Commutator of projectors and of unitary operators

Alain BOUDOU[1] and Sylvie VIGUIER-PLA[2&1]

Abstract We define and study the concept of commutator for two projectors, for a projector and a unitary operator, and for two unitary operators. Then we state several properties of these commutators. We recall that projectors and unitary operators are linked with the spectral elements of stationary processes. We establish relations between these commutators and some other tools related to the proximity between processes.

1 Introduction

This work relates to the field of the operatorial domain, dealing with projectors and unitary operators. These operators take a large place in the statistics of stationary processes. For example, the shift operator is a unitary operator, and a unitary operator is a linear combination of projectors. We define and study the concepts of commutator for two projectors, for a projector and a unitary operator, and for two unitary operators. These concepts are developped in the hilbertian frame, and when the $\mathbb{C}-$Hilbert space H is of the type $L^2(\Omega, \mathscr{A}, P)$, our results apply to stationary processes. The commutativity of two stationary processes is a generalization of the notion of stationary correlation. When there is not a complete commutativity, we may extend the notion of commutativity, asking how to retrieve the part of each process which commute. The commutator proposes an answer to this question. We recall that the product of convolution of spectral measures, such as defined in Boudou and Romain [6], needs an hypothesis of commutativity. Our work uses tools defined in Boudou and Viguier-Pla [9], such as the $r-$convergence and the distance between projectors.

[1]Equipe de Stat. et Proba., Institut de Mathématiques, UMR5219, Université Paul Sabatier, 118 Route de Narbonne, F-31062 Toulouse Cedex 9, Francee-mail: boudou@math.univ-toulouse.fr ·
[2] Université de Perpignan via Domitia, LAMPS, 52 av. Paul Alduy, 66860 Perpignan Cedex 9, France e-mail: viguier@univ-perp.fr

Obviously, the commutator of two projectors is linked with the canonical analysis of the spaces generated by these projectors. When these spaces are complex, the practical interest of it is the domain of stationary processes, as seen above. Several authors work on spectral elements of processes, as, for example, in the large deviation field (Gamboa and Rouault [11]), the autoregressive processes (Bosq [2]), and for reduction of dimension (Brillinger [3], Boudou [4]). The joint study of two processes may lead to the comparison of these processes, by the way of the commutators. When these spaces are real, applications may be forseen by the search of common and specific subspaces of two spaces. Such problematics have been largely developped with other tools, as, for example, in the works of Flury and Gautschi [10], Benko and Kneip [1], Viguier-Pla [12].

2 Prerequisites, recalls and notation

In this text, H is a $\mathbb{C}-$Hilbert space. The set of the orthogonal projectors of a $\mathbb{C}-$Hilbert H' is denoted by $\mathscr{P}(H')$. When (E,ξ) is a measurable space,

a random measure (r.m.) Z, defined on ξ and taking values in H is a vector measure such that $<ZA,ZB>=0$, for any pair (A,B) of disjoint elements of ξ.

Then it is easy to verify that

the application $\mu_Z : A \in \xi \mapsto \|ZA\|^2 \in \mathbb{R}^+$ is a bounded measure.

The stochastic integral, relatively to the r.m. Z, can be defined as

the unique isometry from $L^2(E,\xi,\mu_Z)$ onto $H_Z = \overline{vect}\{ZA; A \in \mathscr{B}\}$ which, with A, associates ZA, for any A of ξ. The image of an element φ of $L^2(E,\xi,\mu_Z)$, by this isometry, denoted by $\int \varphi dZ$, is named integral of φ with respect to the r.m. Z.

A series $(X_n)_{n\in\mathbb{Z}}$ of elements of H is said to be

stationary when, for any pair (n,m) of elements of $\mathbb{Z}$, we have $<X_n,X_m>=<X_{n-m},X_0>$.

If Z is a r.m. defined on $\mathscr{B}$, Borel $\sigma-$field of $[-\pi;\pi[$, taking values in H, then

$(\int e^{i.n} dZ)_{n\in\mathbb{Z}}$ is a stationary series.

Conversally,

with any stationary series $(X_n)_{n\in\mathbb{Z}}$ of elements of H, we can associate a r.m. Z, and only one, defined on $\mathscr{B}$, taking values in H, such that $X_n = \int e^{i.n} dZ$, for any n of $\mathbb{Z}$.

Two stationary series $(X_n)_{n\in\mathbb{Z}}$ and $(Y_n)_{n\in\mathbb{Z}}$ are said to be

stationarily correlated when $<X_n,Y_m>=<X_{n-m},Y_0>$, for any pair (n,m) of elements of $\mathbb{Z}$.

When (E,ξ) is a measurable space,

a spectral measure (s.m.) on ξ for H is an application $\mathscr{E}$ from ξ on $\mathscr{P}(H)$ such that $\mathscr{E}E = I_H$, $\mathscr{E}(A\cup B) = \mathscr{E}A + \mathscr{E}B$, for any pair (A,B) of disjoint elements of ξ, and such that $lim_n \mathscr{E}A_n X = 0$, for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements of ξ which decreasingly converges to $\emptyset$ and for any X of H.

We then show that

for any X of H, the application $Z^X_{\mathscr{E}} : A \in \xi \mapsto \mathscr{E}AX \in H$ is a r.m..

When $\mathscr{E}$ is a s.m. on $\mathscr{B}$ for H,

the application $X \in H \mapsto \int e^{i.n} dZ_{\mathscr{E}}^X \in H$ *is a unitary operator.*

Conversally,

with any unitary operator U *of* H*, we can associate one, and only one, s.m.* $\mathscr{E}$*, on* $\mathscr{B}$ *for* H*, such that* $UX = \int e^{i.1} dZ_{\mathscr{E}}^X$*, for any* X *of* H*.*

When α is a s.m. on $\mathscr{B} \otimes \mathscr{B}$ for H,

the application $S : (\lambda, \lambda') \in [-\pi;\pi[\times [-\pi;\pi[\mapsto \lambda + \lambda' - 2\pi \left[\frac{\lambda+\lambda'+\pi}{2\pi}\right] \in [-\pi;\pi[$*, where* $[x]$ *designates the integer part of* x*, is measurable;*

the application $S\alpha : A \in \mathscr{B} \mapsto S\alpha^{-1}A \in \mathscr{P}(H)$ *is a s.m. on* $\mathscr{B}$ *for* H*, named image of* α *by* S*.*

When

U *is a unitary operator of* H *of associated s.m.* $\mathscr{E}$*,* $(U^n X)_{n\in\mathbb{Z}}$ *is a stationary series of associated r.m.* $Z_{\mathscr{E}}^X$*.*

When $\mathscr{E}_1$ and $\mathscr{E}_2$ are two s.m.'s on $\mathscr{B}$ for H which commute, that is such that the projectors $\mathscr{E}_1 A$ and $\mathscr{E}_2 B$ commute, for any A and B of $\mathscr{B}$, then

there exists a s.m., and only one, $\mathscr{E}_1 \otimes \mathscr{E}_2$*, on* $\mathscr{B} \otimes \mathscr{B}$ *for* H*, such that* $\mathscr{E}_1 \otimes \mathscr{E}_2 A \times B = \mathscr{E}_1 A \mathscr{E}_2 B$*, for any pair* (A,B) *of elements of* $\mathscr{B}$*. We name product of convolution of* $\mathscr{E}_1$ *and* $\mathscr{E}_2$*, and we note* $\mathscr{E}_1 * \mathscr{E}_2$*, the image of* $\mathscr{E}_1 \otimes \mathscr{E}_2$ *by* S*.*

We show that

two unitary operators U_1 *and* U_2*, of respective associated s.m.'s* $\mathscr{E}_1$ *and* $\mathscr{E}_2$*, commute if and only if* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *commute,* $\mathscr{E}_1 * \mathscr{E}_2$ *is the s.m. associated with the unitary operator* $U_1 U_2$*.*

For developments of these notions, the reader can refer to Boudou [5], Boudou and Romain [6], and Boudou and Romain [7].

We will end this section by recalls concerning a relation of partial order defined on $\mathscr{P}(H)$.

We say that a projector P *is smaller than a projector* Q*, and we note* $P \ll Q$*, when* $P = PQ = QP$*. Then we have* $\|PX\| \leq \|QX\|$*, for any* X *of* H*.*

The relation $\ll$ is a relation of partial order, but it has the advantage that any family $\{P_\lambda; \lambda \in \Lambda\}$ of projectors, finite or not, countable or not, has got a larger minorant, that is a lower bound, denoted by $\inf\{P_\lambda; \lambda \in \Lambda\}$, and a smaller majorant, that is an upper bound, denoted $\sup\{P_\lambda; \lambda \in \Lambda\}$. Then we have the following properties:

$Im\, inf\{P_\lambda; \lambda \in \Lambda\} = \cap_{\lambda\in\Lambda} Im\, P_\lambda$;

$(sup\{P_\lambda; \lambda \in \Lambda\})^\perp = inf\{P_\lambda^\perp; \lambda \in \Lambda\}$;

$(inf\{P_\lambda; \lambda \in \Lambda\})^\perp = sup\{P_\lambda^\perp; \lambda \in \Lambda\}$;

if P_1 *and* P_2 *are two projectors which commute, then* $inf\{P_1; P_2\} = P_1 P_2$*.*

When $(P_n)_{n\in\mathbb{N}}$ is a sequence of projectors, it is possible to define its upper bound, $\limsup (P_n)_{n\in\mathbb{N}} = \inf\{\sup\{P_m; m \geq n\}; n \in \mathbb{N}\}$, and its lower bound, $\liminf (P_n)_{n\in\mathbb{N}} = \sup\{\inf\{P_m; m \geq n\}; n \in \mathbb{N}\}$. We have then

$lim\,inf (P_n)_{n\in\mathbb{N}} \ll lim\,sup (P_n)_{n\in\mathbb{N}}$;

when $lim\,inf (P_n)_{n\in\mathbb{N}} = lim\,sup (P_n)_{n\in\mathbb{N}} = P$*, we say that* $(P_n)_{n\in\mathbb{N}}$ $r-$*converges to* P*, and we note it* $lim_n^r P_n = P$*.*

The $r-$convergence implies the point by point convergence, but the converse is not true.

For any pair (P_1,P_2) of elements of $\mathscr{P}(H)$, we define
$d(P_1,P_2) = sup(P_1,P_2) - inf(P_1,P_2)$.

This notion presents a great analogy with a distance, however, it is not a distance, as $d(P_1,P_2)$ is a projector. Its interest lies on the following property.

A sequence $(P_n)_{n\in\mathbb{N}}$ of projectors $r-$converges to P if and only if $lim^r_n d(P_n,P) =$ 0.

We show that
for any pair (P_1,P_2) of projectors, $Im(d(P_1,P_2))^\perp = Ker(P_1-P_2)$.

When U_1 and U_2 are two unitary operators,
a projector P is an equalizator of U_1 and U_2 when $U_1P = U_2P = PU_1 = PU_2$; the upper bound of the family of the equalizators of U_1 and U_2 is an equalizator of U_1 and U_2, we name it the maximal equalizator of U_1 and U_2 and we note it R_{U_1,U_2}. We have $Im R_{U_1,U_2} = \cap_{n\in\mathbb{Z}} Ker(U_1^n - U_2^n)$.

These notions are developped in Boudou and Viguier-Pla [9].

3 Commutator of two projectors

Let us first define this notion of commutator.

Definition 3.1. *A projector K is a commutator of the projectors P_1 and P_2 when it commutes with P_1 and P_2 and when $P_1KP_2 = P_2KP_1$.*

We can establish the following properties.

Proposition 3.1. *Let P_1 and P_2 be two projectors. Then*
i) the upper bound of a family of commutators of the projectors P_1 and P_2 is a commutator of the projectors P_1 and P_2;
ii) 0 is a commutator of the projectors P_1 and P_2;
iii) the upper bound of the family of the commutators of the projectors P_1 and P_2 is the projector on $Ker(P_1P_2-P_2P_1)$, that is $inf\{P_1,P_2\} + inf\{P_1,P_2^\perp\} + inf\{P_1^\perp,P_2\} + inf\{P_1^\perp,P_2^\perp\}$.

So we have the following definition.

Definition 3.2. *Let P_1 and P_2 be two projectors. We call maximal commutator of the projectors P_1 and P_2 the projector $C_{P_1,P_2} = inf\{P_1,P_2\} + inf\{P_1,P_2^\perp\} + inf\{P_1^\perp,P_2\} + inf\{P_1^\perp,P_2^\perp\}$.*

Of course, it is easy to establish that $C_{P_1,P_2} = I$ if and only if P_1 and P_2 commute.

The maximal commutator is a tool for measuring the degree of commutativity of the projectors P_1 and P_2, the larger it is, the larger $\mathrm{Ker}(P_1P_2-P_2P_1)$ is. It is clear that when X belongs to $\mathrm{Ker}(P_1P_2-P_2P_1) = \mathrm{Im}\, C_{P_1,P_2}$, $\|C^\perp_{P_1,P_2}X\| = 0$. So what can we speculate when X is close to $\mathrm{Ker}(P_1P_2-P_2P_1)$, that is when $\|C^\perp_{P_1,P_2}X\|$ is small ? We will bring an answer to this question, with the following property.

Proposition 3.2. *For any pair (P_1,P_2) of projectors, and for any X of H, we have $\|P_1P_2X - P_2P_1X\| \leq 2\|C^\perp_{P_1,P_2}X\|$.*

This means that when X is close to $\mathrm{Ker}(P_1P_2 - P_2P_1)$, then P_1P_2X is close to P_2P_1X.

Let us end this section with a property of continuity of the maximal commutator.

Proposition 3.3. *If the projector K commutes with each of the elements of the sequences of projectors $(P_n^1)_{n\in\mathbb{N}}$ and $(P_n^2)_{n\in\mathbb{N}}$, which respectively $r-$converge to P_1 and P_2, then $KP_1 = P_1K$, $KP_2 = P_2K$ and $lim_n^r C_{KP_n^1,KP_n^2} = C_{KP_1,KP_2}$.*

4 Commutator of a projector and of a unitary operator

In a same way as we have defined a commutator of two projectors, we can define a commutator of a projector and of a unitary operator.

Definition 4.1. *A projector K is a commutator of the projector P and of the unitary operator U when it commutes with P and U, and when $PKU = UKP$.*

We have got similar properties as those of the previous section.

Proposition 4.1. *Let P be a projector and U a unitary operator. We can affirm that*
i) the upper bound of a family of commutators of the projector P and of the unitary operator U is a commutator of the projector P and of the unitary operator U;
ii) 0 is a commutator of the projector P and of the unitary operator U;
iii) the upper bound of the family of commutators of the projector P and of the unitary operator U is the projector on $\cap_{n\in\mathbb{Z}} Ker(PU^n - U^nP)$.

The following definition is a consequence of these properties.

Definition 4.2. *Let P be a projector and U a unitary operator. We name maximal commutator of the projector P and of the unitary operator U, and we note it $C_{P,U}$, the projector on $\cap_{n\in\mathbb{Z}} Ker(PU^n - U^nP)$.*

Of course, it is easy to verify that P and U commute if and only if $C_{P,U} = I$. The association “unitary operator-s.m.” being biunivocal, all these properties can express by means of the s.m. which is associated with a unitary operator. We get then a relation between commutator of a projector and of a unitary operator and a family of commutators of two projectors.

Proposition 4.2. *If P is a projector and U a unitary operator of associated s.m. $\mathscr{E}$, we can affirm that*
i) a projector K is a commutator of the projector P and of the unitary operator U if and only if, for any A of $\mathscr{B}$, K is a commutator of the projectors P and $\mathscr{E}A$;
ii) $Im C_{P,U} = \cap_{A\in\mathscr{B}} Ker(P\mathscr{E}A - \mathscr{E}AP)$;
iii) $C_{P,U} = inf\{C_{P,\mathscr{E}A}; A \in \mathscr{B}\}$.

The last two points come from the fact that $\cap_{n\in\mathbb{Z}}\mathrm{Ker}(PU^n - U^nP) = \cap_{A\in\mathscr{B}}\mathrm{Ker}(P\mathscr{E}A - \mathscr{E}AP)$, and that $\mathrm{Im}\, C_{P,\mathscr{E}A} = \mathrm{Ker}(P\mathscr{E}A - \mathscr{E}AP)$.

If we remark that $\{U^{-n}PU^n; n \in \mathbb{Z}\}$ is a family of projectors and that $\mathrm{Ker}(PU^n - U^nP) = \mathrm{Ker}(U^{-n}PU^n - P) = (\mathrm{Im}\, d(U^{-n}PU^n, P))^{\perp}$, we can give to $C_{P,U}$ an ergodic definition.

Proposition 4.3. *For any projector P and for any unitary operator U, we have $C_{P,U} = inf\{(d(U^{-n}PU^n,P))^\perp; n \in \mathbb{Z}\}$.*

This last result can have the following interpretation. If all the elements of the family $\{U^{-n}PU^n; n \in \mathbb{Z}\}$ are close to P, that is, if for any n of $\mathbb{Z}$, $d(U^{-n}PU^n,P)$ is small, or evenmore, for any n of $\mathbb{Z}$, $(d(U^{-n}PU^n,P))^\perp$ is large, then it is the same for the lower bound $C_{P,U}$. This means that P and U are near to commute.

Proposition 4.3 lets us write $d(U^{-n}PU^n,P) \ll C^\perp_{P,U}$, so

$\|(PU^n - U^nP)X\| = \|U^{-n}PU^nX - PX\| \leq 2\|d(U^{-n}PU^n,P)X\| \leq 2\|C^\perp_{P,U}X\|$,

because for any pair of projectors (P,P'), we have $\|PX - P'X\| \leq 2\|d(P,P')X\|$ (cf. Boudou and Viguier-Pla [9]).

Thanks to a similar approach, Propositions 3.2 and 4.2 let us affirm that

$$\|PZ^X_{\mathscr{E}}A - Z^{PX}_{\mathscr{E}}A\| = \|P\mathscr{E}AX - \mathscr{E}APX\| \leq 2\|C^\perp_{P,\mathscr{E}A}X\| \leq 2\|C^\perp_{P,U}X\|.$$

Then the following stands.

Proposition 4.4. *For any projector P and for any unitary operator U of associated s.m. $\mathscr{E}$, for any X of H, we have*
i) $\|PU^nX - U^nPX\| \leq 2\|C^\perp_{P,U}X\|$;
ii) $\|PZ^X_{\mathscr{E}}A - Z^{PX}_{\mathscr{E}}A\| \leq 2\|C^\perp_{P,U}X\|$, for any A of $\mathscr{B}$.

So, if X is close to $\operatorname{Im} C_{P,U}$, that is if $\|C^\perp_{P,U}X\|$ is small, then the series $(PU^nX)_{n\in\mathbb{Z}}$ is “almost stationary”, in such a way it is close to the stationary series $(U^nPX)_{n\in\mathbb{Z}}$. As for the application $P \circ Z^{PX}_{\mathscr{E}}$, it is almost a r.m., close to $Z^{PX}_{\mathscr{E}}$, r.m. associated with the stationary series $(U^nPX)_{n\in\mathbb{Z}}$.

Let us end this section by the resolution of the following problem:

let $(X_n)_{n\in\mathbb{Z}}$ be a stationary series, of associated r.m. Z, and P be a projector. We wish to define all the stationary series, stationarily correlated with $(X_n)_{n\in\mathbb{Z}}$, included in $\operatorname{Im} P$. Such series will be named “solution series”. Then we have the following.

Proposition 4.5. *If U is a unitary operator whose associated s.m. is such that $Z^{X_0}_{\mathscr{E}} = Z$, then, for any X of $\operatorname{Im} C_{P,U}$, we can affirm that $(U^nPX)_{n\in\mathbb{Z}}$ is a “solution series”. Any “solution series” is of this type.*

We remember that when Z is a r.m. defined on $\mathscr{B}$, taking values in H, there exists at least one s.m. $\mathscr{E}$ on $\mathscr{B}$ for H such that $Z^{X_0}_{\mathscr{E}} = Z$, where $X_0 = \int e^{i.0} dZ$ (cf. Boudou [5]).

5 Commutator of two unitary operators

When two unitary operators U and V commute, the s.m. which is associated with the unitary operator UV is the product of convolution of the s.m.’s respectively associated with U and V. But what happens when $UV \neq VU$? The maximal commutator will bring a partial solution to this question.

Definition 5.1. *A projector K is a commutator of the unitary operators U and V when it commutes with U and V, and when $UKV = VKU$.*

Then we can establish the following properties.

Proposition 5.1. *Let U and V be two unitary operators. We can affirm that*
i) the upper bound of a family of commutators of the unitary operators U and V is a commutator of the unitary operators U and V;
ii) 0 is a commutator of the unitary operators U and V;
iii) the upper bound of the family of commutators of the unitary operators U and V is the projector on $\cap_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} Ker(U^nV^m - V^mU^n)$.

So we can define the following.

Definition 5.2. *Let U and V be two unitary operators. We name maximal commutator of the unitary operators U and V, and we note it $C_{U,V}$, the projector on the space $\cap_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} Ker(U^nV^m - V^mU^n)$.*

Of course, it is easy to verify that U and V commute if and only if $C_{U,V} = I$. The reader will notice the similarities between Definitions 3.1, 4.1 and 5.1, between Propositions 3.1, 4.1 and 5.1, and between Definitions 3.2, 4.2 and 5.2.

The commutator of two unitary operators can be defined from the associated s.m.'s.

Proposition 5.2. *If U and V are two unitary operators of respective associated s.m.'s $\mathscr{E}$ and α, we can affirm that*
i) a projector K is a commutator of U and V if and only if, for any pair (A,B) of elements of $\mathscr{B}$, K is a commutator of the projectors $\mathscr{E}A$ and αB;
ii) $Im C_{U,V} = \cap_{(A,B)\in\mathscr{B}\times\mathscr{B}} Ker(\mathscr{E}A\alpha B - \alpha B\mathscr{E}A)$;
iii) $C_{U,V} = inf\{C_{\mathscr{E}A,\alpha B}; (A,B) \in \mathscr{B}\times\mathscr{B}\} = inf\{C_{\alpha B,U}; B \in \mathscr{B}\}$.

To establish the last two points, we must notice that

$$\cap_{(n,m)\in\mathbb{Z}\times\mathbb{Z}}\mathrm{Ker}(U^nV^m - V^mU^n) = \cap_{(A,B)\in\mathscr{B}\times\mathscr{B}}\mathrm{Ker}(\mathscr{E}A\alpha B - \alpha B\mathscr{E}A) = \cap_{(A,B)\in\mathscr{B}\times\mathscr{B}}\mathrm{Im}\, C_{\mathscr{E}A,\alpha B}.$$

Point *iii)* provides a relation between the three types of maximal commutators which we study. We can also establish a relation between the maximal commutator of two unitary operators and the maximal equalizator of two unitary operators.

Proposition 5.3. *Let U and V be two unitary operators. We have $C_{U,V} = inf\{R_{V,U^{-n}VU^n}; n \in \mathbb{Z}\}$.*

For the proof, we have just to notice that

$$\mathrm{Im}\, C_{U,V} = \cap_n \cap_m \mathrm{Ker}(U^nV^m - V^mU^n) = \cap_n \cap_m \mathrm{Ker}(V^m - U^{-n}V^mU^n)$$
$$= \cap_n \cap_m \mathrm{Ker}(V^m - (U^{-n}VU^n)^m) = \cap_n \mathrm{Im}\, R_{V,U^{-n}VU^n} = \mathrm{Im}\inf\{R_{V,U^{-n}VU^n}; n \in \mathbb{Z}\}.$$

Let us now approach the questions suggested at the beginning of the section. Let us denote by C the maximal commutator of the unitary operators U and V. Let L be the application $X \in \mathrm{Im}\, C \mapsto X \in H$. Then we have $L^*(X) = CX$, for any X of H, $L^*L = I_{\mathrm{Im}C}$, $L^*L = C$, $L^*C = L^*$, and $CL = L$. Let $\mathscr{E}$ be the s.m. which is associated with the unitary operator U. As $CU = UC$, we can prove that (Boudou and Viguier [9])

$U' = L^*UL$ is a unitary operator of $Im C$;
for any A of $\mathscr{B}$, $\mathscr{E}'A = L^*\mathscr{E}AL$ is a projector of $Im C$;
the application $\mathscr{E}' : A \in \mathscr{B} \mapsto \mathscr{E}'A \in \mathscr{P}(Im C)$ is the s.m. which is associated with the unitary operator U'.

With obvious notation, we also show that

$V' = L^*VL$ is a unitary operator of $Im C$;
for any A of $\mathscr{B}$, $\alpha' A = L^*\alpha AL$ is a projector of $Im C$;

the application $\alpha' : A \in \mathscr{B} \mapsto \alpha' A \in \mathscr{P}(ImC)$ *is the s.m. which is associated with the unitary operator* V'.

From the fact that $U'V' = L^*ULL^*VL = L^*UCVL = L^*VCUL = L^*VLL^*UL = V'U'$, we can consider the s.m. $\mathscr{E}' \otimes \alpha'$ on $\mathscr{B} \otimes \mathscr{B}$ for $\mathrm{Im}C$ (as the s.m.'s $\mathscr{E}'$ and α' commute). For any pair (A,B) of elements of $\mathscr{B}$, we have

$$\mathscr{E}' \otimes \alpha'(A \times B) = \mathscr{E}'A\alpha'B = \inf\{\mathscr{E}'A, \alpha'B\} = \inf\{L^*\mathscr{E}AL, L^*\alpha BL\} = L^*\inf\{\mathscr{E}A, \alpha B\}L.$$

If we notice that $U'V' = L^*UVL = L^*VUL$, we have the following.

Proposition 5.4. *There exists one s.m., and only one,* $\mathscr{E}' \otimes \alpha'$*, on* $\mathscr{B} \otimes \mathscr{B}$ *for* ImC*, such that* $\mathscr{E}' \otimes \alpha'(A \times B) = L^* inf\{\mathscr{E}A, \alpha B\}L$*, for any pair* (A,B) *of elements of* $\mathscr{B}$*. Its image by* S *is the s.m. associated with the unitary operator* $L^*UVL = L^*VUL$*.*

Of course, when U and V commute, that is when $C = I$, we have $L = L^* = I$, $U' = U$, $V' = V$, $\mathscr{E}' = \mathscr{E}$, $\alpha' = \alpha$ and $\mathscr{E} \otimes \alpha(A \times B) = \inf\{\mathscr{E}A, \alpha B\} = \mathscr{E}A\alpha B$, for any (A,B) of $\mathscr{B} \times \mathscr{B}$.

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