Chapter 12

Application of convergent sequences of projectors to stationary processes

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Abstract We develop a measure of the gap between two projectors, based on the convergence of sequences of projectors. This allows us to define the gap between two spectral measures, and develop several properties associated with the latter. In particular, we establish the continuity of the convolution product.

12.1 Introduction

The set of projectors plays an important role in statistics and probability (cf. Romain [6]). It can be equipped (cf. Halmos [5]) with a relation of partial order. From this one, we define a pseudo-distance between projectors and a new type of convergence of projectors which mutually legitimate one another: a sequence of projectors $(P_n)_{n\in\mathbb{N}}$ converges to the projector P if and only if $(d(P_n, P))_{n\in\mathbb{N}}$, sequence of the pseudo-distances between P and P_n, converges to 0.
What precedes allows us to define the gap between two spectral measures $\mathcal{E}$ and $\mathcal{E}'$, projector-valued, that is the upper bound of the family of the projectors $\mathrm{d}(\mathcal{E}A, \mathcal{E}'A)$. We can then establish several properties of continuity associated with the spectral measures. These spectral measures are very often used tools in mathematics of stationary processes.
Then, we examine the notion of maximal egalizator of two unitary operators U and U'. Briefly speaking, it can be defined as the projector on the greater vector subspace on which the operators U and U' are equal. If we denote it by R, we show that $I - R$ is the gap, as previously defined, between the spectral measures $\mathcal{E}$ and $\mathcal{E}'$ respectively associated with the unitary operators U and U'. This gives us a close relation between two concepts defined independently from one another. We recall that the shift operators associated with a stationary process are unitary operators.
To a spectral measure we can associate (cf. Boudou [1]) a family of unit operators which has got a group structure. We show that if $\{U_g; g \in G\}$, family of the unitary operators associated with the spectral measure $\mathcal{E}$, is close to $\{U'_g; g \in G\}$,

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family of the unitary operators associated with the spectral measure $\mathcal{E}'$, then the gap between the spectral measures $\mathcal{E}$ and $\mathcal{E}'$ is small. This way of measuring the gap between spectral measures is close to the problem of measuring proximity between functional data with semi-metrics. We can find a large litterature on this subject in Ferraty and Vieu [4].
We end the presentation by defining the set of the stationary processes both filtered from the stationary processes $(X_g)_{g\in G}$ and $(X'_g)_{g\in G}$.

12.2 Recalls and preliminaries

By H we will designate a $\mathbb{C}$–Hilbert, by $\mathcal{P}(H)$ the family of orthogonal projectors, and by G a locally abelian compact group, which dual $\widehat{G}$ is of countable basis. The σ–field of the borelians of $\widehat{G}$ is denoted $\mathcal{B}_{\widehat{G}}$. When $G = \mathbb{Z}$, $\widehat{G}$ is identified with $\Pi = [-\pi, \pi[$.

A random measure (r.m.) Z, on $\mathcal{B}_{\widehat{G}}$ for H, is a vector measure such that $< ZA, ZB >= 0$, when $A \cap B = \emptyset$.

Then we show the following.

The application $\mu_Z : A \in \mathcal{B}_{\widehat{G}} \mapsto \|ZA\|^2 \in \mathbb{R}_+$ is a bounded measure.
The stochastic integral, relatively to the r.m. Z, can be defined as the unique isometry from $L^2(\widehat{G}, \mathcal{B}_{\widehat{G}}, \mu_Z)$ onto $H_Z = \overline{vect}\{ZA; A \in \mathcal{B}_{\widehat{G}}\}$ which to 1_A associates ZA, this for any A of $\mathcal{B}_{\widehat{G}}$.
The image of φ, element of $L^2(\widehat{G}, \mathcal{B}_{\widehat{G}}, \mu_Z)$, by this isometry, denoted $\int \varphi dZ$, is the integral of φ relatively to the r.m. Z.

Let us recall that, with any stationary continuous random function (c.r.f.) we can associate one, and only one, r.m. from which it is the Fourier Transform.

A spectral measure (s.m.) $\mathcal{E}$, on $\mathcal{B}_{\widehat{G}}$ for H, is an application from $\mathcal{B}_{\widehat{G}}$ onto $\mathcal{P}(H)$ such that
$\mathcal{E}(G) = I$;
$\mathcal{E}(A \cup B) = \mathcal{E}A + \mathcal{E}B$, for any pair (A, B) of disjoint elements of $\mathcal{B}_{\widehat{G}}$;
$lim_n \mathcal{E}A_n X = 0$, for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}_{\widehat{G}}$ which decreasingly converges to $\emptyset$ and for any X of H.

If G' is a second group similar to G, and if f is a measurable application from $\widehat{G}$ into $\widehat{G}'$, we can define the image of a s.m..

The measure image of $\mathcal{E}$ by f is the application $f(\mathcal{E}) : A \in \mathcal{B}_{\widehat{G}'} \mapsto \mathcal{E}(f^{-1}A) \in \mathcal{P}(H)$; it is a s.m., on $\mathcal{B}_{\widehat{G}'}$ for H.

To a s.m. we can associate a family of r.m.'s as follows.

If $\mathcal{E}$ is a s.m. on $\mathcal{B}_{\widehat{G}'}$ for H, then for any X of H, the application $Z_{\mathcal{E}}^X : A \in \mathcal{B}_{\widehat{G}} \mapsto \mathcal{E}AX \in H$ is a r.m..

To any unitary operator U we can associate biunivocally a s.m. $\mathcal{E}$ on $\mathcal{B}_{\Pi}$ for H such that

$UX = \int e^{i.1} dZ_{\mathcal{E}}^X$, for any X of H.

For any g of G, we denote by h_g the measurable application from $\widehat{G}$ in Π such that $e^{ih_g(\gamma)} = (\gamma, g)_{\widehat{G}G}$, this for any γ of $\widehat{G}$. So the following stands.

If U_g is the unitary operator of H of associated s.m. $h_g\mathcal{E}$, where $\mathcal{E}$ is a s.m. on $\mathcal{B}_{\widehat{G}}$ for H, then $\{U_g; g \in G\}$ is the group of the unitary operators of H deduced from the s.m. $\mathcal{E}$.

When $\mathcal{E}_1$ and $\mathcal{E}_2$ are two s.m.'s, on $\mathcal{B}_{\widehat{G}}$ for H, which commute, that is when they are such that, for any pair (A_1, A_2) of elements of $\mathcal{B}_{\widehat{G}}$, the projectors $\mathcal{E}_1A_1$ and $\mathcal{E}_2A_2$ commute, we have the following.

There exists one, and only one, s.m., denoted $\mathcal{E}_1 \otimes \mathcal{E}_2$, on $\mathcal{B}_{\widehat{G}} \otimes \mathcal{B}_{\widehat{G}} = \mathcal{B}_{\widehat{G\times G}}$ for H, such that $\mathcal{E}_1 \otimes \mathcal{E}_2(A_1 \times A_2) = \mathcal{E}_1A_1\mathcal{E}_2A_2$, for any (A_1, A_2) of $\mathcal{B}_{\widehat{G}} \times \mathcal{B}_{\widehat{G}}$. The image of $\mathcal{E}_1 \otimes \mathcal{E}_2$ by the measurable application $S : (\gamma_1, \gamma_2) \in \widehat{G} \times \widehat{G} \mapsto \gamma_1 + \gamma_2 \in \widehat{G}$, denoted $\mathcal{E}_1 * \mathcal{E}_2$, is named convolution product of $\mathcal{E}_1$ and $\mathcal{E}_2$.

If we denote by δ_0 the Dirac measure concentrated on 0, the following is easy to verify.

The application $\mathcal{E}_{\widehat{G}} : A \in \mathcal{B}_{\widehat{G}} \mapsto \delta_0(A)I \in \mathcal{P}(H)$, is a s.m. on $\mathcal{B}_{\widehat{G}}$ for H, which commutes with any s.m. $\mathcal{E}$, on $\mathcal{B}_{\widehat{G}}$ for H, and is such that $\mathcal{E} * \mathcal{E}_{\widehat{G}} = \mathcal{E}$.

The application $w : \gamma \in \widehat{G} \mapsto -\gamma \in \widehat{G}$ is measurable and we have the following.

For any s.m. $\mathcal{E}$, on $\mathcal{B}_{\widehat{G}}$ for H, we have $\mathcal{E} * w\mathcal{E} = \mathcal{E}_{\widehat{G}}$.

These recalls can be found in Boudou and Romain ([2] and [3]).

12.3 Relation of order, convergence of a sequence of projectors, gap between two projectors

Let P and Q be two elements of $\mathcal{P}(H)$, the set of orthogonal projectors of the $\mathbb{C}$–Hilbert H.

We say that "P is smaller than Q", and we denote it $P \ll Q$, when $P = PQ$.

This defines a partial relation of order on $\mathcal{P}(H)$. Any family $\{P_\lambda; \lambda \in \Lambda\}$ of elements of $\mathcal{P}(H)$ has got, for this relation of order, a greater minorant, named lower bound of the family, and denoted $\inf\{P_\lambda; \lambda \in \Lambda\}$, and a smaller majorant, named upper bound of the family, and denoted $\sup\{P_\lambda; \lambda \in \Lambda\}$. We have the following.

$Im\, inf\{P_\lambda; \lambda \in \Lambda\} = \cap_{\lambda\in\Lambda}(Im\, P_\lambda)$.

When P is an element of $\mathcal{P}(H)$, if we denote by $P^\perp$ the projector $I - P$, then we get the following dual properties.

$(inf\{P_\lambda; \lambda \in \Lambda\})^\perp = sup\{P_\lambda^\perp; \lambda \in \Lambda\}$;
$(sup\{P_\lambda; \lambda \in \Lambda\})^\perp = inf\{P_\lambda^\perp; \lambda \in \Lambda\}$.

Obviously, if $(P_n)_{n\in\mathbb{N}}$ is a sequence of projectors, we can define the limit superior and the limit inferior of this sequence as following.

$lim\, sup\,(P_n)_{n\in\mathbb{N}} = inf\{sup\{P_m; m \geq n\}; n \in \mathbb{N}\}$;
$lim\, inf(P_n)_{n\in\mathbb{N}} = sup\{inf\{P_m; m \geq n\}; n \in \mathbb{N}\}$.

Of course, we have the following relation.

$lim\,inf (P_n)_{n\in\mathbb{N}} \ll lim\,sup\,(P_n)_{n\in\mathbb{N}}$.

When the two limits are equal, we say that there is a convergence.

Definition 12.1. *A sequence of projectors $(P_n)_{n\in\mathbb{N}}$ is r−convergent to the projector P, what we write $lim^r_n P_n = P$, when $P = lim\,inf (P_n)_{n\in\mathbb{N}} = lim\,sup\,(P_n)_{n\in\mathbb{N}}$.*

A sequence of projectors which is r−convergent is not necessarily convergent as a sequence of $\mathcal{L}(H)$, $\mathbb{C}$−Banach of the set of bounded endomorphisms of H. Conversally, there exists sequences of projectors which are not r−convergent, although they are Cauchy sequences of $\mathcal{L}(H)$. Let us note that, however, both convergences imply the point by point convergence.

Definition 12.2. *Let consider two projectors P and Q. We call "gap between P and Q", and we denote it $d(P, P')$, the projector $sup\,\{P, P'\} - inf\,\{P, P'\}$.*

Beyond the definition, this gap offers a great analogy with the classical distance in $\mathbb{R}$. In particular, we show the following.

Proposition 12.3. *Two projectors P and Q are such that $d(P, Q) = 0$ if and only if $P = Q$.*

Proposition 12.4. *If P is a projector and $(P_n)_{n\in\mathbb{N}}$ a sequence of elements of $\mathcal{P}(H)$, then $lim^r_n P_n = P$ if and only if $lim^r_n d(P_n, P) = 0$.*

12.4 Continuity of the spectral measure

For a sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}_{\hat{G}}$, the limit superior and the limit inferior are defined as follows.

$lim\,sup(A_n)_{n\in\mathbb{N}} = \cap_{n\in\mathbb{N}} \cup_{m\geq n} A_m$;
$lim\,inf(A_n)_{n\in\mathbb{N}} = \cup_{n\in\mathbb{N}} \cap_{m\geq n} A_m$.

We can then show that

$lim\,inf(A_n)_{n\in\mathbb{N}} \subset lim\,sup(A_n)_{n\in\mathbb{N}}$.

Of course,

when $lim\,inf(A_n)_{n\in\mathbb{N}} = lim\,sup(A_n)_{n\in\mathbb{N}} = A$, we say that the sequence $(A_n)_{n\in\mathbb{N}}$ converges to A, what we write $lim_n A_n = A$.

The following result is a continuity property of the s.m..

Proposition 12.5. *When $\mathcal{E}$ is a s.m. on $\mathcal{B}_{\hat{G}}$ for H, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{B}_{\hat{G}}$ which converges to A, then $lim^r_n \mathcal{E}A_n = \mathcal{E}A$.*

12.5 Maximal egalizator of two unitary operators

If a projector D commutes which a unitary operator U, then $U(\mathrm{Im}\,D) = \mathrm{Im}\,D$, and we can prove that the application $X \in \mathrm{Im}\,D \mapsto UX \in \mathrm{Im}\,D$ is a unitary operator of the $\mathbb{C}$−Hilbert Im D.
If the projector D commutes with a second unitary operator U', then the application $X \in \mathrm{Im}\,D \mapsto U'X \in \mathrm{Im}\,D$ is a second unit operator. If both operators, such obtained, are equal, we say that D is an egalizator of U and U'. Hence the following definition.

Definition 12.6. *A projector D is an egalizator of the unitary operators U and U' when D commutes with U and U' and when $UD = U'D$.*

We show the following.

Proposition 12.7. *The upper bound of the family of the egalizators of the unitary operators U and U' is also an egalizator of U and U', named maximal egalizator of the unitary operators U and U'.*

This notion is a new way of dealing with the proximity between two unitary operators : two unitary operators are close together when the maximal egalizator is projecting on a great subspace. So, the unitary operators $I - 2h \otimes h$ and I ($\|h\| = 1$) have $(h \otimes h)^{\perp}$ as maximal egalizator, they are equal except on a subspace of dimension 1, and can be considered close together, unless $\|(I - 2h \otimes h) - I\|_{\mathcal{L}} = 2$.

12.6 Gap between two spectral measures

Definition 12.8. *We name gap between two s.m.'s $\mathcal{E}$ and $\mathcal{E}'$, on $\mathcal{B}_{\widehat{G}}$ for H, and we denote it $E_{\mathcal{E},\mathcal{E}'}$, the upper bound of the family of projectors $\{d(\mathcal{E}A, \mathcal{E}'A); A \in \mathcal{B}_{\widehat{G}}\}$.*

It is easy to verify that two s.m.'s $\mathcal{E}$ and $\mathcal{E}'$, on $\mathcal{B}_{\widehat{G}}$ for H, are equal if and only if $E_{\mathcal{E},\mathcal{E}'} = 0$, and that $E_{\mathcal{E},\mathcal{E}_{\widehat{G}}} = \mathcal{E}\complement\{0\}$.
The gap between two s.m.'s is invariant by convolution.

Proposition 12.9. *If two s.m.'s $\mathcal{E}$ and $\mathcal{E}'$, on $\mathcal{B}_{\widehat{G}}$ for H, commute with a third s.m. α, then $E_{\mathcal{E},\mathcal{E}'} = E_{\mathcal{E}*\alpha,\mathcal{E}'*\alpha}$.*

Consequently, we can show the following.

Proposition 12.10. *If two s.m.'s $\mathcal{E}$ and $\mathcal{E}'$, on $\mathcal{B}_{\widehat{G}}$ for H, commute, we have $E_{\mathcal{E},\mathcal{E}'} = (w\mathcal{E}) * \mathcal{E}'\complement\{0\}$.*

The convolution product is continuous, with respect to this measure of the gap between two s.m.'s. More precisely, we have the following.

Proposition 12.11. *If $\mathcal{E}$ and α are two s.m.'s, on $\mathcal{B}_{\widehat{G}}$ for H, and if $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is a sequence of s.m.'s, on $\mathcal{B}_{\widehat{G}}$ for H, which commute with α and such that $lim_n^r E_{\mathcal{E}_n,\mathcal{E}} = 0$, we can affirm that the s.m.'s $\mathcal{E}$ and α commute and that $lim_n^r E_{\mathcal{E}_n*\alpha,\mathcal{E}*\alpha} = 0$.*

The maximal egalizator and the gap between two s.m.'s are two different expressions for the same reality.

Proposition 12.12. *If $\mathcal{E}$ and $\mathcal{E}'$ are the s.m.'s respectively associated with the unitary operators U and U', then $(E_{\mathcal{E},\mathcal{E}'})^{\perp}$ is the maximal egalizator of U and U'.*

This last result can be generalized in the following way.

Proposition 12.13. *If $\{U_g; g \in G\}$ and $\{U'_g; g \in G\}$ are the groups of the unitary operators on H respectively deduced from the s.m.'s, on $\mathcal{B}_{\widehat{G}}$ for H, $\mathcal{E}$ and $\mathcal{E}'$, then $E_{\mathcal{E},\mathcal{E}'} = sup\{R_g^{\perp}; g \in G\}$, where $R_g^{\perp}$ is the maximal egalizator of the unitary operators U_g and U'_g.*

Hence, from this result, a proximity between the families $\{U_g; g \in G\}$ and $\{U'_g; g \in G\}$ implies a proximity between the s.m.'s $\mathcal{E}$ and $\mathcal{E}'$, and conversally.

12.7 Filtered of two stationary continuous random functions

If Z is the r.m. associated with the stationary c.r.f. $(X_g)_{g\in G}$, and if $\mathcal{E}$ is a s.m., on $\mathcal{B}_{\widehat{G}}$ for H, compatible with the r.m. Z, that is such that $\mathcal{E}AZ\widehat{G} = ZA$, for any A of $\mathcal{B}_{\widehat{G}}$ (there exists an infinity of such r.m.'s, cf. Boudou [1]), we can show the following.

For any X of H_Z, $(U_gX)_{g\in G}$ is a stationary c.r.f. filtered from $(X_g)_{g\in G}$.

Let us recall that a stationary c.r.f. filtered from $(X_g)_{g\in G}$ is a stationary c.r.f. of the type $(\int(.,g)\varphi(.)\mathrm{d}Z)_{g\in G}$, where φ is an element of $L^2(\widehat{G}, \mathcal{B}_{\widehat{G}}, \mu_Z)$.
Let us consider two stationary c.r.f.'s $(X_g)_{g\in G}$ and $(X'_g)_{g\in G}$, of respective associated r.m.'s Z and Z', we wish to determine all the stationary c.r.f.'s, both filtered from $(X_g)_{g\in G}$ and from $(X'_g)_{g\in G}$.
For this purpose, let us consider two s.m.'s $\mathcal{E}$ and $\mathcal{E}'$, on $\mathcal{B}_{\widehat{G}}$ for H, respectively compatible with the r.m.'s Z and Z'. Let $\{U_g; g \in G\}$ and $\{U'_g; g \in G\}$ be the groups of the unitary operators on H respectively deduced from the s.m.'s $\mathcal{E}$ and $\mathcal{E}'$. If X belongs to $H_Z \cap H_{Z'} \cap \mathrm{Im}\, E^{\perp}_{\mathcal{E},\mathcal{E}'}$, then $(U_gX)_{g\in G}$ and $(U'_gX)_{g\in G}$ are, from what precedes, stationary c.r.f.'s respectively filtered from $(X_g)_{g\in G}$ and $(X'_g)_{g\in G}$, but as $\mathrm{Im}\, E^{\perp}_{\mathcal{E},\mathcal{E}'} = \cap_{g\in G}\mathrm{Ker}(U_g - U'_g)$, they are equal, hence the following result.

Proposition 12.14. *For any X of $H_Z \cap H_{Z'} \cap \mathrm{Im}\, E^{\perp}_{\mathcal{E},\mathcal{E}'}$, $(U_gX)_{g\in G} = (U'_gX)_{g\in G}$ is a stationary c.r.f. filtered from $(X_g)_{g\in G}$ and from $(X'_g)_{g\in G}$.*

We show that any filtered function both from $(X_g)_{g\in G}$ and from $(X'_g)_{g\in G}$ is of this type.

Bibliography

[1] Boudou, A. (2007). Groupe d'opérateurs unitaires déduit d'une mesure spectrale - une application. *C. R. Acad. Sci. Paris, Sér. I Math* **344** 791-794.

[2] Boudou, A. and Romain, Y. (2002). On spectral and random measures associated to continuous and discrete time processes. *Statist. Probab. Lett.* **59** 145-157.

[3] Boudou, A. and Romain, Y. (2011). *On product measures associated with stationary processes*. The Oxford handbook of functional data analysis, 423-451, Oxford Univ. Press, Oxford.

[4] Ferraty, F. and Vieu, P. (2006). *Nonparametric Functional Data Analysis. Theory and Practice* Springer Series in Statistics, Springer, New-York.

[5] Halmos, P. (1951). *Introduction to Hilbert Space and the Theory of Spectral Multiplicity.* Chelsea Publishing Company, New-York.

[6] Romain, Y. (2011). *An invitation to operator-based statistics*. The Oxford handbook of functional data analysis, 452-482, Oxford Univ. Press, Oxford.