Frequency Analysis of a Cyclostationary Random Function

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We develop and extend the principal components analysis (PCA) of a cyclostationary random function $(X_t)_{t\in\mathbb{R}}$, that is a function such that $\mathrm{cov}(X_t, X_{t'}) = \mathrm{cov}(X_{t+\Delta}, X_{t'+\Delta})$, for any (t, t') of $\mathbb{R}\times\mathbb{R}$. This property of cyclostationarity for random functions, also refered as periodically correlated, is encountered in various phenomena where some statistics present a periodicity. We can find examples of study of this property in Gardner [5] for telecommunications, Randall et al. [9] for mechanic transmission, Weber et al. [11] for radioastronomy, Zakaria [13] for locomotion, or Roussel [10] for medicine. The earliest mention of such processes can be found in Voychishin et al [12], which gives an english translation of articles first published ind 1957 and 1960. The mathematical formulation has first been given by Gladyshev [6], and then largely developped by Hurd [7], Hurd et al. [8], and more recently by Bouleux et al. [4], in a multidimentional context.

In this presentation, from such a random function, we define a series $(Y_n)_{n\in\mathbb{N}}$ of random functions, where $Y_n = (X_{n+t})_{t\in[0;\Delta[}$. This series is multidimentional, it takes values in the Hilbert space $L^2([0;\Delta[)$, and it is stationary. This stationarity lets us proceed to Principal Components Analysis in the frequency domain, which is more powerful than Principal Components Analysis of each of the random vectors Y_n.

In Boudou and Viguier-Pla [3], this method of PCA is exposed for most common cyclostationary functions. In this presentation, we make use of more complex mathematical tools for an extension of the field of work.

We generalize the set of the indexes $\mathbb{R}$ to an abelian group G_1, which can be $\mathbb{Z}$, $\mathbb{Z}/k\mathbb{Z}$, $\mathbb{Z}^k$, $\mathbb{R}^k$, and as for the subgroup $\Delta\mathbb{Z}$ of $\mathbb{R}$, it becomes a subgroup G of G_1. Then we say that a random function $(X_g)_{g\in G_1}$ is cyclostationary when $\mathrm{cov}(X_{g_1}, X_{g_2}) = \mathrm{cov}(X_{g_1+g}, X_{g_2+g})$, for any (g_1, g_2, g) of $G_1\times G_1\times G$.

Of course, the type of cyclostationarity is linked with the choice of the sub-group G of G_1. We get, as particular cases, various forms of cyclostationarity more or less known.

In this work, the new skill is based on the fact that with a cyclostationary function we associate a stationary multidimensional random function, and then lets us use the powerful tool of reduction of the data which is PCA in the frequency domain (cf. Brillinger [1] and Boudou and Dauxois [2]). Let us finaly mention a result which interest is only theoretical: with cyclostationary random function we associate, in a biunivoque way, a spectral measure, as well as with a stationary continuous random function, we associate a random measure from which it is the Fourier transform.

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