A conservative well-balanced hybrid SPH scheme for the shallow-water model

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HYDROCEAN

Introduction

- hyperbolic system of conservation laws with source term in one space dimension:

$$\partial_t \Phi + \partial_x F(\Phi) = S(\Phi), \text{ with}$$

  - $\Phi : \mathbb{R} \times \mathbb{R}_+ \to \Omega \subset \mathbb{R}^d$, the unknown *vector of conserved variables*
  - $F : \Omega \to \mathbb{R}^d$, the *flux function*
  - $\Omega \subset \mathbb{R}^d$, the *set of admissible states*
- **example**: the shallow-water equations with topography

$$\Phi = \begin{pmatrix} h \\ hu \end{pmatrix}, \; F(\Phi) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}, \; S(\Phi) = \begin{pmatrix} 0 \\ -gh\partial_x Z \end{pmatrix}$$

  - $h \geq 0$ denotes the water height
  - u denotes the horizontal water velocity
  - $g > 0$ denotes the gravity constant
  - Z denotes the smooth topography
- **objectives**: propose a numerical scheme that:
  - uses the SPH method for space discretization
  - preserves the lake at rest steady state
  - is conservative

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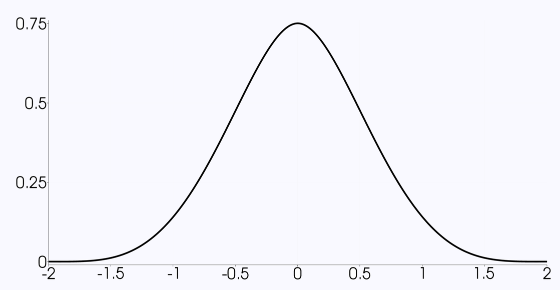
1. The SPH method

The continuous particle approximation

- $f(x) = (f * \delta)(x)$, with $f : \mathbb{R} \mapsto \mathbb{R}$, δ the Dirac distribution
$$= \int_{\mathbb{R}} f(y)\delta(x-y)\,dy$$
- $\Pi^h(f)(x) = (f * W)(x)$, with W a regularizing kernel
$$= \int_K f(y)W(x-y,h)\,dy \simeq f(x)$$
- $$\Pi^h(f')(x) = \int_K f'(y)W(x-y,h)\,dy = \int_K f(y)W'(x-y,h)\,dy \simeq f'(x)$$

The kernel

$$W(r,h) = \frac{C_\theta}{h}\theta\left(\frac{|r|}{h}\right)$$



the Wendland kernel

- θ cut-off function
- C_θ normalization constant

1. bell-shaped even function of class C^∞
2. compact support K
3. bell parameters: r (position) and h (width)
4. $\int_{\mathbb{R}} W(r,h)\,dr = 1$
5. $\int_{\mathbb{R}} W'(r,h)\,dr = 0$

The discrete particle approximation

quadrature formula: $\int_{\mathbb{R}} f(y)\,dy \simeq \sum_{j\in\mathbb{Z}} \omega(x_j)f(x_j) = \sum_{j\in\mathbb{Z}} \omega_j f_j$

where x_j are the quadrature points, or *particles*.

- $\Pi^h(f)(x) = \int_K f(y)W(x-y,h)\,dy$

  becomes $\Pi^h(f)_i = \sum_{j\in\mathcal{P}} \omega_j f_j W_{ij} \simeq f_i$
- $\Pi^h(f')(x) = \int_K f(y)W'(x-y,h)\,dy$

  becomes $\Pi^h(f')_i = \sum_{j\in\mathcal{P}} \omega_j f_j W'_{ij} \simeq f'_i$

main issue: loss of the consistency; indeed,

- $\sum_{j\in\mathcal{P}} \omega_j W_{ij} \neq 1$ and it becomes an approximation of 1
- $\sum_{j\in\mathcal{P}} \omega_j W'_{ij} \neq 0$ and it becomes an approximation of 0

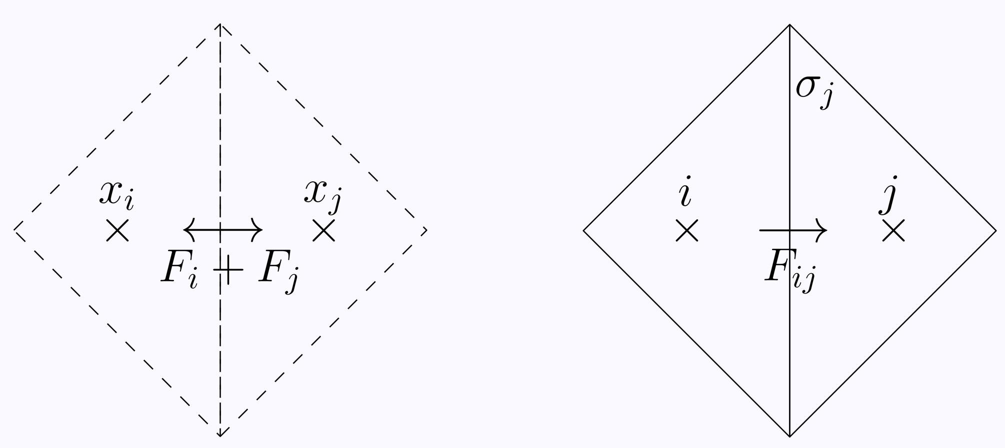
We instead use the following conservative operator:

$$\Pi^h(f')_i = \sum_{j\in\mathcal{P}} \omega_j (f_i + f_j) W'_{ij}.$$

Hybridization with Finite Volumes

aim: derive a SPH approximation of $\partial_x F(\Phi)$

SPH — Finite Volumes



$$\partial_x F_i \simeq \sum_{j\in\mathcal{P}} \omega_j (F_i + F_j) W'_{ij} \qquad \partial_x F_i \simeq \sum_{j\in\gamma(i)} \sigma_j F_{ij}$$

FV-SPH hybridization [2]: $\partial_x F_i \simeq \sum_{j\in\mathcal{P}} \omega_j 2F_{ij} W'_{ij}$

where F_{ij} is **any** conservative FV flux (for instance HLL)

Application to the shallow-water model

Applied to $\partial_t \Phi + \partial_x F(\Phi) = S(\Phi)$ with an explicit Euler time discretization, the hybrid scheme reads:

$$\frac{1}{\Delta t}\left(\Phi_i^{n+1} - \Phi_i^n\right) + \sum_{j\in\mathcal{P}} 2\omega_j F_{ij} W'_{ij} = S_i^n,$$

with Δt the time step and $\Phi_i^n = \Phi(x_i, t^n)$.

Applying this scheme to the shallow-water equations yields:

$$\begin{cases} \dfrac{h_i^{n+1} - h_i^n}{\Delta t} + \displaystyle\sum_{j\in\mathcal{P}} 2\omega_j (hu)_{ij} W'_{ij} = 0 \\ \dfrac{h_i^{n+1}u_i^{n+1} - h_i^n u_i^n}{\Delta t} + \displaystyle\sum_{j\in\mathcal{P}} 2\omega_j \left(hu^2 + \frac{1}{2}gh^2\right)_{ij} W'_{ij} = s_i. \end{cases}$$

s_i is a discretization of the source term, which will be determined later in order to ensure the well-balancedness.

2. A well-balanced scheme

Reformulation of the shallow-water model

recall the lake at rest steady state, where $u = 0$ and $h + Z = \text{cst}$, and introduce the following reformulation [1]:

- $H = h + Z$: the free surface
- $X = \dfrac{h}{H}$: the water volume fraction

rewrite the shallow-water equations for weak solutions:

$$\begin{cases} \partial_t h + \partial_x (X(Hu)) = 0, \\ \partial_t (hu) + \partial_x \left(X\left(Hu^2 + \frac{1}{2}gH^2\right)\right) = \frac{g}{2}\partial_x(hZ) - gh\partial_x Z \end{cases}$$

for the lake at rest, note that $\begin{pmatrix} H \\ Hu \end{pmatrix} = \begin{pmatrix} H \\ 0 \end{pmatrix} = \text{cst}$

SPH discretization

rewrite the scheme for the shallow-water equations using this reformulation :

$$\begin{cases} \dfrac{h_i^{n+1} - h_i^n}{\Delta t} + \displaystyle\sum_{j\in\mathcal{P}} 2\omega_j X_{ij} (Hu)_{ij} W'_{ij} = 0 \\ \dfrac{h_i^{n+1}u_i^{n+1} - h_i^n u_i^n}{\Delta t} + \displaystyle\sum_{j\in\mathcal{P}} 2\omega_j X_{ij} \left(Hu^2 + \frac{1}{2}gH^2\right)_{ij} W'_{ij} = \left(\frac{g}{2}\partial_x(hZ) - gh\partial_x Z\right)_i \end{cases}$$

Source term discretization

to get a *well-balanced* and *conservative* scheme, we adopt the following consistent source term discretization:

$$\left(\frac{g}{2}\partial_x(hZ) - gh\partial_x Z\right)_i = \frac{g}{2}\sum_{j\in\mathcal{P}} 2\omega_j X_{ij} H_{ij}^2 (1 - X_{ij}) W'_{ij} - g\bar{X}_i\bar{H}_i \sum_{j\in\mathcal{P}} 2\omega_j (1 - X_{ij}) W'_{ij} + 2g\bar{H}_i^2 \bar{X}_i (1 - \tilde{X}_i) \sum_{j\in\mathcal{P}} \omega_j W'_{ij}$$

averages H_{ij}, $\bar{H}_i$, $\bar{X}_i$, $\tilde{X}_i$ and X_{ij} still need to be determined

Main results

- Assume both free surface averages H_{ij} and $\bar{H}_i$ to satisfy:

$$H_{ij} = \bar{H}_i = H, \qquad \text{as soon as } H_i = H_j = H.$$

  Assume $\bar{X}_i$ is defined by the following expression (consistent with $X = h/H$):

$$\bar{X}_i = \frac{1}{2}\frac{\sum_{j\in\mathcal{P}} \omega_j X_{ij}^2 W'_{ij}}{\sum_{j\in\mathcal{P}} \omega_j (X_{ij} - 1) W'_{ij} + (\tilde{X}_i - 1)\sum_{j\in\mathcal{P}} \omega_j W'_{ij}}.$$

  Then the scheme defined by the SPH hybridization and source term discretization *preserves the lake at rest*, for any expression of X_{ij} and $\tilde{X}_i$ consistent with X.
- In addition, *whatever the averages*, the scheme is conservative. We use upwind averages for X_{ij} and H_{ij}, and set $\bar{H}_i = H_i$ and $\tilde{X}_i = X_i$.

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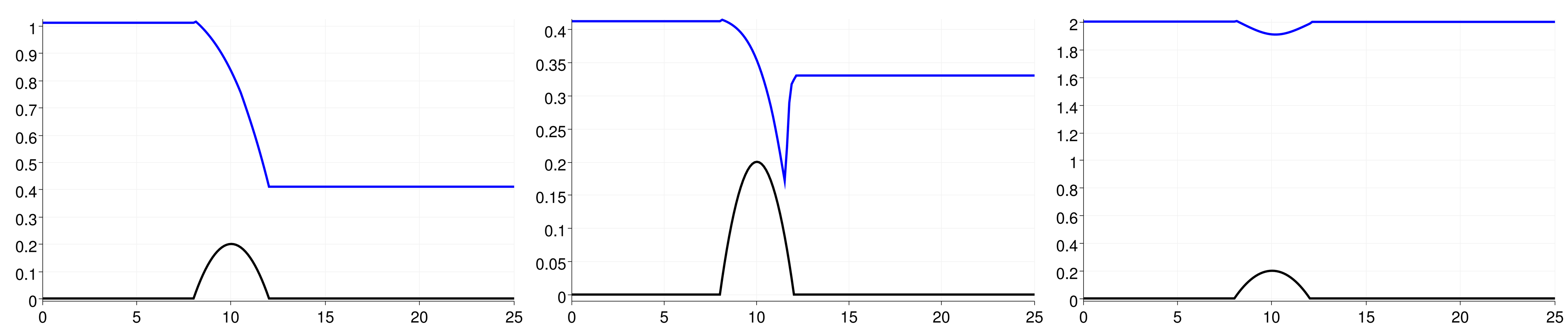
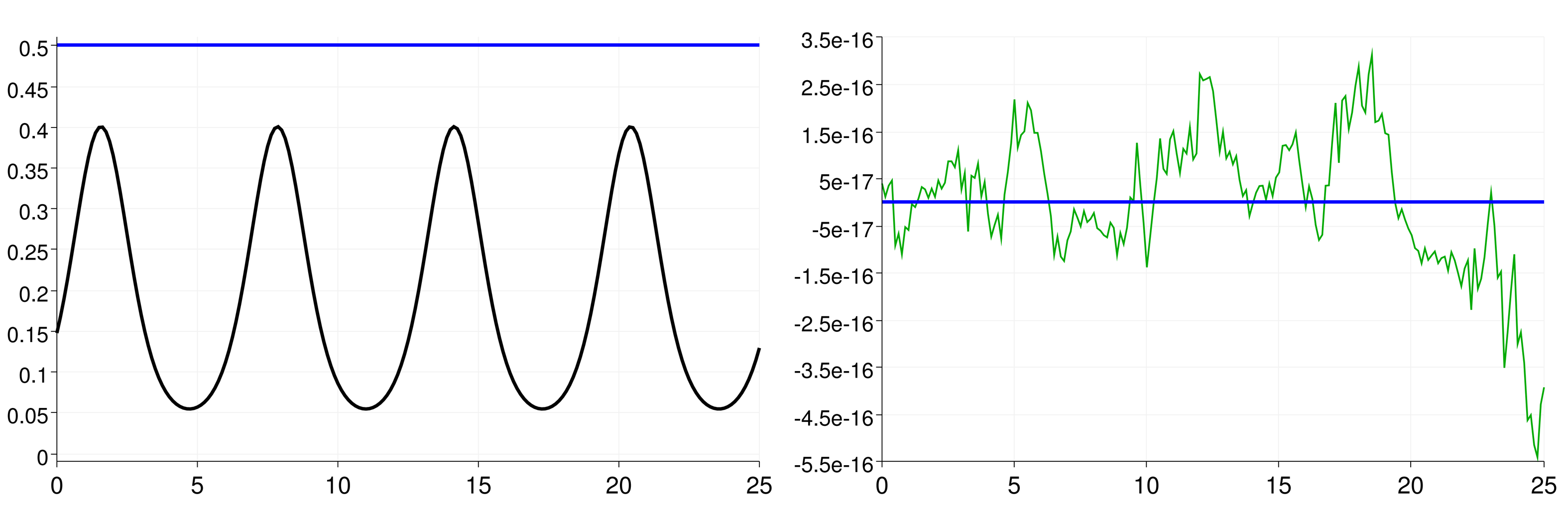
Numerical experiments





validation: steady test cases (above) and well-balancedness (below)

to see those experiments in motion, one may scan the following QR codes: