THÈSE

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Présentée et soutenue le *29/09/2023* **par :**

Viviana Grasselli

Étude de la résolvante pour l'opérateur de Schrödinger perturbé

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Thèse présentée par **Viviana Grasselli**

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Discipline **Mathématiques**

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Étude de la résolvante pour l'opérateur de Schrödinger perturbé

Résumé

Dans cette thèse, nous étudions les propriétés spectrales de l'opérateur de Schrödinger sur des domaines non bornés munis de métriques riemanniennes. On s'intéresse plus particulièrement au comportement asymptotique de sa résolvante au bord de l'ensemble résolvant, dans différents régimes en fréquence.
La première partie de notre analyse a pour but d'étendre au cadre riemannien des résultats connus dans le cas euclidien. Nous commençons par traiter le régime basse fréquence, dans le cas de variétés asymptotiquement coniques. On prouve l'existence des limites de la résolvante et de ses puissances pour un opérateur de Schrödinger avec potentiel. Ce résultat permet notamment de retrouver la décroissance de l'énergie locale de la partie basse fréquence des solutions aux équations de Schrödinger, des ondes et de Klein-Gordon. Du point de vue technique, on utilise la théorie de Mourre pour prouver le principe d'absorption limite, ce qui nécessite un calcul pseudo-différentiel adapté de manière à traiter des opérateurs dépendants du paramètre spectral. Puis, on traite le régime haute fréquence, dans un cadre plus général que précédemment. D'une part on considère une classe de variétés incluant non seulement le cas asymptotiquement conique mais aussi asymptotiquement hyperbolique. D'autre part, on traite des perturbations à l'ordre un de l'opérateur de Schrödinger avec potentiel. Sous ces hypothèses, on obtient une estimation optimale de la résolvante sur la partie non compacte de la variété : celle-ci est bornée par l'inverse de la racine carrée du paramètre spectral. De plus, ces estimations sont obtenues dans des normes de type Besov, ce qui permet de considérer des topologies plus fortes que celles proposées dans la littérature. Pour finir, on traite la région compacte avec des résultats issus des inégalités de Carleman.
Dans le dernier chapitre, on se place dans l'espace euclidien de dimension trois et quatre et on considère l'opérateur de Schrödinger avec un potentiel dont on suppose uniquement qu'il appartient à un espace de Lorentz. Plus précisément, on étudie la nature de la fréquence zéro ainsi que les propriétés de décroissance des états associés. On prouve que tout état résonnant appartenant à l'espace de Sobolev homogène d'ordre un est aussi dans un espace de Lebesgue faible. De plus, sous des hypothèses classiques d'orthogonalité entre l'état résonnant et le potentiel, on obtient l'intégrabilité L^2, $L^{1,\infty}$ et enfin L^1 de l'état résonnant.

Mots clés : opérateur de Schrödinger, théorie du scattering, perturbation par métrique, Laplacien magnétique, variété asymptotiquement conique, variété asymptotiquement hyperbolique, principe d'absorption limite, état résonnant en zéro

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Study of the resolvent of the perturbed Schrödinger operator

Abstract

In this thesis we study spectral properties of the Schrödinger operator on non compact domains equipped with Riemannian metrics. More precisely, we are interested in the asymptotic behavior of the resolvent on the boundary of the resolvent set, in different frequency regimes.
The first part of our analysis it is aimed at extending to the Riemannian framework results which are known in the Euclidean case. We start by treating the low frequency regime in the case of asymptotically conical manifolds. We prove the existence of the limiting resolvents and of its powers for a Schrödinger operator with potential. Notably, this result allows to recover the local energy decay of the low frequency part of solutions to the Schrödinger, wave and Klein-Gordon equations. From the point of view of technique, we employ Mourre theory to prove the limiting absorption principle, which requires an adapted pseudodifferential calculus in order to treat operators depending on the spectral parameter. Next, we treat the high frequency regime, in a more general framework with respect to the previous one. On the one hand we consider a class of manifolds including not only the asymptotically conical case, but also the asymptotically hyperbolic one. On the other hand, we treat order one perturbations of the Schrödinger operator with potential. Under these assumptions, we obtain an optimal estimate for the resolvent on the non compact part of the manifold: it is bounded by the inverse square root of the spectral parameter. Moreover, these estimates are obtained using Besov type norms, making it possible to consider stronger topologies than what is often used in the literature. To conclude, we treat the compact region via Carleman estimates.
In the last chapter, we consider the Schrödinger operator in the Euclidean space in dimensions three and four and with a potential lying in a Lorentz space. More precisely, we study the nature of the frequency zero as well as the decaying properties of the associated states. We prove that any resonant state belonging to the homogeneous Sobolev space of order one is also in a weak Lebesgue space. Under classical assumptions of orthogonality between the resonant state and the potential, we obtain L^2, $L^{1,\infty}$ and finally L^1 integrability of the resonant state.

Keywords: Schrödinger operator, scattering theory, metric perturbation, magnetic Laplacian, asymptotically conical manifold, asymptotically hyperbolic manifold, limiting absorption principle, zero resonant state

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Chapter 1

Introduction

Outline of the current chapter

The subject of this thesis is the study of spectral properties of Schrödinger operators on different types of non Euclidean settings and at different regimes.

1.1 The Schrödinger operator

The rise of quantum mechanics in the first half of the 20th century was due to the observation of new phenomena at the microscopic atomic level. In this framework the state of a particle can not be described by its position $x(t)$ at time t, as it was the case in classical mechanics. Indeed, experiments on subatomical particles like electrons showed that we can not predict its exact position, but rather that their behavior is similar to the one of propagating waves.

To formalize all this, contrary to the classical setting, in quantum mechanics we describe the state of a particle via a probability distribution, which is a unit vector $\psi(t)$ in the Hilbert space $\mathcal{H}$ (the space of states). We often choose L^2 as the space of states. The state ψ describes the likelihood of finding the particle in a certain position at a certain time t. How this probability distribution evolves in time is determined by Schrödinger's equation

$$\begin{cases} i\partial_t\psi = P\psi(t) \\ \psi(0) = \psi_0 \end{cases} \tag{1.1.1}$$

where P is the Schrödinger operator and is a linear operator which models the kinetic and potential energy. Often, this operator has the form $P = P_0 + V$ with P_0 the kinetic energy and V the potential representing the interaction between particles, so a system with $V = 0$ is called a free system and P_0 the free operator.

In physical terms, P is an observable of the system, that is a quantity which is measurable in physical experiments. As we pointed out we can only expect to know the probability distribution of a particle, analogously when we measure the observable P of a particle in state ψ we can only determine its expectation value

$$\langle \psi, P\psi\rangle_{\mathcal{H}}$$

The results of these measurements must be real values and since they satisfy the relation

$$\langle \psi, P\psi\rangle_{\mathcal{H}} = \mu = \mu\langle \psi, \psi\rangle_{\mathcal{H}},$$

we see that P must have real eigenvalues. Indeed, P is a selfadjoint operator and hence its spectrum is a subset of $\mathbb{R}$.

Remark 1.1.1. Symmetry alone would provide real eigenvalues for P, however selfadjointness is needed to have existence and uniqueness of a solution for (1.1.1). Indeed, it is only for selfadjoint operators that we can define the associated unitary group e^{-itP}, which in particular maps ψ_0 to a solution of the system (1.1.1).

The central interest in quantum dynamics is the study of the time evolution of solutions to (1.1.1), which as we just said are of the form $e^{-itP}\psi_0$. It turns out that spectral properties of P give us useful information for the study of the dynamics of not only $e^{-itP}\psi_0$, but also evolution operators associated to other partial differential equations.

Different parts of the spectrum have different interests. A first subdivision of the spectrum is in discrete and essential

$$\sigma(P) = \sigma_d(P) \cup \sigma_d(P)^c =: \sigma_d(P) \cup \sigma_{ess}(P),$$

with $\sigma_d(P)$ the set of isolated eigenvalues of finite multiplicity. In this thesis we will be interested in the essential spectrum, $\sigma_{ess}(P)$, and in some particular subsets of it that we will define in Section 1.2.2.

Remark 1.1.2. We say that μ is an eigenvalue if the operator $P - \mu$ is not injective and therefore not invertible. Since we are in the selfadjoint case, for $\mu \in \sigma(P)$ not an eigenvalue the only other possible situation is that $P - \mu$ is not surjective, but $Ran(P - \mu)$ is nevertheless dense.

1.2 Limiting absorption principle

As we said P is a selfadjoint operator on $\mathcal{H}$, its resolvent set then contains $\mathbb{C} \setminus \mathbb{R}$ and the resolvent at $\lambda^2 \mp i\varepsilon$ is well defined for $\lambda, \varepsilon \in \mathbb{R}$ and $\varepsilon \neq 0$. The objects we will be interested in are the limits

$$(P - \lambda^2 \pm i0)^{-1} = \lim_{\varepsilon \to 0} (P - \lambda^2 \pm i\varepsilon)^{-1}. \tag{1.2.1}$$

The existence of the limits is not trivial as they are clearly not well defined in the L^2 operator topology (we recall the bound on the operator norm of the resolvent $(P - z)^{-1}$ is in $|Imz|^{-1}$). Proving that these limits exist as bounded operators in the appropriate L^2 weighted spaces is usually referred to as limiting absorption principle and was first established in the Euclidean case in [Agm75] and [Kur73] for the Schrödinger operator with short range potentials.

The name derives from the fact that these limits were first considered in the study of the propagation and absorption of electromagnetic waves, where the imaginary part of the spectral parameter was linked to absorption.

The fact that the limits (1.2.1) exist directly tells us that if f belongs to the appropriate weighted space then $(P - \lambda^2 \pm i0)^{-1} f$ will be solution to the Helmholtz equation $(P - \lambda^2)u = f$ and will have a certain decaying behavior (radiation condition).

This principle has also several applications in the study of the spectral properties of P or of the dynamics of certain systems. We give here some examples in relation to the interests of this work. A large review of references and consequences of limiting absorption principle can be found in [RT15].

1.2.1 Dynamics and limiting resolvents

The limiting resolvents in (1.2.1) are present in the well known Stone's formula. This formula gives an alternative way of writing the functional calculus, which in the first place can be defined via the spectral theorem.

For a selfadjoint operator P, we define the spectral measure associated to $u, v \in \mathcal{H}$ as the measure $d(u, E_\lambda(P)v)$ such that

$$(u, Pv) = \int_{\mathbb{R}} \lambda \; d(u, E_\lambda(P)v),$$

this relation can be extended from the identity to any bounded Borel function f. In short we will write

$$f(P) = \int_{\mathbb{R}} f(\lambda) \; dE_\lambda(P). \tag{1.2.2}$$

Stone's formula states that the density of this measure with respect to the Lebesgue measure is given by the imaginary part of the limiting resolvent, that is

$$\frac{dE_\lambda(P)}{d\lambda} = \frac{1}{\pi} Im(P - \lambda + i0)^{-1} = \frac{1}{2\pi i}((P - \lambda + i0)^{-1} - (P - \lambda - i0)^{-1})$$

Remark 1.2.1. In particular the limits of the resolvents approaching the real axis from above and from below do not agree (the previous formula would otherwise not make sense). This can be seen explicitly for example in the $\mathbb{R}^n$ case with $P = -\Delta$ by writing the resolvent via its integral kernel and computing the integrals as $\varepsilon \to 0$ by taking contour deformations that avoid the singularity on the real axis.

As we have seen we can write any bounded Borel function of P via the operators (1.2.1). In particular one can consider the Schrödinger propagator which dictates the behavior of the quantum states as for the previous section. Other quantities that are physically relevant are the wave and Klein-Gordon propagators

$$\cos(tP^{1/2}), \qquad \frac{\sin(tP^{1/2})}{P^{1/2}}, \qquad e^{it(P^{1/2}+1)}$$

which describe solutions of the wave and Klein-Gordon equations. Thanks to (1.2.2) we can represent them as integrals of oscillating functions against limiting values of the resolvent.

Having local information on the behavior of (1.2.1), we can use this to study the integrals defining the evolution operators and hence derive results on the local behavior of solutions to the wave and Klein-Gordon equations for large times. This is the approach we use in Chapter 2.

Remark 1.2.2. Here local refers to the fact that the results we obtain on (1.2.1) hold in weighted spaces. This forces us to study the propagators conjugated by some decaying weights, which localize (in a more or less weak way) the solutions away from infinity.

Another problem to which limiting resolvents are linked is the study of wave operators, defined via Schrödinger propagators. In this case, e^{-itP} is put in relation to the propagator of another Schrödinger operator P_0, usually easier to study and referred to as the free Hamiltonian, with the aim to recover information on e^{-itP} thanks to e^{-itP_0}. Looking at the problem from the point view of (1.2.1), one can shift the attention from large time limits of wave operators to resolvent properties. In a way, one can think of the study of the resolvent as a stationary approach to scattering theory.

1.2.2 Spectrum and limiting resolvents

The limiting absorption principle also helps us to obtain information on the essential spectrum, such as the absence of singular continuous spectrum and of embedded eigenvalues. In particular we are able to prove bounds on the norms of (1.2.1), hence the limiting absorption principle can be seen as a quantification of the absence of such eigenvalues. Being able to rule out this type of eigenvalues is useful as it is one of the first steps in scattering theory.

In general, consider the decomposition of the space of states $\mathcal{H}$ determined by the nature of the spectral measure, that is

$$\mathcal{H} = \mathcal{H}_{pp}(P) \oplus \mathcal{H}_{ac}(P) \oplus \mathcal{H}_{sc}(P).$$

with

$$\begin{aligned} \mathcal{H}_{pp}(P) =& \{v \in \mathcal{H} \mid d(v, E_\lambda(P)v) \text{ is a pure point measure}\}, \\ \mathcal{H}_{sc}(P) =& \{v \in \mathcal{H} \mid d(v, E_\lambda(P)v) \text{ is singular w.r.t. the Lebesgue measure}\}, \\ \mathcal{H}_{ac}(P) =& \{v \in \mathcal{H} \mid d(v, E_\lambda(P)v) \text{ is absolutely continuous w.r.t. the Lebesgue measure}\}. \end{aligned}$$

Corresponding to these subspaces one can define subsets of the spectrum of a selfadjoint operator P as

$$\sigma_{ac}(P) := \sigma(P|_{\mathcal{H}_{ac}}), \qquad \sigma_{sc}(P) := \sigma(P|_{\mathcal{H}_{sc}}),$$

while the pure point spectrum is rather defined as

$$\sigma_{pp}(P) = \{ \ \lambda \mid \lambda \text{ is an eigenvalue } \}.$$

From the definition of $\sigma_{pp}(P)$ and (1.2.2) we can infer a pretty straightforward fact on the behavior of a solution to (1.1.1) with initial data in $\sigma_{pp}(P)$. Indeed, when $dE_\lambda(P)$ is a measure concentrated at a point we see that the solution $e^{-itP}\psi_0$ will be an oscillating function of t, hence not exhibiting time decay.

Remark 1.2.3. These three subsets of $\sigma(P)$ do not give an exact partition of the spectrum, since we did not define the pure point spectrum as $\sigma(P|_{\mathcal{H}_{pp}})$. However we can recover the full spectrum just considering $\sigma(P) = \overline{\sigma_{pp}(P)} \cup \sigma_{ac}(P) \cup \sigma_{sc}(P)$.

Remark 1.2.4. We recall that the discrete spectrum $\sigma_d(P)$ contains only part of the eigenvalues (the isolated and finite multiplicity ones), hence in general we only have

$$\sigma_d(P) \subset \sigma_{pp}(P)$$

and $\sigma_{pp}(P) \cap \sigma_{ess}(P)$ may be non empty.

When the limiting absorption principle holds, by definition the spectral measure has a well defined density with respect to the Lebesgue measure. So, if (1.2.1) exist for all λ then $\mathcal{H}_{sc}(P) = \emptyset$ and hence there is no singular continuous spectrum; if the limits exist only for $\lambda \in J$ then we can nevertheless conclude that $\sigma_{sc}(\mathbb{1}_J(P)P) = \emptyset$.

Another aspect of scattering theory is the subject of eigenvalues embedded in the essential spectrum. The limiting absorption principle implies that there are none of such eigenvalues, which are somehow problematic in scattering theory. As we mentioned earlier the idea in scattering theory is to compare the so-called interacting system generated by $P = P_0 + V$ with a free one generated by P_0. However, eigenvalues in the essential spectrum generate states which do not scatter, that is they do not resemble the dynamics of a free state for large times and are unstable. For example, in [SW98] we can see that the free Hamiltonian has embedded eigenvalues in $\mathbb{R}^+$ that, when adding a perturbation by potential to obtain P, move out of the real line to become complex. This means that near a certain energy level the free Hamiltonian and the interaction one have states which behave in a completely different way, which is undesirable in the scope of wanting to perform perturbation theory.

Example 1.2.5 (Absence of embedded eigenvalues). Consider the case $P = -\Delta + V$ on $\mathbb{R}^n$ and $P_0 = -\Delta$, with V bounded multiplicative potential, decaying at infinity like $|x|^{-1-\nu}, \nu > 0$. It is known that under these conditions

$$\sigma_{ess}(P) = [0, \infty), \qquad \sigma_d(P) \cap [0, \infty) = \emptyset.$$

So the only eigenvalues that might be embedded must be either accumulation points or of infinite multiplicity. We can rule out the latter case right away. Indeed, a potential like the one mentioned here falls into the class of so called short range potentials considered in [Agm75]. In this case positive eigenvalues must have finite multiplicity (Theorem 3.1 [Agm75]).
Moreover, in a limiting absorption interval we can also rule out the presence of accumulation points. This is a byproduct of the strategy we describe in the next Section 1.2.3. In fact, we obtain that, wherever the proof of the limiting absorption principle holds there are no accumulation points of $\sigma_{pp}(P)$ (see Remark 1.2.7). In this case we can conclude that if the limiting absorption principle holds on all of the positive real line

$$\sigma_{pp}(P) \cap [0, \infty) = \emptyset \qquad \sigma_{pp}(P) = \sigma_d(P), \qquad \sigma(P) = \overline{\sigma_{pp}(P)} \cup \sigma_{ac}(P).$$

This means that there are no embedded eigenvalues in the essential spectrum $\mathbb{R}^+$, that happens to coincide with the continuous spectrum, and all eigenvalues are negative, isolated and of finite multiplicity.

1.2.3 Mourre theory

A popular approach in the proof of the existence of limits (1.2.1) is Mourre theory and this will be the tool used in Chapter 2. To be more precise, in the mentioned chapter we will apply a more abstract version of Mourre theory due to [Gé08].

The method, introduced in [Mou81], involves finding $\mathcal{A}$ selfadjoint which satisfies two conditions:

1. the map
$$\mathbb{R} \ni t \mapsto e^{it\mathcal{A}}(P-z)^{-1}e^{-it\mathcal{A}}u \in D(P) \tag{1.2.3}$$
is C^1 for any $u \in L^2(\mathcal{H})$, $z \in \mathbb{C} \setminus \mathbb{R}$, in this case we say that P is $C^1(\mathcal{A})$,
2. we have a positive commutator estimate, that is the inequality
$$\mathbb{1}_I(P)[P, i\mathcal{A}]\mathbb{1}_I(P) \geqslant c\mathbb{1}_I(P) + \mathbb{1}_I(P)K\mathbb{1}_I(P) \quad c > 0 \tag{1.2.4}$$
with K compact and I an interval in the spectrum of P containing $Re\ z$. It tells us that the commutator is positive up to a compact remainder K, provided it is spectrally localized on I.

Other technical assumptions on the domains of P and $\mathcal{A}$ are also required. When P and $\mathcal{A}$ satisfy this set of assumptions we say that $\mathcal{A}$ is **conjugate** to P.

Remark 1.2.6. Here commutators are meant to be interpreted in the sense of quadratic forms. In other words they are defined in a weak sense, via the equality
$$\langle u, [P, i\mathcal{A}]v\rangle_{\mathcal{H}} = i\langle Pu, \mathcal{A}v\rangle_{\mathcal{H}} - i\langle \mathcal{A}u, Pv\rangle_{\mathcal{H}}$$
for all $u, v \in D(P) \cap D(\mathcal{A})$.

With the conditions 1. and 2. satisfied, we obtain that (1.2.1) exist when conjugated with appropriate operator weights, that is the operator
$$\langle \mathcal{A}\rangle^{-s}(P-z)^{-1}\langle \mathcal{A}\rangle^{-s} \tag{1.2.5}$$
is bounded in L^2, uniformly with respect to Imz, for all $z \in \mathbb{C}$ with $Rez \in I$ and $Imz \neq 0$ and $s > 1/2$ (Theorem 1 [Gé08]). Here $\langle \mathcal{A}\rangle^{-s} = (\mathcal{A}^2+1)^{-s/2}$.

Remark 1.2.7. Let $J \subset \sigma(P)$, if for any point in J there exists a neighborhood I in which inequality (1.2.4) holds this implies that the eigenvalues are not dense in J. Indeed, if any $\lambda \in J$ is an accumulation point of a sequence of eigenvalues then in any neighborhood of λ there exists a λ^* and function f such that $Pf = \lambda^* f$. For such f the Virial theorem (Proposition II.4 in [Mou81]) implies $(f, [P, \mathcal{A}]f) = 0$ and this would violate (1.2.4). We can deduce that in an interval where a positive commutator estimate holds there are neither isolated eigenvalues nor accumulation points of $\sigma_{pp}(P)$.

In the case of the operator $P = -\Delta + V$ mentioned in Example 1.2.5, we have discussed how $\sigma_{ess}(-\Delta + V) = \sigma_{ess}(-\Delta) = [0, \infty)$ implies that there are no isolated eigenvalues in the positive half line. The validity of (1.2.4) adds the information that there are no eigenvalues which are accumulation points.

Example 1.2.8. To apply the theory on the classical setting of the Laplacian on $\mathbb{R}^n$ we can choose a suitable $\mathcal{A}$ in order for $[-\Delta, i\mathcal{A}]$ to be easy to compute. In particular what we consider is the generator of dilations
$$\mathcal{A} = \frac{x\cdot\nabla + \nabla\cdot x}{2i}$$
which satisfies the relation
$$[-\Delta, i\mathcal{A}] = 2(-\Delta). \tag{1.2.6}$$

We can see how this technique is well suited for the intermediate and low frequency regimes, that is when we want to prove existence of (1.2.1) for fixed λ or for arbitrarily small λ.

For the intermediate frequency regime one can consider without loss of generality the energy 1, hence proving the existence of $(P-1+i0)^{-1}$. To do so we need (1.2.4) to hold on a fixed interval I containing 1. In the free Euclidean case mentioned in Example 1.2.8 this is pretty straightforward since the inequality reduces to
$$\mathbb{1}_I(-\Delta)i[-\Delta, \mathcal{A}]\mathbb{1}_I(-\Delta) = \mathbb{1}_I(-\Delta)2(-\Delta)\mathbb{1}_I(-\Delta) \geqslant c\mathbb{1}_I(-\Delta) \tag{1.2.7}$$

if I is a neighborhood of one. Adding a perturbation by potential one needs

$$\mathbb{1}_I(-\Delta)i[-\Delta+V,\mathcal{A}]\mathbb{1}_I(-\Delta)=\mathbb{1}_I(-\Delta)(2(-\Delta)+i[V,\mathcal{A}])\mathbb{1}_I(-\Delta) \qquad (1.2.8)$$

to be bounded from below on an interval I around 1, where the first term is obviously bounded from below as we have just seen. The potential term is of order zero and more precisely it is given by

$$[V,\mathcal{A}]=-(x\cdot\nabla)V.$$

Example 1.2.9 (Repulsive potential). To obtain a bound from below one could directly assume that $(x\cdot\nabla)V\leqslant 0$, in this case we say that V is a repulsive potential. So, for a potential of this type we have a positive commutator estimate and, as seen in Remark 1.2.7, one can prove that there are no embedded eigenvalues in any compact region of the positive half line.

Example 1.2.10 (Long range potential). We can also consider V, decaying potential, that belongs to a symbol class. For example if $V\in S^{-\nu}$ this translates to

$$|V(x)|\leqslant O(\langle x\rangle^{-\nu}),\qquad |\nabla V(x)|\leqslant O(\langle x\rangle^{-1-\nu}),$$

where we use the notation

$$\langle x\rangle=(1+|x|^2)^{1/2}.$$

Then

$$\mathbb{1}_I(-\Delta)i[V,\mathcal{A}]\mathbb{1}_I(-\Delta)=-i\mathbb{1}_I(-\Delta)\langle x\rangle^{\nu}(x\cdot\nabla)V(\langle x\rangle^{-\nu}\mathbb{1}_I(-\Delta))$$

is a compact operator since it is the composition of the compact operator $\langle x\rangle^{-\nu}\mathbb{1}_I(-\Delta)$ with a bounded one. This, together with the fact that $\mathbb{1}_I(-\Delta)[-\Delta,i\mathcal{A}]\mathbb{1}_I(-\Delta)\geqslant c\mathbb{1}_I(-\Delta)$ provides us inequality (1.2.4).

Remark 1.2.11. In the case of an elliptic operator with non constant coefficients (1.2.6) does not hold anymore. This will be the situation in the manifold setting, when we will introduce a perturbation of the metric. However, the starting point for the construction of the conjugate operator will still be the generator of dilations. Property (1.2.6) can not be exactly recovered in such setting, this implies that on top of the potential perturbative term due to $[V,\mathcal{A}]$, we have additional terms due to the fact that (1.2.6) holds up to some remainder. This will of course make the proof of (1.2.4) less straightforward, while regularity of the map (1.2.3) can still be proved with a standard argument.

In the low frequency case we consider the resolvent at energy $\lambda$for λ arbitrarily small, then just by taking the basic example in (1.2.7) we can not lower bound the commutator by a positive constant since I must contain $\lambda\ll 1$ and therefore it must be arbitrarily close to zero. However, we can reduce ourselves to a fixed frequency problem. Instead of a variable frequency λ we can always factorize to bring the problem back to energy one, that is we consider instead

$$\lambda^{-1}(\lambda^{-1}P-1+i0)^{-1}.$$

Now the operator in consideration is $\lambda^{-1}P$, so we actually just shifted on the operator the dependence on the parameter approaching zero. However, we can use the tools of symbolic calculus to obtain properties which are uniform in λ. This is what we will need to do in Chapter 2.

We conclude that for the fixed and the low frequencies Mourre theory can be developed in a somehow similar way. In the high frequency case this is not true, unless one adds some assumptions on the geodesic flow, so in general this technique does not apply to both settings. Indeed, the different frequency regimes are influenced by different aspects of the problem, as we will describe in Section 1.3, where we will see that properties of the geodesic flow play a central role in the high frequency theory.

1.2.4 Multiple commutators and time decay

In [JMP84] the previous technique is pushed forward to obtain smoothness with respect to λ of the limiting resolvents. It can be proved that differentiating $(P-\lambda+i0)^{-1}$ with respect to λ is equivalent to adding powers to the resolvent.

So to prove resolvent smoothness means to obtain the existence of the operators $(P-\lambda\pm i0)^{-l}$ in the appropriate weighted spaces. In [JMP84] it is explained how to do so by basically iterating the method in [Mou81].

To have information on the derivatives of $(P-\lambda+i0)^{-1}$ allows us to prove time decay properties on the propagators we mentioned in Section 1.2.1. Indeed, heuristically speaking formula (1.2.2) for $f(\lambda)=e^{it\lambda}$ allows us to write the Schrödinger propagator in the form

$$e^{-itP}=\frac{1}{\pi}\int e^{-it\lambda}Im(P-\lambda+i0)^{-1}d\lambda \tag{1.2.9}$$

that is e^{-itP} and $Im(P-\lambda^2+i0)^{-1}$ are linked via a Fourier transform in the spectral variable with dual variable t. Since the Fourier transform exchanges differentiation with multiplication, taking the previous integral this time against $\partial_\lambda^l((P-\lambda^2+i0)^{-1})$ will result in $t^l e^{-itP}$. That is

$$t^l e^{-itP}=\frac{1}{\pi}\int e^{-it\lambda}(P-\lambda+i0)^{-l}d\lambda$$

and to obtain boundedness results on $t^l e^{-itP}$ means proving time decay with a polynomial rate for the Schrödinger propagator. To study the operator $t^l e^{-itP}$ we need first of all existence of the powers $(P-\lambda^2+i0)^{-l}$ and this, we said, can be proved thanks to [JMP84]. The main additional assumptions we need with respect to the ones in Section 1.2.3 are:

- higher regularity for the map (1.2.3) (that is $P\in C^k(\mathcal{A})$ for $k>1$),
- the iterated commutators, defined in the sense of quadratic forms, are bounded from below and closable.

By iterated commutators we mean the operators

$$i[i[P,\mathcal{A}],\mathcal{A}],\qquad i[i[i[P,\mathcal{A}],\mathcal{A}],\mathcal{A}],\qquad \dots$$

defined as quadratic forms in the sense of Remark 1.2.6.

Example 1.2.12. In the case of $\mathbb{R}^n$ we can see pretty easily what these iterated commutators look like. In the free case iterating does not change the result since from equation (1.2.6)

$$i[i[-\Delta,A],A]=2i[-\Delta,A]=4(-\Delta)$$

so as for the first commutator, all commutators are bounded from below. In the potential case

$$i[i[-\Delta+V,A],A]=i[2(-\Delta)-i(x\cdot\nabla)V,A]=4(-\Delta)-(x\cdot\nabla)^2V$$

tells us that we need to add assumptions on the higher order derivatives of V. Taking V in a symbol class as in Example 1.2.10 gives us information also on the derivatives. We can then proceed in an analogous way since we still have the property $(x\cdot\nabla)^2V=O(\langle x\rangle^{-\nu})$.

With these additional properties one can obtain existence of the powers of the resolvents together with the fact that powers coincide with derivatives with respect to λ. The higher the power the more iterated commutators one needs to control. We recall that the limiting resolvents exist only in weighted spaces and powers of the resolvent require a stronger decay. This translates to the fact that a faster time decay on e^{-itP} requires stronger weights in the spatial domain.

Given expression (1.2.4) the existence of the powers $(P-\lambda+i0)^{-l}$ simply implies the existence of the propagator, however to obtain boundedness we need a better quantification of the operator norm of $(P - \lambda + i0)^{-l}$, intuitively a bound which is L^2 in λ in order to control the L^2 norm of $t^l e^{-itP}$. Results of this type are what we obtain in Theorem 1.5.5.

1.3 Frequency regimes

In this thesis we will focus on specific instances of Hamiltonians $P = P_0 + V$ and on different regions of the spectrum. More precisely, in Chapters 2 and 3 P_0 will be the Laplace-Beltrami operator on some classes of non compact manifolds with ends of infinite volume, which we define in Section 1.4.1. We will allow P_0 to have a potential perturbation V and in Chapter 3 also a perturbation of order one. In the last chapter, instead, we will consider the Euclidean Laplacian $P_0 = -\Delta$ with a potential perturbation. In each case we will be interested in different regions of the spectrum. For now, we summarize the objects of study in Table 1.1. We will state our assumptions and results at the end of this introduction, while in this section we discuss separately the study of the limiting absorption principle in the high and low part of the spectrum. For the intermediate region, that is $(P - \lambda^2 \pm i0)^{-1}$ with λ in a bounded interval of $\mathbb{R}^+$, the limiting absorption principle is well known as it was proven in an abstract setting in [JMP84].

	Hamiltonian	Region of the spectrum
Chap. 1	$P = P_0 + V$, P_0=Laplace-Beltrami (L. B.)	low frequencies
Chap. 2	$P_m = P_{0,m} + V$, $P_{0,m}$=L. B.+ order 1 perturbation	high frequencies
Chap. 3	$-\Delta + V$ on $\mathbb{R}^3, \mathbb{R}^4$	zero

Table 1.1 – Content of chapters

1.3.1 Low frequencies ($\lambda \ll 1$)

In the low frequency regime we are interested in the limiting resolvents $(P - \lambda \pm i0)^{-1}$ for $\lambda \ll 1$.

In the Euclidean unperturbed case results for small λ can be deduced by the behavior of the resolvent at fixed frequency one via a rescaling argument. Indeed on $\mathbb{R}^n$ the resolvent $(\Delta - \lambda \pm i0)^{-1}$ is unitarily equivalent to the renormalized resolvent $(\Delta/\lambda - 1 \pm i0)^{-1}$ via the unitary operator of dilations. The two resolvents then enjoy the same estimates in L^2 spaces.

In the perturbed case the situation is different. For a perturbation by potential the behavior of the resolvent is influenced by the presence of zero eigenfunctions or resonances, this phenomenon can be observed in the resolvent expansions described for example in [JK79]. For a metric perturbation the results depend on the asymptotic behavior of the metric. In [Mor20] it is proved, in dimension three, that the higher the decay rate of the metric the faster the time decay of solutions. However, it is not clear whether this strong decay of the metric is necessary or not.

Remark 1.3.1. In [JK79] one can see directly the influence of a zero resonance or eigenfunction which are responsible for singular terms in the asymptotic expansion of the resolvent. More generally, in the low frequency limit, the constants bounding the operator norms of $(P - \lambda \pm i0)^{-1}$ blow up like negative powers of λ. This blow up is necessary even when there are no zero resonances or eigenvalues, as it is shown in the example in Proposition 1.21 [RT15].

Low frequency estimates are interesting for example in the study of the asymptotic behavior in time of operators like e^{-itP}, as we discussed in Section 1.2.1. By local energy decay we mean decay in time of the weighted L^2 norms of, for example, the operator e^{-itP}. In particular, it is the asymptotics of the resolvent for small λ that determines the time decay rate. The decay in time in the high frequency regime is arbitrarily fast, up to a loss of regularity on the initial data or up to non trapping conditions.

When studying perturbation by potentials in scattering theory we usually consider decaying potentials. This approach is a suitable one in dimension three and higher, as it is the case in this thesis. In this setting the smallness of V allows us to treat it in a perturbative way. In smaller dimensions however, even small potentials can generate infinitely many negative bound states (cf. Theorem XIII.11 in [RS78]). The assumptions which are usually taken on the decay of the potential differ in different frequency regimes and are basically determined by the renormalization of the resolvent.

Remark 1.3.2 (Decay of the potential I). As we have done in previous occasions we can rewrite the limiting resolvent $(P_0 + V - \lambda + i0)^{-1}$ as

$$(\lambda^{-1}P_0 + \lambda^{-1}V - 1 + i0)^{-1},$$

where $\lambda^{-1} >> 1$. In the Euclidean case $P_0 = -\Delta$ consider the change variable $\widetilde{x} = \lambda^{1/2}x$, the new operator is

$$(-\Delta + \lambda^{-1}V\left(\frac{x}{\lambda^{1/2}}\right) - 1 + i0)^{-1},$$

so if we assume V to have some inverse power behavior at infinity, like $V(x) \simeq |x|^{-\alpha}$, we have a small perturbation of the Laplacian if and only if $\alpha = 2 + \nu, \nu > 0$. This is why a decay rate stronger than the inverse square is what is usually considered in low frequency results and what we will assume in Chapter 2. We also remark that a potential with this behavior can be decomposed as $V = V_1V_2$ with $V_1 = signV\sqrt{V}$, $V_2 = \sqrt{V}$ where both V_1, V_2 decay like $|x|^{-1-\nu/2}$. This is an interesting feature since functions decaying faster than $|x|^{-1}$ are $-\Delta$ smooth and such decomposition is used in the proofs of Strichartz estimates by [RS04] that we describe in Section 1.3.4. Smooth perturbations in general enjoy many nice properties like existence and completeness of wave operators.

1.3.1.1 Some references

A fundamental paper in the domain of low frequency resolvent estimates is [JK79]. The authors obtain resolvent series expansions and apply it to prove local energy decay, for which they give rates of decay under spectral assumptions on the zero energy. If there are no zero resonances or eigenvalues the rate of decay is $O(t^{-3/2})$ or $O(t^{-1/2})$ in the presence of a zero resonance. The approach by power series expansion was generalized to several different settings. Among others: in higher dimension by Jensen himself in [Jen80] and [Jen84] and for general elliptic operators on $\mathbb{R}^n$ by [Mur82]. With this technique the assumptions on the decay of the potential V are more demanding than what we described in Remark 1.3.2 the more one goes further in the expansion. This is natural since to obtain a power series they use the analytic expansion that we can write for the free resolvent. Indeed, $(-\Delta - z^2)^{-1}$ on $\mathbb{R}^3$ has kernel

$$\frac{e^{iz|x-y|}}{4\pi|x-y|},$$

where $e^{iz|x-y|}$ can be expanded in power series. However, the series for $e^{iz|x-y|}$ has coefficients growing polynomially in x hence one needs V to compensate for this growth. A seminal paper in the area of time decay estimates is also [Rau78], where the author considers exponential decay on the potential and obtains a decay rate of $O(t^{-1/2})$ for the Schrödinger propagator.

Perturbations on the geometry where initially treated in the study of obstacle problems, as it was first done in [Mor61] for the wave equation on $\mathbb{R}^3$. For an application to time decay, the study of limiting resolvents was generalized to a large class of domains. For compact perturbations of the flat metric in [Bur98] and for long range perturbations of euclidean metrics in [BH08], [Bou11] and [BB21]. In the case of scattering manifolds the problem was studied in [GHS13a] with a rather geometric approach and applied to time decay for the wave and Schrödinger propagators.

Scattering metrics were also studied in [VW09], [GH08] and [GH09], while in [Wan06] or [RT15] the authors consider the more general case of asymptotically conical metric, where in [Wan06] the approach is via a series expansion of the resolvent. Sharp resolvent estimates on asymptotically conical manifolds can also be found in [BR14b].

1.3.2 High frequencies ($\lambda \gg 1$)

For high frequencies we need to consider the resolvents $(P - \lambda^2 \pm i0)^{-1}$ when λ is arbitrarily large. In this case the behavior is influenced by the geodesic flow, in particular by trapped trajectories.

Let $p(x, \xi)$ the principal symbol of P and $\boldsymbol{x}$ the classical trajectory given by the Hamilton flow of p, that is

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \nabla_\xi p(\boldsymbol{x}(t), \boldsymbol{\xi}(t)), & \boldsymbol{x}(0) = x_0 \\ \dot{\boldsymbol{\xi}}(t) = -\nabla_x p(\boldsymbol{x}(t), \boldsymbol{\xi}(t)), & \boldsymbol{\xi}(0) = \xi_0. \end{cases}$$

One says that an interval of frequencies $J \subset (0, +\infty)$ is non trapping if for any subinterval I any trajectory $\boldsymbol{x}$ with initial energy in I escapes any ball of radius R provided one waits long enough (depending on I and R).

Remark 1.3.3. In the free Euclidean setting the symbol is $|\xi|^2$, independent of x, which yields bicharacteristic curves of the form $(\boldsymbol{x}(t), \boldsymbol{\xi}(t)) = (2\xi t + x_0, \xi_0)$. This implies that all classical trajectories escape at infinity both in the past and in the future. Once we change the geometry of our domain by considering a metric perturbation new phenomena can occur and therefore generate some trapped trajectories. For instance one can think of a domain where we smoothly glue a sphere to a plane and the trajectories which enter the sphere will not escape at infinity.

Intuitively, the mass of a solution to the Schrödinger or wave equation tends to move along geodesics. So if there is a trajectory which gets captured in a bounded region we can not see the energy decay phenomenon previously described in Section 1.3.1. In [Wan87] and [Wan88] the author showed how in the perturbed Euclidean case non trapping is not only necessary, but also sufficient to obtain time decay for the propagator $e^{it(-\Delta+V)}$. The following example is developed in detail in [Wan88].

Example 1.3.4. Consider the Euclidean case and the semiclassical operator $-\lambda^{-2}\Delta + V$. Operating the rescaling $\widetilde{x} = \lambda^{1/2}x$ the operator becomes $\lambda^{-1}\Delta + V(\lambda^{1/2}\cdot)$ which is unitarily equivalent to $-\lambda^{-2}\Delta + V$ and therefore the two propagators must have equivalent decay properties. If we want some decay in time for the localized propagator $\chi(-\lambda^{-2}\Delta + V)e^{it\lambda(-\lambda^{-2}\Delta+V)}$ in a weighted L^2_s space the same must hold for the propagator of the rescaled operator in a space with rescaled weights. Let $(\boldsymbol{x}(t), \boldsymbol{\xi}(t))$ the Hamiltonian flow of $|\xi|^2 + V$ with initial condition (x_0, ξ_0). One can show that, if the rescaled propagator has a decay rate of $O(\langle t\rangle^{-\nu})$ for some $\nu > 0$ then

$$|\boldsymbol{x}(t)|^{-s}\langle x_0\rangle^{-s}\chi(|\xi_0|^2 + V(x_0)) \lesssim O(\langle t\rangle^{-\nu}).$$

Hence one must have

$$|\boldsymbol{x}(t)| \to \infty \text{ as } |t| \to \infty \qquad \text{whenever} \qquad \chi(|\xi_0|^2 + V(x_0)) \neq 0$$

which is verified when χ is a cutoff on a non trapping interval.

As we have seen in several examples up to now, resolvent estimates are tightly linked to the behavior of the operator e^{-itP} for large times, since trapped trajectories prevent time decay they also have an influence on the resolvent behavior. In the high frequency limit, resolvent bounds are affected, at worse, by an exponential loss in the sense that the constants bounding the operator norm of $(P - \lambda^2 + i0)^{-1}$ blow up exponentially in λ, this is due to the presence of trapped geodesics, as one can see in the example of Proposition 1.14 [RT15]. Compared to what we described in Remark 1.3.1 then, the situation is worse in the limit $\lambda \to \infty$ than in the asymptotic regime $\lambda \to 0$.

The fact that in the general case we only have an exponential bound has consequences on energy decay results, however it guarantees a rate of decay which is at least logarithmic ([Bur98], [Bur02a]). In Chapter 3 we will be interested in the limiting resolvent in the high frequency regime, however in our approach we will not make explicit assumptions on the geodesic flow and therefore we will see this exponential loss.

Remark 1.3.5 (Decay of the potential II). In Remark 1.3.2 we discussed the fact that we consider decaying potentials and that the assumptions needed differ based on the region of the spectrum we are interested in. We have already seen how, if we factorize the spectral parameter, we end up with a normalized resolvent at energy one, that is

$$(h^2 P_0 + h^2 V - 1 + i\varepsilon)^{-1}$$

where $h = \lambda^{-1}$. For high frequencies $h << 1$, so our new operator $h^2 P_0 + h^2 V$ is a small perturbation of the free semiclassical operator whenever V has some decay, hence we can simply consider $V(x) = O(\langle x \rangle^{-\nu})$ for some $\nu > 0$, as opposed to the low frequency case were we assume a decay at infinity like $|x|^{-2-\nu}$.

1.3.2.1 Some references

We have discussed how the presence of trapping is responsible for the worsening of the constants in the limiting absorption principle. On the contrary, in the non trapping case the constants have the same blowup of the low frequency case. A bound of $O(\lambda^{-1})$ was proved in the Euclidean setting in [RT87] with potentials whose decay depend on the dimension and in [GM89], [Wan87] in the case of long range potentials. In this last paper the result is then applied to prove local energy decay and the rate of decay is arbitrarily fast. As discussed in Section 1.3.1 it is the low frequency limit which gives constraints on the rate of decay.

To find domains that exhibit trapping we need to consider non Euclidean geometries, in this case the bounds in the limiting absorption principle are exponential at worse and the best case is obtained when assuming non trapping. For the non trapping condition we mention [Bur02b] for compact metric perturbations of the Laplacian, [VZ00] for the semiclassical resolvent in a scattering metric, or [Rob92] for non compact perturbations. In the latter the author proves equivalence between non trapping and inverse power bounds in the limiting absorption principle. Exponential bounds can be found in [Bur98] for compact perturbations of the Laplacian and later in [Bur02a] which also includes a perturbation by potential. In these works the author proves that the resolvent $(P - \lambda^2 + i0)^{-1}$ can be extended in the lower half plane and that there are no poles in a strip of exponential width. Similar results were obtained for short range perturbations of the flat metric in [Vod00]. For a generic selfadjoint operator satisfying the black box scattering assumptions, the semiclassical resolvent was studied in [BP00]. The authors show that if in the trapped region the resolvent has an exponential bound then one can extend this estimate to the whole domain. For other types of geometry we mention the works [Vod02] and [CV02]. In [Vod02] the metric has a separation of variables at infinity, which is something we will assume also in this thesis, while this is not true for [CV02] and in both cases these results include manifolds with cusps.

In [RT15] in the case of asymptotically conical manifolds the authors prove in Theorem 1.7 a unified limiting absorption principle on the whole positive half line. We remark that they require their potential to decay like $|x|^{-1-\nu}$ which is more than what we stated in Remark 1.3.5. This is due to the fact that they want to consider not only the region $\lambda \gg 1$, but also the case of bounded λ. Hence the condition $V = O(|x|^{-1-\nu})$ at infinity is needed to exclude the case of the Wigner-Von Neumann potential, which decays like $|x|^{-1}$ and has an eigenvalue at one that would destroy the limiting absorption principle.

1.3.3 Bottom of the spectrum and 0 resonance

In the previous section we have seen how it is necessary to make a distinction between high and low frequencies since in the two cases different problems arise. On top of this, in the low frequency picture an important role is the one of zero. For P with a sufficiently nice potential perturbations, the essential spectrum is the positive half line $[0, \infty)$, hence zero represents the bottom of the essential spectrum. The study of the nature of the frequency zero will be the object of Chapter 4.

Assume we have an eigenvalue λ embedded in the essential spectrum, then there exists a normalized function $f_0 \in D(P)$ such that $Pf_0 = \lambda f_0$. This implies that a solution to the Schrödinger equation with initial data f_0 is given by $f = e^{-itP} f_0 = e^{-it\lambda} f_0$ with $\lambda \geqslant 0$. Then f is a stationary solution and enjoys the following property: for any $\varepsilon > 0$ there is sufficiently large R such that

$$\inf_t \int_{|x| \leqslant R} |f(t,x)|^2 dx \geqslant 1 - \varepsilon,$$

that is almost all the mass of the solution is in a bounded region for any given time. Similarly to the case of trapped geodesics, this is the opposite than what we need if we want to prove some results of local energy decay and for this reason the limiting absorption principle, which rules out the existence of embedded eigenvalues (see Example 1.2.5), is an important tool in the proof of energy decay estimates.

Another object that is an obstacle to energy decay are resonances. A selfadjoint operator on a compact domain has a discrete spectrum, on unbounded domains we can think of resonances as a generalization of eigenvalues. As time decay is tightly linked to resolvent properties we can expect eigenvalues and resonances to be relevant quantities when studying resolvents. At the beginning of Section 1.3 we have seen that the limiting absorption principle for intermediate frequencies is well known and hence, for well-behaved potentials, we do not have embedded eigenvalues in any bounded sub interval of $(0, \infty)$. The case of zero is more delicate and actually there are fairly easy examples for which we have a zero eigenvalue.

Example 1.3.6 (Eigenvalue at zero). Consider on $\mathbb{R}^3$ the potential $V = -c\mathbb{1}_{B(0,1)}$. We look for an L^2 solution of $(-\Delta + V)u = 0$, in radial coordinates we take u with separate variables of the form

$$u(r, \omega) = r^{-1} v(r) Y_{lm}(\omega)$$

where Y_{lm} is a spherical harmonic (i.e. $-\Delta Y_{lm} = l(l+1)Y_{lm}$). The eigenvalue equation then becomes

$$-v'' + r^{-2} l(l+1)v + Vv = 0.$$

In the region $\{r > 1\}$ where V vanishes the function r^{-l} is a solution. Then one can construct a solution which is regular in the region $\{r \leqslant 1\}$ and that agrees with r^{-l} in the region $\{r > 1\}$ and is therefore square integrable, making u a zero eigenfunction.

More generally, an eigenvalue at 0 can occur for potentials decaying arbitrarily fast at infinity, as the one in the example, also in higher dimensions (see Example 8.4 in [Jen80]).

The obstruction generated by zero eigenvalues or resonances can be seen clearly in [JK79] and all the works which take a series expansion approach, where a zero resonance or eigenvalue generates singular terms in the expansion. In these works, estimates on the limiting resolvents are used to prove decay in time of the Schrödinger evolution operator on $\mathbb{R}^n$. There, the existence of a resonance at energy zero is primarily seen from the point of view of preventing local energy decay so the authors use the definition of zero resonance which is better suited to this point of view. The condition

$$\limsup_{\lambda \to 0} \|(-\Delta + V - \lambda^2 \pm i0)^{-1}\|_{L^2_w \to L^2_w} < \infty$$

with L^2_w a weighted L^2 space is equivalent to the absence of a zero resonance or eigenvalue and hence weighted L^2 spaces are used to define resonances, which is a quite common approach. As we have mentioned, resonances can be thought of as generalizations of eigenvalues, so if a zero eigenfunction is an L^2 function satisfying the equation

$$Pu = 0 \tag{1.3.1}$$

a zero resonant state is a solution of (1.3.1) which is not in L^2, but rather in a weighted space. In the case of L^2 weighted spaces on $\mathbb{R}^n$ with $n \geqslant 3$, zero is said to be a resonance for the operator $P_0 + V = -\Delta + V$ if there exists f such that

$$(-\Delta + V)f = 0, \quad f \notin L^2 \quad \text{but} \quad \langle x\rangle^{-\sigma} f \in L^2 \ \sigma > \frac{1}{2}. \tag{1.3.2}$$

This definition is used for example in the already mentioned [JK79] and [Rau78] and in many other works in the area of energy decay estimates (for example in [RS04], [Gol06a], [Gol06b], [Aaf21], [SW22]). It is however limiting in the fact that weighted spaces are not scaling invariant. Moreover in most applications where this definition is considered, the potential is required to have some pointwise behavior, which is also not a scaling invariant condition. An interesting question is then to see if one can replace this pointwise decay with integrability assumptions. The results in Chapter 4 go in this direction, where we make integrability assumptions on the potential to recover integrability properties on the zero resonant state and eigenstate. Details about such results can be found in Section 1.5.3. In particular we can see in the following example that a zero eigenstate or resonance is not necessarily integrable.

Example 1.3.7 ([Sch07]). Let $n \geqslant 3$ and consider the Aubin-Talenti function

$$\phi(x) = (n(n-2))^{\frac{n-2}{4}} \langle x\rangle^{2-n}.$$

Since it satisfies the relation $-\Delta\phi = \phi^{\frac{n+2}{n-2}}$ it is solution of the equation

$$(-\Delta + V)\phi = 0 \quad V = -\phi^{4/(n-2)}.$$

The potential V behaves at infinity like $|x|^{-4}$ and is therefore in $L^{n/2}$. Moreover $\phi \in \dot{H}^1$, since $|\nabla\phi| = O(1)|x|\langle x\rangle^{-n}$, and it is never in L^1. The function ϕ behaves at infinity like $|x|^{2-n}$ which is not square integrable only for $n = 3, 4$ hence it is a zero resonance for $n = 3, 4$ only. Actually, in this example there can not be zero resonances in higher dimensions, we explain why in the following Remark 1.3.8. We have just showed that in dimensions three and four there is a 0 resonant state which is not in L^1.

We also observe that

$$(-\Delta + \widetilde{V})\partial_j\phi = 0 \quad \widetilde{V} = -\frac{n+2}{n-2}\,\phi^{4/(n-2)}$$

where $\partial_j\phi = O(1)x_j\langle x\rangle^{-n} \in L^2$ hence $\partial_j\phi$ is a 0 eigenfunction which is not in L^1.

While for a 0 eigenvalue we have given a pretty easy example, resonances are considered to be a less frequent occurrence. Moreover, at least in the Euclidean case that we will consider in Chapter 4, in the presence of sufficiently nice potentials it is a phenomenon limited to low dimensions as we explain in the following remark. On the contrary, in the non Euclidean case there can be zero resonances in every dimension, as one can see for example in [Wan03].

Remark 1.3.8 ($\mathbb{R}^n$, $n \geqslant 5$). Consider a potential $V(x) \in L^{n/2}(\mathbb{R}^n)$ with $n \geqslant 5$, we can then decompose V into the sum $V = V_1 + V_2$ with

$$V_1 \in C_0^\infty(\mathbb{R}^n), \quad \|V_2\|_{L^{n/2}} \ll 1.$$

Assume f is a non L^2 solution of $(-\Delta + V)f = 0$, we take f belonging to the homogeneous

Sobolev space $\dot{H}^1$. Using the decomposition of V, we can rewrite the equation as

$$(I + (-\Delta)^{-1}V_2)f = -(-\Delta)^{-1}V_1 f.$$

We recall the Lorentz spaces $L^{p,q}$ defined by the quantities

$$\|f\|_{L^{p,q}} := p^{1/q}\left(\int_0^\infty t^{q-1}(d_f(t))^{q/p}dt\right)^{1/q}$$

for $q < \infty$ or

$$\|f\|_{L^{p,\infty}} := \sup_{t\geqslant 0} t d_f(t)^{1/p}$$

otherwise. Thanks to Hardy-Littlewood-Sobolev and Hölder inequalities

$$V_2 : L^2 \to L^{\frac{2n}{4+n},2}, \qquad (-\Delta)^{-1} : L^{\frac{2n}{4+n},2} \to L^2$$

and since

$$\|(-\Delta)^{-1}V_2\|_{L^2\to L^2} \lesssim \|V_2\|_{L^{n/2}} \ll 1$$

the operator $(I + (-\Delta)^{-1}V_2)$ is invertible from L^2 to L^2 via a Neumann series. This implies that f is given by

$$f = -(I + (-\Delta)^{-1}V_2)^{-1}(-\Delta)^{-1}V_1 f$$

and

$$\|f\|_{L^2} \lesssim \|(-\Delta)^{-1}V_1 f\|_{L^2} \simeq \|V_1 f\|_{\dot{H}^{-2}}.$$

Now since $n \geqslant 5$ we have the Sobolev embeddings

$$\dot{H}^1 \hookrightarrow L^{\frac{2n}{n-2},2}, \qquad \dot{H}^2 \hookrightarrow L^{\frac{2n}{n-4},2}$$

and the dual embedding

$$L^{\frac{2n}{n+4},2} \hookrightarrow \dot{H}^{-2}.$$

We obtain that f is square integrable from

$$\begin{aligned}\|f\|_{L^2} &\lesssim \|V_1 f\|_{\dot{H}^{-2}} \lesssim \|V_1 f\|_{L^{\frac{2n}{n+4},2}} \lesssim \|V_1\|_{L^{n/3,\infty}}\|f\|_{L^{\frac{2n}{n-2},2}}\\ &\lesssim \|V_1\|_{L^{n/3,\infty}}\|f\|_{\dot{H}^1} < \infty.\end{aligned}$$

with $V_1 \in L^{n/3}(\mathbb{R}^n)$ since it is in $C_0^\infty(\mathbb{R}^n)$. We conclude that for a potential in $L^{n/2}(\mathbb{R}^n)$, $n \geqslant 5$ there are no zero resonances. For this reason in Chapter 4 we only consider dimensions three and four.

Remark 1.3.9. We remark that if we take the potential in a slightly larger class than $L^{n/2}$ we can find solutions of the equation $(-\Delta + V)f = 0$ that are not L^2, nor in any weighted L^2 space. Indeed, the equation

$$(-\Delta - \frac{n(n-4)}{4|x|^2})f = 0$$

is solved by

$$f(x) = |x|^{-n/2},$$

where f is not in L^2, it would be in any weighted space $L^2_\sigma = L^2(\mathbb{R}^n, \langle x\rangle^{-\sigma}dx)$ for any $\sigma > 0$ but only near infinity, and instead is in the weak Lebesgue space $L^{2,\infty}$. While f is a non L^2 solution of the 0 eigenvalue equation, this example is not in contrast with the previous remark for two simple reasons: the potential is not in $L^{n/2}$, but rather in the larger class $L^{n/2,\infty}$, and in the previous remark we were looking for $\dot{H}^1$ solutions, while f is not in $\dot{H}^1$ (since $|\nabla f| = |x|^{-n/2-1}$).

1.3.4 Connected problems

Although we will not treat these aspects, we mention some other subjects related to resolvent estimates.

Strichartz inequalities Strichartz inequalities, are bounds of the form

$$\|e^{-itP}u\|_{L^p_t,L^q_x} \leqslant \|u\|_{L^2} \tag{1.3.3}$$

for admissible couples (p,q) verifying $\frac{n}{q}+\frac{2}{p}=\frac{n}{2}$ $p\in(2,\infty]$. Let us compare this to the local energy decay estimates of the form

$$\|\langle x\rangle^{-\nu}e^{-itP}\langle x\rangle^{-\nu}\|_{L^2\to L^2} \leqslant O(t^{-n/2}).$$

Strichartz estimates can be seen as a time-averaged version of local energy decay estimates, where the presence of weights, like $\langle x\rangle^{-\nu}$, in the local energy results is reinterpreted as L^q integrability in the Strichartz case. We recall that the Schrödinger propagator is linked to the limiting resolvents via a Fourier transform where t is the dual variable to λ (cfr. (1.2.9)). Hence, intuitively, the need in Strichartz estimates of properties which are global in time induces a localization in spectral parameter λ. In this way, inequalities like (1.3.3) are connected to low frequency results.

In [RS04] the authors introduced a way to pass from limiting resolvents to Strichartz estimates in the perturbed Euclidean case, $P=-\Delta+V$. This is done writing the solution to the Schrödinger equation via Duhamel formula,

$$e^{-it(-\Delta+V)} = e^{it\Delta} - i\int_0^t e^{i(t-s)\Delta}Ve^{-is(-\Delta+V)}\,ds.$$

Then $e^{it\Delta}$ can be bounded via Strichartz estimates for the free operator $-\Delta$. Moreover inhomogeneous Strichartz estimates imply that if F is the source term in the free Schrödinger equation then

$$\|\int_0^t e^{i(t-s)\Delta}F(s)\,ds\|_{L^p_tL^q_x} \lesssim \|F\|_{L^{\widetilde{p}}_tL^{\widetilde{q}}_x},$$

under suitable conditions on $p,q,\widetilde{p}$ and $\widetilde{q}$. Using this inequality with source term $Ve^{-is(-\Delta+V)}$ reduces the problem to the evaluation of the operator $Ve^{-is(-\Delta+V)}$ from L^2 to $L^{\widetilde{p}}_tL^{\widetilde{q}}_x$. As usual and thanks to the spectral theorem, $e^{-is(-\Delta+V)}$ as a function of the Hamiltonian $-\Delta+V$ can be studied via the mapping properties of $Im(-\Delta+V-\lambda^2+i0)^{-1}$. In [BM a] for example, the authors prove global Strichartz estimates with this method in the non Euclidean case and use the resolvent estimates proved in [BR14b].

Smoothing effects The Schrödinger propagator can have a smoothing effect on the initial data, by this we mean that a solution $e^{-itP}u$ with initial condition u in L^2 is $H^{1/2}$ for almost every t. For example in the Euclidean case, we have a global smoothing effect when the map

$$L^2(\mathbb{R}^n) \ni u \mapsto \langle x\rangle^{-1/2-\nu}e^{-itP}u \in L^2(\mathbb{R}_t, H^{1/2}(\mathbb{R}^n))$$

is continuous. Again, one needs to take an L^2 average in time, in this case of the $L^2\to H^{1/2}$ norm of the propagator. As for Strichartz estimates, global in time results for the propagator are linked to low frequency results for the limiting resolvent. Local smoothing was proved for example in [BaK92], or in [CS88] for more general operators with symbols behaving like $|\xi|^m$ at infinity. In the case of local smoothing cutoff resolvent estimates are sufficient as it is shown in [Chr08].

Loss due to trapping In Section 1.3.2 we have seen that trapping can cause resolvent estimates to have an exponential loss, at worst. Improvements on these results can be obtained by allowing only a little amount of trapping (see for example [NZ09], [Dat09], [Chr09]).

The geodesic flow also has an impact on the smoothing effect we described in the previous section. In [Doi00], [Doi96] the author shows that the set of non smoothing point contains all closed orbit and smoothing fails if and only if there exists a trapped complete geodesic. An ε loss of smoothing is also showed in [Bur02b] for trapping obstacles, where the author also proves a necessity result similar to Doi in the case of exterior obstacle Cauchy problems. This loss of smoothing effect was quantified for example in [CW13] and [CM14] based on the type of trapping. On the contrary, for Strichartz estimates the case seems to be that it is the dimension of the trapped set, and not the type of trapping, which determines the loss ([Chr13]).

Complex resonances In Section 1.3.3 we talked about how a zero resonance has an impact on resolvent properties and hence on energy decay. Complex resonances also generate a particular dynamical behavior. In the case of an embedded eigenvalue λ we have seen that λ determines the rate of oscillation of the solution which corresponds to taking the eigenstate as initial data. In unbounded domains there is the added feature of decay, and since we have said that resonances somehow correspond to eigenvalues in the unbounded framework, we can see this decay when taking a solution with resonant initial data. Indeed, let ψ_0 an initial datum satisfying

$$P\psi_0 = (\lambda - i\Gamma/2)\psi_0.$$

Then the corresponding solution of the Schrödinger equation is

$$\psi(t,x) = e^{-it\lambda - t\Gamma/2}\psi_0(x),$$

while the real part λ still represents the rate of oscillation of the wave function, Γ gives the decay in time.

From a physical point of view this means that resonances are interpreted as metastable states, in the sense that they behave like a bound state for a finite amount of time (when the bound state structure $e^{-it\lambda}\psi_0(x)$ is prevalent) and eventually breaks out changing its behavior to the one of a scattering state (when $e^{-t\Gamma/2}\psi_0(x)$ prevails).

In Section 1.3.3 we have defined resonances as solutions of eigenvalue equations which do not belong to L^2. Another point of view can also be used. Indeed in [DZ19] for example, resonances are defined as the poles of the meromorphic continuation of $(P - z^2)^{-1}$ (or equivalently of the Green function) to the lower half plane. This is the definition used for example in [Bur98] where the author obtains boundedness for the resolvent thanks to a stripe where its extension is pole-free.

1.4 Manifold setting

The study of partial differential equations on manifolds is a field which attracts a considerable amount of interest with the hopes of generalizing the features which are known on $\mathbb{R}^n$ to more and more general domains. This interest has its origin in physics, where one can think, for example, to some classical phenomenon like the propagation of a wave outside of an obstacle. Another important example of non Euclidean background is the curved space-time in general relativity, where the metric tensor is given by a solution of the Einstein field equations and different solutions represent the metric induced by different configurations of mass, momentum and stress.

Except in Chapter 4, in this thesis we will consider problems set on a curved background.

More precisely we will consider a non compact Riemannian manifold (M, G_M) of the form

$$M = K \cup (M \setminus K), \qquad K \text{ compact}$$

where $M \setminus K$ is the infinite end. The assumptions we will make will be on the asymptotic behavior of the metric G_M on the infinite end.

Some basic examples that our assumptions will include are $\mathbb{R}^n$ outside of an obstacle, or more generally domains that outside of a compact set are close, in a sense that we specify later, to $\mathbb{R}^n$ or an Euclidean cone. In particular this includes compact perturbations of the euclidean metric or a perturbed hyperbolic or conical metric.

1.4.1 Definitions

Let (M, G_M) an n dimensional Riemannian manifold and $K \subset M$ a compact subset. We will prescribe, up to diffeomorphism, the metric on the manifold end $M \setminus K$ which will be given by the metric on a product manifold.

Definition 1.4.1. *Let $(S, \overline{g})$ an $n-1$-dimensional closed Riemannian manifold and $R > 0$, we define a metric half cone as the product manifold*

$$((R, +\infty) \times S,\ dr^2 + r^2\overline{g}).$$

We will refer to S as the angular manifold.

If M has an infinite end $M \setminus K$ which is isometric to a metric half cone then we say that M is ***exactly conical at infinity****.*

Example 1.4.2 ($\mathbb{R}^n$ outside of a compact set). The most common example of a cone is given by taking

$$(S, \overline{g}) = (\mathbb{S}^{n-1}, \sigma)$$

where σ os the usual metric on $\mathbb{S}^{n-1}$. We obtain $(R, +\infty) \times \mathbb{S}^{n-1}$ with the metric

$$dr^2 + r^2\sigma \tag{1.4.1}$$

which is the usual representation of the Euclidean metric in radial coordinates. In this case the half cone would be a description of $\mathbb{R}^n \setminus B(0, R)$ with the usual flat metric.

Instead of an exact cone as the one defined above, we can also consider a perturbed cone by replacing the exact metric $\overline{g}$ with a perturbed one $g(r)$, also depending on the radial variable. We then need to specify in which sense we assume $g(r)$ to be a perturbation of $\overline{g}$.

For fixed r, $g(r)$ is a metric on S and we have the smooth family of metrics

$$\begin{aligned} g : (R_0, \infty) &\to \Gamma(T_2^0(S)) \\ r &\mapsto g(r) \end{aligned}$$

where $\Gamma(T_2^0(S))$ are the sections of the tensor bundle $T_2^0(S) = T^*(S) \otimes T^*(S)$ and $g(r, \omega)$ is a bilinear form on the tangent space $T_\omega(S)$. In the sections $\Gamma(T_2^0(S))$ we can define a topology given by the seminorms

$$N_{m,J}(f) = \sum_{|\alpha| \leqslant m} \|\partial^\alpha f_{i,j}\|_{L^\infty(J)}$$

where $f_{i,j}$ are the coefficients representing $f \in \Gamma(T_2^0(S))$ around a point $\omega \in S$ on a basis of $T_2^0(S)$ and J is a compact subset of the coordinate patch corresponding to the point ω.

Definition 1.4.3. *We say that a function $f(r)$ taking values in the space of sections $\Gamma(T_2^0(S))$ is in $S^{-\nu}$ if*

$$N_{m,J}(\partial_r^l f) \lesssim \langle r \rangle^{-\nu - l} \quad \text{for any } m, l \in \mathbb{N},\ J \text{ compact set}$$

where

$$\langle r \rangle = (1 + r^2)^{1/2}.$$

Definition 1.4.4. *Let $(S, g(r))$ an $n-1$-dimensional closed Riemannian manifold and $R > 0$, let $\overline{g}$ another Riemannian metric on S. We define a perturbed half cone as the product manifold*

$$((R, +\infty) \times S, \ dr^2 + r^2 g(r)).$$

where $g(r) - \overline{g} \in S^{-\nu}$ with $\nu > 0$.

If M has an infinite end $M \setminus K$ which is isometric to a perturbed half cone we say that M is ***asymptotically conical****.*

This definition is what we will consider in Chapter 2.

Example 1.4.5 (Asymptotically flat metric). As in Example 1.4.2 we take as the angular manifold the sphere, but this time with a perturbed metric $\widetilde{\sigma}(r)$. Then the perturbed cone

$$((R, +\infty) \times \mathbb{S}^{n-1}, \ dr^2 + r^2 \widetilde{\sigma}(r))$$

will represent $\mathbb{R}^n \setminus B(0, R)$ with a perturbed metric. With the assumption that $\widetilde{\sigma}(r)$ gets close to σ as $r \to \infty$ the perturbed metric will approach the flat Euclidean metric near infinity.

Remark 1.4.6. The asymptotically conical end we just defined includes also the so-called scattering metrics, terminology introduced by Melrose [Mel95] and used in several works (for example [MZ96], [GHS13a], [GHS13b], [VW09], [HW08]). These type of metrics are defined as a class of Riemannian metrics on the interior of a compact manifold X of the form

$$g_s = \frac{dx^2}{x^4} + \frac{h}{x^2}$$

with h a smooth metric on the boundary ∂X and x a boundary defining function (that is $x \to 0$ on ∂X). Consider Example 1.4.2, where $r = |\mathbf{x}|$ and $\mathbf{v} = \frac{\mathbf{x}}{|\mathbf{x}|}$. With the change of variable $r = x^{-1}$ the metric (1.4.1) becomes a scattering metric. More in general, g_s is a particular case of the metric in Definition 1.4.4 when $g(r)$ is the Tayolr expansion of $h(x) = h(r^{-1})$ around $x = 0$.

By changing the volume factor when taking the product metric we can construct other types of ends, on top of the conical ones we just showed.

Definition 1.4.7. *Let S an $n-1$-dimensional closed manifold, $\overline{g}$ and $g(r)$ two Riemannian metrics on S and $R > 0$. We define an hyperbolic end as the product manifold*

$$((R, +\infty) \times S, \ dr^2 + e^{2r} \overline{g})$$

and a perturbed hyperbolic end as

$$((R, +\infty) \times S, \ dr^2 + e^{2r} g(r))$$

where $g(r) - \overline{g} \in S^{-\nu}$ with $\nu > 0$.

If M has an infinite end $M \setminus K$ isometric to a perturbed hyperbolic end we will say that M is ***asymptotically hyperbolic****.*

Remark 1.4.8. In the same way of Remark 1.4.6, asymptotically hyperbolic manifolds include the model of manifolds defined as the interior of a compact manifold via a boundary defining function.

For example in Chapter 5 in [DZ19] the authors define an asymptotically hyperbolic manifold (X, g) as the interior of a compact manifold $\overline{X} = X \cup \partial X$ with a boundary defining function x, that is $x \to 0$ on ∂X. In the canonical product structure, the metric is of the form

$$\frac{dx^2 + h(x)}{x^2} \tag{1.4.2}$$

with h a metric on the boundary ∂X, this is a special case of a perturbed hyperbolic end. Indeed, from (1.4.2) we can recover a metric of the form $dr^2 + e^{2r}g(r)$ after the change of variable $r = -\ln x$ and when $g(r)$ is the Taylor expansion of $h(x) = h(e^{-r})$ around $x = 0$. We can see this for example for the hyperbolic cylinder, given by

$$(\mathbb{R} \times \mathbb{S}^1,\ dr^2 + (\cosh r)^2\sigma),$$

with σ the usual metric on the sphere. The canonical structure is obtained by passing from the boundary defining function $(\cosh r)^{-1}$ to e^{-r}. This means we are performing exactly the change of variable $x = e^{-r}$ we just mentioned and hence the canonical structure of an hyperbolic cylinder is a perturbed hyperbolic end in the sense of Definition 1.4.7, namely

$$((R, +\infty) \times \mathbb{S}^1,\ dr^2 + e^{2r}\sigma).$$

Remark 1.4.9. Both a (perturbed) conical end and an hyperbolic one have infinite volume. We will not consider the case of infinite ends with finite volume, such as cusps.

In Chapter 3 we will actually consider a more general type of end which in some sense ranges from the conical type to the hyperbolic one, that is a product manifold of the form

$$((R, +\infty) \times S,\ dr^2 + l(r)^{-2}g(r)) \tag{1.4.3}$$

with $l(r) : (R, +\infty) \to \mathbb{R}^+_*$ and such that

$$O(1) \geqslant -\frac{l'(r)}{l(r)} \geqslant \frac{c}{r} \tag{1.4.4}$$

for a constant $c > 0$. The conical case corresponds to $l(r) = r^{-1}$ and the hyperbolic one to $l(r) = e^{-r}$.

1.4.2 The Laplace-Beltrami operator

On (M, G_M) we will consider the Laplace-Beltrami operator with some lower order perturbations.

Example 1.4.10. On $\mathbb{R}^n$ the Laplacian as a selfadjoint operator can be defined via the Friedrich extension induced by the quadratic form

$$q_0(u, v) = \langle \frac{1}{i}\nabla u, \frac{1}{i}\nabla v\rangle_{L^2(\mathbb{R}^n)} = \int \langle \frac{1}{i}\nabla u, \overline{\frac{1}{i}\nabla v}\rangle_{\mathbb{R}^n} dx$$

defined on the domain

$$D(q_0) := \{\text{closure of } C_0^\infty(\mathbb{R}^n) \text{ with respect to } \sqrt{q_0(u,u) + \|u\|^2_{L^2(\mathbb{R}^n)}}\ \}.$$

On $D(q_0)$ the quantity $\sqrt{q_0(u,u) + \|u\|^2_{L^2(\mathbb{R}^n)}}$ defines a norm (the H^1 norm in this case). With an integration by parts we see that q_0 is nonnegative, moreover it is closed. Then by Theorem VIII.5 in [RS81] it is the quadratic form of a unique selfadjoint operator. We take as domain for the operator $-\Delta$ the set of u such that $q_0(u, \cdot)$ is continuous in the L^2 topology, that is

$$D(-\Delta) = \{u \in D(q_0) : |q_0(u, v)| \leqslant C(u)\|v\|^2_{L^2(\mathbb{R}^n)}\}.$$

Then for any u in such set $q_0(u,\cdot)$ can be extended to a continuous map on $L^2(\mathbb{R}^n)$ and the Friedrich extension of $-\Delta$ applied to u is the unique element $-\Delta u \in L^2(\mathbb{R}^n)$ (given by the Riesz lemma) verifying

$$q_0(u,v) = (-\Delta u, v)_{L^2(\mathbb{R}^n)}.$$

One defines the operator on a manifold in an analogous way replacing the scalar products of $\mathbb{R}^n$ and $L^2(\mathbb{R}^n)$ with the suitable ones induced by the metric G_M.

In general, we will assume that the metric on $M \setminus K$ is given, up to diffeomorphism, by the one of (1.4.3). So when restricted to this region the Laplace-Beltrami operator will agree with the one induced by the metric $dr^2 + l(r)^{-2}g(r)$.

We now give some explicit expressions in local coordinates for the objects which are involved in the definition of the quadratic form. The scalar product on S is defined via this bilinear form as

$$g(r,\omega)(u,v) = \sum_{i,j=1}^{n-1} u_i g_{i,j}(r,\theta) v_j$$

with $(g_{i,j}(r,\theta))_{i,j}$ a symmetric positive definite matrix, while the inverse of this matrix, $(g^{i,j}(r,\theta))_{i,j}$, defines the gradient

$$\nabla_{g(r)} = (g^{i,j}(r,\theta))_{i,j} \begin{pmatrix} \partial_{\theta_1} \\ \vdots \\ \partial_{\theta_{n-1}} \end{pmatrix} = (\sum_{k=1}^{n-1} g^{i,k}(r,\theta)\partial_{\theta_k})_i.$$

To define the generic product metric on (1.4.3) we take the usual flat metric on the first factor $(R,+\infty)$ and the metric $g(r)$ on the second factor. In local coordinates, the scalar product and gradient are respectively

$$G(r,\omega)(u,v) = u_0 v_0 + \frac{1}{l^2(r)} \sum_{i,j=1}^{n-1} u_i g_{i,j}(r,\theta) v_j,$$

$$\nabla_G = (\partial_r, (l^2(r) g^{i,j}(r,\theta))_{i,j} \begin{pmatrix} \partial_{\theta_1} \\ \vdots \\ \partial_{\theta_{n-1}} \end{pmatrix})^T = \begin{pmatrix} \partial_r, \\ (l^2(r)\sum_{k=1}^{n-1} g^{i,k}(r,\theta)\partial_{\theta_k})_i \end{pmatrix}.$$

In analogy to Example 1.4.10, we then define the Laplace-Beltrami operator on $((R,+\infty)\times S,\ dr^2 + l(r)^{-2}g(r))$ via the quadratic form

$$q(u,v) = \langle \frac{1}{i}\nabla_G u, \frac{1}{i}\nabla_G v\rangle_{L^2_{G_M}} = \int G(r,\omega)(\frac{1}{i}\nabla_G u, \overline{\frac{1}{i}\nabla_G v})\ dvol_{G_M}.$$

Although the Laplacian is an ubiquitous object of study in mathematics, other differential operator also have their interest. In this thesis in particular we will also be interested in a more general differential operator where we allow perturbations of order one.

Example 1.4.11. In Example 1.4.10 we have showed how to define the Laplacian via a quadratic form. Let

$$A = (A_0, \dots, A_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$$

be a smooth vector field. If we consider instead

$$q_m(u,v) = \langle (D-A)u, (D-A)v\rangle, \quad D = \frac{1}{i}\nabla$$

on $C_0^\infty(\mathbb{R}^n)$, by integration by parts one can directly check that q_m is symmetric and positive.

Taking as a domain for q_m the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$(q_m(u,u) + \|u\|^2_{L^2(\mathbb{R}^n)})^{1/2}$$

then q_m is also a closed form and a core for q_m is exactly $C_0^\infty(\mathbb{R}^n)$. Again, by Theorem VIII.15 in [RS81], q_m is the quadratic form of a unique selfadjoint operator P_m that we will call the magnetic Laplacian, in this case on $\mathbb{R}^n$. Since it is defined via q_m, the operator is given by $(D-A)^2$, where the addition of the vector field A generates perturbations of order one.

The vector field A is linked to the magnetic field B by the operation of exterior derivative. For example, in dimension three the relation is

$$B = rotA = \nabla \times A,$$

we say that B is the magnetic field generated by A.

In the manifold setting we define the magnetic Laplacian taking the obvious generalizations. Given A a vector field on $(R,+\infty)\times S$, that is $A(r,\omega) \in \mathbb{R} \times T_\omega(S)$ locally around a point $(r,\omega) \in (R,+\infty)\times S$, the operator will be defined via the quadratic form

$$q_m(u,v) = \langle (D_G - A)u, (D_G - A)v \rangle_{L^2_{G_M}}, \quad D = \frac{1}{i}\nabla_G. \tag{1.4.5}$$

The magnetic field is then obtained by taking the exterior derivative of A.

1.5 Results

In this section we present the results proved in this thesis.

1.5.1 Low frequencies on asymptotically conical manifolds

Chapter 2 is dedicated to the study of limiting resolvents of the Laplace-Beltrami operator with a potential perturbation on manifolds with asymptotically conical end (Definition 1.4.4) and in the low frequency regime. We apply these results to obtain time decay estimates for the Schrödinger, wave and Klein-Gordon propagators in weighted L^2 spaces. The content of this chapter is the object of [Gra23] and generalizes some of the results in [BB21] to a larger class of metric perturbations.

1.5.1.1 Time decay

The objects we are interested in are the evolution operators of the Schrödinger, wave and Klein-Gordon equations. These operators map the initial condition to the solution of the equation. If we consider the Schrödinger case, given the selfadjointness of P the operator e^{-itP} is unitary, so if we look at it as an operator from L^2 to L^2 its norm is constant in t. However, if we consider compactly supported initial data and focus our attention on a compact set we can obtain smallness of the L^2 norm after a sufficiently long time interval. Let K a compact set and $\mathbb{1}_K f$ the compactly supported initial data, if $\mathcal{E}_t$ is the evolution operator what we just described can be modeled by the asymptotic behavior

$$\|\mathbb{1}_K \mathcal{E}_t \mathbb{1}_K f\|_{L^2} \leqslant c(t)\|f\|_{L^2} \quad \text{with} \quad c(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Remark 1.5.1. The term dispersive estimates usually denotes $L^1 \to L^\infty$ estimates for the cutoff evolution operator, first studied in [JSS91]. In this thesis we will only consider the weighted $L^2 \to L^2$ setting.

Before giving our result we recall the behavior of the evolution operators on the Euclidean setting.

- If $\mathcal{E}_t = e^{-itP}$ is the Schrödinger evolution operator
$$\|\mathbb{1}_K e^{-itP}\mathbb{1}_K f\|_{L^2\to L^2} \leqslant O(t^{-n/2}).$$
- If $\mathcal{E}_t = (\cos(t\sqrt{P}), \frac{\sin(t\sqrt{P})}{\sqrt{P}})$ is the wave evolution operator
$$\|\mathbb{1}_K \cos(t\sqrt{P})\mathbb{1}_K f\|_{L^2\to L^2} \leqslant O(t^{-n}), \qquad \|\mathbb{1}_K \frac{\sin(t\sqrt{P})}{\sqrt{P}}\mathbb{1}_K f\|_{L^2\to L^2} \leqslant O(t^{1-n}).$$
- If $\mathcal{E}_t = (\cos(t\sqrt{P+1}), \frac{\sin(t\sqrt{P+1})}{\sqrt{P+1}})$ is the Klein-Gordon evolution operator
$$\|\mathbb{1}_K (\cos(t\sqrt{P+1})\mathbb{1}_K f\|_{L^2\to L^2} + \|\mathbb{1}_K \frac{\sin(t\sqrt{P+1})}{\sqrt{P+1}}\mathbb{1}_K f\|_{L^2\to L^2} \leqslant O(t^{-n/2}).$$

Remark 1.5.2. More generally, instead of cutoffs $\mathbb{1}_K$ we can consider smooth functions whose influence vanishes at infinity, in this way we give less weight to the behavior of the evolution operator at spatial infinity. As we anticipated the results on the evolution are an application of the ones on the resolvents and we have already remarked how the limiting resolvents can not exist as bounded operators simply on L^2. The presence of these weights in the time decay estimates corresponds to the need for weighted spaces in the proof of the limiting absorption principle.

In the asymptotically conical setting we are able to replicate the same decay rate known on $\mathbb{R}^n$, up to the wave equation in odd dimension. Before stating the result we recall some details on the objects involved.

- M is a manifold with an infinite end that we assume of product form $(R, +\infty) \times S$, so outside of a compact set we have separation of variables into radial and angular domain.
- The infinite end $(R, +\infty) \times S$ is of asymptotically conical type: we introduce a metric perturbation on the angular component S and the metric is $dr^2 + r^2 g(r)$ with $g(r)$ perturbation of a generic fixed (r independent) metric. The reader can refer to Definition 1.4.4 for a precise statement.
- V is a potential that on the unbounded region $(R, +\infty) \times S$ decays in the radial direction with a prescribed rate.
- We recall the notation $\langle x\rangle = (1 + |x|^2)^{1/2}$.

Theorem 1.5.3 (Theorem 2.1.1)**.** *Let (M, G_M) an asymptotically conical manifold of dimension $n \geqslant 3$ and P_0 the Laplace-Beltrami operator on M. Let $f \in C_0^\infty(\mathbb{R})$ and V a non negative multiplicative potential that on $M \setminus K$ agrees with $O(\langle r\rangle^{-2-\nu})$ with $\nu > 0$. Then*

i) Schrödinger flow:
$$\|\langle r\rangle^{-\alpha} f(P_0+V)e^{it(P_0+V)}\langle r\rangle^{-\alpha}\|_{L^2} \lesssim \langle t\rangle^{-\frac{n}{2}}$$
for $\alpha > \left[\frac{n}{2}\right] + 2$.

ii) Vave flow:
$$\|\langle r\rangle^{-\alpha} f(P_0+V)\frac{\sin(t\sqrt{P_0+V})}{\sqrt{P_0+V}}\langle r\rangle^{-\alpha}\|_{L^2} \lesssim \langle t\rangle^{1-n}$$
and
$$\|\langle r\rangle^{-\alpha} f(P_0+V)e^{it\sqrt{P_0+V}}\langle r\rangle^{-\alpha}\|_{L^2} \lesssim \langle t\rangle^{-n}$$
for $\alpha > n+1$.

iii) Klein-Gordon flow:
$$\|\langle r\rangle^{-\alpha} f(P_0+V)e^{it\sqrt{P_0+V+1}}\langle r\rangle^{-\alpha}\|_{L^2} \lesssim \langle t\rangle^{-\frac{n}{2}}$$

for $\alpha > [\frac{n}{2}] + 2$.

This theorem generalizes Theorem 1.4 [BB21]: here we consider a metric whose asymptotic behavior is $dr^2 + r^2 g(r)$ with $g(r)$ perturbation of a generic fixed metric, as opposed to [BB21] where the authors consider perturbations of the flat euclidean metric only.

Remark 1.5.4. The function f plays the role of a spectral cutoff, localizing in the low part of the spectrum since in this chapter we are only interested in low frequencies. We remark that it is this region of the spectrum which gives the constraints in terms of time decay rates.

This result is obtained writing the flow of the equations via the spectral measure as we have seen in Section 1.2.1, for example

$$e^{it(P_0+V)} = \int e^{it\lambda} dE_\lambda(P_0 + V)$$

for the Schrödinger flow. In Section 1.2.4 we explained how the time decay is linked to having information on the derivatives of the spectral measure, or equivalently on the powers of the resolvent. The actual translation, via Fourier transform, of the results from a limiting resolvent frame to a time decay one can be done exactly as in [BB21].

We fix a small parameter $\lambda_0 > 0$ which delimits the portion of the spectrum under investigation. What is needed to run the arguments in [BB21] is the following result on regularity of the spectral measure.

Theorem 1.5.5 (Theorems 2.1.4, 2.1.5)**.** *Let* $n \geqslant 3$ $\lambda_0 > 0$, $l \in \mathbb{N}$, $\alpha > l$ *and* P_0, V *as in Theorem 1.5.3. There exists a constant* C *such that for every* $\lambda \in (0, \lambda_0]$ *it holds*

$$\|\langle r\rangle^{-\alpha}(P_0 + V - \lambda \pm i0)^{-l}\langle r\rangle^{-\alpha}\|_{L^2\to L^2} \leqslant C\lambda^{\min\{0,n/2-l\}} \tag{1.5.1}$$

if $l \neq \frac{n}{2}$ *and*

$$\|\langle r\rangle^{-\alpha}(P_0 + V - \lambda \pm i0)^{-\frac{n}{2}}\langle r\rangle^{-\alpha}\|_{L^2\to L^2} \leqslant C|\log\lambda| \tag{1.5.2}$$

if n *is even and* $l = \frac{n}{2}$.

Moreover, let E_λ *the spectral measure of* $P_0 + V$. *The function*

$$\lambda \mapsto \langle r\rangle^{-\alpha}\frac{dE_\lambda}{d\lambda}\langle r\rangle^{-\alpha}$$

is of class $C^{l-1}((0, \lambda_0])$ *and if* $\alpha > \frac{n}{2}$ *then*

$$\left\|\langle r\rangle^{-\alpha}\frac{d^j}{d\lambda^j}E_\lambda\langle r\rangle^{-\alpha}\right\|_{L^2\to L^2} \lesssim \lambda^{\frac{n}{2}-j-1}$$

for all $j = 1, \ldots, l-1$.

The second part of the statement regarding the derivatives of the spectral measure is a direct consequence of the first part, since Mourre theory ensures that the powers of the resolvent actually coincide with the derivatives of E_λ.

Remark 1.5.6. These resolvent results hold for a potential with a rate of decay which is better than what can be found in some previous works, such as [Wan06] or [VW09]. In the first case, the approach taken is different and the author provides as asymptotic expansion of the resolvent. While in this case the knowledge on the resolvent is more refined, it nevertheless requires a faster decay in the potential. The bounds (1.5.1) and (1.5.2) are also sharper than what was proved in [BR14b], this is necessary in order to obtain sharp time decay.

To prove the inequality on the resolvent powers we will only need to concentrate on the resolvent spectrally localized around 0 (Theorem 2.2.7, Proposition 2.2.8), since for intermediate frequencies $\lambda \in [\lambda_1, \lambda_0]$ the resolvent is simply uniformly bounded in λ via the spectral theorem.

1.5.1.2 Limiting absorption principle and quantification of resolvents

The main point of Chapter 2 is to prove Theorem 1.5.5. We recall the notation

$$P = P_0 + V.$$

The setting is similar to the one in [BB21], where we have two spatial regions with different properties: a compact part with no information on the metric and an unbounded region on which we can use a perturbative approach.

Remark 1.5.7 (Curved geometry at infinity). In our case the infinite end of the manifold is a perturbation of a curved geometry so we need to interpret P_0 as a perturbation of another Laplace-Beltrami operator. On the contrary in [BB21] the authors consider an infinite end which is a perturbation of a flat domain so the operator of reference is simply the Euclidean Laplacian.

Remark 1.5.8 (Normalized operator). As explained in Section 1.2.3 in the low frequency regime it is convenient to consider the resolvent of the normalized operator P/λ at energy one, rather than the resolvent of P at energy λ. It is what we will do here. To this renormalization in the energy it will correspond a rescaling in the radial variable by $\lambda^{1/2}$. This is in the same spirit of when, to the Euclidean Laplacian $\lambda^{-1}(-\Delta)$, we associate the spatial rescaling $\widetilde{x} = \lambda^{1/2}x$ so that the operator in the new variable $\widetilde{x}$ is independent of λ. In our case we rescale only in the radial direction, which is the only one possible since rescaling on the compact angular manifold S would not make sense.

We apply Mourre theory described in Section 1.2.3 to prove the existence of the limiting resolvents.

Conjugate operator $\mathcal{A}^\lambda$ From what we already explained we need a conjugate operator $\mathcal{A}^\lambda$ for P/λ (as the operator itself is λ dependent, so will be the conjugate operator). This is constructed keeping in mind the generator of dilations we used when we described the theory in the Euclidean case. Here we write the operator in radial coordinates

$$\frac{rD_r + D_r r}{2} = \frac{1}{2i} - ir\partial_r,$$

to exploit the separation of variables we have in $[R, \infty) \times S$. We will define a conjugate operator acting only at spatial infinity, precisely we will define it as the generator of a unitary group which will act as the identity on the compact part of the manifold. Let $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 in $B(0, R)$, then for λ small the function $(1-\chi)(\lambda^{1/2}r)$ localizes in a region near infinity. We use this function to localize the generator of dilations in the region $\{r \gg 1\}$, obtaining

$$\mathcal{A}_1^\lambda = (1-\chi)(\lambda^{1/2}r)\left(\frac{1}{2i} - ir\partial_r\right) + \frac{i}{2}r\partial_r\chi(\lambda^{1/2}r).$$

We will see that this operator is the generator of a strongly continuous unitary group of transformations and hence selfadjoint, but with respect to the fixed metric. We recall the situation is

	$(S, \overline{g})$	$(S, g(r))$
$(R, +\infty) \times S$	$dr^2 + r^2\overline{g}$, fixed	$dr^2 + r^2 g(r)$, perturbed

We can then obtain an operator which is selfadjoint with respect to the perturbed metric by considering a conjugated unitary group acting on the infinite end and whose generator will be

$$\mathcal{A}_2^\lambda = (1-\chi)(\lambda^{1/2}r)\left(\frac{n}{2i} - ir\partial_r + \frac{1}{2i}r\frac{\partial_r|g(r,\theta)|}{|g(r,\theta)|}\right) + \frac{i}{2}r\partial_r\chi(\lambda^{1/2}r).$$

Finally $\mathcal{A}^\lambda$ will be an operator on the whole manifold, which is 0 in a compact region and agrees with A_2^λ on the infinite end.

Once we have a candidate conjugate operator to do Mourre theory with, the main steps are

- regularity of the map (1.2.3), $P \in C^k(\mathcal{A}^\lambda)$,
- a positive commutator estimate,
- convert the operator weights $\langle \mathcal{A}^\lambda \rangle$ of (1.2.5) into physical weights $\langle r \rangle$.

Smoothness with respect to $\mathcal{A}^\lambda$ To obtain that the map (1.2.3) is C^1 amounts to proving that the operators

$$[P/\lambda, i\mathcal{A}^\lambda](P/\lambda + i)^{-1}, \quad [[P/\lambda, i\mathcal{A}^\lambda], i\mathcal{A}^\lambda](P/\lambda + i)^{-1} \tag{1.5.3}$$

are bounded. Similarly, to obtain higher regularity we need to consider iterated commutators of higher order.

To prove properties of this type we use symbolic calculus. In particular, we can construct a **parametrix** for $(P/\lambda + i)^{-1}$ whose symbols have seminorms which are uniformly bounded in λ. We manage to do so by considering pseudodifferential operators with symbols whose radial variable is rescaled by $\lambda^{1/2}$ to take into account the renormalization described in Remark 1.5.8. More precisely, we consider the usual quantization of a symbol on $\mathbb{R}^n$ given by

$$Op(a)f(r,\theta) = \frac{1}{(2\pi)^n} \int\int e^{i(r-r')\rho + i(\theta-\theta')\eta} a(r,\theta,\rho,\eta) f(r',\theta') dr' d\theta' \; d\rho d\eta,$$

so ρ is the momentum variable corresponding to r and η to θ. Then let $\mathcal{A} = \frac{n}{2i} - ir\partial_r$ the generator of dilations in radial coordinates and $\tau = \ln(\lambda^{\frac{1}{2}})$. We will use rescaled pseudodifferential operators defined as

$$Op_\lambda(a) := e^{i\tau\mathcal{A}} Op(a) e^{-i\tau\mathcal{A}} = Op(a_\lambda),$$

where in a_λ we have exactly a rescaling of the spatial variable by $\lambda^{\frac{1}{2}}$ (and consequently a rescaling of the momentum variable ρ by $\lambda^{-\frac{1}{2}}$).

We then apply symbolic calculus (or more precisely an adapted calculus for pseudodifferential operators on manifolds and of rescaled type) to evaluate the compositions in (1.5.3)

We can actually obtain $P/\lambda \in C^k(\mathcal{A}^\lambda)$ for any k since the iterated commutators have always the same symbolic structure. We have seen in Example 1.2.12 how in the Euclidean case the iterated commutators stay exactly the same. In the metric perturbation case this is not true, but in pseudodifferential terms the operators do not change.

Positive commutator estimate In Section 2.3 (Proposition 2.3.9) we prove that there exists I open bounded interval containing 1 such that

$$\mathbb{1}_I(P/\lambda) i[P/\lambda, \mathcal{A}^\lambda] \mathbb{1}_I(P/\lambda) \geqslant \mathbb{1}_I(P/\lambda). \tag{1.5.4}$$

for all $\lambda \in (0, \lambda_0]$. This, together with smoothness of P with respect to $\mathcal{A}^\lambda$, implies

$$\sup_{\lambda\in(0,\lambda_0]} \sup_{\varepsilon>0} \|\langle \mathcal{A}^\lambda \rangle^{-s} (P/\lambda - 1 \pm i\varepsilon)^{-l} \langle \mathcal{A}^\lambda \rangle^{-s}\|_{L^2\to L^2} < \infty \tag{1.5.5}$$

for $s > l - \frac{1}{2}$, that is the limiting absorption principle.

To prove inequality (1.5.4) we exploit a property which is somehow linked to the **uncertainty principle**. The property is the following: for all $\alpha > 0$ and $\varepsilon > 0$ there exist $\lambda_0 > 0$ and $f \in C_0^\infty(\mathbb{R})$ equal to 1 in a neighborhood of 1 such that

$$\|\langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha} f(P/\lambda)\| \leqslant \varepsilon \tag{1.5.6}$$

for all $\lambda \in (0, \lambda_0]$. The uncertainty principle imposes a lower bound when one wants to localize both in the space and frequency domains. Inequality (1.5.6) morally tells us that, even when we do so (i.e. we localize in frequency through f and in space through the decaying weight $\langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha}$) we still have smallness provided the spectral localization is strong enough.

With this property in mind, to bound from below the commutator $[P, i\mathcal{A}^\lambda]$ (plus a spectral localization) we split it in the regions $\{r \gg 1\}$ and $\{r \simeq 1\}$. Inequality (1.5.6) is used to deal with the bounded region $\{r \simeq 1\}$. In the region at infinity we said that we treat the problem as a perturbative one hence we **compare P with the Laplace-Beltrami operator corresponding to the fixed metric $dr^2 + r^2\bar{g}$.**

Remark 1.5.9. Thanks to the construction of the conjugate operator the fixed Laplace-Beltrami operator is stable under commutators with $\mathcal{A}^\lambda$, in other words it enjoys a property analogous to $[-\Delta, i\mathcal{A}] = 2(-\Delta)$, that we have seen in the Euclidean case (Section 1.2.3).

Passing to physical weights To obtain bounds in the weighted L^2 spaces of Theorem 1.5.5 we need to replace the operators $\langle \mathcal{A}^\lambda \rangle^{-s}$ with the decaying weights $\langle r \rangle^{-\alpha}$. We manage to do so using multiple tools, such as symbolic calculus, the parametrix for P/λ and estimates on powers of the resolvent like $(P/\lambda + 1)^{-N}$ that we derive via **heat kernel estimates**. We study the behavior of the heat semigroup e^{-tP} thanks to the fact that we can obtain a version of the **Nash inequality** on the manifold M, which we prove in Appendix 2.B.

Uncertainty principle Proving that $\|\langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha} f(P/\lambda)\|$ is small is one of the main step in Chapter 2. As we have already done previously, we divide the contributions of the regions $\{r \gg 1\}$ and $\{r \simeq 1\}$, where at radial infinity again we compare P with $-\Delta_0$, the Laplace-Beltrami operator of $([R, \infty) \times S, dr^2 + r^2\bar{g})$.

To evaluate the functions $f(P/\lambda)$ and $f(-\Delta_0/\lambda)$ we use **Helffer Sjöstrand** formula, so that we reduce ourselves to studying the difference between the resolvents $(P/\lambda - z)^{-1}$ and $(-\Delta_0/\lambda - z)^{-1}$. By the second resolvent identity, comparing the resolvents of $P = -\Delta_0 + Q$ and $-\Delta_0$ can be done having information on the difference Q. Locally speaking, the behavior of the difference $Q = P - (-\Delta_0)$ is induced by the behavior of the metric perturbation $g(r)$ with respect to $\bar{g}$.

Remark 1.5.10 (Perturbation by potential V). In Chapter 2 we will give all the details of the proofs for the case $V = 0$. The addition of the non negative decaying potential does not add any major difficulty. This is because most computations are carried out using symbolic calculus and the addition of V does not perturb the symbolic structure of the operator (we require that V has decay in the radial direction of order $\langle r \rangle^{-2-\varepsilon}$). Moreover the Nash inequality we mentioned before is stable for non negative potentials.

1.5.2 High frequencies for order one perturbations of the Schrödinger operator on infinite volume ends

In Chapter 3 we consider

$$P_m,$$

the Laplace-Beltrami operator with an order one perturbation, on top of a potential one, and in the high frequency regime. With respect to Chapter 2 we also consider a more general class of manifolds with ends of infinite volume, but not necessarily perturbations of cones. The content of this chapter generalizes some aspects of [CV02], where the authors consider the Laplace-Beltrami operator on manifolds with ends of infinite volume and cusps and prove resolvent estimates in the high frequency regime. Despite not allowing cusps, our main contributions are the fact that we include **differential perturbations** and we consider appropriate functional spaces which allow us to use **sharper weights**. Moreover, to treat the contribution of the infinite volume end we use a different approach for which the proof holds also for intermediate frequencies. This feature can be a first step into obtaining a proof which hold in the whole spectrum except away from the low frequencies.

Let A a vector field on the infinite end $(R, +\infty) \times S$. We recall that we can define the magnetic Laplacian starting from the quadratic form (1.4.5), that is

$$q_m(u, v) = \langle (D_G - A)u, (D_G - A)v \rangle_{L^2_{G_M}} \tag{1.5.7}$$

where we add a perturbation A with respect to the usual quadratic form that defines the Laplace-Beltrami operator. We consider an infinite end of the form

$$((R, +\infty) \times S,\ dr^2 + l(r)^{-2} g(r)) \tag{1.5.8}$$

with a volume factor l satisfying

$$O(1) \geqslant -\frac{l'(r)}{l(r)} \geqslant \frac{c}{r}.$$

Writing the metric in the more general form $dr^2 + l(r)^{-2}g(r)$ allows us to treat a generic infinite volume end including, at the same the asymptotically conical and asymptotically hyperbolic cases. The structure is as before: a bounded region where we do not have information on the metric and an infinite end of perturbative form.

For the region $\{r \gg 1\}$ we exploit the explicit expression of the operator that we have thanks to the explicit information on the metric. The quadratic form (1.5.7) produces an operator which is selfadjoint with respect to the metric perturbed at infinity, namely a metric of the type $dr^2 + l(r)^{-2}g(r)$ with the angular metric $g(r)$ depending on r. After a suitable conjugation and translation by Λ the operator in local coordinates that we consider near infinity is

$$h^2(D_r - A_0)^2 + M(r) + h^2(V_m - \Lambda) \tag{1.5.9}$$

where:

- $h = \lambda^{-1}$: we will renormalize the resolvent $(P_m - \lambda^2 + i\varepsilon')$ to bring it back to energy one by factorizing λ, hence we introduce the parameter h;
- $M(r)$ includes the part of operator acting in the angular variables as well as the order one perturbation due to the addition of the vector field A;
- V_m includes an effective potential and the perturbation by decaying potential;
- Λ is a non negative constant, it is equal to 0 in the asymptotically conical case and to $\frac{(n-1)^2}{4}$ in the asymptotically hyperbolic one;
- the operator is now formally selfadjoint with respect to the fixed metric $dr^2 + l(r)^{-2}\overline{g}$.

Remark 1.5.11. We denoted by $M(r)$ the angular part of the perturbed Laplace-Beltrami operator, so $M(r)^{1/2}$ in some sense plays the role of an angular gradient.

We now state the main result of Chapter 3, the objects involved are

- P_m, the Laplace-Beltrami operator on M with the magnetic perturbation, near infinity it agrees with (1.5.9);
- $\|\cdot\|_{H^1(\cdot)}$, H^1 norm on the manifold M induced by the metric G_M, that outside of a compact subset we recall is of the form (1.5.8);
- $\|\cdot\|_{B_{>R}}, \|\cdot\|_{B^*_{>R}}$, weighted norms on the manifold end $(R, +\infty) \times S$;
- for a function f supported on the manifold end $(R, +\infty) \times S$, $\|f\|_{H^1, B^*_{>R}}$ is a shorthand for the $B^*_{>R}$ norms of $f, (D_r - A_0)f$ and $M(r)^{1/2}f$;
- X_a for $a > R$ is a non compact manifold $X_a := (a, \infty) \times S$.

Theorem 1.5.12 (Theorem 3.1.1)**.** *Let $u \in H^2(M)$, $\lambda \gg 1$, and $R_1, R_2 \in \mathbb{R}$ independent of λ and verifying $R < R_1 < R_2$, then for any $\varepsilon' > 0$*

$$\|u\|^2_{H^1(M\setminus X_{R_1})} + \|u\|^2_{H^1,B^*_{>R}} \leqslant O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon')u\|^2_{L^2(M\setminus X_{R_2})} + O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon')u\|^2_{B_{>R}}$$

for some constant $C > 0$ independent of λ and ε'.

In the left hand side of the inequality we have the norm of u on a compact region $(M \setminus X_{R_1})$ and on a complementary unbounded region, so we are bounding the norm of u on the whole manifold, but we use this distinction to prove the result. Indeed, we consider separately the two regions $M \setminus X_{R_1}$ and X_{R_1}, the latter one is where the $B_{>R}, B^*_{>R}$ norms are defined.

In particular we use the following intermediate result

Lemma 1.5.13 (Propositions 3.2.1, 3.3.1 and (3.3.1))**.** *Let $u \in H^2(M)$, $\lambda \geqslant \lambda_0 > 0$ and $R_1, R_2 \in \mathbb{R}$ independent of λ and verifying $R < R_1 < R_2$. Let also K, U bounded regions in X_{R_1} and $\gamma_0 \in (0,1)$. Then for any $\varepsilon' > 0$ the following inequalities hold*

$$\|u\|^2_{H^1,B^*_{>R}} \leqslant O(\lambda^{-2})\|(P_m - \lambda^2 + i\varepsilon')u\|^2_{B_{>R}} + O(1)\|u\|^2_{H^1(K)} \tag{1.5.10}$$

and

$$\|u\|_{H^1(M\setminus X_{R_1})} \leqslant O(e^{\lambda/\gamma_0})\|(P_m - \lambda^2 + i\varepsilon')u\|_{L^2(M\setminus X_{R_2})} + O(e^{\lambda/\gamma_0})\|u\|_{H^1(U)}. \tag{1.5.11}$$

First of all we remark that this lemma holds for all frequencies $\lambda > \lambda_0$ and we do not need to assume λ large. The inequalities we just stated give us a right hand side that, up to remainder terms, is as the one needed in Theorem 1.5.12. In the process of combining the two contributions (1.5.10) and (1.5.11) we use the assumption $\lambda \gg 1$ to manage the remainders $\|u\|_{H^1(K)}$ and $O(e^{\lambda/\gamma_0})\|u\|_{H^1(U)}$.

We also remark that in (1.5.10) the contribution of the region at infinity provides us a bound in $O(\lambda^{-1})$ and indeed we can obtain the following result.

Corollary 1.5.14 (Corollary 3.1.2)**.** *Let $u \in H^2(M)$, $\lambda \gg 1$ and R_1, R_2 as in Theorem 1.5.12 and $R_3 > R_2$. Let χ a smooth cutoff such that $\chi \equiv 0$ on $M \setminus X_{R_2}$, $\chi \equiv 1$ on X_{R_3}. Then for all $\varepsilon' > 0$*

$$\|\chi u\|^2_{B^*_{>R}} \leqslant O(\lambda^{-2})\|(P_m - \lambda^2 + i\varepsilon')\chi u\|^2_{B_{>R}}.$$

In particular

$$\|\chi r^{-1/2-\mu}(P_m - \lambda^2 + i\varepsilon')^{-1} r^{-1/2-\mu}\chi\|_{L^2\to L^2} = O(\lambda^{-1})$$

with $\mu > 0$.

With this corollary we recover, for the magnetic Laplacian, the known bound for the cutoff resolvent of the unperturbed Laplace-Beltrami operator, proved for example in [CV02] in the presence of cusps.

Our result has also the interest of providing a uniform bound for the high part of the spectrum without requiring any additional knowledge on the geodesic flow.

Remark 1.5.15. In general, the weights used to prove the limiting absorption principle are $\langle\cdot\rangle^{-s}$ with $s > 1/2$. This is due to the fact that studying the behavior of the resolvents is like looking for a solution of the equation $(P - \lambda^2 + i0)u = f$. In the Euclidean case this is equivalent to solving $(|\xi|^2 - \lambda^2)\widehat{u} = \widehat{f}$. If $\widehat{f}$ vanishes on the sphere $\{|\xi| = \lambda\}$ we can directly define $\widehat{u}$ as $\widehat{u} = (|\xi|^2 - \lambda^2)^{-1}\widehat{f}$. Then thanks to Theorem IX.41 [RS75] one can bound u in a weighted space $L^2_{-1/2-\nu} = L^2(\langle x\rangle^{-1/2-\nu}dx)$ if $f \in L^2_{1/2+\nu}$.

The norms $\|\cdot\|_{B_{>R}}$ and $\|\cdot\|_{B^*_{>R}}$ we consider here are defined by partitioning the radial domain in dyadic intervals and by taking on each interval the weight $r^{-1/2}$. In this sense the weights we can consider here are sharper. These norms were first introduced in [Agm75] to study the Schrödinger operator on $\mathbb{R}^n$. With the strategy used in Chapter 3 we combine the Carleman point of view for the compact region, with sharper weighted spaces to take care of the part of domain at infinity.

We now give some ideas on how we obtain inequalities (1.5.10) and (1.5.11) and on the origin of the remainder terms therein.

1.5.2.1 Bound at spatial infinity

As we said we can obtain (1.5.10) not only for the part at infinity of the spectrum, but also the intermediate one. To do so we need to consider $u, (D_r - A_0)u$ and $M(r)^{1/2}u$ and we want to bound these quantities by f, the source term in the equation $(P_m - \lambda^2 + i\varepsilon')u = f$ solved by u. We use the equation itself to obtain expressions for the desired quantities. To be more precise we must use the equation $(P_m - \lambda^2 + i\varepsilon')(\phi u) = f$ where ϕ localizes on the infinite end since it is in this region only that P_m agrees with (1.5.9).

Let us give the example of how we obtain the norm of the angular gradient, since it also shows how we use the property $-l'/l \gtrsim r^{-1}$ of the metric. We use the following relation

$$Im(Bu, Cu) = (u, \frac{[B, C]}{2i}u)$$

which holds for two symmetric operators B, C and u in a dense subset of $D(B) \cap D(C)$.

Let's consider the case $C = P_m - \lambda^2 + i\varepsilon'$ and ϕ a cutoff near radial infinity

$$Im(B(\phi u), (P_m - \lambda^2 + i\varepsilon')(\phi u)) = (\phi u, \frac{[B, P_m]}{2i}(\phi u)) + \varepsilon'(B(\phi u), \phi u), \tag{1.5.12}$$

then in the left hand side, up to applying Cauchy-Schwarz inequality, we already have the desired source term f. It turns out that to obtain $M(r)^{1/2}(\phi u)$ in the right hand side a suitable choice is $B = D_r - A_0$. With this B we then need to consider

$$\frac{1}{i}[D_r, M(r)] = -M'(r)$$

that we can bound from below by the square root of $M(r)$. Indeed, thanks to the property $-l'(r)/l(r) \geqslant \frac{c}{r}$ we can lower bound the following scalar product in the right hand side of (1.5.12) by

$$(\phi u, -M'(r)(\phi u)) \geqslant O(1)\|M(r)^{1/2}(\phi u)\|^2_{B^*_{>R}} = O(1)\|M(r)^{1/2}u\|^2_{B^*_{>R}}.$$

We prove this in Lemmas 3.2.4 and 3.2.5. From (1.5.12) we can then derive an upper bound on $\|M(r)^{1/2}u\|_{B^*_{>R}}$.

Remark 1.5.16 (Remainder term I). Inequality (1.5.10) is the final result of Section 3.2, where we have the term in the right hand side, $\|u\|^2_{H^1(K)}$, which is not present in the statement of Theorem 1.5.12. This is due to the fact that to obtain (1.5.10) an intermediate step is

$$\|u\|^2_{H^1, B^*_{>R}} \leqslant O(\lambda^{-2})\|(P_m - \lambda^2 + i\varepsilon')u\|^2_{B_{>R}} + O(1)\|r^{-\mu}u\|^2_{H^1, B^*_{>R}}.$$

If we consider the norm of $r^{-\mu}u$ in the region $\{r \gg 1\}$ the factor $r^{-\mu}$ can be used as an arbitrarily small prefactor to absorb the term in the left hand side, the remaining quantity that is the norm on $\{r \simeq 1\}$ that we can not absorb on the left, is exactly $\|u\|^2_{H^1(K)}$.

For the norms of u and $(D_r - A_0)u$ we repeat similar computations, using either the real part or the imaginary part of the scalar product of equation solved by ϕu.

1.5.2.2 Bound in the compact region

To obtain a bound on $\|u\|_{H^1(M\setminus X_{R_1})}$ we apply the following result due to Lebeau and Robbiano, first appeared in [LR95] in the Euclidean case. Thanks to Theorem 9.1 in [LRLR22] we can prove

Proposition 1.5.17 (Proposition 3.3.2)**.** *Let (M_0, g_0) an n dimensional Riemannian manifold, T the Laplace-Beltrami operator and $\mathcal{R}$ a differential operator of order one. Let*

$$U_0 \Subset V_0 \Subset M_0 \qquad V_0' \Subset V_0 \Subset M_0 \qquad \overline{V}_0 \cap \partial M_0 = \emptyset,$$

$\alpha \in (0, 1/2)$ and $z \in \mathbb{C}$ with $Rez > z_0 > 0$, $|Imz| \neq 0$. Then there exists $c(z_0) > 0$ and $\gamma_0 \in (0, 1)$ such that

$$\|u\|_{H^1(V_0')} \leqslant c(z_0)e^{|z|/\gamma_0}(\|(T + \mathcal{R} - z^2)u\|_{L^2(V_0)} + \|u\|_{L^2(U_0)}).$$

for all $u \in H^2(V_0)$.

In our case

$$T + \mathcal{R} = P_m, \qquad z^2 = \lambda^2 - i\varepsilon'.$$

To apply the inequality to bound the norm of u on $M \setminus X_{R_1}$ we choose a smooth cutoff χ_0 and consider $\chi_0 u$. If χ_0 vanishes on U_0 we do not have the contribution of the control term $\|u\|_{L^2(U_0)}$ on the right. We also choose $\chi_0 \equiv 1$ on V_0', a region containing $M \setminus X_{R_1}$, in order to have exactly $\|u\|_{H^1(M\setminus X_{R_1})}$ on the left. An application of the proposition gives us

$$\|u\|_{H^1(M\setminus X_{R_1})} \leqslant O(e^{\lambda/\gamma_0})\|(P_m - \lambda^2 + i\varepsilon')(\chi_0 u)\|_{L^2(M\setminus X_{R_2})}.$$

Remark 1.5.18 (Remainder term II)*.* As we can see from the previous inequality, the remainder term we see in (1.5.11) is due to the commutator $[P_m, \chi_0]$. So $O(e^{\lambda/\gamma_0})\|u\|_{H^1(U)}$ is supported in a bounded region of the form $X_{R_2} \setminus X_{R_2+1}$. To eliminate this contribution, we remark that for a generic operator T

$$\|e^{\lambda\varphi}f\| \leqslant \|e^{\lambda\varphi}Tf\|$$

is equivalent to

$$\|f\| \leqslant \|e^{\lambda\varphi}Te^{-\lambda\varphi}f\|.$$

Hence, to find a bound for $e^{\lambda/\gamma_0}u$ we consider the conjugate operator $e^{\lambda\varphi}(P_m - \lambda^2 + i\varepsilon')e^{-\lambda\varphi}$ for an appropriate weight that we can derive from [CV02]. This will also allow us to bound the remainder $\|u\|^2_{H^1(K)}$ given in Remark 1.5.16. In [CV02] the authors use a weight function φ depending on the spectral parameter, using in turn an idea coming from [Bur02a]. In particular the relevant features of φ will be that $\varphi' = O(\lambda^{-1}r^{-1})$ for sufficiently large r and that $\varphi \leqslant 0$ on a compact region.

1.5.3 Properties of zero resonances on $\mathbb{R}^n$, $n = 3, 4$

In Chapter 4 we consider the Laplacian with a potential perturbation on $\mathbb{R}^n$ for $n = 3, 4$. We study the bottom of the essential spectrum, that for the Laplacian is the positive half line $[0, \infty)$.

The essential spectrum is stable under suitable potential perturbations. This will be the case for the potentials considered here, where we assume V to belong to a Lorentz space

$$V \in L^{n/2,1}(\mathbb{R}^n) \subset L^{n/2}(\mathbb{R}^n). \tag{1.5.13}$$

In particular we are interested in the nature of the point 0 in the spectrum, whether it is a resonance or an eigenvalue, and in the properties of the corresponding eigenfunction or resonant state. As we have seen in Section 1.3.3, this influences the rate of decay of the Schrödinger propagator.

To study the zero state we will use some conditions that already appear in [JK79], where the authors study a smaller class of potentials with pointwise decay. In our case, condition (1.5.13) on V is more general than the most common assumptions of pointwise decay, that we can find for example in the previously mentioned [JK79], [GS04], [ES04], and it is also less strict than requiring $L^{3/2-\varepsilon}(\mathbb{R}^3) \cap L^{3/2+\varepsilon}(\mathbb{R}^3)$ as in [Gol06a], [Gol06b]. We also underline that condition (1.5.13) includes potentials decaying at infinity like $|x|^{-2-\nu}$, $\nu > 0$ which is a slightly faster decay than the critical potential $|x|^{-2}$. Indeed, the inverse square potential is a critical one for the validity of uniform resolvent estimates which are explicitly linked to the nature of 0 by the work [JK79].

We limit ourselves to dimensions three and four since they are the only cases for which there is at most one linearly independent resonant state (see Remark 1.3.8 for dimensions five and higher and [JN01] for dimension two). In dimension three some of the results were proved in [Bec16] in a slightly less general framework. In addition to the more relaxed assumptions we use, we are also able to **extend our results to dimension four**.

Remark 1.5.19 (Scaling invariance). In the study of the resolvent $(-\Delta + V - \lambda^2 + i0)^{-1}$ at energy λ, it is somehow useful, for example as in Section 1.2.3, to reduce the problem to energy one by considering $(-\lambda^{-2}\Delta + \lambda^{-2}V - 1 + i0)^{-1}$. In this case, after the rescaling $\widetilde{x} = \lambda x$ the potential perturbation is $\widetilde{V} = \lambda^{-2}V(\lambda^{-1}\cdot)$. Assumption (1.5.13) has also the interest of being invariant under this rescaling. Indeed, given the dilation identity

$$\|f(\alpha\ \cdot)\|_{L^{p,q}} = \alpha^{-n/p}\|f\|_{L^{p,q}}$$

we have $\|\widetilde{V}\|_{L^{n/2,1}} = \lambda^{-2}\|V(\lambda^{-1}\cdot)\|_{L^{n/2,1}} = \|V\|_{L^{n/2,1}}$.

The results proved in Chapter 4 are the following.

Theorem 1.5.20 (Theorem 4.1.2)**.** *Let $n = 3, 4$, $V \in L^{n/2,1}(\mathbb{R}^n)$ and $\psi \in \dot{H}^1(\mathbb{R}^n)$ a solution of the equation $(-\Delta + V)\psi = 0$. The following properties hold:*

i) *$|x|^{n-2}\psi$ has a finite limit as $|x| \to \infty$, hence for $n = 3$*

$$\psi \in L^{3,\infty}(\mathbb{R}^3)$$

and for $n = 4$

$$\psi \in L^{2,\infty}(\mathbb{R}^4).$$

ii) *ψ is a zero eigenfunction, that is $\psi \in L^2(\mathbb{R}^n)$, if and only if $\int V\psi = 0$, in particular $\psi = O(\frac{1}{|x|^{n-1}})$ near infinity.*

iii) *If $\int V\psi = \int y_k V\psi = 0$ for all $k = 1, \dots, n$, then ψ is a zero eigenfunction and $\psi \in L^{1,\infty}(\mathbb{R}^n)$, in particular $\psi = O(\frac{1}{|x|^n})$ near infinity.*

iv) *If $\int V\psi = \int y_k V\psi = \int y_k y_l V\psi = 0$ for all $k, l = 1, \dots, n$, then ψ is a zero eigenfunction and $\psi \in L^1(\mathbb{R}^n)$, in particular $\psi = O(\frac{1}{|x|^{n+1}})$ near infinity.*

Remark 1.5.21. We underline that it is actually sufficient to consider $\psi \in L^{\frac{2n}{n-2},\infty}(\mathbb{R}^n)$, as this is the only property that we will need in the proofs and is a weaker assumption since by Sobolev embeddings we have the inequality

$$\|u\|_{L^{p^*}} \lesssim \|\nabla u\|_{L^p} \qquad p^* = \frac{np}{n-p},$$

which implies $\dot{H}^1(\mathbb{R}^n) \subset L^{\frac{2n}{n-2}}(\mathbb{R}^n) \subset L^{\frac{2n}{n-2},\infty}(\mathbb{R}^n)$. We state the theorem for functions in $\dot{H}^1$ since in Remark 1.3.8 we proved there are no zero resonances in higher dimension taking as a starting space $\dot{H}^1$. Moreover it also represents a natural class of non L^2 functions.

First we remark that thanks to this theorem a zero resonant state belongs to a weak Lebesgue space, rather than a weighted L^2 space as we described in Section 1.3.3, and this provides us with a **scaling invariant condition on the resonant state**. Moreover, Theorem 1.5.20 gives us information on the rate of decay of the solution to the Helmholtz equation from which we derive further **integrability**. Additional assumptions are necessary to reach the L^1 space, since we have examples of zero eigenfunctions, like the Aubin-Talenti functions in Example 1.3.7, which are not in L^1.

Remark 1.5.22. The conditions on the integral of ψ against V had already appeared in previous works such as [Bec16], [JK79], [Jen84]. For example $\int V\psi = 0$ had already been used to prove that a resonant state is an eigenfunction. With respect to the works of Jensen and Kato here we apply such condition for a much more general class of potentials, without requiring anything on the pointwise behavior of V; while compared to [Bec16] we do not need to assume integrability on V to recover $\psi \in L^2$. The quantities $\int y_k V\psi$ appear in the description of the coefficients in the resolvent expansion in [JK79], in particular in the term due to the presence of a zero eigenfunction. In [Bec16], where there are no pointwise decay assumptions, this quantity, together with the integral $\int y_k y_l V\psi$, is used to derive the asymptotic behavior of ψ. Nevertheless, in Theorem 1.5.20 we obtain results on the zero eigenfunction with condition (1.5.13) only, as opposed to Lemmas 2.3 and 2.5 in [Bec16].

Remark 1.5.23. In our proof we take advantage of the fact that simple functions are dense in the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ when $q < \infty$. In particular this allows us to decompose V in the sum

$$V = W + K$$

with K a simple function and W arbitrarily small in $L^{n/2,1}(\mathbb{R}^n)$. For this we need to consider $L^{n/2,1}(\mathbb{R}^n)$ and can not allow $L^{n/2,\infty}(\mathbb{R}^n)$, which includes the critical potential $|x|^{-2}$. However, assumption (1.5.13) still covers the case of potentials decaying like $|x|^{-2-\nu}$, $\nu > 0$.

The proof of Theorem 1.5.20 has two steps. First we construct a Green function for the Laplacian perturbed by the small part of the potential. We use this to prove item $i)$ where we actually have more precise information than what is stated in the previous theorem, since we obtain an explicit expression of the limit of $|x|^{n-2}\psi$. Then starting from the fact that $|x|^{n-2}\psi$ is bounded near infinity we use a fixed point argument to prove that ψ has the faster decay described in items $ii) - iv)$.

1.5.3.1 Green function

The function ψ we want to study is solution of the equation $(-\Delta + V)\psi = 0$, which, using the decomposition $V = W + K$, can be restated as

$$(-\Delta + W)\psi = -K\psi \tag{1.5.14}$$

where the operator

$$(-\Delta + W) = (-\Delta)(I + (-\Delta)^{-1}W)$$

is a small perturbation of the identity in the space of bounded operators from $\dot{H}^1(\mathbb{R}^n)$ to $\dot{H}^{-1}(\mathbb{R}^n)$. We can see this thanks to the Sobolev inclusion and its dual

$$\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n), \qquad (L^{\frac{2n}{n-2}}(\mathbb{R}^n))^* \hookrightarrow \dot{H}^{-1}(\mathbb{R}^n)$$

since W, which has small norm, maps $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ to $L^{\frac{2n}{n+2}}(\mathbb{R}^n) = (L^{\frac{2n}{n-2}}(\mathbb{R}^n))^*$. We can then invert $(I + (-\Delta)^{-1}W)$.

On top of it being invertible, we can also find an expression for the inverse following a strategy from [Pin88]. Since we can invert $(-\Delta)(I + (-\Delta)^{-1}W)$ via a Neumann series, we define its integral kernel by the series of the kernels of $((-\Delta)^{-1}W)^j(-\Delta)^{-1}$, namely

$$G_0(x,y) = c_n \frac{1}{|x-y|^{n-2}}, \qquad G_j(x,y) = c_n \int \frac{1}{|x-z|^{n-2}} W(z) G_{j-1}(z,y) dz$$

with $c_n = n(n-2)|B(0,1)|$ a constant depending on the dimension.

We are then able to prove the following

Proposition 1.5.24 (Theorem 4.2.2)**.** *Let $W \in L^{n/2,1}(\mathbb{R}^n)$ with $\|W\|_{L^{n/2,1}} \ll 1$, then*

$$G(x,y) := \sum_{j\geqslant 0} (-1)^j G_j(x,y)$$

is the Green function of $-\Delta + W$ and is such that $|G(x,y)| \lesssim \frac{1}{|x-y|^{n-2}}$.

We remark that the proposition tells us that we have a pointwise bound on the integral kernel of $(-\Delta + W)^{-1}$ by the integral kernel of the free operator $(-\Delta)^{-1}$.

The proof of this result relies on the fact that $W \in L^{n/2,1}(\mathbb{R}^n)$ is in the dual of $L^{\frac{n}{n-2},\infty}(\mathbb{R}^n)$, which contains $|x - \ \cdot|^{2-n}$.

1.5.3.2 Properties of zero state

We then pass to the proof of the properties of the zero state ψ. Here the main idea is to prove that ψ has the suitable rate of decay outside of a compact set containing the support of K.

For example, for item $i)$ we prove directly that $|x|^{n-2}\psi$ is bounded outside of a compact set using the equation $(-\Delta + W)\psi = -K\psi$ and the Green function G which allows us to write

$$\psi(x) = -\int G(x,y)K(y)\psi(y)dy.$$

Studying the behavior of $|x|^{n-2}G(x,y)$ at infinity we prove

Proposition 1.5.25 (Proposition 4.3.1)**.** *The function $|x|^{n-2}\psi$ is bounded outside of a compact set. Moreover it has a finite limit when $|x|$ tends to infinity, namely*

$$|x|^{n-2}\psi(x) \to -c_n \int V(y)\psi(y)dy \qquad |x| \to \infty.$$

This lemma implies in particular $\psi \in L^{\frac{n}{n-2},\infty}(\mathbb{R}^n)$, which is item $i)$ in Theorem 1.5.20. We underline that the value of the limit is finite since we first prove that $|x|^{n-2}\psi$ is bounded outside of a compact set and this alone implies $\psi \in L^{\frac{n}{n-2},\infty}(\mathbb{R}^n) = (L^{n/2,1}(\mathbb{R}^n))^*$.

Remark 1.5.26. From the expression of the limit of $|x|^{n-2}\psi$ we can directly see that if $\int V\psi \neq 0$ then ψ can not be in L^2. So ψ orthogonal to V is a necessary condition to have an eigenfunction. It is also sufficient as we show in item $ii)$ of Theorem 1.5.20.

To prove the following items in Theorem 1.5.20 we consider the spaces

$$\mathcal{B}_\alpha = |x|^{-\alpha} L^\infty(B(0,R)^c)$$

with $B(0,R)$ a large enough compact set (in particular *supp* $K \subset B(0,R)$). We already know from item $i)$ that $\psi \in \mathcal{B}_{n-2}$ and for item $ii)$, for example, we need to prove $\psi \in \mathcal{B}_{n-1}$. We do so using the assumption $\int V\psi = 0$ and the equation $(-\Delta)\psi = -V\psi$ solved by ψ to write ψ as the solution of a fixed point equation

$$\psi = f + \mathcal{S}\psi. \tag{1.5.15}$$

Thanks to the fact that the expression of $\mathcal{S}$ contains the small potential W we can prove that $\mathcal{S}$ is a contraction both on $\mathcal{B}_{n-2}$ and $\mathcal{B}_{n-1}$. From what we proved in item $i)$, ψ is the unique solution of the fixed point problem (1.5.15) in the space $\mathcal{B}_{n-2}$, but since $\mathcal{B}_{n-1} \subset \mathcal{B}_{n-2}$ the solution of (1.5.15) in $\mathcal{B}_{n-1}$ must coincide with ψ, therefore implying $\psi \in \mathcal{B}_{n-1}$.

For items *iii*) and *iv*) we repeat an analogous argument using the vanishing integrals in the assumptions to define operators which are contractions on the couple of spaces $\mathcal{B}_{n-1}, \mathcal{B}_n$ and $\mathcal{B}_n, \mathcal{B}_{n+1}$ respectively.

Chapter 2

Dispersive equations on asymptotically conical manifolds: time decay in the low frequency regime

Outline of the current chapter

The content of this chapter is the object of the publication [Gra23].

2.1 Introduction

In this chapter we consider the Laplace-Beltrami operator on a quite general class of non compact manifolds with ends, which includes, among others, all compact perturbation of the euclidean metric. The geometric setting is the one of asymptotically conical manifolds that over the years has attracted the interest of a substantial community with the aim of recovering some of the properties that hold in the flat case, such as resolvent or local energy decay estimates.

Let $P = P_0 + V$ with P_0 the Laplace-Beltrami operator on an asymptotically conical manifold and $V \geqslant 0$ a decaying multiplicative potential. Broadly speaking we are interested in estimates on the operators

$$(P - \lambda \pm i0)^{-l} = \lim_{\varepsilon \to 0} (P - \lambda \pm i\varepsilon)^{-l}. \tag{2.1.1}$$

and the related evolutions $e^{it(P)}$, $e^{it\sqrt{P}}$, $e^{it\sqrt{P+1}}$, $\frac{\sin(t\sqrt{P})}{\sqrt{P}}$.

In particular, in this work we give some decay properties in weighted L^2 spaces for the resolvent and the spectral measure of the Laplace-Beltrami operator, which are then applied to recover local energy decay. Our results are closely related to the ones presented in [BB21] where the geometry is the one of $\mathbb{R}^n$, up to an obstacle, with an asymptotically Euclidean metric. Indeed, we shall prove here that even in the case of a manifold with an asymptotically conic end the same results as Theorem 1.2 and 1.3 in [BB21] hold. As one can see in Section 5 in [BB21], these properties can then be used to prove decay on the evolution operators.

Let M an n dimensional manifold with $n \geqslant 3$. We assume M to be of the form $M = K \cup (M \setminus K)$ with K compact and $M \setminus K$ an infinite end which is asymptotically conical. For formal statements see Definitions 2.1.7 and 2.1.3. For the moment we just say that r is the radial coordinate on the manifold end, $\langle r \rangle$ is a positive smooth decaying function which is $O(r)$ for $r \gg 1$ and $\| \cdot \|$ the norm of operators on $L^2(M)$.

The main result of the chapter is the following.

Theorem 2.1.1. *Let (M, G_M) an asymptotically conical manifold of dimension $n \geqslant 3$ and P_0 the Laplace-Beltrami operator on M. Let $f \in C_0^\infty(\mathbb{R})$, $\varepsilon > 0$ and V a non negative multiplicative potential that on $M \setminus K$ agrees with a function in $S^{-2-\varepsilon}$ in the sense of Definition 2.1.7. Let $P = P_0 + V$, then*

i) Schrödinger flow:

$$\| \langle r \rangle^{-\alpha} f(P) e^{it(P)} \langle r \rangle^{-\alpha} \| \lesssim \langle t \rangle^{-\frac{n}{2}}$$

for $\alpha > [\frac{n}{2}] + 2$.

ii) Wave flow:

$$\| \langle r \rangle^{-\alpha} f(P) \frac{\sin(t\sqrt{P})}{\sqrt{P}} \langle r \rangle^{-\alpha} \| \lesssim \langle t \rangle^{1-n}$$

and

$$\| \langle r \rangle^{-\alpha} f(P) e^{it\sqrt{P}} \langle r \rangle^{-\alpha} \| \lesssim \langle t \rangle^{-n}$$

for $\alpha > n + 1$.

iii) Klein-Gordon flow:

$$\| \langle r \rangle^{-\alpha} f(P) e^{it\sqrt{P+1}} \langle r \rangle^{-\alpha} \| \lesssim \langle t \rangle^{-\frac{n}{2}}$$

for $\alpha > [\frac{n}{2}] + 2$.

Remark 2.1.2. In this work we only focus on the flow of the equations in the low frequency regime. Although this might seem a restriction, we point out that it is the low part of the spectrum which dictates the decay rate of the solutions. Indeed, for high frequency up to non trapping assumptions on the geodesic flow one can obtain arbitrary fast decay in time. See for example [Wan06] and [Vod04].

Remark 2.1.3. We will show in detail all the proofs in the case $V = 0$, since the addition of V requires only minor adaptations for which the reader can refer to Section 2.5.

Proving local energy decay for these equations, especially the wave and Klein-Gordon ones on manifolds, is a fundamental question in scattering theory which dates back to the work by Morawetz [Mor61], in which the author considers the flat wave equation outside of an obstacle. The topic is still the subject of recent works, such as the previously mentioned [BB21] or [Mor20], [MW21], in the latter the focus is on the influence of the decay rate of the metric on the decay rate of the solution. Indeed, the full picture of how the energy of the wave equation should decay for long range perturbations is not yet clear and it is therefore desirable to find approaches which are robust enough to allow for this type of perturbations.

We describe how our work compares to known results in this setting, while for an overview of results in the case of an asymptotically euclidean geometry the reader can refer to the introduction of [BB21].

In [GHS13a] the authors consider a manifold with a scattering metric defined via a family of smooth metrics h. After the change of variable $r = 1/x$ and Taylor expansion of h the scattering metric results in a particular case of Definition 2.1.3. More precisely, $g(r)$ would be given by the Taylor expansion of $h(1/r)$ around 0, ∂M corresponds to S and $h(0)$ to $\overline{g}$. There the authors take a geometric approach to obtain an expression for (2.1.1) as a sum of pseudodifferential operators and Legendre distributions. From this, they derive an explicit expansion for the Schwartz kernel of the spectral measure as $\lambda \to 0$, which is applied to obtain long time expansions for the Schrödinger and wave operators. The decay rate depends on the spectrum of the operator at infinity. In particular, using the notation of Section 2.1.1 of the present article, on the smallest eigenvalue of the Laplace-Beltrami operator on $(S, \overline{g})$ and the decay of the potential at infinity. In [GHS13a] the potential is allowed to be negative, however with a control on the negative part (see condition (1.2) therein) and with a decay rate of -2. In this work the authors make the assumption of no zero resonance nor eigenvalue. While [GHS13a] gives sharper estimates (and results in relation to Price's law) the geometric framework is more restrictive than ours.

A similar approach is used in [Wan06]. Here the result provides an asymptotic expansion for the resolvent, however allowing for the presence of zero eigenfunctions or resonant states, which unlike the Euclidean case are still present in higher dimension. The author first considers the operator on a manifold with an exactly conical end and then uses this result to treat the case of a perturbation by metric and by potential. The result also allows a decaying term of order -2, which can not be dealt with using perturbative arguments. We remark that to treat the model operator on the exact cone it is used a diagonalisation on the angular manifold similar to the one we present in Appendix 2.A. In [Wan06] the author starts with a potential with decay as in the present work (condition (1.5) in [Wan06]), but a stronger decay and stronger weights are required to write the expansion of the Schrödinger evolution (cfr. Theorems 6.3 and 6.4). The improvement of our work is the fact that we can allow a long range decay.

In [VW09], instead, in the case of a scattering manifold the authors take a similar approach to the one of the present chapter using positive commutator estimates to prove dispersive properties for the flow of the wave equation. Although the positive commutator estimate is proved for potentials in the same class as the present work, decay of the flow is only recovered for potentials with a stronger decay.

The method used to prove Theorem 2.1.1 is the same as [BB21] and it relies on results on the regularity of the spectral measure and on quantification of estimates on the operators (2.1.1) (and its powers). We now state the two theorems from which Theorem 2.1.1 follows. We will prove these in Section 2.2, while the derivation of Theorem 2.1.1, being analogous to what is presented in Section 5 of [BB21], is omitted here.

First, recall the definition of spectral measure. Let E_Ω the indicator function of a set $\Omega \subset \mathbb{R}$, then for every $u, v \in L^2(M)$ the map $\Omega \mapsto (u, E_\Omega(P)v)$ is a well defined Borel measure. To say that we integrate λ with respect to this measure we write $d(u, E_\lambda(P)v)$. We call this measure the spectral measure of P associated to u and v. It satisfies the property

$$(u, f(P)v) = \int f(\lambda) d(u, E_\lambda(P)v)$$

for any bounded Borel function f, or in short

$$f(P) = \int f(\lambda)\, dE_\lambda.$$

The following theorem gives us regularity results on E_λ. These can be used to prove Theorem 2.1.1, after writing the flow of the equation as an oscillatory integral against the spectral measure.

Theorem 2.1.4. *Let $n \geqslant 3, \lambda_0 > 0$, $\alpha > k$ and $P = P_0 + V$ with P_0, V as in Theorem 2.1.1. Let E_λ the spectral measure of P. The function*

$$\lambda \mapsto \langle r \rangle^{-\alpha} \frac{dE_\lambda}{d\lambda} \langle r \rangle^{-\alpha}$$

is of class $C^{k-1}((0, \lambda_0])$. Moreover if $\alpha > \frac{n}{2}$ then

$$\left\| \langle r \rangle^{-\alpha} \frac{d^j}{d\lambda^j} E_\lambda \langle r \rangle^{-\alpha} \right\| \lesssim \lambda^{\frac{n}{2}-j-1}$$

for all $j = 1, \ldots, k-1$.

The strategy to prove Theorem 2.1.4 is to use Stone's formula to write the spectral measure in terms of the limiting values of the resolvent and then use the following uniform resolvent estimates.

Theorem 2.1.5. *Let $n \geqslant 3$ $\lambda_0 > 0$, $l \in \mathbb{N}$, $\alpha > l$ and $P = P_0 + V$ with P_0, V as in Theorem 2.1.1. There exists a constant C such that for every $\lambda \in (0, \lambda_0]$ it holds*

$$\| \langle r \rangle^{-\alpha} (P - \lambda \pm i0)^{-l} \langle r \rangle^{-\alpha} \| \leqslant C \lambda^{\min\{0, n/2 - l\}}$$

if $l \neq \frac{n}{2}$ and

$$\| \langle r \rangle^{-\alpha} (P - \lambda \pm i0)^{-\frac{n}{2}} \langle r \rangle^{-\alpha} \| \leqslant C |\log \lambda|$$

if n is even and $l = \frac{n}{2}$.

Remark 2.1.6. Some low frequency estimates on the resolvent in weighted L^2 spaces can be found in [BR14b]. The bounds we recover in Theorem 2.1.5 hold for all powers of the resolvent and are sharp with respect to the behaviour in λ, unlike the ones presented in [BR14b] which only provide boundedness with respect to λ.

The method in the proof of Theorem 2.1.5 follows similar steps as the one presented in [BB21] up to the proof of low frequency exact Mourre estimates (Proposition 2.3.9). We refer to Section 2.3 for more details, only mentioning here that key point is to get rid of the compact remainder in a parameter dependent Mourre estimate.

Indeed, the arguments used in [BB21] rest on the underlying Euclidean geometry and can not be applied to this more general context. In our case the fact that the operator has non constant coefficients will not allow us to commute derivatives with resolvents. Even though on the exact cone we will be able to recover some useful features of the flat case, like the fact that $\partial^{j,k}(-\Delta)^{-1}$ is a bounded operator, we will need a more careful spectral analysis to be able to apply similar properties. For example, in the exact conic case we will reduce the problem to dimension one thanks to separation of variables and we will see how we can control our resolvent by studying the one dimensional resolvent corresponding to the spherical Laplacian.

This is also the reason for our assumption on the dimension, that we take grater or equal than three (as opposed to [BB21] where all dimensions greater or equal than two are covered). This will allow us to use Hardy inequalities on L^2 that will be necessary, for example in Appendix 2.D.

The rest of the chapter is organised as follows: we conclude this introduction with precise formulations of the properties we require on the infinite end $M \setminus K$ and with definitions of the rescaled pseudodifferential operators we will use in the computations; in Section 2.2 we give the proofs of Theorems 2.1.4 and 2.1.5 under some conditions that we then prove in Sections 2.3 and 2.4; finally Section 2.5 describes how the arguments adapt to the case of addition of the potential V.

2.1.1 Definitions

Let $n \geqslant 3$, in the following we will consider $(S, \overline{g})$ an $n-1$ dimensional closed Riemannian manifold with local coordinates $(\theta_1, \dots, \theta_{n-1})$, we will use these objects to define the angular part of conical manifolds.

We also need to define some notion of decay with respect to the radial variable.

Definition 2.1.7. *Let $f(r)$ a smooth function of r with values in the space of (h,k) tensor fields (i.e. sections of the (h,k) tensor bundle $(\otimes^h TS) \otimes (\otimes^k T^*S)$). Let θ local coordinates in a patch around a point $\omega \in S$ and $f^{i_1,\dots,i_h}_{j_1\dots j_k}(r,\theta)$ the coefficients of f with respect to a basis of $(\otimes^h T_\omega S) \otimes (\otimes^k T^*_\omega S)$. Then $f(r)$ is in the class $S^{-\nu}$ if*

$$|\partial_r^l \partial_\theta^\alpha f^{i_1,\dots,i_h}_{j_1\dots j_k}(r,\theta)| \lesssim \langle r \rangle^{-\nu - l}$$

locally uniformly in θ, for all $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{n-1}$.

We will equip S either with the fixed metric $\overline{g}$ or with a metric $g(r)$ depending on the radial coordinate and which is a perturbation (in a $S^{-\nu}$ sense) of $\overline{g}$, meaning that we assume

$$g(r) - \overline{g} \in S^{-\nu}. \tag{2.1.2}$$

The geometrical setting for all our analysis will be the following.

Definition 2.1.8 (Asymptotically conical manifold)**.** *Let (M, G_M) a manifold of dimension n with $K \subset M$ compact. M is said to be asymptotically conical if there exist $R > 0$ and a diffeomorphism*

$$\begin{aligned} \Omega : M \setminus K &\to (R, +\infty) \times S \\ m &\mapsto (r(m), \omega(m)) \end{aligned}$$

such that $r : M \to [R, +\infty)$ is a proper function and the metric G_M is given by

$$G_M = \Omega^*(dr^2 + r^2 g(r)).$$

Remark 2.1.9. Although with different notation, this is the same geometric framework as the one used in [IN10].

If $\kappa : U_\kappa \subset S \to V_\kappa \subset \mathbb{R}^{n-1}$ are the coordinate charts on S we will denote by $\Pi_\kappa, \Pi_\kappa^{-1}$ the pullback and pushforward on $[R, +\infty) \times S$; moreover if $(\varphi_\kappa)_\kappa$ is a partition of unity on S and φ a smooth cutoff on $[R, +\infty)$ we will make use of the functions

$$\psi_\kappa(r, \omega) := \varphi(r)\varphi_\kappa(\omega) \in C_0^\infty([R, +\infty) \times U_\kappa)$$

which verify $\sum_\kappa \psi_\kappa \equiv 1$ for large enough r.

Notation. A function on $M \setminus K$ can be identified with a function on $(R, +\infty) \times S$ thanks to Ω. As we will basically always consider the corresponding quantities on $(R, +\infty) \times S$ we will drop the composition by Ω, which rigorously is the one that allows to pass from a point on the manifold to a point on $(R, +\infty) \times S$. This means that we will simply use the notation (r, ω) for a point of $M \setminus K$ and still denote by ψ_κ, Π_κ or Π_κ^{-1} the corresponding functions defined on $M \setminus K$.

In the subsequent table we group the different notations we introduce for the manifolds with their respective metrics, Hilbert spaces and the associated Laplace-Beltrami operators.

manifold	metric	Hilbert space	L-B operator
M	$G_M = \Omega^*(dr^2 + r^2 g(r))$	$L^2(M)$	P_0
$[R, +\infty) \times S$	$\overline{G} = dr^2 + r^2 \overline{g}$	$L^2_{\overline{G}}$	$-\Delta_0$

We also define

$$G = dr^2 + r^2 g(r) \tag{2.1.3}$$

the perturbed metric on $[R, +\infty) \times S$ and $\|\cdot\|_{L^2_G}$ the L^2 norm with respect to this metric. We recall that $\|\cdot\|$ denotes the norm of the operators on $L^2(M)$.

We remark that having quantified in (2.1.2) how much the metric $g(r)$ deviates from $\overline{g}$ we can compare the two operators $-\Delta_0$ and P_0 and also obtain that the norms of L^2_G and $L^2(M)$ are comparable, meaning that their quotient is bounded by constants from above and from below.

Throughout the whole chapter we will have to consider convenient rescaled operators as follows. Since we are interested in resolvents such as $(P - \lambda \pm i0)^{-1}$ we consider the operator P/λ which it is convenient to study using rescaled pseudodifferential operators, that we now define.

Definition 2.1.10. *A function $a(r, \theta, \rho, \eta)$ is in $\widetilde{S}^{m,\mu}(\mathbb{R}^{2n})$ if and only if for every $j, k \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^{n-1}$ there exists a constant C such that*

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\eta^\beta a(r, \theta, \rho, \eta)| \leqslant C \langle r \rangle^{m-j-|\beta|} \left(\langle \rho \rangle + \frac{\langle \eta \rangle}{\langle r \rangle} \right)^{\mu-k-|\beta|} \tag{2.1.4}$$

with

$$\langle r \rangle := \begin{cases} 1 & \text{if } r \text{ is in a compact set,} \\ r & \text{if } r \gg 1. \end{cases}$$

The seminorms of the space are given by the smallest constants verifying the inequality.

Remark 2.1.11. Although we are using the same notation as the radial coordinate on the manifold, here r is simply meant to denote the first variable of $\mathbb{R}^n$.

We consider the usual quantization of a symbol defined as

$$Op(a)f(r, \theta) = \frac{1}{(2\pi)^n} \int \int e^{i(r-r')\rho + i(\theta-\theta')\eta} a(r, \theta, \rho, \eta) f(r', \theta') dr' d\theta' \; d\rho d\eta$$

and we introduce the dilation operator with respect to r and its generator $\mathcal{A}$, namely

$$e^{it\mathcal{A}} u(r, \theta) = e^{\frac{tn}{2}} u(e^t r, \theta), \qquad \mathcal{A} := \frac{n}{2i} - ir\partial_r.$$

For a symbol a defined on $\mathbb{R}^{2n}$ a rescaled pseudodifferential operator is defined as

$$Op_\lambda(a) := e^{i\tau\mathcal{A}} Op(a) e^{-i\tau\mathcal{A}}. \tag{2.1.5}$$

Taking $\tau = \ln(\lambda^{\frac{1}{2}})$ implies a rescaling of the spatial variable by $\lambda^{\frac{1}{2}}$, in other words

$$Op_\lambda(a) = Op(a_\lambda)$$

with $a_\lambda(\breve{r}, \theta, \breve{\rho}, \eta)$ and $\breve{r} := \lambda^{\frac{1}{2}} r$, $\breve{\rho} := \lambda^{-\frac{1}{2}} \rho$.

Analogously we define rescaled pseudodifferential operators on manifolds as

$$Op_{\lambda,\kappa}(a) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega) := \Pi_\kappa Op_\lambda(a) \Pi_\kappa^{-1} \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega) \tag{2.1.6}$$

for a symbol a supported in $[R, +\infty) \times V_\kappa \times \mathbb{R}^n$. We remark that $Op_{\lambda,\kappa}(a)\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)$ maps $C_0^\infty(M)$ in the set of functions supported in $[R, +\infty) \times U_\kappa$.

For example, near infinity P_0 agrees with the Laplace-Beltrami operator on $(R + \infty) \times S$ and therefore in local coordinates it is given by

$$P_0/\lambda = -\frac{\partial_r^2}{\lambda} - \frac{n-1}{r} \frac{\partial_r}{\lambda} - \frac{1}{\lambda r^2} \Delta_{g(r)} - \frac{\partial_r |g(r, \theta)|}{|g(r, \theta)|} \frac{\partial_r}{\lambda}. \tag{2.1.7}$$

In terms of pseudodifferential operators this can be written as

$$\frac{P_0}{\lambda}u = \sum_\kappa Op_{\lambda,\kappa}(a_{0,\lambda} + a_{1,\lambda})\psi_\kappa u \tag{2.1.8}$$

where the symbols are

$$\begin{aligned} a_{0,\lambda}(\breve{r},\theta,\breve{\rho},\eta) :=& \breve{\rho}^2 + \frac{1}{\breve{r}^2} g^{j,k}(\lambda^{-\frac{1}{2}}\breve{r},\theta)\eta_j\eta_k, \\ a_{1,\lambda}(\breve{r},\theta,\rho,\eta) :=& -i\frac{n-1}{\breve{r}}\breve{\rho} - \lambda^{-\frac{1}{2}} w(\lambda^{-\frac{1}{2}}\breve{r},\theta)\breve{\rho} - \lambda^{-1} w_k(\lambda^{-\frac{1}{2}}\breve{r},\theta)\eta_k \end{aligned}$$

for some w, w_k depending on the metric $g(r)$ and such that $w \in S^{-1-\nu}$ and $w_k \in S^{-2}$.

Remark 2.1.12. Considering rescaled pseudodifferential operators is convenient since it allows us to obtain a decay which is uniform with respect to λ, meaning that the symbols in $Op_\lambda(\cdot)$ will belong to λ-independent subsets of $\widetilde{S}^{m,\mu}(\mathbb{R}^{2n})$ for some m and μ. Indeed for $\breve{r} \gtrsim 1$ we obtain

$$a_{0,\lambda}(\breve{r},\theta,\breve{\rho},\eta) \in \widetilde{S}^{0,2}, \quad a_{1,\lambda}(\breve{r},\theta,\breve{\rho},\eta) \in \widetilde{S}^{-1,1},$$

where the bounds on the seminorms are uniform in λ.

Notation. When using symbolic calculus we will often be interested in the decay properties of the symbols, rather than in their explicit expression. For this reason we will use the shorthand

$$Op_{\lambda,\kappa}(\widetilde{S}^{m,\mu})$$

to denote a rescaled pseudodifferential operator with symbol in $\widetilde{S}^{m,\mu}$.

2.2 Main results

As mentioned in the introduction, we focus on the case $V = 0$ and details about the proof for $P = P_0 + V$ can be found in Section 2.5.

In this section we see how to prove the results in Theorems 2.1.4 and 2.1.5, under some conditions whose proof is postponed to Section 2.3. There we shall prove that there exists an operator $\mathcal{A}^\lambda$ selfadjoint on $(L^2(M), G_M)$ which satisfies Proposition 2.3.9, that is

$$\mathbb{1}_I(P_0/\lambda)i[P_0/\lambda, \mathcal{A}^\lambda]\mathbb{1}_I(P_0/\lambda) \geqslant \mathbb{1}_I(P_0/\lambda)$$

for small enough positive λ and I an open neighborhood of 1.

For the construction and precise definition of $\mathcal{A}^\lambda$ we refer to Section 2.3. For the moment we only point out that $\mathcal{A}^\lambda$ is the generator of a unitary group and that in symbolic form it is given by

$$\mathcal{A}^\lambda = \sum_\kappa Op_{\lambda,k}(\widetilde{S}^{1,1}). \tag{2.2.1}$$

Applying Mourre theory in the following section we will be able to prove

$$(\mathcal{A}^\lambda + i)^{-s}(P_0/\lambda - 1 \pm i0)^{-l}(\mathcal{A}^\lambda + i)^{-s} \in \mathcal{L}(L^2(M)) \tag{2.2.2}$$

for any natural $s > l - \frac{1}{2}$ with operator norm uniformly bounded in λ for $\lambda \in (0, \lambda_0]$. Moreover we also obtain that the map

$$\tau \mapsto (\mathcal{A}^\lambda + i)^{-s}(P_0/\lambda - \tau \pm i0)^{-l}(\mathcal{A}^\lambda + i)^{-s} \tag{2.2.3}$$

is of class C^{l-1} in the interior of an interval where the positive commutator estimate holds.

The first step into proving Theorem 2.1.5 will be to look at the resolvent with a spectral localisation, such as

$$\|\langle r\rangle^{-\alpha} f(P_0/\lambda)(P_0-\lambda\pm i0)^{-l}\langle r\rangle^{-\alpha}\|$$

with $f\in C_0^\infty(\mathbb{R})$. To obtain this using (2.2.2) we will need to bound $\langle r\rangle^{-\alpha} f(P_0/\lambda)(\mathcal{A}^\lambda+i)^s$ and we will use Helffer-Sjöstrand formula to evaluate $f(P_0/\lambda)$. The following property of the resolvent will then be useful.

Theorem 2.2.1. *Let $\psi,\widetilde{\psi},\widetilde{\widetilde{\psi}}\in S^0$, supported in $(R,+\infty)\times U_k$ such that*

$$\widetilde{\psi}\psi=\psi,\qquad \widetilde{\widetilde{\psi}}\widetilde{\psi}=\widetilde{\psi}.$$

and $z\in\mathbb{C}\setminus[0,\infty)$. Then for any $N\in\mathbb{N}$ there exist families of symbols $b_{l,\lambda,z}\in\widetilde{S}^{-l,-2-l}$ and $r_{N,\lambda,z}\in\widetilde{S}^{-N,-N}$ such that

$$\psi(\lambda^{\frac12}r,\omega)(P_0/\lambda-z)^{-1}=\sum_{l=0}^{N-1}\psi(\lambda^{\frac12}r,\omega)Op_{\lambda,k}(b_{l,\lambda,z})\widetilde{\psi}(\lambda^{\frac12}r,\omega)+R^N_{\lambda,z}$$

with

$$R^N_{\lambda,z}=\psi(\lambda^{\frac12}r,\omega)Op_{\lambda,k}(r_{N,\lambda,z})\widetilde{\widetilde{\psi}}(\lambda^{\frac12}r,\omega)(P_0/\lambda-z)^{-1}.$$

Moreover, all of the symbols have seminorms uniformly bounded in λ.

The proof is simply by standard techniques for the construction of a parametrix. We use the previous result in the following lemma.

Lemma 2.2.2. *Let $a\in C_0^\infty(\mathbb{R})$, $\lambda_0>0$ and $s,N\in\mathbb{N}$. There exist a family of symbols $(\phi^\kappa_{s,\lambda})_{\lambda\in(0,\lambda_0]}\in\widetilde{S}^{s,-N}$, a family of uniformly bounded operators $(B_\lambda)_{\lambda\in(0,\lambda_0]}$ and $(\widetilde{\psi}_\kappa)_\kappa\in C^\infty([R,+\infty)\times S)$ supported in $(R,+\infty)\times U_\kappa$ with $\widetilde{\psi}_\kappa\psi_\kappa\equiv 1$ such that*

$$(\mathcal{A}^\lambda+i)^s a(P_0/\lambda)=\sum_\kappa Op_{\lambda,\kappa}(\phi^\kappa_{s,\lambda})\widetilde{\psi}_\kappa(\lambda^{\frac12}r,\omega)+B_\lambda(P_0/\lambda+1)^{-N}$$

for $\lambda\in(0,\lambda_0]$.

Proof. By Helffer-Sjöstrand formula we write $a(P_0/\lambda)$ in terms of its resolvent. Indeed, this formula yields

$$a(P_0/\lambda)=\frac{1}{2\pi}\int\overline{\partial}_z\widetilde{a}(z)(P_0/\lambda-z)^{-1}L(dz)\tag{2.2.4}$$

where $\widetilde{a}$ is an almost analytic extension of a, that is an extension of a to the plane $\mathbb{R}^2$ with the following properties: $\widetilde{a}\in C_0^\infty(\mathbb{R}^2)$ and $\overline{\partial}_z\widetilde{a}=(\partial_x+i\partial_y)a$ vanishes to infinite order on $\{y=0\}$. Moreover, we will use the following property

$$\frac{1}{2\pi}\int\overline{\partial}_z\widetilde{a}(z)(\mu-z)^{-1-j}L(dz)=\frac{(-1)^j}{j!}a^{(j)}(\mu).\tag{2.2.5}$$

In (2.2.4) we apply Theorem 2.2.1 and replace the resolvent with the parametrix. Thanks to the expression of the symbols in the parametrix and using (2.2.5) we obtain symbols in $a(P_0/\lambda)$ that have negative decay in space and compact support in the angular part. Namely, we can write for any $M\in\mathbb{N}$

$$\psi_\kappa(\lambda^{\frac12}r,\omega)a(P_0/\lambda)=Op_{\lambda,\kappa}(\widetilde{S}^{0,-M})\widetilde{\psi}_\kappa(\lambda^{\frac12}r,\omega)+R_{\lambda,z}(P_0/\lambda+1)^{-N}\tag{2.2.6}$$

where $R_{\lambda,z}$ includes the integral of the remainder part given by the parametrix. Next we need to compose (2.2.6) on the left with powers $(\mathcal{A}^\lambda)^j$ of order $j\leqslant s$. By choosing the appropriate M

$(M = s + N)$, we can conclude observing that $(\mathcal{A}^\lambda)^j$ will have symbols in $\widetilde{S}^{j,j}$ (see (2.2.1)) and that $R_{\lambda,z}$ is

$$R_{\lambda,z} = \frac{1}{\pi i}\int \overline{\partial}_z \widetilde{a}(z) Op_{\lambda,\kappa}(r^K_{\lambda,z})(P_0/\lambda - z)^{-1}(P_0/\lambda + 1)^N L(dz),$$

where $r^K_{\lambda,z} \in \widetilde{S}^{-K,-K}$ has seminorms growing polynomially in $1/|Imz|^K$ for any $K \in \mathbb{N}$.

□

Remark 2.2.3. In the previous proof we used the fact that symbols in $\widetilde{S}^{0,0}$ correspond to bounded operators of $L^2(M)$. For a proof in the case of the rescaled pesudodifferential operators we are using here see Proposition 3.4 in [BM a] and in particular inequality (3.13).

We now derive some useful properties in order to handle powers of the resolvent, as the one in the statement of Lemma 2.2.2.

Lemma 2.2.4. *Let $p \in [1,2]$ and an integer $N \geqslant 1 + \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$. There exists $C > 0$ such that*

$$\|(P_0/\lambda + 1)^{-N}\|_{L^p(M)\to L^2(M)} \leqslant C\lambda^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}$$

for all $\lambda > 0$.

The result is derived thanks to the behaviour of the flow of the heat equation e^{-tP_0}

$$\|e^{-tP_0}\|_{L^p(M)\to L^2(M)} \leqslant Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}, \qquad p \in [1,2] \tag{2.2.7}$$

which in turn is due to the fact that a Nash type inequality holds. Namely, for all $u \in C_0^\infty(M)$

$$\|u\|^{1+\frac{2}{n}}_{L^2(M)} \leqslant C_n \|u\|^{\frac{2}{n}}_{L^1(M)} \|P_0^{\frac{1}{2}} u\|_{L^2(M)} \tag{2.2.8}$$

for some $C_n > 0$.

The inequality is proved in detail in Appendix 2.B, here we briefly record that it is done by first considering the full cone $\mathbb{R}^+ \times S$ with fixed metric $\overline{G}$ so that locally we can apply the Nash inequality on $\mathbb{R}^n$. The result still holds on M since on the compact part K we can use a finite covering to reduce ourselves to $\mathbb{R}^n$, while on the manifold end we use the inequality obtained for the cone.

To derive (2.2.7) we can interpolate $\|e^{-tP_0}\|_{L^2(M)\to L^2(M)} \lesssim 1$, given by the Hille-Yosida theorem, and $\|e^{-tP_0}\|_{L^1(M)\to L^2(M)} \lesssim t^{-\frac{n}{4}}$ obtained from (2.2.8) and the fact that e^{-tP_0} preserves the sign and the L^1 norm.

Proof of Lemma 2.2.4. We can follow the same proof described in Lemma 3.2 of [BB21]. Writing the resolvent via the heat kernel

$$(P_0/\lambda + 1)^{-N} = \frac{1}{N!}\int_0^\infty e^{-t(P_0/\lambda+1)} t^{N-1} dt$$

we apply (2.2.7) and the fact that, given the assumption on N, $e^{-t}t^{N-1-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}$ is integrable on $\mathbb{R}^+$.

□

Thanks to Lemma 2.2.4 we also easily obtain polynomial decay for powers of the resolvent. This lemma will also be used extensively in Section 2.4.

Lemma 2.2.5. *Let $n \geqslant 3$, for all $s \in [0, \frac{n}{4}] \cap [0, N)$ and $\frac{\sigma}{2} > s$ then there exists $C > 0$ such that*

$$\|(P_0/\lambda + 1)^{-N}\langle r\rangle^{-\sigma}\| \leqslant C\lambda^s$$

for all $\lambda > 0$.

Proof. The statement follows from Lemma 2.2.4 and Hölder inequality to bound $\langle r\rangle^{-\sigma}$ as an operator from $L^2(M)$ to $L^p(M)$. □

Now, using the expression given in Lemma 2.2.2 we can easily derive the following proposition.

Proposition 2.2.6. *Let $a \in C_0^\infty(\mathbb{R})$, $\alpha \geqslant l$, $\lambda_0 > 0$ and $s \in (0, \frac{n}{4}] \cap (0, \frac{\alpha}{2})$. Then*

$$\|(\mathcal{A}^\lambda + i)^l a(P_0/\lambda)\langle r\rangle^{-\alpha}\| \lesssim \lambda^s$$

for $\lambda \in (0, \lambda_0]$.

Proof. From Lemma 2.2.2 and the fact that operators with symbols in $\widetilde{S}^{0,0}$ are bounded, paired with Lemma 2.2.5 to control $\|B_\lambda(P_0/\lambda+1)^{-N}\langle r\rangle^{-\alpha}\|$. We underline that a power $\lambda^{\frac{\alpha}{2}}$ is generated from the terms $Op_{\lambda,\kappa}(\phi^\kappa_{s,\lambda})\widetilde{\psi}_\kappa(\lambda^{\frac{1}{2}}r,\omega)\langle r\rangle^{-\alpha}$ when moving the factor $\langle r\rangle^{-\alpha}$ into the rescaled pseudodifferential operator. □

Combining (2.2.2) with Proposition 2.2.6 we can straightforwardly obtain bounds on the spectrally localised resolvent.

Theorem 2.2.7. *Let $f \in C_0^\infty(\mathbb{R})$, $\lambda_0 > 0$ and $l \in \mathbb{N}$. If $\alpha \geqslant l$ and $s \in [0, \frac{n}{4}] \cap (0, \frac{\alpha}{2})$ then*

$$\|\langle r\rangle^{-\alpha} f(P_0/\lambda)(P_0 - \lambda \pm i0)^{-l}\langle r\rangle^{-\alpha}\| \lesssim \lambda^{2s-l}$$

for $\lambda \in (0, \lambda_0]$.

With the aid of this theorem we obtain a bound on the resolvent which is still localised, but in a weaker way. The proof, being analogous to Proposition 4.4 in [BB21], is omitted here.

Proposition 2.2.8. *Let $\lambda_0 > 0$, there exists $F \in C_0^\infty(\mathbb{R})$ equal to 1 near $[0, \lambda_0]$ such that for $\alpha > l$ and $l \in \mathbb{N} \setminus \{\frac{n}{2}\}$*

$$\|\langle r\rangle^{-\alpha} F(P_0)(P_0 - \lambda \pm i0)^{-l}\langle r\rangle^{-\alpha}\| \lesssim \lambda^{\min\{0,n/2-l\}}$$

for all $\lambda \in (0, \lambda_0]$. If $l = \frac{n}{2}$

$$\|\langle r\rangle^{-\alpha} F(P_0)(P_0 - \lambda \pm i0)^{-\frac{n}{2}}\langle r\rangle^{-\alpha}\| \lesssim |\log \lambda|$$

for all $\lambda \in (0, \lambda_0]$.

Thanks to these preliminary steps we are now ready to prove the estimate without any localisation on the resolvent.

Proof of Theorem 2.1.5 (Case $V = 0$). Pick F as in the previous proposition, the result then follows from Proposition 2.2.8 since $(1 - F(P_0))(P_0 - \lambda \pm i0)^{-l}$ is uniformly bounded in λ by the spectral theorem.

□

As for the result on the spectral measure, we recall that thanks to Stone's formula

$$\frac{dE(\lambda)}{d\lambda} = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left((P_0 - \lambda - i\varepsilon)^{-1} - (P_0 - \lambda + i\varepsilon)^{-1}\right),$$

we can equivalently consider outgoing and incoming resolvents so to use the result we just established in Theorem 2.2.7.

Proof of Theorem 2.1.4 (Case $V=0$). Let $f\in C_0^\infty(\mathbb{R})$ bounded and supported around 1 then we have

$$(1-f(P_0/\lambda))\left((P_0-\lambda-i\varepsilon)^{-1-j}-(P_0-\lambda+i\varepsilon)^{-1-j}\right)\to 0$$

in the strong topology as ε goes to 0. It then suffices to consider the terms

$$\langle r\rangle^{-\alpha}f(P_0/\lambda)\left((P_0-\lambda-i0)^{-1}-(P_0-\lambda+i0)^{-1}\right)\langle r\rangle^{-\alpha},$$

thanks to the regularity of the map (2.2.3) and Lemma 2.2.2 we deduce that

$$\langle r\rangle^{-\alpha}f(P_0/\lambda)\frac{d}{d\lambda}(P_0-\lambda-i0)^{-1}\langle r\rangle^{-\alpha}=\langle r\rangle^{-\alpha}(P_0-\lambda-i0)^{-2}\langle r\rangle^{-\alpha}.$$

In general, for higher derivatives we have

$$\frac{1}{2\pi ij!}\langle r\rangle^{-\alpha}f(P_0/\lambda)\left((P_0-\lambda-i0)^{-1-j}-(P_0-\lambda+i0)^{-1-j}\right)\langle r\rangle^{-\alpha}. \tag{2.2.9}$$

For $j=0,\dots,k-1$ Theorem 2.2.7 applies thanks to the assumption that $\alpha>k$. □

2.3 Limiting absorption principle

The section will be devoted to the proof of the existence of the limits $(P_0/\lambda-1\pm i0)^{-l}$ in weighted L^2 spaces thanks to a limiting absorption principle (specifically Theorem 1 in [Gé08]). Consequently, the section mainly concerns the construction of a conjugate operator $\mathcal{A}^\lambda$ (Remark 2.3.1) and the proof of a positive commutator estimate (Proposition 2.3.9). This will be possible thanks to the condition stated in Assumption 2.1. Proving that this condition holds will be the aim of Section 2.4.

We look for $\mathcal{A}^\lambda$, a conjugate operator for P_0/λ, that is a selfadjoint operator which verifies some positive commutator estimate and such that $P_0/\lambda\in C^2(\mathcal{A}^\lambda)$, meaning that for all $u\in L^2(M)$ the map

$$\mathbb{R}\ni t\mapsto e^{it\mathcal{A}^\lambda}(P_0/\lambda+i)^{-1}e^{-it\mathcal{A}^\lambda}u\in D(P_0) \tag{2.3.1}$$

is of class C^2.

To get selfadjointness we will construct $\mathcal{A}^\lambda$ as the generator of a unitary group. Let $\chi\in C_0^\infty(\mathbb{R})$ equal to 1 in a large enough neighborhood of 0. Consider the generator of dilations $\frac{rD_r+D_rr}{2}$, after localisation in the region $\{|r|\geqslant\lambda^{-\frac{1}{2}}R\}$ we obtain

$$\mathcal{A}_1^\lambda:=\frac{(1-\chi)(\lambda^{\frac{1}{2}}r)rD_r+D_rr(1-\chi)(\lambda^{\frac{1}{2}}r)}{2}=(1-\chi)(\lambda^{\frac{1}{2}}r)\left(\frac{1}{2i}-ir\partial_r\right)+\frac{i}{2}r\partial_r\chi(\lambda^{\frac{1}{2}}r).$$

We define the group of transformations

$$U_t^\lambda\varphi(r,\theta)=\left|\det(\text{Jac }\phi_t^\lambda(r,\theta))\right|^{\frac{1}{2}}\varphi(\phi_t^\lambda(r,\theta)),$$

where ϕ_t^λ is the flow of the complete vector field $((1-\chi)(\lambda^{\frac{1}{2}}r)r,0,\dots,0)$. Thanks to Theorem VIII.10 in [RS81] we can conclude that $\mathcal{A}_1^\lambda$ is essentially selfadjoint on $C_0^\infty([R,+\infty)\times S)$ with respect to the measure induced by the metric $\overline{G}$ and that its closure is the infinitesimal generator of U_t^λ.

Moreover conjugating U_t^λ by the function

$$y_S(r,\theta):=\frac{|g(\theta)|}{r^{n-1}|g(r,\theta)|}$$

we obtain a group $y_S^{\frac{1}{2}} U_t^\lambda y_S^{-\frac{1}{2}}$ which is unitary with respect to the metric G defined in (2.1.3) and whose generator will be

$$\begin{aligned}\mathcal{A}_2^\lambda =&(1-\chi)(\lambda^{\frac{1}{2}}r)\left(\frac{n}{2i} - ir\partial_r + \frac{1}{2i}r\frac{\partial_r|g(r,\theta)|}{|g(r,\theta)|}\right) + \frac{i}{2}r\partial_r\chi(\lambda^{\frac{1}{2}}r)\\ =:&(1-\chi)(\lambda^{\frac{1}{2}}r)\widetilde{\mathcal{A}} + \frac{i}{2}r\partial_r\chi(\lambda^{\frac{1}{2}}r). \qquad (2.3.2)\end{aligned}$$

Remark 2.3.1 (Definition of $\mathcal{A}^\lambda$). To define a unitary group which acts on the Hilbert space where P_0 is defined, that is $L^2(M)$, we set

$$e^{it\mathcal{A}^\lambda}u := e^{it\mathcal{A}_2^\lambda}\chi_{M\setminus K}u + \chi_K u$$

whose generator is the operator $\mathcal{A}^\lambda$, which is selfadjoint on $L^2(M)$ and non zero only on the manifold end where it coincides with $\mathcal{A}_2^\lambda$.

For the continuity of P_0/λ with respect to $\mathcal{A}^\lambda$, that is continuity of (2.3.1), it is enough to prove that the operators

$$[P_0/\lambda, i\mathcal{A}^\lambda](P_0/\lambda+i)^{-1}, \qquad [[P_0/\lambda, i\mathcal{A}^\lambda], i\mathcal{A}^\lambda](P_0/\lambda+i)^{-1} \qquad (2.3.3)$$

are bounded, where the commutators are appropriately defined in the sense of quadratic forms. Indeed, in general given an Hilbert space $\mathcal{H}$ with T, A selfadjoint and T bounded for the map

$$\mathbb{R} \ni t \mapsto e^{itA}Te^{-itA}u =: B(t)u \in \mathcal{H}$$

to be $C^k(\mathbb{R})$ it is enough that k-th derivative $\frac{d^k}{dt^k}B(t_0)$ is a bounded operator of $\mathcal{L}(\mathcal{H})$ for some fixed t_0. This in turn is implied if

$$T,\ ad_A^0(T) := [T, iA],\quad ad_A^j(T) := [ad_A^{j-1}(T), iA]$$

are bounded operators, in the sense of quadratic forms for $j = 1, \dots, k-1$.

In our specific case we need to control only the commutator and the first iterated commutator with $T = (P_0/\lambda+i)^{-1}$ and $A = \mathcal{A}^\lambda$. With some algebraic manipulations we see that it is equivalent to the boundedness of the operators in (2.3.3). To prove this we will exploit symbolic calculus and Theorem 2.2.1.

Before going on with the computations we will state a useful property. Briefly, commutators between rescaled pseudodifferential operators essentially behave like commutators between differential operators, when we write them in symbolic form. We now give a formal description of the result, while the proof, being quite technical, is postponed to Appendix 2.C.

Definition 2.3.2 (Negligible operator of order N)**.** *Let $N \in \mathbb{N}$ and set*

$$Q_N := \langle\lambda^{\frac{1}{2}}r\rangle^N\left(\frac{P_0}{\lambda}+1\right)^N = \sum_\kappa Op_{\lambda,\kappa}(\widetilde{S}^{N,2N}),$$

we say that an operator is negligible of order N if it is of the form $Q_N^{-1}\mathcal{B}Q_N^{-1}$ for some bounded operator $\mathcal{B}$ depending on λ.

Remark 2.3.3. The operator $\mathcal{B}$ depends on λ since it will be the result of the composition of a rescaled pseudodifferential operator of negative order with Q_N. However, the symbols in Q_N have seminorms uniformly bounded with respect to λ and therefore so will $\mathcal{B}$.

Proposition 2.3.4. *Let m, m', μ, μ' real numbers and the operators A, B on $[R,+\infty)\times S$ defined as*

$$A := \sum_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa, \qquad B := \sum_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa$$

with symbols $a^\kappa \in \widetilde{S}^{m,\mu}$ and $b^\kappa \in \widetilde{S}^{m',\mu'}$ spatially supported in $[R,+\infty)\times V_\kappa$. Then for the commutator it holds

$$[A,B] = \sum_\kappa Op_{\lambda,\kappa}(\widetilde{S}^{m+m'-1,\mu+\mu'-1})\psi_\kappa + \mathcal{R}_N$$

with $\mathcal{R}_N$ an operator which is negligible of order N for any $N \in \mathbb{N}$.

Remark 2.3.5. Let $C = Op_{\lambda,\kappa}(c^\kappa)$ with $c^\kappa \in \widetilde{S}^{m,\mu}$, from Proposition 2.3.4 we can always find N large enough such that $\mathcal{R}_N C$ and $C\mathcal{R}_N$ are still bounded. Indeed, the interest of Definition 2.3.2 for N arbitrary is that Q_N^{-1} provides infinite decay both in the spatial and phase variables. In practice we will consider compositions of commutators with resolvents (as in (2.3.3) or Section 2.4) and the remainder term will always stay bounded as we just remarked.

Remark 2.3.6. Let $\mathcal{B}$ bounded and $M > 0$. In the computations we will write negligible operators in the forms

$$\mathcal{B}\sum_\kappa Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M}), \qquad \sum_\kappa Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M})\mathcal{B}$$

when we will need decay only on the right or the left respectively.

Remark 2.3.7. Although we interpreted commutators as derivatives of the map $e^{itA}(P_0/\lambda + i)^{-1}e^{-itA}$, to perform computations we will rather use their symbolic form, (see (2.3.4) below). Indeed, on $C_0^\infty(M)$ (which is dense in $D(P_0)$) we can prove that the derivative of $e^{itA}(P_0/\lambda + i)^{-1}e^{-itA}$ is the commutator $[(P_0/\lambda+i)^{-1}, i\mathcal{A}^\lambda]$, which we can rewrite in terms of the commutator between P_0/λ and $i\mathcal{A}^\lambda$. Now on smooth functions the action of P_0/λ and $\mathcal{A}^\lambda$ is the one of differential operators, this allows us to write the symbolic form used in (2.3.4).

Writing $\mathcal{A}^\lambda$ in its symbolic form as

$$\mathcal{A}^\lambda = \sum_\kappa Op_{\lambda,k}(\widetilde{S}^{1,1} + \lambda^{\frac{\nu}{2}}\widetilde{S}^{-\nu,0} + C_0^\infty(\mathbb{R}\setminus\{0\}))$$

and using Proposition 2.3.4 as well as Remark 2.3.6 we first find that

$$[P_0/\lambda, i\mathcal{A}^\lambda] = [\sum_\kappa Op_{\lambda,\kappa}(\widetilde{S}^{0,2}), \sum_l Op_{\lambda,l}(\widetilde{S}^{1,1})] = \sum_\kappa (Op_{\lambda,\kappa}(\widetilde{S}^{0,2}) + \mathcal{B}Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M})) \quad (2.3.4)$$

for some bounded operator $\mathcal{B}$ and some positive M.

Combining with the information provided by the parametrix we obtain that $[P_0/\lambda, i\mathcal{A}^\lambda](P_0/\lambda+i)^{-1}$ is indeed a sum of bounded operators (recall Remark 2.2.3).

An analogous result holds for the iterated commutator since we can still write it in the form

$$[[P_0/\lambda, i\mathcal{A}^\lambda], i\mathcal{A}^\lambda] = \sum_\kappa (Op_{\lambda,\kappa}(\widetilde{S}^{0,2}) + \mathcal{B}Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M})) \quad (2.3.5)$$

and reason in the same way as before.

Remark 2.3.8. Actually, we remark here that we can iterate the argument as many times as needed. Indeed, continuing from (2.3.5), any iterated commutator is of the form

$$ad^k_{\mathcal{A}^\lambda}(P_0/\lambda) = \sum_\kappa (Op_{\lambda,\kappa}(\widetilde{S}^{0,2}) + \mathcal{B}Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M})).$$

We can therefore apply Theorem 2.2.1 as before to conclude that $ad^k_{\mathcal{A}^\lambda}(P_0/\lambda)(P_0/\lambda + i)^{-1}$ is bounded, which implies that $P_0/\lambda \in C^k(\mathcal{A}^\lambda)$ for any k.

Next, we will prove a positive commutator estimate for P_0/λ for which we need the following property that will be checked in Section 2.4.

Assumption 2.1. *For all $\alpha > 0$ and $\varepsilon > 0$ there exist $\lambda_0 > 0$ and $f \in C_0^\infty(\mathbb{R})$ equal to 1 in a neighborhood of 1 such that*

$$\|\langle\lambda^{\frac{1}{2}}r\rangle^{-\alpha} f(P_0/\lambda)\| \leqslant \varepsilon$$

for all $\lambda \in (0, \lambda_0]$.

With this we can prove the desired inequality.

Proposition 2.3.9. *Let* $\lambda_0 > 0$ *small enough, if Assumption 2.1 holds there exists* I *open bounded interval containing 1 such that*

$$\mathbb{1}_I(P_0/\lambda) i[P_0/\lambda, \mathcal{A}^\lambda] \mathbb{1}_I(P_0/\lambda) \geqslant \mathbb{1}_I(P_0/\lambda).$$

for all $\lambda \in (0, \lambda_0]$.

Before giving the proof of the positive commutator estimate, we point out that thanks to this inequality coupled with the fact that $P_0/\lambda \in C^2(\mathcal{A}^\lambda)$ we can apply Theorem 1 in [Gé08]. As a result we have

$$\sup_{\lambda \in (0,\lambda_0]} \sup_{\varepsilon > 0} \| \langle \mathcal{A}^\lambda \rangle^{-s} (P_0/\lambda - 1 \pm i\varepsilon)^{-1} \langle \mathcal{A}^\lambda \rangle^{-s} \| < \infty \tag{2.3.6}$$

for $s > \frac{1}{2}$, or equivalently

$$(\mathcal{A}^\lambda + i)^{-s} (P_0/\lambda - 1 \pm i\varepsilon)^{-1} (\mathcal{A}^\lambda + i)^{-s} \in \mathcal{L}(L^2(M))$$

with operator norms uniformly bounded in λ and we take $s \in \mathbb{N}$. Finally, thanks to the higher regularity of P_0/λ stated in Remark 2.3.8 similar bounds can be proved for powers of the resolvent, therefore obtaining

$$(\mathcal{A}^\lambda + i)^{-s} (P_0/\lambda - 1 \pm i0)^{-l} (\mathcal{A}^\lambda + i)^{-s} \in \mathcal{L}(L^2(M)) \tag{2.3.7}$$

for any natural $s > l - \frac{1}{2}$ with norms uniformly bounded in λ. Indeed, our conjugate operator $\mathcal{A}^\lambda$ is, in particular, **uniformly conjugate** to P_0/λ according to Definition 5.1 in [BR14a] and **uniformly** ∞**-smooth** with respect to $\mathcal{A}^\lambda$ (see Definition 5.3 in [BR14a]). Following the ideas of [Jen85], it is proved in [BR14a] that with this smoothness properties we have estimates for powers of the resolvent analogous to the ones in (2.3.6). This then implies (2.3.7). Finally, Theorem 5.8 in [BR14a] gives us the regularity of the map mentioned in (2.2.3).

To prove Proposition 2.3.9 we will split the commutator in the part at infinity, where P_0 is of the form (2.1.8) and $\mathcal{A}^\lambda = \widetilde{\mathcal{A}}$ (see (2.3.2)), and treat the rest as a compactly supported perturbation (we recall that on the compact part of the manifold $\mathcal{A}^\lambda$ is simply zero). Namely, we can write

$$i[P_0/\lambda, \mathcal{A}^\lambda] = (1 - \widetilde{\chi})(\lambda^{\frac{1}{2}} r) i[P_0/\lambda, \widetilde{\mathcal{A}}] + \widetilde{\chi}(\lambda^{\frac{1}{2}} r) i[P_0/\lambda, \mathcal{A}^\lambda] \tag{2.3.8}$$

where $\widetilde{\chi}$ is a smooth cutoff equal to one on the support of χ. In local coordinates the Laplace-Beltrami operator $-\Delta_0$ on the fixed half cone $([R, +\infty) \times S, \overline{G})$ is

$$-\partial_r^2 - \frac{n-1}{r}\partial_r - \frac{1}{r^2}\Delta_{\overline{g}}, \tag{2.3.9}$$

for more details on the definition of the operator see Appendix 2.A. Recalling the local coordinates expression in (2.1.7) we notice that on the manifold end we can write P_0 in function of $-\Delta_0$. In doing so, thanks to the fact that $g(r)$ is a perturbation of $\overline{g}$ we can quantify the decay of the remaining part.

Proposition 2.3.10. *Let* $\lambda_0 > 0$ *and* $\lambda \in (0, \lambda_0]$*, then*

$$\begin{aligned}(1 - \widetilde{\chi})(\lambda^{\frac{1}{2}} r) P_0/\lambda =& (1 - \widetilde{\chi})(\lambda^{\frac{1}{2}} r)(-\Delta_0/\lambda) \\ &+ \sum_\kappa \left(\lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\widetilde{S}^{-\nu,2}) + Op_{\lambda,\kappa}(\widetilde{S}^{-1,1}) \right) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)\end{aligned}$$

with symbols belonging to bounded subsets of $\widetilde{S}^{-\nu,2}$ *and* $\widetilde{S}^{-1,1}$ *respectively.*

Notably this will be useful since we are taking the commutator with $\widetilde{\mathcal{A}}$ given by

$$\widetilde{\mathcal{A}} = \mathcal{A} + \sum_{\kappa} \lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\widetilde{S}^{-\nu,0})\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega). \tag{2.3.10}$$

Indeed, writing P_0/λ in terms of $-\Delta_0/\lambda$ and $\widetilde{\mathcal{A}}$ in terms of $\mathcal{A} = \frac{n}{2i} - ir\partial_r$ allows us to take advantage of the identity $[-\Delta_0, i\mathcal{A}] = 2(-\Delta_0)$. This last property can be checked by direct computations given the expression in (2.3.9).

We will also write $\widetilde{\mathcal{A}}$ in the form

$$\widetilde{\mathcal{A}} = \sum_{\kappa} Op_{\lambda,\kappa}(\widetilde{S}^{1,1})\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega) \tag{2.3.11}$$

which will be useful to treat the commutators of $\widetilde{\mathcal{A}}$ with the perturbative terms.

Proof of Proposition 2.3.9. Given (2.3.9) we have $-\Delta_0/\lambda = \sum_\kappa Op_{\lambda,\kappa}(\widetilde{S}^{0,2})\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)$. By Proposition 2.3.10, (2.3.10) and (2.3.11) we can compute

$$\begin{aligned}
[P_0/\lambda, \widetilde{\mathcal{A}}] =& \sum_\kappa [-\Delta_0/\lambda, \mathcal{A} + \lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\widetilde{S}^{-\nu,0})\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)] \\
&+ \sum_\kappa [(\lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\widetilde{S}^{-\nu,2}) + Op_{\lambda,\kappa}(\widetilde{S}^{-1,1}))\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega), Op_{\lambda,\kappa}(\widetilde{S}^{1,1})\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)] \\
=& 2(-\Delta_0/\lambda) + \sum_\kappa (\lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\widetilde{S}^{-\nu,2}) + Op_{\lambda,\kappa}(\widetilde{S}^{-1,1}))\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega) \\
&+ \sum_\kappa \mathcal{B} Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M}))\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)
\end{aligned}$$

(we apply here the calculus rules given by Proposition 2.3.4 and the observation in Remark 2.3.6). On the support of $\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)$

$$\lambda^{\frac{\nu}{2}} Op_{\lambda,k}(\widetilde{S}^{-\nu,2}) = \langle \lambda^{\frac{1}{2}} r\rangle^{-\nu} Op_{\lambda,k}(\langle \lambda^{\frac{1}{2}} r\rangle^{\nu} \widetilde{S}^{-\nu,2}) = \langle \lambda^{\frac{1}{2}} r\rangle^{-\nu} Op_{\lambda,k}(\widetilde{S}^{0,2}),$$

and similarly

$$Op_{\lambda,k}(\widetilde{S}^{-1,1}) = \langle \lambda^{\frac{1}{2}} r\rangle^{-1} Op_{\lambda,k}(\langle r\rangle \widetilde{S}^{-1,1}) = \langle \lambda^{\frac{1}{2}} r\rangle^{-1} Op_{\lambda,k}(\widetilde{S}^{0,1}),$$

so the quantity in (2.3.8) is given by

$$\begin{aligned}
i[P_0/\lambda, \mathcal{A}^\lambda] =& (1-\widetilde{\chi})(\lambda^{\frac{1}{2}} r) \left(2P_0/\lambda + \sum_\kappa \langle \lambda^{\frac{1}{2}} r\rangle^{-\nu} Op_{\lambda,k}(\widetilde{S}^{0,2}) + \langle \lambda^{\frac{1}{2}} r\rangle^{-1} Op_{\lambda,k}(\widetilde{S}^{0,1}) \right) \\
&+ (1-\widetilde{\chi})(\lambda^{\frac{1}{2}} r) \sum_\kappa \mathcal{B} Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M}) \\
&+ \widetilde{\chi}(\lambda^{\frac{1}{2}} r)[P_0/\lambda, i\mathcal{A}^\lambda].
\end{aligned}$$

Moreover, up to a compactly supported perturbation we can commute $(1-\widetilde{\chi})$ with any differential operator and in particular the pseudodifferential operators in the sum above are differential (they are the result of a commutator between differential operators). We have obtained

$$\begin{aligned}
i[P_0/\lambda, \mathcal{A}^\lambda] - 2P_0/\lambda =& -2\widetilde{\chi}(\lambda^{\frac{1}{2}} r) P_0/\lambda \\
&+ \sum_\kappa \langle \lambda^{\frac{1}{2}} r\rangle^{-\nu} Op_{\lambda,k}(\widetilde{S}^{0,2})(1-\widetilde{\chi})(\lambda^{\frac{1}{2}} r)
\end{aligned}$$

$$+\sum_{\kappa}\langle\lambda^{\frac{1}{2}}r\rangle^{-1}Op_{\lambda,k}(\widetilde{S}^{0,1})(1-\widetilde{\chi})(\lambda^{\frac{1}{2}}r)$$
$$+(1-\widetilde{\chi})(\lambda^{\frac{1}{2}}r)\sum_{\kappa}\mathcal{B}Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M})$$
$$+\psi(\lambda^{\frac{1}{2}}r)$$

for $\psi\in C_0^\infty(\mathbb{R}\setminus\{0\})$ with $\psi(\lambda^{\frac{1}{2}}r)$ which includes the term $\widetilde{\chi}(\lambda^{\frac{1}{2}}r)[P_0/\lambda,i\mathcal{A}^\lambda]$. Take f satisfying Assumption 2.1 and compose $i[P_0/\lambda,\mathcal{A}^\lambda]-2P_0/\lambda$ on the right and on the left with $f(P_0/\lambda)$. Noticing that by Theorem 2.2.1 and the spectral theorem

$$Op_{\lambda,k}(\widetilde{S}^{0,2})(1-\widetilde{\chi})(\lambda^{\frac{1}{2}}r)(P_0/\lambda+i)^{-1},\qquad Op_{\lambda,k}(\widetilde{S}^{0,1})(1-\widetilde{\chi})(\lambda^{\frac{1}{2}}r)(P_0/\lambda+i)^{-1},$$

and

$$(P_0/\lambda+i)f(P_0/\lambda),\qquad P_0/\lambda(P_0/\lambda+i)f(P_0/\lambda)$$

are all bounded, we have the estimate

$$\|f(P_0/\lambda)\left(i[P_0/\lambda,\mathcal{A}^\lambda]-2P_0/\lambda\right)f(P_0/\lambda)\|\lesssim\|f(P_0/\lambda)\widetilde{\chi}(\lambda^{\frac{1}{2}}r)(P_0/\lambda+i)^{-1}\| \tag{2.3.12}$$
$$+\|f(P_0/\lambda)\langle\lambda^{\frac{1}{2}}r\rangle^{-\nu}\|+\|f(P_0/\lambda)\langle\lambda^{\frac{1}{2}}r\rangle^{-1}\| \tag{2.3.13}$$
$$+\sum_{\kappa}\|f(P_0/\lambda)\mathcal{B}Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M})\| \tag{2.3.14}$$
$$+\|f(P_0/\lambda)\psi(\lambda^{\frac{1}{2}}r)(P_0/\lambda+i)^{-1}\|. \tag{2.3.15}$$

Thanks Assumption 2.1 we can make all the terms in the right hand side arbitrarily small. Indeed, the assumption applies directly to the terms in (2.3.13), for (2.3.14) the decay in r is provided by $Op_{\lambda,\kappa}(\widetilde{S}^{-M,-M})$, while for (2.3.12) and (2.3.15) we observe that we have compact support in r thanks to $\widetilde{\chi}$ and ψ. In particular it holds

$$\|f(P_0/\lambda)\left(i[P_0/\lambda,\mathcal{A}^\lambda]-2P_0/\lambda\right)f(P_0/\lambda)\|\leqslant\frac{1}{2}$$

for all $\lambda\in(0,\lambda_0]$. Now we simply have

$$f(P_0/\lambda)i[P_0/\lambda,\mathcal{A}^\lambda]f(P_0/\lambda)\geqslant 2f(P_0/\lambda)P_0/\lambda f(P_0/\lambda)-\frac{1}{2}\geqslant\frac{3}{2}f^2(P_0/\lambda)-\frac{1}{2}$$

with the last inequality obtained thanks to $2f^2(x)x\geqslant\frac{3}{2}f^2(x)$ (this is always true for f with small enough support as in Assumption 2.1). At last, we choose $I\subset\sigma(P_0)$ an open bounded interval containing 1 and small enough such that f is constantly 1 on I. Then $f(x)\mathbb{1}_I(x)=\mathbb{1}_I(x)$ and applying $\mathbb{1}_I(P_0/\lambda)$ on the right and left of the previous inequality we have

$$\mathbb{1}_I(P_0/\lambda)i[P_0/\lambda,\mathcal{A}^\lambda]\mathbb{1}_I(P_0/\lambda)\geqslant\mathbb{1}_I^2(P_0/\lambda)=\mathbb{1}_I(P_0/\lambda),$$

concluding the proof.

□

2.4 Proof of Assumption 2.1

A crucial step in the work presented up to now was to obtain the positive commutator estimate which allowed us to state that the outgoing and ingoing resolvents exist. Our main concern now is to prove that the Assumption 2.1 we made to obtain this result is valid for the operator P_0 we are considering.

We will split the analysis into several steps by spatially localizing the operator $f(P_0/\lambda)$ as follows

$$f(P_0/\lambda) = \chi f(P_0/\lambda) + (1-\chi)f(P_0/\lambda)\chi + (1-\chi)f(-\Delta_0/\lambda)(1-\chi) \\ + (1-\chi)f(P_0/\lambda)(1-\chi) - (1-\chi)f(-\Delta_0/\lambda)(1-\chi)$$

where $\chi = \chi(r)$ is a smooth cutoff which is constantly 1 on K and zero for large r.

Remark 2.4.1. The difference in the second line of the expression is well defined. Indeed, thanks to the cutoff on the right we are restricting ourselves to functions supported on $M \setminus K$ that can be identified with functions on $(R, +\infty) \times S$ which is where both the actions of $f(-\Delta_0/\lambda)$ and $f(P_0/\lambda)$ make sense.

We recall that in Assumption 2.1 it is stated that for any α the norm of $\langle \lambda^{\frac{1}{2}} r\rangle^{-\alpha} f(P_0/\lambda)$ can be made arbitrarily small up to spectrally localizing P_0/λ close to one. We will summarize here how each term is treated and where the relative statement can be found.

i) $\chi f(P_0/\lambda)$, $(1-\chi)f(P_0/\lambda)\chi$: their norm can be made arbitrarily small up to choosing λ sufficiently small. See Proposition 2.4.2, via Lemmas 2.2.4 and 2.2.5.

ii) $(1-\chi)f(-\Delta_0/\lambda)(1-\chi)$: thanks to the multiplication by the decaying factor $\langle \lambda^{\frac{1}{2}} r\rangle^{-\alpha}$ the norm can be made arbitrarily small up to choosing f with small enough support. See Proposition 2.4.6, via rescaling argument and Lemma 2.4.5.

iii) $(1-\chi)f(P_0/\lambda)(1-\chi) - (1-\chi)f(-\Delta_0/\lambda)(1-\chi)$: the norm can be made arbitrarily small up to choosing λ sufficiently small. In Section 2.4.2 see (2.4.14). Via Helffer-Sjöstrand formula, Lemma 2.4.8 and (2.4.13) (with intermediate steps in Lemmas 2.4.9, 2.4.11 and 2.4.14, where Lemma 2.2.5 is extensively used).

Notation. In all of Section 2.4 α is a positive scalar, $\alpha > 0$.

2.4.1 Model operator and compact perturbations

In this first subsection we focus on the compactly supported terms of item *i)* and on the term given by the model operator $-\Delta_0/\lambda$ on the fixed cone, that is item *ii)*.

We will start by showing how to bound $\chi f(P_0/\lambda)$ and $(1-\chi)f(P_0/\lambda)\chi$.

Proposition 2.4.2. *Let $f \in C_0^\infty(\mathbb{R})$ and χ a smooth cutoff on K, then*

$$\|\langle \lambda^{\frac{1}{2}} r\rangle^{-\alpha}(\chi f(P_0/\lambda) + (1-\chi)f(P_0/\lambda)\chi)\| \lesssim \lambda^{\frac{n}{4}}$$

for all $\lambda > 0$.

Proof. By the spectral theorem and Lemma 2.2.4 with $p = 1$ we have

$$\|f(P_0/\lambda)\chi\| \lesssim \|(P_0/\lambda + 1)^{-N}\chi\| \lesssim \|(P_0/\lambda+1)^{-N}\|_{L^1(M)\to L^2(M)} \lesssim \lambda^{\frac{n}{4}}.$$

Same holds for $\|\chi f(P_0/\lambda)\|$ and we conclude simply bounding $\langle \lambda^{\frac{1}{2}} r\rangle^{-\alpha}$ by 1. ☐

In the term $(1-\chi)f(-\Delta_0/\lambda)(1-\chi)$ we can take advantage of the fact that on the exact cone $([0,+\infty) \times S, \overline{G})$ we have scaling invariance.

Lemma 2.4.3 (Rescaling on the fixed cone)**.** *Let $\|\cdot\|_{L^2_{\overline{G}}(cone)}$ the norm with respect to the metric $\overline{G}$ on the full cone $[0,+\infty) \times S$. Then for all $\lambda > 0$*

$$\|\langle \lambda^{\frac{1}{2}} r\rangle^{-\alpha} f(-\Delta_0/\lambda)\|_{L^2_{\overline{G}}(cone)\to L^2_{\overline{G}}(cone)} = \|\langle r\rangle^{-\alpha} f(-\Delta_0)\|_{L^2_{\overline{G}}(cone)\to L^2_{\overline{G}}(cone)}.$$

Proof. Let λ_k^2 the k-th eigenvalue of $-\Delta_{\overline{g}}$ and p_k

$$p_k = -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r}\partial_r + \frac{1}{r^2}\mu_k^2$$

as defined in (2.A.8). By the results in Appendix 2.A we can reduce ourselves to the half line $(0, +\infty)$ and prove equivalently that

$$\|\langle \lambda^{\frac{1}{2}} r\rangle^{-\sigma} f(p_k/\lambda)\|_{\mathcal{L}(L^2(\mathbb{R}^+, r^{n-1}dr))} = \|\langle r\rangle^{-\sigma} f(p_k)\|_{\mathcal{L}(L^2(\mathbb{R}^+, r^{n-1}dr))}.$$

The equality follows showing

$$\langle \lambda^{\frac{1}{2}} r\rangle^{-\sigma} f(p_k/\lambda) = T_\lambda \langle r\rangle^{-\sigma} f(p_k) T_\lambda^* \tag{2.4.1}$$

with T_λ unitary operator.

□

Here and later we will need to compare the norm on the exact cone with the L^2 norm on $\mathbb{R}^n$ with respect to the Lebesgue measure, we give further details in the following remark.

Remark 2.4.4 (The flat norm on $\mathbb{R}^n$ and the one on the cone are comparable). The idea is to partition the angular part S into open sets that are diffeomorphic to open sets of $\mathbb{S}^{n-1}$ and take advantage of the fact that the L^2 norm on $(0, +\infty) \times \mathbb{S}^{n-1}$, with respect to the usual metric $dr^2 + r^2 d\sigma$, is equivalent to the norm on $\mathbb{R}^n$.

Let $(\varphi_\kappa)_\kappa$ partition of unity on S and $\kappa_j : U_j \to V_j \subset \mathbb{R}^{n-1}$ the subordinate coordinate charts. In the same way let U an open set of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $\varphi : U \to V \subset \mathbb{R}^{n-1}$ its coordinate chart.

Without loss of generality we can assume $V_j \subset V$ so that it is well defined the diffeomorphism

$$\kappa = \varphi^{-1} \circ \kappa_j : S \supset U_j \to U \subset \mathbb{S}^{n-1}$$

through which $u \in C_0^\infty(S)$ supported in U_j can be identified with $u \circ \kappa^{-1} \in C_0^\infty(\mathbb{S}^{n-1})$.

The metric tensors on S and $\mathbb{S}^{n-1}$ are represented respectively by positive definite matrices such that

$$\overline{c}^{-1} I \leqslant (\overline{g}^{j,k}(\theta))_{j,k} \leqslant \overline{c} I, \qquad c^{-1} I \leqslant (\sigma^{j,k}(x))_{j,k} \leqslant c I.$$

This means that up to multiplication by some bounded function we can pass from one metric to the other. We will say that integrals with respect to $d\overline{g}$ or $d\sigma$ are comparable and write

$$\int_{U_j \subset S} |\varphi_j u| \, d\overline{g} \simeq \int_{U \subset \mathbb{S}^{n-1}} |(\varphi_j u) \circ \kappa^{-1}| \, d\sigma$$

for u smooth on $(0, +\infty) \times S$.

We remark here that $\kappa^{-1} = \kappa_j^{-1} \circ \varphi$ is only defined on $\varphi^{-1}(V_j) \subset U$ since we need to require that φ maps elements into $V_j \subset V$ which is where κ_j^{-1} is defined. However when we consider $(\varphi_j \circ \kappa^{-1})(u \circ \kappa^{-1})$, we can extend it to U by setting it 0 outside of $\varphi^{-1}(V_j)$, since in this case $(\varphi_j \circ \kappa^{-1})$ cuts off near the boundary of $\varphi^{-1}(V_j)$.

Considering the norm on the cone we have found that

$$\int_{(0,+\infty)\times U_j} \varphi_j u \, d\overline{G} \simeq \int_{(0,+\infty)\times U} (\varphi_j u \circ \kappa^{-1}) \, r^{n-1} dr d\sigma \simeq \int_{(0,+\infty)\times U \subset \mathbb{R}^n} (\varphi_j u) \circ \kappa^{-1} \, dx$$

where we still denote by κ the diffeomorphism $(r, \omega) \mapsto (r, \kappa(\omega))$ through which we can identify a function on the cone (suitably supported) with a function on $(0, +\infty) \times \mathbb{S}^{n-1}$. In particular we see that the L^2 norm with respect to the metric $\overline{G}$ on the cone is equivalent to the one on $\mathbb{R}^n$ with the Lebesgue measure.

Having got rid of the dependence on λ thanks to Lemma 2.4.3, we can prove convergence in norm as the support of f shrinks to 1 and therefore write

Lemma 2.4.5. *For any $\varepsilon > 0$ there exists $f \in C_0^\infty(\mathbb{R})$ equal to one in a neighborhood of one and small enough support such that*

$$\|\langle r\rangle^{-\alpha} f(-\Delta_0)\|_{L^2_{\overline{G}}(cone)\to L^2_{\overline{G}}(cone)} \leqslant \varepsilon.$$

Proof. Let $\widetilde{f} \in C_0^\infty(\mathbb{R})$ a fixed function such that $\widetilde{f}$ is 1 near the support of f so that we can write

$$\langle r\rangle^{-\sigma} f(-\Delta_0) = \langle r\rangle^{-\sigma}\widetilde{f}(-\Delta_0)f(-\Delta_0).$$

If the support of f shrinks to $\{1\}$ then $f(-\Delta_0)$ converges strongly to 0, given that 1 is not an eigenvalue. Moreover $\langle r\rangle^{-\sigma}\widetilde{f}(-\Delta_0)$ is a fixed compact operator and therefore the composition converges to 0 in norm.

To prove compactness first let $g \in C_0^1(\mathbb{R})$ supported around 0 and $(\rho_\kappa)_\kappa$ a partition of unity of $[0,+\infty)\times S$ with $supp\rho_\kappa \subset [0,+\infty)\times U_\kappa$. If $(u_n)_n$ is a sequence of uniformly bounded functions in $L^2_{\overline{G}}(cone)$ we have

$$g(r)\widetilde{f}(-\Delta_0)u_n = \sum_\kappa g(r)\rho_\kappa\widetilde{f}(-\Delta_0)u_n \in H_0^1([0,+\infty)\times S)$$

where each term is supported in an open bounded set $(0,\overline{R})\times U_\kappa$ thanks to the supports of g and ρ_κ. Here, by $H_0^1([0,+\infty)\times S)$ we mean the space defined by performing the closure of C_0^∞ functions as in (2.A.3). The Sobolev regularity of $g(r)\rho_\kappa\widetilde{f}(-\Delta_0)u_n$ is given by the fact that $\widetilde{f}(-\Delta_0)$ has image in the domain of $-\Delta_0$ which is contained in the Sobolev space (see (2.A.4)). By the spectral theorem and the uniform bound on $(u_n)_n$

$$\|g(r)\rho_\kappa\widetilde{f}(-\Delta_0)u_n\|_{L^2((0,\overline{R})\times U_\kappa)} \lesssim \|g(r)\rho_\kappa\widetilde{f}\|_\infty\|u_n\|_{L^2(0,\overline{R})\times U_\kappa)} \lesssim 1.$$

Moreover by definition $\nabla_{\overline{G}} = (\partial_r, 1/r\nabla_{\overline{g}})$ and from the fact that ρ_κ is a function of the angular variables only we obtain

$$\begin{aligned}|\nabla_{\overline{G}}\ (g(r)\rho_\kappa\widetilde{f}(-\Delta_0)u_n)|^2 \lesssim &|\nabla_{\overline{G}}(g\rho_\kappa)\widetilde{f}(-\Delta_0)u_n|^2 + |g\rho_\kappa\nabla_{\overline{G}}(\widetilde{f}(-\Delta_0)u_n)|^2\\ \lesssim &\|g'\|^2_\infty|\rho_\kappa\widetilde{f}(-\Delta_0)u_n|^2 + \|g\|_\infty|(\nabla_{\overline{g}}\rho_\kappa)1/r\widetilde{f}(-\Delta_0)u_n|^2\\ &+\|g\|^2_\infty|\rho_\kappa\nabla_{\overline{G}}(\widetilde{f}(-\Delta_0)u_n)|^2.\end{aligned}$$

We can bound $\|1/r\widetilde{f}(-\Delta_0)u_n\|_{L^2((0,\overline{R})\times U_\kappa)}$ with the L^2 norm of the gradient by Hardy inequality which paired with the equality $\|\nabla_{\overline{G}}u\|_{L^2_{\overline{G}}} = \|(-\Delta_0)^{\frac{1}{2}}u\|_{L^2_{\overline{G}}}$ gives

$$\begin{aligned}\|\nabla_{\overline{G}}\ (g(r)\rho_\kappa\widetilde{f}(-\Delta_0)u_n)\|^2_{L^2((0,\overline{R})\times U_\kappa)} &\lesssim \|\widetilde{f}(-\Delta_0)u_n\|^2_{L^2((0,\overline{R})\times U_\kappa)}\\ &\quad+\|(-\Delta_0)^{\frac{1}{2}}\widetilde{f}(-\Delta_0)u_n\|^2_{L^2((0,\overline{R})\times U_\kappa)}\\ &\lesssim \|u_n\|^2_{L^2((0,\overline{R})\times U_\kappa)} \qquad (2.4.2)\\ &\lesssim 1\end{aligned}$$

where to get to (2.4.2) we use again the spectral theorem.

So for each fixed κ the sequence $(g(r)\widetilde{f}(-\Delta_0)u_n\rho_\kappa)_n$ is uniformly bounded in $H_0^1((0,\overline{R})\times U_\kappa)$. By Remark 2.4.4 it will then be diffeomorphic to a uniformly bounded sequence of $H_0^1(\Omega)$ with Ω an open bounded subset of $\mathbb{R}^n$. By compact Sobolev embedding we can then extract a subsequence converging on $L^2(\Omega)$ and composing with the right diffeomorphism we can recover a subsequence of $g(r)\widetilde{f}(-\Delta_0)u_n\rho_\kappa$ that converges in $L^2((0,\overline{R})\times U_\kappa)$. We have therefore proved compactness of $g(r)\widetilde{f}(-\Delta_0)$.

Since the set of compact operators is closed with respect to norm convergence, considering $\phi \in C_0^\infty(\mathbb{R})$ equal to 1 on $B(0,1)$ we write

$$\langle r\rangle^{-\sigma}\widetilde{f}(-\Delta_0) = \langle r\rangle^{-\sigma}\phi\left(\frac{r}{R}\right)\widetilde{f}(-\Delta_0) + \langle r\rangle^{-\sigma}(1-\phi)\left(\frac{r}{R}\right)\widetilde{f}(-\Delta_0)$$

for some large R. The first term is compact and the second one converges to 0 in norm as R tends to ∞, hence $\langle r\rangle^{-\sigma}\widetilde{f}(-\Delta_0)$ is a compact operator. □

Since on the support of $(1-\chi)$ the manifold is diffeomorphic to $(R,+\infty)\times S$, and therefore the norm of $L^2(M)$ is comparable with the one of $L^2_{\overline{G}}$, the previous propositions imply the result for the norm of operators on $L^2(M)$.

Proposition 2.4.6. *For any $\varepsilon > 0$ there exists $f \in C_0^\infty(\mathbb{R})$ equal to one in a neighborhood of one and with small enough support such that*

$$\|(1-\chi)\langle\lambda^{\frac{1}{2}}r\rangle^{-\alpha}f(-\Delta_0/\lambda)(1-\chi)\| \leqslant \varepsilon$$

for any $\lambda > 0$.

2.4.2 Perturbative terms on the infinite end

The rest of the section will be dedicated to the analysis of the term localized on the end of the manifold, that is

$$D_f(\lambda) := (1-\chi)f(P_0/\lambda)(1-\chi) - (1-\chi)f(-\Delta_0/\lambda)(1-\chi), \tag{2.4.3}$$

in particular we will prove that $D_f(\lambda)$ converges to 0 for any $f \in C_0^\infty(\mathbb{R})$ as λ goes to 0. (We recall the term is well defined, see Remark 2.4.1.)

Using Helffer-Sjöstrand formula to compute functional calculus we can reduce ourselves to comparing two resolvents, in particular setting

$$R_z(\lambda) := (1-\chi)\left(\frac{P_0}{\lambda} - z\right)^{-1}(1-\chi) - (1-\chi)\left(\frac{-\Delta_0}{\lambda} - z\right)^{-1}(1-\chi)$$

we rewrite

$$D_f(\lambda) = \frac{1}{\pi i}\int_{\mathbb{C}} \overline{\partial}_z\widetilde{f}(z)R_z(\lambda)L(dz).$$

As we have seen in Proposition 2.3.10, on the support of $(1-\chi)$ we can compare P_0 with $-\Delta_0$. More precisely, by expanding the expressions of $\Delta_{g(r)}$ and $\Delta_{\overline{g}}$ we can decompose the operator in a part on the fixed half cone and a differential operator with decaying coefficients. Namely,

$$(1-\chi)P_0/\lambda = (1-\chi)\left(-\Delta_0 - Q\right)/\lambda,$$

where Q is of the form

$$Q := \sum_\kappa \varphi_\kappa\Pi_\kappa\left(a_\kappa(r,\theta)\partial_r + \sum_l b_\kappa^l(r,\theta)\frac{1}{r}\partial_l + \sum_{j,l} c_\kappa^{j,l}(r,\theta)\frac{1}{r^2}\partial^2_{j,l}\right)\Pi_\kappa^{-1}$$

with

$$a_\kappa \in S^{-1-\nu}, \quad b_\kappa^l \in S^{-1-\nu}, \quad c_\kappa^{j,l} \in S^{-\nu}. \tag{2.4.4}$$

Here we recall that ∂_l, ∂_j are the partial derivatives with respect to θ_l, θ_j, where θ is the local coordinate on S.

With this in mind, we will split $D_f(\lambda)$ into several terms, since by algebraic manipulations we obtain

$$\begin{aligned} R_z(\lambda) = &- \frac{1}{\lambda} (P_0/\lambda - z)^{-1} [P_0, \chi] (P_0/\lambda - z)^{-1} (1-\chi) \\ &+ \frac{1}{\lambda} (P_0/\lambda - z)^{-1} [P_0, \chi] (-\Delta_0/\lambda - z)^{-1} (1-\chi) \\ &+ \frac{1}{\lambda} (P_0/\lambda - z)^{-1} (1-\chi) Q (-\Delta_0/\lambda - z)^{-1} (1-\chi) \\ =: &\, R_z^1(\lambda) + R_z^2(\lambda) + R_z^Q(\lambda). \end{aligned} \tag{2.4.5}$$

Remark 2.4.7. For $R_z^1(\lambda)$ and $R_z^2(\lambda)$ we can take advantage of the fact that the commutator is a differential operator of order one in the spatial variable only with compactly supported coefficients. This allows us to use Lemma 2.2.5 to obtain a bound by a positive power of λ, see Lemma 2.4.8. Lemmas 2.4.9, 2.4.11 and 2.4.14 are the main results providing a bound for $R_z^Q(\lambda)$ which allows to conclude obtaining (2.4.13) and consequently (2.4.14).

Lemma 2.4.8. *Let $f \in C_0^\infty(\mathbb{R})$, there exists $\delta > 0$ such that*

$$\left\| \int_{\mathbb{C}} \overline{\partial}_z \widetilde{f}(z) R_z^i(\lambda) L(dz) \right\| \lesssim \lambda^\delta$$

for $i = 1, 2$ and for all $\lambda > 0$.

Notation. To make the notation lighter, in the sequel we will omit the pullback and pushforward in the expression in local coordinates of Q and $[P_0, \chi]$, meaning for example that we will still denote by $a\partial_r$ the operator **on the manifold** that, on an open set of $\mathbb{R}^n$, corresponds to the derivative with respect to the radial variable and to the multiplication by a.

Proof. The commutator $[P_0, \chi]$ is supported away from the compact part of the manifold, here we recall the expression of the operator in local coordinates is (2.1.7).The angular derivatives commute with χ, so first of all we have

$$[P_0, \chi] = \sum_\kappa \varphi_\kappa (f_{1,\kappa}(r,\theta) + f_{2,\kappa}(r,\theta)\partial_r)$$

with $f_{1,\kappa}, f_{2,\kappa} \in C_0^\infty(\mathbb{R}^n)$. Then using this relation in the definition of $R_z^1(\lambda)$ we apply Lemma 2.2.5 to each term. First, let $\sigma \in \mathbb{R}$ and consider

$$\left\| \frac{1}{\lambda} (P_0/\lambda - z)^{-1} \langle r \rangle^{-\sigma} \varphi_\kappa f_{1,\kappa} \langle r \rangle^{2\sigma} \langle r \rangle^{-\sigma} (P_0/\lambda - z)^{-1} (1-\chi) \right\|.$$

By Lemma 2.2.5 we can have estimates of the type

$$\|(P_0/\lambda - z)^{-1} \langle r \rangle^{-\sigma}\| \lesssim \frac{\langle z \rangle}{|Imz|} \lambda^s \tag{2.4.6}$$

for all $s \in [0, \frac{n}{4}] \cap [0, 1)$ such that $s < \frac{\sigma}{2}$ and we also remark that

$$\|\varphi_\kappa f_{1,\kappa} \langle r \rangle^{2\sigma}\| \lesssim \|f_{1,\kappa} \langle r \rangle^{2\sigma}\|_{L^\infty} \lesssim 1$$

for any σ, given the compact support of $f_{1,\kappa}$. We can therefore freely choose the exponent σ and picking $\sigma > 1$ allows us to find $s \in \left(\frac{1}{2}, \frac{\sigma}{2}\right)$ such that (2.4.6) holds. We observe that the $\langle z \rangle / |Imz|$ factor in (2.4.6) is provided by

$$\|(P_0/\lambda - z)^{-1} (P_0/\lambda + 1)\| \lesssim \frac{\langle z \rangle}{|Imz|}.$$

Collecting all this information together yields

$$\left\| \frac{1}{\lambda} \left(P_0/\lambda - z\right)^{-1} \varphi_\kappa f_{1,\kappa} \left(P_0/\lambda - z\right)^{-1} (1-\chi) \right\| \lesssim \lambda^{2s-1} \frac{\langle z \rangle^2}{|Imz|^2}$$

with $2s-1>0$. We treat similarly the f_2 term where this time we apply Lemma 2.2.5 only to the resolvent on the left. We we still have

$$\|\varphi_\kappa f_{2,\kappa} \langle r \rangle^\sigma\| \lesssim \|f_{2,\kappa}\langle r \rangle^\sigma\|_{L^\infty} \lesssim 1$$

and we choose again $\sigma > 1$ so to obtain (2.4.6) for any $s \in (\frac{1}{2}, \frac{\sigma}{2})$. Picking $\widetilde{f}_{2,\kappa}$ such that $f_{2,\kappa}\widetilde{f}_{2,\kappa} \equiv 1$

$$\begin{aligned} \left\| \widetilde{f}_{2,\kappa} \partial_r \left(P_0/\lambda - z\right)^{-1} \right\| &\lesssim \| \widetilde{f}_{2,\kappa} \partial_r P_0^{-\frac{1}{2}} \| \| P_0^{\frac{1}{2}} \left(P_0/\lambda - z\right)^{-1} \| \\ &\lesssim \lambda^{\frac{1}{2}} \| \nabla_G P_0^{-\frac{1}{2}} \|_{L^2_G \to L^2_G} \\ &\lesssim \lambda^{\frac{1}{2}}. \end{aligned}$$

We conclude the proof, since

$$\left\| \frac{1}{\lambda} \left(P_0/\lambda - z\right)^{-1} \varphi_\kappa f_{2,\kappa} \partial_r \left(P_0/\lambda - z\right)^{-1} (1-\chi) \right\| \lesssim \lambda^{s+\frac{1}{2}-1} \frac{\langle z \rangle}{|Imz|}$$

with $s+\frac{1}{2}-1>0$ and $\overline{\partial}_z \widetilde{f}(z) = O(|Imz|^M)$ for any $M \geqslant 0$.

The proof for $R_z^2(\lambda)$ carries out in the same way. □

As opposed to $R_z^1(\lambda)$ and $R_z^2(\lambda)$, in the case of $R_z^Q(\lambda)$ we have Q which is a differential operator whose coefficients have only finite order decay. In particular the fact that $\|\langle r \rangle^\alpha a_\kappa\|_{L^\infty}$ and $\|\langle r \rangle^\alpha b_\kappa^l\|_{L^\infty}$ are finite only for $\alpha \leqslant 1+\nu$ will limit our choice of exponents when applying Lemma 2.2.5.

However this will not be a source of difficulty in the first order terms of Q since we can still choose $\sigma = \nu + 1 > 1$ as in Lemma 2.4.8 and get additional powers of λ by bounding the operators $\partial_r(-\Delta_0)^{-1}$ and $\frac{1}{r}\partial_l(-\Delta_0)^{-1}$.

On the contrary, for the second order term we only have $\|\langle r \rangle^\alpha c_\kappa^{j,l}\| \lesssim 1$ for $\alpha \leqslant \nu$. This will limit us to $\sigma = \nu > 0$ and moreover we will only be able to obtain the boundedness of $\partial^2_{j,l}(-\Delta_{\overline{g}})^{-1}$ (projecting away from the 0 eigenspace and using an elliptic parametrix in Lemma 2.4.12). This represents an additional difficulty since we will then need to control operators like $(-\Delta_{\overline{g}})/r^2 \left(-\Delta_0/\lambda - z\right)^{-1}$.

Lemma 2.4.9 (Bound on first order terms I)**.** *Let I_1 defined by*

$$I_1 = \frac{1}{\lambda} \left(P_0/\lambda - z\right)^{-1} (1-\chi)\varphi_\kappa(a_\kappa \partial_r) \left(\frac{-\Delta_0}{\lambda} - z \right)^{-1} (1-\chi),$$

there exists $\delta_1 > 0$ such that

$$\|I_1\| \lesssim \lambda^{\delta_1} \frac{\langle z \rangle}{|Imz|}$$

for all $\lambda > 0$.

Proof. We start by mimicking the proof of Lemma 2.4.8, hence writing I_1 as

$$\left\| \frac{1}{\lambda} \left(P_0/\lambda - z\right)^{-1} \langle r \rangle^{-\sigma} (1-\chi)\varphi_\kappa \langle r \rangle^{\sigma} a_\kappa \partial_r \left(-\Delta_0/\lambda - z\right)^{-1} (1-\chi) \right\|. \tag{2.4.7}$$

We recall that $a_\kappa \in S^{-1-\nu}$ so the fact that $\|\langle r\rangle^{\nu+1} a_\kappa\|_{L^\infty} \lesssim 1$ suggests that this time we choose $\sigma = \nu+1$ in Lemma 2.2.5, therefore giving us

$$\| \left(P_0/\lambda - z\right)^{-1} \langle r\rangle^{-\nu-1} \| \lesssim \frac{\langle z\rangle}{|Imz|}\lambda^s \tag{2.4.8}$$

for any $s \in (\frac{1}{2}, \frac{\nu+1}{2})$. We then proceed similarly to the previous proof, that is we estimate the quantity

$$\begin{aligned}
\left\| \partial_r \left(\frac{-\Delta_0}{\lambda} - z\right)^{-1} (1-\chi) \right\| \lesssim & \|\partial_r (-\Delta_0)^{-\frac{1}{2}}\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} \left\| (-\Delta_0)^{\frac{1}{2}} \left(\frac{-\Delta_0}{\lambda} - z\right)^{-1} \right\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} \\
\lesssim & \lambda^{\frac{1}{2}} \|\nabla_{\overline{G}} (-\Delta_0)^{-\frac{1}{2}}\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} \left\| \left(\frac{-\Delta_0}{\lambda}\right)^{\frac{1}{2}} \left(\frac{-\Delta_0}{\lambda} - z\right)^{-1} \right\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} \\
\lesssim & \lambda^{\frac{1}{2}}.
\end{aligned}$$

The statement is proved since the bound on (2.4.7) is

$$\|I_1\| \lesssim \lambda^{s+\frac{1}{2}-1} \frac{\langle z\rangle}{|Imz|}$$

with $s + \frac{1}{2} - 1 > 0$.

□

Once we have established

Lemma 2.4.10. *Let φ_κ a term of the partition of unity of S, then $\varphi_\kappa \frac{1}{r}\partial_l(-\Delta_0)^{-\frac{1}{2}}$ is a bounded operator on $L^2_{\overline{G}}$.*

we can bound the remaining first order part of $R^Q_z(\lambda)$ with the exact same reasoning of Lemma 2.4.9.

Lemma 2.4.11 (Bound on first order terms II)**.** *Let I_2 defined by*

$$I_2 = \frac{1}{\lambda}\left(P_0/\lambda - z\right)^{-1}(1-\chi)\varphi_\kappa b^l_\kappa \frac{1}{r}\partial_l \left(-\Delta_0/\lambda - z\right)^{-1}(1-\chi),$$

then there exists $\delta_2 > 0$ such that

$$\|I_2\| \lesssim \lambda^{\delta_2} \frac{\langle z\rangle}{|Imz|}$$

for all $\lambda > 0$.

Proof of Lemma 2.4.10. By ellipticity of the operator $-\Delta_{\overline{g}}$, locally on coordinate patches we have the following lower bound

$$|\nabla_{\overline{g}} u|^2_{\overline{g}} = \sum_{l,j} \overline{g}^{l,j}(\theta)\partial_l u \partial_j u \geqslant C_0 \sum_j |\partial_j u|^2 \geqslant C_0 |\partial_l u|^2$$

for some $C_0 > 0$. Consequently for the operator on the manifold it holds

$$\int_S |\varphi_\kappa \partial_l u|^2 \, d\mathrm{vol}_S = \int_{V_\kappa} |\partial_l u|^2 |\overline{g}(\theta)| d\theta \leqslant \frac{1}{C_0}\int_{V_\kappa} |\nabla_{\overline{g}} u|^2_{\overline{g}} |\overline{g}(\theta)| d\theta \leqslant \frac{1}{C_0}\int_S |\varphi_\kappa \nabla_{\overline{g}} u|^2_{\overline{g}} d\mathrm{vol}_S.$$

We can conclude, since we have found

$$\|\varphi_\kappa 1/r \partial_l u\|_{L^2_{\overline{G}}} \lesssim \|\varphi_\kappa 1/r \nabla_{\overline{g}} u\|_{L^2_{\overline{G}}} \lesssim \|\nabla_{\overline{G}} u\|_{L^2_{\overline{G}}} = \|(-\Delta_0)^{\frac{1}{2}} u\|_{L^2_{\overline{G}}}.$$

□

Now passing to consider the second order part in $R^Q_\lambda(z)$, that is

$$I_3 := \frac{1}{\lambda}\left(P_0/\lambda - z\right)^{-1}(1-\chi)\varphi_\kappa c^{j,l}_\kappa \frac{1}{r^2}\partial^2_{j,l}\left(-\Delta_0/\lambda - z\right)^{-1}(1-\chi)$$

we first remark a useful property.

Lemma 2.4.12. *Let $-\Delta_{\overline{g}}$ the Laplace-Beltrami operator on $(S,\overline{g})$, if φ_κ is a term of the partition of unity of S then $\varphi_\kappa\partial^2_{j,l}(-\Delta_{\overline{g}})^{-1}$ is a bounded operator on $L^2(S,d\overline{g})$.*

Proof. Let $u \in D(-\Delta_{\overline{g}})$ and Π_0 the projection on $\ker(-\Delta_{\overline{g}})$. The kernel is spanned by 1 and consequently $\partial^2_{j,l}\Pi_0 u = 0$. Moreover Π_0 can be written as $f(-\Delta_{\overline{g}})$ for some $f \in C^\infty_0(\mathbb{R})$ supported around 0 and with $f(0) = 1$, we consider

$$\begin{aligned}\varphi_\kappa\partial^2_{j,l}u =& \varphi_\kappa\partial^2_{j,l}(u - f(-\Delta_{\overline{g}})u)\\ =& \varphi_\kappa\partial^2_{j,l}(-\Delta_{\overline{g}}+1)^{-1}\frac{(-\Delta_{\overline{g}}+1)}{(-\Delta_{\overline{g}})}(-\Delta_{\overline{g}})(1-f)(-\Delta_{\overline{g}})u\\ =& \varphi_\kappa\partial^2_{j,l}(-\Delta_{\overline{g}}+1)^{-1}\frac{(-\Delta_{\overline{g}}+1)(1-f)(-\Delta_{\overline{g}})}{(-\Delta_{\overline{g}})}(-\Delta_{\overline{g}})u. \end{aligned} \tag{2.4.9}$$

With standard computations we can find a parametrix for the elliptic operator $-\Delta_{\overline{g}}$, namely there exist a family of symbols $q^\kappa \in S^{-2}(\mathbb{R}^{n-1}_\theta)$ supported in open subsets of $\mathbb{R}^{n-1}$ and $\mathcal{R}_N$ pseudodifferential operator with symbol in $S^{-N}(\mathbb{R}^{n-1}_\theta)$ such that

$$(-\Delta_{\overline{g}}+1)\left(\sum_\kappa \Pi_\kappa Op(q^\kappa)\Pi_\kappa^{-1}\right) = I + \mathcal{R}_N.$$

On the support of φ_κ the resolvent $(-\Delta_{\overline{g}}+1)^{-1}$ is a pseudodifferential operator of order minus two and the composition with the order two differential operator $(\partial^2_{j,l})$ results in a bounded operator of $L^2(S)$. By the spectral theorem $\frac{(-\Delta_{\overline{g}}+1)(1-f)(-\Delta_{\overline{g}})}{(-\Delta_{\overline{g}})}$ is bounded and therefore the statement follows from (2.4.9) which yields

$$\|\varphi_\kappa\partial^2_{j,l}u\|_{L^2(S)} \lesssim \|(-\Delta_{\overline{g}})u\|_{L^2(S)}.$$

□

As usual, we want to apply Lemma 2.2.5 to I_3 and we will do so by taking advantage of the fact that $\|\langle r\rangle^\nu c^{j,l}_\kappa\| \lesssim 1$ thanks to (2.4.4). However, applying Lemma 2.2.5 with $\sigma = \nu > 0$ would provide a bound by λ^s with $s \in (0, \frac{\nu}{2})$ which is worse than what we gained in the estimations of I_1 and I_2, where taking $\sigma = \nu + 1$ produced a higher power of λ, namely with exponent $s > \frac{1}{2}$.

We will then proceed differently by considering separately low and high angular frequencies. Let ϕ a smooth cutoff function such that $\phi \equiv 1$ on $[0, n-1]$.

1. Consider
$$\frac{1}{\lambda}(P_0/\lambda - z)^{-1}c^{j,l}_\kappa\varphi_\kappa\partial^2_{j,l}\phi(-\Delta_{\overline{g}})(1-\chi)\frac{1}{r^2}\left(\frac{-\Delta_0}{\lambda} - z\right)^{-1}$$
where we have bounded contributions given by
$$\langle r\rangle^\nu c^{j,l}_\kappa, \quad (1-\chi)(r)\frac{\langle r\rangle^2}{r^2}, \quad \varphi_\kappa\partial^2_{j,l}(-\Delta_{\overline{g}})^{-1}\phi(-\Delta_{\overline{g}})(-\Delta_{\overline{g}})$$

(see (2.4.4), Lemma 2.4.12 and the support of ϕ) and we are left to consider

$$(P_0/\lambda - z)^{-1}\langle r\rangle^{-\nu-1} \quad \text{and} \quad \frac{\langle r\rangle^{-1}}{\lambda}\left(-\Delta_0/\lambda - z\right)^{-1}.$$

By Lemma 2.2.5 and Hardy inequality (see Propositions 2.2 in [BR14b]) we can handle these two remaining terms obtaining

$$\left\|(P_0/\lambda - z)^{-1}\langle r\rangle^{-\nu-1}\frac{1}{\lambda\langle r\rangle}\left(-\Delta_0/\lambda - z\right)^{-1}\right\| \lesssim \lambda^{s-\frac{1}{2}}\frac{\langle z\rangle^2}{|Imz|^2}$$

where we can choose $s \in (\frac{1}{2}, \frac{\nu+1}{2})$.

We notice here that we have used Lemma 2.2.5 in the same way as in the proof of Lemma 2.4.9, where we had obtained (2.4.8). However here $c_\kappa^{j,l}$ has less decay than a_κ which leaves us with an extra growing term $\langle r\rangle$ to handle (we cannot bound $\langle r\rangle^{\nu+1}c_\kappa^{j,l}$, but only $\langle r\rangle^{\nu}c_\kappa^{j,l}$). For this reason we take advantage of the localization $(1-\chi)$ and we use it to write a bounded term where we collect all the factors depending on the radial variable.

2. The part localized at high angular frequencies is

$$\frac{1}{\lambda}(P_0/\lambda - z)^{-1}c_\kappa^{j,l}\varphi_\kappa\partial_{j,l}^2(1-\phi)(-\Delta_{\overline{g}})(1-\chi)\frac{1}{r^2}\left(-\Delta_0/\lambda - z\right)^{-1}$$

where the operators

$$\langle r\rangle^{\nu}c_\kappa^{j,l}, \qquad \varphi_\kappa\partial_{j,l}^2(-\Delta_{\overline{g}})^{-1}$$

are bounded independently of λ thanks to (2.4.4) and Lemma 2.4.12. Additionally, by Lemma 2.2.5

$$\left\|(P_0/\lambda - z)^{-1}\langle r\rangle^{-\nu}\right\| \lesssim \lambda^{s}\frac{\langle z\rangle}{|Imz|} \tag{2.4.10}$$

for some $s \in (0, \frac{\nu}{2})$. At this point we are left with

$$\left\|(1-\phi)(-\Delta_{\overline{g}})\frac{(-\Delta_{\overline{g}})}{\lambda r^2}\left(\frac{-\Delta_0}{\lambda} - z\right)^{-1}\right\| \lesssim \frac{\langle z\rangle}{|Imz|}\left\|(1-\phi)(-\Delta_{\overline{g}})\frac{(-\Delta_{\overline{g}})}{\lambda r^2}\left(\frac{-\Delta_0}{\lambda} + 1\right)^{-1}\right\|$$

where, as opposed to item 1, the localization by $(1-\phi)$ requires some extra care.

Lemma 2.4.13. *Let $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi \equiv 1$ on $[0, n-1]$, then for $\lambda \in (0, \lambda_0]$*

$$\left\|(1-\phi)(-\Delta_{\overline{g}})\frac{(-\Delta_{\overline{g}})}{\lambda r^2}\left(\frac{-\Delta_0}{\lambda} + 1\right)^{-1}\right\|_{L^2_{\overline{G}}\to L^2_{\overline{G}}} \lesssim_n 1.$$

Proof. By the results of Appendix 2.A (i.e. Proposition 2.A.2) we can rather consider the one dimensional problem of bounding

$$\sup_{\mu_k^2 > n-1}\left\|\frac{\mu_k^2}{r^2}\left(p_k + \lambda\right)^{-1}\right\|_{L^2((R,+\infty),r^{n-1}dr)\to L^2((R,+\infty),r^{n-1}dr)} \tag{2.4.11}$$

where we recall

$$p_k := -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r}\partial_r + \frac{1}{r^2}\mu_k^2$$

and (μ_k^2, e_k) are eigenpairs of $-\Delta_{\overline{g}}$. To bound (2.4.11) we will use an estimate on an analogous quantity where μ_k^2 is replaced by the eigenvalue of the Laplace-Beltrami operator on the unit sphere (see Appendix 2.D). Indeed, we can separate the values μ_k^2 with the eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$, for whom we have the explicit expression $\sigma_{j,n}^2 = j(j+n-2)$. That is, once we fix k

$$\mu_k^2 \in (\sigma_{l,n}^2, \sigma_{l+1,n}^2] \quad \text{for a unique } l = l(k),$$

recalling that we are only considering eigenvalues $\mu_k^2 > n-1$. We can then rewrite the operator p_k as

$$p_k = \underbrace{-\partial_r^2 - \frac{(n-1)}{r}\partial_r + \frac{\sigma_{l,n}^2}{r^2}}_{p_{l,n}} + \underbrace{\frac{\mu_k^2 - \sigma_{l,n}^2}{r^2}}_{v}$$

with $v > 0$ and $p_{l,n}$ which is the equivalent of p_k where the values μ_k^2 are replaced by $\sigma_{l,n}^2$. If we express the resolvent in terms of the heat semigroup we have

$$(p_k + \lambda)^{-1} = \int_0^\infty e^{-t\lambda} e^{-tp_k} dt = \int_0^\infty e^{-t\lambda} e^{-t(p_{l,n}+v)} dt, \tag{2.4.12}$$

since v and $p_{l,n}$ do not commute of course $e^{-t(p_{l,n}+v)} \neq e^{-tp_{l,n}} e^{-tv}$. However from Trotter product formula (see Theorem VIII.31 in [RS81]) we know that

$$e^{-t(p_{l,n}+v)} = \lim_{m\to+\infty} (e^{-\frac{t}{m}p_{l,n}} e^{-\frac{t}{m}v})^m$$

taking the limit in the strong sense. Here we can bound the kernel of $(e^{-\frac{t}{m}p_{l,n}} e^{-\frac{t}{m}v})^m$ with the one of $e^{-tp_{l,n}}$ thanks to the non negativity of v, hence obtaining a pointwise upper bound on $(e^{-\frac{t}{m}p_{l,n}} e^{-\frac{t}{m}v})^m$ by $e^{-tp_{l,n}}$. Consequently, given that now $e^{-t(p_{l,n}+v)}$ is bounded by $e^{-tp_{l,n}}$, by (2.4.12) we see that we control $(p_k+\lambda)^{-1}$ with $(p_{l,n}+\lambda)^{-1}$ for which the result in Corollary 2.D.2 holds.

In particular from such corollary we obtain first that

$$\left\| \frac{\sigma_{l,n}^2}{r^2}(p_{l,n}+\lambda)^{-1} \right\|_{L^2((R,+\infty),r^{n-1}dr)\to L^2((R,+\infty),r^{n-1}dr)} \lesssim_n 1$$

uniformly in λ. Then since $1 \leqslant \frac{\sigma_{l+1,n}^2}{\sigma_{l,n}^2} \leqslant m_n$ for some constant m_n it follows that $\frac{\mu_k^2}{\sigma_{l,n}^2} \leqslant m_n$, going back in (2.4.11) we have found

$$\sup_{\mu_k^2 > n-1} \left\| \frac{\mu_k^2}{r^2}(p_k+\lambda)^{-1} \right\| \lesssim_{m_n} \left\| \frac{\sigma_{l,n}^2}{r^2}(p_{l,n}+\lambda)^{-1} \right\| \lesssim_n 1$$

where we are considering again the norm of operators on $L^2((R,+\infty),r^{n-1}dr)$.

□

We are now able to bound the second order term.

Lemma 2.4.14 (Bound on second order term)**.** *Let I_3 defined by*

$$I_3 = \frac{1}{\lambda}(P_0/\lambda - z)^{-1}(1-\chi)\varphi_\kappa c_\kappa^{j,l} \frac{1}{r^2}\partial_{j,l}^2 (-\Delta_0/\lambda - z)^{-1}(1-\chi),$$

there exists $\delta_3 > 0$ such that

$$\|I_3\| \lesssim \lambda^{\delta_2} \frac{\langle z \rangle^2}{|Imz|^2}$$

for all $\lambda > 0$.

Proof. Take ϕ a spectral localization on the interval $[0, n-1)$ and split I_3 with the partition $\phi(-\Delta_{\bar{g}})$ and $(1-\phi)(-\Delta_{\bar{g}})$. Conclude by using item 1 (page 60) on the term localized on the angular frequencies $[0, n-1)$ and item 2 (page 61) together with Lemma 2.4.13 on the part localized at high angular frequencies. $\square$

Eventually, thanks to Lemmas 2.4.9, 2.4.11 and 2.4.14 and the properties of almost analytic extensions we have

$$\left\| \int_{\mathbb{C}} \overline{\partial}_z \widetilde{f}(z) R_z^Q(\lambda) L(dz) \right\| \lesssim \lambda^\delta \tag{2.4.13}$$

for some $\delta > 0$ and $\lambda > 0$. Recalling the result of Lemma 2.4.8 and the definition of $D_f(\lambda)$ in (2.4.3) we have found that

$$\|D_f(\lambda)\| \to 0 \quad \text{as} \quad \lambda \to 0 \tag{2.4.14}$$

which concludes the proof of Assumption 2.1.

2.5 Adding a potential

We explain here how to obtain Proposition 2.3.9, and consequently Theorems 2.1.5 and 2.1.4, when P_0 is replaced by P_0.

Let

$$P := P_0 + V$$

with $V \geqslant 0$ a multiplicative potential that on $M \setminus K$ is the multiplication by a function V belonging to $S^{-2-\varepsilon}$. The robustness of the approach lies in the fact that the symbolic structure of the operator is not altered by the addition of such a potential. In particular, in local coordinates V can be represented by a pseudodifferential operator $Op_\lambda(\widetilde{S}^{-2-\varepsilon,0})$ and if χ is a cutoff on the compact part of the manifold we can still write the operator under the form

$$(1-\chi)P/\lambda = \sum_\kappa Op_{\lambda,\kappa}(a_{0,\lambda} + a_{1,\lambda}^V) \tag{2.5.1}$$

with $a_{0,\lambda} \in \widetilde{S}^{0,2}, a_{1,\lambda}^V \in \widetilde{S}^{-1,1}$ that have seminorms uniformly bounded with respect to λ.

We sketch the main steps to obtain the results corresponding to the ones in Sections 2.2, 2.3 and 2.4.

- Provided we have existence of the outgoing and ingoing resolvents, all the proofs of Section 2.2 carry on in the exact same way for P.
- Since P has same symbolic structure as P_0 the results of Section 2.3 hold with analogous proofs, provided we assume the equivalent of Assumption 2.1 replacing P_0 by P. Therefore, by limiting absorption principle we obtain

  $$(\mathcal{A}^\lambda + i)^{-s}(P/\lambda - 1 \pm i0)^{-l}(\mathcal{A}^\lambda + i)^{-s} \in \mathcal{L}(L^2(M))$$

  for $s > l - \frac{1}{2}$ and $s \in \mathbb{N}$.
- Lemmas 2.2.4 and 2.2.5 were crucial properties to be able to prove Assumption 2.1 and we have remarked that they derive from the behaviour of the heat flow.

  Equivalents of Lemmas 2.2.4 and 2.2.5 hold for the operator P. Indeed, as we have done before we can use Trotter product formula (Theorem VIII.31 in [RS81]) to write

  $$e^{-tP} = e^{-t(P_0+V)} = \lim_{m\to+\infty} (e^{-\frac{t}{m}P_0} e^{-\frac{t}{m}V})^m,$$

  since $V \geqslant 0$ we can bound the kernel of $e^{-\frac{t}{m}P_0}e^{-\frac{t}{m}V}$ with the one of $e^{-\frac{t}{m}P_0}$. Therefore, if $K_0(x,y,t)$ and $K_V(x,y,t)$ are the kernels of e^{-tP_0} and e^{-tP} respectively

  $$K_V(x,y,t) \leqslant K_0(x,y,t)$$

which allows us to recover for e^{-tP} the same kind of estimates that we had for e^{-tP_0}.

Remark 2.5.1. Alternatively, we can recover the heat flow estimates by noticing that they follow from the Nash inequality which also holds true for P. Thanks to the non negativity of V and the selfadjointess of P_0 we can easily see that

$$\|P^{\frac{1}{2}}u\|^2 = \langle (P_0+V)u,u\rangle = \langle P_0^{\frac{1}{2}}u, P_0^{\frac{1}{2}}u\rangle + \langle Vu,u\rangle \geqslant \|P_0^{\frac{1}{2}}u\|^2$$

and therefore use the Nash inequality for P_0 (namely (2.2.8)) to prove

$$\|u\|_{L^2(M)}^{1+\frac{2}{n}} \lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \|P^{\frac{1}{2}}u\|_{L^2(M)}.$$

We notice that $\|P^{\frac{1}{2}}u\| \geqslant \|P_0^{\frac{1}{2}}u\|$ implies that P has no 0 eigenvalue nor resonance, since 0 it is not an eigenvalue nor a resonance for P_0 either.

- Once we get to the proof of Assumption 2.1 the only relevant difference is when considering the term

$$D_f^V(\lambda) := (1-\chi)f(P/\lambda)(1-\chi) - (1-\chi)f(-\Delta_0/\lambda)(1-\chi).$$

Applying Helffer Sjöstrand formula we pass to comparison between resolvents and we have an additional term involving V that is

$$R_z^V(\lambda) := \frac{1}{\lambda}\,(P/\lambda - z)^{-1}\,(1-\chi)V(-\Delta_0/\lambda - z)^{-1}(1-\chi), \tag{2.5.2}$$

while all the other terms can be bounded with the analogous of Lemma 2.2.5. This leads to results equivalent to Lemma 2.4.8 and (2.4.13).
For (2.5.2) we exploit the boundedness of $\|\langle r\rangle^{2+\varepsilon}V\|_{L^\infty}$ and applying Lemma 2.2.5 with $\sigma = 1+\varepsilon$ we obtain

$$\left\| \int_{\mathbb{C}} \overline{\partial}_z \widetilde{f}(z) R_z^V(\lambda) L(dz) \right\| \lesssim \lambda^\delta$$

for some positive δ and $\lambda > 0$. We can therefore conclude that $\|D_f^V(\lambda)\|$ converges to 0 as λ goes to 0, as in the conclusion of Section 2.4.

Appendices

2.A Operator on the exact cone and separation of variables

Let $-\Delta_0$ the Friedrichs extension of the Laplace-Beltrami operator on the half cone $([R,+\infty)\times S, \overline{G})$ with Dirichlet boundary condition. In local coordinates this looks like

$$-\Delta_0 = -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{\overline{g}}. \tag{2.A.1}$$

The quadratic from which we derive the Friedrichs extension is

$$q_0(u,v) := (\partial_r u, \partial_r v)_{L^2_{\overline{G}}} + (1/r(-\Delta_{\overline{g}})^{\frac{1}{2}}u, 1/r(-\Delta_{\overline{g}})^{\frac{1}{2}}v)_{L^2_{\overline{G}}}, \tag{2.A.2}$$

defined on elements of the space

$$\{\text{closure of } C_0^\infty([R,+\infty)\times S) \text{ with respect to } \|u\|_+ := (q_0(u,u) + \|u\|^2_{L^2_{\overline{G}}})^{\frac{1}{2}}\ \} \tag{2.A.3}$$

that we will denote by $H_0^1([R,+\infty)\times S) = H_0^1$. The domain of the Friedrichs extension $-\Delta_0$ then is

$$D(-\Delta_0) = \{u \in H_0^1 \mid |q_0(u,v)| \leqslant C(u)\|v\|_{L^2_{\overline{G}}}\ \forall v \in H_0^1\} \tag{2.A.4}$$

Since $-\Delta_{\overline{g}}$ is selfadjoint on $L^2(S)$, we can consider an orthonormal basis $(e_k)_k$ of the space such that

$$-\Delta_{\overline{g}} e_k = \mu_k^2 e_k, \qquad 0 = \mu_0 \leqslant \mu_1 \leqslant \dots,$$

we can then decompose any function $u \in L^2(S)$ on this basis

$$u(\omega) = \sum_{k \geqslant 0} \left(\int_S \overline{e_k} u \ \mathrm{dvol}_S \right) e_k(\omega).$$

A function of $L^2_{\overline{G}}$ is then of the form

$$u(r,\omega) = \sum_{k \geqslant 0} \left(\int_S \overline{e_k(\cdot)} u(r,\cdot) \ \mathrm{dvol}_S \right) e_k(\omega) =: \sum_{k \geqslant 0} u_k(r) e_k(\omega). \tag{2.A.5}$$

In particular we can identify a function in $L^2_{\overline{G}}$ with its one dimensional coefficients. It is straightforward to obtain the following.

Proposition 2.A.1. *The map between Hilbert spaces*

$$\begin{aligned} L^2_{\overline{G}} &\longrightarrow \bigoplus_{k \geqslant 0} L^2((R,+\infty), r^{n-1} dr) \\ u &\mapsto (u_k)_k \end{aligned}$$

is an isometry with

$$\|u\|^2_{L^2_{\overline{G}}} = \sum_{k \geqslant 0} \|u_k\|^2_{L^2((R,+\infty), r^{n-1} dr)}. \tag{2.A.6}$$

Moreover, prescribing the action of one dimensional operators on the coefficients u_k gives origin to well defined operators on the Hilbert space $L^2_{\overline{G}}$.

Proposition 2.A.2. *Let $(A_k)_k$ a bounded sequence of bounded operators on $L^2((R,+\infty), r^{n-1} dr)$. The operator A defined by*

$$Au(r,\omega) = \sum_k (A_k u_k)(r) e_k(\omega). \tag{2.A.7}$$

is well defined on $L^2_{\overline{G}}$ with norm

$$\|A\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} = \sup_k \|A_k\|_{L^2((R,+\infty), r^{n-1} dr) \to L^2((R,+\infty), r^{n-1} dr)}.$$

Proof. Follows directly from Proposition 2.A.1. □

With Proposition 2.A.2 in mind we want to reduce $-\Delta_0$ to the action of suitable one dimensional operators. Indeed, once a function is represented with respect to the orthonormal basis $(e_k)_k$ as in (2.A.5), the action of $-\Delta_{\overline{g}}$ becomes multiplication by a scalar so going back to (2.A.1) this suggests we set

$$p_k := -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r} \partial_r + \frac{1}{r^2} \mu_k^2. \tag{2.A.8}$$

We can prove that the sequence of one dimensional operators $(p_k)_k$ corresponds exactly to $-\Delta_0$.

Proposition 2.A.3. *Let $u \in H^1_0([R,+\infty) \times S)$, then $u \in D(-\Delta_0)$ if and only if $u_k \in D(p_k)$ for every k and*

$$\sum_k \|p_k u_k\|^2_{L^2((R,+\infty), r^{n-1} dr)} < +\infty. \tag{2.A.9}$$

Moreover

$$(-\Delta_0 u)_k = p_k u_k, \quad \textit{from which} \quad \|(-\Delta_0)u\|^2_{L^2_{\overline{G}}} = \sum_k \|p_k u_k\|^2_{L^2((R,+\infty),r^{n-1}dr)}.$$

Proof. Can be proved by direct computations, given that defining p_k via Friedrichs extension its associated quadratic form is

$$q_k(f,g) = \int_R^{+\infty} \left(\overline{f'}g' + \frac{1}{r^2}\mu_k^2 \overline{f} g \right) \, r^{n-1} dr,$$

with domain

$$h_{+1,k} := \left\{ \text{closure of } C_0^\infty([R,+\infty)) \text{ with respect to } \left(q_k(f,f) + \|f\|^2_{L^2((R,+\infty),r^{n-1}dr)} \right)^{\frac{1}{2}} \right\},$$

and the domain of p_k is

$$D(p_k) = \{ f \in h_{+1,k} \mid |q_k(f,g)| \leqslant C(f)\|g\|_{L^2((R,+\infty),r^{n-1}dr)} \; \forall g \in h_{+1,k} \}.$$

□

The result of Proposition 2.A.3 extends to the case of functions of $-\Delta_0$, as we will see below. This will be of use to simplify the argument in estimations of norms in Section 2.4.

Proposition 2.A.4. *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *then*

$$\varphi(-\Delta_0)u(r,\omega) = \sum_k (\varphi(p_k)u_k)(r) e_k(\omega)$$

for any $u \in L^2_{\overline{G}}$.

Proof. The statement can be proved directly when $\varphi(x) = (x-z)^{-1}$ with $z \in \mathbb{C} \setminus \mathbb{R}$ and can be generalized to any $\varphi \in C_0^\infty(\mathbb{R})$ applying Helffer-Sjöstrand formula to compute the right hand side of the equality. □

2.B Nash inequality

We show in this appendix how to obtain an inequality of the type

$$\|u\|^{1+\frac{2}{n}}_{L^2(M)} \lesssim_n \|u\|^{\frac{2}{n}}_{L^1(M)} \|P_0^{\frac{1}{2}} u\|_{L^2(M)}$$

using the analogous result which holds for the free operator $-\Delta$ on $\mathbb{R}^n$, namely

$$\|u\|^{1+\frac{2}{n}}_{L^2(\mathbb{R}^n)} \lesssim_n \|u\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} \|(-\Delta)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^n)} = \|u\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)}. \tag{2.B.1}$$

See [Nas58] p. 936.

We will proceed in two steps: first we will prove the inequality on a pure cone (thanks to (2.B.1)) and next we will pass onto M. On the manifold end, $M \setminus K$, we will use the result that holds on a pure a cone, while on the compact part we will exploit the fact that locally the domain is diffeomorphic to $\mathbb{R}^n$ where (2.B.1) applies.

2.B.1 Inequality on a fixed cone

Since we are considering the fixed cone $((0,\infty) \times S, \overline{G})$ we can proceed as in Remark 2.4.4 to reduce ourselves to norms on $\mathbb{R}^n$ for which (2.B.1) holds. We recall the definition of $\|\cdot\|_{L^2_{\overline{G}}(cone)}$

given in Lemma 2.4.3, that is the norm on the full cone with respect to the fixed metric $\overline{g}$. In the same way we define $\|\cdot\|_{L^1_{\overline{G}}(cone)}$.

Lemma 2.B.1. *Let $u \in C_0^\infty((0,+\infty)\times S)$, then there exists $C_n > 0$ such that*

$$\|u\|^{1+\frac{2}{n}}_{L^2_{\overline{G}}(cone)} \leqslant C_n \|u\|^{\frac{2}{n}}_{L^1((0,+\infty)\times S)} \|\nabla_{\overline{G}} u\|_{L^2((0,+\infty)\times S)}.$$

Proof. Recall the equivalence between the L^2 norm on $((0,\infty)\times S, \overline{G})$ and the usual L^2 norm on $\mathbb{R}^n$ with respect to the Lebesgue measure mentioned in Remark 2.4.4. Then for each term in $u = \sum_j \varphi_j(\omega)u$ we can apply (2.B.1) and get

$$\|\varphi_j(\omega)u\|^{1+\frac{2}{n}}_{L^2_{\overline{G}}(cone)} \simeq \|\varphi_j u\|^{1+\frac{2}{n}}_{L^2(\mathbb{R}^n)} \lesssim \|\varphi_j u\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} \|\nabla_{\mathbb{R}^n}(\varphi_j u)\|_{L^2(\mathbb{R}^n)}$$

with

$$\|\varphi_j u\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} \simeq \|\varphi_j u\|^{\frac{2}{n}}_{L^1_{\overline{G}}(cone)} \lesssim \|u\|^{\frac{2}{n}}_{L^1_{\overline{G}}(cone)}. \tag{2.B.2}$$

Considering polar coordinates on $(0,+\infty)\times\mathbb{S}^{n-1}$ the gradient is $(\partial_r, 1/r\nabla_{\mathbb{S}^{n-1}})$, so

$$\begin{aligned}
\|\nabla_{\mathbb{R}^n}(\varphi_j u)\|^2_{L^2(\mathbb{R}^n)} &\simeq \|\varphi_j \partial_r u\|^2_{L^2((0,+\infty)\times\mathbb{S}^{n-1})} + \|1/r\nabla_{\mathbb{S}^{n-1}}(\varphi_j u)\|^2_{L^2((0,+\infty)\times\mathbb{S}^{n-1})}\\
&\simeq \|\varphi_j(\omega)\partial_r u\|^2_{L^2_{\overline{G}}(cone)} + \|1/r\nabla_{\overline{g}}(\varphi_j(\omega)u)\|^2_{L^2_{\overline{G}}(cone)}\\
&\lesssim \|\partial_r u\|^2_{L^2_{\overline{G}}(cone)} + \|1/r\nabla_{\overline{g}}(\varphi_j(\omega)u)\|^2_{L^2_{\overline{G}}(cone)}.
\end{aligned} \tag{2.B.3}$$

Now observing that $\nabla_{\overline{g}} = (\overline{g}^{j,k}(\theta))_{j,k}\nabla_\theta$ when we rewrite $\nabla_{\overline{g}}(\varphi_j(\omega)u)$ as $[\nabla_{\overline{g}}, \varphi_j] + \varphi_j\nabla_{\overline{g}}$ the commutator

$$[\nabla_{\overline{g}}, \varphi_j(\omega)] = (\overline{g}^{j,k}(\theta))_{j,k}\nabla_\theta(\varphi_j\circ\kappa_j^{-1}) \in C_0^\infty(\mathbb{R}^{n-1}),$$

is the multiplication by a bounded function and hence a bounded operator. So we have found

$$\begin{aligned}
\|1/r\nabla_{\overline{g}}(\varphi_j(\omega)u)\|_{L^2((0,+\infty)\times S)} &\leqslant \|1/r\varphi_j(\omega)\nabla_{\overline{g}}u\|_{L^2_{\overline{G}}(cone)} + \|\ 1/r[\nabla_{\overline{g}}, \varphi_j(\omega)]u\|_{L^2_{\overline{G}}(cone)}\\
&\lesssim \|1/r\nabla_{\overline{g}}u\|_{L^2_{\overline{G}}(cone)} + C\|u/r\|_{L^2_{\overline{G}}(cone)}\\
&\lesssim \|1/r\nabla_{\overline{g}}u\|_{L^2_{\overline{G}}(cone)} + C\|\partial_r u\|_{L^2_{\overline{G}}(cone)}
\end{aligned}$$

thanks to Hardy inequality (Propositions 2.2 and 3.5 in [BR14b]) in the last line. The statement follows combining (2.B.2), (2.B.3) together with this last estimate. □

2.B.2 Inequality on the manifold

We split the analysis into the part near infinity and the compact one. From what we have found in the previous section, if $(1-\chi)$ is a cutoff on the manifold end at first we get

$$\begin{aligned}
\|(1-\chi)u\|^{1+\frac{2}{n}} &\simeq \|(1-\chi)u\|^{1+\frac{2}{n}}_{L^2_{\overline{G}}} \lesssim \|(1-\chi)u\|^{\frac{2}{n}}_{L^1_{\overline{G}}}\|\nabla_{\overline{G}}((1-\chi)u)\|_{L^2_{\overline{G}}}\\
&\simeq \|(1-\chi)u\|^{\frac{2}{n}}_{L^1(M\setminus K)}\|\nabla_{\overline{G}}((1-\chi)u)\|_{L^2_{\overline{G}}}\\
&\leqslant \|u\|^{\frac{2}{n}}_{L^1(M)}\|\nabla_{\overline{G}}((1-\chi)u)\|_{L^2_{\overline{G}}}.
\end{aligned} \tag{2.B.4}$$

The gradient we need to evaluate is

$$|\nabla_{\overline{G}}((1-\chi)u)|^2_{\overline{G}} \lesssim |\chi' u|^2 + |(1-\chi)\partial_r u|^2 + |(1-\chi)/r\nabla_{\overline{g}}u|^2_{\overline{g}} \tag{2.B.5}$$

with χ' compactly supported in $[R,+\infty)$ so that $\|\chi'\langle r\rangle\|_{L^\infty}\lesssim 1$. Evaluating the L^2 norm and using the Hardy inequality mentioned in the previous proof we can first bound the terms

$$\|\chi' u\|^2_{L^2_{\overline{G}}}+\|(1-\chi)\partial_r u\|^2_{L^2_{\overline{G}}}\lesssim\|\langle r\rangle^{-1}u\|^2_{L^2_{\overline{G}}}+\|\partial_r u\|^2_{L^2_{\overline{G}}}\lesssim\|\partial_r u\|^2_{L^2_{\overline{G}}}. \tag{2.B.6}$$

Then we will need to compare the gradient with respect to the fixed metric $\overline{g}$ with the one with respect to the metric $g(r)$ via the following lemma.

Lemma 2.B.2. *Let $\overline{g}$ and $g(r)$ two metrics on a closed manifold S satisfying the property in (2.1.2), then there exists a constant $C>0$ such that*

$$\||\nabla_{\overline{g}}u|_{\overline{g}}\|_{L^2(S,d\overline{g})}\leqslant C\||\nabla_{g(r)}u|_{g(r)}\|_{L^2(S,dg(r))}.$$

for $r\gg 1$ and $u\in H^1_0(S,d\overline{g})\cap H^1_0(S,dg(r))$.

Proof. Given the two gradients $\nabla_{\overline{g}}=(\overline{g}^{j,k}(\theta))_{j,k}\nabla_\theta$ and $\nabla_{g(r)}=(g(r,\theta)^{j,k})_{j,k}\nabla_\theta$ and property (2.1.2) for large r we get

$$\||\nabla_{\overline{g}}u|_{\overline{g}}\|_{L^2(S,d\overline{g})}\lesssim\|\sum_\kappa|\nabla_{\overline{g}}(\varphi_\kappa u)|_{\overline{g}}\|_{L^2(S,d\overline{g})}\lesssim\|\sum_\kappa|\nabla_{g(r)}(\varphi_\kappa u)|_{g(r)}\|_{L^2(S,dg(r))}.$$

Consider the kernel of the operator $-\Delta_{g(r)}$, which is spanned by 1, and let

$$\Pi^0_{g(r)}:=\text{ projection on }\ker_{L^2}(-\Delta_{g(r)}),$$

in particular $\nabla_{\overline{g}}u=\nabla_{\overline{g}}(u-\Pi^0_{g(r)}u)$. Now the gradient of u is

$$\begin{aligned}\||\nabla_{\overline{g}}u|_{\overline{g}}\|_{L^2(S,d\overline{g})}=&\||\nabla_{\overline{g}}(u-\Pi^0_{g(r)}u)|_{\overline{g}}\|_{L^2(S,d\overline{g})}\\ \lesssim&\sum_\kappa\||\nabla_{g(r)}(\varphi_\kappa(u-\Pi^0_{g(r)}u))|_{g(r)}\|_{L^2(S,dg(r))}\\ \lesssim&\sum_\kappa\|\varphi_\kappa|\nabla_{g(r)}(u-\Pi^0_{g(r)}u)|_{g(r)}\|_{L^2(S,dg(r))}\\ &+\sum_\kappa\||\nabla_{g(r)}\varphi_\kappa|_{g(r)}(u-\Pi^0_{g(r)}u)\|_{L^2(S,dg(r))}\\ \lesssim&\||\nabla_{g(r)}u|_{g(r)}\|_{L^2(S,dg(r))}+\|u-\Pi^0_{g(r)}u\|_{L^2(S,dg(r))}.\end{aligned}$$

Write the projection on the 0 eigenspace $\Pi^0_{g(r)}$ as $f(-\Delta_{g(r)})$ with $f\in C^\infty_0(\mathbb{R})$ such that $f(0)=1$ and supported around 0, then by the spectral theorem

$$\begin{aligned}\|u-\Pi^0_{g(r)}u\|_{L^2(S,dg(r))}=&\left\|\frac{(1-f)(-\Delta_{g(r)})}{(-\Delta_{g(r)})^{\frac12}}(-\Delta_{g(r)})^{\frac12}u\right\|_{L^2(S,dg(r))}\\ \lesssim&\|(-\Delta_{g(r)})^{\frac12}u\|_{L^2(S,dg(r))}\\ \simeq&\||\nabla_{g(r)}u|_{g(r)}\|_{L^2(S,dg(r))}\end{aligned}$$

and the statement follows. □

Going back to (2.B.5), thanks to (2.B.6) and Lemma 2.B.2 we obtain

$$\begin{aligned}\|\nabla_{\overline{G}}((1-\chi)u)\|^2_{L^2((0,+\infty)\times S)}\lesssim&\|\partial_r u\|^2_{L^2_{\overline{G}}}+\|(1-\chi)/r|\nabla_{\overline{g}}u|_{\overline{g}}\|^2_{L^2_{\overline{G}}}\\ \lesssim&\|\partial_r u\|^2_{L^2_G}+\|(1-\chi)/r|\nabla_{g(r)}u|_{g(r)}\|^2_{L^2_G}\\ \lesssim&\|\nabla_G u\|^2_{L^2_G}\simeq\|P_0^{\frac12}u\|^2_{L^2(M\setminus K)}.\end{aligned}$$

Applying this estimate to (2.B.4) we have proved the desired inequality on the manifold end

$$\|(1-\chi)u\|^{1+\frac{2}{n}} \lesssim \|u\|^{\frac{2}{n}}_{L^1(M)} \|P_0^{\frac{1}{2}} u\|^2_{L^2(M)}. \tag{2.B.7}$$

Using a partition of unity for the remaining compact part we derive Nash inequality for the full manifold.

Lemma 2.B.3. *Let $u \in C_0^\infty(M)$ then there exists $C_n > 0$ such that*

$$\|u\|^{1+\frac{2}{n}}_{L^2(M)} \leqslant C_n \|u\|^{\frac{2}{n}}_{L^1(M)} \|P_0^{\frac{1}{2}} u\|_{L^2(M)}.$$

Proof. Write u as $(1-\chi)u + \chi u$ and use (2.B.7) on $(1-\chi)u$. Then χu is supported on a compact set on which we can consider a finite covering $(K_j)_j$ with associated partition of unity $(\chi_j)_j$. Each $K_j \subset M$ is diffeomorphic to an open set of $\mathbb{R}^n$, via a diffeomorphism ψ_j and since the metric tensor on M is represented by a positive definite matrix we have

$$\int_M \chi_j(\chi u)\, dG \simeq \int_{\psi_j(K_j)} \chi_j(\chi u) \circ \psi_j^{-1}\, dx.$$

Applying again (2.B.1) it follows

$$\begin{aligned}
\|\chi_j(\chi u)\|^{1+\frac{2}{n}}_{L^2(M)} \simeq \|\chi_j(\chi u)\|^{1+\frac{2}{n}}_{L^2(\mathbb{R}^n)} &\lesssim \|\chi_j(\chi u)\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} \|\nabla_{\mathbb{R}^n}(\chi_j \chi u)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|u\|^{\frac{2}{n}}_{L^1(M)} \left(\|\nabla_{\mathbb{R}^n}(\chi u)\|_{L^2(\mathbb{R}^n)} + \|\chi u\|_{L^2(\mathbb{R}^n)} \right) \\
&\simeq \|u\|^{\frac{2}{n}}_{L^1(M)} \left(\|\nabla_G(\chi u)\|_{L^2(M)} + \|\chi u\|_{L^2(M)} \right) \\
&\lesssim \|u\|^{\frac{2}{n}}_{L^1(M)} \|\nabla_G(\chi u)\|_{L^2(M)} \\
&\lesssim \|u\|^{\frac{2}{n}}_{L^1(M)} \left(\|\nabla_G u\|_{L^2(M)} + \|\chi' u\|_{L^2(M)} \right) \\
&\lesssim \|u\|^{\frac{2}{n}}_{L^1(M)} \|\nabla_G u\|_{L^2(M)}
\end{aligned}$$

where we have used Hardy inequality (Proposition 2.2 of [BR14b]) to estimate $\|\chi u\|_{L^2(M)}$ and (2.4) in [BR14b] to bound $\|\chi' u\|_{L^2(M)}$.

□

2.C Commutators and symbolic calculus

We will see here how to use symbolic calculus to compute commutators between rescaled pseudodifferential operators on a manifold. First of all we point out that for $Op_\lambda(\cdot)$ described in (2.1.5) the usual rules on the composition of pseudodifferential operators apply (for example see Proposition 3.1 in [BM a]).

As defined in (2.1.6), $Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa$ acts on functions supported on $U_\kappa \subset M$ so summing up all the contributions $\sum_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa$ will be a pseudodifferential operator defined on the whole manifold. We also underline that we will consider operators with spatially localized symbols, namely *supp* $a^\kappa \subset [R, +\infty) \times V_\kappa \times \mathbb{R}^n$.

We proceed to the proof of the result mentioned in Proposition 2.3.4.

Proposition 2.C.1. *Let m, m', μ, μ' real numbers and the operators A, B on $[R, +\infty) \times S$ defined as*

$$A := \sum_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa, \qquad B := \sum_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa$$

with $a^\kappa \in \widetilde{S}^{m,\mu}$ *and* $b^\kappa \in \widetilde{S}^{m',\mu'}$ *and both spatially supported in* $[R,+\infty) \times V_\kappa$*. Then*

$$[A,B] = \sum_\kappa Op_{\lambda,\kappa}(\widetilde{S}^{m+m'-1,\mu+\mu'-1})\psi_\kappa + \mathcal{R}_N$$

with $\mathcal{R}_N$ *an operator which is negligible of order* N *for any* $N \in \mathbb{N}$*.*

Before giving the proof, we recall the definition of a negligible operator.

Definition 2.C.2 (Negligible operator of order N)**.** *Let* $N \in \mathbb{N}$ *and set*

$$Q_N := \langle \lambda^{\frac{1}{2}} r\rangle^N \left(\frac{P_0}{\lambda} + 1\right)^N = \sum_\kappa Op_{\lambda,\kappa}(\widetilde{S}^{N,2N}),$$

we say that an operator is negligible of order N *if it is of the form* $Q_N^{-1}\mathcal{B}Q_N^{-1}$ *for some bounded operator* $\mathcal{B}$ *depending on* λ*.*

Proof. In the following $\mathcal{B}$ is a generic bounded operator that will be allowed to change from one line to the other.

The composition AB is given by a double sum, by the support properties of the symbols $Op_{\lambda,\kappa}(b^\kappa)$ will be supported in $[R,+\infty) \times U_\kappa$, while ψ_m localizes in $[R,+\infty) \times U_m$. Hence for any chart such that $U_m \cap U_\kappa = \emptyset$ the corresponding term in the sum, $Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa$, is 0.

We start by looking at the easier case where we are composing two operators localized on the same chart. We denote by $\psi_\kappa^{\mathbb{R}^n}$ the pushforward on $\mathbb{R}^n$ of ψ_κ through any chart l whenever this quantity is well defined, that is $\psi_\kappa^{\mathbb{R}^n} = \Pi_l^{-1}\psi_\kappa$, and we will extensively use the relations

$$\psi_\kappa \Pi_l = \Pi_l \psi_\kappa^{\mathbb{R}^n}, \qquad \Pi_l^{-1}\psi_\kappa = \psi_\kappa^{\mathbb{R}^n}\Pi_l^{-1}.$$

This said, we have

$$\begin{aligned} Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa =& \Pi_\kappa Op_\lambda(a^\kappa)\Pi_\kappa^{-1}(\psi_\kappa\Pi_\kappa)Op_\lambda(b^\kappa)\Pi_\kappa^{-1}\psi_\kappa \\ =& \Pi_\kappa Op_\lambda(a^\kappa)\Pi_\kappa^{-1}\Pi_\kappa(\psi_\kappa^{\mathbb{R}^n} Op_\lambda(b^\kappa))\Pi_\kappa^{-1}\psi_\kappa \\ =& \Pi_\kappa Op_\lambda(a^\kappa)Op_\lambda(\widetilde{b}^\kappa)\Pi_\kappa^{-1}\psi_\kappa \end{aligned}$$

where $\widetilde{b}^\kappa := \psi_\kappa^{\mathbb{R}^n} b^\kappa \in \widetilde{S}^{m',\mu'}$ is still spatially supported on V_κ. Thanks to the multiplication on the right by $\Pi_\kappa^{-1}\psi_\kappa$ the composition $Op_\lambda(a^\kappa)Op_\lambda(\widetilde{b}^\kappa)$ is applied to functions localized on V_κ so we can harmlessly extend the symbols to 0 outside of their support. This, will give us a rescaled pseudodifferential operator on $\mathbb{R}^n$ to which usual composition formulas apply. Consequently

$$\begin{aligned} Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa =& \Pi_\kappa Op_\lambda(a^\kappa \widetilde{b}^\kappa)\Pi_\kappa^{-1}\psi_\kappa + \Pi_\kappa Op_\lambda(\widetilde{S}^{m+m'-1,\mu+\mu'-1})\Pi_\kappa^{-1}\psi_\kappa \\ =& Op_{\lambda,\kappa}(a^\kappa b^\kappa)\psi_\kappa + Op_{\lambda,\kappa}(\widetilde{S}^{m+m'-1,\mu+\mu'-1})\psi_\kappa. \end{aligned} \tag{2.C.1}$$

Obviously the same holds for $Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa$ so that when taking the commutator the term $Op_{\lambda,\kappa}(a^\kappa\widetilde{b}^\kappa)\psi_\kappa$ cancels with $Op_{\lambda,\kappa}(b^\kappa\widetilde{a}^\kappa)\psi_\kappa$, where $b^\kappa\widetilde{a}^\kappa = b^\kappa\psi_\kappa^{\mathbb{R}^n} a^\kappa = a^\kappa\widetilde{b}^\kappa$.

Now for the overlapping terms with $m \neq \kappa$ let $\widetilde{\psi}_\kappa$ equal to 1 on the support of ψ_κ, that we use to move the localization on the left

$$\Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1}\psi_\kappa = \widetilde{\psi}_\kappa \Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1}. \tag{2.C.2}$$

Then in

$$Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa = Op_{\lambda,m}(a^m)\psi_m\widetilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa)$$

the cutoff $\psi_m\widetilde{\psi}_\kappa \in C^\infty([R,+\infty)\times U_m\cap U_\kappa)$ localizes in a region where both the charts κ and m are defined.

We pick smooth cutoffs $\widetilde{\widetilde{\psi}}_\kappa$ and $\widetilde{\psi}_m$ such that

$$\widetilde{\widetilde{\psi}}_\kappa\widetilde{\psi}_\kappa \equiv \widetilde{\psi}_\kappa, \qquad \widetilde{\psi}_m\psi_m \equiv \psi_m$$

and since $\widetilde{\psi}_m^{\mathbb{R}^n} \equiv 1$ on the support of a^m we have

$$\begin{aligned} Op_{\lambda,m}(a^m)\psi_m\widetilde{\psi}_\kappa =& \Pi_m\widetilde{\psi}_m^{\mathbb{R}^n} Op_\lambda(a^m)\Pi_m^{-1}(\psi_m\widetilde{\psi}_\kappa) \\ =& \widetilde{\widetilde{\psi}}_\kappa\widetilde{\psi}_m\Pi_m Op_\lambda(a^m)\Pi_m^{-1}(\psi_m\widetilde{\psi}_\kappa) \\ &+ (1-\widetilde{\widetilde{\psi}}_\kappa)\widetilde{\psi}_m\Pi_m Op_\lambda(a^m)\Pi_m^{-1}(\psi_m\widetilde{\psi}_\kappa). \end{aligned} \tag{2.C.3}$$

On the intersection $U_m\cap U_\kappa$ both the expression in local coordinates given by the chart m and the chart κ are well defined and we can pass from one to another by composing with smooth transition maps, like $\kappa\circ m^{-1}$. We notice that $\Pi_m^{-1}(\psi_m\widetilde{\psi}_\kappa)$ localizes exactly in $m(U_m\cap U_\kappa)$ which is where $\kappa\circ m^{-1}$ is well defined and we have the relation

$$Op_\lambda(a^m)\Pi_m^{-1}(\psi_m\widetilde{\psi}_\kappa) = \Pi_{\kappa\circ m^{-1}}Op_\lambda(\widetilde{a}^\kappa)\Pi^{-1}_{\kappa\circ m^{-1}}\Pi_m^{-1}(\psi_m\widetilde{\psi}_\kappa)$$

where $\widetilde{a}^\kappa$ is a symbol belonging to the same class of a^m. Indeed, we will prove in Proposition 2.C.3 that conjugation by transition maps does not affect the decay of the symbol and that in particular it holds

$$Op_{\lambda,\kappa}(\widetilde{a}^\kappa) = Op_{\lambda,\kappa}(\widetilde{S}^{m,\mu}) + \mathcal{R}_N \tag{2.C.4}$$

with $\mathcal{R}_N$ a negligible operator.

In the following sum we can first use (2.C.2) and then the expression found in (2.C.3)

$$\begin{aligned} \sum_\kappa\sum_m Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa =& \sum_\kappa\sum_m Op_{\lambda,m}(a^m)\psi_m\widetilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa \\ =& \sum_\kappa\sum_m (\widetilde{\widetilde{\psi}}_\kappa\widetilde{\psi}_m\Pi_m(\Pi_{\kappa\circ m^{-1}}Op_\lambda(\widetilde{a}^\kappa)\Pi^{-1}_{\kappa\circ m^{-1}})\Pi_m^{-1}\psi_m \\ & \qquad \widetilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa))\psi_\kappa \\ &+ \sum_\kappa\sum_m (1-\widetilde{\widetilde{\psi}}_\kappa)\widetilde{\psi}_m Op_{\lambda,m}(a^m)(\psi_m\widetilde{\psi}_\kappa)Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa. \end{aligned}$$

Noticing that $\Pi_{\kappa\circ m^{-1}} = \Pi_m^{-1}\Pi_\kappa$ we can simplify some terms

$$\Pi_m\Pi_{\kappa\circ m^{-1}} = \Pi_\kappa, \qquad \Pi^{-1}_{\kappa\circ m^{-1}}\Pi_m^{-1} = \Pi_\kappa^{-1},$$

yielding

$$\begin{aligned} \sum_\kappa\sum_m Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa =& \sum_\kappa\sum_m \widetilde{\widetilde{\psi}}_\kappa\widetilde{\psi}_m\Pi_\kappa Op_\lambda(\widetilde{a}^\kappa)\Pi_\kappa^{-1}\psi_m\widetilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa \\ &+ \sum_\kappa\sum_m (1-\widetilde{\widetilde{\psi}}_\kappa)\widetilde{\psi}_m Op_{\lambda,m}(a^m)(\psi_m\widetilde{\psi}_\kappa)Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa. \end{aligned} \tag{2.C.5}$$

First, thanks to the support properties of $\widetilde{\psi}_m$ and $\widetilde{\widetilde{\psi}}_\kappa$ and since $(\psi_m)_m$ sum up to one

$$\begin{aligned} &\sum_m \widetilde{\widetilde{\psi}}_\kappa\widetilde{\psi}_m\Pi_\kappa Op_\lambda(\widetilde{a}^\kappa)\Pi_\kappa^{-1}\psi_m\widetilde{\psi}_\kappa\Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1} \\ &= \Pi_\kappa Op_\lambda(\widetilde{a}^\kappa)\Pi_\kappa^{-1}\sum_m \psi_m\widetilde{\psi}_\kappa\Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1} \end{aligned}$$

$$= \Pi_\kappa Op_\lambda(\widetilde{a}^\kappa)\widetilde{\psi}_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1} \tag{2.C.6}$$
$$= Op_{\lambda,\kappa}(\widetilde{a}^\kappa b^\kappa)\widetilde{\psi}_\kappa + Op_{\lambda,\kappa}(\widetilde{S}^{m+m'-1,\mu+\mu'-1})\widetilde{\psi}_\kappa \tag{2.C.7}$$

where (2.C.6) falls in the same case of (2.C.1) so we use (2.C.4) and the properties of composition of pseudodifferential operators. For the term in (2.C.5) the part

$$\Pi_m(1-\widetilde{\widetilde{\psi}}{}^{\mathbb{R}^n}_\kappa)\widetilde{\psi}^{\mathbb{R}^n}_m Op_\lambda(a^m)\psi^{\mathbb{R}^n}_m\widetilde{\psi}^{\mathbb{R}^n}_\kappa\Pi_m^{-1}$$

is a composition of pseudodifferential operators with disjoint supports, hence the usual formula for the composition produces the remainder only. This implies that we can write

$$R_m := (1-\widetilde{\widetilde{\psi}}_\kappa)\widetilde{\psi}_m Op_{\lambda,m}(a^m)\psi_m\widetilde{\psi}_\kappa = \widetilde{\widetilde{\psi}}_m Op_{\lambda,m}(r)$$

with $r\in\widetilde{S}^{-4N,-4N}$ for any $N\in\mathbb{N}$ and that

$$\sum_m R_m = Q_N^{-1}\mathcal{B}Q_N^{-1}.$$

Adding the contribution of $Op_\lambda(b^\kappa)$ gives us

$$\begin{aligned}\sum_m(1-\widetilde{\widetilde{\psi}}_\kappa)\widetilde{\psi}_m Op_{\lambda,m}(a^m)(\psi_m\widetilde{\psi}_\kappa)Op_{\lambda,\kappa}(b^\kappa) =& \sum_m R_m Op_{\lambda,\kappa}(b^\kappa)\\ =& Q_N^{-1}\mathcal{B}Q_N^{-1}Op_{\lambda,\kappa}(b^\kappa)\\ =& Q_{N/2}^{-1}Q_{N/2}^{-1}\mathcal{B}Q_N^{-1}Op_{\lambda,\kappa}(b^\kappa)Q_{N/2}Q_{N/2}^{-1}\\ =& Q_{N/2}^{-1}\mathcal{B}Q_{N/2}^{-1}.\end{aligned}$$

Thanks to (2.C.7) we have found

$$\begin{aligned}\sum_\kappa\sum_m Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa =& Op_{\lambda,\kappa}(\widetilde{a}^\kappa b^\kappa)\widetilde{\psi}_\kappa+\\ &+ Op_{\lambda,\kappa}(\widetilde{S}^{m+m'-1,\mu+\mu'-1})\widetilde{\psi}_\kappa + Q_N^{-1}\mathcal{B}Q_{N/2}^{-1}.\end{aligned}$$

We repeat the same procedure when evaluating the double sum produced by the composition BA, where in this case we obtain terms $Op_{\lambda,\kappa}(b^\kappa\widetilde{a}^\kappa)\widetilde{\psi}_\kappa$. Hence taking the difference $AB-BA$ results in the statement.

□

We prove now the invariance of the symbol classes by conjugation with a diffeomorphism. This will imply that passing from one chart to another, which means conjugating with a transition map the operators on $\mathbb{R}^n$, does not alter the decay of the symbols.

Notation. To simplify the notations of the kernels we state the proposition for pseudodifferential operators, instead of the rescaled version $Op_\lambda(a)$. However, given that the kind of diffeomorphism we are considering leaves untouched the radial variable r, which is the only one affected by the rescaling, the result generalizes easily for $Op_\lambda(a)$.

Proposition 2.C.3. *Let $\gamma: V\to W$ a diffeomorphism between open sets of $\mathbb{R}^{n-1}$ with $|\partial^\alpha\gamma|\leqslant C_\alpha$, $|\partial^\alpha\gamma^{-1}|\leqslant c_\alpha$ and set $\psi(r,\theta)=\chi(r)\varphi(\theta)\in C^\infty([R,+\infty)\times V)$ compactly supported in θ and*

$$\begin{aligned}\Pi_\gamma: C^\infty([R,+\infty)\times W)&\to C^\infty([R,+\infty)\times V)\\ v&\mapsto v(r,\gamma(\theta)).\end{aligned}$$

If $Op(a)$ has symbol $a \in \widetilde{S}^{m,\mu}$ with supp $a \subset [R, +\infty) \times W$ then

$$\Pi_\gamma Op(a)\Pi_\gamma^{-1}\psi = Op(a_\gamma)\widetilde{\psi} + \mathcal{R}_V$$

with $a_\gamma \in \widetilde{S}^{m,\mu}$, $\widetilde{\psi} \in C^\infty([R, +\infty) \times V)$ compactly supported in θ and $\mathcal{R}_V$ a pseudodifferential operator of negative order. In particular, $\mathcal{R}_V$ is the pushforward on $\mathbb{R}^n$ of a negligible operator of order N with arbitrary N.

Remark 2.C.4. The integral kernel of $Op(a)$ is

$$K_a(r, \theta, r', \theta') = \frac{1}{(2\pi)^n} \int e^{i(r-r')\rho + i(\theta-\theta')\eta} a(r, \theta, \rho, \eta) d\rho d\eta$$

whereas the integral kernel of $\Pi_\gamma Op(a)\Pi_\gamma^{-1}\psi$ is instead

$$K_\gamma(r, \theta, r', \theta') = \frac{1}{(2\pi)^n} \int \Big(e^{i(r-r')\rho + i(\gamma(\theta)-\gamma(\theta'))\eta} a(r, \gamma(\theta), \rho, \eta)\chi(r')\varphi(\theta') \\ |\mathrm{Jac}\gamma|(\theta') \Big) d\rho d\eta.$$

To show that $\Pi_\gamma Op(a)\Pi_\gamma^{-1}\psi$ is still a pseudodifferential operator, up to some remainder, we will need to write its kernel K_γ as an oscillating integral with phase $i(r - r')\rho + i(\theta - \theta')\eta$ and a symbol a_γ depending only on the variables r, θ, ρ and η.

Proof. As observed in the previous remark, we need the oscillating term in the integral K_γ to be $e^{i(r-r')\rho+i(\theta-\theta')\eta}$, so to linearize with respect to θ we consider the Taylor expansion

$$\gamma(\theta) - \gamma(\theta') = (\theta - \theta') \int_0^1 d\gamma(\theta' + t(\theta - \theta'))dt =: (\theta - \theta')M(\theta, \theta')$$

with $M(\theta, \theta')$ an invertible matrix. Performing a change of variable in K_γ which sends η to $M(\theta, \theta')^{-1}\eta$ yields

$$K_\gamma(r, \theta, r', \theta') = \frac{1}{(2\pi)^n} \cdot \int e^{i(r-r')\rho+i(\theta-\theta')\eta} A(r, \theta, r', \theta', \rho, \eta) d\rho d\eta \tag{2.C.8}$$

where we have set

$$A(r, \theta, r', \theta', \rho, \eta) := a(r, \gamma(\theta), \rho, M(\theta, \theta')^{-1}\eta)\chi(r')\varphi(\theta')\widetilde{M}(\theta, \theta').$$

In (2.C.8) it now appears the oscillatory term in the desired form as commented before. To finally obtain the kernel of a pseudodifferential operator we must get rid of the dependence of A on (r', θ').

Let $\Theta \in C_0^\infty(\mathbb{R}^n)$ a cutoff function such that $\Theta \equiv 1$ near 0, we first consider the kernel K_γ localized around the diagonal $\{r = r', \theta = \theta'\}$, that is

$$\Theta(r - r', \theta - \theta')K_\gamma(r, \theta, r', \theta').$$

We use again a Taylor expansion: we expand A with respect to the variables r', θ' around the point (r, θ) up to order N, hence providing us with a polynomial of order $N - 1$ plus a remainder term. In particular we can write

$$A(r, \theta, r', \theta', \rho, \eta) = A(r, \theta, r, \theta, \rho, \eta) \\ + \sum_{l=1}^{N-1} \sum_{j+|\alpha|=l} \mathcal{O}((r' - r)^j(\theta' - \theta)^\alpha)(\partial_{r'}^j \partial_{\theta'}^\alpha A)(r, \theta, r, \theta, \rho, \eta)$$

$$+\sum_{j+|\alpha|=N}\mathcal{O}((r'-r)^j(\theta'-\theta)^\alpha)R^{j,\alpha}(r,\theta,r',\theta',\rho,\eta).$$

Here $R^{j,\alpha}$ are the terms coming from the Taylor remainder: they are compactly supported in (r',θ') and such that $|\partial_\rho^k\partial_\eta^\beta R^{j,\alpha}|\lesssim\left(\langle\rho\rangle+\frac{\langle\eta\rangle}{\langle r\rangle}\right)^{\mu-k-|\beta|}$ (property which is inherited from a).

We want to use this expansion in (2.C.8) and do integration by parts after observing that

$$(r'-r)^j(\theta'-\theta)^\alpha e^{i(r-r')\rho+i(\theta-\theta')\eta}=D_\rho^jD_\eta^\alpha e^{i(r-r')\rho+i(\theta-\theta')\eta}.$$

Integral (2.C.8) then results in

$$\begin{aligned}\Theta(r-r',\theta-\theta')K_\gamma(r,\theta,r',\theta') =&\Theta(r-r',\theta-\theta')\frac{1}{(2\pi)^n}\\ &\cdot\int e^{i(r-r')\rho+i(\theta-\theta')\eta}A(r,\theta,r,\theta,\rho,\eta)d\rho d\eta\\ &+\Theta(r-r',\theta-\theta')\frac{1}{(2\pi)^n}\\ &\sum_l\sum_{(j,\alpha)}\int e^{i(r'-r)\rho+i(\theta'-\theta)\eta}(D_\rho^jD_\eta^\alpha\partial_{r'}^j\partial_{\theta'}^\alpha A)(r,\theta,r,\theta,\rho,\eta)d\rho d\eta\\ &+\Theta(r-r',\theta-\theta')\frac{1}{(2\pi)^n}\\ &\sum_{j+|\alpha|=N}\int e^{i(r'-r)\rho+i(\theta'-\theta)\eta}D_\rho^jD_\eta^\alpha R^{j,\alpha}(r,\theta,r',\theta',\rho,\eta)d\rho d\eta.\end{aligned}$$

Moreover we notice that by definition A preserves the decay of a with respect to r,ρ and η, meaning that

$$A(r,\theta,r,\theta,\rho,\eta)\in\widetilde{S}^{m,\mu},\qquad(D_\rho^jD_\eta^\alpha\partial_{r'}^j\partial_{\theta'}^\alpha A)(r,\theta,r,\theta,\rho,\eta)\in\widetilde{S}^{m-|\alpha|,\mu-|\alpha|-j}.$$

Up to the remainder term, we have obtained integrals which are the kernels of pseudodifferential operators of symbol $\widetilde{S}^{m,\mu}$ as we wanted.

For the remainders we can prove that they are kernels of negligible operators in the sense of Definition 2.3.2. The same can be proved for the contribution of $(1-\Theta)K_\gamma$ and this will allow us to conclude the proof.

To do this we need to show that composing on the left and right with $Q_S:=\langle r\rangle^S(P_0+1)^S$ the Taylor remainder results in a bounded operator for any fixed large S.

Here the key point is that we are conjugating by Q_S an operator whose kernel is of the form

$$\int\Theta(r-r',\theta-\theta')e^{i(r'-r)\rho+i(\theta'-\theta)\eta}D_\rho^jD_\eta^\alpha R^{j,\alpha}(r,\theta,r',\theta',\rho,\eta)d\rho d\eta.\tag{2.C.9}$$

This kernel is smooth, with derivatives in (r',θ') which are compactly supported and such that in the integral we have arbitrary fast decay in ρ and η, since we recall $j+|\alpha|=N$ implies

$$|D_\rho^jD_\eta^\alpha R^{j,\alpha}|\lesssim\left(\langle\rho\rangle+\frac{\langle\eta\rangle}{\langle r\rangle}\right)^{\mu-N}.$$

Applying Q_S on the right, where $(P_0+1)^S$ is selfadjoint, would lead us to differentiate the kernel in r' and θ', while applying it on the left means taking derivatives with respect to r and θ. The ensemble of these actions still results in a kernel which is bounded together with its derivatives, thanks to the properties we just stated. Notably, the decay in ρ and η allows to compensate the growth which is generated when taking derivatives of the oscillating factor.

We can therefore apply Calderon-Vaillancourt theorem (Theorem 2.8.1 in [Mar02]) to conclude that the corresponding operator is bounded. (Actually here one needs to apply Calderon-Vaillancourt to a suitably conjugated operator in order to have boundedness with respect to the appropriate L^2 norm, that is L^2_G.)

The same reasoning can be adapted to the contribution away from the diagonal, after rewriting it as

$$(1-\Theta)(r-r',\theta-\theta')K_\gamma(r,\theta,r',\theta') = \frac{(1-\Theta)(r-r',\theta-\theta')}{(r-r')^M(\theta-\theta')^\beta} \int e^{i(r-r')\rho+i(\theta-\theta')\eta}(D^M_\rho D^\beta_\eta A)(r,\theta,r',\theta',\rho,\eta)d\rho d\eta$$

where we performed again integration by parts thanks to

$$(r-r')^M(\theta-\theta')^\beta e^{i(r-r')\rho+i(\theta-\theta')\eta} = D^M_\rho D^\beta_\eta e^{i(r-r')\rho+i(\theta-\theta')\eta}.$$

Indeed, $D^M_\rho D^\beta_\eta A$ also decays as

$$|D^M_\rho D^\beta_\eta A| \lesssim \left(\langle\rho\rangle + \frac{\langle\eta\rangle}{\langle r\rangle}\right)^{\mu-M-|\beta|}$$

with arbitrary M and β and hence the same arguments used for (2.C.9) can be replicated to conclude that $Q_S(1-\Theta)K_\gamma Q_S$ is an integral kernel of a bounded operator.

□

2.D A uniform bound for the spherical Laplacian

Consider the Laplace-Beltrami operator on the unit sphere

$$-\frac{\Delta_{\mathbb{S}^{n-1}}}{r^2} = -\Delta_{\mathbb{R}^n} + \partial_r^2 + \frac{(n-1)}{r}\partial_r, \tag{2.D.1}$$

given the relation between the radial and euclidean coordinates $r = |x|$ we can derive

$$\partial_r = \frac{1}{|x|}\sum_j x_j\partial_j, \qquad \partial_r^2 = \frac{1}{|x|^2}\sum_k x_k \sum_j x_j \partial^2_{j,k}$$

and therefore rewrite the operator in terms of euclidean derivatives as

$$-\frac{\Delta_{\mathbb{S}^{n-1}}}{r^2} = -\Delta_{\mathbb{R}^n} + \sum_{j,k=1}^n \frac{x_j x_k}{|x|^2}\partial^2_{j,k} + (n-1)\sum_{j=1}^n \frac{x_j}{|x|^2}\partial_j. \tag{2.D.2}$$

Proposition 2.D.1. *Let $n \geqslant 3$ and $\lambda_0 > 0$, there exists a constant $C_n > 0$ such that*

$$\left\| \frac{-\Delta_{\mathbb{S}^{n-1}}}{r^2}\left(-\Delta_{\mathbb{R}^n}+\lambda\right)^{-1} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leqslant C_n. \tag{2.D.3}$$

for all $\lambda \in (0,\lambda_0]$.

Proof. We use the expression in (2.D.2). By the spectral theorem $-\Delta_{\mathbb{R}^n}\left(-\Delta_{\mathbb{R}^n}+\lambda\right)^{-1}$ is bounded and by elliptic regularity results

$$\left\| \frac{x_j x_k}{|x|^2}\partial^2_{j,k}\left(-\Delta_{\mathbb{R}^n}+\lambda\right)^{-1} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim \left\| \partial^2_{j,k}\left(-\Delta_{\mathbb{R}^n}+\lambda\right)^{-1} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim 1.$$

Finally, since $n \geqslant 3$ we can apply Hardy inequality to conclude

$$\begin{aligned}\left\|\frac{x_j}{|x|^2}\partial_j\left(-\Delta_{\mathbb{R}^n}+\lambda\right)^{-1}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} &\leqslant \left\|\frac{1}{|x|}\left(-\Delta_{\mathbb{R}^n}+\lambda\right)^{-\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\ &\quad\cdot \left\|\left(-\Delta_{\mathbb{R}^n}+\lambda\right)^{-\frac{1}{2}}\partial_j\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\ &\leqslant \frac{2}{n-2}\left\|\nabla\left(-\Delta_{\mathbb{R}^n}+\lambda\right)^{-\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\ &\leqslant \frac{2}{n-2}.\end{aligned}$$

□

Let $(\sigma_{l,n}^2, s_{l,n}(\theta))_j$ the eigenpairs of $-\Delta_{\mathbb{S}^{n-1}}$, representing $u \in L^2(\mathbb{R}^n)$ in spherical coordinates and decomposing the angular part along the basis of eigenfunctions we have

$$\frac{-\Delta_{\mathbb{S}^{n-1}}}{r^2}\left(-\Delta_{\mathbb{R}^n}+\lambda\right)^{-1}u(r,\theta) = \sum_j \frac{\sigma_{l,n}^2}{r^2}\left(-\partial_r^2 - \frac{(n-1)}{r}\partial_r + \frac{\sigma_{l,n}^2}{r^2}+\lambda\right)^{-1} u_{l,n}(r)s_{l,n}(\theta)$$

where we have used for $-\Delta_{\mathbb{R}^n}$ the expression deriving from (2.D.1).

The uniform bound of the previous proposition translates to the following bound on one dimensional operators.

Corollary 2.D.2. *Let $n \geqslant 3$ and $\sigma_{l,n}^2$ the $l-$th eigenvalue of the spherical Laplacian on $\mathbb{S}^{n-1}$, then there exists $C_n > 0$ such that*

$$\left\|\frac{\sigma_{l,n}^2}{r^2}\left(-\partial_r^2 - \frac{(n-1)}{r}\partial_r + \frac{\sigma_{l,n}^2}{r^2}+\lambda\right)^{-1}\right\|_{L^2(\mathbb{R}^+,r^{n-1}dr)\to L^2(\mathbb{R}^+,r^{n-1}dr)} \leqslant C_n$$

for all $\lambda \in (0,\lambda_0]$.

We remark that the operator that appears in the statement is exactly $p_{l,n}$ that we defined in the proof of Lemma 2.4.13.

2.E Notations for the current chapter

Manifolds and metrics

$M = K \cup (M \setminus K)$, $M \setminus K$ infinite end
S angular manifold of dimension $n-1$
$p \in M$, $\omega \in S$, θ local coordinate on S
$\kappa : U_\kappa \subset S \to V_\kappa \subset \mathbb{R}^{n-1}$ coordinate chart on S
$\Pi_\kappa, \Pi_\kappa^{-1}$ pullback and pushforward on S
$\overline{g}, g(r)$ metrics on S
$w = \frac{|g(r,\theta)|}{|\overline{g}(\theta)|}$
$(R, +\infty) \times S$ isometric to $M \setminus K$
$\overline{G} = dr^2 + r^2\overline{g}$ fixed metric on $(R, +\infty) \times S$
$G = dr^2 + r^2 g(r)$ perturbed metric on $(R, +\infty) \times S$
$\psi_\kappa(r, \omega) = \varphi(r)\varphi_\kappa(\omega)$ with $\varphi(r)$ smooth cutoff on $[R, +\infty)$, $(\varphi_\kappa)_\kappa$ partition of unity on S

Norms

$\| \cdot \|_{L^2_G}$ norm with respect to the metric G
$\| \cdot \|$ norm of bounded operators on $L^2(M)$

Operators

P_0 Laplace-Beltrami operator on M
$P = P_0 + V$
$-\Delta_0$ Laplace-Beltrami operator on $((R, +\infty) \times S, \overline{G})$
$\mathcal{A}^\lambda$ conjugate operator in Mourre theory

Pseudodifferential calculus

$e^{i\tau\mathcal{A}}$ with $\tau = \ln(\lambda^{1/2})$ and $\mathcal{A}$ generator of dilations
$\breve{r} = \lambda^{1/2} r$, $\breve{\rho} = \lambda^{-1/2}\rho$
$\widetilde{S}^{m,\mu}$ class of symbols (m order of decay, μ order of differentiation)
$Op_\lambda(a) = e^{i\tau\mathcal{A}} Op(a) e^{i\tau\mathcal{A}}$ rescaled pseudodifferential operator
$Op_{\lambda,\kappa}(a)$ rescaled pseudodifferential operator on the manifold
$\mathcal{B}$ bounded operator
$\mathcal{R}_N$ negligible operator of order N (Definition 2.3.2)

Others

$\langle r \rangle$ weight equal to r for $r \gg 1$
$\chi(r)$ smooth cutoff equal to one in a large neighborhood of zero

Chapter 3

High frequency resolvent estimates for the magnetic Laplacian on smooth manifolds with ends

Outline of the current chapter

3.1 Introduction

In this chapter we consider the Laplace-Beltrami operator with order one and order zero perturbations on non compact manifolds with ends of infinite volume, in particular including asymptotically conical and asymptotically hyperbolic ones. The manifolds we consider are such that outside of a compact set they are isometric to a product manifold whose metric is a long range perturbation of a fixed one.

Let P_m the operator we consider of (M, G_M) including all the perturbations, we will be interested in high frequency estimates for the resolvent as the spectral parameter approaches the positive real axis, hence we consider

$$P_m - \lambda^2 + i\varepsilon' \qquad \lambda \gg 1,\ \varepsilon' > 0.$$

The aim of this chapter is to obtain estimates for large λ in all generality, without assumptions on the geodesic flow. As we discussed in the Introduction in Section 1.3.2, without geodesic information we expect an exponential loss in the estimates with respect to the polynomial blowup of the previous chapter. This is indeed what we will obtain: the resolvent on the entire manifold has an exponential bound and the cutoff resolvent near infinity has an polynomial blowup. This is due to the fact that outside of a sufficiently large compact region there is no trapping effect and hence no loss in the estimates. More precisely, the main result is the following. Precise definitions are postponed to the end of this introduction.

Theorem 3.1.1. *Let (M, G_M) a non compact Riemannian manifold of dimension $n \geqslant 3$ and $K \subset M$ such that $M \setminus K$ is isometric to a product manifold $X_R = (R, +\infty) \times S$. Let P_m be the Laplace-Beltrami operator on M with perturbations of order one and zero. Let $\|\cdot\|_{H^1, B^*_{>R}}$, $\|\cdot\|_{B_{>R}}$ weighted norms on the manifold end $M \setminus K$ (defined in Section 3.1.3).*

Let $u \in H^2(M), \lambda \gg 1$ and $R < R_1 < R_2$, then for any $\varepsilon' > 0$

$$\|u\|^2_{H^1(M\setminus X_{R_1})} + \|u\|^2_{H^1, B^*_{>R}} \leqslant O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon')u\|^2_{L^2(M\setminus X_{R_2})} + O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon')u\|^2_{B_{>R}}$$

for some constant $C > 0$ independent of λ and ε'.

In the theorem we obtain indeed a uniform exponential bound on the resolvent of the Laplace-Beltrami operator with an order one perturbation. Having this type of resolvent estimate can be applied, as in [Vod02] or [CV02], to prove that there is a strip of exponential size around the positive real axis which is free of resonances. In other words, this tells us that resonances can not accumulate exponentially fast and hence gives a lower bound on their width. For complex resonances their imaginary part gives the rate of decay of a solution associated to resonant initial data, hence the width of a resonance carries dynamical information. For example, in [Bur02a] the existence of a resonance free strip is applied to prove logarithmic time decay of solutions to wave equations.

In the proof of the result we can see that the contribution of the exponential terms $O(e^{\lambda C})$ is due only to norms on compact regions of the manifold. We therefore obtain a corollary for the cutoff resolvent.

Corollary 3.1.2. *Under the assumptions of the previous theorem, let χ a smooth cutoff such that $\chi \equiv 1$ on $X_{R_3} = (R_3, +\infty) \times S$ for a sufficiently large R_3. Then for any $\varepsilon' > 0$*

$$\|\chi u\|^2_{B^*_{>R}} \leqslant O(\lambda^{-2})\|(P_m - \lambda^2 + i\varepsilon')u\|^2_{B_{>R}}$$

In particular

$$\|\chi r^{-1/2-\mu}(P_m - \lambda^2 + i\varepsilon)^{-1} r^{-1/2-\mu}\chi\|_{L^2 \to L^2} = O(\lambda^{-1})$$

with $\mu > 0$.

Remark 3.1.3. We will define precisely the weighted norms $\|\cdot\|_{H^1, B^*_{>R}}$ and $\|\cdot\|_{B_{>R}}$ in Section 3.1.3. For the moment we only comment that they are norms of Besov type on the region $(R, +\infty) \times S$ with weights given by $r^{-1/2}$ with $r \in (R, +\infty)$ the radial variable. From this corollary we see that estimates with respect to these norms imply the more classic ones on weighted L^2 spaces with stronger weights with respect to $r^{-1/2}$. Moreover, we obtain the expected bound on the cutoff resolvent in the presence of an order one perturbation.

Unlike the previous chapter, we will not use Mourre theory, indeed this technique is well adapted in the high frequency regime under additional assumptions on the geodesic flow, which we do not make here. The main strategy is the same as in [CV02] which in turn is inspired by the works of Burq ([Bur98], [Bur02a]): we divide the manifold in two parts, a bounded region and an infinite end, and we treat the two separately. In the compact region we use are Carleman estimates, which are stable by order one perturbations and hence suitable for our operator P_m.

In the unbounded region we do not use any complex theory, but we rather exploit the equation $(P_m - \lambda^2 + i\varepsilon')u = f$ and bound the solution u by the source term f and we will use simple identities like

$$Im(Au, Bu) = (u, \frac{[A,B]}{2i}u), \quad Re(Au, Bu) = \Big(\frac{AB+BA}{2}u, u\Big)$$

for A, B symmetric operators.

Remark 3.1.4. For the estimates on the unbounded region $M \setminus K$ we use a slightly different approach than in [CV02]. In particular the strategy used to treat this region (Section 3.2) is frequency independent, in the sense that the proof holds for intermediate and high frequencies. In this section we replace the smallness of a semiclassical parameter $h = \lambda^{-1}$ with the decay of the radial variable. In the last section, when combining the two regions, we will need to assume $\lambda \gg 1$.

On top of considering a more general operator with differential perturbations and Besov type norms with sharper weights, in this work we revisit some of the arguments in [CV02] trying to give a rather complete exposition of the whole strategy. We also remark that we combine the Carleman approach, which is robust enough to be adapted to our perturbed case, to the use of Besov type norms that were introduced for more classical Hamiltonians ([AH76], [AP85]).

The chapter is organized as follows: we conclude the introduction with some definitions, in Section 3.2 we treat the unbounded region, in Section 3.3 we show how to use Carleman estimates to treat the compact region, in Section 3.4 we use an argument presented first in [Bur02a] to conclude with the proofs of Theorem 3.1.1 and Corollary 3.1.2.

3.1.1 Definition of the geometric framework

Let S a compact $n-1$ dimensional Riemannian manifold, around a point $\omega \in S$ we will denote by $\theta \in \mathbb{R}^{n-1}$ the local coordinates. We will equip S with two different metrics $(S, \overline{g})$ and $(S, g(r))$. We will use the product manifold

$$((R, +\infty) \times S, dG) \quad dG = dr^2 + l(r)^{-2}g(r)$$

to model the infinite ends of our manifold M.

We consider a Riemannian manifold M, a compact set $K \subset M$ and a diffeomorphism Ω

$$\Omega : M \setminus K \to (R, +\infty) \times S$$

such that the metric on M is given by

$$G_M := \Omega^*(G) = \Omega^*(dr^2 + l(r)^{-2}g(r))$$

for some smooth function $l : (R, +\infty) \to \mathbb{R}^+$ verifying

$$-\frac{l'(r)}{l(r)} \geqslant \frac{c}{r}, \qquad -\frac{l'(r)}{l(r)} \in L^\infty((R, +\infty) \times S). \tag{3.1.1}$$

Remark 3.1.5. Since we require $-l'/l \geqslant 0$ it follows that $l' \leqslant 0$, so l is a decreasing function. Moreover integrating the inequality in (3.1.1) we obtain

$$l(r) \lesssim \frac{1}{r} \tag{3.1.2}$$

hence $l(r) \to 0$ as $r \to \infty$. Then l' is bounded for large enough r and the L^∞ condition in (3.1.1) implies $|l'(r)|r \in L^\infty((R, +\infty) \times S)$.

Remark 3.1.6. With an abuse of notation we will write G for both the metric on the product $(R, +\infty) \times S$ and the manifold end $M \setminus K$.

Remark 3.1.7. Let $k^* \in \mathbb{N}$. We will denote by ϕ a smooth cutoff on $(R, +\infty)$ for which we will assume $supp\phi \subset (2^{k^*+3}R, +\infty)$. Such a function can be identified via Ω with a cutoff on $M \setminus K$ of the form $\chi = \phi \circ \Omega$.

We assume that $g(r)$, the angular metric, is a perturbation of a fixed metric in the following sense.

Let f a function of r with values in the space of sections $\Gamma(T^p_q(S))$:

$$f : (R, +\infty) \to \Gamma(T^p_q(S))$$
$$r \mapsto f(r),$$

with coefficients $f^{i_1,\dots i_p}_{j_1,\dots,j_q}(r,\theta)$ around a point ω with respect to a basis of the tensor product $(\otimes T_\omega(S))^p \otimes (\otimes T^*_\omega(S))^q$. We define a topology on $\Gamma(T^p_q(S))$ given by the seminorms

$$N^{pq}_{m,J}(f) = \sum_{|\alpha|\leqslant m} \|\partial^\alpha f^{i_1,\dots i_p}_{j_1,\dots,j_q}\|_{L^\infty(J)}$$

with J a compact subset of the coordinate patch on $\mathbb{R}^{n-1}$.

Definition 3.1.8. *Let $f : (R, +\infty) \to \Gamma(T^p_q(S))$ a smooth function, then $f \in S^{-\nu}$ if*

$$N^{pq}_{m,J}(\partial^l_r f(r)) \lesssim \langle r\rangle^{-\nu-l} \text{ for any } m, l \in \mathbb{N}, \ J\text{compact set.}$$

We assume

$$g(r) - \overline{g} \in S^{-\nu} \text{ for some } \nu > 0. \tag{3.1.3}$$

We point out two examples of infinite end the reader should keep in mind throughout this work. All the assumptions we make on l are be satisfied by these two examples.

Example 3.1.9 (Asymptotically conical end). In the particular case of a fixed metric $g(r) = \overline{g}$ and $l(r) = r^{-1}$ we obtain on $(R, +\infty) \times S$ the following metric

$$dr^2 + r^2\overline{g}$$

which is the conic end defined in Definition 1.4.1. If we replace $\overline{g}$ with a perturbation of it $g(r)$ we obtain an asymptotically conical end, as defined in Definition 1.4.4.

Example 3.1.10 (Asymptotically hyperbolic end). Taking $l(r) = e^{-r}$ and again $g(r) = \overline{g}$, if we equip $(R, +\infty) \times S$ with the metric

$$dr^2 + e^{2r}\overline{g}$$

we obtain an hyperbolic end in the sense of Definition 1.4.7. If we take a perturbed metric $g(r)$ we obtain a perturbed hyperbolic end, again defined in Definition 1.4.7.

Remark 3.1.11. Let $-\Delta_{g(r)}$ the Laplace-Beltrami operator on $(S, g(r))$ and we define

$$w(r,\theta) = \frac{|g(r,\theta)|}{|\overline{g}(\theta)|} \tag{3.1.4}$$

which is of the form

$$w(r,\theta) = 1 + \widetilde{w}(r,\theta) \text{ with } \widetilde{w} \in S^{-\nu}. \tag{3.1.5}$$

Since $dg(r) = w(r,\theta)d\overline{g}$, we can use w to conjugate $-\Delta_{g(r)}$ and obtain $w^{\frac{1}{2}}(-\Delta_{g(r)})w^{-\frac{1}{2}}$ which will be symmetric with respect to the measure induced by the fixed metric $\overline{g}$.

3.1.2 The operator

We consider A a vector field on M, thanks to the diffeomorphism Ω, we can identify the restriction to $M \setminus K$ with a map taking values in the product space $\mathbb{R} \times T_\omega(S)$. Namely

$$\begin{aligned} A : M \setminus K \to T_0^1(M) \simeq \mathbb{R} \times T(S) \\ p \mapsto A(p) \simeq A(\Omega(p)) = A(r,\omega) \in \mathbb{R} \times T_\omega(S). \end{aligned}$$

In a coordinate patch around (r, ω) we will denote the components of A as

$$(A_0(r,\theta), A_S(r,\theta)) = (A_0(r,\theta), A_1(r,\theta), \dots A_{n-1}(r,\theta)).$$

We assume $A_0 \in S^{-\nu}$ and

$$r^\nu l^{-2}(r) A_j,\ r^{\nu+1} l^{-2}(r) \partial_r A_j \in L^\infty((R,+\infty) \times S). \tag{3.1.6}$$

We also introduce the notation

$$\widetilde{A}_0(r,\theta) := A_0(r,\theta), \quad \widetilde{A}_i(r,\theta) := l^{-2}(r) \sum_{j=1}^{n-1} g_{i,j}(r,\theta) A_j(r,\theta), \tag{3.1.7}$$

hence by the previous assumption

$$r^\nu \widetilde{A}_j \in L^\infty((R,+\infty) \times S)$$

for all $j = 0, \dots, n-1$.

Notation. Let $D_G = (D_r, \frac{1}{i} l(r)^2 \nabla_{g(r)})$ and $\langle \cdot, \cdot \rangle_G$ the scalar product, between two vectors, induced by the metric G while $(\cdot, \cdot)_{dG}$ denotes the scalar products on $L^2((R,+\infty) \times S, dG)$.

With $(\cdot,\cdot)_{drd\overline{g}}$ we will denote the scalar products on $L^2((R,+\infty) \times S, drd\overline{g})$ and $\| \cdot \|_{L^2(drd\overline{g})}$ will be the induced L^2 norm.

We will also consider the scalar product and relative L^2 norm on a bounded region of $(R, \infty) \times S$, in doing so we will always consider the one induced by the measure $drd\overline{g}$ and we will specify the region by denoting them like $(\cdot,\cdot)_{L^2(\cdot)}$ and $\|\cdot\|_{L^2(\cdot)}$.

Let V a multiplicative potential in the symbol class

$$V \in S^{-\nu}, \nu > 0.$$

The perturbed Laplace-Beltrami operator on (M, G_M) can be defined via a quadratic form. On the manifold end $M \setminus K$, where we have an isometry with $((R,+\infty) \times S, dG)$, it agrees with the one defined by the following quadratic form on smooth compactly supported functions

$$\begin{aligned} q(u,v) =& ((D_G - A)u, (D_G - A)v)_{dG} + (u, Vv)_{dG} \\ =& \int (\langle \overline{(D_G - A)u}, (D_G - A)v \rangle_G + \overline{u} V v) l(r)^{1-n} dr dg(r). \end{aligned} \tag{3.1.8}$$

The operator defined by this quadratic form is symmetric with respect to the scalar product induced by the measure $dG = l(r)^{1-n} dr dg(r)$. After integration by parts and conjugation by $e^{iF} = l(r)^{\frac{1-n}{2}} w^{1/2}$ near infinity we obtain an operator

$$h^2 \widetilde{P}_{\overline{g}} = h^2 (D_r - A_0)^2 + M(r) + h^2 V_m \tag{3.1.9}$$

with $h = \lambda^{-1}$ for some $\lambda > 0$ where V_m is a multiplicative potential including V and the effective potential and

$$M(r) := h^2 l^2(r)(1 + T(r)) \tag{3.1.10}$$

with $T(r)$ a differential operator of order two in the angular variables. More precisely,

$$\begin{aligned} T(r) :&= w^{1/2}(-\widetilde{\Delta}_{g(r)})w^{-1/2} \\ &= w^{1/2}\frac{1}{|g(r,\theta)|}\sum_{i,j=1}^{n-1}(D_i - \widetilde{A}_i)\Big(|g(r,\theta)|g^{i,j}(r,\theta)(D_j-\widetilde{A}_j)\Big)w^{-1/2}. \end{aligned} \tag{3.1.11}$$

We underline that the operator $-\widetilde{\Delta}_{g(r)}$ is the Laplace-Beltrami operator $-\Delta_{g(r)}$ in which we have incorporated the perturbation by A.

Remark 3.1.12. Conjugating by $e^{iF} = l(r)^{\frac{1-n}{2}}w^{1/2}$ has a double effect: with $l(r)^{\frac{1-n}{2}}$ we pass from an operator symmetric with respect to dG to one symmetric with respect to $drdg$; with $w^{1/2}$ we pass the symmetry from $drdg$ to $dr d\overline{g}$. In conclusion, $\widetilde{P_{\overline{g}}}$ is symmetric with respect to the scalar product induced by the measure $dr d\overline{g}$.

Remark 3.1.13. The operator $M(r)$ includes the angular terms of the operator, so all the components act differentially only on the angular variable θ. In particular for any function s of r

$$[M(r), s] = 0.$$

A useful property we will use later on is

$$-M'(r) \gtrsim M(r) - h^2 l^2(r)T'(r)$$

which we can derive thanks to (3.1.1). Here $M'(r)$ and $T'(r)$ denote the differential operators obtained from M and T by differentiating the coefficients with respect to r.

The potential V_m is of the form

$$V_m(r,\theta) = a(r,\theta) + V_0(r,\theta)$$

where

$$a(r,\theta) = \frac{(n-1)^2}{4}\left(\frac{l'}{l}\right)^2 - \frac{(n-1)}{2}\left(\frac{l'}{l}\right)'$$

and V_0 has the properties

$$r^\nu V_0,\ r^{1+\nu}\partial_r V_0 \in L^\infty((R,+\infty)).$$

Moreover we make the following assumptions on a

$$\frac{(n-1)^2}{4}\left(\frac{l'}{l}\right)^2 =: \Lambda + a_1(r)$$

with a constant $\Lambda = \Lambda(l) \geqslant 0$ and $a_1(r) \geqslant 0$ such that $r^\nu a_1(r), r^{1+\nu}a_1'(r) \in L^\infty((R,+\infty))$ and

$$\frac{(n-1)}{2}\left(\frac{l'}{l}\right)' =: a_2(r)$$

is such that $r^\nu a_2(r), r^{1+\nu}a_2'(r) \in L^\infty((R,+\infty))$.

With this notation

$$V_m = \Lambda + a_1(r) + a_2(r) + V_0(r,\theta) \tag{3.1.12}$$

and we can conclude that

$$r^\nu(V_m - \Lambda),\ r^{1+\nu}\partial_r V_m \in L^\infty((R,+\infty)). \tag{3.1.13}$$

We report the two typical cases of Examples 3.1.9 and 3.1.10.

Example 3.1.14. If $l(r) = e^{-r}$ then

$$\Lambda = \frac{(n-1)^2}{4}, \qquad a_1 = a_2 = 0.$$

If $l(r) = r^{-s}$ with $s \geqslant 1$ then

$$\Lambda = 0, \qquad a_1(r) = \frac{(n-1)}{2} \cdot \frac{s}{r^2}, \qquad a_2(r) = \frac{(n-1)^2}{4} \cdot \frac{s}{r^2}.$$

Notation. Since $V_m - \Lambda$ is a decaying potential, without loss of generalization, we define P_m the perturbed Laplace-Beltrami operator translated by Λ. In this way, near infinity we will equivalently to consider

$$P_{\overline{g}} = \widetilde{P}_{\overline{g}} - \Lambda = h^2(D_r - A_0)^2 + M(r) + h^2(V_m - \Lambda)$$

which has a decaying potential $V_m - \Lambda$. We recall that Λ depends on l, that is on the choice of metric and that in the cases $l(r) = r^{-1}$ or $l(r) = e^{-r}$ it represents the bottom of the essential spectrum.

3.1.3 The norms

Let $\psi \in C_0^\infty(\mathbb{R})$ a non negative bump function with $\|\psi\|_\infty \leqslant 1$ and

$$\psi(s) = \begin{cases} 1 & \text{for } |s| \leqslant \frac{1}{2} \\ 0 & \text{for } |s| \geqslant 1. \end{cases}$$

Set $\varphi(s) := \psi(\frac{s}{2}) - \psi(s)$ which is then non negative and

$$supp\varphi \subset \{\frac{1}{2} < |s| < 2\}, \quad \|\varphi\|_\infty \leqslant 2.$$

We rescale φ dyadically in order to construct a partition of unity

$$1 = \psi(s) + \sum_{k \geqslant 0} \varphi(2^{-k}s).$$

Remark 3.1.15. The support of $\varphi(2^{-k}s)$ is

$$D_k := [2^{k-1}, 2^{k+1}],$$

so for fixed s there are only two non vanishing terms in the sum $\sum_{k\geqslant 0} \varphi(2^{-k}s)$ since the interval $[\frac{\ln s}{\ln 2} - 1, \frac{\ln s}{\ln 2} + 1]$ contains at most 2 integers.

We fix a $k^* \in \mathbb{N}$. To partition only the half line $(2^{k^*}R, +\infty)$ we start the sum at

$$k_0 := \frac{\ln R}{\ln 2} + k^* + 2,$$

in this way $[2^{k-1}, 2^{k+1}] \subset (2^{k^*}R, +\infty)$ for all $k \geqslant k_0$. For $r > 2^{k^*}R$

$$\psi(r) + \sum_{k \geqslant k_0} \varphi(2^{-k}r) = \sum_{k \geqslant k_0} \varphi(2^{-k}r) = \begin{cases} \in (0,1) & r \in (2^{k_0-1}, 2^{k_0}], \\ 1 & r > 2^{k_0}. \end{cases}$$

We define the following function's norm on $(2^{k^*+1}R, +\infty) \times S$

$$\|f\|_{B_{>R}} := \sum_{k \geqslant k_0} \|r^{1/2} f\|_{L^2(drd\overline{g}, 2^{k-1} \leqslant r \leqslant 2^{k+1})} \tag{3.1.14}$$

and the dual quantity

$$\|g\|_{B^*_{>R}} := \sup_{k \geqslant k_0} \|r^{-1/2} g\|_{L^2(drd\overline{g}, 2^{k-1} \leqslant r \leqslant 2^{k+1})}. \tag{3.1.15}$$

We also define the shorthand

$$\|g\|^2_{H^1, B^*_{>R}} := \|g\|^2_{B^*_{>R}} + \|h(D_r - A_0)g\|^2_{B^*_{>R}} + \|M(r)^{1/2} g\|^2_{B^*_{>R}}. \tag{3.1.16}$$

Remark 3.1.16. The previous norm is well defined since $M(r)$ is a non negative operator, hence its square root exists. Despite $M(r)^{1/2}$ being a non local operator the norm is still well defined since it is an operator only in the angular variables, hence the non local action is only on the manifold S and we recall that in the definition of the norm we integrate over all S.

We also define an adapted H^1 norm on compact regions of the manifold end, we still use the angular operator $M(r)$ to define the angular gradient but remove the weights, since we are in a compact region: for U a bounded region in $(R, +\infty) \times S$ we write

$$\|f\|^2_{H^1(U)} = \|f\|^2_{L^2(U)} + \|h(D_r - A_0)f\|^2_{L^2(U)} + \|M(r)^{1/2} f\|^2_{L^2(U)}$$

where we recall the L^2 norms are with respect to the measure $drd\overline{g}$.

Remark 3.1.17. The definition of the norms is independent of the choice of dyadic step, in other words the norms

$$\|f\|_{B_{>R},m} := \sum_{k \geqslant (m-1)+k_0} \|r^{1/2} f\|_{L^2(drdg, 2^{k-m} \leqslant r \leqslant 2^{k+m})}$$

are all equivalent. Given this equivalence we will drop the dependence on the index n and denote by the same symbols $\| \cdot \|_{B_{>R}}$ and $\| \cdot \|_{B^*_{>R}}$, any norm regardless of the dyadic step in the definition.

For the same reason, defining

$$\| \cdot \|_{L^2(drd\overline{g}, D_{k,m})} := \| \cdot \|_{L^2(drd\overline{g}, 2^{k-m} \leqslant r \leqslant 2^{k+m})}$$

we will drop the index m in the notation for $D_{k,m}$ and simply write $\| \cdot \|_{L^2(drd\overline{g}, D_k)}$.

Remark 3.1.18. We already commented that on the manifold end, $(R, +\infty) \times S$, the operator P_m agrees with the one defined via the quadratic form (3.1.8), that we will call P_G. This operator is symmetric with respect to the measure $dG = l(r)^{1-n} drdg(r)$. However, in the norms we just defined in this region, $\| \cdot \|_{B_{>R}}, \| \cdot \|_{B^*_{>R}}, \| \cdot \|_{H^1(U)}$, we take the L^2 norm with respect to the $drd\overline{g}$ measure and we use the expression of $P_{\overline{g}}$, symmetric with respect to $drd\overline{g}$, to define $\| \cdot \|_{H^1, B^*_{>R}}$ and $\| \cdot \|_{H^1(U)}$. It is actually equivalent to bound the norms in which we take the measure $drd\overline{g}$ or the corresponding ones defined with respect to the measure dG since the two are linked by the relation

$$\|v\|_{L^2(dG)} = \|e^{iF} v\|_{L^2(drd\overline{g})} \tag{3.1.17}$$

with

$$e^{iF} = l(r)^{\frac{1-n}{2}} w^{1/2} = l(r)^{\frac{1-n}{2}} \frac{|g(r,\theta)|^{1/2}}{|\overline{g}(\theta)|^{1/2}}.$$

Indeed, let u solution of

$$(P_{\overline{g}} - \lambda^2 + i\varepsilon')u = f.$$

We recall that

$$P_{\overline{g}} = e^{iF} P_G e^{-iF}$$

so setting $\widetilde{u} = e^{-iF}u$, $\widetilde{f} = e^{-iF}f$ they are the solution and source term of the equation

$$(P_G - \lambda^2 + i\varepsilon')\widetilde{u} = \widetilde{f}.$$

We denote by $\|\cdot\|_{B^*_{>R},G}$ and $\|\cdot\|_{B_{>R},G}$ the norms defined with the same procedure as above and using the L^2 norm $\|\cdot\|_{L^2(dG)}$. From relation (3.1.17) we then have that inequality

$$\|\widetilde{u}\|_{B^*_{>R},G} \lesssim \|\widetilde{f}\|_{B_{>R},G}$$

is equivalent to

$$\|u\|_{B^*_{>R}} \lesssim \|f\|_{B_{>R}}.$$

Analogous equivalences are true when considering the norms of $h(D_r - A_0)u$ and $M(r)^{1/2}u$.

Notation. For the reasons presented in the above remark, all L^2 norms and scalar products considered in Section 3.2, which treats the manifold end $M \setminus K$, will be with respect to the measure $drd\overline{g}$. Likewise, $P_{\overline{g}}$ will be the operator used in the computations of Section 3.2.

Remark 3.1.19 ($B_{>R}$, $B^*_{>R}$ duality)*.* In the computations of the following sections we will exploit the duality of $\|\cdot\|_{B_{>R}}$ and $\|\cdot\|_{B^*_{>R}}$ in the following way. Let $\chi \in C_0^\infty(\mathbb{R})$ a cutoff in the interval $(\frac{1}{4}, 4)$ and such that $\chi \equiv 1$ on $(\frac{1}{2}, 2)$. Define $\chi_k(r) = \chi(2^{-k}r)$, then

$$\chi_k \equiv 1 \ \text{ on } supp\varphi(2^{-k}\cdot), \qquad supp\chi_k \subset [2^{k-2}, 2^{k+2}]. \tag{3.1.18}$$

We recall that ϕ is supported in $(2^{k^*+3}R, +\infty) = (2^{k_0+1}, +\infty)$ which implies $supp\varphi(2^{-k_0}\cdot) \cap supp\phi = \emptyset$ and

$$\begin{aligned}
(\phi g, f)_{drdg} &= \sum_{k \geqslant k_0+1} (\phi\varphi(2^{-k}\cdot)g, f)_{drdg} \\
&= \sum_{k \geqslant k_0+1} (r^{-1/2}\chi_k\phi\varphi(2^{-k}\cdot)g, r^{1/2}\chi_k f)_{drdg} \\
&\leqslant \sum_{k \geqslant k_0+1} \|r^{-1/2}g\|_{L^2(drd\overline{g},D_k)} \|r^{1/2}f\|_{L^2(drd\overline{g},D_k)} \\
&\leqslant \|g\|_{B^*_{>R}} \|f\|_{B_{>R}} \\
&\leqslant \frac{\delta}{2}\|g\|^2_{B^*_{>R}} + \frac{1}{2\delta}\|f\|^2_{B_{>R}}
\end{aligned} \tag{3.1.19}$$

for some $\delta \in (0, 1)$.

3.2 Estimates on $M \setminus K$

We recall Remark 3.1.18, which tells us that in the region $M \setminus K$ we can equivalently consider the operator

$$P_{\overline{g}} = h^2(D_r - A_0)^2 + M(r) + h^2(V_m - \Lambda)$$

which is symmetric with respect to the scalar product induced by the measure $drd\overline{g}$ and with a potential $V_m - \Lambda = O(r^{-\nu})$ as $r \to \infty$. We are interested in solutions of the equation $(P_m - \lambda^2 + i\varepsilon')u = f$ or equivalently $(P_{\overline{g}} - \lambda^2 + i\varepsilon')u = f$. Hence, after factorizing λ^2 we set

$$\mathcal{P} = h^2 P_{\overline{g}} - 1 + i\varepsilon. \tag{3.2.1}$$

The aim of this section is to bound $\|u\|_{H^1,B^*_{>R}}$ by $\|\mathcal{P}u\|_{B_{>R}}$ uniformly in λ and ε and up to some compactly supported remainder terms which will be treated in Section 3.3. More precisely we obtain

Proposition 3.2.1. *Let $u \in H^2(M)$, $\lambda > \lambda_0 > 0$ and $h = \lambda^{-1}$. For any $\delta \in (0, \lambda_0) \cap (0, 1)$ there exist $c(\delta) > 0$ decreasing function of δ and $K(\delta)$ bounded region of $(R, \infty) \times S$ such that*

$$\|u\|^2_{H^1,B^*_{>R}} \leqslant c\delta\|u\|^2_{H^1,B^*_{>R}} + \frac{c}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}.$$

Since the $\|\cdot\|_{H^1,B^*_{>R}}$ norm contains the contributions of the L^2 norm of the function, of the radial derivative and of the angular derivatives we will proceed in the following way:

- in Section 3.2.1 we bound the norm $\|M(r)^{1/2}u\|_{B^*_{>R}}$,
- in Section 3.2.2 we bound the norm $\|h(D_r - A_0)u\|_{B^*_{>R}}$,
- finally in Section 3.2.3 we bound the norm $\|u\|_{B^*_{>R}}$.

Notation. In all of this chapter c, C and $c(\delta)$ are constants that are allowed to change from line to line. If an integral norm is denoted with the symbol $r \simeq R$ it means that the integral is supported in a compact sub interval of $(2^{k^*}R, +\infty)$ which may vary but does not depend on any parameter. On the contrary $K(\delta)$ will be a bounded region of $(R, \infty) \times S$ depending on the parameter δ. The symbols $\lesssim, \gtrsim$ will denote inequalities holding up to a positive multiplicative constant which is independent of the parameters δ, λ or ε. With the symbol $O(a)$ we mean that there exists a positive constant c such that $O(a) = c \cdot a$.

We will use the following elementary identity that holds between two symmetric operators B, C

$$Im(Bu, Cu) = (u, \frac{[B, C]}{2i}u). \tag{3.2.2}$$

3.2.1 Estimating the angular gradient

In this section we evaluate the norm $\|M(r)^{1/2}u\|_{B^*_{>R}}$, we obtain

Proposition 3.2.2. *Let $\phi(r)$ a smooth cutoff on the interval $(2^{k_0+1}, +\infty)$, $\lambda > \lambda_0 > 0$ and $h = \lambda^{-1}$. For any $\delta \in (0, \lambda_0) \cap (0, 1)$ there exist $c(\delta) > 0$ decreasing function of δ and $K(\delta)$ bounded region of $(R, \infty) \times S$ such that*

$$\|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} \leqslant \delta c\|u\|^2_{H^1,B^*_{>R}} + \frac{c'}{\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}$$

with $c, c' > 0$ constants independent of δ.

Remark 3.2.3. The support of the H^1 norm $\|u\|_{H^1(K(\delta))}$ is a compact interval contained in $(R, +\infty)$ with upper bound depending on δ and growing as δ approaches 0. This will not cause any particular problem since, when applying the results of this section we will fix the parameter δ and this will determine a fixed (potentially large but bounded) interval for r.

We start by applying (3.2.2) to $P_{\overline{g}}$ and $(D_r - A_0)$ which are symmetric with respect to the measure $drd\overline{g}$. By definition of $\mathcal{P} = h^2P_{\overline{g}} - 1 + i\varepsilon$

$$\begin{aligned} Im((D_r - A_0)(\phi u), \mathcal{P}(\phi u))_{drd\overline{g}} =& \varepsilon((D_r - A_0)\phi u, \phi u)_{drd\overline{g}} + Im((D_r - A_0)\phi u, h^2P_{\overline{g}}\phi u)_{drd\overline{g}} \\ =& \varepsilon((D_r - A_0)\phi u, \phi u)_{drd\overline{g}} + (\phi u, \frac{[(D_r - A_0), h^2P_{\overline{g}}]}{2i}\phi u)_{drd\overline{g}} \end{aligned} \tag{3.2.3}$$

where

$$\frac{1}{2i}[(D_r - A_0), h^2 P_{\overline{g}}] = -\frac{1}{2}M'(r) - \frac{1}{2i}[A_0, M(r)] + \frac{1}{2i}h^2 D_r(V_m). \tag{3.2.4}$$

From Remark 3.1.13 we recall that $(-M'(r)(\phi u), \phi u)_{drd\overline{g}}$ can be bounded from below by $(M(r)(\phi u), \phi u)_{drd\overline{g}} = \|M(r)^{1/2}(\phi u)\|^2_{L^2(drd\overline{g})}$ (up to some additional terms). More precisely, we have

Lemma 3.2.4. *Let $(\cdot,\cdot)_{d\overline{g}}$ the scalar product on $L^2(S,\overline{g})$ and $\varphi \in L^2(S,d\overline{g})$, then*

$$(\varphi, -\partial_r M(r)\varphi)_{d\overline{g}} \gtrsim \frac{1}{r}\|M(r)^{1/2}\varphi\|^2_{L^2(d\overline{g})}(1 - O(r^{-\nu})).$$

Recall the definition of $M(r)$

$$M(r) = h^2 l^2(r)(1 + T(r)), \qquad T(r) = w^{1/2}(-\widetilde{\Delta}_{g(r)})w^{-1/2}.$$

We can prove Lemma 3.2.4 thanks to the following equivalence of norms.

Lemma 3.2.5. *Let $\varphi \in L^2(S,d\overline{g})$, then for $r > R$*

$$\|(1 + T(r))^{1/2}\varphi\|_{L^2(d\overline{g})} \simeq \|(1 - \widetilde{\Delta}_{g(r)})^{1/2}\varphi\|_{L^2(d\overline{g})}$$

that is the quotient between the right and left hand sides is bounded from above and from below.

Proof. We can rewrite $T(r)$ as

$$\begin{aligned} 1 + T(r) =& 1 - \widetilde{\Delta}_{g(r)} + w^{1/2}[-\widetilde{\Delta}_{g(r)}, w^{-1/2}] \\ =& (1 - \widetilde{\Delta}_{g(r)})^{1/2}(1 + S(r))(1 - \widetilde{\Delta}_{g(r)})^{1/2} \end{aligned} \tag{3.2.5}$$

where

$$S(r) := (1 - \widetilde{\Delta}_{g(r)})^{-1/2} w^{1/2}[-\widetilde{\Delta}_{g(r)}, w^{-1/2}](1 - \widetilde{\Delta}_{g(r)})^{-1/2}. \tag{3.2.6}$$

We have seen in Remark 3.1.13 that $-\widetilde{\Delta}_{g(r)}$ is a differential operator of order two in the angular variables and is given by $-\Delta_{g(r)}$ plus lower order terms. In particular it has an elliptic principal symbol and lower order terms with decaying coefficients. One can then construct its resolvent via parametrix with a standard procedure and obtain that $(1 - \widetilde{\Delta}_{g(r)})^{-1/2}$ is a pseudodifferential operator of order minus one in the angular variables.

Moreover, we have

$$\begin{aligned} w^{1/2}[-\widetilde{\Delta}_{g(r)}, w^{-1/2}] =& w^{1/2}[-\Delta_{g(r)}, w^{-1/2}] + l^{-2}(r)\sum_{i=1}^{n-1} A_i \frac{D_i w}{w} \\ =& w^{1/2}[-\Delta_{g(r)} + \Delta_{\overline{g}}, w^{-1/2}] + w^{1/2}[-\Delta_{\overline{g}}, w^{-1/2}] + l^{-2}(r)\sum_{i=1}^{n-1} A_i \frac{D_i w}{w} \end{aligned}$$

where all the terms are differential in the angular variables, of order one or zero and with $O(r^{-\nu})$ coefficients. This is due to the fact that w^{-1} is bounded and

$$w^{1/2} D_{i,j}(w^{-1/2}) \in S^{-\nu}, \qquad w^{1/2} D_i(w^{-1/2}) \in S^{-\nu}$$

together with the boundedness of $r^{\nu} l^{-2}(r) A_i$. We have obtained

$$\|S(r)\|_{\mathcal{L}(L^2(S,d\overline{g}))} \simeq r^{-\nu}.$$

First, since $S(r)$ is a bounded operator

$$\begin{aligned}\|(1+T(r))^{1/2}\varphi\|^2_{L^2(d\overline{g})} =&((1+S(r))(1-\widetilde{\Delta}_{g(r)})^{1/2}\varphi,(1-\widetilde{\Delta}_{g(r)})^{1/2}\varphi)_{d\overline{g}}\\ \leqslant&\|1+S(r)\|_{\mathcal{L}(L^2(S))}\|(1-\widetilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(d\overline{g})}\\ \lesssim&\|(1-\widetilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(d\overline{g})}.\end{aligned}$$

We conclude using the lower bound $(S(r)v,v)_{d\overline{g}} \geqslant -\|S(r)\|_{\mathcal{L}(L^2(S,d\overline{g}))}\|v\|^2_{L^2(d\overline{g})}$ in

$$\begin{aligned}\|(1+T(r))^{1/2}\varphi\|^2_{L^2(d\overline{g})} =&\|(1-\widetilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(d\overline{g})}+(S(r)(1-\widetilde{\Delta}_{g(r)})^{1/2}\varphi,(1-\widetilde{\Delta}_{g(r)})^{1/2}\varphi)_{d\overline{g}}\\ \geqslant&\|(1-\widetilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(d\overline{g})}(1-\|S(r)\|_{\mathcal{L}(L^2(S,d\overline{g}))})\end{aligned}$$

and recalling that for large enough r

$$1-\|S(r)\|_{\mathcal{L}(L^2(S,d\overline{g}))} > c > 0$$

with c independent of r.

□

Proof of Lemma 3.2.4. By definition of M

$$\begin{aligned}M'(r) =&h^2 2l'(r)l(r)(1+T(r))+h^2l^2(r)T'(r)\\ =&2\frac{l'(r)}{l(r)}M(r)+h^2l^2(r)T'(r)\end{aligned}$$

so thanks to $-l'(r)/l(r) \gtrsim r^{-1}$

$$\begin{aligned}(\varphi,-\partial_r M(r)\varphi)_{d\overline{g}} =&(\varphi,-2\frac{l'(r)}{l(r)}M(r)\varphi)_{d\overline{g}}+(\varphi,-h^2l^2(r)T'(r)\varphi)_{d\overline{g}}\\ \gtrsim&\frac{2}{r}\|M(r)^{1/2}\varphi\|^2_{L^2(d\overline{g})}+(\varphi,-h^2l^2(r)T'(r)\varphi)_{d\overline{g}}.\end{aligned} \tag{3.2.7}$$

Using the expression in (3.2.5)

$$T'(r)=\partial_r(-\widetilde{\Delta}_{g(r)})+\partial_r(w^{1/2}[-\widetilde{\Delta}_{g(r)},w^{-1/2}]).$$

From the arguments in the proof of the previous lemma, differentiating in r the coefficients of $w^{1/2}[-\widetilde{\Delta}_{g(r)},w^{-1/2}]$ yields a differential operator of order one in the angular variables with coefficients of order $O(r^{-\nu-1})$. Moreover $(-\widetilde{\Delta}_{g(r)})$ has coefficients whose expressions contain $g^{i,j}(r,\theta)$ and $\widetilde{A}_i$. By definition of $\widetilde{A}_i$ we compute

$$\begin{aligned}\partial_r\widetilde{A}_i =&-2l^{-2}(r)\frac{l'(r)}{l(r)}\sum_{i,j}g_{i,j}(r,\theta)A_j+l^{-2}(r)\sum_{i,j}\partial_r g_{i,j}(r,\theta)A_j\\ &+l^{-2}(r)\sum_{i,j}g_{i,j}(r,\theta)\partial_r A_j\end{aligned}$$

where

$$r^{\nu}l^{-2}(r)A_j,\quad r^{1+\nu}\partial_r g_{i,j}(r,\theta),\quad r^{1+\nu}l^{-2}(r)\partial_r A_j$$

are bounded functions. Then, thanks again to $-l'(r)/l(r) \gtrsim r^{-1}$ we can bound from below the derivative of $\widetilde{A}_i$

$$-\partial_r\widetilde{A}_i \gtrsim -\frac{1}{r^{\nu+1}}b(r,\theta)$$

for some $b \in L^\infty((R,+\infty)\times S)$. Since $\partial_r g^{i,j}(r,\theta) \in S^{-\nu-1}$ we can conclude that $-\partial_r(-\widetilde{\Delta}_{g(r)})$ is a differential operator of order two whose coefficients can be bounded from below by $-r^{-\nu-1}b(r,\theta)$ for some bounded b. We can then lower bound the scalar product

$$(\varphi, -h^2 l^2(r) T'(r)\varphi)_{d\overline{g}} \gtrsim -\frac{1}{r^{\nu+1}}(\varphi, T_2 h^2 l^2(r)\varphi)_{d\overline{g}}$$

where T_2 is a differential operator of order two in the angular variables with bounded coefficients. It contains the contributions of $r^{\nu+1}\partial_r(-\widetilde{\Delta}_{g(r)})$ and $r^{\nu+1}\partial_r(w^{1/2}[-\widetilde{\Delta}_{g(r)}, w^{-1/2}])$. Then from

$$(1-\widetilde{\Delta}_{g(r)})^{-1/2} T_2 (1-\widetilde{\Delta}_{g(r)})^{-1/2} \in \mathcal{L}(L^2(S, d\overline{g})) \tag{3.2.8}$$

we obtain

$$\begin{aligned} -(\varphi, h^2 l^2(r) T'(r)\varphi)_{d\overline{g}} &\gtrsim -\frac{1}{r^{\nu+1}}\|hl(r)(1-\widetilde{\Delta}_{g(r)})^{1/2}\varphi\|^2_{L^2(S,d\overline{g})} \\ &\simeq -\frac{1}{r^{\nu+1}}\|M(r)^{1/2}\varphi\|^2_{L^2(d\overline{g})} \end{aligned}$$

up to some constant given by the norm of the operator in (3.2.8) and thanks to Lemma 3.2.5. The norm of (3.2.8) depends on r, however we can obtain a fixed constant by taking the supremum over all r, since the coefficients of T_2 are bounded in r. We conclude by using this lower bound in (3.2.7) which yields

$$(\varphi, -\partial_r M(r)\varphi)_{d\overline{g}} \gtrsim \frac{1}{r}\|M(r)^{1/2}\varphi\|^2_{L^2(d\overline{g})}(1-O(r^{-\nu})).$$

□

We recall identities (3.2.3) and (3.2.4) which yield

$$\begin{aligned} Im((D_r - A_0)(\phi u), \mathcal{P}(\phi u))_{drd\overline{g}} =& \varepsilon((D_r - A_0)\phi u, \phi u)_{drd\overline{g}} + \frac{1}{2}(\phi u, -M'(r)\phi u)_{drd\overline{g}} \\ &- \frac{1}{2i}(\phi u, [A_0, M(r)](\phi u))_{drd\overline{g}} + \frac{1}{2i}(\phi u, h^2 D_r(V_m)\phi u)_{drd\overline{g}}. \end{aligned} \tag{3.2.9}$$

The function $M(r)^{1/2}\phi u$ is supported in $supp\phi \subset (2^{k_0+1},+\infty)$, we apply Lemma 3.2.4 in this region

$$\begin{aligned} (\phi u, -M'(r)\phi u)_{drd\overline{g}} \gtrsim& \int_{2^{k_0+1}}^\infty \frac{1}{r}(1-O(r^{-\nu}))\|M(r)^{1/2}\phi u\|^2_{L^2(d\overline{g})}dr \\ =& \|r^{-1/2}M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g})} - \|r^{-1/2}O(r^{-\nu/2})M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g})}. \end{aligned}$$

For any $k \geqslant k_0$

$$\|r^{-1/2}M(r)^{1/2}\phi u\|_{L^2(drd\overline{g})} \geqslant \|r^{-1/2}M(r)^{1/2}\phi u\|_{L^2(drd\overline{g},D_k)}$$

where we recall $D_k = [2^{k-1}, 2^{k+1}]$ (in particular $[2^{k_0-1}, 2^{k_0+1}] \cap (2^{k_0+1},+\infty) = \emptyset$, $[2^{k_0}, 2^{k_0+2}] \cap (2^{k_0+1},+\infty) \neq \emptyset$ and $[2^{k-1}, 2^{k+1}] \subset (2^{k_0+1},+\infty)$ for all $k \geqslant k_0+2$). We have then obtained

$$\begin{aligned} (\phi u, -M'(r)\phi u)_{drd\overline{g}} \gtrsim& \sup_{k\geqslant k_0} \|r^{-1/2}M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g},D_k)} \\ &- \|r^{-1/2}O(r^{-\nu/2})M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g})} \\ =& \|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} - \|r^{-1/2}O(r^{-\nu/2})M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g})} \end{aligned}$$

which implies, thanks to (3.2.9)

$$\|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} \lesssim Im((D_r - A_0)(\phi u), \mathcal{P}(\phi u))_{drd\overline{g}} - \varepsilon((D_r - A_0)\phi u, \phi u)_{drd\overline{g}} + \frac{1}{2}|(\phi u, [A_0, M(r)]\phi u)_{drd\overline{g}}| + \frac{1}{2}(\phi u, h^2 D_r(V_m)\phi u)_{drd\overline{g}} + \|r^{-1/2}O(r^{-\nu/2})M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g})}. \tag{3.2.10}$$

In order to prove Proposition 3.2.1 we will need a series of intermediate lemmas to bound each term in (3.2.10). Notably, Proposition 3.2.1 is obtained directly from the following inequalities.

Lemma 3.2.6. *Let $\phi(r)$ a smooth cutoff on the interval $(2^{k_0+1}, +\infty)$, $\lambda > \lambda_0 > 0$ and $h = \lambda^{-1}$. Let $\delta \in (0, \lambda_0) \cap (0, 1)$. The following inequalities hold*

i)
$$|Im((D_r - A_0)(\phi u), \mathcal{P}(\phi u))_{drd\overline{g}}| \leqslant \frac{\delta}{2}\|u\|^2_{H^1, B^*_{>R}} + \frac{1}{2\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}},$$

ii)
$$|\varepsilon((D_r - A_0)\phi u, \phi u)_{drd\overline{g}}| \leqslant \delta c\|u\|^2_{B^*_{>R}} + \frac{c}{\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}},$$

iii)
$$|(\phi u, [A_0, M(r)]\phi u)_{drd\overline{g}}| \leqslant \delta c\|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))},$$

iv)
$$|(\phi u, h^2 D_r(V_m)\phi u)_{drd\overline{g}}| \leqslant \delta c\|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{L^2(K(\delta))}$$

We postpone the proofs of these present inequalities to the next section.

Proof of Proposition 3.2.2. Let $\delta < \lambda_0$ as in the statement of Proposition 3.2.2. Thanks to Lemma 3.2.6

$$\|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} \lesssim c\delta\|u\|^2_{H^1, B^*_{>R}} + \frac{c'}{\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))} + \|r^{-\nu/2}r^{-1/2}M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g})}.$$

We use the partition of unity $\varphi(2^{-k}\cdot)$ to decompose the following L^2 norm

$$\|r^{-\nu/2}r^{-1/2}M(r)^{1/2}\phi u\|_{L^2(drd\overline{g})} \leqslant \sum_{k\geqslant k_0+1} \|r^{-\nu/2}r^{-1/2}\chi_k M(r)^{1/2}\phi u\|_{L^2(drd\overline{g})}$$

where we recall $\chi_k \equiv 1$ on the support of $\varphi(2^{-k}\cdot)$. We can write k in the form $k = k_0 + m + 1$ with $m = m(\delta) \geqslant 0$, then on the support of χ_k we have $r = O(2^m)$ (i.e. $r = c2^m$ for a positive constant c) and hence

$$r^{-\nu/2} = O(2^{-\nu m/2}).$$

We can choose $m^*(\delta) \geqslant 0$ such that $\sum_{m\geqslant m^*} 2^{-m\nu/2} < \delta$, then we conclude the proof since

$$\begin{aligned}\|r^{-1/2}O(r^{-\nu/2})M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g})} &\lesssim \sum_{\substack{m\geqslant 0\\ k=k_0+1+m}} 2^{-m\nu}\|r^{-1/2}M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g}, D_k)} \\ &\lesssim \sum_{\substack{m=0\\ k=k_0+1+m}}^{m^*} 2^{-m\nu}\|r^{-1/2}M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g}, D_k)} \\ &\quad + \sum_{\substack{m\geqslant m^*\\ k=k_0+1+m}} 2^{-m\nu}\|r^{-1/2}M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g}, D_k)}\end{aligned}$$

$$\lesssim (\delta \|M(r)^{1/2}\phi u\|^2_{B^*_{>R}} + c(\delta)\|M(r)^{1/2}\phi u\|^2_{L^2(drd\overline{g},K(\delta))})$$
$$\lesssim c\delta\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}$$

where the norm on the compact set $K(\delta)$ contains all the contributions of the sum for m from zero to m^*. □

The following lemma allows us to evaluate the commutator between $\mathcal{P}$ and ϕ which will yield our final bound.

Lemma 3.2.7. *Let $\phi(r)$ a smooth cutoff on the interval $(2^{k_0+1}, +\infty)$, $\lambda > \lambda_0 > 0$ and $h = \lambda^{-1}$. Then*

$$\|\mathcal{P}(\phi u)\|_{B_{>R}} \leqslant \|\mathcal{P}u\|_{B_{>R}} + ch^2\|u\|_{H^1(r\simeq R)},$$

for a constant c depending on R and on the size of the interval $\{r : \ \phi(r) \in (0,1)\}$.

We recall that $M(r)^{1/2}$ commutes with functions of r. Applying the previous result and Proposition 3.2.2 we find that for any $\delta \in (0, \lambda_0)$ there exists $c(\delta) > 1$ such that

$$\|M(r)^{1/2}u\|^2_{B^*_{>R}} \leqslant \frac{c'}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + \delta c\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))} \tag{3.2.11}$$

for constants $c, c' > 0$ independent of δ.

Proof of Lemma 3.2.7. Since ϕ is a function of r only

$$[\mathcal{P}, \phi] = [h^2 P_{\overline{g}}, \phi] = [h^2(D_r - A_0)^2, \phi] = h^2(D_r^2(\phi) + 2D_r(\phi)D_r - 2A_0 D_r(\phi))$$

with $D_r(\phi), D_r^2(\phi) \in C_0^\infty((2^{k_0+1}, +\infty))$. In particular in

$$\begin{aligned}\|\mathcal{P}(\phi u)\|_{B_{>R}} \leqslant& \|\mathcal{P}u\|_{B_{>R}} + \|[\mathcal{P}, \phi]u\|_{B_{>R}} \\ \leqslant& \|\mathcal{P}u\|_{B_{>R}} + \| + \sum_{k\geqslant k_0} \|h^2 r^{1/2}(D_r^2(\phi) - 2A_0 D_r(\phi))u\|_{L^2(drd\overline{g},D_k)} \\ &+ \sum_{k\geqslant k_0} \|h^2 r^{1/2} 2D_r(\phi)D_r u\|_{L^2(drd\overline{g},D_k)}\end{aligned}$$

the sums only have a finite number on non vanishing terms, since the supports of $D_r(\phi)$ and $D_r^2(\phi)$ intersect only a finite number of intervals $[2^{k-1}, 2^{k+1}]$. So there exists $\alpha > 0$ (depending on the fixed size of the interval $\{r : \ \phi(r) \in (0,1)\}$) such that in the above L^2 norms

$$r \leqslant 2^{k_0+\alpha+1} = 2^{k^*+\alpha}R.$$

Hence bounding $r^{1/2}$ and the L^∞ functions $D_r(\phi), D_r^2(\phi), A_0$ we have

$$\|\mathcal{P}(\phi u)\|_{B_{>R}} \lesssim \|\mathcal{P}u\|_{B_{>R}} + h^2(\|u\|_{L^2(r\simeq R)} + \|D_r u\|_{L^2(r\simeq R)}).$$

□

3.2.1.1 Auxiliary lemmas

In this subsection we prove the inequalities stated in Lemma 3.2.6. Given the statement of Proposition 3.2.2 the terms allowed in the bound are going to be $\|\mathcal{P}(\phi u)\|_{B^*_{>R}}$, compactly supported terms (like $c(\delta)\|u\|_{H^1(r\simeq R)}$) and absorbable ones (like $\delta c\|u\|^2_{H^1,B^*_{>R}}$). We recall that everywhere in this section ϕ is a smooth cutoff on the interval $(2^{k_0+1}, \infty)$ and which is equal to one on $(2^{k_0+2}, \infty)$.

We start with the first result which we obtain simply by duality of the norms.

Lemma 3.2.8. *Let* $\delta \in (0,1)$,

$$|Im((D_r - A_0)(\phi u), \mathcal{P}(\phi u))_{dr d\overline{g}}| \leqslant \frac{\delta}{2}\|u\|^2_{H^1, B^*_{>R}} + \frac{1}{2\delta h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}.$$

Proof. We multiply and divide by $h^2 r^{1/2}$ and since $supp(D_r - A_0)(\phi u) \subset supp\ \phi u$ we can reason by duality as in (3.1.19). □

For $\varepsilon((D_r - A_0)\phi u, \phi u)_{dr d\overline{g}}$ we will bound separately $\varepsilon\|\phi u\|_{L^2(dr d\overline{g})}$ and $\varepsilon\|h(D_r - A_0)(\phi u)\|_{L^2(dr d\overline{g})}$. For example in

$$(\phi u, \mathcal{P}(\phi u))_{dr d\overline{g}} = i\varepsilon(\phi u, \phi u)_{dr d\overline{g}} - \|\phi u\|^2_{L^2(dr d\overline{g})} + (\phi u, h^2 P_{\overline{g}}(\phi u))_{dr d\overline{g}}$$

$(\phi u, h^2 P_{\overline{g}}(\phi u))_{dr d\overline{g}}$ is real since $P_{\overline{g}}$ is symmetric with respect to the measure $dr d\overline{g}$. Hence from

$$Im(\phi u, \mathcal{P}(\phi u))_{dr d\overline{g}} = \varepsilon\|\phi u\|^2_{L^2(dr d\overline{g})}$$

we can prove again by duality of the norms

Lemma 3.2.9. *Let* $\delta \in (0,1)$, *then*

$$\varepsilon\|\phi u\|^2_{L^2(dr d\overline{g})} \leqslant \frac{\delta}{2}\|u\|^2_{B^*_{>R}} + \frac{1}{2\delta}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}.$$

Proof. The proof is analogous to Lemma 3.2.8. □

On the other hand, taking the real part we use

$$(\phi u, h^2 P_{\overline{g}}(\phi u))_{dr d\overline{g}} = Re(\phi u, \mathcal{P}(\phi u))_{dr d\overline{g}} + \|\phi u\|^2_{L^2(dr d\overline{g})} \tag{3.2.12}$$

to prove the following.

Lemma 3.2.10. *For any* $\delta \in (0,1)$, *then*

$$\varepsilon\|h(D_r - A_0)(\phi u)\|^2_{L^2(dr d\overline{g})} \leqslant \delta c\|u\|^2_{B^*_{>R}} + \frac{1}{\delta}c\|\mathcal{P}(\phi u)\|^2_{B_{>R}}$$

with $c > 0$ *independent of* δ.

Proof. By definition of $P_{\overline{g}} = h^2(D_r - A_0)^2 + M(r) + h^2(V_m - \Lambda)$

$$\begin{aligned}(\phi u, h^2 P_{\overline{g}}(\phi u))_{dr d\overline{g}} =& \|h(D_r - A_0)(\phi u)\|^2_{L^2(dr d\overline{g})} + \|M(r)^{1/2}(\phi u)\|^2_{L^2(dr d\overline{g})} \\ &+ (\phi u, h^2(V_m - \Lambda)(\phi u))_{dr d\overline{g}}.\end{aligned}$$

hence from (3.2.12)

$$\begin{aligned}\varepsilon\|h(D_r - A_0)(\phi u)\|^2_{L^2(dr d\overline{g})} \leqslant& \varepsilon Re(\phi u, \mathcal{P}(\phi u))_{dr d\overline{g}} + \varepsilon\|\phi u\|^2_{L^2(dr d\overline{g})} \\ &- \varepsilon(\phi u, h^2(V_m - \Lambda)\phi u))_{dr d\overline{g}}.\end{aligned} \tag{3.2.13}$$

Again by the duality in (3.1.19)

$$\varepsilon|Re(\phi u, \mathcal{P}(\phi u))_{dr d\overline{g}}| \leqslant \frac{\varepsilon\delta}{2}\|u\|^2_{B^*_{>R}} + \frac{\varepsilon}{2\delta}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}$$

while $\varepsilon\|\phi u\|^2_{L^2(dr d\overline{g})}$ can be bounded via Lemma 3.2.9, which we also use in

$$|\varepsilon(\phi u, h^2(V_m - \Lambda)\phi u)_{dr d\overline{g}}| \leqslant h^2\varepsilon\|V_m - \Lambda\|_\infty\|\phi u\|^2_{L^2(dr d\overline{g})}.$$

□

With Lemmas 3.2.9 and 3.2.10 we bound the term $\varepsilon((D_r - A_0)\phi u, \phi u)_{dr d\bar{g}}$.

Lemma 3.2.11. *For any $\mu \in (0,1)$ such that $\mu < \lambda_0$*

$$|\varepsilon((D_r - A_0)\phi u, \phi u)_{dr d\bar{g}}| \leqslant \mu c \|u\|^2_{B^*_{>R}} + \frac{c}{\mu h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}$$

for some $c > 0$ independent of h, μ.

Proof. Applying Lemmas 3.2.9 and 3.2.10 with $\delta = h\mu$, which belongs to $(0,1)$ since $\mu < \lambda_0$

$$\begin{aligned}|\varepsilon((D_r - A_0)\phi u, \phi u)_{dr d\bar{g}}| \leqslant& \frac{\varepsilon}{2h}\|h(D_r - A_0)(\phi u)\|^2_{L^2(dr d\bar{g})} + \frac{\varepsilon}{2h}\|\phi u\|^2_{L^2(dr d\bar{g})}\\ \leqslant& \mu c \|u\|^2_{B^*_{>R}} + \frac{c}{\mu h^2}\|\mathcal{P}(\phi u)\|^2_{B_{>R}}\end{aligned}$$

where c_1 given by the previous lemma is bounded uniformly in h. □

We now need to bound the term

$$(\phi u, [A_0, M(r)]\phi u)_{dr d\bar{g}} = (\phi u, h^2 l^2(r)[A_0, T(r)]\phi u)_{dr d\bar{g}}$$

In the proof of Lemma 3.2.5 we remarked some useful properties on $T(r)$ in particular that

$$T(r) = -\widetilde{\Delta}_{g(r)} + w^{1/2}[-\widetilde{\Delta}_{g(r)}, w^{-1/2}]$$

where $-\widetilde{\Delta}_{g(r)}$ is the sum of $-\Delta_{g(r)}$ with lower order perturbations, and $w^{1/2}[-\widetilde{\Delta}_{g(r)}, w^{-1/2}]$ is a differential operator of order one with coefficients decaying like $O(r^{-\nu})$.

Lemma 3.2.12. *For any $\delta \in (0,1)$ there exists $c(\delta) > 0$ decreasing function of δ such that*

$$|(\phi u, [A_0, M(r)]\phi u)_{dr d\bar{g}}| \leqslant \delta c \|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}$$

where $c > 0$ is independent of δ.

Proof. We need to evaluate the commutator of A_0 with $-\Delta_{g(r)}$ and with differential operators of order one in the angular variables whose coefficients decay radially. First of all

$$[A_0, -\Delta_{g(r)}]$$

is a differential operator of order one with coefficients in $S^{-\nu}$, since $A_0 \in S^{-\nu}$, while the commutator of A_0 with a differential operator of order one with bounded coefficients is going to be a function in $S^{-\nu}$.

As we pointed out in the proof of Lemma 3.2.5 $(1 - \widetilde{\Delta}_{g(r)})^{-1/2}$ is a pseudodifferential operator of order minus one in the angular variables. Then

$$\|r^\nu[A_0, -\Delta_{g(r)}](1 - \widetilde{\Delta}_{g(r)})^{-1/2}\|_{\mathcal{L}(L^2(dr d\bar{g}))} = O(1) \tag{3.2.14}$$

as well as

$$\|r^\nu[A_0, -\widetilde{\Delta}_{g(r)} + \Delta_{g(r)}]\|_{\mathcal{L}(L^2(dr d\bar{g}))} = O(1) \tag{3.2.15}$$

and

$$\|r^\nu[A_0, w^{1/2}[-\widetilde{\Delta}_{g(r)}, w^{-1/2}]]\|_{\mathcal{L}(L^2(dr d\bar{g}))} = O(1).. \tag{3.2.16}$$

Recalling $l(r) \lesssim r^{-1}$

$$\begin{aligned}
|(\phi u, h^2 l^2(r)[A_0, -\Delta_{g(r)}]\phi u)_{dr d\overline{g}}| &\leqslant |(r^{-\nu} r^{-1/2} \phi u, h^2 r^{-1/2} r^{\nu} [A_0, -\Delta_{g(r)}] l(r) \phi u)_{dr d\overline{g}}| \\
&\leqslant \sum_{k \geqslant k_0+1} (|r^{-1/2-\nu} \varphi(2^{-k} r) \chi_k \phi u|, |h^2 r^{-1/2} r^{\nu} [A_0, -\Delta_{g(r)}] l(r) \chi_k \phi u|)_{dr d\overline{g}}
\end{aligned}$$

where we introduce χ_k, functions which are equal to one on the support of $\varphi(2^{-k}\cdot)$. Write $k \geqslant k_0$ in the form

$$k = k_0 + m + 1 \qquad m = m(k) \geqslant 0$$

recalling that *supp* $\chi_k = [2^{k-2}, 2^{k+2}]$, inequality

$$r^{-1} \leqslant 2^{-k_0} 2^{-m+1} \tag{3.2.17}$$

holds on the support of χ_k. Inserting $(1 - \widetilde{\Delta}_{g(r)})^{-1/2}$ and recalling (3.2.14)

$$\begin{aligned}
|(\phi u, h^2 l^2(r)[A_0, -\Delta_{g(r)}]\phi u)_{dr d\overline{g}}| &\lesssim h \sum_{\substack{m \geqslant 0 \\ k = k_0+1+m}} \Big(2^{-m\nu+\nu} \|r^{-1/2} u\|_{L^2(dr d\overline{g}, D_k)} \\
&\qquad \cdot \|r^{-1/2} h l (1 - \widetilde{\Delta}_{g(r)})^{1/2} \phi u\|_{L^2(dr d\overline{g}, D_k)} \Big) \\
&\lesssim h \sum_{\substack{m \geqslant 0 \\ k = k_0+1+m}} \Big(2^{-m\nu+\nu} \|r^{-1/2} u\|_{L^2(dr d\overline{g}, D_k)} \\
&\qquad \cdot \|r^{-1/2} M(r)^{1/2} \phi u\|_{L^2(dr d\overline{g}, D_k)} \Big)
\end{aligned}$$

where we used Lemma 3.2.5 in the last inequality. We can choose $m^* = m^*(\delta)$ such that $\sum_{m \geqslant m^*} 2^{-m\nu} < \delta$ in order to bound the tail of the series by the $B^*_{>R}$ norms, while the remaining terms will be norms supported on compact intervals (depending on $m^*(\delta)$)

$$\begin{aligned}
|(\phi u, h^2 l^2(r)[A_0, -\Delta_{g(r)}]\phi u)_{dr d\overline{g}}| \lesssim & h\delta (\|u\|^2_{B^*_{>R}} + \|M(r)^{1/2} \phi u\|^2_{B^*_{>R}}) \\
& + h c(\delta) (\|u\|^2_{L^2(K(\delta))} + \|M(r)^{1/2} \phi u\|^2_{L^2(K(\delta))}) \\
\lesssim & h\delta \|u\|^2_{H^1, B^*_{>R}} \\
& + h c(\delta) (\|u\|^2_{L^2(K(\delta))} + \|M(r)^{1/2} \phi u\|^2_{L^2(K(\delta))})
\end{aligned}$$

Here the constant $c(\delta)$ includes the sum $\sum_{m=0}^{m^*} 2^{-m\nu}$, hence it is a constant which grows as δ approaches 0.

Thanks to (3.2.15) and (3.2.16) we can proceed analogously to bound

$$|(\phi u, h^2 l^2(r)[A_0, -\widetilde{\Delta}_{g(r)} + \Delta_{g(r)}]\phi u)_{dr d\overline{g}}|$$

and

$$|(\phi u, h^2 l^2(r)[A_0, w^{1/2}[-\widetilde{\Delta}_{g(r)}, w^{-1/2}]]\phi u)_{dr d\overline{g}}|$$

where this time we do not need to insert $(1 - \widetilde{\Delta}_{g(r)})^{-1/2}$ since the commutators are of order zero. This yields

$$|(\phi u, h^2 l^2(r)[A_0, -\widetilde{\Delta}_{g(r)} + \Delta_{g(r)}]\phi u)_{dr d\overline{g}}| \lesssim h R^{-\nu} (\delta \|u\|^2_{B^*_{>R}} + c(\delta) \|u\|^2_{L^2(K(\delta))})$$

and

$$|(\phi u, h^2 l^2(r)[A_0, w^{1/2}[-\widetilde{\Delta}_{g(r)}, w^{-1/2}]]\phi u)_{drd\overline{g}}| \lesssim h R^{-\nu}(\delta \|u\|^2_{B^*_{>R}} + c(\delta)\|u\|^2_{L^2(K(\delta))})$$

which conclude the proof.

□

We conclude the section with the proof of the last item in 3.2.6.

Lemma 3.2.13. *For any $\delta \in (0,1)$ there exists $c(\delta) > 0$ with $c(\delta)$ decreasing function of δ such that*

$$|(\phi u, h^2 D_r(V_m)\phi u)_{drd\overline{g}}| \leqslant \delta c \|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{L^2(K(\delta))}$$

for some constant $c > 0$ independent of δ.

Proof. We proceed similarly to the previous proof. We write $k \geqslant k_0$ in the form $k = k_0 + m + 1$ for $m = m(k) \geqslant 0$ and since $r^{1+\nu} D_r(V_m)$ is bounded thanks to (3.1.13) we first have

$$\begin{aligned}
|(\phi u, h^2 D_r(V_m)\phi u)_{drd\overline{g}}| \leqslant & h^2 \sum_{k \geqslant k_0+1} |(r^{-\nu}\varphi(2^{-k}\cdot) r^{-1/2}\chi_k \phi u, r^{1+\nu} D_r(V_m) r^{-1/2}\chi_k \phi u)_{drd\overline{g}}| \\
\lesssim & h^2 2^\nu R^{-\nu} \sum_{\substack{m \geqslant 0 \\ k = k_0+1+m}} 2^{-m\nu} \|r^{-1/2} u\|^2_{L^2(drd\overline{g}, D_k)}
\end{aligned}$$

where we have used as in (3.2.17) that $r \leqslant O(2^{-m})$. Choosing $m^* = m^*(\delta)$ such that $\sum_{m \geqslant m^*} 2^{-m\nu} < \delta$ we have

$$|(\phi u, h^2 D_r(V_m)\phi u)_{drd\overline{g}}| \lesssim h^2 2^\nu R^{-\nu}(\delta \|u\|^2_{B^*_{>R}} + c(\delta)\|u\|^2_{L^2(K(\delta))})$$

□

3.2.2 Estimating the radial derivative

In this section we give estimates on the radial part of the $H^1, B^*_{>R}$ norm, that is $\|h(D_r - A_0)u\|_{B^*_{>R}}$. More precisely we prove

Proposition 3.2.14. *For any $\delta \in (0,1)$ there exist $c(\delta)$ and a set $K(\delta)$ such that*

$$\begin{aligned}
\|h(D_r - A_0)u\|^2_{B^*_{>R}} \leqslant & c\delta \|u\|^2_{H^1, B^*_{>R}} + c' \frac{1}{\delta h^2} \|\mathcal{P} u\|^2_{B_{>R}} \\
& + c(\delta)\|u\|^2_{H^1(r \simeq R)} + \|u\|^2_{H^1(K(\delta))}
\end{aligned}$$

with $c(\delta) > 1$ and $K(\delta)$ which increase as δ approaches 0.

We start again by considering the imaginary part of a scalar product. Let a a function of r only. We apply property (3.2.2) to the symmetric operators $P_{\overline{g}}$ and

$$\frac{a(D_r - A_0) + (D_r - A_0)a}{2} = a(D_r - A_0) + \frac{a'}{2i}.$$

This yields

$$\begin{aligned}
Im\left(\frac{a(D_r - A_0) + (D_r - A_0)a}{2} u, \mathcal{P} u\right)_{drd\overline{g}} = & \varepsilon \left(\frac{a(D_r - A_0) + (D_r - A_0)a}{2} u, u\right)_{drd\overline{g}} \\
& + \left(u, \frac{1}{2i}[\frac{a(D_r - A_0) + (D_r - A_0)a}{2}, h^2 P_{\overline{g}}]u\right)_{drd\overline{g}},
\end{aligned} \tag{3.2.18}$$

where we remark that

$$\left[\frac{a(D_r-A_0)+(D_r-A_0)a}{2}, h^2(D_r-A_0)^2\right] = [a, h^2(D_r-A_0)^2](D_r-A_0) + \frac{1}{2i}[a', h^2(D_r-A_0)^2]$$

is a differential operator of order two with coefficients depending on a, a', a'', a'''. In particular

$$\begin{aligned}\left[\frac{a(D_r-A_0)+(D_r-A_0)a}{2}, h^2(D_r-A_0)^2\right] &= 2a''h^2(D_r-A_0) - 2\frac{a'}{i}h^2(D_r-A_0)^2 + \frac{1}{2i}h^2a''' \\ &= -\frac{2}{i}D_r(a')h^2(D_r-A_0) - \frac{2}{i}a'h^2(D_r-A_0)^2 + \frac{1}{2i}h^2a''' \\ &= -\frac{2}{i}h(D_r-A_0)(a'h(D_r-A_0)) + \frac{1}{2i}h^2a''',\end{aligned}$$

so that if we were to integrate the commutator against u, as in (3.2.18),

$$\begin{aligned}(u, \frac{1}{2i}\left[\frac{a(D_r-A_0)+(D_r-A_0)a}{2}, h^2(D_r-A_0)^2\right]u)_{drd\overline{g}} &= (u, h(D_r-A_0)(a'h(D_r-A_0))u)_{drd\overline{g}} \\ &\quad - \frac{h^2}{4}(u, a'''u)_{drd\overline{g}} \\ &= \|(a')^{1/2}h(D_r-A_0)u\|^2_{L^2(drd\overline{g})} \\ &\quad - \frac{h^2}{4}(u, a'''u)_{drd\overline{g}} \end{aligned} \tag{3.2.19}$$

provided that a' is non negative.

Choosing a such that a' is a cutoff on a dyadic interval the previous identity provides us with $\|h(D_r-A_0)u\|_{L^2(drd\overline{g},D_k)}$ which are the norms we need to consider to estimate $\|h(D_r-A_0)u\|_{B^*_{>R}}$.

Given this remark we consider a_k the primitive of χ_k^2, we recall that χ_k is supported on $[2^{k-2}, 2^{k+2}]$ and equal to one on the support of $\varphi(2^{-k}\cdot)$. Choosing $a = a_k$ in (3.2.18) and using (3.2.19)

$$\begin{aligned}\left(u, \frac{1}{2i}[\frac{a_k(D_r-A_0)+(D_r-A_0)a_k}{2}, h^2P_{\overline{g}}]u\right)_{drd\overline{g}} &= (u, h^2(D_r-A_0)(a_k'(D_r-A_0))u)_{drd\overline{g}} \\ &\quad - \frac{h^2}{4}(u, a_k'''u)_{drd\overline{g}} \\ &\quad + (u, -\frac{1}{2}a_kM'(r)u)_{drd\overline{g}} \\ &\quad - (u, \frac{1}{2i}a_k[A_0, M(r)]u)_{drd\overline{g}} \\ &\quad + \frac{1}{2i}(u, h^2a_kD_r(V_m)u)_{drd\overline{g}} \\ &\geqslant \|\chi_kh(D_r-A_0)u\|^2_{L^2(drd\overline{g})} \\ &\quad - \frac{h^2}{4}(u, \chi_k''u)_{drd\overline{g}} - (u, \frac{1}{2i}a_k[A_0, M(r)]u)_{drd\overline{g}} \\ &\quad + (u, \frac{1}{2i}h^2a_kD_r(V_m)u)_{drd\overline{g}}\end{aligned} \tag{3.2.20}$$

where $(u, -a_kM'(r)u)_{drd\overline{g}} \geqslant 0$ thanks to Lemma 3.2.4 and the non negativity of a_k. Additionally

$$0 \leqslant a_k(r) = \int \chi_k^2(s)ds = \int_{2^{k-2}}^{2^{k+2}} \chi_k^2(s)ds \leqslant \frac{15}{4}2^k \tag{3.2.21}$$

and without loss of generality we can assume

$$supp\ a_k \subset [2^{k-2}, +\infty)$$

since $supp \chi_k \subset [2^{k-2}, 2^{k+2}]$.

Remark 3.2.15. We recall that

$$\sum_{l \geqslant k_0} \varphi(2^{-l}r) = 1 \quad \text{on} \quad (2^{k_0}, +\infty)$$

and $supp\ a_k \subset [2^{k-2}, +\infty) \subset [2^{k_0}, +\infty)$ for any $k \geqslant k_0 + 2$ on top of the fact that $\chi_l \equiv 1$ on the support of $\varphi(2^{-l}\cdot)$. For all $k \geqslant k_0 + 2$ we can then write

$$(a_k \cdot, \cdot)_{drd\overline{g}} \leqslant \sum_{l \geqslant k_0} (\varphi(2^{-l}r) a_k \cdot, \cdot)_{drd\overline{g}} \leqslant (\chi_{k_0} a_k \cdot, \chi_{k_0} \cdot)_{drd\overline{g}} + \sum_{l \geqslant k_0+1} (\varphi(2^{-l}r)\chi_l a_k \cdot, \chi_l \cdot)_{drd\overline{g}} \quad (3.2.22)$$

On the other hand $supp\ a_{k_0+1} \subset [2^{k_0-1}, +\infty)$ and

$$\sum_{k \geqslant k_0} \chi_k^2 \geqslant 1 \quad \text{on} \quad [2^{k_0-1}, +\infty),$$

hence we also have

$$\begin{aligned}(a_{k_0+1} \cdot, \cdot)_{drd\overline{g}} \leqslant &((\chi_{k_0} + \chi_{k_0+1}) a_{k_0+1} \cdot, (\chi_{k_0} + \chi_{k_0+1}) \cdot)_{drd\overline{g}} \\ &+ \sum_{k \geqslant k_0+2} (\varphi(2^{-k}r)\chi_k a_{k_0+1} \cdot, \chi_k \cdot)_{drd\overline{g}}\end{aligned}$$

since $supp \chi_k \subset (2^{k_0}, +\infty)$ for all $k \geqslant k_0 + 2$.

Remark 3.2.16. To obtain (3.2.20) we have used the following expressions for the commutators with $M(r)$ and $V_m - \Lambda$:

$$\left[\frac{a(D_r - A_0) + (D_r - A_0)a}{2}, M(r)\right] = a\frac{M'(r)}{i} - a[A_0, M(r)],$$

$$\left[\frac{a(D_r - A_0) + (D_r - A_0)a}{2}, h^2(V_m - \Lambda)\right] = h^2 a D_r(V_m).$$

Going back to expression (3.2.18) and given the contribution of (3.2.20)

$$\begin{aligned}\|h\chi_k(D_r - A_0)u\|^2_{L^2(drd\overline{g})} \lesssim & Im\left(a_k(D_r - A_0)u + \frac{a_k'}{2i}u, \mathcal{P}u\right)_{drd\overline{g}} \\ &- \varepsilon\left(\frac{a_k(D_r - A_0) + (D_r - A_0)a_k}{2}u, u\right)_{drd\overline{g}} \\ &+ \frac{h^2}{4}(u, \chi_k'' u)_{drd\overline{g}} + (u, \frac{1}{2i}a_k[A_0, M(r)]u)_{drd\overline{g}} \\ &- (u, \frac{1}{2i}h^2 a_k D_r(V_m)u)_{drd\overline{g}} \\ = & Im\left(a_k(D_r - A_0)u + \frac{\chi_k^2}{2i}u, \mathcal{P}u\right)_{drd\overline{g}} - \varepsilon Re(a_k u, (D_r - A_0)u)_{drd\overline{g}} \\ &+ \frac{h^2}{4}(u, (\chi_k^2)'' u)_{drd\overline{g}} + (u, \frac{1}{2i}a_k[A_0, M(r)]u)_{drd\overline{g}}\end{aligned}$$

$$-(u,\frac{1}{2i}h^2a_kD_r(V_m)u)_{drd\overline{g}} \tag{3.2.23}$$

where we have used the relation

$$Re(Bu,Cu)=\left(\frac{BC+CB}{2}u,u\right) \tag{3.2.24}$$

for B,C symmetric.

We recall that the $B^*_{>R}$ norm is defined as a supremum over all dyadic intervals, hence we need to bound the quantity $h\chi_k(D_r-A_0)u$ uniformly in k for all $k\geqslant k_0+1$. To prove Proposition 3.2.14 we use a series of inequalities that we collect here and that are proven in the following subsection.

Lemma 3.2.17. *Let $k\geqslant k_0+1$ of the form $k=k_0+m+1$ for some $m=m(k)\geqslant 1$ then for any $\delta\in(0,1)$ we have the following inequalities*

i)

$$\sum_{l\geqslant k_0+1} Im\Big(\varphi(2^{-l}r)\chi_l|(D_r-A_0)u|,\chi_l|\mathcal{P}u|\Big)_{drd\overline{g}}+|Im\left(\frac{\chi_k^2}{2i}u,\mathcal{P}u\right)_{drd\overline{g}}|$$
$$\leqslant\frac{c}{h^2\delta}\|\mathcal{P}u\|^2_{B_{>R}}+c\delta\|u\|^2_{H^1,B^*_{>R}},$$

ii)

$$\varepsilon|Re(a_ku,(D_r-A_0)u)_{drd\overline{g}}|\leqslant O(2^m)(\delta c\|u\|^2_{B^*_{>R}}+\frac{1}{\delta h}c\|\mathcal{P}u\|^2_{B_{>R}})$$
$$+O(2^m)\frac{c}{\delta}\|u\|_{H^1(r\simeq R)},$$

iii)

$$|\frac{h^2}{4}(u,(\chi_k^2)''u)_{drd\overline{g}}|\leqslant\delta c\|u\|^2_{B^*_{>R}},\quad\text{or}\quad|\frac{h^2}{4}(u,(\chi_k^2)''u)_{drd\overline{g}}|\leqslant c\|u\|^2_{L^2(K(\delta))},$$

iv)

$$\sum_{l\geqslant k_0+1}(\varphi(2^{-l}r)\chi_l|u|,\frac{1}{2i}\chi_l|[A_0,M(r)]u|)_{drd\overline{g}}\lesssim(c\delta\|u\|^2_{H^1,B^*_{>R}}+c(\delta)\|u\|^2_{H^1(K(\delta))}),$$

v)

$$\sum_{l\geqslant k_0+1}(\varphi(2^{-l}r)\chi_l|u|,\frac{1}{2i}h^2\chi_l|D_r(V_m)u|)_{drd\overline{g}}\lesssim(c\delta\|u\|^2_{H^1,B^*_{>R}}+c(\delta)\|u\|^2_{H^1(K(\delta))}).$$

Proof of Proposition 3.2.14. Given the support of χ_k, we will bound the norm $\|\cdot\|_{B^*_{>R},2}$ and hence we will need to consider the supremum over all $k\geqslant k_0+1$ (see Remark 3.1.17). Write

$$k=k_0+1+m,\qquad m=m(k)\geqslant 0$$

then on the support of χ_k

$$R^{-1}\geqslant r^{-1}2^{m+k^*}$$

and

$$R^{-1}\|h\chi_k(D_r-A_0)u\|^2_{L^2(drd\overline{g})}\geqslant O(2^m)\|r^{-1/2}h\chi_k(D_r-A_0)u\|^2_{L^2(drd\overline{g})}. \tag{3.2.25}$$

Dividing by R the right hand side of (3.2.23) we have

$$\begin{aligned}O(2^m)\|r^{-1/2}h\chi_k(D_r-A_0)u\|^2_{L^2(dr d\overline{g})} \lesssim & |R^{-1}Im\left(a_k(D_r-A_0)u+\frac{\chi_k^2}{2i}u,\mathcal{P}u\right)_{dr d\overline{g}}| \\ & +|R^{-1}\varepsilon Re(a_k u,(D_r-A_0)u)_{dr d\overline{g}}| \\ & +|R^{-1}\frac{h^2}{4}(u,(\chi_k^2)''u)_{dr d\overline{g}}|+|R^{-1}(u,\frac{1}{2i}a_k[A_0,M(r)]u)_{dr d\overline{g}}| \\ & +|R^{-1}(u,\frac{1}{2i}h^2a_kD_r(V_m)u)_{dr d\overline{g}}|. \end{aligned} \tag{3.2.26}$$

We divide the proof in two steps.

1. Consider a_k with $k \geqslant k_0+2$. Thanks to Remark 3.2.15 for all $k \geqslant k_0+2$ we can use inequality (3.2.22). Take for example the first term in the right hand side of (3.2.26), inserting the partition of unity we obtain

$$\begin{aligned}|R^{-1}Im\Big(a_k(D_r-A_0)u+\frac{\chi_k^2}{2i}u,\mathcal{P}u\Big)_{dr d\overline{g}}| \lesssim & Im(a_k\chi_{k_0}|(D_r-A_0)u|,\chi_{k_0}|\mathcal{P}u|)_{dr d\overline{g}} \\ & +\sum_{l\geqslant k_0+1} Im\left(a_k\varphi(2^{-l}r)\chi_l|(D_r-A_0)u|,\chi_l|\mathcal{P}u|\right)_{dr d\overline{g}} \\ & +|Im\left(\frac{\chi_k^2}{2i}u,\mathcal{P}u\right)_{dr d\overline{g}}| \\ \lesssim & O(2^m)Im(\chi_{k_0}|(D_r-A_0)u|,\chi_{k_0}|\mathcal{P}u|)_{dr d\overline{g}} \\ & +O(2^m)\sum_{l\geqslant k_0+1} Im\left(\varphi(2^{-l}r)\chi_l|(D_r-A_0)u|,\chi_l|\mathcal{P}u|\right)_{dr d\overline{g}} \\ & +|Im\left(\frac{\chi_k^2}{2i}u,\mathcal{P}u\right)_{dr d\overline{g}}| \\ \lesssim & O(2^m)\|u\|^2_{H^1(r\simeq R)}+O(2^m)\|\mathcal{P}u\|^2_{B_{>R}} \\ & +O(2^m)\frac{c}{h^2\delta}\|\mathcal{P}u\|^2_{B_{>R}}+O(2^m)c\delta\|u\|^2_{H^1,B^*_{>R}},\end{aligned}$$

   where we used item $i)$ of Lemma 3.2.17. We can then eliminate the unbounded factor $O(2^m)$, coming from the bound $a_k \leqslant O(2^k)$, with the one on the left hand side of (3.2.26). Doing the same with all the other terms in the right hand side of (3.2.26) and applying the inequalities in Lemma 3.2.17 we conclude that for all $k \geqslant k_0+2$

$$\begin{aligned}\|r^{-1/2}h\chi_k(D_r-A_0)u\|^2_{L^2(dr d\overline{g})} \leqslant & c\delta\|u\|^2_{H^1,B^*_{>R}}+c\frac{1}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} \\ & +c(\delta)\|u\|^2_{H^1(r\simeq R)}+\|u\|^2_{H^1(K(\delta))}\end{aligned} \tag{3.2.27}$$

   with $c(\delta)>1$ and $K(\delta)$ which increase as $\delta \to 0$.

2. Let $k=k_0+1$, we recall that for a_{k_0+1} we have the following bound

$$\begin{aligned}(a_{k_0+1}\cdot,\cdot)_{dr d\overline{g}} \leqslant & ((\chi_{k_0}+\chi_{k_0+1})a_{k_0+1}\cdot,(\chi_{k_0}+\chi_{k_0+1})\cdot)_{dr d\overline{g}} \\ & +\sum_{k\geqslant k_0+2}(\varphi(2^{-k}r)\chi_k a_{k_0+1}\cdot,\chi_k\cdot)_{dr d\overline{g}}.\end{aligned}$$

   We can then repeat the same argument as in the previous step and obtain

$$\|r^{-1/2}h\chi_{k_0+1}(D_r-A_0)u\|^2_{L^2(dr d\overline{g})} \leqslant c\delta\|u\|^2_{H^1,B^*_{>R}}+c\frac{1}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}}$$

$$+ c(\delta)\|u\|^2_{H^1(r\simeq R)} + \|u\|^2_{H^1(K(\delta))} \tag{3.2.28}$$

Thanks to (3.2.27) and (3.2.28) we have the desired bound on

$$\sup_{k\geqslant k_0+1} \|r^{-1/2}\chi_k h(D_r - A_0)u\|^2_{L^2(drd\overline{g})}$$

and we conclude the proof.

□

3.2.2.1 Auxiliary lemmas

In this section we provide the results needed in Lemma 3.2.17.

Lemma 3.2.18. *For any $\delta \in (0,1)$ the following inequalities hold*

$$\sum_{l\geqslant k_0+1} Im\Big(\varphi(2^{-l}r)\chi_l|(D_r - A_0)u|, \chi_l|\mathcal{P}u|\Big)_{drd\overline{g}} \leqslant \frac{c}{h^2\delta}\|\mathcal{P}u\|^2_{B_{>R}} + c\delta\|u\|^2_{H^1,B^*_{>R}}$$

and

$$|Im\left(\frac{\chi_k^2}{2i}u, \mathcal{P}u\right)_{drd\overline{g}}| \leqslant \frac{c}{\delta}\|\mathcal{P}u\|^2_{B_{>R}} + c\delta\|u\|^2_{H^1,B^*_{>R}}$$

with $c > 0$ independent of δ.

Proof. Thanks to the duality of the norms $B^*_{>R}, B_{>R}$ we can insert the weights $r^{1/2}$, $r^{-1/2}$ and h, h^{-1}, then by Cauchy-Schwarz inequality

$$\begin{aligned}\sum_{l\geqslant k_0+1} Im\Big(\varphi(2^{-l}r)\chi_l|(D_r - A_0)u|, \chi_l|\mathcal{P}u|\Big)_{drd\overline{g}} \leqslant &\sum_{l\geqslant k_0+1} \|r^{-1/2}\chi_k h(D_r - A_0)u\|_{L^2(drd\overline{g},D_k)} \\ &\frac{1}{h}\|r^{1/2}\mathcal{P}u\|_{L^2(drd\overline{g},D_k)} \\ \lesssim &\|h(D_r - A_0)u\|_{B^*_{>R}}\frac{1}{h}\|\mathcal{P}u\|_{B_{>R}} \\ \lesssim &\delta\|h(D_r - A_0)u\|^2_{B^*_{>R}} + \frac{1}{h^2\delta}\|\mathcal{P}u\|^2_{B_{>R}}.\end{aligned}$$

We do the same for the second inequality

$$|Im\left(\frac{\chi_k^2}{2i}u, \mathcal{P}u\right)_{drd\overline{g}}| = |Re\left(\frac{\chi_k}{2}r^{-1/2}u, \chi_k r^{1/2}\mathcal{P}u\right)_{drd\overline{g}}| \lesssim \frac{\delta}{4}\|u\|^2_{B^*_{>R}} + \frac{1}{4\delta}\|\mathcal{P}u\|^2_{B_{>R}}.$$

□

Lemma 3.2.19. *For any $\delta \in (0,1)$*

$$\begin{aligned}\varepsilon|Re(a_k u, (D_r - A_0)u)_{drd\overline{g}}| \leqslant &O(2^m)(\delta c\|u\|^2_{B^*_{>R}} + \frac{1}{\delta h}c\|\mathcal{P}u\|^2_{B_{>R}}) \\ &+ O(2^m)\frac{c}{\delta}\|u\|_{H^1(r\simeq R)}.\end{aligned}$$

with $c > 0$ independent of δ and ε.

Proof. Let $\widetilde{\phi}$ a smooth function which is one on *supp* a_k, we can apply Lemmas 3.2.9 and 3.2.10, which yield

$$|R^{-1}\varepsilon Re(a_k u, (D_r - A_0)u)_{drd\overline{g}}| \lesssim O(2^m)\frac{\varepsilon}{h}\|\widetilde{\phi}u\|_{L^2(drd\overline{g})}\|h(D_r - A_0)(\widetilde{\phi}u)\|_{L^2(drd\overline{g})}$$

$$+ O(2^m)\frac{\varepsilon}{h}\|\widetilde{\phi}u\|_{L^2(drd\overline{g})}\|D_r(\widetilde{\phi})u\|_{L^2(drd\overline{g})}$$
$$\lesssim O(2^m)\frac{1}{h}(\delta c\|u\|^2_{B^*_{>R}} + \frac{1}{\delta}c'\|\mathcal{P}(\widetilde{\phi}u)\|^2_{B_{>R}})$$

Choosing $\delta = h\delta'$ and applying Lemma 3.2.7 we conclude

$$\begin{aligned}|R^{-1}\varepsilon Re(a_k u, (D_r - A_0)u)_{drd\overline{g}}| &\lesssim O(2^m)(\delta' c\|u\|^2_{B^*_{>R}} + \frac{c'}{\delta' h^2}\|\mathcal{P}(\widetilde{\phi}u)\|^2_{B_{>R}}) \\ &\lesssim O(2^m)(\delta' c\|u\|^2_{B^*_{>R}} + \frac{c'}{\delta' h^2}\|\mathcal{P}u\|^2_{B_{>R}} \\ &\quad + \frac{c'}{\delta'}\|u\|^2_{H^1(r\simeq R)}).\end{aligned}$$

□

Lemma 3.2.20. *Let $k \geqslant k_0 + 1$ of the form $k = k_0 + m + 1$ for some $m = m(k) \geqslant 1$, then for any $\delta \in (0,1)$ we either have*

$$|\frac{h^2}{4}(u, (\chi_k^2)'' u)_{drd\overline{g}}| \leqslant \delta c\|u\|^2_{B^*_{>R}}$$

for some constant $c > 0$ independent of δ or there exists a compact set $K(\delta)$ depending on δ such that

$$|\frac{h^2}{4}(u, (\chi_k^2)'' u)_{drd\overline{g}}| \leqslant c\|u\|^2_{L^2(K(\delta))}$$

for some constant $c > 0$ independent of δ.

Proof. By definition

$$(\chi_k^2)'' = 2^{1-2k}(\chi'(2^{-k}\cdot))^2 + 2^{1-2k}\chi_k\chi''(2^{-k}\cdot)$$

where $2^{-2k} = O(2^{-2m})$ and

$$|\frac{h^2}{4}(u, (\chi_k^2)'' u)_{drd\overline{g}}| \leqslant h^2 O(2^{-2m})(\|\chi'\|_\infty + \|\chi''\|_\infty)\|u\|^2_{L^2(drd\overline{g}, D_k)}.$$

Let $\delta \in (0,1)$, if m is large enough such that $O(2^{-2m}) < \delta$ then we directly have

$$|\frac{h^2}{4}(u, (\chi_k^2)'' u)_{drd\overline{g}}| \leqslant \delta c\|u\|^2_{B^*_{>R}}.$$

Otherwise for all m such that $2^{-2m} > \delta$ the interval D_k is a bounded one, albeit depending on δ. Hence

$$|\frac{h^2}{4}(u, (\chi_k^2)'' u)_{drd\overline{g}}| \leqslant c\|u\|^2_{L^2(K(\delta))}.$$

□

Lemma 3.2.21. *For any $\delta \in (0,1)$ there exist $c(\delta) > 1$ and a bounded region $K(\delta)$ such that*

$$\sum_{l \geqslant k_0+1}(\varphi(2^{-l}r)\chi_l|u|, \frac{1}{2i}\chi_l|[A_0, M(r)]u|)_{drd\overline{g}} \lesssim (c\delta\|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}).$$

Proof. The proof is analogous to the one of Lemma 3.2.12.

□

Lemma 3.2.22. *For any $\delta \in (0,1)$ there exist $c(\delta) > 1$ and a bounded region $K(\delta)$*

$$\sum_{l \geqslant k_0+1} (\varphi(2^{-l}r)\chi_l|u|, \frac{1}{2i}h^2\chi_l|D_r(V_m)u|)_{drd\overline{g}} \lesssim (c\delta\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}).$$

Proof. The proof is analogous to the one of Lemma 3.2.13. □

3.2.3 Estimating u

In this section we give bounds on the norm $\|u\|_{B^*_{>R}}$.

Let χ_k as defined in (3.1.18), taking the real part

$$Re(\chi_k^2 u, \mathcal{P}u)_{drd\overline{g}} = -\|\chi_k u\|^2_{L^2(drd\overline{g})} + Re(\chi_k^2 u, h^2 P_{\overline{g}} u)_{drd\overline{g}}.$$

As in the previous section we write $k = k_0 + m + 1$ for a certain $m = m(k) \geqslant 0$. With this notation, on the support of χ_k

$$2^{k_0-1}2^m \leqslant r \leqslant 2^{k_0+3}2^m \tag{3.2.29}$$

where $k_0 = \frac{\ln R}{\ln 2} + k^* + 2$ and k^* is a fixed natural number. In particular

$$2^{k^*+1+m}R \leqslant r \leqslant 2^{k^*+5+m}R \tag{3.2.30}$$

which implies for some positive c

$$\begin{aligned} 2^m c\|r^{-1/2}\chi_k u\|^2_{L^2(drd\overline{g})} \leqslant & R^{-1}\|\chi_k u\|^2_{L^2(drd\overline{g})} \\ \leqslant & R^{-1}|Re(\chi_k^2 u, h^2 P_{\overline{g}} u)_{drd\overline{g}}| \quad (3.2.31) \\ & + R^{-1}|(r^{-1/2}\chi_k u, r^{1/2}\chi_k \mathcal{P}u)_{drd\overline{g}}|. \quad (3.2.32) \end{aligned}$$

The main result of this section is:

Proposition 3.2.23. *For all $\delta \in (0,1)$ there exist $c(\delta)$, and a set $K(\delta)$ such that*

$$\|u\|^2_{B^*_{>R}} \leqslant \frac{c'}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + \delta c\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(r\simeq R)} + \|u\|^2_{H^1(K(\delta))}$$

with $c(\delta) > 1$ and $K(\delta)$ which increase as $\delta \to 0$.

Proof. For term (3.2.32) we can plainly use the duality of the norms $B^*_{>R}$, $B_{>R}$. Term (3.2.31) is made of several parts. First we consider the one involving $h^2(D_r - A_0)^2$, using (3.2.24)

$$\begin{aligned} Re(\chi_k^2 u, h^2(D_r - A_0)^2 u)_{drd\overline{g}} =& (u, \frac{h^2}{2}\left(\chi_k^2(D_r - A_0)^2 + (D_r - A_0)^2\chi_k^2\right) u)_{drd\overline{g}} \\ =& (u, h^2(D_r - A_0)(\chi_k^2(D_r - A_0))u)_{drd\overline{g}} - h^2(u, (\chi_k^2)'' u)_{drdg} \end{aligned}$$

Since (3.2.30) holds on the support of χ_k we also have for some positive constant c

$$R^{-1} \leqslant c 2^m r^{-1}$$

and hence

$$\begin{aligned} |R^{-1}Re(\chi_k^2 u, h^2(D_r - A_0)^2 u)_{drd\overline{g}}| \lesssim & 2^m\|h(D_r - A_0)u\|^2_{B^*_{>R}} + h^2R^{-1}|(u, (\chi_k^2)'' u)_{drd\overline{g}}| \\ \lesssim & 2^m(\frac{c}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + \delta c\|u\|^2_{H^1,B^*_{>R}} \\ & + c(\delta)\|u\|^2_{H^1(r\simeq R)} + \|u\|^2_{H^1(K(\delta))}) \quad (3.2.33) \end{aligned}$$

thanks to Proposition 3.2.14 and Lemma 3.2.20. Applying (3.2.11) we also obtain

$$|R^{-1}(\chi_k u, \chi_k M(r)u)_{drd\overline{g}}| \lesssim 2^m \|r^{-1/2} M(r)^{1/2} u\|_{L^2(drd\overline{g}, D_k)} \leqslant 2^m \Big(\frac{c}{\delta h^2} \|\mathcal{P}u\|^2_{B_{>R}} + \delta c \|u\|^2_{H^1, B^*_{>R}} + c(\delta) \|u\|^2_{H^1(K(\delta))}\Big). \tag{3.2.34}$$

Finally, we can exploit the decay of $V_m - \Lambda$ given by (3.1.13) and obtain

$$R^{-1}|(\chi_k u, \chi_k h^2 (V_m - \Lambda)u)_{drd\overline{g}}| \lesssim 2^m |(r^{-1/2}\chi_k u, r^{-1/2-\nu}\chi_k h^2 u)_{drd\overline{g}}| \lesssim 2^{m - m\nu} h^2 \|r^{-1/2} u\|^2_{L^2(drd\overline{g}, D_k)}.$$

Now for any $\delta \in (0,1)$ there exists an $m^* = m^*(\delta) \geqslant 0$ such that for all $m \geqslant m^*$ we have $2^{-m\nu} < \delta$ and the previous inequality, if k is large enough to satisfy $m(k) \geqslant m^*$, renders

$$R^{-1}|(\chi_k u, \chi_k h^2 (V_m - \Lambda)u)_{drd\overline{g}}| \lesssim 2^m \delta c \|u\|^2_{B^*_{>R}}. \tag{3.2.35}$$

Otherwise, the scalar product is bounded by the L^2 norm of u over a compact interval of r, depending on δ. Thanks to (3.2.33), (3.2.34) and (3.2.35) we have

$$2^m \|r^{-1/2}\chi_k u\|^2_{L^2(drd\overline{g})} \lesssim 2^m \Big(\frac{c}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + \delta c \|u\|^2_{H^1, B^*_{>R}} + c(\delta)\|u\|^2_{H^1(r \simeq R)} + \|u\|^2_{H^1(K(\delta))}\Big)$$

for any $k \geqslant k_0 + 1$ which yields the statement.

□

3.3 Estimates in the compact region: unique continuation

By the result of Proposition 3.2.1 in the previous section we have

$$\|u\|^2_{H^1, B^*_{>R}} \leqslant c\delta \|u\|^2_{H^1, B^*_{>R}} + \frac{c}{\delta h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta)\|u\|^2_{H^1(drd\overline{g}, K(\delta))}.$$

with $K(\delta)$ a bounded region in $(2^{k^*}R, +\infty) \times S$. Now fix δ_0 such that $c\delta_0 < 1$, then

$$\|u\|^2_{H^1, B^*_{>R}} \leqslant \frac{c}{\delta_0 h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta_0)\|u\|^2_{H^1(K(\delta_0))}, \tag{3.3.1}$$

that is we can bound the $H^1, B^*_{>R}$ norm of u (which is a norm on the manifold end) by the operator $\mathcal{P} = h^2 P_{\overline{g}} - 1 + i\varepsilon$ up to a compactly supported term.

In this section we show how to bound the H^1 norm on a compact region of the manifold by applying unique continuation results.

Notation. We define the notation

$$X_R := (R, +\infty) \times S \simeq M \setminus K$$

and in general

$$X_a := (a, +\infty) \times S$$

so that $K(\delta_0) \subset X_{2^{k^*}R}$. Let $a > R$ such that $K(\delta_0) \subset X_a$ and without loss of generality we can assume $K(\delta_0) \subset X_a \setminus X_{a+2}$.

The main result of this section is

Proposition 3.3.1. *Let $u \in H^2(M)$, $\lambda > \lambda_0 > 0$, $h = \lambda^{-1}$ and $a > R$ defined as above. There exists $\gamma_0 \in (0,1)$ and U bounded region of $(2^{k^*}R, +\infty) \times S$ such that*

$$\|u\|_{H^1(M \setminus X_{2^{k^*}R})} \leqslant O(e^{\lambda/\gamma_0})(\|(P_m - \lambda^2 + i\varepsilon')u\|_{L^2(M\setminus X_{a+3})} + \|u\|_{H^1(U)}).$$

The following proposition is a direct application of unique continuation.

Proposition 3.3.2. *Let (M_0, g_0) an n dimensional Riemannian manifold, T the Laplace-Beltrami operator and $\mathcal{R}$ a differential operator of order one. Let*

$$U_0 \Subset V_0 \Subset M_0 \qquad V_0' \Subset V_0 \Subset M_0 \quad \overline{V}_0 \cap \partial M_0 = \emptyset,$$

$\alpha \in (0, 1/2)$ and $z \in \mathbb{C}$ with $Rez > z_0 > 0$, $|Imz| \neq 0$. Then there exists $c(z_0) > 0$ and $\gamma_0 \in (0,1)$ such that

$$\|u\|_{H^1(V_0')} \leqslant c(z_0)e^{|z|/\gamma_0}(\|(T + \mathcal{R} - z^2)u\|_{L^2(V_0)} + \|u\|_{L^2(U_0)}).$$

for all $u \in H^2(V_0)$.

Proof. Define

$$\begin{aligned} M_1 = (-1,1) \times M_0, \quad & \widetilde{\sigma} = (-1 + 2\alpha, 1 - 2\alpha) \times U_0 \\ & \sigma = (-1,1) \times U_0 \\ & U = (-1 + \alpha, 1 - \alpha) \times V_0' \\ & \widetilde{V} = (-1 + \alpha/2, 1 - \alpha/2) \times V_0 \\ & V = (-1,1) \times V_0, \end{aligned}$$

then $\overline{U} \cap \partial M_1 = \emptyset$, $\widetilde{\sigma}$ is an open subset of U and $U \Subset V \Subset M_1$. Let us also consider

$$T - \partial_t^2 + \mathcal{R}, \qquad v(t,m) = e^{tz}u(m) \in H^2(V)$$

and f such that $(T - \partial_t^2 + \mathcal{R})v = f$.

We apply Theorem 9.1 in [LRLR22] to the sets $\widetilde{\sigma}, U, \widetilde{V}$ and the operator $T - \partial_t^2 + \mathcal{R}$. Hence there exist $c > 0$ and $\gamma_0 \in (0,1)$ for which

$$\begin{aligned} \|v\|_{H^1(U)} &\leqslant c\|v\|_{H^1(\widetilde{V})}^{1-\gamma_0}(\|(T - \partial_t^2 + \mathcal{R})v\|_{L^2(\widetilde{V})} + \|v\|_{L^2(\widetilde{\sigma})})^{\gamma_0} \\ &\leqslant c\|v\|_{H^1(V)}^{1-\gamma_0}(\|(T - \partial_t^2 + \mathcal{R})v\|_{L^2(V)} + \|v\|_{L^2(\sigma)})^{\gamma_0} \end{aligned} \tag{3.3.2}$$

where

$$(T - \partial_t^2 + \mathcal{R})v = e^{tz}(T + \mathcal{R} - z^2)u(m).$$

Computing the integrals with respect to t in (3.3.2) yields

$$\begin{aligned} (|z|^2\|u\|_{L^2(V_0')}^2 + \|\nabla_{g_0}u\|_{L^2(V_0')}^2 + \|u\|_{L^2(V_0')}^2)^{\gamma_0/2} \leqslant & c\frac{e^{Rez} + e^{-Rez}}{e^{Rez(1-\alpha)} - e^{-Rez(1-\alpha)}} \\ & (\|(T + \mathcal{R} - z^2)u\|_{L^2(V_0)} + \|u\|_{L^2(U_0)})^{\gamma_0}, \end{aligned}$$

where we used the relation $b^{1/2} - b^{-1/2} \leqslant (b - b^{-1})^{1/2} \leqslant b^{1/2} + b^{-1/2}$ which holds for $b \geqslant 1$. The left hand side can be bounded from below by $O(1)\|u\|_{H^1(V_0)}^{\gamma_0}$ (since $\min\{|z|^2, 1\}$ is a strictly positive constant). On the other hand since $Rez(1-\alpha) > z_0/2 > 0$ there exists $c' = c'(z_0)$ such that $e^{Rez(1-\alpha)} - e^{-Rez(1-\alpha)} > c'$. We conclude that

$$\|u\|_{H^1(V_0')} \leqslant c\frac{2e^{|z|/\gamma_0}}{c'(z_0)}(\|(T + \mathcal{R} - z^2)u\|_{L^2(V_0)} + \|u\|_{L^2(U_0)}).$$

□

With these result we can now prove the main result of this section.

Proof of Proposition 3.3.1. We apply Proposition 3.3.2. Choose

$$U_0 \Subset X_{a+2} \setminus X_{a+3},\ V_0' = M \setminus X_{a+2}\ V_0 = M \setminus X_{a+3},\ M_0 = M \setminus X_{a+4}$$

and we apply Proposition 3.3.2 to the function $\chi_0 u$ with

$$\chi_0 = \begin{cases} 1 & \text{on } M \setminus X_{2^{k^*}R} \\ \in (0,1) & \text{on } U = X_{2^{k^*}R} \setminus X_{a+2} \\ 0 & \text{on } X_{a+3} \end{cases} \tag{3.3.3}$$

so that $\chi_0 u \equiv 0$ on U_0 and

$$K(\delta_0) \subset U. \tag{3.3.4}$$

Taking $T + \mathcal{R} - z^2 = P_m - \lambda^2 + i\varepsilon'$ results in

$$\|u\|_{H^1(M \setminus X_{2^{k^*}R})} \leqslant O(e^{\lambda/\gamma_0})(\|(P_m - \lambda^2 + i\varepsilon')u\|_{L^2(M\setminus X_{a+3})} + \|u\|_{H^1(U)}). \tag{3.3.5}$$

□

Thanks to (3.3.4) we can bound the perturbative term in (3.3.1) by $\|u\|_{H^1(U)}$, so (3.3.1) becomes

$$\|u\|^2_{H^1,B^*_{>R}} \leqslant \frac{c}{\delta_0 h^2}\|\mathcal{P}u\|^2_{B_{>R}} + c(\delta_0)\|u\|^2_{H^1(U)}$$

and a combination with Proposition 3.3.1 yields

$$\begin{aligned}\|u\|^2_{H^1(M\setminus X_{2^{k^*}R})} + \|u\|^2_{H^1,B^*_{>R}} \leqslant & O(h^{-2}e^{\lambda/\gamma_0})\|\mathcal{P}u\|^2_{L^2(M\setminus X_{a+2})} \\ & + O(h^{-2})\|\mathcal{P}u\|^2_{B_{>R}} + O(e^{\lambda/\gamma_0})\|u\|^2_{H^1(U)}.\end{aligned} \tag{3.3.6}$$

Notation. In the previous section we had actually set $\mathcal{P} = h^2 P_{\overline{g}} - 1 + i\varepsilon$. With an abuse of notation we use the same symbol to denote the corresponding quantity on the whole manifold, that is $h^2 P_m - 1 + i\varepsilon$.

The rest of the paper will be devoted to eliminate the perturbative term of exponential size $O(e^{\lambda/\gamma_0})\|u\|^2_{H^1(U)}$.

3.4 Estimates on the exponential remainder

In this section we consider $\lambda \gg 1$ in

$$P - \lambda^2 + i\varepsilon',$$

we recall the (slight abuse of) notation

$$\mathcal{P} = h^2 P_m - 1 + i\varepsilon$$

with $\varepsilon = O(h^2)$ in which now $h = \lambda^{-1} \ll 1$. The aim of this section is to prove the following result, which implies Theorem 3.1.1.

Theorem 3.4.1. *Let $u \in H^2(M)$, $\lambda \gg 1$, $R < a < 2^{k^*}R < a+3$, then*

$$\|u\|^2_{H^1(M\setminus X_{2^{k^*}R})} + \|u\|^2_{H^1,B^*_{>R}} \leqslant O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon)u\|^2_{L^2(M\setminus X_{a+3})}$$

$$+ O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon)u\|^2_{B_{>R}}$$

for some constant $C > 0$ independent of λ and ε.

More precisely, what we will be able to prove is

$$\begin{aligned}\|u\|^2_{L^2(M\setminus X_{a+3})} + O(e^{\lambda C})\|u\|^2_{B^*_{>R}} \leqslant & O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon)u\|^2_{L^2(M\setminus X_{a+3})} \\ & + O(\lambda^{-2}e^{\lambda C})\|(P_m - \lambda^2 + i\varepsilon)u\|^2_{B_{>R}}, \end{aligned} \tag{3.4.1}$$

see Remark 3.4.7 for further details. By simply considering a function supported sufficiently far at radial infinity we can then derive Corollary 3.1.2, that is

Corollary 3.4.2. *Let $u \in H^2(M)$, $\lambda \gg 1$, $R < a < 2^{k^*}R < a + 3$ and χ a smooth cutoff such that $\chi \equiv 0$ on $M \setminus X_{a+3}$, $\chi \equiv 1$ on X_{a+4}. Then*

$$\|\chi u\|^2_{B^*_{>R}} \leqslant O(\lambda^{-2})\|(P_m - \lambda^2 + i\varepsilon)\chi u\|^2_{B_{>R}}.$$

In particular

$$\|r^{-1/2-\mu}\chi(P_m - \lambda^2 + i\varepsilon)^{-1}\chi r^{-1/2-\mu}\|_{L^2\to L^2} = O(\lambda^{-1})$$

with $\mu > 0$.

Proof. The $B_{>R} \to B^*_{>R}$ bound follows directly from inequality (3.4.1) thanks to the support of χ. To recover the norm in the weighted L^2 space we just remark the inclusions

$$L^2_{1/2+\mu} \hookrightarrow B_{>R}, \quad B^*_{>R} \hookrightarrow L^2_{-1/2-\mu}$$

which can be proved by direct computations. □

As we pointed out above, the main concern is now to take care of the exponentially large remainder in (3.3.6). To do so we can exploit the weight φ constructed in Section 2 [CV02]. In particular let $a > R$ such that

$$\varphi'(r) = \frac{1}{\lambda r} \quad r \geqslant a. \tag{3.4.2}$$

Moreover, $\varphi > \gamma_0^{-1}$ for all $r \geqslant R + 2$ for a parameter $\gamma_0 > 0$ independent of λ.

Remark 3.4.3. For $r \geqslant a$ we have

$$\varphi(r) - \varphi(a) = \frac{1}{\lambda}\ln\left(\frac{r}{a}\right)$$

and hence the quantity

$$e^{\lambda(\varphi(r)-\varphi(a))} = \frac{r}{a} \quad r \geqslant a$$

is independent of λ.

Remark 3.4.4. The subset U defined in (3.3.3) is contained in X_{R+2}, so $\varphi > \gamma_0^{-1}$ on U and therefore

$$e^{2\lambda/\gamma_0}\|u\|^2_{H^1(U)} = \|e^{\lambda(1/\gamma_0-\varphi)}e^{\lambda\varphi}u\|^2_{H^1(U)} \leqslant O(e^{-c\lambda})\|e^{\lambda\varphi}u\|_{H^1(X_{b_2}\setminus X_{a+2})}.$$

We will use the following properties of φ which are due to [CV02].

Lemma 3.4.5 (Lemma 2.1 [CV02])**.** *Let $\delta \in (0, 1)$. The following inequalities hold for $\lambda > \lambda(\delta) \gg 1$ and $r > R$.*

$$C\lambda^{-1}r^{-1} \leqslant \varphi',$$

$$
\begin{gathered}
-\varphi'\varphi'' \leqslant C\delta r^{-1} \\
|\varphi''| \leqslant C\lambda^{1/2} r^{-1}\varphi', \qquad (\varphi'')^2 \leqslant C\lambda^{1/2} r^{-1}\varphi' \\
|\varphi'''| \leqslant C\lambda r^{-1}\varphi', \qquad |\varphi'''| \leqslant C\lambda^{1/2} r^{-1}, \\
|\varphi^{(4)}| \leqslant C\lambda^{3/2} r^{-1}\varphi'
\end{gathered}
$$

We will conclude the proof of Theorem 3.4.1 thanks to the following proposition.

Proposition 3.4.6. *Let $v \in H^2(X_R \setminus X_{a+4})$ such that $v = \partial_r v = 0$ on ∂X_R, $\lambda \gg 1$ and $a > R_0$ such that $\varphi' = \lambda^{-1} r^{-1}$ for $r \geqslant a$. Then*

$$
\begin{aligned}
\|e^{\lambda(\varphi-\varphi(a))} v\|^2_{H^1(X_R\setminus X_{a+3})} \leqslant & O(h^{-2})\|e^{\lambda(\varphi-\varphi(a))}\mathcal{P}v\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2})\|\mathcal{P}v\|_{L^2(X_a\setminus X_{a+4})} \\
& + O(h^{-2})\|\mathcal{P}v\|^2_{B_{>R}}.
\end{aligned}
$$

The proof, being quite technical, will be postponed to the end of this section, we first show how its application allows us to pass from (3.3.6) to the result in Theorem 3.4.1.

Let v such that the assumptions of Proposition 3.4.6 are satisfied and recall $K(\delta_0) \subset U$. In particular $r > a$ on $K(\delta_0)$ and $\varphi' \geqslant 0$ implies $\varphi - \varphi(a) > 0$ on $K(\delta_0)$. Then there is a positive constant c such that

$$
\|v\|_{H^1(K(\delta_0))} \leqslant e^{-\lambda c}\|e^{\lambda(\varphi-\varphi(a))}v\|_{H^1(K(\delta_0))} \leqslant e^{-\lambda c}\|e^{\lambda(\varphi-\varphi(a))}v\|_{H^1(X_R\setminus X_{a+2})}. \tag{3.4.3}
$$

since $K(\delta_0) \subset X_R \setminus X_{a+2}$ and hence

$$
e^{2\lambda\varphi(a)}\|v\|^2_{H^1(K(\delta_0))} \leqslant e^{-\lambda c}\|e^{\lambda\varphi}v\|^2_{H^1(X_R\setminus X_{a+2})} \tag{3.4.4}
$$

From Proposition 3.4.6 we obtain

$$
\begin{aligned}
\|e^{\lambda\varphi} v\|^2_{H^1(X_R\setminus X_{a+3})} \leqslant & O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}v\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2}e^{\lambda\varphi(a)})\|\mathcal{P}v\|_{L^2(X_a\setminus X_{a+4})} \\
& + O(h^{-2}e^{\lambda\varphi(a)})\|\mathcal{P}v\|^2_{B_{>R}},
\end{aligned}
$$

nevertheless we can replace the left hand side with $\|e^{\lambda\varphi}v\|^2_{H^1(X_R\setminus X_{a+3})} + e^{2\lambda\varphi(a)}\|v\|^2_{H^1,B^*_{>R}}$. Indeed, applying (3.3.1) to v

$$
e^{2\lambda\varphi(a)}\|v\|^2_{H^1,B^*_{>R}} \leqslant e^{2\lambda\varphi(a)}\frac{c}{\delta_0 h^2}\|\mathcal{P}v\|^2_{B_{>R}} + c(\delta_0)e^{2\lambda\varphi(a)}\|v\|^2_{H^1(K(\delta_0))}
$$

and (3.4.4) implies we can absorb the remainder term. We have obtained

$$
\begin{aligned}
\|e^{\lambda\varphi} v\|^2_{H^1(X_R\setminus X_{a+3})} + e^{2\lambda\varphi(a)}\|v\|^2_{H^1,B^*_{>R}} \leqslant & O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}v\|^2_{L^2(X_R\setminus X_a)} \\
& + O(h^{-2}e^{2\lambda\varphi(a)})\|\mathcal{P}v\|^2_{L^2(X_a\setminus X_{a+4})} \\
& + O(h^{-2}e^{2\lambda\varphi(a)})\|\mathcal{P}v\|^2_{B_{>R}}.
\end{aligned} \tag{3.4.5}
$$

Proof of Theorem 3.4.1. We define

$$
\widetilde{\chi}_1 = \begin{cases} 0 & M \setminus X_{b_1} \\ \in (0,1) & X_{b_1} \setminus X_{b_2} \\ 1 & X_{b_2} \end{cases}
$$

with $R < b_1 < b_2 < R+1$ such that $\varphi < -c < 0$ on $[b_1, b_2]$ and apply (3.4.5) to $\widetilde{\chi}_1 u$, yielding

$$
\|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+3})} + e^{2\lambda\varphi(a)}\|u\|^2_{H^1,B^*_{>R}} \leqslant O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}u\|^2_{L^2(X_R\setminus X_a)}
$$

$$
\begin{aligned}
&+ O(h^{-2}e^{2\lambda\varphi(a)})\|\mathcal{P}u\|^2_{L^2(X_a\setminus X_{a+4})}\\
&+ O(h^{-2}e^{2\lambda\varphi(a)})\|\mathcal{P}u\|^2_{B_{>R}}\\
&+ O(e^{-c\lambda})\|u\|_{H^1(X_{b_1}\setminus X_{b_2})}.
\end{aligned}
$$

In this inequality we have another compactly supported remainder term that we wish to absorb, since the prefactor is a small one. To do so we need to add a term on the left hand side that is supported in a region containing $X_{b_1}\setminus X_{b_2}$, for example $M\setminus X_{2k^*R}$. We can now use (3.3.6) and Remark 3.4.4 to add the contribution of this region

$$
\begin{aligned}
\|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+3})} &+ \|u\|^2_{H^1(M\setminus X_{2k^*R})} + e^{2\lambda\varphi(a)}\|u\|^2_{H^1,B^*_{>R}}\\
&\leqslant O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}u\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2})e^{2\lambda\varphi(a)}\|\mathcal{P}u\|^2_{L^2(X_a\setminus X_{a+4})}\\
&\quad + O(h^{-2})e^{2\lambda\varphi(a)}\|\mathcal{P}u\|^2_{B_{>R}} + O(h^{-2})e^{2\lambda/\gamma_0}\|\mathcal{P}u\|^2_{L^2(M\setminus X_{a+3})}\\
&\quad + O(e^{-c\lambda})\|u\|^2_{H^1(X_{b_1}\setminus X_{b_2})} + O(e^{-c\lambda})\|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+2})}\\
&\leqslant O(h^{-2})\|e^{\lambda\varphi}\mathcal{P}u\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2})e^{2\lambda\varphi(a)}\|\mathcal{P}u\|^2_{B_{>R}}\\
&\quad + O(h^{-2})e^{2\lambda/\gamma_0}\|\mathcal{P}u\|^2_{L^2(M\setminus X_{a+3})}\\
&\quad + O(e^{-c\lambda})\|u\|^2_{H^1(X_{b_1}\setminus X_{b_2})} + O(e^{-c\lambda})\|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+2})}.
\end{aligned}
\tag{3.4.6}
$$

Both terms in (3.4.6) can be absorbed to the left hand side by $\|u\|^2_{H^1(M\setminus X_{2k^*R})}$ and $\|e^{\lambda\varphi}u\|^2_{H^1(X_{b_2}\setminus X_{a+2})}$ respectively. First of all, we remark that after absorption of the remainders this last inequality implies (3.4.1). Then, thanks to the properties of φ we have

$$
e^{\lambda(\varphi-\varphi(a))}\leqslant 1 \ \text{ on } X_R\setminus X_a, \qquad e^{2\lambda/\gamma_0-2\lambda\varphi(a)}\leqslant 1
$$

so dividing everything by $e^{2\lambda\varphi(a)}$

$$
e^{-2\lambda\varphi(a)}(\|u\|^2_{H^1(M\setminus X_{2k^*R})} + \|u\|^2_{H^1,B^*_{>R}}) \leqslant O(h^{-2})\|\mathcal{P}u\|^2_{B_{>R}} + O(h^{-2})\|\mathcal{P}u\|^2_{L^2(M\setminus X_{a+3})}. \tag{3.4.7}
$$

This proves the statement since we recall the rescaling

$$
h^{-1}\mathcal{P} = h^{-1}(h^2P_m - 1 + i\varepsilon) = \lambda^{-1}(P_m-\lambda^2+i\varepsilon').
$$

□

Remark 3.4.7. We can obtain inequality (3.4.1) directly from the computations in the previous proof. The right hand side in inequality (3.4.6) also bounds

$$
\|u\|^2_{L^2(X_{2k^*R}\setminus X_{a+3})} + \|u\|^2_{L^2(M\setminus X_{2k^*R})} + e^{2\lambda\varphi(a)}\|u\|^2_{B^*_{>R}},
$$

and hence we obtain

$$
\begin{aligned}
\|u\|^2_{L^2(M\setminus X_{a+3})} + e^{2\lambda\varphi(a)}\|u\|^2_{B^*_{>R}} &\leqslant O(\lambda^{-2})e^{2\lambda\varphi(a)}\|(P_m-\lambda^2+i\varepsilon')u\|^2_{B_{>R}}\\
&\quad + O(\lambda^{-2}e^{\lambda C})\|(P_m-\lambda^2+i\varepsilon')u\|^2_{L^2(M\setminus X_{a+3})}\\
&\quad + O(\lambda^{-2})\|e^{\lambda\varphi}(P_m-\lambda^2+i\varepsilon')u\|^2_{L^2(X_R\setminus X_a)}
\end{aligned}
$$

from which (3.4.1) follows.

As announced earlier, we conclude the section with the proof of Proposition 3.4.6. We will need first the following lemma, which is the equivalent of Proposition 2.3 [CV02] in our case of a Schrödinger operator with order one perturbation. We nevertheless include the proof at the end of this section for the sake of clarity and completeness.

Lemma 3.4.8. *Let $v \in H^2(X_R \setminus \partial X_a)$ such that $v = \partial_r v = 0$ on $\partial X_R \cup \partial X_a$, $\lambda \gg 1$ and $a > R_0$ such that $\varphi' = \lambda^{-1} r^{-1}$ for $r \geqslant a$. Then*

$$\|(\varphi'/r)^{1/2} v\|_{H^1(X_R\setminus X_a)} \leqslant O(\lambda^{1/2}) \|\mathcal{P}_\varphi v\|_{L^2(X_R\setminus X_a)}$$

where $\mathcal{P}_\varphi = e^{\lambda\varphi}\mathcal{P}e^{-\lambda\varphi}$.

Proof of Proposition 3.4.6. Let v as in the statement of Proposition 3.4.6 and define

$$\chi_1 = \begin{cases} 1 & \text{on } M \setminus X_{a+3}, \\ \in (0,1) & \text{on } X_{a+3} \setminus X_{a+4}, \\ 0 & \text{on } X_{a+4}. \end{cases}$$

We apply Lemma 3.4.8 to $e^{\lambda\varphi}\chi_1 v$ which vanishes, together with its radial derivative, on $\partial X_R \cup \partial X_{a+4}$. This yields

$$\|(\varphi'/r)^{1/2} e^{\lambda\varphi}\chi_1 v\|_{H^1(X_R\setminus X_{a+4})} \leqslant O(h^{-1/2}) \|e^{\lambda\varphi}\mathcal{P}(\chi_1 v)\|_{L^2(X_R\setminus X_{a+4})},$$

where $(\varphi'/r)^{1/2} \geqslant h^{1/2} r^{-1}$ on $X_R \setminus X_{a+4}$ thanks to the inequality $\varphi' \geqslant C\lambda^{-1}r^{-1}$. Then

$$\begin{aligned} \|e^{\lambda\varphi} v\|_{H^1(X_R\setminus X_{a+3})} \leqslant& O(h^{-1})\|e^{\lambda\varphi}\mathcal{P}v\|_{L^2(X_R\setminus X_{a+4})} + O(h^{-1})\|e^{\lambda\varphi}[\mathcal{P},\chi_1]v\|_{L^2(X_{a+3}\setminus X_{a+4})} \\ \leqslant& O(h^{-1})\|e^{\lambda\varphi}\mathcal{P}v\|_{L^2(X_R\setminus X_{a+4})} + O(h) e^{\lambda\varphi(a+4)}\|v\|_{H^1(X_{a+3}\setminus X_{a+4})} \end{aligned}$$

since $[\mathcal{P},\chi_1] = [h^2(D_r - A_0)^2, \chi_1]$ is supported on the set $\{\chi_1 \in (0,1)\}$ and is an operator of order one in the radial variable. Dividing by $e^{\lambda\varphi(a)}$ and thanks to Remark 3.4.3

$$\begin{aligned} \|e^{\lambda(\varphi-\varphi(a))} v\|^2_{H^1(X_R\setminus X_{a+3})} \leqslant& O(h^{-2})\|e^{\lambda(\varphi-\varphi(a))}\mathcal{P}v\|^2_{L^2(X_R\setminus X_a)} + O(h^{-2})\|\mathcal{P}v\|^2_{L^2(X_a\setminus X_{a+4})} \\ &+ O(1)\|v\|^2_{H^1(X_{a+3}\setminus X_{a+4})}. \end{aligned}$$

The norm of v can be bounded by the inequality (3.3.1) on the region at infinity and recalling (3.4.3) we obtain

$$\|v\|_{H^1(X_{a+3}\setminus X_{a+4})} \leqslant O(h^{-2})\|\mathcal{P}v\|_{B_{>R}} + e^{-\lambda c}\|e^{\lambda(\varphi-\varphi(a))}v\|_{H^1(X_R\setminus X_{a+2})}$$

from which the statement follows since $e^{-\lambda c}\|e^{\lambda(\varphi-\varphi(a))}v\|_{H^1(X_R\setminus X_{a+2})}$ is an absorbable term. □

Proof of Lemma 3.4.8. The conjugated operator is given by

$$\mathcal{P}_\varphi = \mathcal{P} - (\varphi')^2 + h\varphi'' + 2i\varphi' h D_r - 2iA_0\varphi'. \tag{3.4.8}$$

Let $\psi \in C^\infty([R,a])$ real valued, we consider the scalar product

$$Re(\psi\mathcal{P}_\varphi v, v)_{L^2(X_R\setminus X_a)} = Re\int_R^a \int_S \overline{\psi\mathcal{P}_\varphi v}\; v \; dr d\overline{g}.$$

By integration by parts

$$\begin{aligned} Re(\psi h^2(D_r - A_0)^2 v, v)_{L^2(X_R\setminus X_a)} =& Re(\psi h(D_r - A_0)v, h(D_r - A_0)v)_{L^2(X_R\setminus X_a)} \\ &+ Re(h(D_r - A_0)v, hD_r(\psi)v)_{L^2(X_R\setminus X_a)}. \end{aligned}$$

In this expression we notice

$$Re(hA_0 v, hD_r(\psi)v)_{L^2(X_R\setminus X_a)} = Re\frac{1}{i}\int_R^a\int_S h^2 A_0\psi'|v|^2 \; dr d\overline{g} = 0$$

and

$$\begin{aligned} Re(hD_r v, hD_r(\psi)v)_{L^2(X_R\setminus X_a)} =& Re\ h^2 \int_R^a \int_S \partial_r \overline{v}\psi' v\ dr d\overline{g} \\ =& \frac{h^2}{2}\int_R^a \int_S \partial_r |v|^2 \psi'\ dr d\overline{g} = -(\frac{h^2}{2}\psi'' v, v)_{L^2(X_R\setminus X_a)}. \end{aligned}$$

Hence we have

$$\begin{aligned} Re(\psi h^2 (D_r - A_0)^2 v, v)_{L^2(X_R\setminus X_a)} =& Re(\psi h(D_r - A_0)v, h(D_r - A_0)v)_{L^2(X_R\setminus X_a)} \\ &- (\frac{h^2}{2}\psi'' v, v)_{L^2(X_R\setminus X_a)}. \end{aligned} \tag{3.4.9}$$

Moreover

$$\begin{aligned} Re(\psi 2i\varphi' h D_r v, v)_{L^2(X_R\setminus X_a)} =& Re\ 2h \int_R^a \int_S \partial_r \overline{v} v \psi\varphi'\ dr d\overline{g} \\ =& h \int_R^a \int_S \partial_r(|v|^2)\psi\varphi'\ dr d\overline{g} \\ =& -h \int_R^a \int_S |v|^2(\psi'\varphi' + \psi\varphi'')\ dr d\overline{g} \end{aligned} \tag{3.4.10}$$

so if we evaluate again the scalar product $Re(\psi\mathcal{P}_\varphi v, v)_{L^2(X_R\setminus X_a)}$ we have

$$\begin{aligned} Re(\psi\mathcal{P}_\varphi v, v)_{L^2(X_R\setminus X_a)} =& Re(\psi h(D_r - A_0)v, h(D_r - A_0)v)_{L^2(X_R\setminus X_a)} \\ &+ (\psi M(r)v, v)_{L^2(X_R\setminus X_a)} \\ &- ((\psi + \psi(\varphi')^2 - \psi h^2(V_m - \Lambda) + h\varphi'\psi' + \frac{h^2}{2}\psi'')v, v)_{L^2(X_R\setminus X_a)}. \end{aligned} \tag{3.4.11}$$

We define

$$F(r) := -((M(r) - 1 + W)v_r, v_r)_{L^2(S)} + \|h(D_r - A_0)v_r\|^2_{L^2(S)} \tag{3.4.12}$$

where $v_r = v(r, \cdot)$ and

$$W := h^2(V_m - \Lambda) - (\varphi')^2 + h\varphi''.$$

and by definition of $\mathcal{P}_\varphi$

$$-\mathcal{P}_\varphi = -h^2(D_r - A_0)^2 - M(r) + (1 - W) - 2ih\varphi' h(D_r - A_0) - i\varepsilon. \tag{3.4.13}$$

We need to compute F', so

$$\begin{aligned} \partial_r \|h(D_r - A_0)v_r\|^2_{L^2(S)} =& 2Re(h(D_r - A_0)\partial_r v_r, h(D_r - A_0)v_r)_{L^2(S)} \\ &+ 2Re([\partial_r, h(D_r - A_0)]v_r, h(D_r - A_0)v_r)_{L^2(S)} \\ =& -2Re(h(D_r - A_0)hD_r v_r, (\partial_r - iA_0)v_r)_{L^2(S)} - 2Re(hA_0' v_r, h(D_r - A_0)v_r)_{L^2(S)} \\ =& -2Re(h^2(D_r - A_0)^2 v_r, (\partial_r - iA_0)v_r)_{L^2(S)} - 2Re(h^2(D_r - A_0)(A_0 v_r), (\partial_r - iA_0)v_r)_{L^2(S)} \\ &- 2Re(hA_0' v_r, h(D_r - A_0)v_r)_{L^2(S)}. \end{aligned}$$

After commuting A_0 with D_r and noticing that

$$Re(h^2 A_0(D_r - A_0)v_r, i(D_r - A_0)v_r)_{L^2(S)} = 0$$

we obtain

$$-2Re(h^2(D_r - A_0)(A_0 v_r), (\partial_r - iA_0)v_r)_{L^2(S)} = 2Re(hA_0' v_r, h(D_r - A_0)v_r)_{L^2(S)},$$

so that the last two terms in $\partial_r \|h(D_r - A_0)v_r\|^2_{L^2(S)}$ cancel. This gives us the final expression

$$\partial_r \|h(D_r - A_0)v_r\|^2_{L^2(S)} = -2Re(h^2(D_r - A_0)^2 v_r, (\partial_r - iA_0)v_r)_{L^2(S)}.$$

We can now compute F'

$$\begin{aligned} F'(r) =& 2Re((-h^2(D_r - A_0)^2 - M(r) + (1 - W))v_r, \partial_r v_r)_{L^2(S)} \\ &- 2Re(h^2(D_r - A_0)^2 v_r, -iA_0 v_r)_{L^2(S)} \\ &- ([\partial_r, M(r)]v_r, v_r)_{L^2(S)} - (W' v_r, v_r)_{L^2(S)} \end{aligned}$$

and adding the suitable terms

$$\begin{aligned} F'(r) &+ 2Re(M(r)v_r, iA_0 v_r)_{L^2(S)} - 2Re((1 - W)v_r, iA_0 v_r)_{L^2(S)} \\ &= 2Re((-h^2(D_r - A_0)^2 - M(r) + (1 - W))v_r, (\partial_r - iA_0)v_r)_{L^2(S)} \\ &\quad - ([\partial_r, M(r)]v_r, v_r)_{L^2(S)} - (W' v_r, v_r)_{L^2(S)} \\ &= - 2Re(\mathcal{P}_\varphi v_r, (\partial_r - iA_0)v_r)_{L^2(S)} + 4h^{-1}\varphi' \|h(D_r - A_0)v_r\|^2_{L^2(S)} \\ &\quad + 2\varepsilon Im(v_r, (\partial_r - iA_0)v_r)_{L^2(S)} \\ &\quad - ([\partial_r, M(r)]v_r, v_r)_{L^2(S)} - (W' v_r, v_r)_{L^2(S)}. \end{aligned} \tag{3.4.14}$$

where we used (3.4.13). Integrating against φ' we find

$$\begin{aligned} \int_R^a \varphi' F' \, dr =& - 2Re \int_R^a (\varphi' \mathcal{P}_\varphi v_r, (\partial_r - iA_0)v_r)_{L^2(S)} + 4h^{-1} \int_R^a (\varphi')^2 \|h(D_r - A_0)v_r\|^2_{L^2(S)} dr \\ &+ 2\varepsilon Im \int_R^a (\varphi' v_r, (\partial_r - iA_0)v_r)_{L^2(S)} dr \\ &- \int_R^a \varphi'([\partial_r, M]v_r, v_r)_{L^2(S)} dr - \int_R^a (\varphi' W' v_r, v_r)_{L^2(S)} dr \\ &- 2Re \int_R^a \varphi'(M(r)v_r, iA_0 v_r)_{L^2(S)} dr + 2Re \int_R^a \varphi'((1 - W)v_r, iA_0 v_r)_{L^2(S)} dr. \end{aligned} \tag{3.4.15}$$

Doing integration by parts, we can rewrite $\int_R^a \varphi' F' \, dr$ in terms of the integral of $\varphi'' F$ and in this regard expression (3.4.11) with $\psi = \varphi''$ gives us

$$\begin{aligned} \int_R^a Re(\varphi'' \mathcal{P}_\varphi v_r, v_r)_{L^2(S)} dr =& 2 \int_R^a \varphi'' \|h(D_r - A_0)v_r\|^2_{L^2(S)} dr - \int_R^a \varphi'' F \, dr \\ &- \int_R^a ((h(\varphi'')^2 + h\varphi'\varphi''' + \frac{h^2}{2}\varphi^{(4)})v_r, v_r)_{L^2(S)} dr \end{aligned}$$

where we have used $1 - h^2(V_m - \Lambda) + (\varphi')^2 = 1 - W + h\varphi''$. Using this relation

$$\begin{aligned} \int_R^a \varphi' F' \, dr =& - \int_R^a \varphi'' F \, dr \\ =& Re \int_R^a (\varphi'' \mathcal{P}_\varphi v_r, v_r)_{L^2(S)} dr - 2 \int_R^a \varphi'' \|h(D_r - A_0)v_r\|^2_{L^2(S)} dr \\ &+ \int_R^a ((h(\varphi'')^2 + h\varphi'\varphi''' + \frac{h^2}{2}\varphi^{(4)})v_r, v_r)_{L^2(S)} dr. \end{aligned} \tag{3.4.16}$$

So finally, coupling (3.4.15) and (3.4.16)

$$
\begin{aligned}
2\int_R^a ((2h^{-1}(\varphi')^2+\varphi'')h(D_r-A_0)v_r, h(D_r-A_0)v_r)_{L^2(S)} - \int_R^a \varphi'([\partial_r, M]v_r, v_r)_{L^2(S)}dr \\
= 2Re\int_R^a (\varphi'\mathcal{P}_\varphi v_r, (\partial_r - iA_0)v_r)_{L^2(S)}dr + Re\int_R^a(\varphi''\mathcal{P}_\varphi v_r, v_r)_{L^2(S)}dr \\
- 2\varepsilon Im\int_R^a (\varphi' v_r, (\partial_r - iA_0)v_r)_{L^2(S)}dr \\
+ \int_R^a ((\varphi' W' + h(\varphi'')^2 + h\varphi'\varphi''' + \frac{h^2}{2}\varphi^{(4)})v_r, v_r)_{L^2(S)}dr \quad (3.4.17) \\
+ 2Re\int_R^a \varphi'(M(r)v_r, iA_0 v_r)_{L^2(S)}dr - 2Re\int_R^a \varphi'((1-W)v_r, iA_0v_r)_{L^2(S)}dr. \quad (3.4.18)
\end{aligned}
$$

With the exception of the last two terms in (3.4.18) all the other terms can be treated as in Proposition 2.3 in [CV02]. For the left hand side we have on one hand

$$
2h^{-1}(\varphi')^2 + \varphi'' \geqslant C'\frac{\varphi'}{r} \qquad (3.4.19)
$$

thanks $\varphi' = hr^{-1}$ and Lemma 3.4.5, on the other hand

$$
-\varphi'([\partial_r, M]v_r, v_r)_{L^2(S)} \gtrsim \frac{\varphi'}{r}(M(r)v_r, v_r)_{L^2(S)} \qquad (3.4.20)
$$

from Lemma 3.2.4 on $[R, a]$. Thus we have the following lower bound on the left hand side

$$
\begin{aligned}
2\int_R^a ((2h^{-1}(\varphi')^2+\varphi'')h(D_r-A_0)v_r, h(D_r-A_0)v_r)_{L^2(S)} - \int_R^a \varphi'([\partial_r, M]v_r, v_r)_{L^2(S)}dr \\
\gtrsim 2\int_R^a \|(\varphi'/r)^{1/2}h(D_r-A_0)v\|_{L^2(S)}dr + \int_R^a \|(\varphi'/r)^{1/2}M(r)^{1/2}v\|_{L^2(S)}dr
\end{aligned}
$$

In the right hand side we use the inequality

$$
|\varphi''| \leqslant C\lambda^{1/2}\frac{\varphi'}{r}.
$$

coming from Lemma 3.4.5 from which

$$
\begin{aligned}
|Re\int_R^a(\varphi''\mathcal{P}_\varphi v_r, v_r)_{L^2(S)}dr| \leqslant & O(h^{-1}\delta^{-1})\int_R^a \|\mathcal{P}_\varphi v_r\|^2_{L^2(S)}dr \\
& + O(\delta)\int_R^a \|(\varphi'/r)^{1/2}v_r\|^2_{L^2(S)}dr. \qquad (3.4.21)
\end{aligned}
$$

Moreover $r^{1+\nu}W'$ is bounded from the properties of Lemma 3.4.5 and the fact that $r^{1+\nu}\partial_r V_m$ is also bounded. This allows to bound (3.4.17) by

$$
O(h^{1/2})\int_R^a \|(\varphi'/r)^{1/2}v\|^2_{L^2(S)}dr.
$$

For (3.4.18) we have

$$
\begin{aligned}
|Re\int_R^a \varphi'(M(r)v_r, & iA_0v_r)_{L^2(S)}dr| \\
& \leqslant \int_R^a h|l(r)((\varphi'/r)^{1/2}M^{1/2}(r)v_r, [(1+T(r))^{1/2}, A_0](\varphi'/r)^{1/2}v_r)_{L^2(S)}|dr
\end{aligned}
$$

$$\leqslant O(h) \int_R^a \|(\varphi'/r)^{1/2} M^{1/2} v_r\|^2_{L^2(S)} dr + O(h) \int_R^a \|(\varphi'/r)^{1/2} v_r\|^2_{L^2(S)} dr \tag{3.4.22}$$

since $(M^{1/2}(r)v_r, iA_0 M^{1/2}(r)v_r)_{L^2(S)}$ is pure complex and $[(1+T)^{1/2}, A_0]$ acts as multiplication by a bounded function. Moreover in $1 - W = 1 - h^2(V_m - \Lambda) + (\varphi')^2 - h\varphi''$ the quantity $1 - h^2\Lambda + (\varphi')^2 - h\varphi''$ is real and V_m, A_0 are bounded hence

$$|Re \int_R^a \varphi'((1-W)v_r, iA_0 v_r)_{L^2(S)} dr| \leqslant h^2 \int_R^a \|(\varphi'/r)^{1/2} v_r\|^2_{L^2(S)} dr. \tag{3.4.23}$$

The remaining terms can be bounded by the Cauchy-Schwarz inequality recalling that $\varepsilon = O(h^2)$. From (3.4.19)-(3.4.23) we have obtained

$$\int_R^a \|(\varphi'/r)^{1/2} h(D_r - A_0)v_r\|^2_{L^2(S)} dr + \int_R^a \|(\varphi'/r)^{1/2} M^{1/2} v_r\|^2_{L^2(S)} dr$$
$$\leqslant O(h^{-1}\delta^{-1}) \int_R^a \|\mathcal{P}_\varphi v_r\|^2_{L^2(S)} dr + O(\delta) \int_R^a \|(\varphi'/r)^{1/2} v_r\|^2_{L^2(S)} dr$$

for a small parameter δ.

On the other hand applying (3.4.11) with $\psi = r^{-1}\varphi'$, thanks to Lemma 3.4.5 and the fact that $r^\nu(V_m - \Lambda)$ is a bounded function we can recover

$$\int_R^a \|(\varphi'/r)^{1/2} v_r\|^2_{L^2(S)} dr \leqslant \int_R^a \|(\varphi'/r)^{1/2} h(D_r - A_0)v_r\|^2_{L^2(S)} dr$$
$$+ \int_R^a \|(\varphi'/r)^{1/2} M^{1/2} v_r\|^2_{L^2(S)} dr + \int_R^a \|\mathcal{P}_\varphi v_r\|^2_{L^2(S)} dr.$$

The statement follows combining the last two inequalities.

□

Appendix

3.A Notations for the current chapter

Manifolds and metrics
$M = K \cup (M \setminus K)$, $M \setminus K$ infinite end
S angular manifold of dimension $n-1$
$p \in M$, $\omega \in S$, θ local coordinate on S
$\overline{g}, g(r)$ metrics on S
$X_R = (R, +\infty) \times S$, isometric to $M \setminus K$
$l(r)$ volume factor
$G = dr^2 + l(r)^{-2} g(r)$ metric on $(R, +\infty) \times S$
$w = \frac{\lvert g(r,\theta)\rvert}{\lvert \overline{g}(\theta)\rvert}$
$(\cdot,\cdot)_{drd\overline{g}}$ scalar product with respect to the measure $drd\overline{g}$
$(\cdot,\cdot)_{L^2(\cdot)}$ scalar product in a bounded region of $(R, +\infty) \times S$ with respect to the measure $drd\overline{g}$
Operators
P_m magnetic Laplacian
$P_{\overline{g}}$ operator on $(R, \infty) \times S$, symmetric with respect to $drd\overline{g}$
$\mathcal{P} = h^2 P_m - 1 + i\varepsilon$ with $h = \lambda^{-1}$
A vector field on M, with $(A_0(r,\theta), \dots, A_{n-1}(r,\theta))$ local coordinates on $M \setminus K$
$(-\widetilde{\Delta}_{g(r)})$ Laplace-Beltrami operator with order one perturbation
$M(r)$ angular operator, $T(r) = w^{1/2}(-\widetilde{\Delta}_{g(r)})w^{-1/2}$
$V_m = V +$ effective potential
Norms
$k^* \in \mathbb{N}$, $k_0 = \frac{\ln R}{\ln 2} + k^* + 2$
$\varphi(2^{-k}r)$ partition of unity on $(2^{k^*}R, +\infty)$, $supp\varphi(2^{-k}r) \subset [2^{k-1}, 2^{k+1}]$
χ_k smooth cutoff equal to one on $supp\varphi(2^{-k}r)$, $supp\chi_k \subset [2^{k-2}, 2^{k+2}]$
$D_k = [2^{k-1}, 2^{k+1}]$
$\lVert f\rVert_{B_{>R}} = \sum_{k\geqslant k_0} \lVert r^{1/2} f\rVert_{L^2(drd\overline{g}, D_k)}$
$\lVert g\rVert_{B^*_{>R}} = \sup_{k\geqslant k_0} \lVert r^{-1/2} g\rVert_{L^2(drd\overline{g}, D_k)}$
$\lVert \cdot \rVert_{L^2(\cdot)}$ norm in a bounded region of $(R, +\infty) \times S$ with respect to the measure $drd\overline{g}$
Others
$\delta, \delta_0, \gamma_0 \in (0,1)$
$\phi(r)$ smooth cutoff equal to one on $(2^{k_0+2}, \infty)$
$K(\delta_0), U$ bounded regions in $(R, +\infty) \times S$, $K(\delta_0) \subset U$
$\varphi(r)$ λ- dependent weight, $\varphi' = \lambda^{-1} r^{-1}$ for $r \geqslant a$, and $\varphi < 0$ for $b_1 < r < b_2$

Chapter 4

On the definition of zero resonances for the Schrödinger operator with optimal scaling potentials

Outline of the current chapter

4.1 Introduction

We consider the Schrödinger operator $-\Delta+V$ on $\mathbb{R}^n$ with $n=3,4$. The evolution of solutions of the time-dependent Schrödinger equation is influenced by the spectrum of the operator $-\Delta+V$, in particular by the bottom of it and the nature of the 0 state: whether it is in the spectrum and if it is an eigenstate or a resonance. Roughly speaking, a resonance at 0 is a solution to the equation $(-\Delta+V)\psi=0$ which does not decay fast enough to be in L^2 and is usually assumed to belong in some kind of weighted space.

In this chapter we recover properties of 0 resonances and eigenfunctions described in the seminal paper [JK79] under weaker assumptions on V and with optimal behavior of such states as $|x|\to\infty$. We do not exhibit any exotic or new behavior but rather give what we consider to be a fairly simple proof of classical results for a more general class of potentials (essentially the optimal one). Our result also has the advantage of generalizing a characterization found by Beceanu in [Bec16] as we will see more in detail later on.

In much of the literature, resonances are usually found to belong to weighted Sobolev or Lebesgue spaces where there is no scaling invariance. For example, in [JK79] the authors define a resonance as a solution to $(-\Delta+V)\psi=0$ where the operator $-\Delta+V$ is meant to be extended to the weighted Sobolev space $H_{1,-s}$ with weight $\langle x\rangle^{-s}$ and a suitable $s>1/2$. A similar definition can also be found in [RS04],[Gol06a], [Gol06b] and [GS04], where the authors define resonances as functions that belong to the intersection of weighted L^2 spaces given by $\cap_{s>1/2}L^{2,-s}$ with weight $\langle x\rangle^{-s}$. Even in more recent papers ([Aaf21] ,[SW22]) with much stricter assumptions on the potentials than the one considered herein, the framework used to define resonances is the one of weighted L^2 spaces.

The fact that we can consider a 0 resonance as a function in a weak Lebesgue space is a very useful feature and it comes at a fairly small cost. Indeed, in various applications (like [JK79], [GS04], [ES04], [Aaf21] or [SW22]) the potentials must have a prescribed decay at infinity, that is $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 2$. Our assumption will be much less strict than requiring a specific pointwise decay and the results presented here only require scaling invariant assumptions on the potentials. This can be a useful feature when working with nonlinear problems and we aim to further investigate properties on the bottom of the spectrum for Schrödinger operators with potentials that satisfy only scaling invariant assumptions.

More precisely, let $L^{p,q}(\mathbb{R}^n)$ the Lorentz space defined on $\mathbb{R}^n$ with $n = 3, 4$. The Lorentz space is defined as the space of functions such that the quasinorm denoted by $\|\cdot\|_{p,q}$ is finite. We define the quasinorm via the distribution function $d_f(t) = |\{|f(x)| > t\}|$ as

$$\|f\|_{p,q} := p^{1/q}\left(\int_0^\infty t^{q-1}(d_f(t))^{q/p}dt\right)^{1/q}$$

for $q < \infty$ or

$$\|f\|_{p,\infty} := \sup_{t\geqslant 0} t d_f(t)^{1/p}$$

otherwise. As for the Lebesgue spaces, we identify two functions that are equal almost everywhere. (For some properties that we will use throughout the reader can refer to Appendix 4.A).

As mentioned before, for a prescribed decay the most general assumption is $V(x)$ decaying like $|x|^{-2-\varepsilon}$ at infinity (notice that $\langle x\rangle^{-2-\varepsilon} \in L^{n/2}(\mathbb{R}^n)$). From now on, we will take V in a slightly smaller space than $L^{n/2}$, that is we will assume

$$V \in L^{n/2,1}(\mathbb{R}^n) \tag{4.1.1}$$

and the chain of inclusions (4.A.2) directly gives us $L^{n/2,1}(\mathbb{R}^n) \subset L^{n/2}(\mathbb{R}^n)$. Assuming (4.1.1) is a little more restrictive than taking the potential in a Lebesgue space or in a Kato class, like in [BDH16] and [BG12]. However, it is still less strict than what is assumed in [Gol06a], [Gol06b], or [Gol10], where $L^{3/2-\varepsilon} \cap L^{3/2+\varepsilon} \subset L^{3/2,1}$ from Proposition 4.A.2, and much more general than imposing a power-like decay, which we have seen is the custom in several cases.

As we can see in the following remark, our assumption is satisfied from classes of potentials that have been of common use for a long time in the theory of Schrödinger operators. Nonetheless, it does not seem to have been clearly observed that they give the right framework to describe zero resonances and eigenfunctions more simply and optimally than in the usual approach of [JK79]. Indeed, the result presented in Theorem 4.1.2 may already be known to some experts, but we couldn't find any reference in the literature.

Remark 4.1.1. Assumption (4.1.1) covers the typical case of $V(x) = O(\langle x\rangle^{-2-\varepsilon})$. To check this we need to evaluate the following integral

$$\|\langle x\rangle^{-2-\varepsilon}\|_{\frac{n}{2},1} = \int_0^\infty |\{x : \langle x\rangle^{-2-\varepsilon} \geqslant t\}|^{2/n}dt.$$

First of all, since $\langle x\rangle^{-2-\varepsilon}$ is bounded by 1 the integral will be non zero only over $(0, 1)$ and secondly for t in such interval

$$\begin{aligned}|\{x : \langle x\rangle^{-2-\varepsilon} \geqslant t\}| =&|\{x : \frac{1}{(1+|x|^2)^{\frac{2+\varepsilon}{2}}} \geqslant t\}| = |\{x : |x|^2 \leqslant \frac{1}{t^{\frac{2}{2+\varepsilon}}} - 1\}|\\ \leqslant&|B(0, t^{-1/(2+\varepsilon)})| \simeq t^{-n/(2+\varepsilon)} \in L^{2/n}([0, 1]).\end{aligned}$$

We will be interested in $\dot{H}^1(\mathbb{R}^n)$ solutions of the equation

$$(-\Delta + V)\psi = 0 \tag{4.1.2}$$

with $V \in L^{n/2,1}(\mathbb{R}^n)$ and $\dot{H}^1(\mathbb{R}^n)$ the homogeneous Sobolev space (see (4.2.1) for a definition). We will state our result for a function in $\dot{H}^1(\mathbb{R}^n)$ which gives a simple and natural class of non (necessarily) L^2 functions where to look for solutions of (4.1.2). Moreover, $-\Delta$ is an isometry from $\dot{H}^1(\mathbb{R}^n)$ to $\dot{H}^{-1}(\mathbb{R}^n)$, a feature that we shall extensively use in Section 4.2.

We also comment that by inverting $-\Delta$ equation (4.1.2) is equivalent to solving $\psi + (-\Delta)^{-1}V\psi = 0$. In his paper [Bec16] Beceanu solves this equation in $L^\infty(\mathbb{R}^3)$. As we just said, here we will rather solve it in $\dot{H}^1(\mathbb{R}^n)$, or actually in $L^{\frac{2n}{n-2},\infty}(\mathbb{R}^n)$, in particular without seeking a priori bounded solutions.

The main result we will prove is the following.

Theorem 4.1.2. *Let $n = 3, 4$, $V \in L^{n/2,1}(\mathbb{R}^n)$ and $\psi \in \dot{H}^1(\mathbb{R}^n)$ a solution of the equation $(-\Delta + V)\psi = 0$. The following properties hold:*

i) $|x|^{n-2}\psi$ *has a finite limit as* $|x| \to \infty$*, hence for* $n = 3$

$$\psi \in L^{3,\infty}(\mathbb{R}^3)$$

and for $n = 4$

$$\psi \in L^{2,\infty}(\mathbb{R}^4).$$

ii) ψ *is a zero eigenfunction, that is* $\psi \in L^2(\mathbb{R}^n)$*, if and only if* $\int V\psi = 0$*, in particular* $\psi = O(\frac{1}{|x|^{n-1}})$ *near infinity.*

iii) If $\int V\psi = \int y_k V\psi = 0$ *for all* $k = 1, \dots, n$*, then* ψ *is a zero eigenfunction and* $\psi \in L^{1,\infty}(\mathbb{R}^n)$*, in particular* $\psi = O(\frac{1}{|x|^n})$ *near infinity.*

iv) If $\int V\psi = \int y_k V\psi = \int y_k y_l V\psi = 0$ *for all* $k, l = 1, \dots, n$*, then* ψ *is a zero eigenfunction and* $\psi \in L^1(\mathbb{R}^n)$*, in particular* $\psi = O(\frac{1}{|x|^{n+1}})$ *near infinity.*

Let us give a few comments about this result:

- in dimension three the simple assumption $V \in L^{3/2,1}(\mathbb{R}^3)$ generalizes the result stated by Beceanu in [Bec16] (Lemma 2.3) where the necessary and sufficient condition for ψ to be an eigenfunction is recovered only for potentials in $L^{3/2,1}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Regarding the behavior at infinity, we give a more precise statement than just $\psi \in \langle x\rangle^{-1}L^\infty(\mathbb{R}^3)$ since in Proposition 4.3.1 we give an explicit expression for the limit of $|x|\psi$ for $|x| \to \infty$.
- As we commented earlier, $\dot{H}^1(\mathbb{R}^n)$ is a pretty natural class of non L^2 functions to consider for this problem. Indeed, in higher dimension $n \geqslant 5$, there exist no $\dot{H}^1(\mathbb{R}^n)$ solutions of (4.1.2) which do not belong to L^2 as we explained in Remark 1.3.8. Therefore, with respect to the existence of resonances $\dot{H}^1$ represents an optimal class of functions where to solve (4.1.2). However, from the point of view of regularity it is actually enough to require $\psi \in L^{\frac{2n}{n-2},\infty}(\mathbb{R}^n)$ and indeed we will perform all proofs for ψ in this larger class.
- The conditions of orthogonality between $V\psi$ and various other polynomial functions are not new ones. Indeed, they can also be found in [JK79], [Jen84] or [Bec16]. In particular in [JK79],[Jen84] it was already observed that the condition $\int \psi V = 0$ is the right one to discriminate between a zero resonance or eigenvalue. However, in these works the authors consider conditions of pointwise decay on the potential that are more strict than our assumption.

Notation. We will drop from the notations the dependence on the underlying space $\mathbb{R}^n$ unless the situation requires it to make it explicit.

We define the function a

$$a(x) = \frac{1}{|x|^{n-2}}$$

which belongs to $L^{\frac{n}{n-2},\infty}$.

The main steps will be the following:

1. We can use the density of simple functions in $L^{n/2,1}$ to decompose the potential V in

$$V = W + K$$

where $W \in L^{n/2,1}$ is such that $\|W\|_{n/2,1}$ is arbitrarily small and K is a simple function, hence compactly supported and in any Lorentz space.

2. Thanks to the smallness of W in $L^{n/2,1}$ we will be able to define and estimate G, the Green function of $(-\Delta + W)$.

3. We will then solve in $L^{\frac{2n}{n-2},\infty}$ the equation

$$(-\Delta + W)\psi = -K\psi$$

where $(-\Delta + W) : \dot{H}^1 \to \dot{H}^{-1}$ is invertible and we will use the Green function computed in the previous step to write

$$\psi(x) = -\int G(x,y)K(y)\psi(y)dy.$$

In the rest of the paper ψ will be a $L^{\frac{2n}{n-2},\infty}$ solution of $(-\Delta + V)\psi = 0$.

4. We will show that $|x|^{n-2}\psi$ has finite limit at infinity and that such limit is given by $\int V\psi$. Therefore, the value of such integral determines wether the function is in L^2 or not (see Proposition 4.3.4).

5. With additional orthogonality conditions on $V\psi$ we can repeat the argument of the previous step to prove that ψ has faster, but limited, decay, until we can reach the L^1 space (see Propositions 4.3.6 and 4.3.8).

4.2 Green function for a small potential

Thanks to the decomposition $V = W+K$, instead of looking for the solution of $(-\Delta+V)\psi = 0$, in this section we will study ψ as solution of the equation

$$(-\Delta + W)\psi = -K\psi.$$

and we will show how to construct the Green function of the operator $-\Delta+W$ where $W \in L^{n/2,1}$ is a potential with sufficiently small quasinorm. We consider the operator defined on

$$-\Delta + W : \dot{H}^1 \to \dot{H}^{-1} \tag{4.2.1}$$

where $-\Delta$ is an isometry and W is a bounded operator. Indeed by definition of the homogeneous Sobolev space

$$\dot{H}^s := \{u \ : \ \|u\|_{\dot{H}^s} := \||\cdot|^s\widehat{u}\|_{L^2} < \infty\}$$

and since $-\Delta = \mathcal{F}^{-1}(|\xi|^2)\mathcal{F}$ it is straightforward to see that $\|(-\Delta)u\|_{\dot{H}^{-1}} = \|u\|_{\dot{H}^1}$.

As for W, we recall that from Sobolev embeddings (see Theorem 1.38 in [BCD11])

$$\dot{H}^p \hookrightarrow L^{p^*} \tag{4.2.2}$$

for $p* = \frac{np}{n-p}$. Hence in the case $p = 2$ we have a continuous embedding

$$\dot{H}^1 \hookrightarrow L^{\frac{2n}{n-2}}$$

which by duality implies $L^{\frac{2n}{n+2}} \hookrightarrow \dot{H}^{-1}$. Therefore for $u \in \dot{H}^1$ we have

$$Wu \in L^{n/2,1} \cdot L^{\frac{2n}{n-2}} \subset L^{n/2} \cdot L^{\frac{2n}{n-2}} \subset L^{\frac{2n}{n+2}} \subset \dot{H}^{-1} \tag{4.2.3}$$

and moreover

$$\|W\|_{\dot{H}^1 \to \dot{H}^{-1}} \leqslant \|W\|_{\frac{n}{2},1}$$

thanks to the continuous embeddings.

Remark 4.2.1. The chain of inclusions (4.2.3) is still valid when just considering $u \in L^{\frac{2n}{n-2},\infty}$ since $L^{n/2,1} \cdot L^{\frac{2n}{n-2},\infty} \subset L^{\frac{2n}{n+2},1} \subset L^{\frac{2n}{n+2}} \subset \dot{H}^{-1}$.

Moreover, $-\Delta + W$ maps $\dot{H}^1$ into $\dot{H}^{-1}$ also for $W \in L^{n/2,2}$ since $\dot{H}^1$ also embeds in the smaller space $L^{\frac{2n}{n-2},2}$. However, the space $L^{n/2,1}$ has the interest of being the dual of $L^{\frac{n}{n-2},\infty}$, this will allow us to integrate W against the kernel of $(-\Delta)^{-1}$ and to state that $\int V\psi$, limit of $|x|^{n-2}\psi$ at infinity, is finite.

To construct the Green function we first invert the operator via a Neumann series. Indeed, writing

$$(-\Delta + W)^{-1} = (-\Delta)^{-1}(I + W(-\Delta)^{-1})^{-1}$$

with W small enough such that $\|W(-\Delta)^{-1}\|_{\dot{H}^{-1} \to \dot{H}^{-1}} < 1$ we can write, at least formally,

$$(-\Delta + W)^{-1} = \sum_{j \geqslant 0} (-1)^j (-\Delta)^{-1} (W(-\Delta)^{-1})^j \tag{4.2.4}$$

where the series is convergent thanks to the smallness of W. More precisely, it is enough to assume $\|W\|_{n/2,1} < 1$ since $(-\Delta)^{-1}$ is an isometry from $\dot{H}^{-1}$ to $\dot{H}^1$ and as we have seen the operator norm of W is bounded by its $L^{n/2,1}$ quasinorm.

We use an idea from [Pin88] and use the operator series (4.2.4) to construct the Green function. We can do so by recurrence defining

$$G_0(x,y) = c_n \frac{1}{|x-y|^{n-2}}, \; G_j(x,y) = c_n \int \frac{1}{|x-z|^{n-2}} W(z) G_{j-1}(z,y) dz$$

where $c_n = n(n-2)|B(0,1)|$ is a constant depending on the dimension. Here G_0 is the kernel of $(-\Delta)^{-1}$, G_1 that is given by

$$G_1(x,y) = c_n \int \frac{1}{|x-z|^{n-2}} W(z) c_n \frac{1}{|z-y|^{n-2}} dz$$

is the kernel of $(-\Delta)^{-1} W (-\Delta)^{-1}$ and so on, G_j will be the kernel of $(-\Delta)^{-1}(W(-\Delta)^{-1})^j$.

Theorem 4.2.2. *Let $W \in L^{n/2,1}$ with $\|W\|_{n/2,1} < 1$ sufficiently small, then*

$$G(x,y) := \sum_{j \geqslant 0} (-1)^j G_j(x,y) \tag{4.2.5}$$

is the Green function of $-\Delta + W$ and is such that $|G(x,y)| \lesssim \frac{1}{|x-y|^{n-2}}$.

Remark 4.2.3. The theorem gives us a pointwise bound on the integral kernel of $(-\Delta + W)^{-1}$ by the integral kernel of $(-\Delta)^{-1}$. We deduce that $(-\Delta + W)^{-1}$ inherits any $L^p \to L^q$ or $L^{p,q} \to L^{p',q'}$ bound that $(-\Delta)^{-1}$ enjoys.

To bound the integrals defining G_j we first remark a useful inequality.

Lemma 4.2.4. *Let $a(x) := \frac{1}{|x|^{n-2}}$. For $x \neq y$ it holds*

$$\int \frac{1}{|x-z|^{n-2}}|W(z)|\frac{1}{|z-y|^{n-2}}dz \leqslant \|a\|_{\frac{n}{n-2},\infty}\|W\|_{n/2,1}\frac{2^{n-1}}{|x-y|^{n-2}}.$$

Proof. We split the integral into the regions $\{|z-y| \leqslant \frac{|x-y|}{2}\}$ and $\{|z-y| > \frac{|x-y|}{2}\}$ so that $z \in B(y, \frac{|x-y|}{2})$ implies $|x-z| \geqslant \frac{|x-y|}{2}$ and we have

$$\begin{aligned}\int \frac{1}{|x-z|^{n-2}}|W(z)|\frac{1}{|z-y|^{n-2}}dz \leqslant & \frac{2^{n-2}}{|x-y|^{n-2}}\int_{B(y,\frac{|x-y|}{2})}|W(z)|\frac{1}{|z-y|^{n-2}}dz \\ &+ \frac{2^{n-2}}{|x-y|^{n-2}}\int_{B(y,\frac{|x-y|}{2})^c}\frac{1}{|x-z|^{n-2}}|W(z)|dz \\ \leqslant & \frac{2^{n-2}}{|x-y|^{n-2}}\|W\|_{n/2,1}\||\cdot - y|\|_{\frac{n}{n-2},\infty} \\ &+ \frac{2^{n-2}}{|x-y|^{n-2}}\|W\|_{n/2,1}\||\cdot - x|\|_{\frac{n}{n-2},\infty} \\ \leqslant & \|a\|_{\frac{n}{n-2},\infty}\|W\|_{n/2,1}\frac{2^{n-1}}{|x-y|^{n-2}}.\end{aligned}$$

The last inequality follows from the fact that $L^{p,q}$ quasinorm are invariant by translation. □

Proof of Theorem 4.2.2. By the previous Lemma it is straightforward to bound G_1 by

$$|G_1(x,y)| \leqslant c_n^2\|a\|_{\frac{n}{n-2},\infty}\|W\|_{n/2,1}\frac{4}{|x-y|^{n-2}} = 4c_n\|a\|_{\frac{n}{n-2},\infty}\|W\|_{n/2,1}G_0(x,y)$$

and setting $C = 4c_n\|a\|_{\frac{n}{n-2},\infty}\|W\|_{n/2,1}$ we obtain by induction

$$|G_j(x,y)| \leqslant C^j G_0(x,y).$$

Indeed, assuming $|G_{j-1}(x,y)| \leqslant C^{j-1}G_0(x,y)$ and applying again Lemma 4.2.4 we directly obtain

$$\begin{aligned}|G_j(x,y)| \leqslant & c_n \int \frac{1}{|x-z|^{n-2}}|W(z)|\frac{C^{j-1}}{|z-y|^{n-2}}dz \\ \leqslant & c_n C^{j-1}\|a\|_{\frac{n}{n-2},\infty}\|W\|_{n/2,1}\frac{4}{|x-y|^{n-2}} = C^j G_0(x,y).\end{aligned}$$

The constant C is less than one thanks to the smallness of W and hence the series (4.2.5) is convergent. The bound on G follows directly from the one on G_j.

Finally, we check that G is indeed the kernel of $(-\Delta + W)^{-1}$. Let $\varphi, \psi \in C_0^\infty$ two test functions and $\langle\ ,\ \rangle$ the scalar product of L^2

$$\begin{aligned}\langle \psi, (-\Delta + W)^{-1}\varphi\rangle =& \langle \psi, \sum_{j\geqslant 0}(-1)^j(-\Delta)^{-1}(W(-\Delta)^{-1})^j\varphi\rangle \\ =& \sum_{j\geqslant 0}(-1)^j\langle \psi, (-\Delta)^{-1}(W(-\Delta)^{-1})^j\varphi\rangle \qquad (4.2.6) \\ =& \sum_{j\geqslant 0}(-1)^j\int \psi(x)G_j(x,y)\varphi(y)\ dxdy \\ =& \int \psi(x)\sum_{j\geqslant 0}(-1)^jG_j(x,y)\varphi(y)\ dxdy \qquad (4.2.7)\end{aligned}$$

$$= \int \psi(x)G(x,y)\varphi(y)\, dxdy$$

where to obtain (4.2.6) we used the fact that the series (4.2.4) is convergent with respect to the topology of $\mathcal{B}(\dot{H}^{-1}, \dot{H}^1)$ and for (4.2.7) we used the absolute convergence of the series (4.2.5).

□

4.3 Properties of a zero resonant state

The aim of this section is to prove Theorem 4.1.2. Using the decomposition $V = W + K$ we said that we can write the resonance ψ as solution of $(-\Delta + W)\psi = -K\psi$ and by Theorem 4.2.2 this solution is defined by

$$\psi(x) = -\int G(x,y)K(y)\psi(y)dy. \tag{4.3.1}$$

First of all we obtain that ψ is in $L^{\frac{2n}{n-2},\infty}$, more precisely we obtain the value of the limit of $|x|^{n-2}\psi$ as $|x|$ tends to infinity.

Proposition 4.3.1. *The function $|x|^{n-2}\psi(x)$ has finite limit when $|x| \to +\infty$ and therefore ψ is in $L^{\frac{n}{n-2},\infty}$. More precisely, the following holds*

$$\lim_{|x|\to\infty} |x|^{n-2}\psi(x) = -c_n \int V(y)\psi(y)dy < \infty.$$

Proof. We recall from Theorem 4.2.2 that $|G(x,y)| \lesssim \frac{1}{|x-y|^{n-2}}$, then using (4.3.1) for large enough $|x|$

$$|x|^{n-2}\psi(x) \lesssim |x|^{n-2} \int \frac{1}{|x-y|^{n-2}} K(y)\psi(y)dy \lesssim \int K(y)\psi(y)dy < \infty.$$

The integral of ψK is finite. Indeed, K has compact support, so we can equivalently consider the integral of $K\mathbb{1}_{supp\ K}\psi$. Here $\psi \in L^{\frac{2n}{n-2},\infty}$ and $\mathbb{1}_{supp\ K} \in L^{p,q}$ for any p and q, in particular $\mathbb{1}_{supp\ K} \in L^{\frac{2n}{n-2},\infty}$ implies $\mathbb{1}_{supp\ K}\psi \in L^{\frac{n}{n-2},\infty} = (L^{n/2,1})^*$ and hence $K\mathbb{1}_{supp\ K}\psi \in L^1$.

We can then obtain $\psi \in L^{\frac{n}{n-2},\infty}$. Let χ be a smooth cutoff which is equal to 1 on a large enough compact set then as we have seen above for the compact part it holds $\psi\chi \in L^{\frac{n}{n-2},\infty}$. For the part at infinity we can bound $\psi(1-\chi)$ by $\frac{1}{|x|^{n-2}} \in L^{\frac{n}{n-2},\infty}$. So applying (4.A.1) we conclude

$$\|\psi\|_{\frac{n}{n-2},\infty} \lesssim_n \|\chi\psi\|_{\frac{n}{n-2},\infty} + \|(1-\chi)\psi\|_{\frac{n}{n-2},\infty} < \infty.$$

Now to determine the value of the limit we need to study the behavior of $|x|^{n-2}G(x,y)$ for large $|x|$ and y that ranges in a compact set (the support of K). Using the second resolvent identity

$$(-\Delta + W)^{-1} = (-\Delta)^{-1} - (-\Delta)^{-1}W(-\Delta + W)^{-1} \tag{4.3.2}$$

we can write

$$|x|^{n-2}G(x,y) = c_n \frac{|x|^{n-2}}{|x-y|^{n-2}} - c_n \int \frac{|x|^{n-2}}{4\pi|x-z|^{n-2}} W(z)G(z,y)dz. \tag{4.3.3}$$

The first term in $|x|^{n-2}G(x,y)$ converges to c_n, for the second term we split the integral in the regions $B(0,|x|/2)$ and its complementary. First, taking $|x|$ large enough if $z \in B(0,|x|/2)^c$ then we will have

$$|z-y| \geqslant ||z| - |y|| = |z| - |y| \geqslant \frac{|x|}{2} - |y| > 0.$$

Using this bound we obtain

$$\begin{aligned}\int_{B(0,|x|/2)^c}\frac{|x|^{n-2}}{4\pi|x-z|^{n-2}}|W(z)G(z,y)|dz &\lesssim \int_{B(0,|x|/2)^c}\frac{1}{|x-z|^{n-2}}|W(z)|\frac{|x|^{n-2}}{(\frac{|x|}{2}-|y|)^{n-2}}dz\\ &\lesssim \int_{B(0,|x|/2)^c}\frac{1}{|x-z|^{n-2}}|W(z)|dz\\ &\lesssim \|a\|_{\frac{n}{n-2},\infty}\|W\mathbb{1}_{B(0,|x|/2)^c}\|_{n/2,1}\xrightarrow{|x|\to\infty}0 \qquad (4.3.4)\end{aligned}$$

where the convergence to 0 is due to the fact that the superlevel of $W\mathbb{1}_{B(0,|x|/2)^c}$ tends to the empty set as $|x|\to\infty$ and we can pass the limit in the integral defining $\|W\mathbb{1}_{B(0,|x|/2)^c}\|_{n/2,1}$ thanks to the domination

$$d_{W\mathbb{1}_{B(0,|x|/2)^c}}(t)^{2/n}\leqslant d_W(t)^{2/n}\in L^1(\mathbb{R}^+).$$

On the other hand, for $|x|\to\infty$ we have the pointwise convergence of

$$\mathbb{1}_{B(0,|x|/2)}(z)c_n\frac{|x|^{n-2}}{|x-z|^{n-2}}W(z)G(z,y)\to c_nW(z)G(z,y)$$

and since the points $z\in B(0,\frac{|x|}{2})$ satisfy $|x-z|>\frac{|x|}{2}$ we have the domination

$$\mathbb{1}_{B(0,|x|/2)}(z)\frac{|x|^{n-2}}{4\pi|x-z|^{n-2}}|W(z)G(z,y)|\lesssim|W(z)|\frac{1}{|z-y|^{n-2}}\in L^1.$$

So again by dominated convergence

$$c_n\int_{B(0,|x|/2)}\frac{|x|^{n-2}}{4\pi|x-z|^{n-2}}W(z)G(z,y)dz\xrightarrow{|x|\to\infty}c_n\int W(z)G(z,y)dz. \qquad (4.3.5)$$

Summing together (4.3.4) and (4.3.5) in (4.3.3) we obtain

$$\lim_{|x|\to\infty}|x|^{n-2}G(x,y)=c_n-c_n\int W(z)G(z,y)dz$$

and since $|x|^{n-2}G(x,y)\lesssim 1$ and $K\psi\in L^1$ we can pass the limit in the integral in (4.3.1). This yields

$$\lim_{|x|\to\infty}|x|^{n-2}\psi(x)=-c_n\int K(y)\psi(y)dy+c_n\int\left(\int W(z)G(z,y)dz\right)K(y)\psi(y)dy.$$

Now exchanging the order of integration in the second term and using relation (4.3.1) we have

$$\begin{aligned}\int\left(\int W(z)G(z,y)dz\right)K(y)\psi(y)dy&=\int W(z)\int G(z,y)K(y)\psi(y)dydz\\ &=-\int W(z)\psi(z)dz\end{aligned}$$

from which the statement follows since $V=W+K$. □

Thanks to the previous proposition we can derive the behavior of ψ at ∞. If $\int V\psi$ is non zero, then ψ is asymptotic to $\frac{1}{|x|^{n-2}}$ and therefore not in L^2. Otherwise we have the opposite result.

We will prove further results on the integrability of ψ thanks to the decay properties of ψ, hence a main step in the following proofs will be to prove that $|x|^\alpha\psi$ is bounded at infinity for a suitable α. We will let $R>0$ such that *supp* $K\subset B(0,R)$ and study the behavior of ψ in

$B(0,R)^c$. To do so we define the spaces

$$\mathcal{B}_\alpha = |x|^{-\alpha} L^\infty(B(0,R)^c) = \{ f \ : \ |x|^\alpha f \in L^\infty(B(0,R)^c) \ \} \tag{4.3.6}$$

with the natural norm

$$\|f\|_{\mathcal{B}_\alpha} = \| |x|^\alpha f \ \mathbb{1}_{B(0,R)^c} \|_\infty .$$

We will use a series of elementary but useful inequalities. The reader can find a proof of the following two lemmas in Appendix 4.B.

Lemma 4.3.2. *Let $x, y \in \mathbb{R}^n$, then*

a)

$$\left| \frac{1}{|x|} - \frac{1}{|x-y|} \right| \lesssim \frac{|y|}{|x||x-y|},$$

b)

$$\left| \frac{1}{|x|} + \frac{x \cdot y}{|x|^3} - \frac{1}{|x-y|} \right| \lesssim \frac{|y|^2}{|x|^2|x-y|},$$

c)

$$\left| \frac{1}{|x|} + \frac{x \cdot y}{|x|^3} + \frac{3(x \cdot y)^2}{2|x|^5} - \frac{|y|^2}{2|x|^3} - \frac{1}{|x-y|} \right| \lesssim \frac{|y|^3}{|x|^3|x-y|}.$$

Lemma 4.3.3. *Let $x, y \in \mathbb{R}^n$, then*

i)

$$\left| \frac{1}{|x|^2} - \frac{1}{|x-y|^2} \right| \leqslant \frac{|y|^2}{|x|^2|x-y|^2} + 2\frac{|y|}{|x||x-y|^2},$$

ii)

$$\left| \frac{1}{|x|^2} + \frac{2x \cdot y}{|x|^4} - \frac{1}{|x-y|^2} \right| \lesssim \frac{|y|^2}{|x|^2|x-y|^2} + \frac{2|y|^3}{|x|^3|x-y|^2}$$

iii)

$$\left| \frac{1}{|x|^2} + \frac{2x \cdot y}{|x|^4} - \frac{|y|^2}{|x|^4} + \frac{(2x \cdot y)^2}{|x|^6} - \frac{1}{|x-y|^2} \right| \lesssim \frac{|y|^3}{|x|^3|x-y|^2} + \frac{|y|^4}{|x|^4|x-y|^2}$$

Proposition 4.3.4. *If $\int V(y)\psi(y)dy = 0$ then $\psi \in L^2$.*

We obtain this result via the following lemma.

Lemma 4.3.5. *If $\int V(y)\psi(y)dy = 0$ then $|x|^{n-1}\psi$ is bounded outside of a compact set, in other words $\psi \in \mathcal{B}_{n-1}$.*

Proof of Proposition 4.3.4. From the previous lemma $|x|\psi \lesssim |x|^{2-n}$ as $|x|$ tends to infinity which implies

$$|x|\psi \in L^{\frac{n}{n-2},\infty}.$$

We can then write

$$\psi(x) = \frac{1}{|x|} b(x) \quad \text{with} \quad \frac{1}{|x|} \in L^{n,\infty}, b \in L^{\frac{n}{n-2},\infty}$$

that by Hölder inequality (4.A.3) implies $\psi \in L^{\frac{n}{n-1},\infty}$ with $\frac{n}{n-1} < 2$. Combining with Proposition 4.3.1 up to now we have

$$\psi \in L^{\frac{n}{n-1},\infty} \cap L^{\frac{n}{n-2},\infty} = \begin{cases} L^{3/2,\infty}(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3) & n = 3, \\ L^{4/3,\infty}(\mathbb{R}^4) \cap L^{2,\infty}(\mathbb{R}^4) & n = 4. \end{cases}$$

In dimension three we can conclude $\psi \in L^2$ by interpolation since $2 \in (3/2, 3)$. For the case $n = 4$ we remark that with the same reasoning as in the proof of Proposition 4.3.1 we can show that $|x|\psi$ has a finite limit in $\mathbb{R}^4$ when $|x|$ tends to infinity, which in turn implies $\psi \in L^{4,\infty}(\mathbb{R}^4)$ and we can again conclude that $\psi \in L^2(\mathbb{R}^4)$ by interpolation since $2 \in (4/3, 4)$.

□

Proof of Lemma 4.3.5. We recall that ψ is solution to $(-\Delta + V)\psi = 0$, so given the kernel of Δ^{-1} on $\mathbb{R}^n$ we can write

$$\psi(x) = -c_n \int \frac{1}{|x-y|^{n-2}} V(y)\psi(y)dy.$$

By adding the quantity $\int V\psi = 0$ we obtain

$$\psi(x) = c_n \int \left(\frac{1}{|x|^{n-2}} - \frac{1}{|x-y|^{n-2}} \right) V(y)\psi(y)dy.$$

We recall the decomposition of the potential $V = W + K$ and the radius $R > 0$ such that $suppK \subset B(0, R)$. We split the previous integral obtaining

$$\psi(x) = \underbrace{c_n \int_{|y|<R} \left(\frac{1}{|x|^{n-2}} - \frac{1}{|x-y|^{n-2}} \right) V(y)\psi(y)dy}_{=:f(x)} + c_n \int_{|y|\geqslant R} \left(\frac{1}{|x|^{n-2}} - \frac{1}{|x-y|^{n-2}} \right) W(y)\psi(y)dy.$$

Defining the operator

$$\mathcal{S} : \varphi \mapsto c_n \int_{|y|\geqslant R} \left(\frac{1}{|x|^{n-2}} - \frac{1}{|x-y|^{n-2}} \right) W(y)\varphi(y)dy \tag{4.3.7}$$

we can rewrite the previous identity as

$$\psi = f + \mathcal{S}\psi. \tag{4.3.8}$$

We will prove that the operator $\mathcal{S}$ is a contraction in the spaces $\mathcal{B}_{n-1}, \mathcal{B}_{n-2}$ and that $f \in \mathcal{B}_{n-1} \subset \mathcal{B}_{n-2}$, hence the equation

$$\varphi = f + \mathcal{S}\varphi \tag{4.3.9}$$

has a unique solution in $\mathcal{B}_{n-1}$ and one in $\mathcal{B}_{n-2}$.

Identity (4.3.8) together with Proposition 4.3.1 tell us that ψ is the unique solution in $\mathcal{B}_{n-2}$ of the fixed point problem (4.3.9). Moreover we also have uniqueness of the solution of (4.3.9) in $\mathcal{B}_{n-1} \subset \mathcal{B}_{n-2}$, so ψ must coincide with the solution in $\mathcal{B}_{n-1}$, implying $\psi \in \mathcal{B}_{n-1}$ that is the statement.

To prove the needed properties on f and $\mathcal{S}$ we will use item $a)$ Lemma 4.3.2 and item $i)$ Lemma 4.3.3. Given the different form of the right hand sides in such inequalities, we differentiate the rest of the proof for the case $n = 3$ and $n = 4$ since the computations differ in some places.

- **Case** $n = 3$**.** We need to show that $|x|^2 f$ is bounded in $B(0, R)^c$. Thanks to Lemma 4.3.2 for $|x| \geqslant 2R$

$$|f(x)| \leqslant c_3 \frac{R}{|x|(|x| - R)} \left(\int |V\psi| \right)$$

and hence $|x|^2 f$ is bounded for $|x| \geqslant 2R$. Moreover, $|x|^2 f$ is also bounded for $R \leqslant |x| < 2R$

rewriting f as

$$f(x) = c_n \frac{1}{|x|^{n-2}} \int_{|y|<R} V(y)\psi(y)dy + \psi(x) + c_n \int_{|y|\geqslant R} \frac{1}{|x-y|^{n-2}} V(y)\psi(y)dy \qquad (4.3.10)$$

for which in the case $n = 3$ it holds

$$|f(x)| \leqslant c_3 \frac{1}{|x|} \int |V\psi| + |\psi(x)| + c_3 \int_{|y|\geqslant R} \frac{1}{|x-y|} |V(y)\psi(y)|dy.$$

Here $|x|^2|\psi|$ is bounded for $R \leqslant |x| \leqslant 2R$, thanks to the fact that $|x|\psi$ converges to 0 as $|x|$ tends to infinity, and

$$|x|^2 \int_{|y|\geqslant R} \frac{1}{|x-y|} |V(y)\psi(y)|dy \lesssim \int \frac{1}{|x-y|} |V(y)|dy \lesssim \|V\|_{3/2,1}\|a\|_{3,\infty}.$$

We have obtained $f \in \mathcal{B}_2 \subset \mathcal{B}_1$.

Now we only need to show that $\mathcal{S}$ is a contraction on $\mathcal{B}_1$ and $\mathcal{B}_2$. On $\mathcal{B}_1$ it is straightforward to obtain

$$\begin{aligned} |\mathcal{S}\varphi(x)| &\leqslant \int_{|y|\geqslant R} \frac{|y||\varphi(y)|}{|x||x-y|} |W(y)|dy \leqslant \frac{1}{|x|} \|\varphi\|_{\mathcal{B}_1} \int \frac{|W(y)|}{|x-y|} dy \\ &\leqslant \frac{1}{|x|} \|\varphi\|_{\mathcal{B}_1} \|W\|_{3/2,1} \|a\|_{3,\infty} \end{aligned}$$

which implies $\|\mathcal{S}\varphi\|_{\mathcal{B}_1} \leqslant \|W\|_{3/2,1}\|a\|_{3,\infty}\|\varphi\|_{\mathcal{B}_1}$ where the constant $\|W\|_{3/2,1}\|a\|_{3,\infty}$ is smaller than one thanks to the smallness of W. We have obtained that $\mathcal{S}$ is a contraction on $\mathcal{B}_1$.

For $\mathcal{B}_2$ we split the domain of integration in the regions

$$\begin{aligned} \{|y| \geqslant R\} &= \left\{|y| \geqslant R \text{ and } |y| < \frac{|x|}{2}\right\} \sqcup \left\{|y| \geqslant R \text{ and } \frac{|x|}{2} \leqslant |y| \leqslant 2|x|\right\} \\ &\quad \sqcup \{|y| \geqslant R \text{ and } |y| > 2|x|\} \\ &=: E_< \sqcup E_\approx \sqcup E_>. \end{aligned} \qquad (4.3.11)$$

Both in $E_<$ and $E_>$ we have the lower bound $|x-y| \gtrsim |x|$ so using Lemma 4.3.2-a) again

$$\begin{aligned} \left| \int_{E_< \cup E_>} \left(\frac{1}{|x|} - \frac{1}{|x-y|} \right) W(y)\varphi(y)dy \right| &\leqslant \frac{2}{|x|^2} \int_{E_< \cup E_>} |W(y)\varphi(y)||y|dy \\ &\leqslant \frac{2}{|x|^2} \|\varphi\|_{\mathcal{B}_2} \int \frac{|W(y)|}{|y|} dy \\ &\leqslant \frac{2}{|x|^2} \|\varphi\|_{\mathcal{B}_2} \|W\|_{3/2,1} \|a\|_{3,\infty}. \end{aligned} \qquad (4.3.12)$$

On $E_\approx$ instead

$$|\varphi(y)| \leqslant \frac{\|\varphi\|_{\mathcal{B}_2}}{|y|^2} \leqslant 2 \frac{\|\varphi\|_{\mathcal{B}_2}}{|x||y|}$$

which we use in

$$\begin{aligned} \left| \int_{E_\approx} \left(\frac{1}{|x|} - \frac{1}{|x-y|} \right) W(y)\varphi(y)dy \right| &\leqslant \int_{E_\approx} \frac{|y|}{|x||x-y|} |W(y)\varphi(y)|dy \\ &\leqslant \frac{2}{|x|^2} \|\varphi\|_{\mathcal{B}_2} \int \frac{|W(y)|}{|x-y|} dy \end{aligned}$$

$$\leqslant \frac{2}{|x|^2}\|\varphi\|_{\mathcal{B}_2}\|W\|_{3/2,1}\|a\|_{3,\infty}. \tag{4.3.13}$$

Combining (4.3.12) and (4.3.13) we have

$$|x|^2|\mathcal{S}\varphi(x)| \lesssim \|\varphi\|_{\mathcal{B}_2}\|W\|_{3/2,1}\|a\|_{3,\infty}, \qquad |x| \geqslant R$$

from which $\mathcal{S}$ is a contraction on $\mathcal{B}_2$ and we conclude the proof for the three dimensional case.

- **Case** $n = 4$**.** The proof is analogous to the case $n = 3$, but now we use item $i)$ in Lemma 4.3.3 to prove the necessary inequalities.

  First we show that $f \in \mathcal{B}_3$: if $|x| \geqslant 2R$ from Lemma 4.3.3

$$|f(x)| \leqslant c_4\left(\frac{R^2}{|x|^2(|x|-R)^2} + 2\frac{R}{|x|(|x|-R)^2}\right)\int |V(y)\psi(y)|dy,$$

  hence $|x|^3|f(x)|$ is bounded for $|x| \geqslant 2R$. If $R \leqslant |x| \leqslant 2R$ we obtain $|x|^3|f(x)|$ bounded by using again (4.3.10) and the fact that $|x|^2\psi$ converges to zero in $\mathbb{R}^4$.

  We are left with the proof of the fact that $\mathcal{S}$ is a contraction on $\mathcal{B}_2$ and $\mathcal{B}_3$.

  First we remark that

$$\int_{|y|\geqslant R} \frac{|y|^2}{|x|^2|x-y|^2}|W(y)||\varphi(y)|dy \leqslant \frac{1}{|x|^2}\|\varphi\|_{\mathcal{B}_2}\|W\|_{2,1}\|a\|_{2,\infty}. \tag{4.3.14}$$

  Moreover, on $E_>$ and $E_\approx$ we have $|y|^{-1} \lesssim |x|^{-1}$, yielding as above

$$\int_{E_>\cup E_\approx} \frac{|y|}{|x||x-y|^2}|W(y)||\varphi(y)|dy \lesssim \frac{1}{|x|^2}\|\varphi\|_{\mathcal{B}_2}\|W\|_{2,1}\|a\|_{2,\infty}, \tag{4.3.15}$$

  viceversa on $E_<$ we use the inequality $|x|^{-1} \lesssim |y|^{-1}$ and Lemma 4.2.4 to obtain

$$\int_{E_<} \frac{|y|}{|x||x-y|^2}|W(y)||\varphi(y)|dy \lesssim \|\varphi\|_{\mathcal{B}_2}\int \frac{|W(y)|}{|y|^2|x-y|^2}dy \lesssim \frac{1}{|x|^2}\|\varphi\|_{\mathcal{B}_2}\|W\|_{2,1}\|a\|_{2,\infty}. \tag{4.3.16}$$

  Combining equations (4.3.14)-(4.3.16) and thanks to Lemma 4.3.3 we have

$$|x|^2|\mathcal{S}\varphi(x)| \lesssim \|\varphi\|_{\mathcal{B}_2}\|W\|_{2,1}\|a\|_{2,\infty}, \qquad |x| \geqslant R$$

  and $\mathcal{S}$ is a contraction on $\mathcal{B}_2$ thanks to the smallness of W.

  On $\mathcal{B}_3$ we consider again the partition $E_<, E_<, E_\approx$. On $E_<$

$$\begin{aligned}\int_{E_<} \frac{|y|^2}{|x|^2|x-y|^2}|W(y)||\varphi(y)|dy &\lesssim \frac{1}{|x|}\int_{E_<} \frac{|W(y)|}{|y|^2|x-y|^2}|y|^3|\varphi(y)|dy\\ &\lesssim \frac{1}{|x|^3}\|\varphi\|_{\mathcal{B}_3}\|W\|_{2,1}\|a\|_{2,\infty},\end{aligned}$$

  since $|x-y| \geqslant |x|/2 \geqslant |y|$, and analogously

$$\begin{aligned}\int_{E_<} \frac{|y|}{|x||x-y|^2}|W(y)||\varphi(y)|dy &= \frac{1}{|x|}\int_{E_<} \frac{|W(y)|}{|y|^2|x-y|^2}|y|^3|\varphi(y)|dy\\ &\lesssim \frac{1}{|x|^3}\|\varphi\|_{\mathcal{B}_3}\|W\|_{2,1}\|a\|_{2,\infty}.\end{aligned}$$

On $E_>$ we recall $|y|^{-1} \lesssim |x|^{-1}$ hence

$$\int_{E_>} \left(\frac{|y|^2}{|x|^2|x-y|^2} + \frac{|y|}{|x||x-y|^2} \right) |W(y)||\varphi(y)|dy \lesssim \frac{1}{|x|^3} \|\varphi\|_{\mathcal{B}_3} \int \frac{|W(y)|}{|x-y|^2} dy$$
$$\lesssim \frac{1}{|x|^3} \|\varphi\|_{\mathcal{B}_3} \|W\|_{2,1} \|a\|_{2,\infty}$$

Finally on $E_\approx$ we use the inequalities

$$|y|^2|\varphi(y)| \lesssim \frac{\|\varphi\|_{\mathcal{B}_3}}{|x|} \quad \text{and} \quad |y||\varphi(y)| \lesssim \frac{\|\varphi\|_{\mathcal{B}_3}}{|x|^2}.$$

We conclude again that

$$|x|^3|\mathcal{S}\varphi(x)| \lesssim \|\varphi\|_{\mathcal{B}_3}\|W\|_{2,1}\|a\|_{2,\infty}, \qquad |x| \geqslant R$$

and therefore $\mathcal{S}$ is a contraction on $\mathcal{B}_3$.

□

Up to now we have found the following properties

- $\psi \in L^{\frac{n}{n-2},\infty}$,
- $\int V\psi = 0$ if and only if $\psi \in L^2$.

We will now prove, under suitable assumptions, that a zero eigenfunction is in the weak Lebesgue space $L^{1,\infty}$.

Proposition 4.3.6. *If $\int y_k V\psi dy = \int V\psi = 0$ for all $k = 1, \dots, n$ then $\psi \in L^{1,\infty}$.*

As for the case of square integrability we obtain the result thanks to the behavior at infinity of ψ, more precisely the proposition follows directly from the following lemma.

Lemma 4.3.7. *If $\int y_k V\psi dy = \int V\psi = 0$ for all $k = 1, \dots, n$ then $|x|^n\psi$ is bounded outside of a compact set, or in other words $\psi \in \mathcal{B}_n$.*

Proof. The proof uses the same approach of Lemma 4.3.4, as in we will write ψ as the solution of a fixed point problem, namely

$$\begin{aligned}\psi(x) =& c_n \int \left(\frac{1}{|x|^{n-2}} + (n-2)\frac{x \cdot y}{|x|^n} - \frac{1}{|x-y|^{n-2}} \right) V(y)\psi(y)dy \\ =& \underbrace{c_n \int_{|y|\leqslant R} \left(\frac{1}{|x|^{n-2}} + (n-2)\frac{x \cdot y}{|x|^n} - \frac{1}{|x-y|^{n-2}} \right) V(y)\psi(y)dy}_{=:g(x)} \\ &+ c_n \int_{|y|\geqslant R} \left(\frac{1}{|x|^{n-2}} + (n-2)\frac{x \cdot y}{|x|^n} - \frac{1}{|x-y|^{n-2}} \right) W(y)\psi(y)dy. \end{aligned} \tag{4.3.17}$$

where we added quantities which are zero by assumption. We will use item *b*) in Lemma 4.3.2 and item *ii*) in Lemma 4.3.3. Here we define

$$\mathcal{T}\varphi := c_n \int_{|y|\geqslant R} \left(\frac{1}{|x|^{n-2}} + (n-2)\frac{x \cdot y}{|x|^n} - \frac{1}{|x-y|^{n-2}} \right) W(y)\varphi(y)dy$$

and we will prove that the fixed point problem

$$\varphi = g + \mathcal{T}\varphi$$

has a unique solution in $\mathcal{B}_n$ and $\mathcal{B}_{n-1}$ by showing that $\mathcal{T}$ is a contraction on both spaces and $g \in \mathcal{B}_n \subset \mathcal{B}_{n-1}$. From Lemma 4.3.4 and (4.3.17) we already know that ψ is the unique solution in $\mathcal{B}_{n-1}$, then the inclusion $\mathcal{B}_n \subset \mathcal{B}_{n-1}$ implies $\psi \in \mathcal{B}_n$ as we wished.

As in Lemma 4.3.4 we separate the cases $n = 3$ and $n = 4$ to do the computations, which for a large part are analogous to Lemma 4.3.4.

- **Case** $n = 3$**.** We can show that $|x|^3|g(x)|$ is bounded in $B(0,R)^c$ with the same reasoning of the previous lemma. We separate the regions $|x| \geqslant 2R$ and $R \leqslant |x| \leqslant 2R$ and use Lemma 4.3.2-b) and the fact that $|x|^2\psi$ is bounded in $B(0,R)^c$.

  That $\mathcal{T}$ is a contraction on $\mathcal{B}_2$ is a direct consequence of inequality b) in Lemma 4.3.2. For the proof in $\mathcal{B}_3$ we consider the usual division $E_<, E_>, E_\approx$. In $E_<$ and $E_>$ we have $|x-y| \gtrsim |x|$ and hence, using Lemma 4.3.2,

$$\Big|\int_{E_<\cup E_>}\Big(\frac{1}{|x|}+\frac{x\cdot y}{|x|^3}-\frac{1}{|x-y|}\Big)W(y)\varphi(y)dy\Big| \lesssim \frac{1}{|x|^3}\Big|\int_{E_<\cup E_>}\frac{|W(y)|}{|y|}|y|^3|\varphi(y)|dy$$
$$\lesssim \frac{1}{|x|^3}\|\varphi\|_{\mathcal{B}_3}\|W\|_{3/2,1}\|a\|_{3,\infty}.$$

  Finally in $E_\approx$ we obtain the same bound using Lemma 4.3.2 and

$$|y|^2|\varphi| = \frac{\|\varphi\|_{\mathcal{B}_3}}{|y|} \lesssim \frac{\|\varphi\|_{\mathcal{B}_3}}{|x|}. \tag{4.3.18}$$

  Combining the contributions of the three regions $E_<, E_>$ and $E_\approx$ we have proved

$$|x|^3|\mathcal{T}\varphi(x)| \lesssim \|\varphi\|_{\mathcal{B}_3}\|W\|_{3/2,1}\|a\|_{3,\infty}, \qquad |x| \geqslant R$$

  and therefore that $\mathcal{T}$ is a contraction on $\mathcal{B}_3$ concluding the proof in $\mathbb{R}^3$.

- **Case** $n = 4$**.** The proof that $g \in \mathcal{B}_4$ is analogous to the three dimensional case thanks to Lemma 4.3.3 and to the property $\psi \in \mathcal{B}_3$. Then, $\mathcal{T}$ is a contraction on $\mathcal{B}_3$ since

$$\int_{|y|\geqslant R}\frac{|y|^3}{|x|^3|x-y|^2}|W(y)||\varphi(y)|dy \leqslant \frac{1}{|x|^3}\|\varphi\|_{\mathcal{B}_3}\|W\|_{2,1}\|a\|_{2,\infty}$$

  and

$$\int_{E_>\cup E_\approx}\frac{|y|^2}{|x|^2|x-y|^2}|W(y)||\varphi(y)|dy \lesssim \frac{1}{|x|^3}\|\varphi\|_{\mathcal{B}_3}\|W\|_{2,1}\|a\|_{2,\infty}$$

  thanks to $|y|^{-1} \lesssim |x|^{-1}$, while by Lemma 4.2.4

$$\int_{E_<}\frac{|y|^2}{|x|^2|x-y|^2}|W(y)||\varphi(y)|dy \lesssim \frac{1}{|x|}\|\varphi\|_{\mathcal{B}_3}\int\frac{|W(y)|}{|x-y|^2|y|^2}dy$$
$$\lesssim \frac{1}{|x|^3}\|\varphi\|_{\mathcal{B}_3}\|W\|_{2,1}\|a\|_{2,\infty}.$$

  Now we just need to prove that $\mathcal{T}$ is a contraction on $\mathcal{B}_4$ to conclude. First of all we remark

$$\int_{|y|\geqslant R}\frac{|y|^2}{|x|^2|x-y|^2}|W(y)||\varphi(y)|dy \lesssim \frac{1}{|x|^2}\|\varphi\|_{\mathcal{B}_4}\int_{|y|\geqslant R}\frac{|W(y)|}{|y|^2|x-y|^2}dy$$
$$\lesssim \frac{1}{|x|^4}\|\varphi\|_{\mathcal{B}_4}\|W\|_{2,1}\|a\|_{2,\infty}.$$

On $E_>$ and $E_\approx$ we use the inequality

$$|y|^3|\varphi(y)| \lesssim \frac{\|\varphi\|_{\mathcal{B}_4}}{|x|}$$

to bound

$$\int_{E_> \cup E_\approx} \frac{|y|^3}{|x|^3|x-y|^2}|W(y)\varphi(y)|dy \lesssim \frac{1}{|x|^4}\|\varphi\|_{\mathcal{B}_4}\|W\|_{2,1}\|a\|_{2,\infty}.$$

On $E_<$ we use again Lemma 4.2.4 and the bound $|x|^{-1} \lesssim |y|^{-1}$:

$$\int_{E_<} \frac{|y|^3}{|x|^3|x-y|^2}|W(y)||\varphi(y)|dy \lesssim \frac{1}{|x|^2}\|\varphi\|_{\mathcal{B}_4}\int \frac{|W(y)|}{|x-y|^2|y|^2}dy \lesssim \frac{1}{|x|^4}\|\varphi\|_{\mathcal{B}_4}\|W\|_{2,1}\|a\|_{2,\infty}.$$

We have just obtained

$$|x|^4|\mathcal{T}\varphi(x)| \lesssim \|\varphi\|_{\mathcal{B}_4}\|W\|_{2,1}\|a\|_{2,\infty}, \qquad |x| \geqslant R$$

which proves that $\mathcal{T}$ is a contraction on $\mathcal{B}_4$.

□

Finally, we conclude the argument by proving that ψ is in L^1.

Proposition 4.3.8. *If $\int y_k y_l V\psi dy = \int y_k V\psi dy = \int V\psi = 0$ for all $k, l = 1, \ldots, n$ then $\psi \in L^1$.*

This follows from the asymptotic behavior of ψ:

Lemma 4.3.9. *If $\int y_k y_l V\psi dy = \int y_k V\psi dy = \int V\psi = 0$ for all $k, l = 1, \ldots, n$ then $|x|^{n+1}\psi$ is bounded outside of a compact set, or in other words $\psi \in \mathcal{B}_{n+1}$.*

Proof of Proposition 4.3.8. By the fact that $|x|^{n+1}\psi$ is bounded outside of a compact set we obtain $|x|\psi \in L^{1,\infty}$ and

$$\psi(x) = \frac{1}{|x|}c(x) \quad \text{with } \frac{1}{|x|} \in L^{n,\infty},\ c \in L^{1,\infty}.$$

By Hölder inequality $\psi \in L^{\frac{n}{n+1},\infty}$. Since $\psi \in L^{\frac{n}{n+1},\infty} \cap L^{\frac{n}{n-2},\infty}$ we can conclude by interpolation, given that $1 \in (\frac{n}{n+1}, \frac{n}{n-2})$. □

The main argument to prove Lemma 4.3.9 is the same as Lemmas 4.3.5 and 4.3.7. However, the definition of the contraction is slightly different in dimension three and four, so we separate the two proofs from the beginning.

Proof of Lemma 4.3.9 in the case $n = 3$. By assumption

$$\int \left(\frac{3(x\cdot y)^2}{2|x|^5} - \frac{|y|^2}{2|x|^3}\right)V\psi = 0,$$

so we can rewrite $\psi = -(-\Delta)^{-1}V\psi$ as

$$\begin{aligned}\psi(x) &= \int \left(\frac{1}{|x|} + \frac{x\cdot y}{|x|^3} + \frac{3(x\cdot y)^2}{2|x|^5} - \frac{|y|^2}{2|x|^3} - \frac{1}{|x-y|}\right)V(y)\psi(y)dy \\ &= \underbrace{\int_{|y|<R} \left(\frac{1}{|x|} + \frac{x\cdot y}{|x|^3} + \frac{3(x\cdot y)^2}{2|x|^5} - \frac{|y|^2}{2|x|^3} - \frac{1}{|x-y|}\right)V(y)\psi(y)dy}_{:=h(x)}\end{aligned}$$

$$+\int_{|y|\geqslant R}\left(\frac{1}{|x|}+\frac{x\cdot y}{|x|^3}+\frac{3(x\cdot y)^2}{2|x|^5}-\frac{|y|^2}{2|x|^3}-\frac{1}{|x-y|}\right)V(y)\psi(y)dy.$$

As in the previous cases we define

$$\mathcal{U}\varphi=\int_{|y|\geqslant R}\left(\frac{1}{|x|}+\frac{x\cdot y}{|x|^3}+\frac{3(x\cdot y)^2}{|x|^5}+\frac{|y|^2}{|x|^3}-\frac{1}{|x-y|}\right)W(y)\varphi(y)dy$$

and we prove that the equation

$$\varphi=h+\mathcal{U}\varphi$$

has a unique solution in $\mathcal{B}_3$ and $\mathcal{B}_4$ by proving that $\mathcal{U}$ is a contraction in these spaces and $h\in\mathcal{B}_4\subset\mathcal{B}_3$. Since we already know that ψ is the unique solution in $\mathcal{B}_3$ we can conclude as usual that $\psi\in\mathcal{B}_4$ since $\mathcal{B}_4\subset\mathcal{B}_3$.

To see that $h\in\mathcal{B}_4$ we use Lemma 4.3.2-*c*). If $|x|\geqslant 2R$ then $|x-y|\geqslant|x|-R$ and the quantity

$$|x|^4|h(x)|\lesssim|x|^4\int_{|y|<R}\frac{|y|^3}{|x|^3(|x|-R)}|V\psi|dy\lesssim\frac{|x|^4}{|x|^3(|x|-R)}\|V\|_{3/2,1}\|\psi\|_{3,\infty}$$

is bounded. In the region $R\leqslant|x|\leqslant 2R$ we write h as

$$h(x)=\int_{|y|<R}\left(\frac{1}{|x|}+\frac{x\cdot y}{|x|^3}+\frac{(x\cdot y)^2}{|x|^5}+\frac{|y|^2}{|x|^3}\right)V(y)\psi(y)dy+\psi(x)+\int_{|y|\geqslant R}\frac{1}{|x-y|}V\psi dy. \tag{4.3.19}$$

Thanks to the property $\psi\in\mathcal{B}_3$ and $V\psi\in L^1$ we can deduce that $|x|^4|h(x)|$ is bounded also for $R\leqslant|x|\leqslant 2R$.

We can directly obtain that $\mathcal{U}$ is a contraction on $\mathcal{B}_3$ by using Lemma 4.3.2 which implies

$$|\mathcal{U}\varphi|\lesssim\frac{1}{|x|^3}\int_{|y|\geqslant R}\frac{|W(y)|}{|x-y|}|y|^3|\varphi(y)|dy\leqslant\frac{1}{|x|^3}\|\varphi\|_{\mathcal{B}_3}\|W\|_{3/2,1}\|a\|_{3,\infty}$$

where we recall $\|W\|_{3/2,1}$ is arbitrarily small. In $\mathcal{B}_4$ we separate as usual $E_<,E_>,E_\approx$. On $E_<$ and $E_>$, $|x-y|\gtrsim|x|$ so

$$\begin{aligned}\int_{E_<\cup E_>}\frac{|y|^3}{|x|^3|x-y|}|W(y)\varphi(y)|&\lesssim\frac{1}{|x|^4}\int_{E_<\cup E_>}\frac{|W(y)|}{|y|}|y|^4|\varphi(y)|dy\\&\lesssim\frac{1}{|x|^4}\|\varphi\|_{\mathcal{B}_4}\|W\|_{3/2,1}\|a\|_{3,\infty}.\end{aligned}$$

On $E_\approx$ we use

$$|y|^3\varphi=\frac{\|\varphi\|_{\mathcal{B}_4}}{|y|}\lesssim\frac{\|\varphi\|_{\mathcal{B}_4}}{|x|}$$

to obtain the same bound. We have found

$$|x|^4|\mathcal{U}\varphi(x)|\lesssim\|\varphi\|_{\mathcal{B}_4}\|W\|_{3/2,1}$$

with $\|W\|_{3/2,1}\ll 1$ hence $\mathcal{U}$ is a contraction on $\mathcal{B}_4$, which concludes the proof.

□

Proof of Lemma 4.3.9 in the case $n=4$. By assumption the quantities

$$\int V\psi,\quad\int(x\cdot y)V\psi,\quad\int|y|^2V\psi,\quad\int(x\cdot y)^2V\psi$$

are all equal to zero, hence we can rewrite ψ as

$$\psi(x) = \int \left(\frac{1}{|x|^2} + \frac{2x\cdot y}{|x|^4} - \frac{|y|^2}{|x|^4} + \frac{(2x\cdot y)^2}{|x|^6} - \frac{1}{|x-y|^2} \right) V(y)\psi(y)dy$$
$$= \underbrace{\int_{|y|<R} \left(\frac{1}{|x|^2} + \frac{2x\cdot y}{|x|^4} - \frac{|y|^2}{|x|^4} + \frac{(2x\cdot y)^2}{|x|^6} - \frac{1}{|x-y|^2} \right) V(y)\psi(y)dy}_{=:j(x)}$$
$$+ \int_{|y|\geqslant R} \left(\frac{1}{|x|^2} + \frac{2x\cdot y}{|x|^4} - \frac{|y|^2}{|x|^4} + \frac{(2x\cdot y)^2}{|x|^6} - \frac{1}{|x-y|^2} \right) W(y)\psi(y)dy.$$

We need to show that

$$\mathcal{V}\varphi := \int_{|y|\geqslant R} \left(\frac{1}{|x|^2} + \frac{2x\cdot y}{|x|^4} - \frac{|y|^2}{|x|^4} + \frac{(2x\cdot y)^2}{|x|^6} - \frac{1}{|x-y|^2} \right) W(y)\varphi(y)dy$$

is a contraction on $\mathcal{B}_4$ and $\mathcal{B}_5$ as well as $j \in \mathcal{B}_5 \subset \mathcal{B}_4$. This will imply that

$$\varphi = j + \mathcal{V}\varphi$$

has a unique solution in $\mathcal{B}_4$ and $\mathcal{B}_5$. Since ψ is the solution in $\mathcal{B}_4$ it implies $\psi \in \mathcal{B}_5$. This can be done thanks to inequality *iii*) in Lemma 4.3.3. Indeed, $j \in \mathcal{B}_5$ since for $|x| \geqslant 2R$ we have

$$\frac{1}{|x|^3|x-y|^2} \leqslant \frac{1}{|x|^3(|x|-R)^2} \qquad \frac{1}{|x|^4|x-y|^2} \leqslant \frac{1}{|x|^4(|x|-R)^2},$$

while for $R \leqslant |x| \leqslant 2R$ we rewrite j as

$$j(x) = \int_{|y|<R} \left(\frac{1}{|x|^2} + \frac{2x\cdot y}{|x|^4} - \frac{|y|^2}{|x|^4} + \frac{(2x\cdot y)^2}{|x|^6} \right) V(y)\psi(y)dy + \psi(x)$$
$$+ \int_{|y|\geqslant R} \frac{1}{|x-y|^2} V(y)\psi(y)dy$$

where both x and y are in bounded regions and $\psi \in \mathcal{B}_4$ by Lemma 4.3.7.

Now we prove that $\mathcal{V}$ is a contraction on $\mathcal{B}_4$, using item *iii*) we need to consider

$$\int_{|y|\geqslant R} \frac{|y|^4}{|x|^4|x-y|^2} |W(y)\varphi(y)|dy \leqslant \frac{1}{|x|^4} \|\varphi\|_{\mathcal{B}_4} \|W\|_{2,1} \|a\|_{2,\infty} \tag{4.3.20}$$

and

$$\int_{|y|\geqslant R} \frac{|y|^3}{|x|^3|x-y|^2} |W(y)\varphi(y)|dy.$$

For the latter term we separate the regions $E_>, E_\approx$, where $|y|^{-1} \lesssim |x|^{-1}$ allows us to conclude as in (4.3.20), and $E_<$ where $|x|^{-1} \lesssim |y|^{-1}$ leads us to use Lemma 4.2.4

$$\int_{E_<} \frac{|y|^3}{|x|^3|x-y|^2} |W(y)\varphi(y)|dy \lesssim \frac{1}{|x|^2} \|\varphi\|_{\mathcal{B}_4} \int_{E_<} \frac{|W(y)|}{|x-y|^2|y|^2} dy \lesssim \frac{1}{|x|^4} \|\varphi\|_{\mathcal{B}_4} \|W\|_{2,1} \|a\|_{2,\infty}.$$

The inequality we obtain

$$|x|^4 |\mathcal{V}\varphi(x)| \lesssim \|\varphi\|_{\mathcal{B}_4} \|W\|_{2,1} \|a\|_{2,\infty}, \qquad |x| \geqslant R$$

implies that $\mathcal{V}$ is a contraction on $\mathcal{B}_4$. On $\mathcal{B}_5$ thanks to Lemma 4.2.4 we first remark that

$$\begin{aligned}\int_{|y|\geqslant R}\frac{|y|^3}{|x|^3|x-y|^2}|W(y)\varphi(y)|dy \lesssim &\frac{1}{|x|^3}\int_{|y|\geqslant R}\frac{|W(y)|}{|x-y|^2|y|^2}|y|^5|\varphi(y)|dy\\ \lesssim &\frac{1}{|x|^5}\|\varphi\|_{\mathcal{B}_5}\|W\|_{2,1}\|a\|_{2,\infty}.\end{aligned} \tag{4.3.21}$$

Then on $E_>$ and $E_\approx$ where $|y|^{-1}\lesssim|x|^{-1}$

$$\int_{E_>\cup E_\approx}\frac{|y|^4}{|x|^4|x-y|^2}|W(y)\varphi(y)|dy \lesssim \frac{1}{|x|^5}\|\varphi\|_{\mathcal{B}_5}\int_{E_>}\frac{|W(y)|}{|x-y|^2}dy \lesssim \frac{1}{|x|^5}\|\varphi\|_{\mathcal{B}_5}\|W\|_{2,1}\|a\|_{2,\infty}, \tag{4.3.22}$$

while on $E_<$, where $|x|^{-1}\lesssim|y|^{-1}$, we use Lemma 4.2.4

$$\int_{E_<}\frac{|y|^4}{|x|^4|x-y|^2}|W(y)\varphi(y)|dy \lesssim \frac{1}{|x|^3}\|\varphi\|_{\mathcal{B}_5}\int_{E_<}\frac{|W(y)|}{|x-y|^2|y|^2}dy \lesssim \frac{1}{|x|^5}\|\varphi\|_{\mathcal{B}_5}\|W\|_{2,1}\|a\|_{2,\infty}. \tag{4.3.23}$$

Combining inequalities (4.3.21)-(4.3.23) we have

$$|x|^5|\mathcal{V}\varphi(x)|\lesssim\|\varphi\|_{\mathcal{B}_5}\|W\|_{2,1}\|a\|_{2,\infty}, \qquad |x|\geqslant R$$

and hence $\mathcal{P}$ is a contraction on $\mathcal{B}_5$ which concludes the proof. □

Appendices

4.A Facts about Lorentz spaces

We collect here a few properties of Lorentz spaces that we have used. The following statements hold on $\mathbb{R}^n$ for any n.

First of all we recall the definition of the quasinorm

$$\|f\|_{p,q} := p^{1/q}\left(\int_0^\infty t^{q-1}(d_f(t))^{q/p}dt\right)^{1/q}$$

for $q<\infty$ or

$$\|f\|_{p,\infty} := \sup_{t\geqslant 0} t^{1/p}d_f(t) < \infty$$

otherwise. Observe that $L^{p,p}=L^p$ and that $L^{p,\infty}$ is the weak L^p space. The quantity we just defined is only a quasinorm, since it holds

$$\|f+g\|_{p,q}\lesssim_{p,q}\|f\|_{p,q}+\|g\|_{p,q} \tag{4.A.1}$$

(see inequality (1.4.9) in [Gra14]). Only for $p>1$ and any $q\in[1,\infty]$ the space $L^{p,q}$ is normable ([Hun66]).

Remark 4.A.1. We recall that the quasinorm of the Lorentz space can also be defined via the decreasing rearrangement $f^*(t)=\inf\{s>0 \mid d_f(s)<t\}$ as

$$\|f\|_{p,q} := \left(\int_0^\infty t^{q/p-1}f^*(t)^q dt\right)^{1/q} < \infty$$

for $q<\infty$ or

$$\|f\|_{p,\infty} := \sup_{t\geqslant 0} t^{1/p}f^*(t) < \infty$$

otherwise.

Lorentz spaces are growing spaces with respect to the second index, in particular we have the chain of inclusions

$$L^{p,q_1} \subset L^{p,q_2} \quad \text{for any } q_1 < q_2 \tag{4.A.2}$$

(see Proposition 1.4.10 in [Gra14]).

We also have a two indexed Hölder inequality

$$\|fg\|_{L^{p,q}} \leqslant \|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}} \tag{4.A.3}$$

for any p_1, q_1, p_2, q_2 such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$ and $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Inequality (4.A.3) can be easily proved using the definitions given in Remark 4.A.1. For $q=\infty$ it follows directly from the definition of $\|\cdot\|_{p,\infty}$, while for $q\in(0,\infty)$ it is obtained using inequality $\int f^\alpha g^\beta \frac{dt}{t} \leqslant (\int f\frac{dt}{t})^\alpha(\int g\frac{dt}{t})^\beta$ for α and β which sum to 1. In particular taking $\alpha=\frac{q_2}{q_1+q_2}$ and $\beta=\frac{q_1}{q_1+q_2}$

$$\begin{aligned}\|fg\|_{p,q}^q &= \int (t^{\frac{1}{p}} f^* g^*)^q \frac{dt}{t} = \int [(t^{\frac{1}{p_1}} f^*)^{q_1}]^{\frac{q_2}{q_1+q_2}}[(t^{\frac{1}{p_2}} f^*)^{q_2}]^{\frac{q_1}{q_1+q_2}}\frac{dt}{t}\\ &\leqslant (\int (t^{\frac{1}{p_1}} f^*)^{q_1}\frac{dt}{t})^{\frac{q_2}{q_1+q_2}}(\int (t^{\frac{1}{p_2}} f^*)^{q_2}\frac{dt}{t})^{\frac{q_1}{q_1+q_2}}\end{aligned}$$

from which(4.A.3) follows taking the power $\frac{1}{q}=\frac{q_1+q_2}{q_1 q_2}$

The following property is a sort of interpolation over the first index of the space.

Proposition 4.A.2. *Let $f \in L^{p_0,\infty}\cap L^{p_1,\infty}$, then $f\in L^{p,q}$ for any $p\in(p_0,p_1)$ and any $q\in(0,\infty]$.*

Proof. We first consider $q=\infty$ then

$$\|f\|_{p,\infty} = \max\{\sup_{d_f(t)\geqslant 1} td_f(t)^{1/p}, \sup_{0\leqslant d_f(t)\leqslant 1} td_f(t)^{1/p}\}. \tag{4.A.4}$$

For the case $d_f(t)\leqslant 1$ since $p<p_1$ we have

$$d_f(t)^{1/p-1/p_1}\leqslant 1$$

while when $d_f(t)\geqslant 1$ then using the fact that $p>p_0$

$$d_f(t)^{1/p-1/p_0}\leqslant 1.$$

We can therefore bound both suprema in (4.A.4) as

$$\sup_{d_f(t)\geqslant 1} td_f(t)^{1/p-1/p_0+1/p_0} \leqslant \sup_{t\geqslant 0} td_f(t)^{1/p_0} = \|f\|_{p_0,\infty}<\infty$$

and

$$\sup_{0\leqslant d_f(t)\leqslant 1} td_f(t)^{1/p-1/p_1+1/p_1} \leqslant \sup_{0\leqslant d_f(t)\leqslant 1} td_f(t)^{1/p_1} = \|f\|_{p_1,\infty}<\infty.$$

Now let $q<\infty$. We have the quantity $td_f(t)^{1/p_1}$ which is bounded at infinity, then there exists a $\bar{t}>0$ such that

$$d_f(t)\lesssim \frac{1}{t^{p_1}} \quad \text{for any} \quad t\geqslant \bar{t}$$

so for large enough t we have

$$td_f(t)^{1/p_0}\lesssim t^{1-p_1/p_0} \quad \text{with} \quad 1-\frac{p_1}{p_0}<0.$$

On the other hand $td_f(t)^{1/p_0}$ is bounded around 0 so there exists $\widetilde{t} > 0$ such that

$$d_f(t) \lesssim \frac{1}{t^{p_0}} \quad \text{for any} \quad 0 < t \leqslant \widetilde{t}$$

from which for small t it holds

$$td_f(t)^{1/p_1} \lesssim t^{1-p_0/p_1} \quad \text{with} \quad 1 - \frac{p_0}{p_1} > 0.$$

Now we can proceed to estimate the $L^{p,q}$ norm, for $\lambda \in (0,1)$ we write $\frac{1}{p} = (1-\lambda)\frac{1}{p_1} + \lambda\frac{1}{p_0}$

$$\begin{aligned}
\|f\|_{p,q} = &\int_0^t (t^{1-\lambda+\lambda} d_f(t)^{(1-\lambda)\frac{1}{p_1}+\lambda\frac{1}{p_0}})^q \frac{dt}{t} = \int_0^{\widetilde{t}} \dots dt + \int_{\widetilde{t}}^{\bar{t}} \dots dt + \int_{\bar{t}}^{\infty} \dots dt \\
\leqslant &\sup_t (td_f(t)^{1/p_0})^{\lambda q} \int_0^{\widetilde{t}} (td_f(t)^{1/p_1})^{(1-\lambda)q} \frac{dt}{t} \\
&+ \int_{\widetilde{t}}^{\bar{t}} \dots dt \\
&+ \sup_t (td_f(t)^{1/p_1})^{(1-\lambda)q} \int_{\bar{t}}^{\infty} (td_f(t)^{1/p_0})^{\lambda q} \frac{dt}{t} \\
\lesssim &\|f\|_{p_0,\infty}^{\lambda q} \int_0^{\widetilde{t}} t^{(1-p_0/p_1)(1-\lambda)q-1} dt \\
&+ \int_{\widetilde{t}}^{\bar{t}} \dots dt \\
&+ \|f\|_{p_1,\infty}^{(1-\lambda)q} \int_{\bar{t}}^{\infty} t^{(1-p_1/p_0)\lambda q-1} dt \\
\simeq &\|f\|_{p_0,\infty}^{\lambda q} \; t^{(1-p_0/p_1)(1-\lambda)q}|_0^{\widetilde{t}} + \int_{\widetilde{t}}^{\bar{t}} \dots dt \\
&+ \|f\|_{p_1,\infty}^{(1-\lambda)q} \; t^{(1-p_1/p_0)\lambda q}|_{\bar{t}}^{\infty}
\end{aligned}$$

where all the terms are finite since $(1-p_0/p_1)(1-\lambda)q > 0$ and $(1-p_1/p_0)\lambda q < 0$. ☐

4.B Proof of some elementary inequalities

We prove in this section Lemmas 4.3.2 and 4.3.3.

Lemma 4.B.1. *Let $x, y \in \mathbb{R}^n$, then*

a)

$$\left| \frac{1}{|x|} - \frac{1}{|x-y|} \right| \lesssim \frac{|y|}{|x||x-y|},$$

b)

$$\left| \frac{1}{|x|} + \frac{x \cdot y}{|x|^3} - \frac{1}{|x-y|} \right| \lesssim \frac{|y|^2}{|x|^2|x-y|},$$

c)

$$\left| \frac{1}{|x|} + \frac{x \cdot y}{|x|^3} + \frac{3(x \cdot y)^2}{2|x|^5} - \frac{|y|^2}{2|x|^3} - \frac{1}{|x-y|} \right| \lesssim \frac{|y|^3}{|x|^3|x-y|}.$$

A proof of the inequalities in this lemma can also be found in Lemma 2.4 [Bec16], we nevertheless prove the bounds here for completeness.

Proof. The first bound follows from a direct application of the triangular inequality. To prove *b*) we need to consider the quantity

$$\frac{1}{|x|}+\frac{x\cdot y}{|x|^3}-\frac{1}{|x-y|}=\frac{|x|^2(|x-y|-|x|)+x\cdot y|x-y|}{|x|^3|x-y|}. \tag{4.B.1}$$

We recall that $||x-y|-|x||\leqslant |y|$, then we have the inequality

$$\left|\frac{2x\cdot y}{|x-y|+|x|}-\frac{x\cdot y}{|x|}\right|=\frac{|x\cdot y(|x|-|x-y|)|}{|x|(|x-y|+|x|)}\leqslant\frac{|y|^2}{|x|}.$$

We can also remark that

$$|x-y|-|x|=\frac{|x-y|^2-|x|^2}{|x-y|+|x|}=\frac{|y|^2}{|x-y|+|x|}-\frac{2x\cdot y}{|x-y|+|x|} \tag{4.B.2}$$

and we use this expression in the following

$$\begin{aligned}\left||x-y|-|x|+\frac{x\cdot y}{|x|}\right|=&\left|\frac{|y|^2}{|x-y|+|x|}-\frac{2x\cdot y}{|x-y|+|x|}+\frac{x\cdot y}{|x|}\right|\\ \leqslant&\frac{|y|^2}{|x-y|+|x|}+\left|\frac{2x\cdot y}{|x-y|+|x|}-\frac{x\cdot y}{|x|}\right|\leqslant\frac{|y|^2}{|x|}.\end{aligned} \tag{4.B.3}$$

We now go back to bound the numerator in (4.B.1)

$$\begin{aligned}||x|^2(|x-y|-|x|)+x\cdot y|x-y||=&||x|^2(|x-y|-|x|+\frac{x\cdot y}{|x|})+x\cdot y|x-y|-x\cdot y|x||\\ \leqslant&|x|^2\left||x-y|-|x|+\frac{x\cdot y}{|x|}\right|+|x\cdot y(|x-y|-|x|)|\\ \leqslant&|x|^2\frac{|y|^2}{|x|}+|x||y|^2=2|x||y|^2,\end{aligned}$$

dividing by $|x|^3|x-y|$ we obtain the statement.

To prove *c*) we set

$$I:=\frac{1}{|x|}+\frac{x\cdot y}{|x|^3}+\frac{3(x\cdot y)^2}{2|x|^5}-\frac{|y|^2}{2|x|^3}-\frac{1}{|x-y|}.$$

We use again (4.B.1) to compute I

$$\begin{aligned}I=&\frac{|x|^2(|x-y|-|x|+\frac{x\cdot y}{|x|}+\frac{(x\cdot y)^2}{2|x|^3}-\frac{|y|^2}{2|x|})+|x|^2(-\frac{(x\cdot y)^2}{2|x|^3}+\frac{|y|^2}{2|x|})+x\cdot y(|x-y|-|x|)}{|x|^3|x-y|}\\ &+\frac{3(x\cdot y)^2}{2|x|^5}-\frac{|y|^2}{2|x|^3}\\ =&\frac{|x|^2(|x-y|-|x|+\frac{x\cdot y}{|x|}+\frac{(x\cdot y)^2}{2|x|^3}-\frac{|y|^2}{2|x|})}{|x|^3|x-y|}\\ &+\frac{-\frac{(x\cdot y)^2}{2|x|}+\frac{|y|^2(|x|-|x-y|)}{2}+x\cdot y(|x-y|-|x|+\frac{x\cdot y}{|x|})-\frac{(x\cdot y)^2}{|x|}+\frac{3(x\cdot y)^2|x-y|}{2|x|^2}}{|x|^3|x-y|}\\ =&\frac{|x|^2(|x-y|-|x|+\frac{x\cdot y}{|x|}+\frac{(x\cdot y)^2}{2|x|^3}-\frac{|y|^2}{2|x|})}{|x|^3|x-y|}\end{aligned} \tag{4.B.4}$$

$$+\frac{\frac{|y|^2(|x|-|x-y|)}{2}+x\cdot y(|x-y|-|x|+\frac{x\cdot y}{|x|})+\frac{3(x\cdot y)^2(|x-y|-|x|)}{2|x|^2}}{|x|^3|x-y|}. \tag{4.B.5}$$

For the terms in (4.B.5) we remark that thanks to (4.B.3)

$$\frac{|y|^2||x|-|x-y||}{2}\lesssim|y|^3,\quad |x\cdot y|\left||x-y|-|x|+\frac{x\cdot y}{|x|}\right|\lesssim|x||y|\frac{|y|^2}{|x|}\lesssim|y|^3$$
$$\frac{|(x\cdot y)|^2||x-y|-|x||}{2|x|^2}\lesssim|y|^3.$$

In (4.B.4) we can prove that

$$II:=|x-y|-|x|+\frac{x\cdot y}{|x|}+\frac{(x\cdot y)^2}{2|x|^3}-\frac{|y|^2}{2|x|}$$

has modulus bounded by $|y|^3/|x|^2$. We take the expression in (4.B.2)

$$\begin{aligned}II=&\frac{|y|^2}{|x-y|+|x|}-\frac{2x\cdot y}{|x-y|+|x|}+\frac{x\cdot y}{|x|}+\frac{(x\cdot y)^2}{2|x|^3}-\frac{|y|^2}{2|x|}\\
=&|y|^2\left(\frac{1}{|x-y|+|x|}-\frac{1}{2|x|}\right)+\frac{x\cdot y(|x-y|-|x|)}{|x|(|x-y|+|x|)}+\frac{(x\cdot y)^2}{2|x|^3}\\
=&|y|^2\left(\frac{1}{|x-y|+|x|}-\frac{1}{2|x|}\right)+\frac{x\cdot y(|x-y|-|x|)}{|x|}\left(\frac{1}{|x-y|+|x|}-\frac{1}{2|x|}\right)\\
&+\frac{x\cdot y(||x-y|-|x|+\frac{x\cdot y}{|x|})}{2|x|^2}.\end{aligned}$$

We obtain the bound on $|II|$ since

$$\left|\frac{1}{|x-y|+|x|}-\frac{1}{2|x|}\right|=\frac{||x|-|x-y||}{2|x|||x-y|+|x||}\lesssim\frac{|y|}{|x|^2}$$

and thanks to (4.B.3).

□

Lemma 4.B.2. *Let $x,y\in\mathbb{R}^n$, then*

i)
$$\left|\frac{1}{|x|^2}-\frac{1}{|x-y|^2}\right|\leqslant\frac{|y|^2}{|x|^2|x-y|^2}+2\frac{|y|}{|x||x-y|^2},$$

ii)
$$\left|\frac{1}{|x|^2}+\frac{2x\cdot y}{|x|^4}-\frac{1}{|x-y|^2}\right|\lesssim\frac{|y|^2}{|x|^2|x-y|^2}+\frac{2|y|^3}{|x|^3|x-y|^2}$$

iii)
$$\left|\frac{1}{|x|^2}+\frac{2x\cdot y}{|x|^4}-\frac{|y|^2}{|x|^4}+\frac{(2x\cdot y)^2}{|x|^6}-\frac{1}{|x-y|^2}\right|\lesssim\frac{|y|^3}{|x|^3|x-y|^2}+\frac{|y|^4}{|x|^4|x-y|^2}$$

Proof. We compute

$$\left|\frac{1}{|x|^2}-\frac{1}{|x-y|^2}\right|=\frac{||x-y|^2-|x|^2|}{|x|^2|x-y|^2},$$

if $|x-y|>|x|$ using triangular inequality on $|x-y|$

$$\begin{aligned}\left|\frac{1}{|x|^2}-\frac{1}{|x-y|^2}\right| =&\frac{|x-y|^2-|x|^2}{|x|^2|x-y|^2}\leqslant\frac{|x|^2+|y|^2+2|x||y|-|x|^2}{|x|^2|x-y|^2}\\ \leqslant&\frac{|y|^2}{|x|^2|x-y|^2}+2\frac{|y|}{|x||x-y|^2}.\end{aligned}$$

On the contrary if $|x-y|<|x|$ then

$$\left|\frac{1}{|x|^2}-\frac{1}{|x-y|^2}\right| =\frac{|x|^2-|x-y|^2}{|x|^2|x-y|^2}$$

here $|x-y|^2\geqslant||x|-|y||^2=|x|^2+|y|^2-2|x||y|$ so using this bound

$$\begin{aligned}\left|\frac{1}{|x|^2}-\frac{1}{|x-y|^2}\right| \leqslant&\frac{|x|^2-|x|^2-|y|^2+2|x||y|}{|x|^2|x-y|^2}\leqslant\frac{2|y|}{|x||x-y|^2}\\ \leqslant&\frac{|y|^2}{|x|^2|x-y|^2}+2\frac{|y|}{|x||x-y|^2}.\end{aligned}$$

For the second bound we compute the sum

$$\begin{aligned}\frac{1}{|x|^2}+\frac{2x\cdot y}{|x|^4}-\frac{1}{|x-y|^2} =&\frac{|x-y|^2|x|^2+2x\cdot y|x-y|^2-|x|^4}{|x|^4|x-y|^2}\\ =&\frac{|x|^2(|x-y|^2-|x|^2)+2x\cdot y|x-y|^2}{|x|^4|x-y|^2}\\ =&\frac{|x|^2(|y|^2-2x\cdot y)+2x\cdot y(|x|^2+|y|^2-2x\cdot y)}{|x|^4|x-y|^2}\\ =&\frac{|x|^2|y|^2+2x\cdot y|y|^2-4(x\cdot y)^2}{|x|^4|x-y|^2}\end{aligned}$$

then taking the modulus we obtain the statement. Analogously we do the computations for the last inequality. We remark that

$$\frac{|x-y|^2}{|x|^2}=1+\frac{|y|^2}{|x|^2}-\frac{2x\cdot y}{|x|^2},$$

so that

$$\begin{aligned}\frac{2x\cdot y}{|x|^4}-\frac{|y|^2}{|x|^4} =&\frac{2x\cdot y\frac{|x-y|^2}{|x|^2}}{|x|^2|x-y|^2}-\frac{|y|^2\frac{|x-y|^2}{|x|^2}}{|x|^2|x-y|^2}\\ =&\frac{2x\cdot y-|y|^2}{|x|^2|x-y|^2}+\frac{2x\cdot y(|y|^2-2x\cdot y)}{|x|^4|x-y|^2}-\frac{|y|^2(|y|^2-2x\cdot y)}{|x|^4|x-y|^2}\end{aligned}\tag{4.B.6}$$

and

$$\frac{(2x\cdot y)^2}{|x|^6}=\frac{(2x\cdot y)^2\frac{|x-y|^2}{|x|^2}}{|x|^4|x-y|^2}=\frac{(2x\cdot y)^2}{|x|^4|x-y|^2}+\frac{(2x\cdot y)^2(|y|^2-2x\cdot y)}{|x|^6|x-y|^2}\tag{4.B.7}$$

We also remark that summing the first and last term in the left hand side of item *iii*)

$$\frac{1}{|x|^2}-\frac{1}{|x-y|^2}=\frac{|y|^2-2x\cdot y}{|x|^2|x-y|^2}.\tag{4.B.8}$$

Summing together (4.B.6), (4.B.7), (4.B.8) we obtain

$$\left|\frac{1}{|x|^2}+\frac{2x\cdot y}{|x|^4}-\frac{|y|^2}{|x|^4}+\frac{(2x\cdot y)^2}{|x|^6}-\frac{1}{|x-y|^2}\right|=\left|\frac{4x\cdot y|y|^2-|y|^4}{|x|^4|x-y|^2}+\frac{(2x\cdot y)^2(|y|^2-2x\cdot y)}{|x|^6|x-y|^2}\right|$$

and by triangular inequality the statement follows. □

4.C Notations for the current chapter

Functions and spaces
$\dot{H}^s(\mathbb{R}^n)=\{f\ :\ \|f\|_{\dot{H}^s}:=\||\cdot|^s\widehat{f}\|_{L^2}<\infty\}$
$d_f(t)$ distribution function of f
$\|\cdot\|_{p,q}$ norm of the Lorentz space $L^{p,q}(\mathbb{R}^n)$
$a(x)=\dfrac{1}{\|x\|^{n-2}}$
$c_n=n(n-2)\|B(0,1)\|$
$V=W+K$
W small part of the potential
K compactly supported part of the potential, $suppK\subset B(0,R)$
$\mathcal{B}_\alpha=\{f\ :\ \|x\|^\alpha f\in L^\infty(B(0,R)^c)\ \}$
$\|f\|_{\mathcal{B}_\alpha}=\|\|x\|^\alpha\mathbb{1}_{B(0,R)^c}\ f\|_\infty$
Operators
$G(x,y)$ integral kernel of $(-\Delta+W)^{-1}$
$\mathcal{S},\mathcal{T},\mathcal{U},\mathcal{V}$ contraction operators

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