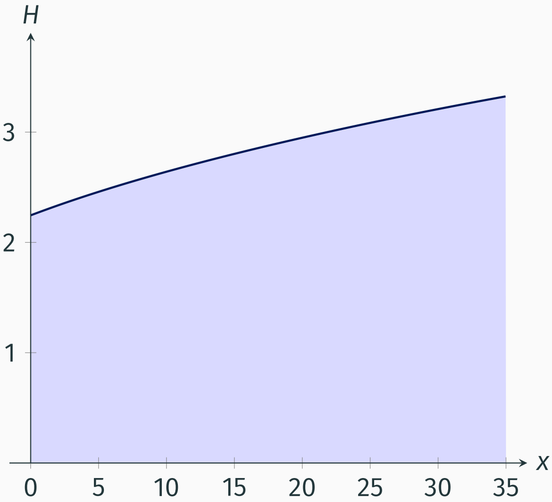

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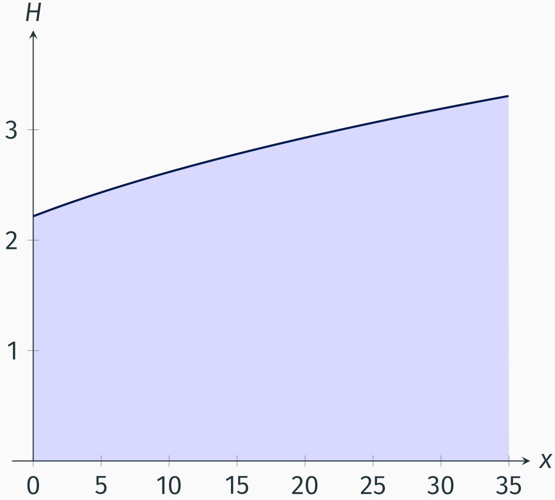

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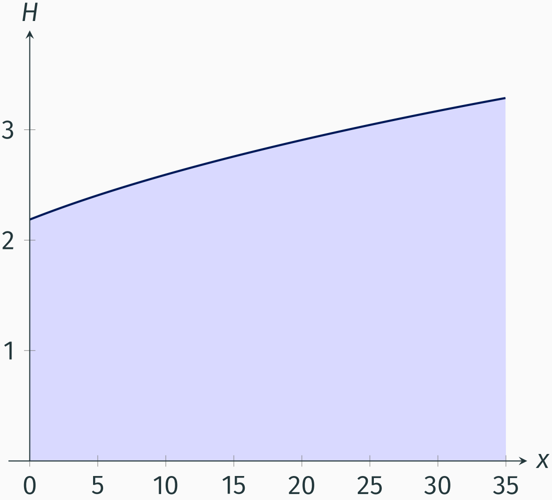

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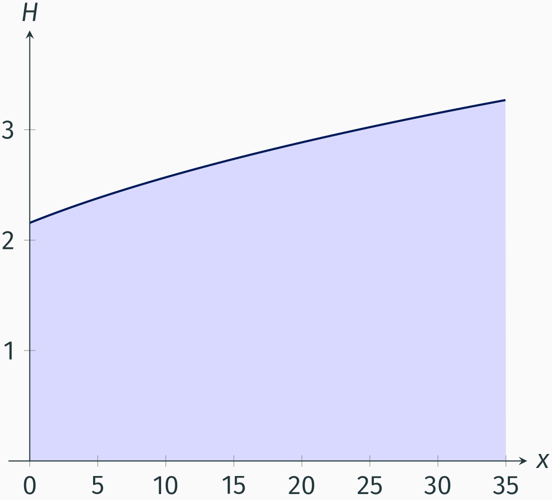

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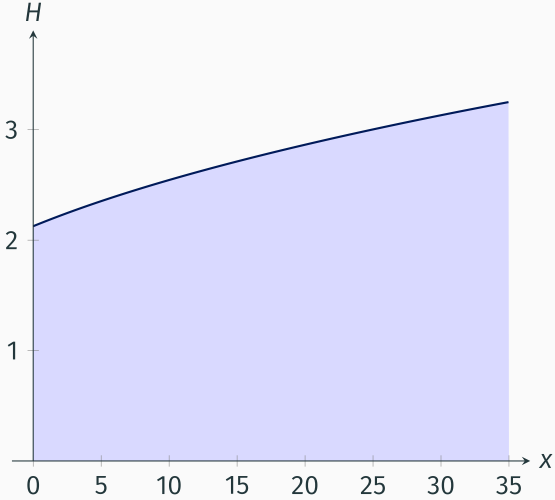

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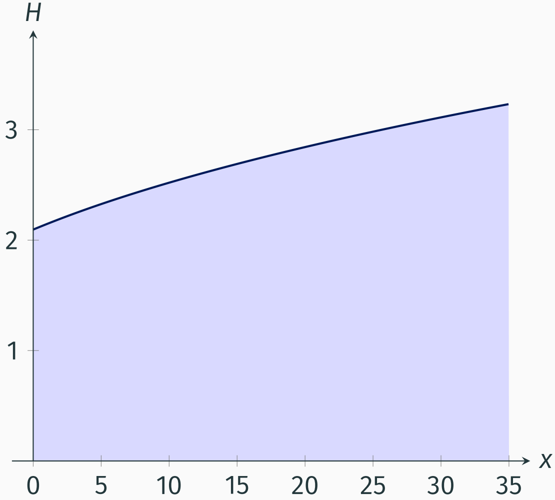

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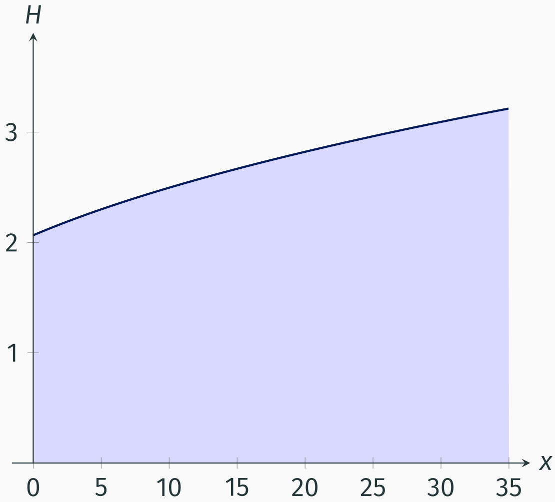

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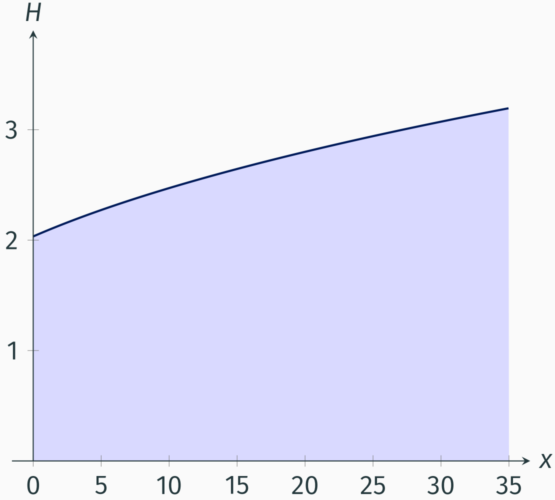

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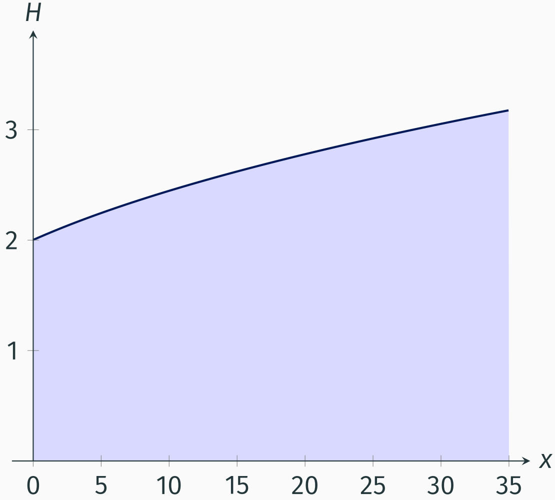

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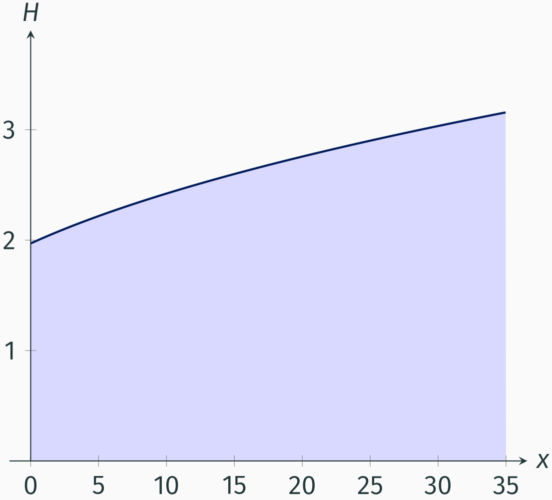

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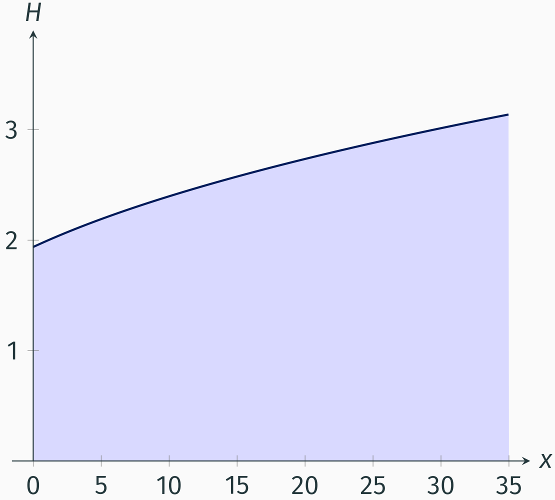

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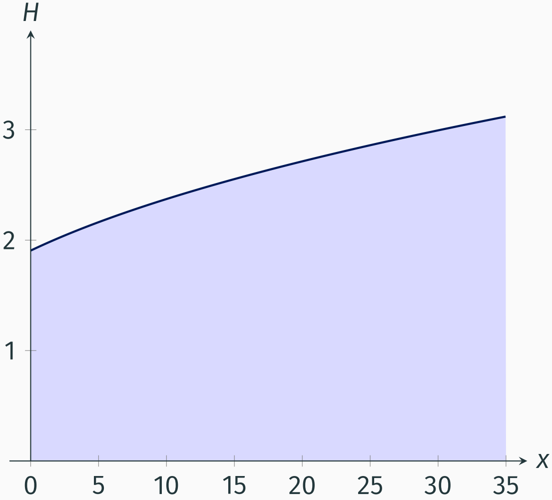

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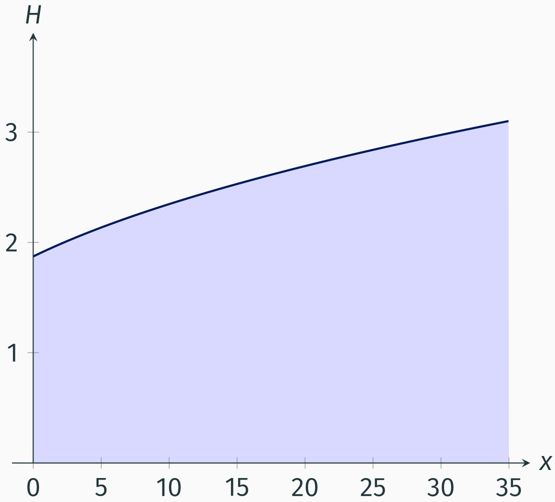

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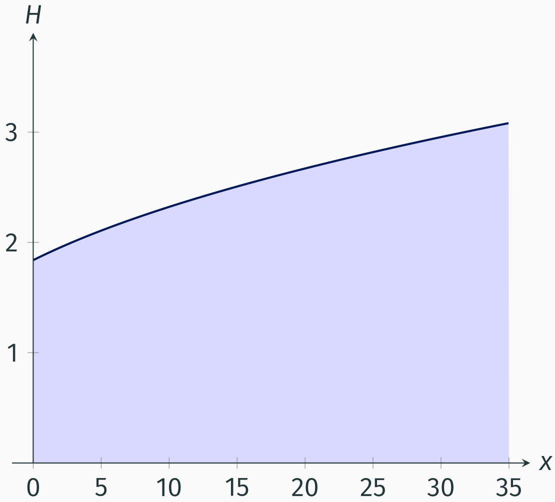

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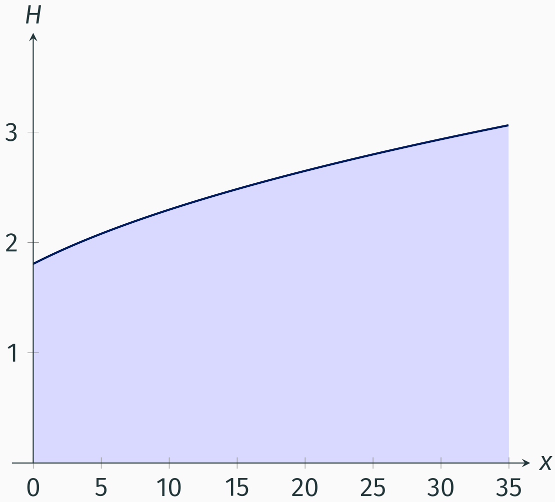

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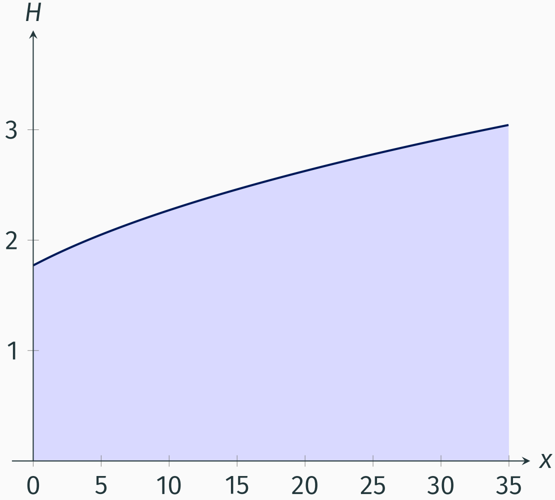

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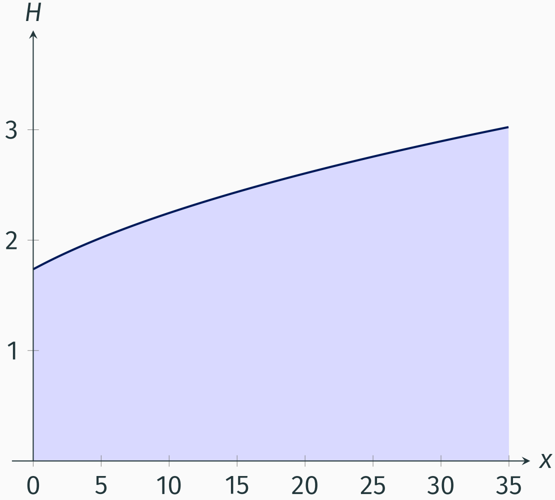

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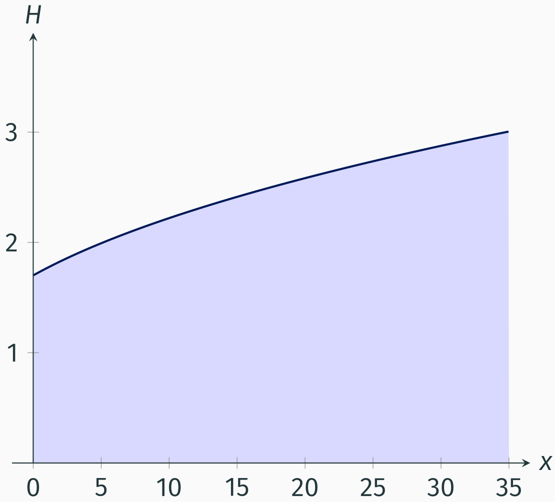

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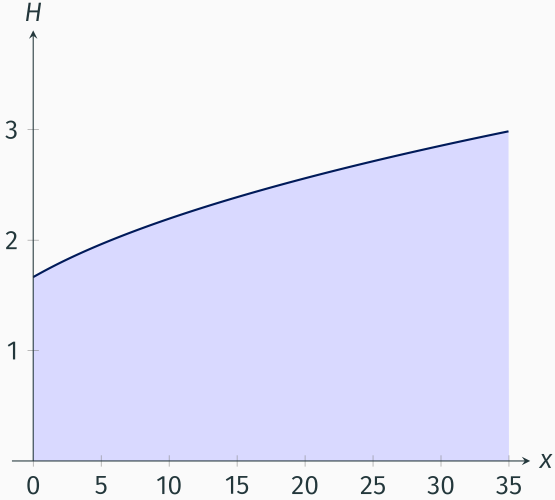

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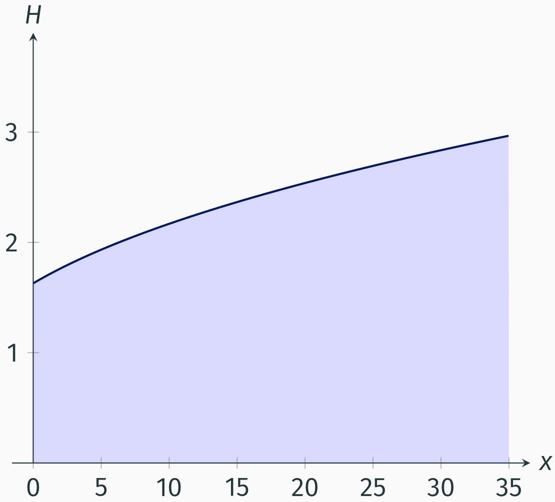

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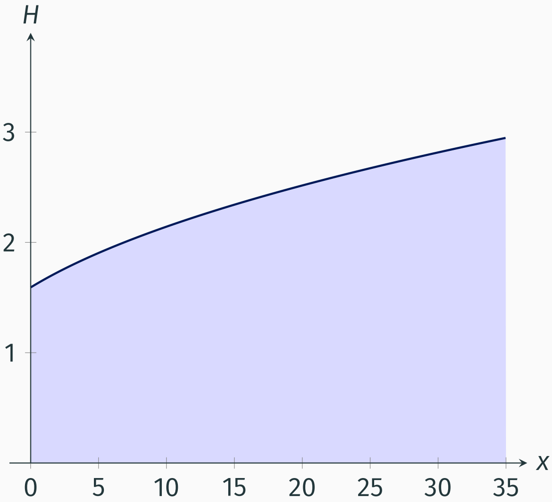

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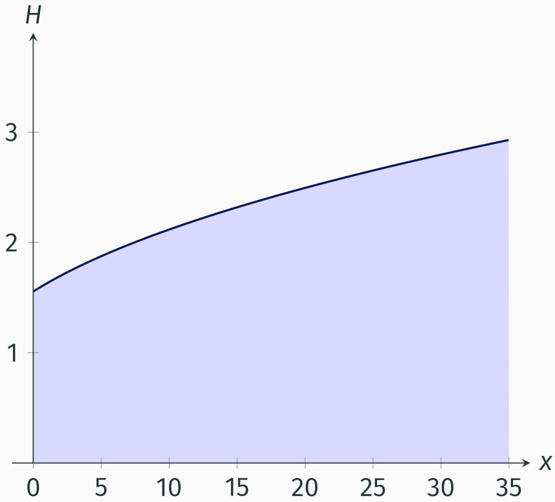

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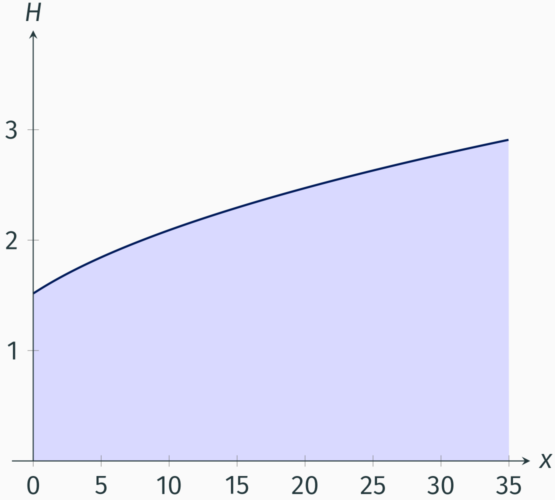

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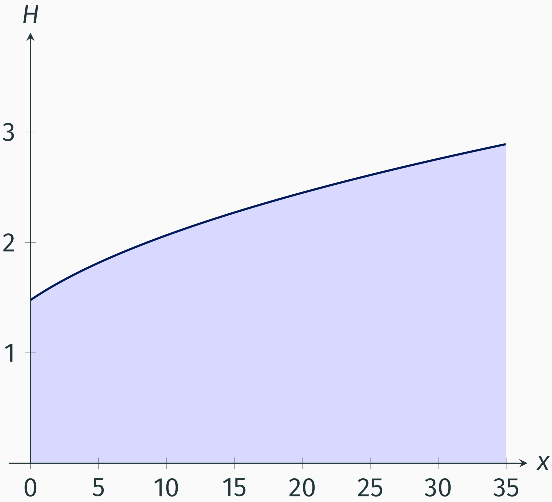

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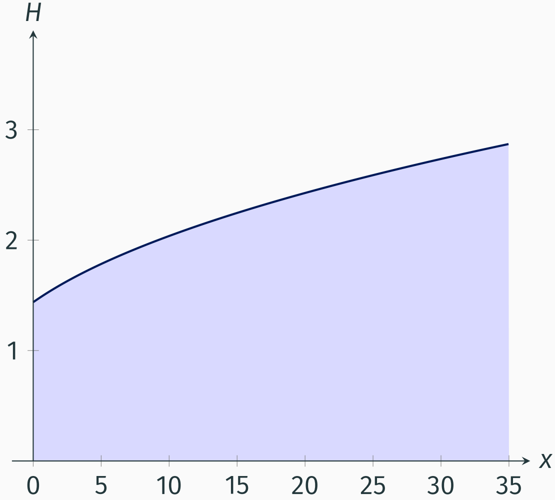

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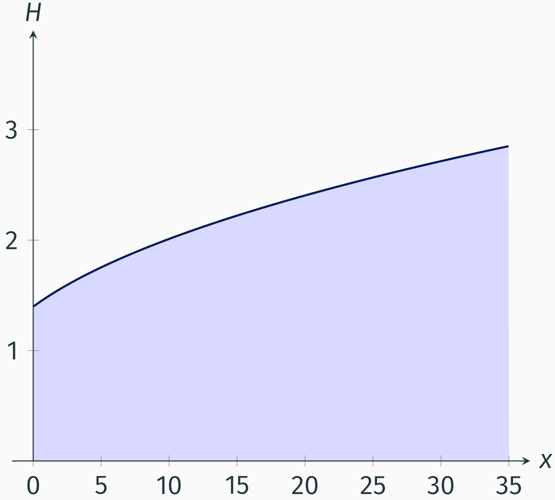

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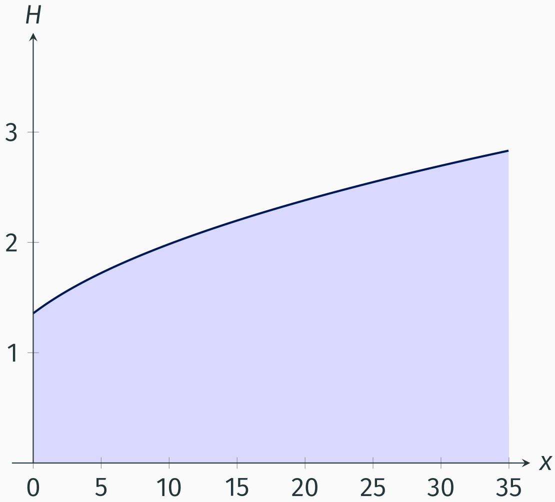

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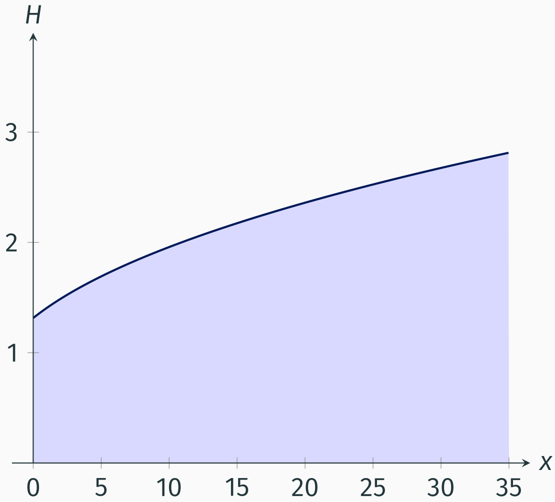

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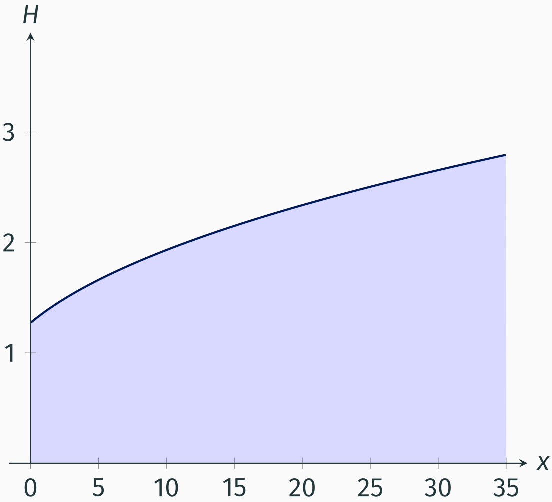

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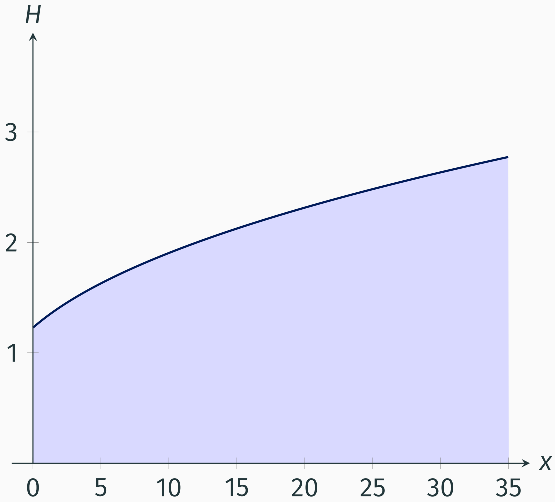

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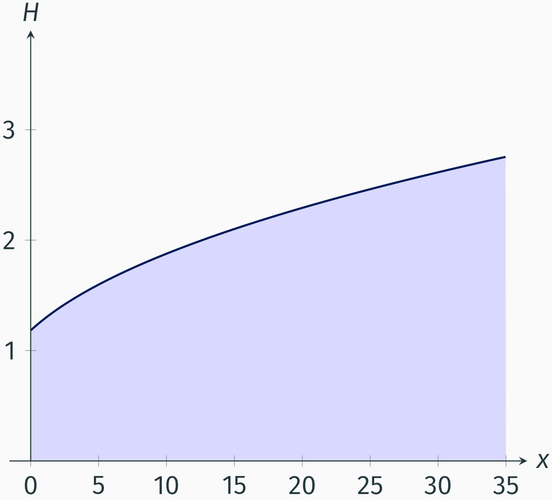

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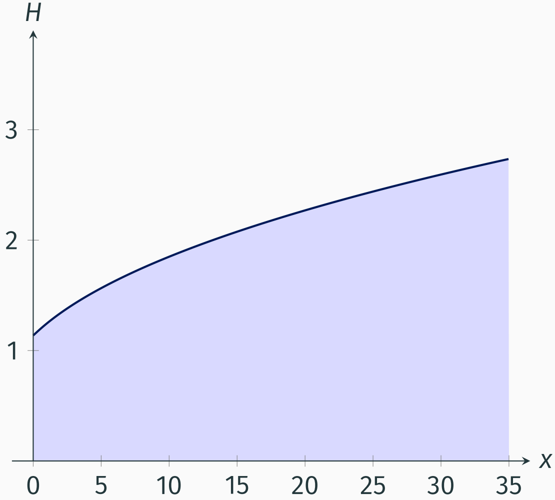

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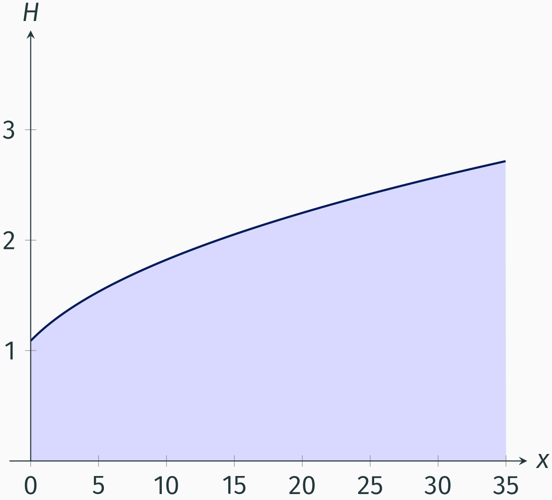

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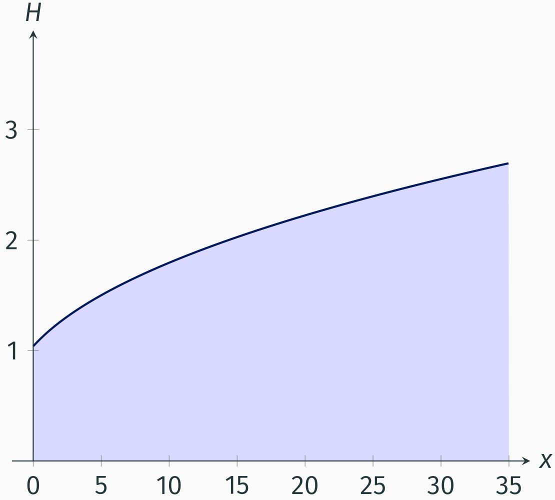

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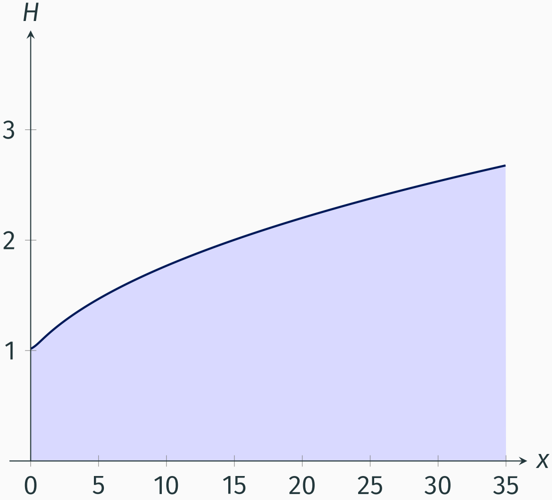

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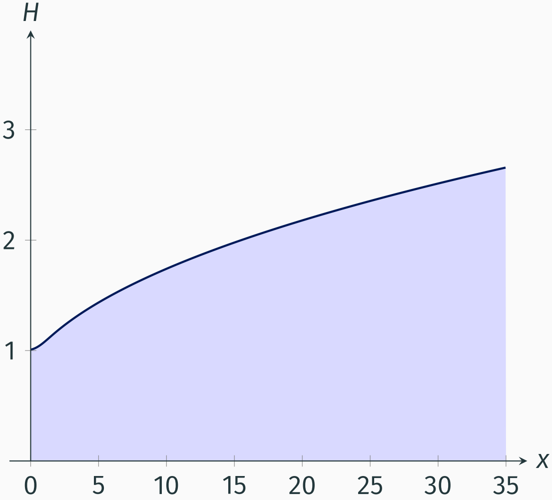

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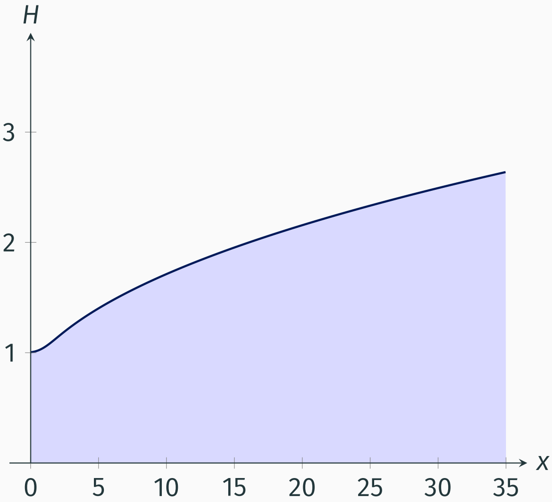

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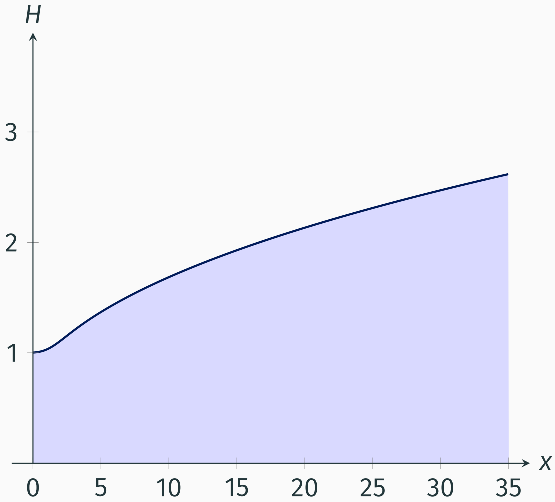

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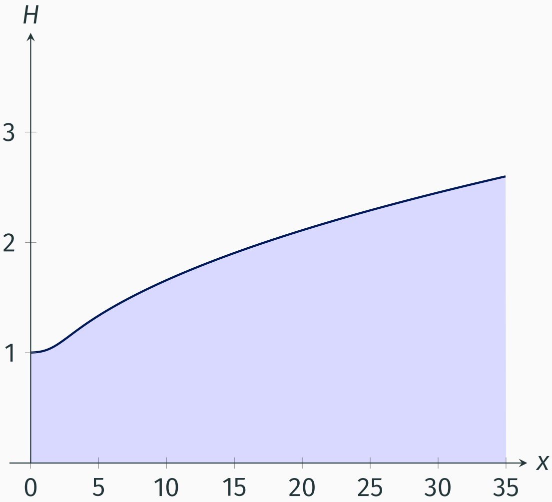

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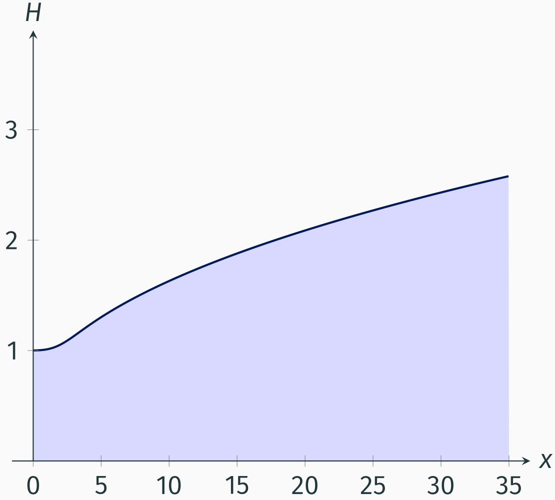

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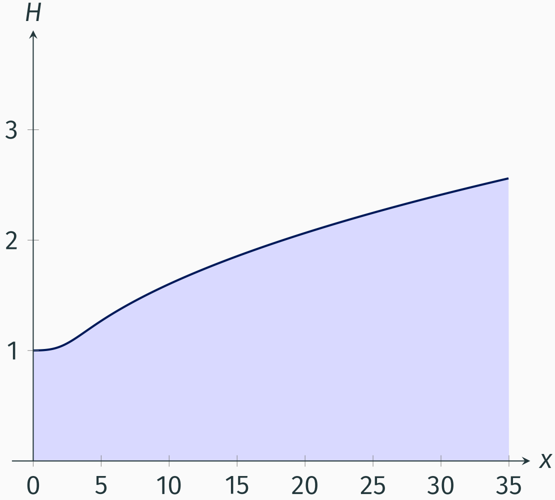

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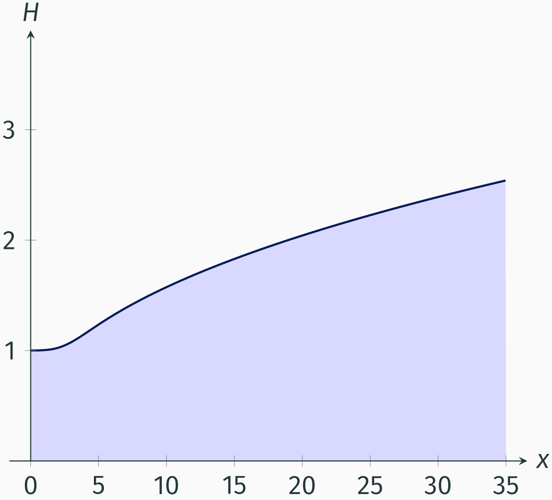

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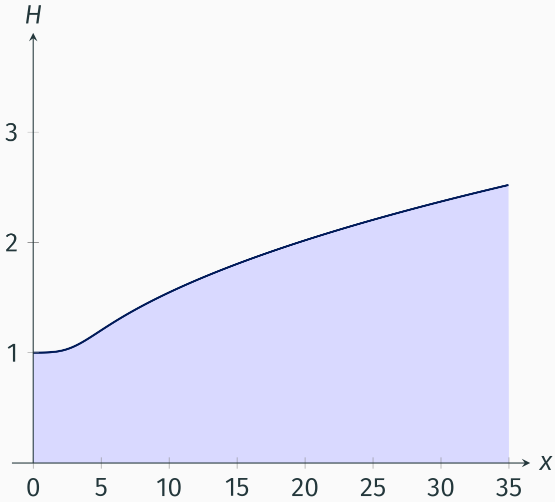

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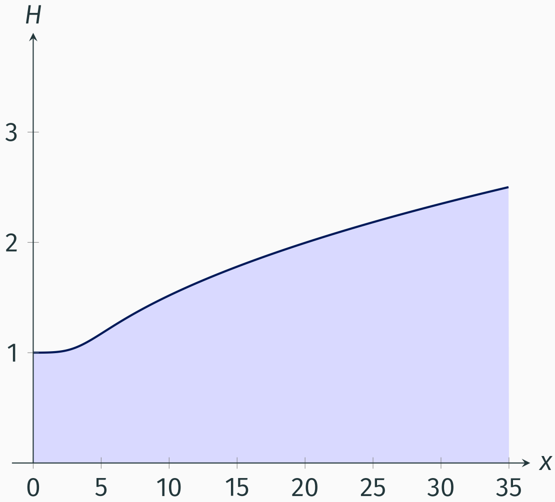

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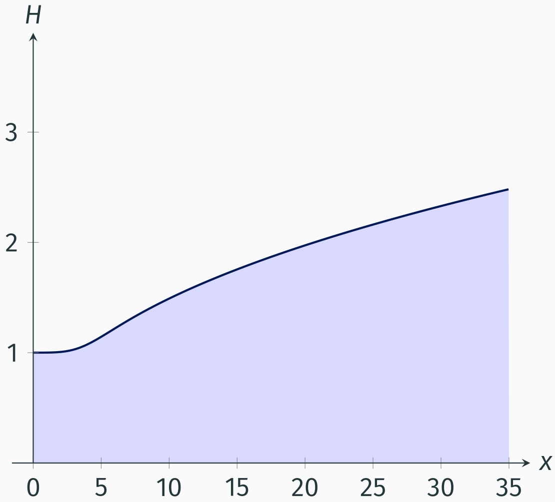

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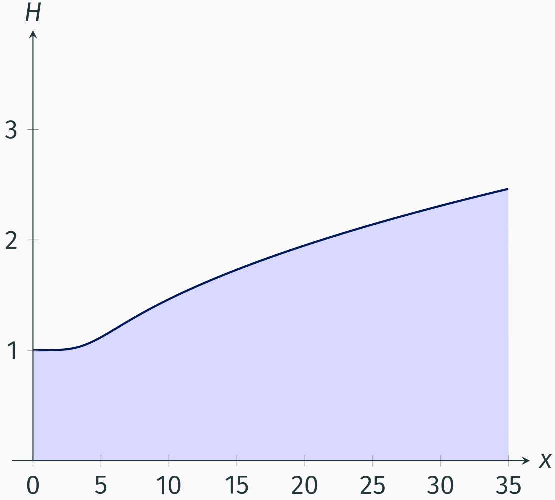

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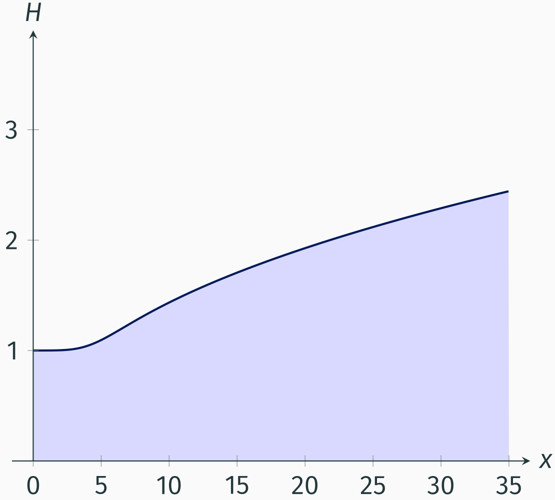

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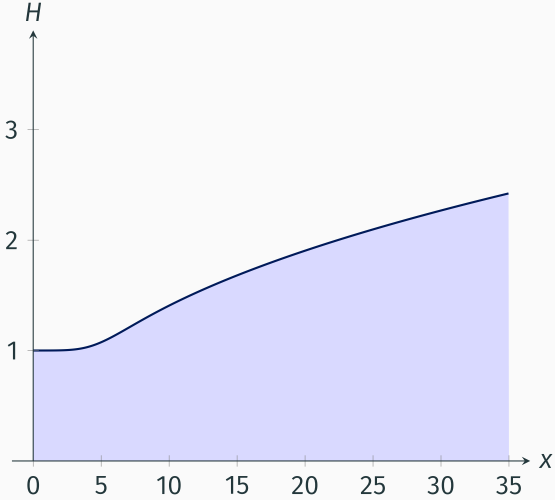

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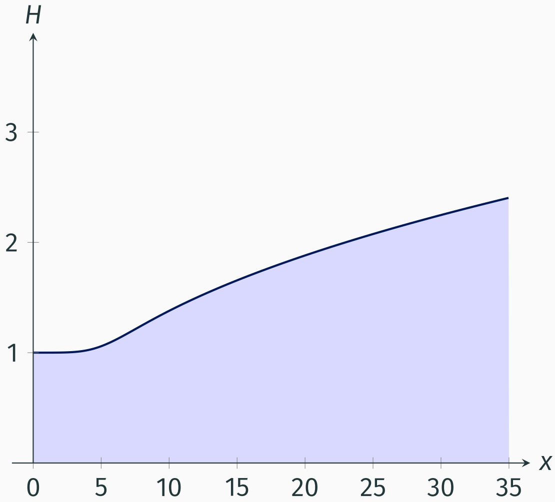

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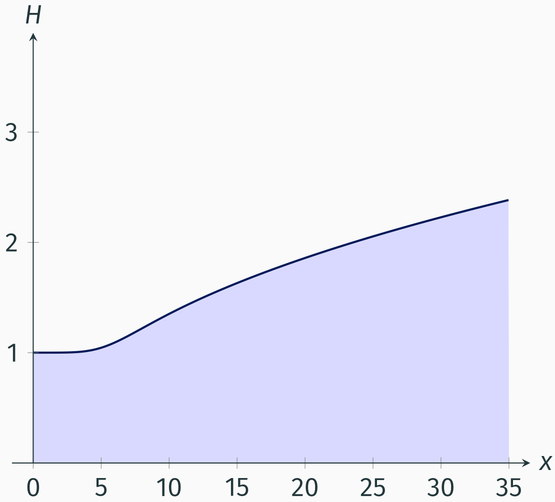

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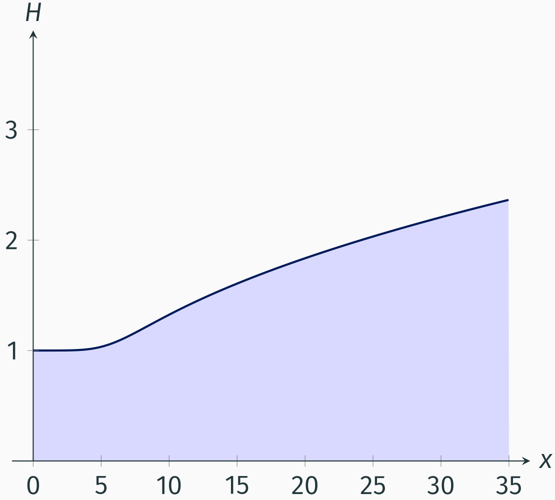

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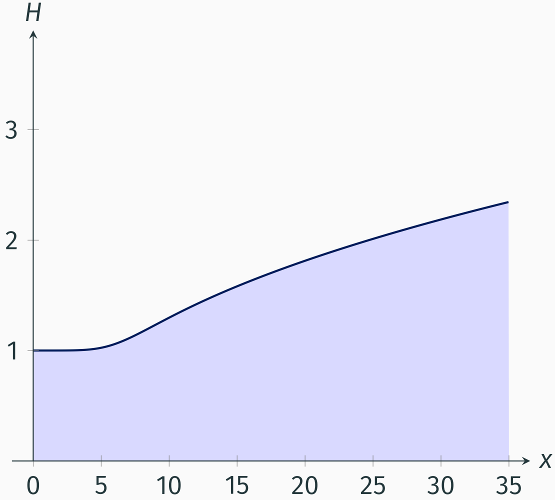

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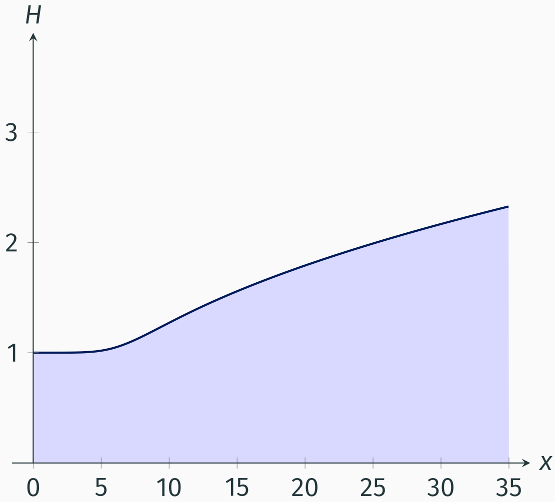

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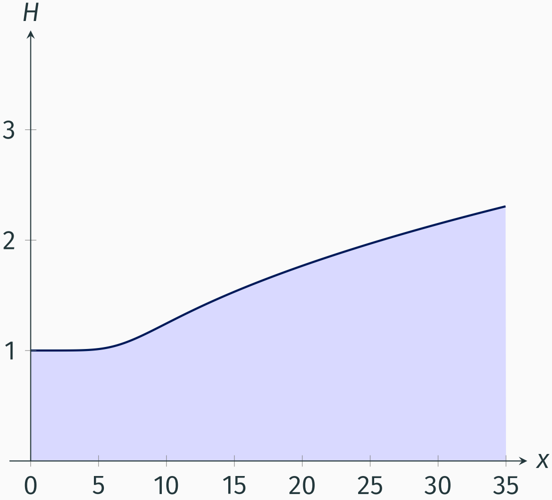

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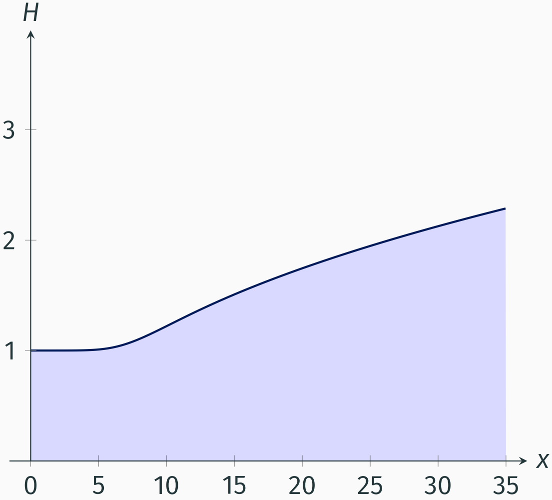

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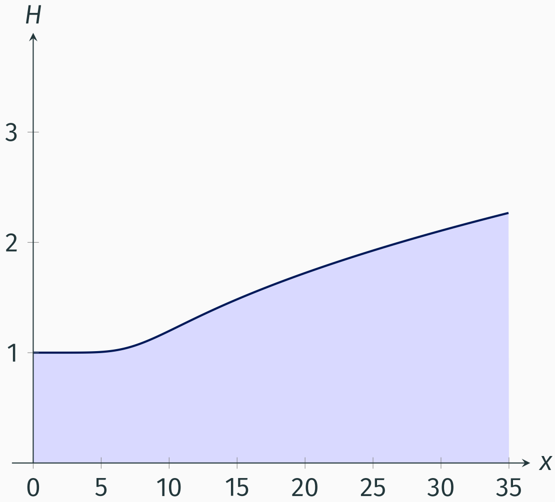

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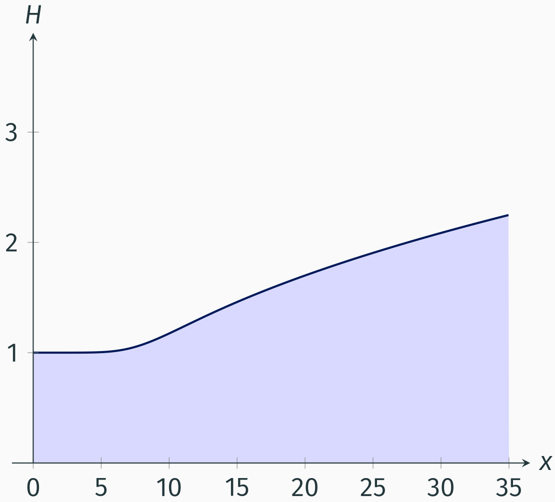

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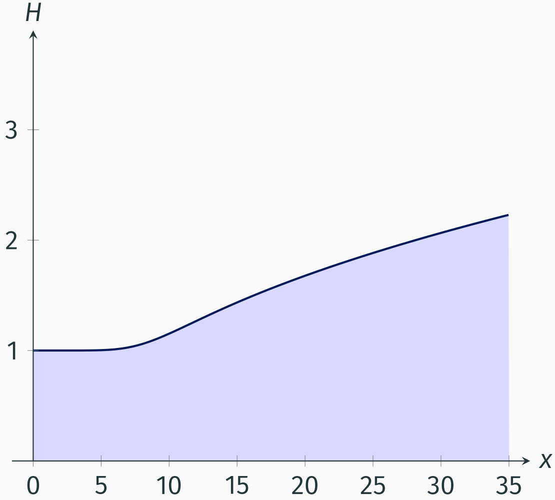

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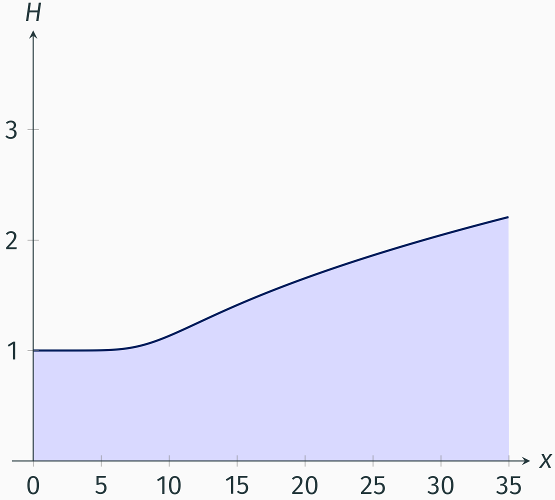

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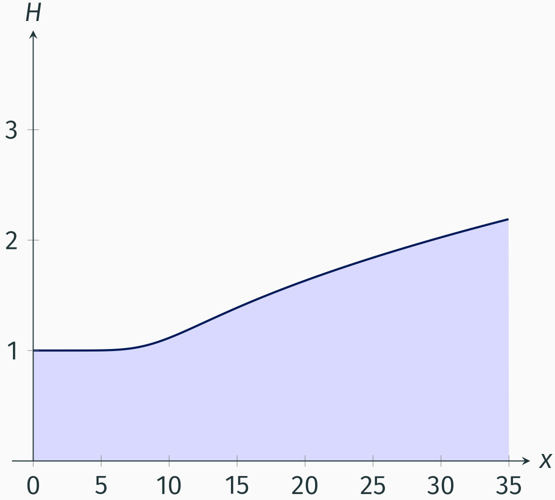

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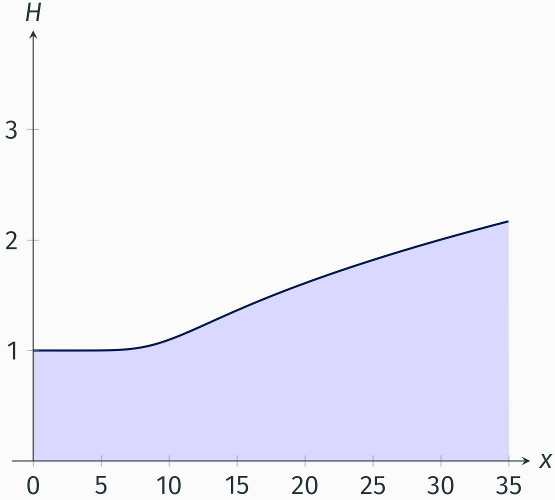

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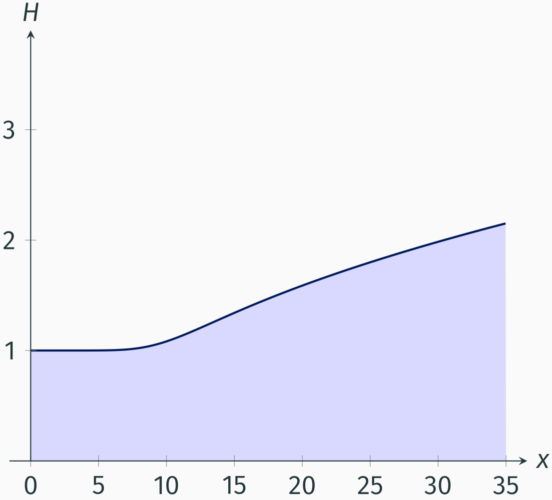

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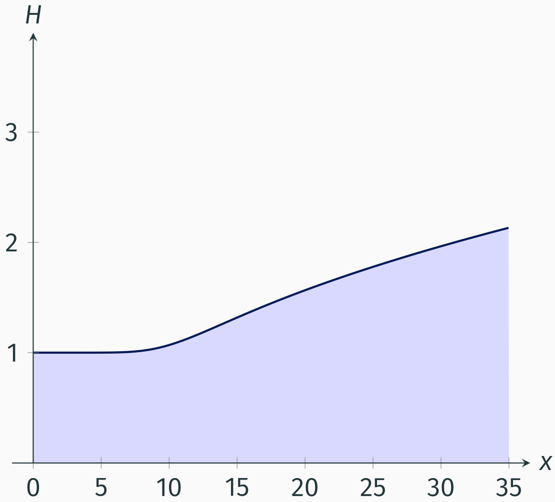

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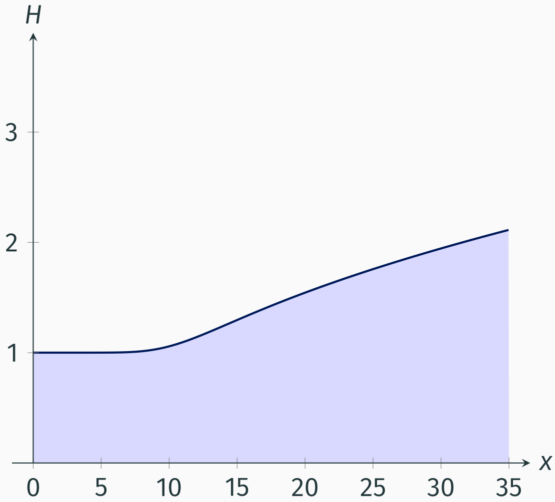

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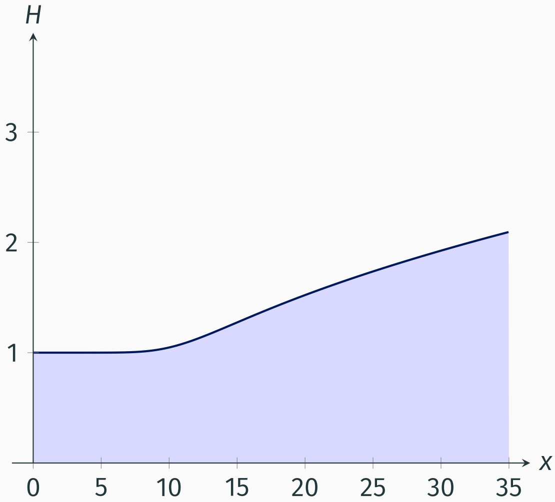

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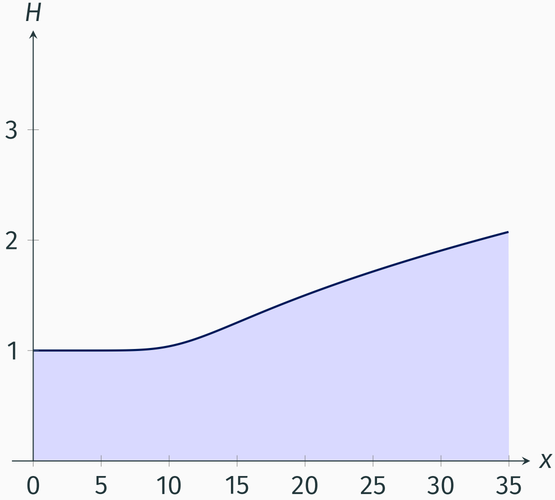

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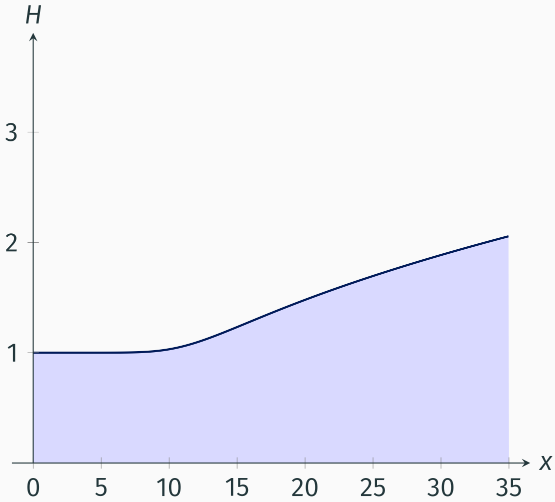

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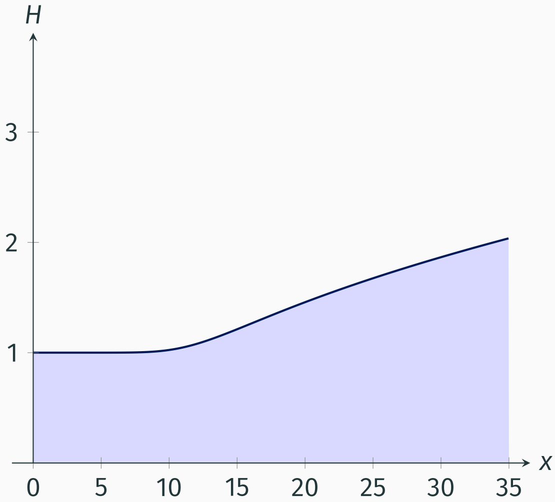

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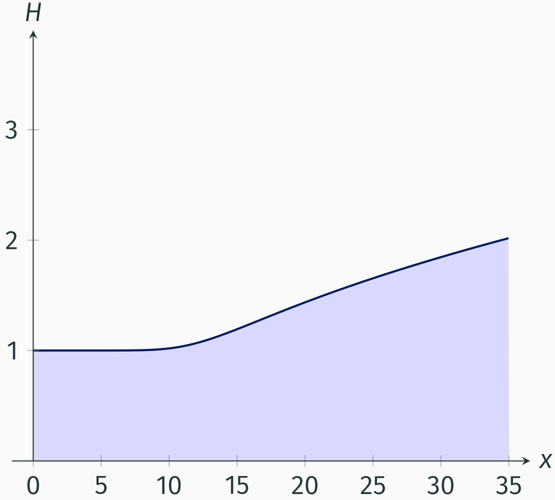

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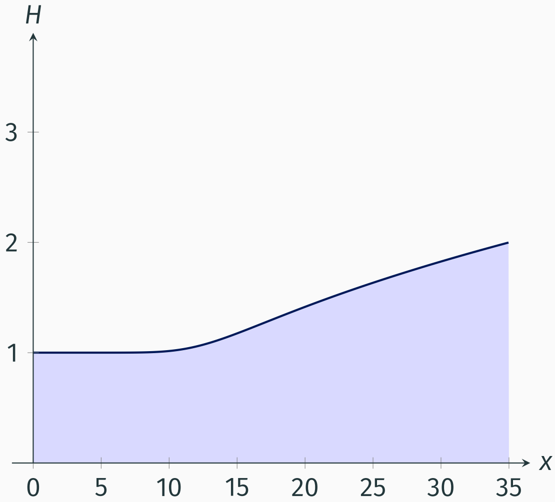

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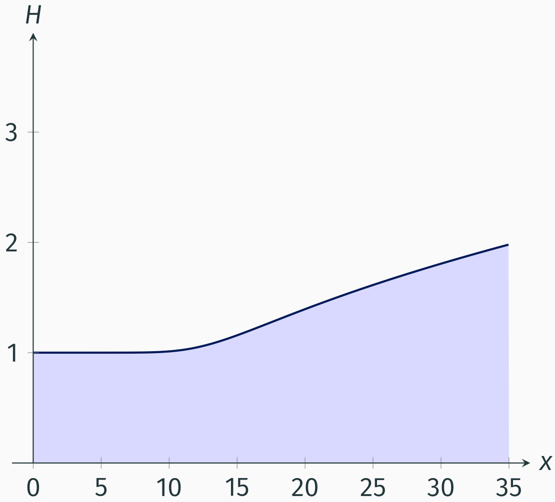

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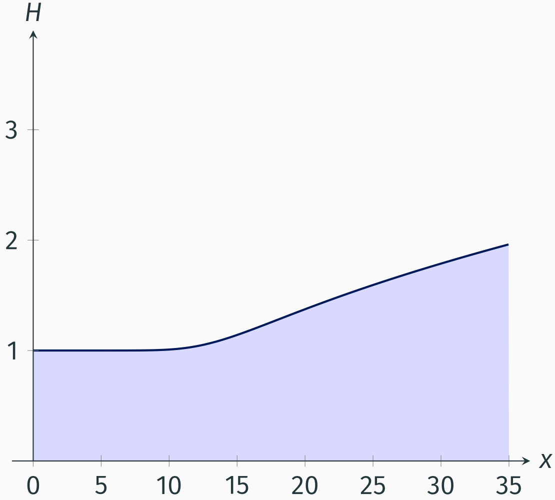

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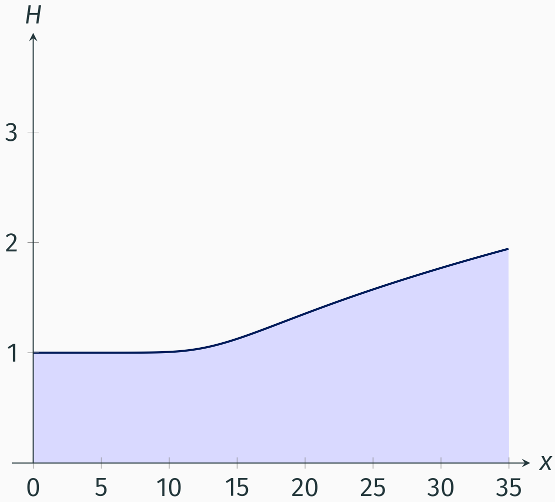

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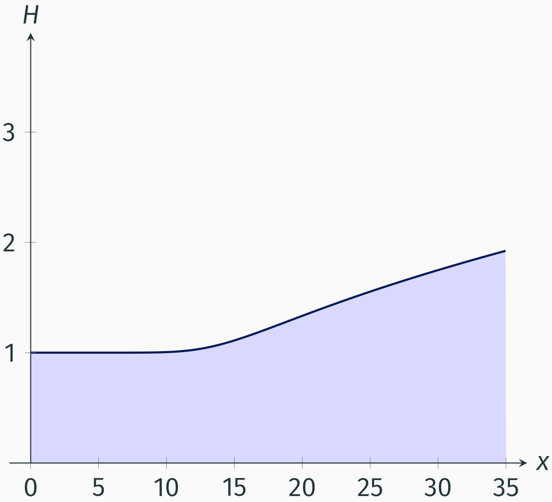

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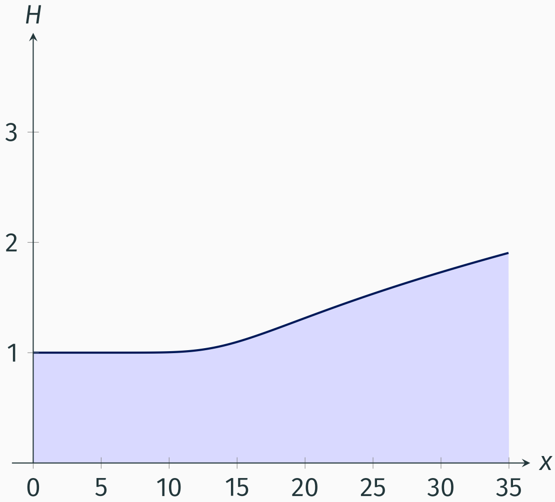

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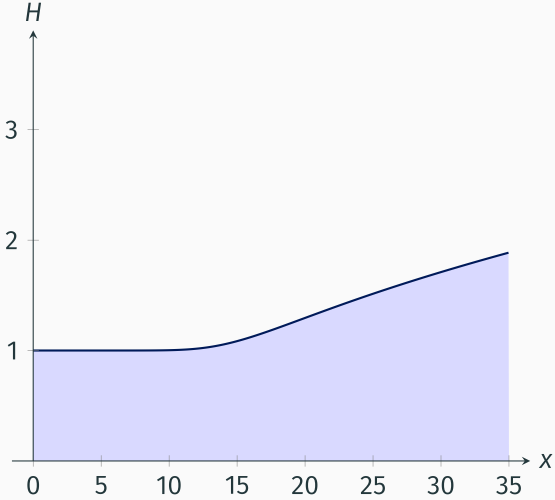

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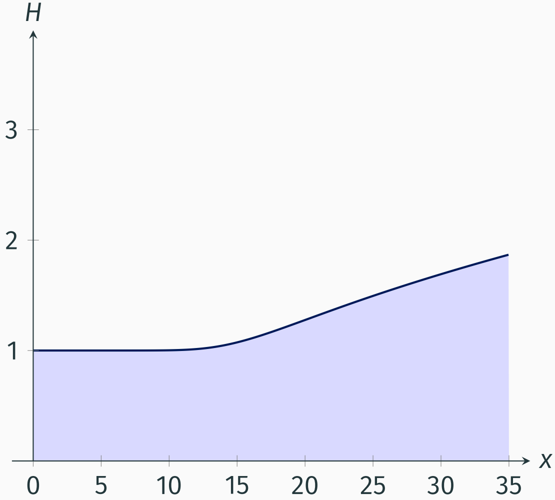

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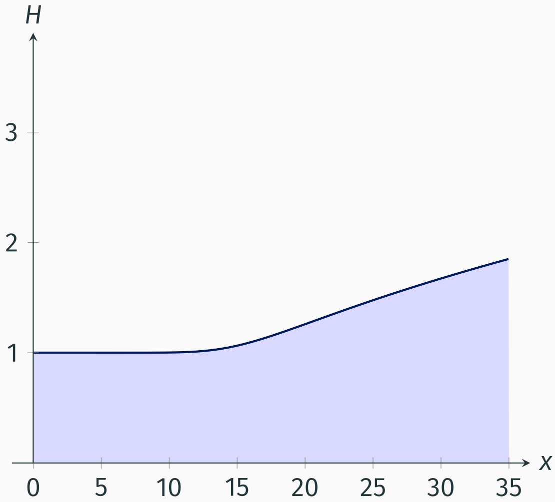

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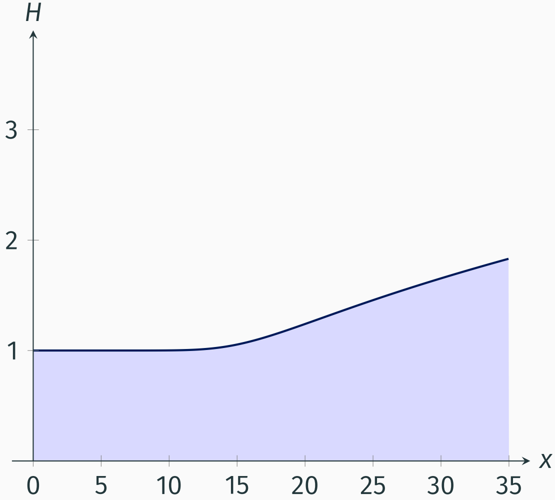

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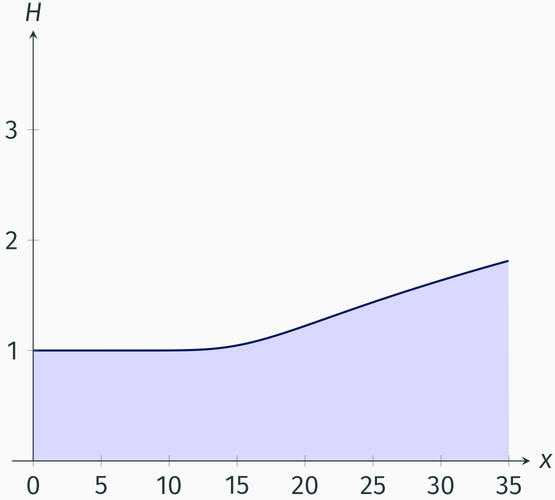

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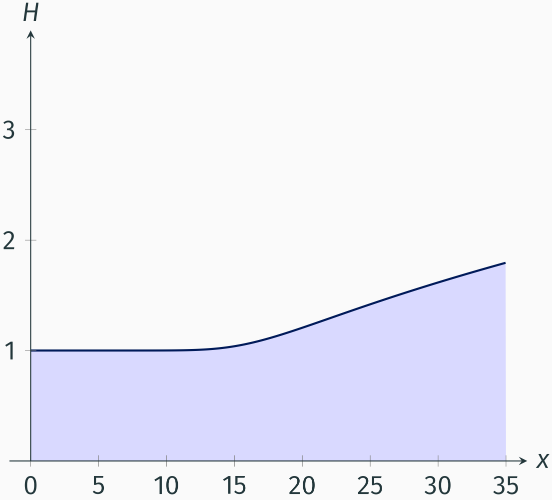

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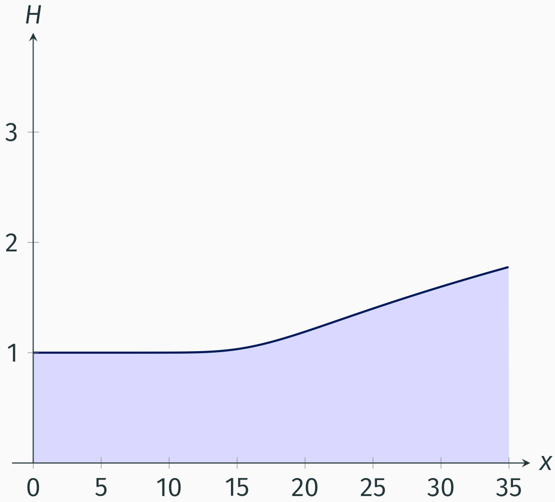

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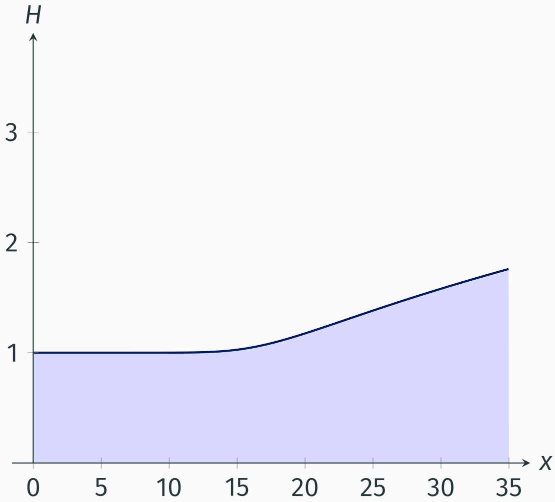

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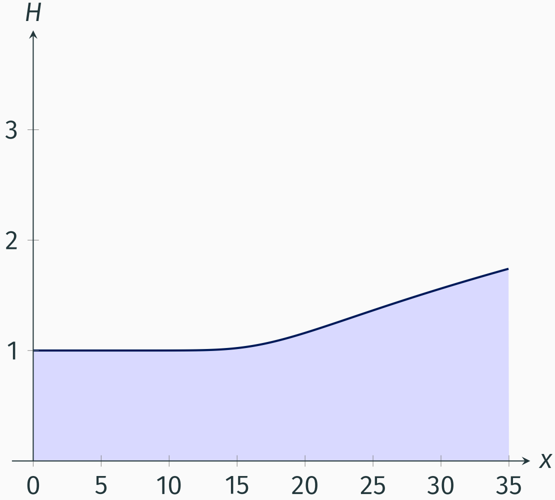
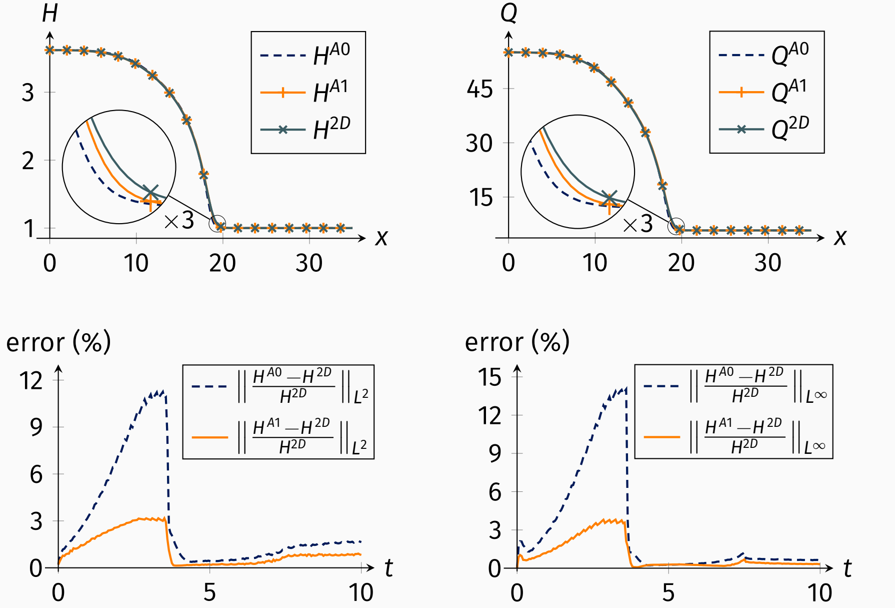


Unsteady flood flow (2D: ref. sol., A0: 0^{th}-order, A1: 1^{st}-order)



1. Governing equations
2. Asymptotic expansions
3. Transverse averaging
4. A zeroth-order model
5. A first-order model
6. Numerical validation of the model
7. Conclusion and perspectives

Conclusion

We have developed a new 1D model, based on the 2D shallow water equations, that is:

- consistent, up to first-order, with the 2D model in the asymptotic regime corresponding to a river flow:
  - the zeroth-order is obtained with a new explicit friction term,
  - the first-order relies on new equations describing the evolution of the enstrophy and the potential;
- hyperbolic and linearly stable;
- easily implementable and numerically validated.

The preprint related to these results is available on HAL:

V. Michel-Dansac, P. Noble et J.-P. Vila, **Consistent section-averaged shallow water equations with bottom friction**, 2018.
https://hal.archives-ouvertes.fr/hal-01962186

Perspectives

Work related to the model:

- improve the treatment of the river meanders by going to the first-order instead of the zeroth-order
- adapt this methodology to treat confluences
- consider a time-dependent topography to model the effects of sedimentation

Work related to the implementation and scientific computation:

- compare the 1D results to the ones given by a fully 2D code, in real test cases (Garonne, Lèze, Gironde, Amazon, ...)
- couple the 1D and 2D equations in the context of the Gironde estuary (collaboration with the SHOM)

Thank you for your attention!

Non-dimensional form of the 2D shallow water system

To emphasize the different scales of the flow, we perform a non-dimensionalization of the 2D system.

We introduce the following dimensionalization scales and related non-dimensional quantities (which are denoted with a bar, like $\bar{x}$):

$$h := \mathcal{H}\bar{h}, \quad u := \mathcal{U}\bar{u}, \quad v := \mathcal{V}\bar{v}, \quad x := \mathcal{X}\bar{x}, \quad y := \mathcal{Y}\bar{y}, \quad t := \mathcal{T}\bar{t}, \quad \mathcal{T} := \frac{\mathcal{X}}{\mathcal{U}}.$$

The mass conservation equation

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

then becomes

$$\frac{\mathcal{H}}{\mathcal{T}}\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\mathcal{H}\mathcal{U}}{\mathcal{X}}\frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{\mathcal{H}\mathcal{V}}{\mathcal{Y}}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0.$$

Non-dimensional form of the 2D shallow water system

The non-dimensional conservation equation is

$$\frac{\mathcal{H}}{\mathcal{T}}\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\mathcal{H}\mathcal{U}}{\mathcal{X}}\frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{\mathcal{H}\mathcal{V}}{\mathcal{Y}}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0, \quad \text{i.e.} \qquad \frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{\mathcal{V}}{\mathcal{U}}\frac{\mathcal{X}}{\mathcal{Y}}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0.$$

We set $R_u := \mathcal{V}/\mathcal{U}$ and $R_x := \mathcal{Y}/\mathcal{X}$, to get

$$\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} + \frac{R_u}{R_x}\frac{\partial \bar{h}\bar{v}}{\partial \bar{y}} = 0.$$

We have

- $\mathcal{V} \ll \mathcal{U}$ (quasi-unidimensional flow) $\implies R_u \ll 1$,
- $\mathcal{Y} \ll \mathcal{X}$ (quasi-unidimensional geometry) $\implies R_x \ll 1$.

We assume $R_u = R_x$ to keep the mass conservation equation unchanged from the dimensional case.

Non-dimensional form of the 2D shallow water system

Regarding the geometry, we assume that $Z(x, y) = b(x) + \phi(x, y)$, where:

- $b(x)$ represents the main longitudinal topography, driving the flow from upstream to downstream;
- $\phi(x, y)$ represents small longitudinal and transverse variations.

The related non-dimensional quantities are

$$b = \mathcal{B}\bar{b}\left(\frac{x}{\mathcal{X}}\right) \quad \text{and} \quad \phi = \mathcal{H}\bar{\phi}\left(\frac{x}{\mathcal{X}}, \frac{y}{\mathcal{Y}}\right).$$

The non-dimensional topography gradient then reads:

$$\boldsymbol{\nabla} Z = \begin{pmatrix} \dfrac{\mathcal{B}}{\mathcal{X}} \dfrac{\partial \bar{b}}{\partial \bar{x}}(\bar{x}) + \dfrac{\mathcal{H}}{\mathcal{X}} \dfrac{\partial \bar{\phi}}{\partial \bar{x}}(\bar{x}, \bar{y}) \\ \dfrac{\mathcal{H}}{\mathcal{Y}} \dfrac{\partial \bar{\phi}}{\partial \bar{y}}(\bar{x}, \bar{y}) \end{pmatrix}.$$

Non-dimensional form of the 2D shallow water system

Regarding the friction, we take $C_h = \mathcal{C}\,\bar{C}(\bar{x},\bar{y})$.

The non-dimensional friction source term then reads:

$$\frac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\,h^p} = \begin{pmatrix} \dfrac{\mathcal{U}}{\mathcal{C}\mathcal{H}^p}\cdot\dfrac{\bar{u}\sqrt{\mathcal{U}^2\bar{u}^2+\mathcal{V}^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} \\ \dfrac{\mathcal{V}}{\mathcal{C}\mathcal{H}^p}\cdot\dfrac{\bar{v}\sqrt{\mathcal{U}^2\bar{u}^2+\mathcal{V}^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} \end{pmatrix} = \begin{pmatrix} \dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\cdot\dfrac{\bar{u}\sqrt{\bar{u}^2+R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} \\ \dfrac{\mathcal{V}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\cdot\dfrac{\bar{v}\sqrt{\bar{u}^2+R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} \end{pmatrix}.$$

Non-dimensional form of the 2D shallow water system

We are finally able to write the non-dimensional form of the 2D shallow water system: from the dimensional system

$$\begin{cases} h_t + \boldsymbol{\nabla} \cdot (h\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} + g\boldsymbol{\nabla} h = g\left(-\boldsymbol{\nabla} Z - \dfrac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\, h^p}\right), \end{cases}$$

we get the following non-dimensional form:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \dfrac{\mathcal{U}^2}{\mathcal{X}}\bar{u}_{\bar{t}} + \dfrac{\mathcal{U}^2}{\mathcal{X}}\bar{u}\bar{u}_{\bar{x}} + \dfrac{\mathcal{U}\mathcal{V}}{\mathcal{Y}}\bar{v}\bar{u}_{\bar{y}} + \dfrac{g\mathcal{H}}{\mathcal{X}}\left(\bar{h} + \bar{\Phi}\right)_{\bar{x}} = g\left(-\dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - \dfrac{\mathcal{B}}{\mathcal{X}}\bar{b}_{\bar{x}}\right), \\ \dfrac{\mathcal{V}\mathcal{U}}{\mathcal{X}}\bar{v}_{\bar{t}} + \dfrac{\mathcal{V}\mathcal{U}}{\mathcal{X}}\bar{u}\bar{v}_{\bar{x}} + \dfrac{\mathcal{V}^2}{\mathcal{Y}}\bar{v}\bar{v}_{\bar{y}} + \dfrac{g\mathcal{H}}{\mathcal{Y}}\left(\bar{h} + \bar{\Phi}\right)_{\bar{y}} = g\left(-\dfrac{\mathcal{V}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p}\right). \end{cases}$$

Non-dimensional form of the 2D shallow water system

We are finally able to write the non-dimensional form of the 2D shallow water system: from the dimensional system

$$\begin{cases} h_t + \nabla \cdot (h\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g\nabla h = g\left(-\nabla Z - \dfrac{\mathbf{u}\|\mathbf{u}\|}{C_h^2\, h^p}\right), \end{cases}$$

we get the following non-dimensional form:

$$\begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\ \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \dfrac{g\mathcal{H}}{\mathcal{U}^2}\left(\bar{h} + \bar{\Phi}\right)_{\bar{x}} = \dfrac{g\mathcal{X}}{\mathcal{U}^2}\left(-\dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - \dfrac{\mathcal{B}}{\mathcal{X}}\bar{b}_{\bar{x}}\right), \\ \bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \dfrac{g\mathcal{H}\mathcal{X}}{\mathcal{V}\mathcal{U}\mathcal{Y}}\left(\bar{h} + \bar{\Phi}\right)_{\bar{y}} = \dfrac{g\mathcal{X}}{\mathcal{U}^2}\left(-\dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}\dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p}\right). \end{cases}$$

Non-dimensional form of the 2D shallow water system

We introduce:

- $F^2 = \dfrac{\mathcal{U}^2}{g\mathcal{H}}$ the reference Froude number,
- $\delta = \dfrac{\mathcal{H}}{\mathcal{X}}$ the shallow water parameter,
- $I_0 = \dfrac{\mathcal{B}}{\mathcal{X}}$ and $J_0 = \dfrac{\mathcal{U}|\mathcal{U}|}{\mathcal{C}\mathcal{H}^p}$ the topography and friction slopes.

With $\dfrac{g\mathcal{X}}{\mathcal{U}^2} = \dfrac{g\mathcal{H}}{\mathcal{U}^2}\dfrac{\mathcal{X}}{\mathcal{H}} = \dfrac{1}{\delta F^2}$ and $\dfrac{g\mathcal{H}\mathcal{X}}{\mathcal{V}\mathcal{U}\mathcal{Y}} = \dfrac{g\mathcal{H}}{\mathcal{U}^2}\dfrac{\mathcal{U}}{\mathcal{V}}\dfrac{\mathcal{X}}{\mathcal{Y}} = \dfrac{1}{R_u^2 F^2}$, we finally get:

$$
\begin{cases}
\bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} + (\bar{h}\bar{v})_{\bar{y}} = 0, \\[2ex]
\bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \dfrac{1}{F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{x}} = \dfrac{1}{\delta F^2}\left(-J_0 \dfrac{\bar{u}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p} - I_0\bar{b}_{\bar{x}}\right), \\[2ex]
\bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \dfrac{1}{R_u^2 F^2}\left(\bar{h} + \bar{\phi}\right)_{\bar{y}} = -\dfrac{J_0}{\delta F^2}\dfrac{\bar{v}\sqrt{\bar{u}^2 + R_u^2\bar{v}^2}}{\bar{C}^2\bar{h}^p}.
\end{cases}
$$

Model with source terms: first-order

The zeroth-order expansions of Ψ and Π are defined by

$$\Psi_{2D}^{(0)} = |\Lambda| \left(\mathcal{M}_2 - \frac{\mathcal{M}_1^2}{\mathcal{M}_0} \right) \quad \text{and} \quad \Pi_{2D}^{(0)} = |\Lambda| \left(\frac{\mathcal{M}_0}{\mathcal{M}_1} \mathcal{M}_2 - \frac{\mathcal{M}_1^2}{\mathcal{M}_0} \right),$$

where $\mathcal{M}_n$ is a shorter notation for

$$\mathcal{M}_n = \int_{y_-}^{y_+} h \left(C\, h^{p/2} \right)^n dy \quad (\text{note that } \mathcal{M}_0 = S \text{ and } \mathcal{M}_1 = Q_{2D}^{(0)}).$$

Mathematical properties

To define the hyperbolicity and the linear stability, we write the system under the condensed form

$$W_t + A(W)W_x = \frac{1}{\varepsilon}S(W).$$

1. Hyperbolicity: We compute the eigenvalues of the matrix $A(W)$. For the system to be hyperbolic, they have to be real-valued.
2. Linear stability: We linearize the system around $W_0 = \text{cst}$ such that $S(W_0) = 0$, by taking $W = W_0 + \overline{W}e^{i(kx-\omega t)}$, with $|\overline{W}| < |W_0|$: the system is linearly stable if $\text{Im}(\omega) \leqslant 0$.
   After linearization, we obtain $(kA(W_0) + i\nabla S(W_0)/\varepsilon - \omega \text{Id})\overline{W} = 0$, and therefore
   $$\omega \text{ is an eigenvalue of } M(k) := kA(W_0) + \frac{i}{\varepsilon}\nabla S(W_0).$$

Hyperbolicity

We study the hyperbolicity of the homogeneous model.
First, the homogeneous classical (S, Q) shallow water system is:

$$
\begin{cases}
S_t + Q_x = 0, \\
Q_t + \left(\dfrac{Q^2}{S} \right)_x + \dfrac{SH_x}{F^2} = 0.
\end{cases}
$$

This system is hyperbolic, with wave velocities

$$
\underbrace{U}_{\substack{\text{material} \\ \text{velocity}}} \pm \underbrace{\sqrt{\frac{S}{F^2 L(H)}}}_{\substack{\text{acoustic} \\ \text{velocity}}}.
$$

Hyperbolicity

We study the hyperbolicity of the homogeneous model.
The zeroth-order homogeneous model (taking $K_1 = K_2 = 0$) reads:

$$
\begin{cases}
S_t + Q_x = 0, \\
Q_t + \left(\dfrac{Q^2}{S} + \Psi \right)_x + \dfrac{S H_x}{F^2} = 0, \\
\left(\dfrac{1}{2} \dfrac{Q^2}{S} + \dfrac{1}{2} \Psi \right)_t + \left(\dfrac{Q}{S} \left(\dfrac{1}{2} \dfrac{Q^2}{S} + \dfrac{1}{2} \Pi \right) \right)_x + \dfrac{Q H_x}{F^2} = 0, \\
\left(\dfrac{1}{2} (\Pi - 3\Psi) \right)_t = 0.
\end{cases}
$$

As mentioned before, this system is hyperbolic, with wave velocities

$$
\underbrace{0}_{\substack{\text{stationary}\\\text{wave}}}, \quad \underbrace{U}_{\substack{\text{material}\\\text{wave}}}, \quad \underbrace{U}_{\substack{\text{material}\\\text{velocity}}} \pm \underbrace{\sqrt{\frac{S}{F^2 L(H)} + \frac{\Pi}{S}}}_{\substack{\text{acoustic}\\\text{velocity}}}.
$$

Hyperbolicity

We study the hyperbolicity of the homogeneous model.
Finally, the first-order homogeneous model is:

$$
\begin{cases}
S_t + Q_x = 0, \\
Q_t + \left(\dfrac{Q^2}{S} + \Psi\right)_x + (1+K_1)\dfrac{SH_x}{F^2} = 0, \\
\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Psi\right)_t + \left(\dfrac{Q}{S}\left(\dfrac{1}{2}\dfrac{Q^2}{S} + \dfrac{1}{2}\Pi\right)\right)_x + \dfrac{QH_x}{F^2} = 0, \\
\left(\dfrac{1}{2}(\Pi - 3\Psi)\right)_t = 0.
\end{cases}
$$

This system is hyperbolic for small enough F, with wave velocities

$$
0,\ U\frac{1+3K_1}{1+K_1} + \mathcal{O}(F^2),\ \frac{U}{1+K_1} \pm \sqrt{\frac{S}{L(H)}}\left(\frac{\sqrt{1+K_1}}{F} + \frac{F}{2}\frac{\Pi + K_1(2\Pi + K_1(\Pi - 3SU^2))}{S(1+K_1)^{5/2}}\right) + \mathcal{O}(F^2).
$$

Thus, a necessary condition for the hyperbolicity is $1 + K_1 > 0$.

Linear stability

Let us first consider the special case $k = 0$.

For the classical shallow water equations in (S, Q) variables, the matrix $M(0)$, whose eigenvalues we seek, is:

$$M(0) = i\,\frac{\mathcal{J}}{\varepsilon}\begin{pmatrix} 0 & 0 \\ \dfrac{\operatorname{sgn}(U_0)}{S_0} & \dfrac{-2}{U_0} \end{pmatrix}.$$

By inspection, the eigenvalues are

$$0 \quad \text{and} \quad -2i\frac{1}{\varepsilon}\frac{\mathcal{J}}{U_0} < 0.$$

The case $k = 0$ is thus treated for the classical shallow water system.

Linear stability

Now, for the model, the eigenvalues for $k = 0$ are

$$\underbrace{\omega_1 = 0, \quad \omega_2 = -2i\frac{1}{\varepsilon}\frac{\mathcal{J}}{U_0},}_{\text{shallow water eigenvalues}} \quad \omega_3 = 2i\frac{1}{\varepsilon}K_1\frac{S_0 U_0}{\Psi_0}, \quad \omega_4 = -2i\frac{1}{\varepsilon}K_2\frac{S_0 U_0}{\Pi_0}.$$

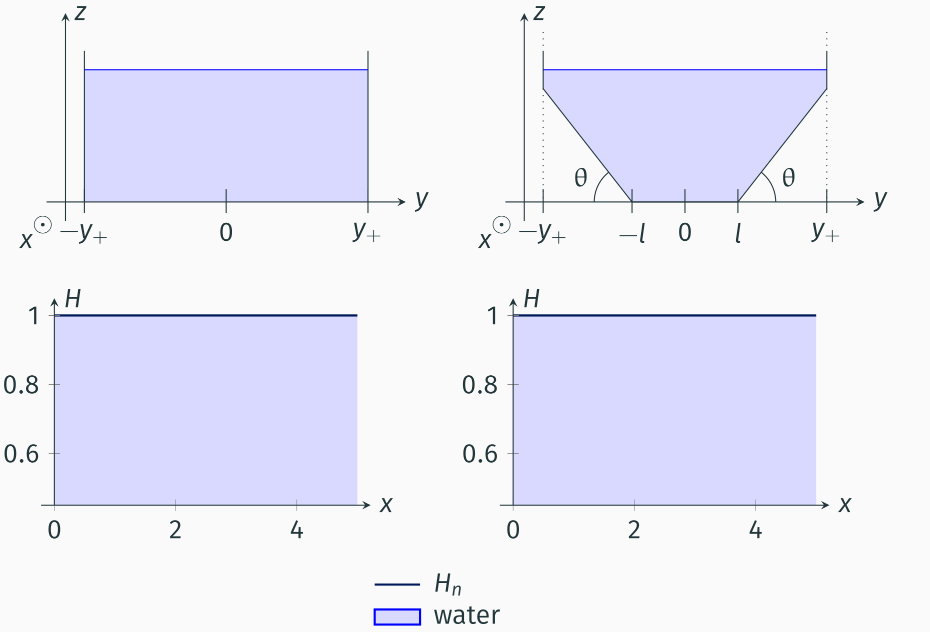
We elect to define K_1 and K_2 by taking $\omega_3 = \omega_2$ and $\omega_4 = \omega_2$, to get:

$$K_1 = -\frac{S\Psi^{(0)}}{\left(Q^{(0)}\right)^2} \quad \text{and} \quad K_2 = \frac{S\Pi^{(0)}}{\left(Q^{(0)}\right)^2}.$$

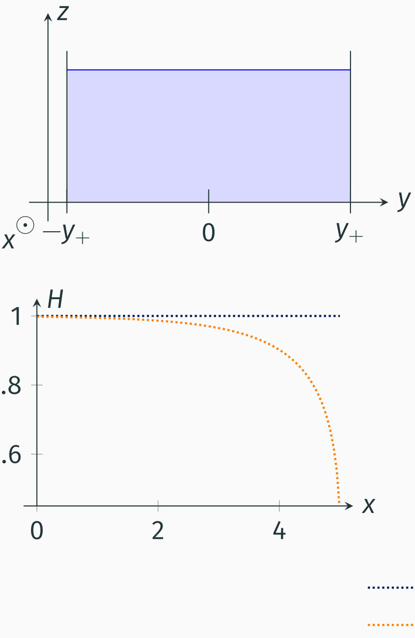
Then, for $k \neq 0$, tedious computations show that, for small enough ε, the following linear stability condition holds:

$$U_0^2 < \frac{4S_0}{L(H)} + \frac{\Pi}{S}, \quad \text{as opposed to } U_0^2 < \frac{4S_0}{L(H)} \text{ for classical SW.}$$

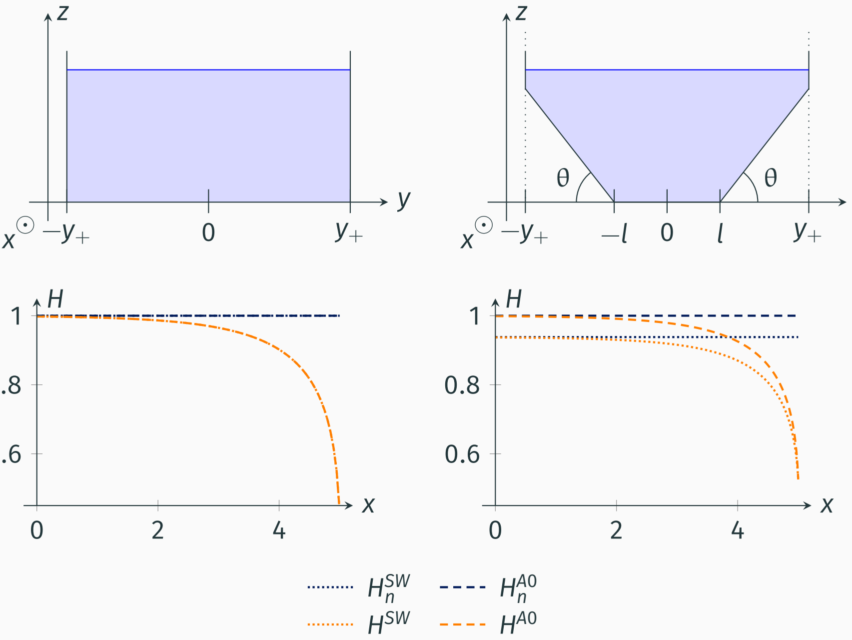
Backwater curves: ⎣⎦-shaped channel versus trapezoidal river



Backwater curves: ⊔-shaped channel versus trapezoidal river



Backwater curves: ⎵-shaped channel versus trapezoidal river



Backwater curves: ⊔-shaped channel versus trapezoidal river