Statistical properties of a random series transmitted by filtering

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Abstract Considering a multidimensional stationary series, this work aims to present how the properties of periodicity in distribution and of ergodicity are stable by filtering.

1 Introduction

The property of weak stationarity of a multidimensional random series is transmitted by linear filtering (see, for example, Brillinger, 1981, Boudou & Dauxois, 1994). We know that the properties of periodicity in distribution and of ergodicity are stable by linear filtering for univariate random series (Priestley,1981, Papoulis & Pillai, 2002). This paper aims at developing a very general theoretical framework for which these properties stand. The random series is not necessarily assumed to have a spectral density and its spectrum may be any. It takes values in a separable Hilbert space, of any dimension, non necessarily finite.

The Hilbert spaces encountered in this text, H, H', H'', of type $L^2_H(\Omega, \mathcal{A}, P)$ (which we denote $L^2_H(\mathcal{A})$ when there is no ambiguity, and $L^2(\mathcal{A})$ when $H = \mathbb{C}$), are assumed to be separable. So they are equiped with an orthonormal basis and we may consider Hilbert-Schmidt operators. Of course, if $\mathcal{A}'$ is a sub-$\sigma-$field of $\mathcal{A}$, $L^2_H(\mathcal{A}')$ is separable as a closed sub-space of $L^2_H(\mathcal{A})$. If a Hilbert space is separable, it is also the case for any isometric Hilbert space. When $\mathcal{X}$ is an element of $\mathcal{L}^2_H(E, \tau, \eta)$, that is a map from E into H, measurable and of $\eta-$integrable squared norm, we note X or $\overline{\mathcal{X}}$, its equivalence class, which is consequently an element of $L^2_H(E, \tau, \eta)$. When the context is obvious, we will not make difference between the two notation.

Theorem 1 *If X is an element of $L^2_H(E, \tau, \eta)$, then*
i) for any y of $\mathcal{L}^2_H(E, \tau, \eta)$, the map $\mathcal{X}(.)y(.)$ is measurable and of $\eta-$integrable

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norm;
ii) the map $\widetilde{X} : y \in L^2(E,\tau,\eta) \mapsto \int \mathcal{X}(.)y(.)d\eta(.) \in H$ *is a Hilbert-Schmidt operator;*
iii) for any h *from* H, $\widetilde{X}^* h$, *element of* $L^2(E,\tau,\eta)$, *has as representative the map* $\langle h, \mathcal{X}(.)\rangle$.

Definition 1 We say that a family $(X_n)_{n\in\mathbb{Z}}$ of elements of $L^2_H(\mathcal{A})$ is a $H-$stationary series when, for any pair (n,m) of elements of $\mathbb{Z}$, we have $\widetilde{X}_n \circ \widetilde{X}^*_m = \int \mathcal{X}_m(.) \otimes \mathcal{X}_n(.)\mathrm{d}P = \widetilde{X}_{n-m} \circ \widetilde{X}^*_0$.

Let us note that when $H = \mathbb{C}$ and when $E(X_n) = 0$, we find the classical definition of the weak stationarity: $\operatorname{cov}(X_n, X_m) = \operatorname{cov}(X_{n-m}, X_0)$.

In this text, we study the transmission by filtering of properties which are statistically of interest, as that of periodicity in distribution, strict stationarity, and ergodicity. As we situate our work in a multidimensional context, these results apply, for example, to Principal Components Analysis in the frequency domain.

We will often use the following result.

Theorem 2 *If* T *is a measurable map from* (E,τ,η) *into* (E',τ'), *then the map* $\overline{f} \in L^2_H(E',\tau',T\eta) \mapsto \overline{f\circ T} \in L^2_H(E,T^{-1}\tau,\eta)$ *is an isometry.*

Remark 1 . $T\eta : A' \in \tau' \mapsto \eta T^{-1}A' \in [0;1]$. Of course, when E' is an Hilbert space, we speak about distribution.

2 The spaces $H^{\mathbb{Z}}$ and $\mathcal{H}^{\mathbb{Z}}$

In this section, we define the $\sigma-$field $\mathcal{B}^H$ of subsets of $H^{\mathbb{Z}}$, and we study the maps $T^H_n : (h_p)_{p\in\mathbb{Z}} \in H^{\mathbb{Z}} \mapsto h_n \in H$ and $\theta^H_n : (h_p)_{p\in\mathbb{Z}} \in H^{\mathbb{Z}} \mapsto (h_{p+n})_{p\in\mathbb{Z}} \in H^{\mathbb{Z}}$, which allow us define later the trajectory of a process.

Lemma 1 *For any pair* (n,m) *of elements of* $\mathbb{Z}$, *we have* $T^H_n \circ \theta^H_m = T^H_{n+m}$ *and* $\theta^H_n \circ \theta^H_m = \theta^H_{n+m}$.

Let us now examine the $\sigma-$field $\mathcal{B}^H$ of subsets of $H^{\mathbb{Z}}$ generated by the family $\{(T^H_n)^{-1}B; (n,B) \in \mathbb{Z}\times\mathcal{B}_H\}$ ($\mathcal{B}_H$ is the Borel $\sigma-$field of H). This $\sigma-$field is sometimes named cylindric measurable $\sigma-$field.

Theorem 3 *i) The* $\sigma-$*field* $\mathcal{B}^H$ *is the smallest* $\sigma-$*field of subsets of* $H^{\mathbb{Z}}$ *which makes the maps* T^H_n *measurable;*
ii) a map f *from* (E,τ) *in* $H^{\mathbb{Z}}$ *is measurable if and only if, for any* n *of* $\mathbb{Z}$, $T^H_n \circ f$ *is measurable;*
iii) for any n *of* $\mathbb{Z}$, $(\theta^H_n)^{-1}\mathcal{B}^H = \mathcal{B}^H$.

The last point is easy to check: from $T^H_n\theta^H_m = T^H_{n+m}$, for any n of $\mathbb{Z}$, we can deduce, from i) and ii), that θ^H_m is measurable. So we can write that $(\theta^H_{-m})^{-1}\mathcal{B}^H \subset \mathcal{B}^H$, so that $(\theta^H_m)^{-1}(\theta^H_{-m})^{-1}\mathcal{B}^H \subset (\theta^H_m)^{-1}\mathcal{B}^H \subset \mathcal{B}^H$, that is $(\theta^H_m)^{-1}\mathcal{B}^H = \mathcal{B}^H$.

Let us now study a bijection from $H^{\mathbb{Z}}$ on $\mathcal{H}^{\mathbb{Z}}$, where $\mathcal{H}$ is the Hilbert space H^p, p integer greater or equal to 2. For this, let us consider the maps
$\overline{p} : n \in \mathbb{Z} \mapsto n - p[\frac{n}{p}] + 1 \in \{1, \ldots, p\}$, where $[x]$ is the integer part of x,
$K_j^H : h \in H \mapsto (\delta_{jl} h)_{l=1,\ldots,p} \in \mathcal{H}$,
$\mathcal{F} : h \in H^{\mathbb{Z}} \mapsto ((\sum_{j=1}^{p} K_j^H \circ T_{pn+j-1}^H) h)_{n\in\mathbb{Z}} \in \mathcal{H}^{\mathbb{Z}}$
and $\mathcal{L} : h \in (\mathcal{H})^{\mathbb{Z}} \mapsto ((K_{\overline{p}n}^H)^* \circ T_{[\frac{n}{p}]}^{\mathcal{H}} h)_{n\in\mathbb{Z}} \in H^{\mathbb{Z}}$.

We first remark that $(K_j^H)^* \circ K_{j'}^H = \delta_{jj'} I_H$, for any pair (j, j') of elements of $\{1, \ldots, p\}$ and that $\sum_{j=1}^{p} K_j^H \circ (K_j^H)^* = I_{\mathcal{H}}$.

If we note that $T_n^{\mathcal{H}} \circ \mathcal{F} = \sum_{j=1}^{p} K_j^H \circ T_{pn+j-1}^H$ and that $T_n^H \circ \mathcal{L} = (K_{\overline{p}n}^H)^* \circ T_{[\frac{n}{p}]}^{\mathcal{H}}$, for any n of $\mathbb{Z}$, we deduce that $\mathcal{F}$ and $\mathcal{L}$ are measurable. Moreover, for any n of $\mathbb{Z}$, we have $T_n^{\mathcal{H}} \circ \mathcal{F} \circ \mathcal{L} = \sum_{j=1}^{p} K_j^H \circ T_{pn+j-1}^H \circ \mathcal{L} = \sum_{j=1}^{p} K_j^H \circ (K_j^H)^* \circ T_n^{\mathcal{H}} = T_n^{\mathcal{H}}$, and $T_n^H \circ \mathcal{L} \circ \mathcal{F} = (K_{\overline{p}n}^H)^* \circ T_{[\frac{n}{p}]}^{\mathcal{H}} \circ \mathcal{F} = \sum_{j=1}^{p} (K_{\overline{p}n}^H)^* \circ K_j^H \circ T_{p[\frac{n}{p}]+j-1}^H = T_n^H$.

Theorem 4 *The maps $\mathcal{F}$ and $\mathcal{L}$ are inverse one each other: $\mathcal{F} \circ \mathcal{L} = I_{\mathcal{H}^{\mathbb{Z}}}$ and $\mathcal{L} \circ \mathcal{F} = I_{H^{\mathbb{Z}}}$. Moreover, $\mathcal{F} \circ \theta_p^H = \theta_1^{\mathcal{H}} \circ \mathcal{F}$ and $\theta_p^H \circ \mathcal{L} = \mathcal{L} \circ \theta_1^{\mathcal{H}}$.*

3 $H-$stationarity

Let us now study the various operators which we can associate with a $H-$stationary series $(X_n)_{n\in\mathbb{Z}}$ and define the derived filtered $H'-$stationary series.

Definition 2 We name shift operator of $(X_n)_{n\in\mathbb{Z}}$ any unitary operator U of $L^2(\Omega, \mathcal{A}, P)$ such that $U \circ \widetilde{X}_n^* = \widetilde{X}_{n+1}^*$, for any n of $\mathbb{Z}$.

Let us note that any $H-$stationary series $(X_n)_{n\in\mathbb{Z}}$ is associated with a shift operator. Indeed, for any pair $((n_1, h_1), (n_2, h_2))$ of elements of $\mathbb{Z} \times H$, we have
$\langle \widetilde{X}_{n_1}^* h_1, \widetilde{X}_{n_2}^* h_2 \rangle = \langle \widetilde{X}_{n_2} \circ \widetilde{X}_{n_1}^* h_1, h_2 \rangle = \langle \widetilde{X}_{n_2+1} \circ \widetilde{X}_{n_1+1}^* h_1, h_2 \rangle = \langle \widetilde{X}_{n_1+1}^* h_1, \widetilde{X}_{n_2+1}^* h_2 \rangle$.

So there exists an isometry (and only one) V from $\overline{\text{vect}}\{\widetilde{X}_n^* h; (n, h) \in \mathbb{Z} \times H\}$ on $\overline{\text{vect}}\{\widetilde{X}_{n+1}^* h; (n, h) \in \mathbb{Z} \times H\}$ such that $V(\widetilde{X}_n^* h) = \widetilde{X}_{n+1}^* h$, for any (n, h) of $\mathbb{Z} \times H$.

The following property comes from the relation $\overline{\text{vect}}\{\widetilde{X}_n^* h; (n, h) \in \mathbb{Z} \times H\} = \overline{\text{vect}}\{\widetilde{X}_{n+1}^* h; (n, h) \in \mathbb{Z} \times H\}$.

Theorem 5 *There exists a unitary operator V of $H' = \overline{vect}\{\widetilde{X}_n^* h; (n, h) \in \mathbb{Z} \times H\}$ such that $V(\widetilde{X}_n^* h) = \widetilde{X}_{n+1}^* h$, for any (n, h) of $\mathbb{Z} \times H$.*

Let P be the projector from $L^2(\mathcal{A})$ on H' and by L the canonical injection $y \in H' \mapsto y \in L^2(\mathcal{A})$. Then $U = P_\perp + L \circ V \circ L^*$ is a unitary operator of $L^2(\mathcal{A})$ such that $U(\widetilde{X}_n^* h) = \widetilde{X}_{n+1}^* h$, for any (n, h) of $\mathbb{Z} \times H$, so $U \circ \widetilde{X}_n^* = \widetilde{X}_{n+1}^*$, for any n of $\mathbb{Z}$.

The shift operator is not unique. Moreover, with a double induction, we can show that $U^n \circ \widetilde{X}_0^* = \widetilde{X}_n^*$, for any n of $\mathbb{Z}$. We name it shift operator because when $(X_n)_{n\in\mathbb{Z}}$ is unidimensional ($H = \mathbb{C}$) and taking real values, then $UX_n = X_{n+1}$.

From the shift operator U and $(X_n)_{n\in\mathbb{Z}}$, we define a whole family of shift operators. For this, we use the notion of ampliation of a bounded operator of $L^2(\mathcal{A})$.

Definition 3 We name ampliation of an element A of $L(L^2(\mathcal{A}))$ the map A^H defined by $K \in \sigma_2(L^2(\mathcal{A}), H) \mapsto K \circ A^* \in \sigma_2(L^2(\mathcal{A}), H)$.

Let us give some properties of the ampliation.

Theorem 6 *i) For any A of $L(L^2(\mathcal{A}))$, A^H is linear and bounded.*
ii) For any A of $L(L^2(\mathcal{A}))$, $(A^H)^* = (A^*)^H$.
iii) For any pair (A, B) of elements of $L(L^2(\mathcal{A}))$, $(A \circ B)^H = A^H \circ B^H$.
iv) $(I_{L^2(\mathcal{A})})^H = I_{\sigma_2}$.

By $\mathcal{I}_{H'}$ we denote the isometry $X' \in L^2_{H'}(\mathcal{A}) \mapsto \widetilde{X'} \in \sigma_2(L^2(\mathcal{A}), H')$. Let us consider $U_{H'} = \mathcal{I}_{H'} \circ U^{H'} \circ \mathcal{I}_{H'}$, which is clearly a unitary operator.

Theorem 7 *i) For any n of $\mathbb{Z}$, $U^n_H X_0 = X_n$.*
ii) For any (K, X') of $L(H', H'') \times L^2_{H'}(\mathcal{A})$, $U_{H''}(K \circ X') = K \circ (U_{H'} X')$.

Let us briefly examine the proves of these properties. They come from the following.

$$U^n_H X_0 = \mathcal{I}^*_H \circ (U^H)^n \circ \mathcal{I}_H X_0 = \mathcal{I}^*_H (\widetilde{X_0} \circ U^{-n}) = \mathcal{I}^*_H (\widetilde{X_n}) = X_n;$$
$$\widetilde{U_{H''}(K \circ X')} = K \circ \widetilde{X'} \circ U^{-1} = K \circ (\widetilde{U_{H'} X'}) = K \circ \widetilde{(U_{H'} X')}.$$

These tools let us define the filter of $(X_n)_{n \in \mathbb{Z}}$.

Definition 4 A filter of $(X_n)_{n\in\mathbb{Z}}$ is a series of the type $(U^n_{H'} X')_{n\in\mathbb{Z}}$, where X' belongs to $L^2_{H'}(\mathcal{A})$.

Theorem 8 *A filter $(U^n_{H'} X')_{n\in\mathbb{Z}}$ of $(X_n)_{n\in\mathbb{Z}}$ is a $H'-$stationary series, and U is its shift operator.*

Indeed, for any n of $\mathbb{Z}$, we have

$$\widetilde{U^n_{H'} X'} = (U^{H'})^n \widetilde{X'} = \widetilde{X'} \circ U^{-n}. \qquad (1)$$

From (1), we deduce that $(U^n_{H'} X')_{n\in\mathbb{Z}}$ is a $H'-$stationary series:
$\widetilde{U^n_{H'} X'} \circ \widetilde{U^m_{H'} X'}^* = \widetilde{X'} \circ U^{-n} \circ U^m \circ \widetilde{X'}^* = \widetilde{X'} \circ U^{m-n} \circ \widetilde{X'}^* = \widetilde{U^{n-m}_{H'} X'} \circ \widetilde{U^0_{H'} X'}^*$,
and that U is a shift operator of $(U^n_{H'} X')_{n\in\mathbb{Z}}$: $U^n \circ \widetilde{X'}^* = U^n \circ \widetilde{U^0_{H'} X'}^* = \widetilde{U^n_{H'} X'}^*$.

Such a filter can have the apearance of a moving average. Indeed, let $\{A_p; p \in \mathbb{Z}\}$ be a family of elements of $L(H, H')$ such that $\{A_p \circ X_{-p}; p \in \mathbb{Z}\}$ is a summable family of elements of $L^2_{H'}(\mathcal{A})$, of sum X'_0, we can affirm that $\{U^n_{H'}(A_p \circ X_{-p}); p \in \mathbb{Z}\}$ is a summable family of sum $U^n_{H'} X'_0$, because $U^n_{H'}$ is an isometry. For any p of $\mathbb{Z}$, we have $U^n_{H'}(A_p \circ X_{-p}) = A_p \circ X_{n-p}$.

4 Trajectory

Let $(\mathcal{X}_n)_{n\in\mathbb{Z}}$ be a series of elements of $\mathcal{L}^2_H(\mathcal{A})$ such that $\overline{\mathcal{X}_n} = X_n$, for any n of $\mathbb{Z}$.

Definition 5 We name trajectory of the $H-$stationary series $(X_n)_{n\in\mathbb{Z}}$ the map $\widehat{X}$: $\omega \in \Omega \mapsto (\mathcal{X}_n(\omega))_{n\in\mathbb{Z}} \in H^{\mathbb{Z}}$.

This trajectory may be considered as a random value, as it is measurable: $T_n^H \circ \widehat{X} = \mathcal{X}_n$. It is relevant to notice that if $(\mathcal{X}'_n)_{n\in\mathbb{Z}}$ is another series of elements of $\mathcal{L}^2_H(\mathcal{A})$ such that $\overline{\mathcal{X}'_n} = X_n$, for any n of $\mathbb{Z}$, that is such that $\mathcal{X}_n = \mathcal{X}'_n$ $P-$almost anywhere, then for $P-$almost any ω of Ω, $\widehat{X}\omega = \widehat{X}'\omega$, and then $\widehat{X}P = \widehat{X}'P$. Thus the image probability $\widehat{X}P$ does not depend on the choice of the representative $\mathcal{X}_n$.

Definition 6 We say that the $H-$stationary series $(X_n)_{n\in\mathbb{Z}}$ has got an order p periodicity in distribution when $\theta_p^H \widehat{X}P = \widehat{X}P$.

Especially, for any n of $\mathbb{Z}$, $\mathcal{X}_{n+p}(P) = T^H_{n+p}\widehat{X}P = T_n^H\theta_p^H\widehat{X}P = T_n^H\widehat{X}P = \mathcal{X}_n(P)$. When $p = 1$, we talk about strict stationarity. Classically, strict stationarity is defined differently, but in an equivalent way.

For any n of $\mathbb{Z}$, we have $(T_n^H)^{-1}\mathcal{B}_H \subset \mathcal{B}^H$, so $\mathcal{X}_n^{-1}\mathcal{B}_H = \widehat{X}^{-1}(T_n^H)^{-1}\mathcal{B}_H \subset \widehat{X}^{-1}\mathcal{B}^H$. Then the family $\{\mathcal{X}_n; n \in \mathbb{Z}\}$ is made of elements of $\mathcal{L}^2_H(\Omega, \widehat{X}^{-1}\mathcal{B}^H, P)$ and the family $\{X_n; n \in \mathbb{Z}\}$ of elements of $L^2_H(\Omega, \widehat{X}^{-1}\mathcal{B}, P)$. From Theorem 2, we can affirm the following.

Theorem 9 *The map* $T_{H'} : \overline{f} \in L^2_{H'}(H^{\mathbb{Z}}, \mathcal{B}^H, \widehat{X}P) \mapsto \overline{f \circ \widehat{X}} \in L^2_{H'}(\widehat{X}^{-1}\mathcal{B}^H)$ *is an isometry.*

5 Transmission of strict stationarity and ergodicity by filtering

We show here that the periodicity in distribution of order 1 and ergodicity are transmitted by filtering. Let $(X_n)_{n\in\mathbb{Z}}$ be a $H-$stationary series such that $\theta_1^H\widehat{X}P = \widehat{X}P$.

As $\theta_1^H\widehat{X}P = \widehat{X}P$ and $(\theta_1^H)^{-1}\mathcal{B}^H = \mathcal{B}^H$, Theorem 2 lets us establish the following.

Corollary 1 *The map* $V : t \in L^2(H^{\mathbb{Z}}, \mathcal{B}^H, \widehat{X}P) \mapsto \overline{t \circ \theta_1^H} \in L^2(H^{\mathbb{Z}}, \mathcal{B}^H, \widehat{X}P)$ *is a unitary operator.*

We know that if T' belongs to $\mathcal{L}^2_{H'}(H^{\mathbb{Z}}, \mathcal{B}^H, \widehat{X}P)$, then $T' \circ \widehat{X}$ belongs to $\mathcal{L}^2_{H'}(\Omega, \widehat{X}^{-1}\mathcal{B}^H, P)$. So we have, between the two element, the following relation.

Lemma 2 *For any* T' *of* $\mathcal{L}^2_{H'}(H^{\mathbb{Z}}, \mathcal{B}^H, \widehat{X}P)$, $\widetilde{\overline{T'}} = \widetilde{\overline{T' \circ \widehat{X}}} \circ T_{\mathbb{C}}$.

Indeed, $\langle h', T'(.)\rangle$ is a representative of $\widetilde{\overline{T'}}^* h'$, and $\langle h', T'(.)\rangle \circ \widehat{X}$ of $T_{\mathbb{C}}(\widetilde{\overline{T'}}^* h')$. But as $\langle h', (T' \circ \widehat{X})(.)\rangle$, that is $\langle h', T'(.)\rangle \circ \widehat{X}$ is a representative of $\widetilde{\overline{T' \circ \widehat{X}}}^* h'$, we can write $\widetilde{\overline{T' \circ \widehat{X}}}^* h' = T_{\mathbb{C}}(\widetilde{\overline{T'}}^* h')$, for any h' of H', so $\widetilde{\overline{T' \circ \widehat{X}}}^* = T_{\mathbb{C}} \circ \widetilde{\overline{T'}}^*$ and then $\widetilde{\overline{T' \circ \widehat{X}}} = \widetilde{\overline{T'}} \circ T_{\mathbb{C}}^*$, that is $\widetilde{\overline{T' \circ \widehat{X}}} \circ T_{\mathbb{C}} = \widetilde{\overline{T'}}$. In the particular case where $T' = T_n^H$, we have the following.

Corollary 2 *For any* n *of* $\mathbb{Z}$, $\widetilde{\overline{T_n^H}} = \widetilde{X_n} \circ T_{\mathbb{C}}$.

If T' belongs to $\mathcal{L}^2_{H'}(H^{\mathbb{Z}}, \mathcal{B}, \widehat{X}P)$, so it is for $T' \circ \theta^H_1$. The following property establishes a relation between the two elements.

Lemma 3 *For any T' of $\mathcal{L}^2_{H'}(H^{\mathbb{Z}}, \mathcal{B}^H, \widehat{X}P)$, $V \circ \widetilde{\widetilde{T'}}^* = \widetilde{\widetilde{T' \circ \theta^H_1}}^*$.*

The element $\widetilde{\widetilde{T'}}^* h'$, of $L^2_{H'}(H^{\mathbb{Z}}, \mathcal{B}, \widehat{X}P)$, has as representative $\langle h', T'(.)\rangle$, so $V \circ \widetilde{\widetilde{T'}}^* h'$ has as representative $\langle h', T'(.)\rangle \circ \theta^H_1$.

As for $\widetilde{\widetilde{T' \circ \theta^H_1}}^* h'$, one of its representatives is $\langle h', T' \circ \theta^H_1(.)\rangle$, but as $\langle h', T'(.)\rangle \circ \theta^H_1 = \langle h', T' \circ \theta^H_1(.)\rangle$, we deduce that $V \circ \widetilde{\widetilde{T'}}^* h' = \widetilde{\widetilde{T' \circ \theta^H_1}}^* h'$. As it is exact for any h' of H', we have $V \circ \widetilde{\widetilde{T'}}^* = \widetilde{\widetilde{T' \circ \theta^H_1}}^*$.

Lemma 3 and corollary 2 let us find a shift operator of the $H-$stationary series $(X_n)_{n\in\mathbb{Z}}$.

Lemma 4 *$T_{\mathbb{C}} \circ V \circ T^*_{\mathbb{C}}$ is a shift operator of $(X_n)_{n\in\mathbb{Z}}$.*

Indeed, $T_{\mathbb{C}} \circ V \circ T^*_{\mathbb{C}} \circ \widetilde{X_n}^* = T_{\mathbb{C}} \circ V \circ \widetilde{\widetilde{T^H_n}}^* = T_{\mathbb{C}} \circ \widetilde{\widetilde{T^H_n \circ \theta^H_1}}^* = T_{\mathbb{C}} \circ \widetilde{\widetilde{T^H_{n+1}}}^* = \widetilde{X_{n+1}}^*$.

From these properties, we can prove the following.

Theorem 10 *For any T' of $\mathcal{L}^2_{H'}(H^{\mathbb{Z}}, \mathcal{B}^H, \widehat{X}P)$, $U_{H'}\overline{(T' \circ \widehat{X})} = \overline{T' \circ \theta^H_1 \circ \widehat{X}}$.*

Indeed, $U_{H'}\overline{(T' \circ \widehat{X})} = \mathcal{I}^*_{H'} \circ (T_{\mathbb{C}} \circ V \circ T^*_{\mathbb{C}})^{H'} \circ \mathcal{I}_{H'}\overline{T' \circ \widehat{X}} = \mathcal{I}^*_{H'}(\widetilde{\overline{T' \circ \widehat{X}}} \circ T_{\mathbb{C}} \circ V^{-1} \circ T^*_{\mathbb{C}}) = \mathcal{I}^*_{H'}(\widetilde{\widetilde{T'}} \circ V^{-1} \circ T^*_{\mathbb{C}}) = \mathcal{I}^*_{H'}(\widetilde{\widetilde{T' \circ \theta^H_1}} \circ T^*_{\mathbb{C}}) = \mathcal{I}^*_{H'}(\widetilde{\overline{T' \circ \theta^H_1 \circ \widehat{X}}}) = \overline{T' \circ \theta^H_1 \circ \widehat{X}}$.

With a double induction, we can generalize the previous result.

Theorem 11 *For any T' of $\mathcal{L}^2_{H'}(H^{\mathbb{Z}}, \mathcal{B}^H, \widehat{X}P)$ and for any n of $\mathbb{Z}$, we have $U^n_{H'}\overline{T' \circ \widehat{X}} = \overline{T' \circ \theta^H_n \circ \widehat{X}}$.*

Now we have got the necessary tools to examine the trajectory $\widehat{X'}$ of a filter $(U^n_{H'}X')_{n\in\mathbb{Z}}$ as a function of the trajectory $\widehat{X}$ of the $H-$stationary series $(X_n)_{n\in\mathbb{Z}}$.

Theorem 12 *There exists a measurable map F from $H^{\mathbb{Z}}$ in $H'^{\mathbb{Z}}$ such that*
i) $\widehat{X'} = F \circ \widehat{X}$;
ii) $\theta^{H'}_1 \circ F = F \circ \theta^H_1$.

Let then $(U^n_{H'}X')_{n\in\mathbb{Z}}$ be a $H'-$stationary series, filter of $(X_n)_{n\in\mathbb{Z}}$. Let T' be an element of $\mathcal{L}^2_{H'}(H^{\mathbb{Z}}, \mathcal{B}^H, \widehat{X}P)$ such that $T_{H'}\overline{T'} = \overline{T' \circ \widehat{X}} = X'_0$.

Let us consider the map $F : h \in H^{\mathbb{Z}} \mapsto (T'\theta^H_n h)_{n\in\mathbb{Z}} \in H'^{\mathbb{Z}}$. This map is measurable, because for any n of $\mathbb{Z}$, we have $T^{H'}_n \circ F = T' \circ \theta^H_n$, and it is such that $\theta^{H'}_1 \circ F = F \circ \theta^H_1$ ($F \circ \theta^H_1 h = (T'\theta^H_n \theta^H_1 h)_{n\in\mathbb{Z}} = (T'\theta^H_{n+1} h)_{n\in\mathbb{Z}} = \theta^{H'}_1(T'\theta^H_n h)_{n\in\mathbb{Z}} = \theta^{H'}_1 F h$).

For any n of $\mathbb{Z}$, we have $U^n_{H'}X'_0 = U^n_{H'}\overline{T' \circ \widehat{X}} = \overline{T' \circ \theta^H_n \circ \widehat{X}}$.

So $T' \circ \theta^H_n \circ \widehat{X}$, element of $\mathcal{L}^2_{H'}(\Omega, \widehat{X}^{-1}\mathcal{B}^H, P)$, is a representative of $U^n_{H'}X'_0$, element of $L^2_{H'}(\Omega, \widehat{X}^{-1}\mathcal{B}^H, P)$. The trajectory $\widehat{X'}$ of $(U^n_{H'}X'_0)_{n\in\mathbb{Z}}$ is then such that $\widehat{X'}(\omega) = (T'\theta^H_1 \widehat{X}\omega)_{n\in\mathbb{Z}} = F\widehat{X}\omega$, for any ω of Ω, so we have $\widehat{X'} = F \circ \widehat{X}$.

This factorization lets us prove the strict stationarity of the filtered series.

Corollary 3 $\theta_1^{H'}\widehat{X'}P = \widehat{X'}P$.

Indeed, $\theta_1^{H'}\widehat{X'}P = \theta_1^{H'}F\widehat{X}P = F\theta_1^{H}\widehat{X}P = F\widehat{X}P = \widehat{X'}P$.

We recall that a $H-$stationary series $(X_n)_{n\in\mathbb{Z}}$ is said to be ergodic when, for any B of $\mathcal{B}^H$, $P(\widehat{X}^{-1}B\Delta(\theta_1^H)^{-1}\widehat{X}^{-1}B) = 0$ implies either $\widehat{X}PB = P\widehat{X}^{-1}B = 0$, either $\widehat{X}PB = 1$, where Δ is the symmetric difference.

Theorem 13 *If $(X_n)_{n\in\mathbb{Z}}$ is ergodic, so it is is for $(U_{H'}^n X_0')_{n\in\mathbb{Z}}$.*

For any B' of $\mathcal{B}^{H'}$, we have $\widehat{X'}P(B'\Delta(\theta_1^{H'})^{-1}B') = F\widehat{X}P(B'\Delta(\theta_1^{H'})^{-1}B') = \widehat{X}P(F^{-1}(B'\Delta(\theta_1^{H'})^{-1}B')) = \widehat{X}P(F^{-1}B'\Delta F^{-1}(\theta_1^{H'})^{-1}B') = \widehat{X}P(F^{-1}B'\Delta(\theta_1^{H})^{-1}F^{-1}B')$.

If $\widehat{X'}P(B'\Delta(\theta_1^{H'})^{-1}B') = 0$, then $\widehat{X}P(F^{-1}B'\Delta(\theta_1^{H})^{-1}F^{-1}B') = 0$.

Either $\widehat{X}PF^{-1}B' = 0$, and then $0 = F\widehat{X}PB' = \widehat{X'}PB'$, either $\widehat{X}PF^{-1}B' = 1$, and then $1 = \widehat{X'}P(B'\Delta(\theta_1^{H'})^{-1}B') = F\widehat{X}PB' = \widehat{X'}PB' = 0$.

6 Deployment

Definition 7 We name deployment of order p of the $H-$stationary series $(X_n)_{n\in\mathbb{Z}}$ the series $(Y_n)_{n\in\mathbb{Z}} = (\sum_{j=1}^p K_j^H \circ X_{pn+j-1})_{n\in\mathbb{Z}}$.

It is easy to establish the following.

Theorem 14 *The deployment of order p of a $H-$stationary series is a $\mathcal{H}-$stationary series.*

From $Y_n(\omega) = \sum_{j=1}^p K_j^H \circ X_{pn+j-1}(\omega) = \sum_{j=1}^p K_j^H \circ T_{pn+j-1}X(\omega)$, we deduce, with obvious notation, that

$$\widehat{Y}(\omega) = (Y_n(\omega))_{n\in\mathbb{Z}} = ((\sum_{j=1}^p K_j^H \circ T_{pn+j-1})X(\omega))_{n\in\mathbb{Z}} = \mathcal{F} \circ \widehat{X}(\omega).$$

Hence the relations between the trajectories of $(X_n)_{n\in\mathbb{Z}}$ with that of its deployment of order p $(Y_n)_{n\in\mathbb{Z}}$:

$$\widehat{Y} = \mathcal{F} \circ \widehat{X} \text{ and } \mathcal{L} \circ \widehat{Y} = \widehat{X}.$$

Theorem 15 *If U is a shift operator of $(X_n)_{n\in\mathbb{Z}}$, then U^p is a shift operator of $(Y_n)_{n\in\mathbb{Z}}$ issued from a deployment of order p.*

Indeed, $U^p \circ \widetilde{Y_n}^* = U^p \circ \sum_{j=1}^p \widetilde{X}_{pn+j-1}^* \circ (K_j^H)^* = \sum_{j=1}^p \widetilde{X}_{pn+p+j-1}^* \circ (K_j^H)^* = \sum_{j=1}^p \widetilde{X}_{p(n+1)+j-1}^* \circ (K_j^H)^* = \widetilde{Y}_{n+1}^*$.

Let us consider a $H'-$stationary series $(X_n')_{n\in\mathbb{Z}}$, filtered from $(X_n)_{n\in\mathbb{Z}}$, $X_n' = U_{H'}^n X_0'$. With obvious notation, $(K_j^{H'}h' = (\delta_{jl}h')_{l=1,\ldots,p})$, its deployment of order p is the series $(Y_n')_{n\in\mathbb{Z}} = (\sum_{j=1}^p K_j^{H'} \circ X_{pn+j-1}')_{n\in\mathbb{Z}}$.

Theorem 16 *The $\mathcal{H}'-$stationary series ($\mathcal{H}' = H'^p$) $(U_{\mathcal{H}'}^n Y_0')_{n\in\mathbb{Z}}$ is a filtered series of the $\mathcal{H}-$stationary series $(Y_n)_{n\in\mathbb{Z}}$.*

Indeed, for any n of $\mathbb{Z}$, we have $U_{\mathcal{H}'}^n Y_0' = (\mathcal{I}_{\mathcal{H}'}^* \circ (U^p)^{\mathcal{H}'} \circ \mathcal{I}_{\mathcal{H}'})^n Y_0' = \mathcal{I}_{\mathcal{H}'}^* \circ ((U^p)^{\mathcal{H}'})^n \circ \widetilde{Y_0'} = \mathcal{I}_{\mathcal{H}'}^*((U^p)^n)^{\mathcal{H}'}\widetilde{Y_0'} = \mathcal{I}_{\mathcal{H}'}^*\widetilde{Y_0'} \circ U^{-pn} = \mathcal{I}_{\mathcal{H}'}^*\widetilde{Y_n'} = Y_n'$.

So the deployment $(Y_n')_{n\in\mathbb{Z}}$ is equal to $(U_{\mathcal{H}'}^n Y_0')_{n\in\mathbb{Z}}$, that is a filtered of the deployment $(Y_n)_{n\in\mathbb{Z}}$. We summarize these result in Figure 1.

$$\begin{array}{ccc} (X_n)_{n\in\mathbb{Z}} & \xrightarrow{\text{deployment}} & (Y_n)_{n\in\mathbb{Z}} \\ \Big\downarrow \text{filter} & & \Big\downarrow \text{filter} \\ (X'_n)_{n\in\mathbb{Z}} & \xrightarrow{\text{deployment}} & (Y'_n)_{n\in\mathbb{Z}} \end{array}$$

Fig. 1 Summary of relations deployment-filtering

7 Transmission of the periodicity in distribution

Let us now assume that $(X_n)_{n\in\mathbb{Z}}$ is periodic of order p in distribution, so $\theta_p^H\widehat{X}P = \widehat{X}P$.

The trajectory $\widehat{Y}$ of the deployment of order p is such that $\widehat{Y} = \mathcal{F}\circ\widehat{X}$.

So we have $\theta_1^{\mathcal{H}}\widehat{Y}P = \theta_1^{\mathcal{H}}\mathcal{F}\widehat{X}P = \mathcal{F}\theta_p^H\widehat{X}P = \mathcal{F}\widehat{X}P = \widehat{Y}P$.

This means that the series $(Y_n)_{n\in\mathbb{Z}}$, deployment of order p of $(X_n)_{n\in\mathbb{Z}}$, is periodic in distribution of order 1, and then so it is for the filter $(Y'_n)_{n\in\mathbb{Z}}$ of $(Y_n)_{n\in\mathbb{Z}}$ (the periodicity in distribution of order 1 is transmitted by filtering).

So we have $\theta_1^{\mathcal{H}'}\widehat{Y'}P = \widehat{Y'}P$ (with obvious notation, $\mathcal{F}'$ is a map from $H'^{\mathbb{Z}}$ into $\mathcal{H}'^{\mathbb{Z}}$ and $\mathcal{L}'$ its reverse). So the trajectory $\widehat{X'}$ of $(X'_n)_{n\in\mathbb{Z}}$ is such that $\widehat{X'} = \mathcal{L}'\circ\widehat{Y'}$.

Then we have $\theta_p^{H'}\widehat{X'}P = \theta_p^{H'}\mathcal{L}'\widehat{Y'}P = \mathcal{L}'\theta_1^{\mathcal{H}'}\widehat{Y'}P = \mathcal{L}'\widehat{Y'}P = \widehat{X'}P$. As a conclusion, the filtered series $(Y'_n)_{n\in\mathbb{Z}}$ is periodic of order p in distribution. The periodicity in distribution of order p is transmitted by filtering. More generally, we can prove that the strict stationarity is also transmitted by filtering for any series $(X_g)_{g\in G}$, where G is an abelian locally compact group (e.g. $\mathbb{R}^k, \mathbb{Z}^k$).

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