Geometric Measure Theory: Analysis and non-smooth objects

Institut de Mathématiques de Toulouse

12–16 September 2016

Program of the week

FROM UNBALANCED OPTIMAL TRANSPORT TO THE CAMASSA-HOLM EQUATION

FRANÇOIS-XAVIER VIALARD

In the first part of the talk, we present a natural extension of the Wasserstein L2 metric on the space of probablility measures to the space of positive Radon measures that do not have the same total mass. This generalization can be seen as a Riemannian analog of the flat norm on the space of measures. We introduce it as the infimal convolution of the Wasserstein and the Fisher-Rao metric using a dynamic formulation. We then show its equivalent static formulation efficient for computations. In the second part, we present, via a generalization of Otto's Riemannian submersion the link between this new metric on the space of probabilities and the Camassa-Holm equation. The Camassa-Holm equation is a geodesic equation for a right-invariant metric on the group of diffeomorphisms. Our point of view gives an isometric embedding of the group in a Hilbert manifold which has interesting analytical consequences.

INTEGRAL MENGER CURVATURE

DAMIAN DĄBROWSKI

In knot theory, knots are equivalence classes of a certain equivalence relation. Each knot class consists of many different representatives, some of them being more complicated, some others – less complicated. In order to find the simplest, optimal shapes for knots, one considers certain functionals known as knot energies. An example of such functional is integral Menger curvature

$$\mathcal{M}_p(\gamma) = \int_\gamma \int_\gamma \int_\gamma \frac{1}{R(x,y,z)^p} d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z),$$

where $R(x,y,z)$ denotes radius of the unique circle passing through x, y, z.

In the talk I will discuss some basic properties of $\mathcal{M}_p$, as well as the connection to Sobolev-Slobodeckij spaces. Stress will be put on the scale invariant case $p = 3$.

DIFFERENTIATION OF REAL FUNCTIONS ALONG RECTANGLES

LAURENT MOONENS

Given a family R of rectangles in $\mathbb{R}^n$ of the form $[0,\alpha_1] \times \cdots \times [0,\alpha_n]$, we let $B := \{\tau(I) : I \in R, \tau \text{ translation}\}$ be the associated (translation-invariant) differentiation basis and define a maximal operator M_R by:

$$M_R f(x) := \sup_{I \in B, I \ni x} \frac{1}{|I|} \int_I |f|.$$

Given $X \subseteq L^1(\mathbb{R}^n)$ an Orlicz space, it is often the case that the two following properties are equivalent:

(A) $M_R f(x) < +\infty$ for a.e. $x \in \mathbb{R}^n$;

(B) R differentiates X in the sense that for all $f \in X$, one has:

$$f(x) = \lim_{x \in I \in R, \operatorname{diam} I \to 0} \frac{1}{|I|} \int_I f,$$

for a.e. $x \in \mathbb{R}^n$.

In this talk we shall discuss some geometrical properties on R that guarantee or not the validity of properties (A) & (B) above for some classical Orlicz spaces X. We shall particularly focus on the case $n = 2$, survey the classical results obtained in this case, and see how things change in the plane when rectangles from R are allowed to rotate around their lower left vertex, with an angle belonging to some small set. If time allows us to do so, we shall also discuss recent results obtained jointly with E. D'Aniello in the n-dimensional case.

DIMENSION REDUCTION FOR OPTIMAL POINT CONFIGURATIONS

MIRCEA PETRACHE

I will present some new techniques developed with L. Betermin for studying the structure of minimising point configurations for long-range interactions, amongst lattices and more general point configurations.

In all cases the problem is simplified by looking at layers within the configurations and reducing the dimension of the problem.

I will mention the links to Computer Science problems, to the Thompson problem, and to crystallization conjectures in Statistical Physics.

VARIFOLDS AND DISCRETE SURFACES

BLANCHE BUET

We aim at connecting tools from geometric measure theory (varifolds) to practical issues in discrete geometry (notion of discrete curvature, geometric motions, surface comparison, etc.).

Varifolds have been introduced by F. Almgren in 1965 to study minimal surfaces. They have been widely used in order to study existence and regularity of solutions to geometric variational problems, but in general for theoretical purpose. The structure of varifold is flexible enough so that both regular surfaces and discrete surfaces (point clouds, triangulated surfaces or volumetric representations for instance) can be provided with a varifold structure, allowing to study surfaces and their different discretizations in a consistent unified setting. In this framework, we propose a notion of discrete mean curvature obtained by regularization of the first variation, which has nice estimation and convergence properties. We illustrate this notion on 2D and 3D examples.

H-DISTRIBUTIONS AND COMPENSATED COMPACTNESS

MARIN MIŠUR

H-measures, introduced independently by Luc Tartar and Patrick Gerard, are matrix Radon measures describing the behaviour of weak limits of quadratic quantities. They proved to be very successful tool in investigations of asymptotic limits of quadratic quantities. However, they turned insufficient for nonlinear problems.

H-distributions were introduced by Antonić and Mitrović as an extension of H-measures to the $\mathrm{L}^p - \mathrm{L}^q$ setting. Their variants have been successfully applied to problems in velocity averaging (Lazar-Mitrović 2012) and compensated compactness with variable coefficients

(Mišur-Mitrović 2015). Unlike H-measures, which are nonnegative Radon measures, H-distributions are distributions in the Schwartz sense, which follows from the standard Schwartz kernel theorem.

To give a precise description of H-distributions, we will introduce the notion of anisotropic distributions – distributions of different order with respect to different coordinate directions. In order to show that H-distributions are anisotropic distributions of finite order with respect to every coordinate direction, we will prove a variant of Schwartz kernel theorem.

In the second part of the talk, we will show a variant of compensated compactness using a variant of H-distributions. Namely, we will investigate conditions under which, for two sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ weakly converging to $\mathbf{u}$ and $\mathbf{v}$ in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^N)$ and $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^N)$, respectively, $1/p+1/q \leq 1$, a quadratic form $q(\mathbf{x};\mathbf{u}_r,\mathbf{v}_r) = \sum_{j,m=1}^N q_{jm}(\mathbf{x})u_{jr}v_{mr}$ converges toward $q(\mathbf{x};\mathbf{u},\mathbf{v})$ in the sense of distributions. The conditions involve fractional derivatives and variable coefficients, and they represent a generalization of the known compensated compactness theory. We will apply the developed techniques to a nonlinear (degenerate) parabolic equation.

This talk will present results of joint works with Nenad Antonić, Marko Erceg and Darko Mitrović.

References

[1] N. Antonić, M. Erceg, M. Mišur, *On H-distributions*, in progress 25 pp.

[2] N. Antonić, M. Mišur, D. Mitrović, *On the First commutation lemma*, submitted 17 pp.

[3] M. Mišur, D. Mitrović, *On a Generalization of Compensated Compactness in the* $\mathrm{L}^p - \mathrm{L}^q$ *setting*, *Journal of Functional Analysis* **268** (2015) 1904–1927.

	Monday 12/9	Tuesday 13/9	Wednesday 14/9	Thursday 15/9	Friday 16/9
9:00 – 10:00			Sławomir Kolasiński Lecture 5/5		Ulrich Menne Lecture 4/5
		Sławomir Kolasiński Lecture 3/5		Ulrich Menne Lecture 2/5	
10:00 – 11:00					
	Welcome coffee		Coffee break		Coffee break
11:00 – 12:00	Sławomir Kolasiński Lecture 1/5	Coffee break	Ulrich Menne Lecture 1/5	Coffee break	Ulrich Menne Lecture 5/5
		Laurent Moonens		Blanche Buet	
12:00 – 13:00	Lunch	Lunch	Lunch	Lunch	Lunch
13:00 – 14:00					
14:00 – 15:00	Sławomir Kolasiński Lecture 2/5	Sławomir Kolasiński Lecture 4/5	Free afternoon	Ulrich Menne Lecture 3/5	IMT Colloquium
15:00 – 16:00					
	Coffee break	Coffee break		Coffee break	
16:00 – 17:00	François-Xavier Vialard	Mircea Petrache		Marin Mišur	
17:00 – 18:00	Damian Dąbrowski				

Social dinner