Asymptotically accurate high-order space and time schemes for the Euler system in the low Mach regime

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Introduction

- **Model under consideration**. We study the compressible isentropic Euler system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho U) = 0, \\ \partial_t (\rho U) + \nabla \cdot (\rho U \otimes U) + \dfrac{1}{\varepsilon} \nabla p(\rho) = 0. \end{cases}$$

  - $\rho > 0$ is the density of the fluid
  - $U \in \mathbb{R}$ is the velocity of the fluid
  - $p(\rho) = \rho^\gamma$ is the pressure
  - $\gamma \geq 1$ is the ratio of specific heats
  - ε is the squared Mach number

- **Incompressible limit**. With well-prepared initial and boundary conditions, the compressible Euler system tends to the following incompressible limit when ε tends to 0:

$$\begin{cases} \rho = \rho_0, \\ \nabla \cdot U = 0, \\ \rho_0 \, \partial_t U + \rho_0 \nabla \cdot (U \otimes U) + \nabla \pi_1 = 0, \end{cases}$$

  where π_1 is the order one correction of the pressure.
  The time singularity of this limit is due to the propagation of the acoustic waves at a velocity proportional to $1/\sqrt{\varepsilon}$.

- **Numerical method**. In [3], Dimarco, Loubère and Vignal propose a numerical scheme to preserve this asymptotic behavior. It is written below in semi-discrete form:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot (\rho U)^{n+1} = 0,$$
$$\frac{(\rho U)^{n+1} - (\rho U)^n}{\Delta t} + \nabla \cdot (\rho U \otimes U)^n + \frac{1}{\varepsilon} \nabla (p(\rho))^{n+1} = 0.$$

  Thanks to the semi-implicitation, this scheme is:
  - *asymptotic preserving* (AP), i.e. it discretizes the incompressible Euler system when ε tends to 0;
  - *uniformly L^∞-stable* providing the space discretization is well-chosen.

- **Objective**. Propose an *asymptotically accurate* extension of this numerical scheme. The following properties must be satisfied:
  - *higher order of accuracy* for all values of ε (including the asymptotic preserving property when $\varepsilon \to 0$);
  - ability to *control the oscillations* induced by the use of high accuracy space/time numerical scheme.

1. A model problem

We consider the following advection equation as a model problem:

$$\partial_t u + c_s \, \partial_x u + \frac{c_f}{\sqrt{\varepsilon}} \partial_x u = 0,$$

where the slow and fast velocities c_s and $c_f/\sqrt{\varepsilon}$ are assumed to be non-negative and of order one.
Similarly to the Euler system, the characteristic velocity of the information is proportional to $1/\sqrt{\varepsilon}$. As a consequence, we consider the following semi-discrete scheme, mimicking the structure of the one proposed in [3]:

$$\frac{u^{n+1} - u^n}{\Delta t} + c_s \, (\partial_x u)^n + \frac{c_f}{\sqrt{\varepsilon}} (\partial_x u)^{n+1} = 0.$$

Since $c_s \geq 0$ and $c_f \geq 0$, we use an upwind discretization in space:

$$\partial_x u \simeq \frac{u_j - u_{j-1}}{\Delta x}.$$

As a consequence, the fully discrete scheme reads:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c_s \frac{u_j^n - u_{j-1}^n}{\Delta x} + \frac{c_f}{\sqrt{\varepsilon}} \frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x} = 0.$$

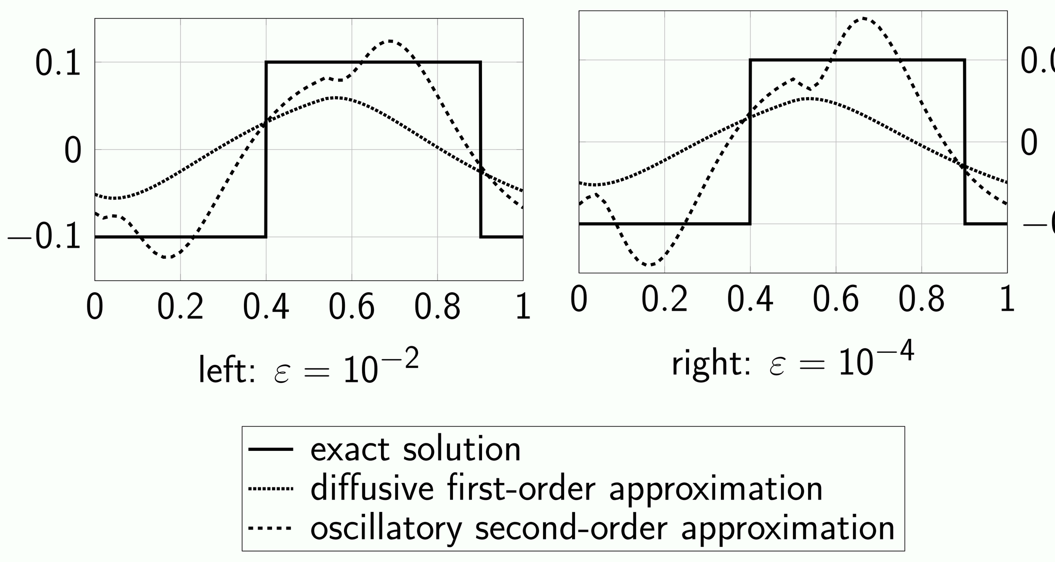
Goal: Propose an **asymptotically accurate** extension of this scheme.

1.1. A more accurate time discretization

This scheme falls within the framework of the **IMEX** (IMplicit-EXplicit) schemes (see [5] for instance). We turn to the two-step second-order in time **ARS(2,2,2)** discretization (see [1]), as follows:

$$\begin{cases} u_j^\star = u_j^n - \beta c_s \dfrac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) - \beta \dfrac{c_f}{\sqrt{\varepsilon}} \dfrac{\Delta t}{\Delta x}\left(u_j^\star - u_{j-1}^\star\right), \\ u_j^{n+1} = u_j^n - (\beta - 1) c_s \dfrac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) - (1-\beta) \dfrac{c_f}{\sqrt{\varepsilon}} \dfrac{\Delta t}{\Delta x}\left(u_j^\star - u_{j-1}^\star\right) \\ \qquad - (2-\beta) c_s \dfrac{\Delta t}{\Delta x}\left(u_j^\star - u_{j-1}^\star\right) - \beta \dfrac{c_f}{\sqrt{\varepsilon}} \dfrac{\Delta t}{\Delta x}\left(u_j^{n+1} - u_{j-1}^{n+1}\right). \end{cases}$$

This discretization preserves the AP property of the scheme. However, it is oscillatory, as displayed below with the advection of a step function.



1.2. A limiting procedure

The implicit part of this IMEX scheme is nothing but an implicit Runge-Kutta discretization. Unfortunately, the following result holds.

Theorem ([4]): There are no strong stability preserving implicit Runge-Kutta schemes of order higher than one.

To tackle this problem and still obtain a scheme more than first-order accurate, we introduce a **limiting procedure**. It consists in **lowering the order** of the scheme if **oscillations** are detected; it belongs to the framework of MOOD techniques (see [2]).

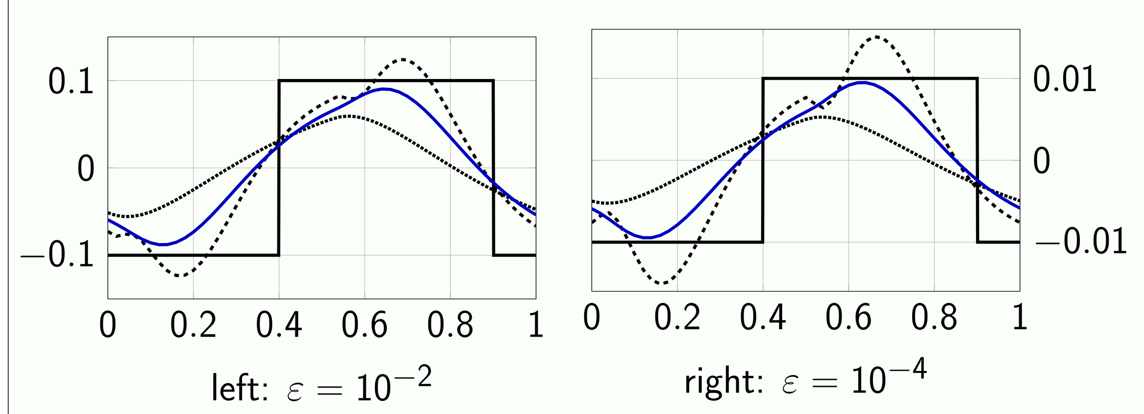
We introduce a **convex combination**, of parameter θ, between the second-order discretization and the first-order discretization, as follows:

$$\begin{aligned} u_j^{n+1} = u_j^n &- \theta(\beta - 1) c_s \frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) - \theta(1-\beta) \frac{c_f}{\sqrt{\varepsilon}} \frac{\Delta t}{\Delta x}\left(u_j^\star - u_{j-1}^\star\right) \\ &- \theta(2-\beta) c_s \frac{\Delta t}{\Delta x}\left(u_j^\star - u_{j-1}^\star\right) - \theta \beta \frac{c_f}{\sqrt{\varepsilon}} \frac{\Delta t}{\Delta x}\left(u_j^{n+1} - u_{j-1}^{n+1}\right) \\ &- (1-\theta) c_s \frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) - (1-\theta) \frac{c_f}{\sqrt{\varepsilon}} \frac{\Delta t}{\Delta x}\left(u_j^{n+1} - u_{j-1}^{n+1}\right). \end{aligned}$$

The following procedure is then applied at each time step:

1. compute a **candidate solution** u^{n+1} with the original ARS(2,2,2) discretization, i.e. with $\theta = 1$;
2. detect if this candidate solution satisfies the following global **maximum principle**: $\|u^{n+1}\|_\infty \leq \|u^n\|_\infty$;
3. if this maximum principle is not satisfied, then take $0 < \theta < 1$ and compute a **new solution** u^{n+1} with the above θ-AP scheme.

Remark: There is a value $0 < \theta < 1$ such that this θ-AP scheme is TVD (Total Variation Diminishing).

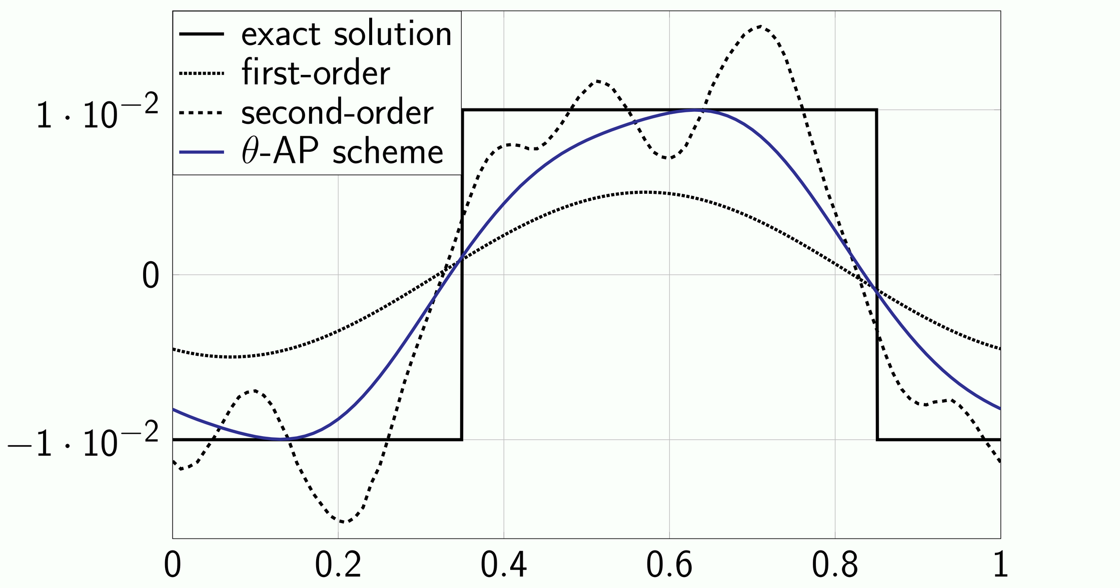


The approximation provided by the θ-AP scheme, which corresponds to the blue curve when considering the advection of a step function, is in-bounds and more accurate than the first-order discretization.

1.3. Space accuracy improvement

To address the issue of the second-order space accuracy, we turn to a classical **MUSCL** method. In each cell, we take a **linear approximation** $u_j^n(x)$ instead of the constant u_j^n.

In the figure below, we compare the different time discretizations at our disposal, coupled with the MUSCL method to increase space accuracy.

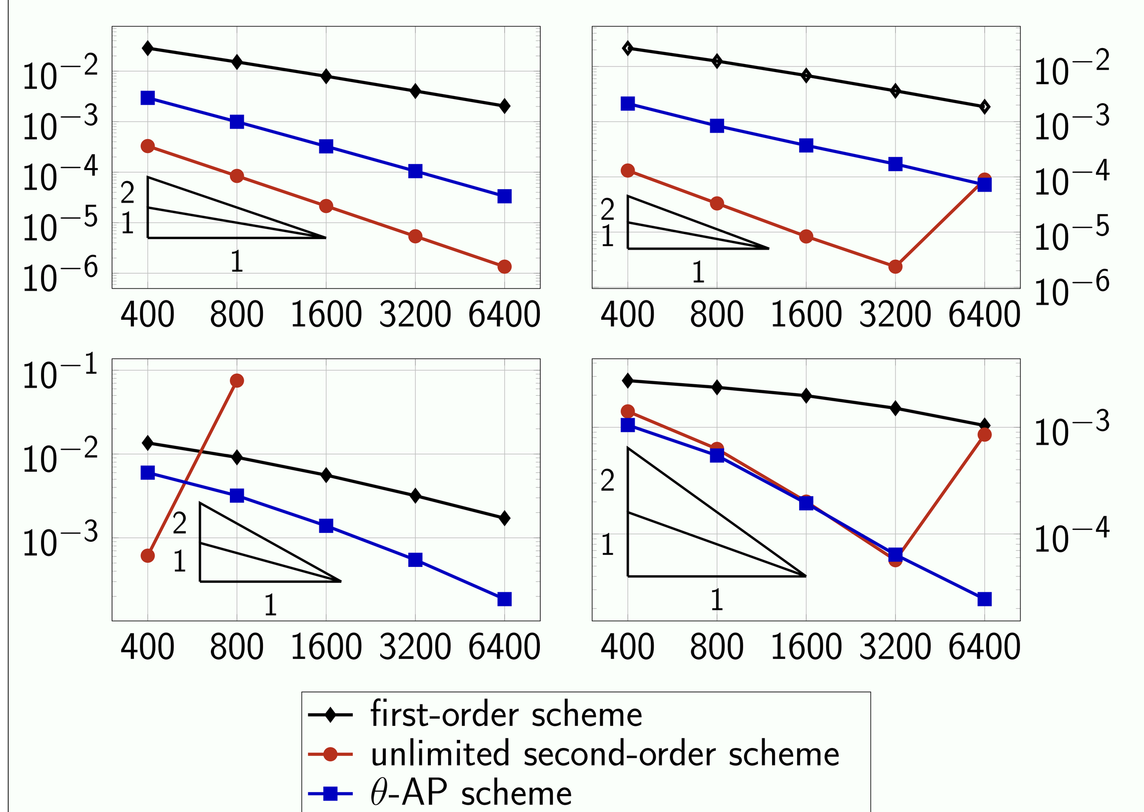


The approximation provided by the θ-AP scheme (i.e. the blue curve, here with $\varepsilon = 10^{-4}$) provides the best solution among the three discretizations.

What about the numerical **order of accuracy**?

Definition: A numerical method is of space (resp. time) order p if its error is proportional to Δx^p (resp. Δt^p) when considering the approximation of a smooth solution.

We thus display the error with respect to the number of points for a **smooth solution** and $\varepsilon \in \{1, 10^{-1}, 10^{-2}, 10^{-4}\}$ (from left to right and top to bottom); the slopes correspond to the order of accuracy.



Remark: The oscillations of the unlimited scheme cause an explosion of the numerical solution when $\Delta x \to 0$.

2. Application to the Euler system

The strategy developed for the model problem is now applied to the Euler system. For the second-order accuracy, we use:

- the *ARS(2,2,2) time discretization*;
- a *linear reconstruction*.

To control the oscillations, we introduce:

- the Euler analogue of the θ-AP scheme;
- a limiter on the reconstruction slopes.

Remark: The Euler variables no longer satisfy a maximum principle.

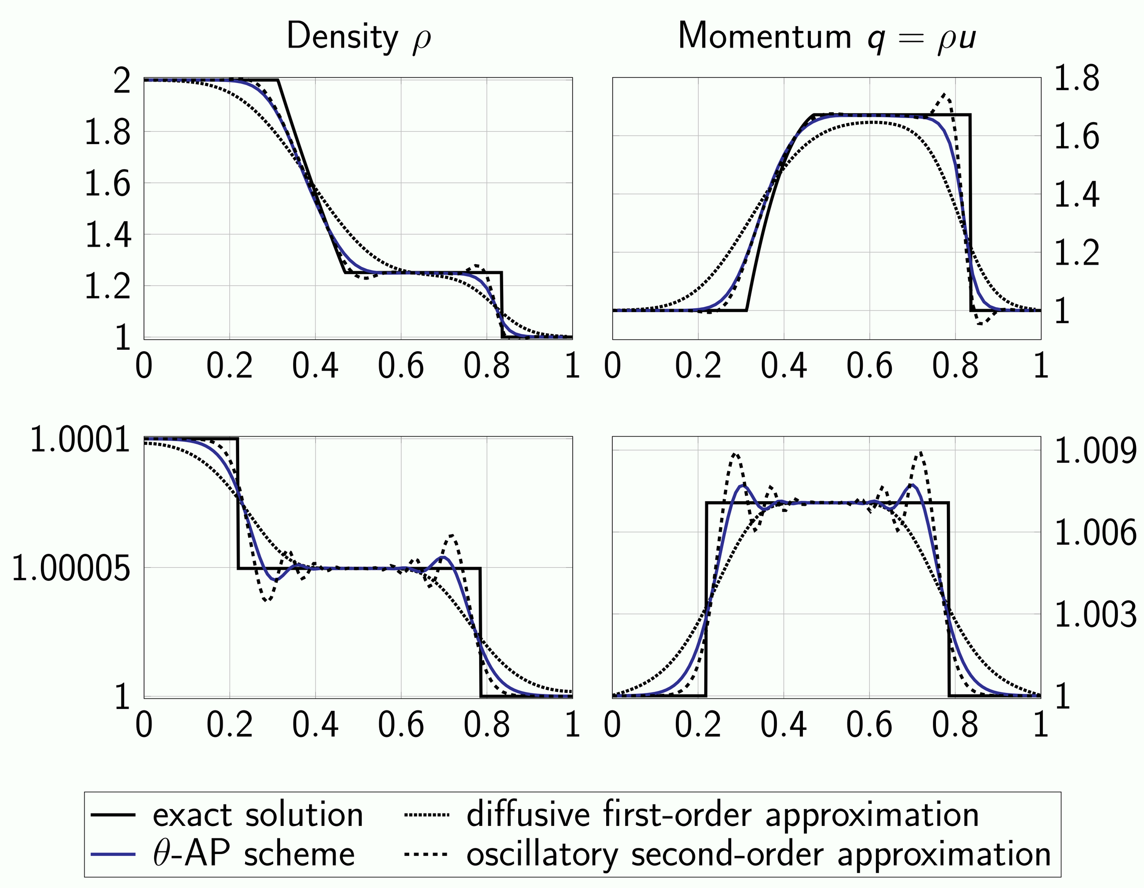
We do not apply the same detection criterion as in the transport case. Instead, we turn to the **Riemann invariants**. Indeed, the Riemann invariants of smooth solutions are transported at the characteristic velocities, and thus they satisfy a **maximum principle**.

First numerical experiment: Riemann problem

We consider a Riemann problem with the following initial data:

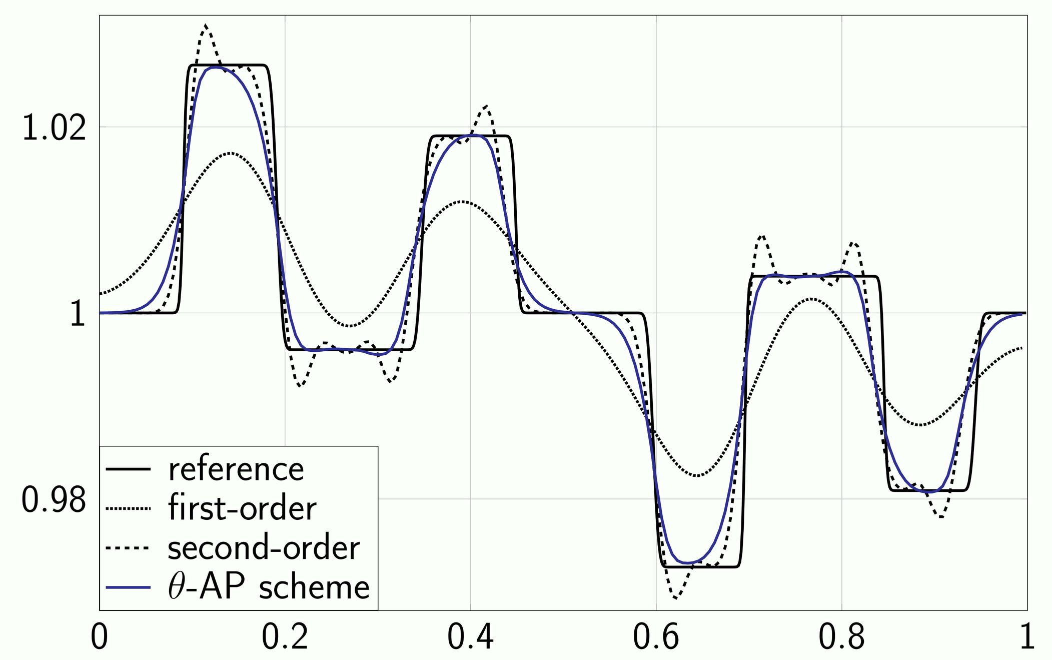
$$\begin{cases} \rho_L = 1 + \varepsilon, \\ \rho_R = 1, \end{cases} \qquad \begin{cases} q_L = h_L u_L = 1, \\ q_R = h_R u_R = 1, \end{cases}$$

with $\varepsilon = 1$ (top panels) and $\varepsilon = 10^{-4}$ (bottom panels). This leads to a left rarefaction wave and a right shock wave, both with characteristic velocities proportional to $1/\sqrt{\varepsilon}$. Note that the amplitude of the rarefaction wave fan is also proportional to $1/\sqrt{\varepsilon}$.



For both values of ε, the θ-AP scheme yields a better approximation than both other schemes: it is less diffusive than the first-order one and less oscillatory than the second-order one.

A more complex numerical experiment



Ongoing works

- develop a relevant criterion to determine a local θ;
- implement the extension of the θ-AP scheme to two space dimensions;
- validate and verify its behavior on the full Euler system.

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