TORUS ACTIONS AND INTEGRABLE SYSTEMS

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Abstract. This is a survey on natural local torus actions which arise in integrable dynamical systems, and their relations with other subjects, including: reduced integrability, local normal forms, affine structures, classical and quantum monodromy, global invariants, integrable surgery, convexity properties of momentum maps, localization formulas.

Contents

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1. Introduction

To say that everything is a torus would be a great exageration, but to say that *everything contains a torus* would not be too far from the truth. According to ancient oriental philosophy, everything can be described by (a combination of) five elemental aspects, or phases: regular, transitive, expansive, chaotic, and contractive, and if we look at these five phases as a whole then they also form cycles.

This survey paper is concerned with regular aspects of things. Mathematically, they correspond to regular dynamics, or integrable dynamical systems. The motto of this paper is: *an integrable system is a local torus action*. The main dynamical property of a regular dynamical system is its quasi-periodic behavior. Mathematically, it means that the system is invariant under a (local) torus action. These torus actions exist near compact regular orbits (Liouville's theorem). To a great extent, they exist near singularities of integrable systems as well, and this is one of the main topics of the paper (Section 3 and Section 4). Even when we don't see any torus at first, they are hidden somewhere: for example, multi-soliton solutions of the Korteweg–de Vries equation can be seen as homo/heteroclinic solutions of another system (the Neumann system) for which there are plenty of tori, see e.g. [60].

Other topics discussed in this paper, which can be seen from the table of contents, include: reduced integrability, proper groupoid actions, intrinsic convexity properties of momentum maps, classical and quantum monodromy, global invariants, localization formulas. Of course, they are all intimately related to local torus actions.

This paper only deals with finite-dimensional dynamical systems, i.e. ordinary differential equations, though some ideas and results can probably be extended to the infinite-dimensional case.

2. Integrability, torus actions, and reduction

2.1. **Integrability à la Liouville.**

Probably the most well-kown notion of integrability in dynamical systems is the notion of *integrability à la Liouville* for Hamiltonian systems on symplectic manifolds. Denote by (M^{2n}, ω) a symplectic manifold of dimension $2n$ with symplectic form ω, and H a function on M^{2n}. Denote by X_H the Hamiltonian vector field of H on M^{2n}:

$$i_{X_H}\omega = -dH \ . \tag{2.1}$$

Definition 2.1. A function H (or the corresponding Hamilonian vector field X_H) on a $2n$-dimensional symplectic manifold (M^{2n}, ω) is called *integrable à la Liouville*, or *Liouville-integrable*, if it admits n functionally independent first integrals in involution. In other words, there are n functions $F_1 = H, F_2, \ldots, F_n$ on M^{2n} such that $dF_1 \wedge \cdots \wedge dF_n \neq 0$ almost everywhere and $\{F_i, F_j\} = 0 \ \forall \ i, j$.

In the above definition, $\{F_i, F_j\} := X_{F_i}(F_j)$ denotes the Poisson bracket of F_i and F_j with respect to the symplectic form ω. The map

$$\mathbf{F} = (F_1, \ldots, F_n) : (M^{2n}, \omega) \to \mathbb{K}^n \tag{2.2}$$

is called the *momentum map* ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). The above definition works in many categories: smooth, real analytic, holomorphic, formal, etc.

The condition $X_H(F_i) = \{H, F_i\} = 0$ implies that the Hamiltonian vector field X_H is tangent to the level sets of $\mathbf{F}$. Let $N = \mathbf{F}^{-1}(c)$ be a regular connected (component of a) level set of $\mathbf{F}$. Then it is a Lagrangian submanifold of M^{2n}: the dimension of N is half the dimension of M^{2n}, and the restriction of ω to N is zero. So we can talk about a (singular) Lagrangian foliation/fibration given by the momentum map.

The classical result of Liouville [53] says that, in the smooth or real analytic case, if a connected level set N is compact and does not intersect with the boundary of M^{2n}, then it is diffeomorphic to a standard torus $\mathbb{T}^n$, and the Hamiltonian system X_H is quasi-periodic on N: in other words, there is a periodic coordinate system $(q_1, \ldots, q_n)$ on N with respect to which the restriction of X_H to N has constant coefficients: $X_H = \sum \gamma_i \partial/\partial q_i$, γ_i being constants. For this reason, N is called a *Liouville torus*.

The description of a Liouville-integrable Hamiltonian system near a Liouville torus is given by the following theorem about the existence of action-angle variables. This theorem is often called *Arnold-Liouville theorem*, but it was essentially obtained by Henri Mineur in 1935 [56, 57]:

Theorem 2.2 (Liouville-Mineur-Arnold)**.** *Let N be a Liouville torus of a Liouville-integrable Hamiltonian system with a given momentum map $\mathbf{F} : (M^{2n}, \omega) \to \mathbb{R}^n$. Then there is a neighborhood $\mathcal{U}(N)$ of N and a smooth symplectomorphism*

$$\Psi : (\mathcal{U}(N), \omega) \to (D^n \times \mathbb{T}^n, \sum_1^n d\nu_i \wedge d\mu_i) \tag{2.3}$$

(ν_i - coordinates of D^n, μ_i (mod 1) - periodic coordinates of $\mathbb{T}^n$) such that $\mathbf{F}$ depends only on $I_i = \phi^* \nu_i$, i.e. $\mathbf{F}$ does not depend on $\phi_i = \phi^* \mu_i$.

The variables (I_i, ϕ_i) in the above theorem are called *action-angle variables.* The map

$$(I_1, \ldots, I_n) : (\mathcal{U}(N), \omega) \to \mathbb{R}^n \tag{2.4}$$

is the momentum map of a Hamiltonian torus $\mathbb{T}^n$-action on $(\mathcal{U}(n), \omega)$ which preserves $\mathbf{F}$. The existence of this Hamiltonian torus action is essentially equivalent to Liouville-Mineur-Arnold theorem: once the action variables are found, angle variables can also be found easily by fixing a Lagrangian section to the foliation by Liouville tori. The quasi-periodicity of the system on N also follows immediately from the existence of this torus action.

The existence of action-angle variables is very important, both for the theory of near-integrable systems (K.A.M. theory), and for the quantization of integrable systems (Bohr-Sommerfeld rule). Actually, Mineur was an astrophysicist, and Bohr-Sommefeld quantization was his motivation for finding action-angle variables.

Mineur [57] also wrote down the following simple formula, which we will call *Mineur–Arnold formula*, for action functions:

$$I_i(z) = \int_{\Gamma_i(z)} \beta \tag{2.5}$$

where z is a point in $\mathcal{U}(N)$, β is a primitive of the symplectic form ω, i.e. $d\beta = \omega$, and $\Gamma_i(z)$ is an 1-cycle on the Liouville torus which contains z (and which depends on z continuously).

In the case of *algebraically integrable systems* (see e.g. [1]), where invariant tori can be identified with (the real part of) Jacobian or Prym varieties of complex curves (*spectral curves* of the system), the integral in Mineur–Arnold formula corresponds to Abelian integrals on complex curves, as observed by Novikov and Veselov [70].

As observed by many people (see e.g. [35, 43] and Section 3), Mineur-Arnold formula is very useful near singularities of the momentum map as well.

2.2. **Generalized Liouville integrability.**

In practice, one often deals with Hamiltonian systems which admit a non-Abelian group of symmetries, or Hamiltonian systems on Poisson (instead of symplectic) manifolds. A typical example is an Euler equation on the dual of a Lie algebra. For such systems, Liouville integrability needs to be replaced by a more general and convenient notion of integrability, which nevertheless retains the main feature of Liouville integrability, namely the existence of local torus actions.

Let (M, Π) be a *Poisson manifold*, with Π being the Poisson structure. It means that Π is a 2-vector field on M such that the following binary operation on the space of functions on M, called the *Poisson bracket*,

$$\{H, F\} = \langle dH \wedge dF, \Pi\rangle \tag{2.6}$$

is a Lie bracket, i.e. it satisfies the Jacobi identity. A symplectic manifold is also a Poisson manifold. Conversely, a Poisson manifold can be seen as a singular foliation by symplectic manifolds, see e.g. [75]

Let H be a function on a Poisson manifold (M, Π), and X_H the corresponding Hamiltonian vector field: $X_H = dH \lrcorner \Pi$. Let $\mathcal{F}$ be a set of first integrals of X_H, i.e. each $F \in \mathcal{F}$ is a function on M which is preserved by X_H (equivalently, $\{F, H\} = 0$). Denote by ddim $\mathcal{F}$ the *functional dimension* of $\mathcal{F}$, i.e. the maximal number of functions in $\mathcal{F}$ which are functionally independent almost everywhere.

We will associate to $\mathcal{F}$ the space $\mathcal{X}_\mathcal{F}$ of Hamiltonian vector fields X_E such that $X_E(F) = 0$ for all $F \in \mathcal{F}$ and E is functionally dependent of $\mathcal{F}$ (i.e. the functional dimension of the union of $\mathcal{F}$ with the function E is the same as the functional dimension of $\mathcal{F}$). Clearly, the vector fields in $\mathcal{X}_\mathcal{F}$ commute pairwise and commute with X_H. Denote by ddim $\mathcal{X}_\mathcal{F}$ the functional dimension of $\mathcal{X}_\mathcal{F}$, i.e. the maximal number of vector fields in X which are linearly independent at almost every point. Note that we always have ddim $\mathcal{F}$ + ddim $\mathcal{X}_\mathcal{F} \leq m$, because the vector fields in $\mathcal{X}_\mathcal{F}$ are tangent to the common level sets of the functions in $\mathcal{F}$.

The following definition is essentially due to Nekhoroshev [61] and Mischenko and Fomenko [59] :

Definition 2.3. A Hamiltonian vector field X_H on an m-dimensional Poisson manifold (M, Π) is called *integrable in generalized Liouville sense* if there is a set of first integrals $\mathcal{F}$ such that ddim $\mathcal{F}$ + ddim $\mathcal{X}_\mathcal{F} = m$.

The above notion of integrability is also called *noncommutative integrability*, due to the fact that the functions in $\mathcal{F}$ do not Poisson-commute in general, and in many cases one may choose $\mathcal{F}$ to be a finite-dimensional non-commutative Lie algebra of functions (under the Poisson bracket). When the functions in $\mathcal{F}$ Poisson-commute and the Poisson structure is nondegenerate, we get back to the usual integrability à la Liouville.

Denote $q = \text{ddim } \mathcal{F}, p = \text{ddim } \mathcal{X}_\mathcal{F}$. Then we can find p Hamiltonian vector fields $X_1 = X_{E_1}, ..., X_p = X_{E_p} \in \mathcal{X}_\mathcal{F}$ and q functions $F_1, ..., F_q \in \mathcal{F}$ such that we have:

$$\begin{array}{c} X_H(F_i) = 0,\ [X_H, X_i] = 0,\ [X_i, X_j] = 0,\ X_i(F_j) = 0\ \forall\ i, j\ , \\ X_1 \wedge \ldots X_p \neq 0 \text{ and } dF_1 \wedge \ldots dF_q \neq 0 \text{ almost everywhere.} \end{array} \tag{2.7}$$

The existence of such a p-tuple $\mathbf{X} = (X_1, ..., X_p)$ of commuting Hamiltonian vector fields and q-tuple $\mathbf{F} = (F_1, ..., F_q)$ of common first integrals with $p + q = m$ is equivalent to the integrability in the generalized Liouville sense. When $p + q = m$, we will say that H is integrable with the aid of $(\mathbf{F}, \mathbf{X})$, and by abuse of language, we will also say that $(\mathbf{F}, \mathbf{X})$ *is an integrable Hamiltonian system* in generalized Liouville sense. The map

$$\mathbf{F} = (F_1, ..., F_q) : (M, \Pi) \to \mathbb{K}^q \tag{2.8}$$

(where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is called the *generalized momentum map.* The (regular) level sets of this map are called *invariant manifolds*: they are invariant with respect to X_H, $\mathbf{X}$ and $\mathbf{F}$. They are of dimension p, lie on the symplectic leaves of M, and are isotropic. When $p < \frac{1}{2}\text{rank } \Pi$, i.e. when the invariant manifolds are isotropic but non-Lagrangian, one also speaks of *degenerate integrability*, or *superintegrability*, see e.g. [30, 61, 63].

Definition 2.4. With the above notations, a Hamiltonian system X_H, on a real Poisson manifold (M, Π), integrable with the aid of $(\mathbf{F}, \mathbf{X})$, is called *proper* if the generalized momentum map $\mathbf{F} : M \to \mathbb{R}^q$ is a proper map from M to its image, and the image of the singular set $\{x \in M, X_1 \wedge X_2 \wedge ... \wedge X_p(x) = 0\}$ of the commuting Hamiltonian vector fields under the momentum map $\mathbf{F} : M \to \mathbb{R}^q$ is nowhere dense in $\mathbb{R}^q$.

Under the properness condition, one get a natural generalization of the classical Liouville-Mineur-Arnold theorem [61, 59]: outside the singular region, the Poisson manifold M is foliated by invariant isotropic p-dimensional tori on which the flow of X_H is quasi-periodic, and there exist local action-angle coordinates. The action variables can still be defined by Mineur-Arnold formula (2.5). There will be p action and p angle variables (so one will have to add $(q - p)$ variables to get a full system of variables). In particular, near every isotropic invariant torus there is a free Hamiltonian $\mathbb{T}^p$-action which preserves the system.

For example, a Hamiltonian $\mathbb{T}^p$-action on a Poisson manifold can be seen as a proper integrable system – the space of first integrals is the space of $\mathbb{T}^p$-invariant functions, and in this case we have a *global* $\mathbb{T}^p$-action which preserves the system. More generally, one can associate to each Hamiltonian compact group action on a Poisson manifold a proper integrable system, see Subsection 2.4 and Subsection 5.2.

There is a natural question: is an integrable system in generalized Liouville sense on a symplectic manifold also integrable à la Liouville ? In general, one expects the answer to be Yes. See e.g. Fomenko [33] for a long discussion on this question, and the related question about the existence of Liouville-integrable systems on given symplectic manifolds.

Remark 2.5. Another natural question is the following. Let $\mathcal{F}_H$ denote the space of all first integrals of H. Suppose that H is integrable in generalized Liouville sense. Is it true that H is integrable with the aid of $(\mathcal{F}_H, \mathcal{X}_{\mathcal{F}_H})$, i.e. ddim $\mathcal{F}_H +$ ddim $\mathcal{X}_{\mathcal{F}_H} = 0$? We expect the answer to be yes for "reasonable" systems. It is easy to see that the answer is Yes in the proper integrable case, under the additional assumption that the orbits of X_H are dense (i.e. its frequencies are incommensurable) on almost every invariant torus (i.e. common level of a given set of first integrals $\mathcal{F}$). In this case $\mathcal{X}_{\mathcal{F}_H}$ consists of the Hamiltonian vector fields whose flow is quasi-periodic on each invariant torus. Another case where the answer is also Yes arises in the study of local normal forms of analytic integrable vector fields, see Section 3.

2.3. **Non-Hamiltonian integrability.**

There are many physical non-Hamiltonian (e.g. non-holonomic) systems, who may naturally be called integrable in a non-Hamiltonian sense, because their behavior is very similar to that of integrable Hamiltonian systems, see e.g. [7, 20]. A simple example is the Chinese top. (It is a spinning top whose lower part looks like a hemisphere and whose upper part is heavy. When you spin it, it will turn upside down after a while). The notion of non-Hamiltonian integrability was probably first introduced by Bogoyavlenskij (see [10] and references therein), who calls it *broad integrability*, though other authors also arrived at it independently, from different points of view, see e.g. [7, 10, 20, 67, 86].

Definition 2.6. A vector field X on a manifold M is called *integrable in non-Hamiltonian sense* with the aid of $(\mathcal{F}, \mathcal{X})$, where $\mathcal{F}$ is a set of functions on M and $\mathcal{X}$ is a set of vector fields on M, if the following conditions are satisfied :
a) $X(F) = 0$ and $Y(F) = 0 \ \forall \ \ F \in \mathcal{F}, Y \in \mathcal{X}$,
b) $[Y, X] = [Y, Z] = 0 \ \ \forall \ Y, Z \in \mathcal{X}$,
d) $\dim M =$ ddim $\mathcal{F} +$ ddim $\mathcal{X}$.
In the real case, if, moreover, there is a p-tuple $\mathbf{X} = (X_1, ..., X_p)$ of vector fields in $\mathcal{X}$ and a q-tuple $\mathbf{F} = (F_1, ..., F_q)$ of functionally independent functions in $\mathcal{F}$, where $p =$ ddim $\mathcal{X}$ and $q =$ ddim $\mathcal{F}$, such that the map $\mathbf{F} : M \to \mathbb{R}^q$ is a proper map from M to its image, and for almost every level set of this map the vector fields $X_1, ..., X_p$ are linearly independent everywhere on this level set, then we say that X is *proper integrable* with the aid of $(\mathbf{F}, \mathbf{X})$, and by abuse of language we will also say that $(\mathbf{F}, \mathbf{X})$ is a proper integrable non-Hamiltonian system of bi-degree (p, q) of freedom.

So non-Hamiltonian integrability is the same as Hamiltonian integrability; except for the fact that the vector fields $X, X_1, \ldots, X_p$ are not required to be Hamiltonian. It is not surprising that Liouville's theorem holds for proper non-Hamiltonian integrable systems as well: each regular invariant manifold (connected level set of $\mathbf{F}$) is a p-dimensional torus on which the system is quasi-periodic, and in a neighborhood of it there is a free $\mathbb{T}^p$-torus action which preserves the system.

If a Hamiltonian system is (proper) integrable in the generalized Liouville sense, then of course it is also (proper) integrable in the non-Hamiltonian sense, though the inverse is not true: it may happen that the invariant tori are not isotropic, see e.g. [10].

Remark 2.7. Remark 2.5 also applies to non-Hamiltonian systems: For an integrable vector field X on a manifold M, denote by $\mathcal{F}_X$ the set of all first integrals of X, and by $\mathcal{X}_X$ the set of all vector fields which commute with X and preserve every function in $\mathcal{F}$. Then a natural question is, do we have the equality $\mathrm{ddim}\ \mathcal{F}_X + \mathrm{ddim}\ \mathcal{X}_X = \dim M$? The answer is similar to the Hamiltonian case. In particular, if the system is proper and the vector field X is nonresonant (i.e. has a dense orbit) on almost every invariant torus, then the answer is yes.

2.4. **Reduced integrability of Hamiltonian systems.**

In the literature, when people speak about integrability of a dynamical system, they often actually mean its *reduced integrability*, i.e. integrability of the reduced (with respect to a natural symmetry group action) system. For example, consider an integrable spinning top (e.g. the Kovalevskaya top). Its configuration space is $SO(3)$, so it is naturally a Hamiltonian system with 3 degrees of freedom. But it is often considered as a 2-degree-of-freedom integrable system (with a parameter), see e.g. [11].

Curiously, to my knowledge, the natural question about the effect of reduction on integrability has never been formally addressed in monographs on dynamical systems. In [88], we studied this question, and showed that, for a Hamiltonian system invariant under the proper action of a Lie group, integrability is essentially equivalent to reduced integrability.

It turns out that the most natural notion of integrability to use here is not the Liouville integrability, but rather the integrability in generalized Liouville sense. Also, since the category of manifolds is not invariant under the operation of taking quotient with respect to a proper group action, we have to replace manifolds by *generalized manifolds*: in this paper, *a generalized manifold is a differentiable space which is locally isomorphic to the quotient of a manifold by a compact group action.* Due to well-kown results about functions invariant under compact group actions, see e.g. [62], one can talk about smooth functions, vector fields, differential forms, etc. on generalized manifolds, and the previous integrability definitions work for them as well.

Let (M, Π) be a Poisson generalized manifold, G a Lie group which acts properly on M, H a function on M which is invariant under the action of G. Then the quotient space M/G is again a Poisson generalized manifold, see e.g. [21]. We will denote the projection of Π, H, X_H on M/G by $\Pi/G, H/G, X_H/G$ respectively. Of course, X_H/G is the Hamiltonian vector field of H/G.

We will assume that the action of G on (M, Π) is Hamiltonian, with an equivariant moment map $\pi : M \to \mathfrak{g}^*$, where $\mathfrak{g}$ denotes the Lie algebra of G, and that the following additional condition is satisfied: Recall that the image $\pi(M)$ of M under the moment map $\pi : M \to \mathfrak{g}^*$ is saturated by symplectic leaves (i.e. coadjoint orbits) of $\mathfrak{g}^*$. Denote by s the minimal codimension in $\mathfrak{g}^*$ of a coadjoint orbit which lies in $\pi(M)$. Then the additional condition is that there exist s functions $f_1, ..., f_s$ on $\mathfrak{g}^*$, which are invariant on the coadjoint orbits which lie in $\pi(M)$, and such that for almost every point $x \in M$ we have $df_1 \wedge ... \wedge df_s(\pi(x)) \neq 0$. For example, when G is compact and M is connected, then this condition is satisfied automatically.

With the above notations and assumptions, we have :

Theorem 2.8 ([88])**.** *If the system $(M/G, X_H/G)$ is integrable in generalized Liouville sense, then the system (M, X_H) also is. Moreover, if G is compact and $(M/G, X_H/G)$ is proper, then (M, X_H) also is.*

Similar results to Theorem 2.8 have been obtained independently by Bolsinov and Jovanovic [12, 46], who used them to construct new examples of integrable geodesic flows, e.g. on biquotients of compact Lie groups.

Examples. 1) The simplest example which shows an evident relationship between reduction and integrability is the classical Euler top : it can be written as a Hamiltonian system on $T^*SO(3)$, invariant under a natural Hamiltonian action of $SO(3)$, is integrable with the aid of a set of four first integrals, and has 2-dimensional isotropic invariant tori. 2) The geodesic flow of a bi-invariant metric on a compact Lie group is also properly integrable : in fact, the corresponding reduced system is trivial (identically zero). More generally, let $H = h \circ \mu$ be a collective Hamiltonian in the sense of Guillemin–Sternberg (see e.g. [40]), where $\mu : M \to \mathfrak{g}^*$ is the momentum map of a Hamiltonian compact group action, and h is a function on $\mathfrak{g}^*$. If h is a Casimir function on $\mathfrak{g}^*$, then H is integrable because its reduction will be a trivial Hamiltonian system.

Since [88] will be available as a preprint only (not submitted for publication as a separate paper), let us include here a full proof of this theorem.

Proof. Denote by $\mathcal{F}'$ a set of first integrals of X_H/G on M/G which provides the integrability of X_H/G, and by $\mathcal{X}' = \mathcal{X}_{\mathcal{F}'}$ the corresponding space of commuting Hamiltonian vector fields on M/G. We have $\dim M/G = p' + q'$ where $p' = \text{ddim } \mathcal{X}'$ and $q' = \text{ddim } \mathcal{F}'$.

Recall that, by our assumptions, there exist s functions $f_1, ..., f_s$ on $\mathfrak{g}^*$, which are functionally independent almost everywhere in $\pi(M)$, and which are invariant on the coadjoint orbits which lie in $\pi(M)$. Here s is the minimal codimension in $\mathfrak{g}^*$ of the coadjoint orbits which lie in $\pi(M)$. We can complete $(f_1, ..., f_s)$ to a set of d functions $f_1, ..., f_s, f_{s+1}, ..., f_d$ on $\mathfrak{g}^*$, where $d = \dim G = \dim \mathfrak{g}$ denotes the dimension of $\mathfrak{g}$, which are functionally independent almost everywhere in $\pi(M)$.

Denote by $\overline{\mathcal{F}}$ the pull-back of $\mathcal{F}'$ under the projection $\mathfrak{p} : M \to M/G$, and by $F_1, ..., F_d$ the pull-back of $f_1, ..., f_d$ under the moment map $\pi : M \to \mathfrak{g}^*$. Note that, since H is G-invariant, the functions F_i are first integrals of X_H. And of course, $\overline{\mathcal{F}}$ is also a set of first integrals of X_H. Denote by $\mathcal{F}$ the union of $\overline{\mathcal{F}}$ with $(F_{s+1}, ..., F_d)$. (It is not necessary to include $F_1, ..., F_s$ in this union, because these functions are G-invariant and project to Casimir functions on M/G, which implies that they are functionally dependent of $\overline{\mathcal{F}}$). We will show that X_H is integrable with the aid of $\mathcal{F}$.

Notice that, by assumptions, the coadjoint orbits of $\mathfrak{g}^*$ which lie in $\pi(M)$ are of generic dimension $d-s$, and the functions $f_{s+1}, ..., f_d$ may be viewed as a coordinate system on a symplectic leaf of $\pi(M)$ at a generic point. In particular, we have

$$\langle df_{s+1} \wedge ... \wedge df_d, X_{f_{s+1}} \wedge ... X_{f_d} \rangle \neq 0,$$

which implies, by equivariance :

$$\langle dF_{s+1} \wedge ... \wedge dF_d, X_{F_{s+1}} \wedge ... X_{F_d} \rangle \neq 0.$$

Since the vector fields $X_{F_{s+1}}, ..., X_{F_d}$ are tangent to the orbits of G on M, and the functions in $\overline{\mathcal{F}}$ are invariant on the orbits of G, it implies that the set $(F_{s+1}, ..., F_d)$ is "totally" functionally independent of $\overline{\mathcal{F}}$. In particular, we have :

$$\text{ddim } \mathcal{F} = \text{ddim } \mathcal{F}' + \text{ddim } (F_{s+1}, ..., F_d) = q' + d - s, \tag{2.9}$$

where $q' = \text{ddim } \mathcal{F}'$. On the other hand, we have

$$\dim M = \dim M/G + (d-k) = p' + q' + d - k,$$

where $p' = \operatorname{ddim} \mathcal{X}_{\mathcal{F}'}$, and k is the dimension of a minimal isotropic group of the action of G on M. Thus, in order to show the integrability condition

$$\dim M = \operatorname{ddim} \mathcal{F} + \operatorname{ddim} \mathcal{X}_{\mathcal{F}},$$

it remains to show that

$$\operatorname{ddim} \mathcal{X}_{\mathcal{F}} = \operatorname{ddim} \mathcal{X}_{\mathcal{F}'} + (s-k). \tag{2.10}$$

Consider the vector fields $Y_1 = X_{F_1}, ..., Y_d = X_{F_d}$ on M. They span the tangent space to the orbit of G on M at a generic point. The dimension of such a generic tangent space is $d-k$. It implies that, among the first s vector fields, there are at least $s-k$ vector fields which are linearly independent at a generic points : we may assume that $Y_1 \wedge ... \wedge Y_{s-k} \neq 0$.

Let $X_{h_1}, ..., X_{h_{p'}}$ be p' linearly independent (at a generic point) vector fields which belong to $\mathcal{X}_{\mathcal{F}'}$, where $p' = \operatorname{ddim} \mathcal{X}_{\mathcal{F}'}$. Then we have

$$X_{\mathfrak{p}^*(h_1)}, ..., X_{\mathfrak{p}^*(h_{p'})}, Y_1, ..., Y_{s-k} \in \mathcal{X}_{\mathcal{F}},$$

and these $p'+s-k$ vector fields are linearly independent at a generic point. (Recall that, at each point $x \in M$, the vectors $Y_1(x), ..., Y_{s-k}(x)$ are tangent to the orbit of G which contains x, while the linear space spanned by $X_{\mathfrak{p}^*(h_1)}, ..., X_{\mathfrak{p}^*(h_{p'})}$ contains no tangent direction to this orbit).

Thus we have $\operatorname{ddim} \mathcal{X}_{\mathcal{F}} \geq p' + s - k$, which means that $\operatorname{ddim} \mathcal{X}_{\mathcal{F}} = p' + s - k$ (because, as discussed earlier, we always have $\operatorname{ddim} \mathcal{F} + \operatorname{ddim} \mathcal{X}_{\mathcal{F}} \leq \dim M$). We have proved that if $(M/G, X_H/G)$ is integrable in generalized Liouville sense then (M, X_H) also is.

Now suppose that G is compact and $(M/G, X_H/G)$ is proper: there are q' functionally independent functions $g_1, ..., g_{q'} \in \mathcal{F}'$ such that $(g_1, ..., g_{q'}) : M/G \to \mathbb{R}^{q'}$ is a proper map from M/G to its image, and p' Hamiltonian vector fields $X_{h_1}, ..., X_{h_{p'}}$ in $\mathcal{X}'$ such that on a generic common level set of $(g_1, ..., g_{q'})$ we have that $X_{h_1} \wedge ... \wedge X_{h_{p'}}$ does not vanish anywhere. Then it is straightforward that

$$\mathfrak{p}^*(g_1), ..., \mathfrak{p}^*(g_{q'}), F_{s+1}, ..., F_d \in \mathcal{F}$$

and the map

$$(\mathfrak{p}^*(g_1), ..., \mathfrak{p}^*(g_{q'}), F_{s+1}, ..., F_d) : M \to \mathbb{R}^{q'+d-s}$$

is a proper map from M to its image. More importantly, on a generic level set of this map we have that the $(q'+s-k)$-vector $X_{\mathfrak{p}^*(h_1)} \wedge ... \wedge X_{\mathfrak{p}^*(h_{p'})} \wedge Y_1 \wedge ... \wedge Y_{s-k}$ does not vanish anywhere. To prove this last fact, notice that $X_{\mathfrak{p}^*(h_1)} \wedge ... \wedge X_{\mathfrak{p}^*(h_{p'})} \wedge Y_1 \wedge ... \wedge Y_{s-k}(x) \neq 0$ for a point $x \in M$ if and only if $X_{\mathfrak{p}^*(h_1)} \wedge ... \wedge X_{\mathfrak{p}^*(h_{p'})}(x) \neq 0$ and $Y_1 \wedge ... \wedge Y_{s-k}(x) \neq 0$ (one of these two multi-vectors is transversal to the G-orbit of x while the other one "lies on it"), and that these inequalities are $G \times \mathbb{R}^{p'}$-invariant properties, where the action of $\mathbb{R}^{p'}$ is generated by $X_{\mathfrak{p}^*(h_1)}, ..., X_{\mathfrak{p}^*(h_{p'})}$. □

Remark 2.9. Recall from Equation (2.10) above that we have $\operatorname{ddim} \mathcal{X}_{\mathcal{F}} - \operatorname{ddim} \mathcal{X}_{\mathcal{F}'} = s-k$, where k is the dimension of a generic isotropic group of the G-action on M, and s is the (minimal) corank in $\mathfrak{g}^*$ of a coadjoint orbit which lies in $\pi(M)$. On the other hand, the difference between the rank of the Poisson structure on M and the reduced Poisson structure on M/G can be calculated as follows :

$$\text{rank } \Pi - \text{rank } \Pi/G = (d-k) + (s-k) \tag{2.11}$$

Here $(d-k)$ is the difference between $\dim M$ and $\dim M/G$, and $(s-k)$ is the difference between the corank of Π/G in M/G and the corank of Π in M. It follows that

$$\text{rank } \Pi - 2\text{ddim } \mathcal{X}_{\mathcal{F}} = \text{rank } \Pi/G - 2\text{ddim } \mathcal{X}_{\mathcal{F}'} + (d-s) \tag{2.12}$$

In particular, if $d-s>0$ (typical situation when G is non-Abelian), then we always have rank $\Pi - 2$ddim $\mathcal{X}_{\mathcal{F}} > 0$ (because we always have rank $\Pi/G - 2$ddim $\mathcal{X}_{\mathcal{F}'} \geq 0$ due to integrability), i.e. the original system is always super-integrable with the aid of $\mathcal{F}$. When G is Abelian (implying $d=s$), and the reduced system is Liouville-integrable with the aid of $\mathcal{F}'$ (i.e. rank $\Pi/G = 2$ddim $\mathcal{X}_{\mathcal{F}'}$), then the original system is also Liouville-integrable with the aid of $\mathcal{F}$.

Remark 2.10. Following Mischenko-Fomenko [59], we will say that a hamiltonian system (M,Π,X_H) is *non-commutatively integrable in the restricted sense* with the aid of $\mathcal{F}$, if $\mathcal{F}$ is a finite-dimensional Lie algebra under the Poisson bracket and (M,Π,X_H) is integrable with the aid of $\mathcal{F}$. In other words, we have an equivariant moment maps $(M,\Pi)\to\mathfrak{f}^*$, where $\mathfrak{f}$ is some finite-dimensional Lie algebra, and if we denote by $f_1,...,f_n$ the components of this moment map, then they are first integrals of X_H, and X_H is integrable with the aid of this set of first integrals. Theorem 2.8 remains true, and its proof remains the same if not easier, if we replace Hamiltonian integrability by non-commutative integrability in the restricted sense. Indeed, if $M\to\mathfrak{g}^*$ is the equivariant moment map of the symmetry group G, and if $M/G\to\mathfrak{h}^*$ is an equivariant moment map which provides non-commutative integrability in the restricted sense on M/G, then the map $M\to\mathfrak{h}^*$ (which is the composition $M\to M/G\to\mathfrak{h}^*$) is an equivariant moment map which commutes with $M\to\mathfrak{g}^*$, and the direct sum of this two maps, $M\to\mathfrak{f}^*$ where $\mathfrak{f}=\mathfrak{g}\bigoplus\mathfrak{h}$, will provide non-commutative integrability in the restricted sense on M.

Theorem 2.8 has the following inverse (see Remark 2.5):

Theorem 2.11. *If G is compact, and if the Hamiltonian system (M,X_H) is integrable with the aid of $\mathcal{F}_H$ (the set of all first integrals of H) in the sense that* ddim $\mathcal{F}_H$ + ddim $\mathcal{X}_{\mathcal{F}_H} = \dim M$, *then the reduced Hamiltonian system $(M/G,X_H)$ is also integrable. Moreover, if (M,X_H) is proper then $(M/G,X_H)$ also is.*

Proof. By assumptions, we have $\dim M = p+q$, where $q=$ ddim $\mathcal{F}_H$ and $p=$ ddim $\mathcal{X}_{\mathcal{F}_H}$, and we can find p first integrals $H_1,...,H_p$ of H such that $X_{H_1},...,X_{H_p}$ are linearly independent (at a generic point) and belong to $\mathcal{X}_{\mathcal{F}_H}$. In particular, we have $X_{H_i}(F)=0$ for any $F\in\mathcal{F}$ and $1\leq i\leq p$.

An important observation is that the functions $H_1,...,H_p$ are G-invariant. In deed, if we denote by $F_1,...,F_d$ the components of the equivariant moment map $\pi: M\to\mathfrak{g}^*$ (via an identification of $\mathfrak{g}^*$ with $\mathbb{R}^d$), then since H is G-invariant we have $\{H,F_j\}=0$, i.e. $F_j\in\mathcal{F}_H$, which implies that $\{F_j,H_i\}=0\ \ \forall 1\leq i\leq d,\ \ 1\leq j\leq p$, which means that H_i are G-invariant.

The Hamiltonian vector fields X_{H_i}/G belong to $\mathcal{X}_{\mathcal{F}_{H/G}}$: Indeed, if $f\in\mathcal{F}_{H/G}$ then $\mathfrak{p}^*(f)$ is a first integral of H, implying $\{H_i,\mathfrak{p}^*(f)\}=0$, or $\{H_i/G,f\}=0$, where $\mathfrak{p}$ denotes the projection $M\to M/G$.

To prove the integrability of X_H/G, it is sufficient to show that

$$\dim M/G \leq \text{ddim } \mathcal{F}_{H/G} + \text{ddim } (X_{H_1}/G, ..., X_{H_q}/G) \tag{2.13}$$

But we denote by r the generic dimension of the intersection of a common level set of p independent first integrals of X_H with an orbit of G in M, then one can check that

$$p - \text{ddim } (X_{H_1}/G, ..., X_{H_q}/G) = \text{ddim } \mathcal{X}_{\mathcal{F}_H} - \text{ddim } (X_{H_1}/G, ..., X_{H_q}/G) = r$$

and

$$q - \text{ddim } \mathcal{F}_{H/G} = \text{ddim } \mathcal{F}_H - \text{ddim } \mathcal{F}_{H/G} \leq (d-k) - r$$

where $(d-k)$ is the dimension of a generic orbit of G in M. To prove the last inequality, notice that functions in $\mathcal{F}_{H/G}$ can be obtained from functions in $\mathcal{F}_H$ by averaging with respect to the G-action. Also, G acts on the (separated) space of common level sets of the functions in $\mathcal{F}_H$, and isotropic groups of this G-action are of (generic) codimension $(d-k)-r$.

The above two formulas, together with $p+q = \dim M = \dim M/G + (d-k)$, implies Inequality (2.13) (it is in fact an equality). The proper case is straightforward. □

2.5. **Non-Hamiltonian reduced integrability.**

One of the main differences between the non-Hamiltonian case and the Hamiltonian case is that reduced non-Hamiltonian integrability does not imply integrability. In fact, in the Hamiltonian case, we can lift Hamiltonian vector fields from M/G to M via the lifting of corresponding functions. In the non-Hamiltonian case, no such canonical lifting exists, therefore commuting vector fields on M/G do not provide commuting vector fields on M. For example, consider a vector field of the type $X = a_1\partial/\partial x_1 + a_2\partial/\partial x_2 + b(x_1, x_2)\partial/\partial x_3$ on the standard torus $\mathbb{T}^3$ with periodic coordinates (x_1, x_2, x_3), where a_1 and a_2 are two incommensurable real numbers $(a_1/a_2 \notin \mathbb{Q})$, and $b(x_1, x_2)$ is a smooth function of two variables. Then clearly X is invariant under the $\mathbb{T}^1$-action generated by $\partial/\partial x_3$, and the reduced system is integrable. On the other hand, for X to be integrable, we must be able to find a function $c(x_1, x_2)$ such that $[X, \partial/\partial x_1 + c(x_1, x_2)\partial/\partial x_3] = 0$. This last equation does not always have a solution (it is a small divisor problem, and depends on a_1/a_2 and the behavior of the coefficients of $b(x_1, x_2)$ in its Fourier expansion), i.e. there are choices of $a_1, a_2, b(x_1, x_2)$ for which the vector field X is not integrable.

However, non-Hamiltonian integrability still implies reduced integrability. Recall from Remark 2.7 that if a vector field X on a (generalized) manifold M is integrable, then under mild additional conditions we have $\text{ddim } \mathcal{X}_X + \text{ddim } \mathcal{F}_X = \dim M$, where $\mathcal{F}_X$ is the set of all first integrals of X, and $\mathcal{X}_X$ is the set of all vector fields which commute with X and preserve every function in $\mathcal{F}$.

Theorem 2.12. *Let X be a smooth non-Hamiltonian proper integrable system on a manifold M with the aid of $(\mathcal{F}_X, \mathcal{X}_X)$, i.e.* $\text{ddim } \mathcal{X}_X + \text{ddim } \mathcal{F}_X = \dim M$, *and G be a compact Lie group acting on M which preserves X. Then the reduced system on M/G is also proper integrable.*

Proof. Let $\mathcal{X}_X^G$ denote the set of vector fields which belong to $\mathcal{X}_X$ and which are invariant under the action of G. Note that the elements of $\mathcal{X}_X^G$ can be obtained from the elements of $\mathcal{X}_X$ by averaging with respect to the G-action.

A key ingredient of the proof is the fact ddim $\mathcal{X}_X^G =$ ddim $\mathcal{X}_X$ (To see this fact, notice that near each regular invariant torus of the system there is an effective torus action (of the same dimension) which preserves the system, and this torus action must necessarily commute with the action of G. The generators of this torus action are linearly independent vector fields which belong to $\mathcal{X}_X^G$ - in fact, they are defined locally near the union of G-orbits which by an invariant torus, but then we can extend them to global vector fields which lie in $\mathcal{X}_X^G$)

Therefore, we can project the pairwise commuting vector fields in $\mathcal{X}_X^G$ from M to M/G to get pairwise commuting vector fields on M/G. To get the first integrals for the reduced system, we can also take the first integrals of X on M and average them with respect to the G-action to make them G-invariant. The rest of the proof of Theorem 2.12 is similar to that of Theorem 2.11. □

3. Torus actions and local normal forms

3.1. **Toric characterization of Poincaré-Birkhoff normal form.**

It is a simple well-known fact that every vector field near an equilibrium point admits a formal Poincaré-Birkhoff normal form (Birkhoff in the Hamiltonian case, and Poincaré-Dulac in the non-Hamiltonian case). What is also very simple but much less well-known is that these normal forms are governed by torus actions.

Let X be a given analytic vector field in a neighborhood of 0 in $\mathbb{K}^m$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with $X(0) = 0$. When $\mathbb{K} = \mathbb{R}$, we may also view X as a holomorphic (i.e. complex analytic) vector field by complexifying it. Denote by

$$X = X^{(1)} + X^{(2)} + X^{(3)} + ... \tag{3.1}$$

the Taylor expansion of X in some local system of coordinates, where $X^{(k)}$ is a homogeneous vector field of degree k for each $k \geq 1$.

In the Hamiltonian case, on a symplectic manifold, $X = X_H$, $m = 2n$, $\mathbb{K}^{2n}$ has a standard symplectic structure, and $X^{(j)} = X_{H^{(j+1)}}$

The algebra of linear vector fields on $\mathbb{K}^m$, under the standard Lie bracket, is nothing but the reductive algebra $gl(m, \mathbb{K}) = sl(m, \mathbb{K}) \oplus \mathbb{K}$. In particular, we have

$$X^{(1)} = X^s + X^{nil}, \tag{3.2}$$

where X^s (resp., X^{nil}) denotes the semi-simple (resp., nilpotent) part of $X^{(1)}$. There is a complex linear system of coordinates (x_j) in $\mathbb{C}^m$ which puts X^s into diagonal form:

$$X^s = \sum_{j=1}^{m} \gamma_j x_j \partial/\partial x_j, \tag{3.3}$$

where γ_j are complex coefficients, called *eigenvalues* of X (or $X^{(1)}$) at 0.

In the Hamiltonian case, $X^{(1)} \in sp(2n, \mathbb{K})$ which is a simple Lie algebra, and we also have the decomposition $X^{(1)} = X^s + X^{nil}$, which corresponds to the decomposition

$$H^{(2)} = H^s + H^{nil} \tag{3.4}$$

There is a complex canonical linear system of coordinates (x_j, y_j) in $\mathbb{C}^{2n}$ in which H^s has diagonal form:

$$H^s = \sum_{j=1}^{n} \lambda_j x_j y_j, \tag{3.5}$$

where λ_j are complex coefficients, called *frequencies* of H (or $H^{(2)}$) at 0.

For each natural number $k \geq 1$, the vector field X^s acts linearly on the space of homogeneous vector fields of degree k by the Lie bracket, and the monomial vector fields are the eigenvectors of this action:

$$[\sum_{j=1}^{m} \gamma_j x_j \partial/\partial x_j, x_1^{b_1} x_2^{b_2} ... x_n^{b_n} \partial/\partial x_l] = (\sum_{j=1}^{n} b_j \gamma_j - \gamma_l) x_1^{b_1} x_2^{b_2} ... x_n^{b_n} \partial/\partial x_l. \tag{3.6}$$

When an equality of the type

$$\sum_{j=1}^{m} b_j \gamma_j - \gamma_l = 0 \tag{3.7}$$

holds for some nonnegative integer m-tuple (b_j) with $\sum b_j \geq 2$, we will say that the monomial vector field $x_1^{b_1} x_2^{b_2} ... x_m^{b_m} \partial/\partial x_l$ is a resonant term, and that the m-tuple $(b_1, ..., b_l - 1, ..., b_l)$ is a resonance relation for the eigenvalues (γ_i). More precisely, a *resonance relation* for the n-tuple of eigenvalues (γ_j) of a vector field X is an m-tuple (c_j) of integers satisfying the relation $\sum c_j \gamma_j = 0$, such that $c_j \geq -1, \sum c_j \geq 1$, and at most one of the c_j may be negative.

In the Hamiltonian case, H^s acts linearly on the space of functions by the Poisson bracket. Resonant terms (i.e. generators of the kernel of this action) are monomials $\prod x_j^{a_j} y_j^{b_j}$ which satisfy the following resonance relation, with $c_j = a_j - b_j$:

$$\sum_{j=1}^{m} c_j \lambda_j = 0 \tag{3.8}$$

Denote by $\mathcal{R}$ the subset of $\mathbb{Z}^m$ (or sublattice of $\mathbb{Z}^n$ in the Hamiltonian case) consisting of all resonance relations (c_j) for a given vector field X. The number

$$r = \dim_{\mathbb{Z}}(\mathcal{R} \otimes \mathbb{Z}) \tag{3.9}$$

is called the *degree of resonance* of X. Of course, the degree of resonance depends only on the eigenvalues of the linear part of X, and does not depend on the choice of local coordinates. If $r = 0$ then we say that the system is *nonresonant* at 0.

The vector field X is said to be in *Poincaré-Birkhoff normal form* if it commutes with the semisimple part of its linear part (see e.g. [14, 64]):

$$[X, X^s] = 0. \tag{3.10}$$

In the Hamiltonian case, the above equation can also be written as

$$\{H, H^s\} = 0. \tag{3.11}$$

The above equations mean that if X is in normal form then its nonlinear terms are resonant. A transformation of coordinates (which is symplectic in the Hamiltonian case) which puts X in Poincaré-Birkhoff normal form is called a *Poincaré-Birkhoff normalization*. It is a classical result of Poincaré, Dulac, and Birkhoff that any analytic vector field which vanishes at 0 admits a *formal* Poincaré-Birkhoff normalization.

Denote by $\mathcal{Q} \subset \mathbb{Z}^m$ the integral sublattice of $\mathbb{Z}^m$ consisting of m-dimensional vectors $(\rho_j) \in \mathbb{Z}^m$ which satisfy the following properties :

$$\sum_{j=1}^{m} \rho_j c_j = 0 \ \forall \ (c_j) \in \mathcal{R} \ , \text{ and } \ \rho_j = \rho_k \ \text{ if } \ \gamma_j = \gamma_k \tag{3.12}$$

(where $\mathcal{R}$ is the set of resonance relations as before). In the Hamiltonian case, $\mathcal{Q}$ is defined by

$$\sum_{j=1}^{n} \rho_j c_j = 0 \ \forall \ (c_j) \in \mathcal{R} \ . \tag{3.13}$$

We will call the number

$$d = \dim_{\mathbb{Z}} \mathcal{Q} \tag{3.14}$$

the *toric degree* of X at 0. Of course, this number depends only on the eigenvalues of the linear part of X, and we have the following (in)equality : $r + d = n$ in the Hamiltonian case (where r is the degree of resonance), and $r + d \leq m$ in the non-Hamiltonian case.

Let $(\rho_j^1), ..., (\rho_j^d)$ be a basis of $\mathcal{Q}$. For each $k = 1, ..., d$ define the following diagonal linear vector field Z_k :

$$Z_k = \sum_{j=1}^{m} \rho_j^k x_j \partial / \partial x_j \tag{3.15}$$

in the non-Hamiltonian case, and $Z_k = X_{F^k}$ where

$$F^k = \sum_{j=1}^{n} \rho_j^k x_j y_j \tag{3.16}$$

in the Hamiltonian case.

The vector fields $Z_1, ..., Z_r$ have the following remarkable properties :

a) They commute pairwise and commute with X^s and X^{nil}, and they are linearly independent almost everywhere.

b) iZ_j is a periodic vector field of period 2π for each $j \leq r$ (here $i = \sqrt{-1}$). What does it mean is that if we write $iZ_j = \Re(iZ_j) + i\Im(iZ_j)$, then $\Re(iZ_j)$ is a periodic real vector field in $\mathbb{C}^n = \mathbb{R}^{2n}$ which preserves the complex structure.

c) Together, $iZ_1, ..., iZ_r$ generate an effective linear $\mathbb{T}^r$-action in $\mathbb{C}^n$ (which preserves the symplectic structure in the Hamiltonian case), which preserves X^s and X^{nil}.

A simple calculation shows that X is in Poincaré-Birkhoff normal form, i.e. $[X, X^s] = 0$, if and only if we have

$$[X, Z_k] = 0 \quad \forall \ k = 1, ..., r. \tag{3.17}$$

The above commutation relations mean that if X is in normal form, then it is preserved by the effective r-dimensional torus action generated by $iZ_1, ..., iZ_r$. Conversely, if there is a torus action which preserves X, then because the torus is a compact group we can linearize this torus action (using Bochner's linearization theorem in the non-Hamiltonian case, and Guillemin–Sternberg–Marle linearization theorem in the Hamiltonian case, see e.g. [18, 39]), leading to a normalization of X. In other words, we have:

Theorem 3.1 ([85, 86]). *A holomorphic (Hamiltonian) vector field X in a neighborhood of 0 in $\mathbb{C}^m$ (or $\mathbb{C}^{2n}$ with a standard symplectic form) admits a locally holomorphic Poincaré-Birkhoff normalization if and only if it is preserved by an effective holomorphic (Hamiltonian) action of a real torus of dimension t, where t is the toric degree of $X^{(1)}$ as defined in (3.14), in a neighborhood of 0 in $\mathbb{C}^m$ (or $\mathbb{C}^{2n}$), which has 0 as a fixed point and whose linear part at 0 has appropriate weights (given by the lattice $\mathcal{Q}$ defined in (3.12,3.13), which depends only on the linear part $X^{(1)}$ of X).*

The above theorem is true in the formal category as well. But of course, any vector field admits a formal Poincaré-Birkhoff normalization, and a formal torus action.

3.2. Some simple consequences and generalizations.

Theorem 3.1 has many important implications. One of them is:

Proposition 3.2 ([85, 86]). *A real analytic vector field X (Hamiltonian or non-Hamiltonian) in the neighborhood of an equilibrium point admits a local real analytic Poincaré-Birkhoff normalization if and only if it admits a local holomorphic Poincaré-Birkhoff normalization when considered as a holomorphic vector field.*

The proof of the above proposition (see [85]) is based on the fact that the complex conjugation induces an involution on the torus action which governs the Poincaré-Birkhoff normalization.

If a dynamical system near an equilibrium point is invariant with respect to a compact group action which fixes the equilibrium point, then this compact group action commutes with the (formal) torus action of the Poincaré-Birkhoff normalization. Together, they form a bigger compact group action, whose linearization leads to a simultaneous Poincaré-Birkhoff normalization and linearization of the compact symmetry group, i.e. we can perform the Poincaré-Birkhoff normalization in an invariant way. This is a known result in dynamical systems, see e.g. [81], but the toric point of view gives a new simple proof of it. The case of equivariant vector fields is similar. For example, one can speak about Poincaré-Dulac normal forms for time-reversible vector fields, see e.g. [51].

Another situation where one can use the toric characterization is the case of isochore (i.e. volume preserving) vector fields. In this case, naturally, the normalization transformation is required to be volume-preserving. Both Theorem 3.1 and Proposition 3.2 remain valid in this case.

One can probably use the toric point of view to study normal forms of Hamiltonian vector field on *Poisson* manifolds as well. For example, let $\mathfrak{g}^*$ be the dual of a semi-simple Lie algebra, equipped with the standard linear Poisson structure, and let $H : \mathfrak{g}^* \to \mathbb{K}$ be a regular function near the origin 0 of $\mathfrak{g}^*$. The corresponding Hamiltonian vector field X_H will vanish at 0, because the Poisson structure itself vanishes at 0. Applying Poincaré-Birkhoff normalization techniques, we can kill the "nonresonant terms" in H (with respect to the linear part of H, or $dH(0)$). The normalized Hamiltonian will be invariant under the coadjoint action of a subtorus of a Cartan torus of the (complexified) Lie group of $\mathfrak{g}$. In the "nonresonant" case, we have a Cartan torus action which preserves the system.

3.3. Convergent normalization for integrable systems.

Though every vector field near an equilibrium admits a formal Poincaré-Birkhoff normalization, the problem of finding a convergent (i.e. locally real analytic or holomorphic) normalization is much more difficult. The usual step by step killing of non-resonant terms leads to an infinite product of coordinate transformations, which may diverge in general, due to the presence of small divisors. Positive results about the convergence of this process are due to Poincaré, Siegel, Bruno and others mathematicians, under Diophantine conditions on the eigenvalues of the linear part of the system, see e.g. [14, 64].

However, when the vector field is analytically integrable (i.e. it is an real or complex analytic vector field, and the additional first integrals and commuting vector fields in question are also analytic), then we don't need any Diophatine or nonresonance condition for the existence of a convergent Poincaré-Birkhoff normalization. More precisely, we have:

Theorem 3.3 ([85, 86])**.** *Let X be a local analytic (non-Hamiltonian, isochore, or Hamiltonian) vector field in $(\mathbb{K}^m, 0)$ (or in $(\mathbb{K}^{2n}, 0)$ with a standard symplectic structure), where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, such that $X(0) = 0$. Then X admits a convergent Poincaré-Birkhoff normalization in a neighborhood of* 0.

Partial cases of the above theorem were obtained earlier by many authors, including Rüssmann [65] (the nondegenerate Hamiltonian case with 2 degrees of freedom), Vey [71, 72] (the nondegenerate Hamiltonian and isochore cases), Ito [42] (the nonresonant Hamiltonian case), Ito [44] and Kappeler et al. [47] (the Hamiltonian case with a simple resonance), Bruno and Walcher [15] (the non-Hamiltonian case with $m = 2$). These authors, except Vey who was more geometric, relied on long and heavy analytical estimates to show the convergence of an infinite normalizing coordinate transformation process. On the other hand, the proof of Theorem 3.3 in [85, 86] is based on the toric point of view and is relatively short.

Following [85], we will give here a proof of the above theorem in the Liouville-integrable case. The other cases are similar, and of course the theorem is valid for Hamiltonian vector fields which are integrable in generalized Liouville sense as well. According to Proposition 3.2, it is enough to show the existence of a holomorphic normalization. We will do it by finding local Hamiltonian $\mathbb{T}^1$-actions which preserve the moment map of an analytically completely integrable system. The Hamiltonian function generating such an action is an *action function*. If we find $(n-q)$ such $\mathbb{T}^1$-actions, then they will automatically commute and give rise to a Hamiltonian $\mathbb{T}^{n-q}$-action.

To find an action function, we will use the Mineur-Arnold formula $P = \int_\Gamma \beta$, where P denotes an action function, β denotes a primitive 1-form (i.e. $\omega = d\beta$ is the symplectic form), and Γ denotes an 1-cycle (closed curve) lying on a level set of the moment map.

To show the existence of such 1-cycles Γ, we will use an approximation method, based on the existence of a formal Birkhoff normalization.

Denote by $\mathbf{G} = (G_1 = H, G_2, ..., G_n) : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ the holomorphic momentum map germ of a given complex analytic Liouville-integrable Hamiltonian system. Let $\epsilon_0 > 0$ be a small positive number such that $\mathbf{G}$ is defined in the ball $\{z = (x_j, y_j) \in \mathbb{C}^{2n}, |z| < \epsilon_0\}$. We will restrict our attention to what happens inside this ball. As in Subsection 3.1, we may assume that in the symplectic coordinate system $z = (x_j, y_j)$ we have

$$H = G_1 = H^s + H^n + H^{(3)} + H^{(4)} + ... \tag{3.18}$$

with

$$H_s = \sum_{k=1}^{n-q} \alpha_k F^k, \ F^k = \sum_{j=1}^{n} \rho_j^k x_j y_j, \tag{3.19}$$

with no resonance relations among $\alpha_1, ..., \alpha_{n-q}$. We will fix this coordinate system $z = (x_j, y_j)$, and all functions will be written in this coordinate system.

The real and imaginary parts of the Hamiltonian vector fields of $G_1, ..., G_n$ are in involution and their infinitesimal $\mathbb{C}^n$-action defines an *associated singular foliation* in the ball $\{z = (x_j, y_j) \in \mathbb{C}^{2n}, |z| < \epsilon_0\}$. Similarly to the real case, the leaves of this foliation are called local *orbits* of the system; they are complex isotropic submanifolds, and generic leaves are Lagrangian and have complex dimension n. For each z we will denote the leaf which contains z by M_z. Recall that the momentum map is constant on the orbits of the system. If z is a point such that $\mathbf{G}(z)$ is a regular value for the momentum map, then M_z is a connected component of $\mathbf{G}^{-1}(\mathbf{G}(z))$.

Denote by

$$S = \{z \in \mathbb{C}^{2n}, |z| < \epsilon_0, dG_1 \wedge dG_2 \wedge ... \wedge dG_n(z) = 0\} \tag{3.20}$$

the singular locus of the moment map, which is also the set of singular points of the associated singular foliation. What we need to know about S is that it is analytic and of codimension at least 1, though for generic integrable systems S is in fact of codimension 2. In particular, we have the following Łojasiewicz-type inequality (see [54]): there exist a positive number N and a positive constant C such that

$$|dG_1 \wedge ... \wedge dG_n(z)| > C(d(z, S))^N \tag{3.21}$$

for any z with $|z| < \epsilon_0$, where the norm applied to $dG_1 \wedge ... \wedge dG_n(z)$ is some norm in the space of n-vectors, and $d(z, S)$ is the distance from z to S with respect to the Euclidean metric. In the above inequality, if we change the coordinate system, then only ϵ_0 and C have to be changed, N (the Łojasiewicz exponent) remains the same.

We will choose an infinite decreasing series of small numbers ϵ_m $(m = 1, 2, ...)$, as small as needed, with $\lim_{m\to\infty} \epsilon_m = 0$, and define the following open subsets U_m of $\mathbb{C}^{2n}$:

$$U_m = \{z \in \mathbb{C}^{2n}, |z| < \epsilon_m, d(z, S) > |z|^m\} \tag{3.22}$$

We will also choose two infinite increasing series of natural numbers a_m and b_m $(m = 1, 2, ...)$, as large as needed, with $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = \infty$. It follows from Birkhoff's formal normalization that there is a series of local holomorphic symplectic coordinate transformations Φ_m, $m \in \mathbb{N}$, such that the following two conditions are satisfied :

a) The differential of Φ_m at 0 is identity for each m, and for any two numbers m, m' with $m' > m$ we have

$$\Phi_{m'}(z) = \Phi_m(z) + O(|z|^{a_m}). \tag{3.23}$$

In particular, there is a formal limit $\Phi_\infty = \lim_{m\to\infty} \Phi_m$.

b) The moment map is normalized up to order b_m by Φ_m. More precisely, the functions G_j can be written as

$$G_j(z) = G_{(m)j}(z) + O(|z|^{b_m}),\ j = 1, ...n, \tag{3.24}$$

with $G_{(m)j}$ such that

$$\{G_{(m)j}, F^k_{(m)}\} = 0 \quad \forall j = 1, ...n,\ k = 1, ..., n - q. \tag{3.25}$$

Here the functions $F^k_{(m)}$ are quadratic functions

$$F^k_{(m)}(x, y) = \sum_{j=1}^{n} \rho^k_j x_{(m)j} y_{(m)j} \tag{3.26}$$

in local symplectic coordinates

$$(x_{(m)}, y_{(m)}) = \Phi_m(x, y). \tag{3.27}$$

Notice that $F^k_{(m)}$ has the same form as F^k, but with respect to a different coordinate system. When considered in the original coordinate system (x, y), $F^k_{(m)}$ is a different function than F^k, but the quadratic part of $F^k_{(m)}$ is F^k.

Denote by $\Gamma^k_m(z)$ the orbit of the real part of the periodic Hamiltonian vector field $X_{iF^k_{(m)}}$ which goes through z. Then for any $z' \in \Gamma^k_m(z)$ we have $G_{(m)j}(z') = G_{(m)j}(z)$ and $|z'| \simeq |z|$, therefore

$$|\mathbf{G}(z') - \mathbf{G}(z)| = O(|z'|^{b_m}). \tag{3.28}$$

(Note that we can choose the numbers a_m and b_m first, then choose the radii ϵ_m of small open subsets to make them sufficiently small with respect to a_m and b_m, so that the equivalence $O(|z'|^{b_m}) \simeq O(|z|^{b_m})$ makes sense).

On the other hand we have

$$\begin{aligned} &|dG_1(z') \wedge ... \wedge dG_n(z')| \\ &= |dG_{(m)1}(z') \wedge ... \wedge dG_{(m)n}(z')| + O(|z|^{b_m-1}) \\ &\simeq |dG_{(m)1}(z) \wedge ... \wedge dG_{(m)n}(z)| + O(|z|^{b_m-1}) \\ &= |dG_1(z) \wedge ... \wedge dG_n(z)| + O(|z|^{b_m-1}) \end{aligned} \tag{3.29}$$

We can assume that $b_m - 1 > N$. Then for $|z| < \epsilon_m$ small enough, the above inequality may be combined with Łojasiewicz inequality (3.21) to yield

$$|dG_1(z') \wedge ... \wedge dG_n(z')| > C_1 d(z, S)^N \tag{3.30}$$

where $C_1 = C/2$ is a positive constant (which does not depend on m).

If $z \in U_m$, and assuming that ϵ_m is small enough, we have $d(z, S) > |z|^m$, which may be combined with the last inequality to yield :

$$|dG_1(z') \wedge ... \wedge dG_n(z')| > C_1 |z|^{mN} \tag{3.31}$$

Assuming that b_m is much larger than mN, we can use the implicit function theorem to project the curve $\Gamma^k_m(z)$ on M_z as follows :

For each point $z' \in \Gamma^k_m(z)$, let $D_m(z')$ be the complex n-dimensional disk centered at z', which is orthogonal to the kernel of the differential of the momentum map G at z', and which has radius equal to $|z'|^{2mN}$. Since the second derivatives of G are locally bounded by a constant near 0, it follows from the definition of $D_m(z')$ that we have we have, for $|z| < \epsilon_m$ small enough :

$$|D\mathbf{G}(w) - D\mathbf{G}(z')| < |z|^{3mN/2} \ \forall w \in D_m(z') \tag{3.32}$$

where $D\mathbf{G}(w)$ denotes the differential of the momentum map at w, considered as an element of the linear space of $2n \times n$ matrices.

Inequality (3.31) together with Inequality (3.32) imply that the momentum map $\mathbf{G}$, when restricted to $D_m(z')$, is a diffeomorphism from $D(z')$ to its image, and the image of $D_m(z')$ in $\mathbb{C}^n$ under $\mathbf{G}$ contains a ball of radius $|z|^{4mN}$. (Because $4mN > 2mN + mN$, where $2mN$ is the order of the radius of $D_m(z')$, and mN is a majorant of the order of the norm of the differential of $\mathbf{G}$. The differential of $\mathbf{G}$ is "nearly constant" on $D_m(z')$ due to Inequality 3.32). Thus, if $b_m > 5mN$ for example, then Inequality 3.28 implies that there is a unique point z'' on $D_m(z')$ such that $\mathbf{G}(z'') = \mathbf{G}(z)$. The map $z' \mapsto z''$ is continuous, and it maps $\Gamma^k_m(z)$ to some close curve $\widetilde{\Gamma}^k_m(z)$, which must lie on M_z because the point z maps to itself under the projection. When b_m is large enough and ϵ_m is small enough, then $\tilde{\Gamma}^k_m(z)$ is a smooth curve with a natural parametrization inherited from the natural parametrization of $\Gamma^k_m(z)$, it has bounded derivative (we can say that its velocity vectors are uniformly bounded by 1), and it depends smoothly on $z \in U_m$.

Define the following action function P^k_m on U_m :

$$P^k_m(z) = \oint_{\widetilde{\Gamma}^k_m(z)} \beta \ , \tag{3.33}$$

where $\beta = \sum x_j dy_j$ (so that $d\beta = \sum dx_j \wedge dy_j$ is the standard symplectic form). This function has the following properties:

i) Because the 1-form $\beta = \sum x_j dy_j$ is closed on each leaf of the Lagrangian foliation of the integrable system in U_m, P^k_m is a holomorphic first integral of the foliation. (This fact is well-known in complex geometry : period integrals of holomorphic k-forms, which are closed on the leaves of a given holomorphic foliation, over p-cycles of the leaves, give rise to (local) holomorphic first integrals of the foliation). The functions $P^1_m, ..., P^{(n-q)}_m$ Poisson commute pairwise, because they commute with the momentum map.

ii) P^k_m is uniformly bounded by 1 on U_m, because $\widetilde{\Gamma}^k_m(z)$ is small together with its first derivative.

iii) Provided that the numbers a_m are chosen large enough, for any $m' > m$ we have that P^k_m coincides with $P^k_{m'}$ in the intersection of U_m with $U_{m'}$. To see this important point, recall that we have

$$P^k_m = P^k_{m'} + O(|z|^{a_m}) \tag{3.34}$$

by construction, which implies that the curve $\Gamma^k_{m'}(z)$ is $|z|^{a_m - 2}$-close to the curve $\Gamma^k_m(z)$ in C^1-norm. If a_m is large enough with respect to mN (say $a_m > 5mN$), then it follows that the complex n-dimensional cylinder

$$V_{m'}(z) = \{w \in \mathbb{C}^{2n} \,|\, d(w, \Gamma^k_{m'}(z)) < |z|^{2m'N}\} \bigcap M_z \tag{3.35}$$

lies inside (and near the center of) the complex n-dimensional cylinder

$$V_m(z) = \{w \in \mathbb{C}^{2n} \,|\, d(w, \Gamma^k_m(z)) < |z|^{2mN}\} \bigcap M_z. \tag{3.36}$$

On the other hand, one can check that $\widetilde{\Gamma}^k_m(z)$ is a retract of $V_m(z)$ in M_z, and the same thing is true for the index m'. It follows easily that $\widetilde{\Gamma}^k_{m'}(z)$ must be homotopic to $\widetilde{\Gamma}^k_m(z)$ in M_z, implying that $P^k_m(z)$ coincides with $P^k_{m'}(z)$.

iv) Since P^k_m coincides with $P^k_{m'}$ in $U_m \bigcap U_{m'}$, we may glue these functions together to obtain a holomorphic function, denoted by P^k, on the union $U = \bigcup_{m=1}^\infty U_m$. Lemma 3.4 in the following subsection shows that if we have a bounded holomorphic function in $U = \bigcup_{m=1}^\infty U_m$ then it can be extended to a holomorphic function in a neighborhood of 0 in $\mathbb{C}^{2n}$. Thus our action functions P^k are holomorphic in a neighborhood of 0 in $\mathbb{C}^{2n}$.

v) P^k is a local periodic Hamiltonian function whose quadratic part is $\sqrt{-1}F^k = \sqrt{-1}\sum \rho^k_j x_j y_j$. To see this, remark that

$$\sqrt{-1}F^k_m(z) = \sqrt{-1}\sum \rho^k_j x_{(m)j} y_{(m)j} = \oint_{\Gamma^k_m(z)} \beta \ , \tag{3.37}$$

for $z \in U_m$. Since the curve $\widetilde{\Gamma}^k_m(z)$ is $|z|^{3mN}$-close to the curve $\Gamma^k_m(z)$ by construction (provided that $b_m > 4mN$), we have that

$$P^k(z) = \sqrt{-1}F^k_m(z) + O(|z|^{3mN}) \tag{3.38}$$

for $z \in U_m$. Due to the nature of U_m (almost every complex line in $\mathbb{C}^{2n}$ which contains the origin 0 intersects with U_m in an open subset (of the line) which surrounds the point 0), it follows from the last estimation that in fact the coefficients of all the monomial terms of order $< 3mN$ of P^k coincide with that of $\sqrt{-1}F^k_m$, i.e. we have

$$P^k(z) = \sqrt{-1}F^k_m(z) + O(|z|^{3mN}) \tag{3.39}$$

in a neighborhood of 0 in $\mathbb{C}^{2n}$. In particular, we have

$$P^k = \lim_{m\to\infty} \sqrt{-1}F^k_m \ , \tag{3.40}$$

where the limit on the right-and side of the above equation is understood as the formal limit of Taylor series, and the left-hand side is also considered as a Taylor series. This is enough to imply that P^k has $\sqrt{-1}\sum \rho^k_j x_j y_j$ as its quadratic part, and that P^k is a periodic Hamiltonian of period 2π because each $\sqrt{-1}F^k_m$ is so. (If a local holomorphic Hamiltonian vector field which vanishes at 0 is formally periodic then it is periodic). Thus we have found analytic action functions and the corresponding Hamiltonian torus action. The rest of the proof is straightforward. □

3.4. **A holomorphic extension lemma.**

The following lemma on holomorphic extension, which is interesting in its own right, implies that the action functions P^k constructed in the previous subsection can be extended holomorphically in a neighborhood of 0.

Lemma 3.4. *Let $U = \bigcup_{m=1}^\infty U_m$, with $U_m = \{x \in \mathbb{C}^n, |x| < \epsilon_m, d(x,S) > |x|^m\}$, where ϵ_m is an arbitrary series of positive numbers and S is a local proper complex analytic subset of $\mathbb{C}^n$ ($codim_{\mathbb{C}} S \geq 1$). Then any bounded holomorphic function on U has a holomorphic extension in a neighborhood of 0 in $\mathbb{C}^n$.*

Proof. Though we suspect that this lemma should have been known to specialists in complex analysis, we could not find it in the literature, so we will provide a proof here. When $n = 1$ the lemma is obvious, so we will assume that $n \geq 2$. We divide the lemma into two steps :

Step 1. The case when S is contained in the union of hyperplanes $\bigcup_{j=1}^{n}\{x_j = 0\}$ where $(x_1, ..., x_n)$ is a local holomorphic system of coordinates. Clearly, U contains a product of non-empty annuli $\eta_j < |x_j| < \eta'_j$, hence f is defined by a Laurent series in $x_1, \cdots, x_n$ there. We will study the domain of convergence of this Laurent series, using the well-known fact that the domain of convergence of a Laurent series is logarithmically convex. More precisely, denote by π the map $(x_1, \cdots, x_n) \mapsto (\log|x_1|, \cdots, \log|x_n|)$ from $(\mathbb{C}^*)^n$ to $\mathbb{R}^n$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, and set

$$E = \{\mathbf{r} = (r_1, ..., r_n) \in \mathbb{R}^n \;\mid\; \pi^{-1}(\mathbf{r}) \subset U\}$$

Denote by $Hull(E)$ the convex hull of E in $\mathbb{R}^n$. Then since the function f is analytic and bounded in $\pi^{-1}(E)$, it can be extended to abounded analytic function on $\pi^{-1}(Hull(E))$. On the other hand, by definition of $U = \bigcup_{m=1}^{\infty} U_m$, there is a series of positive numbers K_m (tending to infinity) such that $E \supset (\bigcup_{m=1}^{\infty} E_m)$, where

$$E_m = \{(r_1, ..., r_n) \in \mathbb{R}^n \;\mid\; (r_j < -K_m \;\forall j)\;,\; (r_j > mr_i \;\forall j \neq i)\}$$

It is clear that the convex hull of $\bigcup_{m=1}^{\infty} E_m$, with each E_m defined as above, contains a neighborhood of $(-\infty, ..., -\infty)$, i.e. a set of the type

$$\{(r_1, ..., r_n) \in \mathbb{R}^n \;\mid\; r_j < -K \;\forall j\}.$$

It implies that the function f can be extended to a bounded analytic function in $\mathcal{U} \bigcap (\mathbb{C}^*)^n$, where $\mathcal{U}$ is a neighborhood of 0 in $\mathbb{C}^n$. Since f is bounded in $\mathcal{U} \bigcap (\mathbb{C}^*)^n$, it can be extended analytically on the whole $\mathcal{U}$. Step 1 is finished.

Step 2. Consider now the case with an arbitrary S. Then we can use Hironaka's desingularization theorem [41] to make it smooth. In fact, since the exceptional divisor will also have to be taken into account, after the desingularization process we will have a variety which may have normal crossings. More precisely, we have the following commutative diagram

$$\begin{array}{ccccc} Q & \subset & S' & \subset & M^n \\ \downarrow & & \downarrow & & \downarrow p \\ 0 & \subset & S & \subset & (\mathbb{C}^n, 0) \end{array} \quad , \tag{3.41}$$

where $(\mathbb{C}^n, 0)$ denotes the germ of $\mathbb{C}^n$ at 0 presented by a ball which is small enough; M^n is a complex manifold; the projection p is surjective, and injective outside the exceptional divisor; S' denotes the union of the exceptional divisor with the smooth proper submanifold of M^n which is desingularization of S – the only singularities in S' are normal crossings; $Q = p^{-1}(0)$ is compact. M^n is obtained from $(\mathbb{C}^n, 0)$ by a finite number of blowing-ups along submanifolds.

Denote by $U' = p^{-1}(U)$ the preimage of U under the projection p. One can pull back f from U to U' to get a bounded holomorphic function on U', denoted by f'. An important observation is that the type of U persists under blowing-ups along submanifolds.(Or equivalently, the type of its complement, which may be called a "sharp-horn-neighborhood" of S because it is similar to "horn-type neighborhoods" used by singularists but it is sharp of arbitrary order, is persistent under blowing-ups). More precisely, for each point $x \in Q$, the complement of U' in a small neighborhood of x is a "sharp-horn-neighborhood" of S' at x. Since S' only has normal crossings, the pair (U', S') satisfies the conditions of Step 1, and therefore we can extend f' holomorphically in a neighborhood of x in M^n. Since $Q = p^{-1}(0)$ is compact, we can extend f' holomorphically in a neighborhood of Q in M'. One can now project this extension of f' back to $(\mathbb{C}^n, 0)$ to get a holomorphic extension of f in a neighborhood of 0. The lemma is proved. □

Remark 3.5. The "sharp-horn" type of the complement of U in the above lemma is essential. If we replace U by U_m (for any number m) then the lemma is false.

3.5. **Torus action near a compact singular orbit.**

Consider a real analytic integrable vector field X on a real analytic manifold M^m of dimension $m = p + q$, with the aid of a p-tuple $\mathbf{X} = (X_1, ..., X_p)$ of commuting analytic vector fields and a q-tuple $\mathbf{F} = (F_1, ..., F_q)$ of analytic common first integrals: $[X, X_i] = [X_i, X_j] = 0, X(F_j) = X_i(F_j) = 0\ \forall i, j$. In the Hamiltonian case, when there is an analytic Poisson structure on M^m, we suppose that the system is integrable in generalized Liouville sense, i.e. the vector fields $X, X_1, ..., X_p$ are Hamiltonian.

The commuting vector fields $X_1, ..., X_p$ generate an infinitesimal $\mathbb{R}^p$-action on M – as usual, its orbits will be called orbits of the system. The map $\mathbf{F} : M^m \to \mathbb{R}^q$ is constant on the orbits of the system. Let $O \subset M^m$ be a singular orbit of dimension r of the system, $0 \le r < p$. We suppose that O is a compact submanifold of M^m (or more precisely, of the interior of M^m if M^m has boundary). Then O is a torus of dimension r. Denote by N the connected component of $\mathbf{F}^{-1}(\mathbf{F}(O))$ which contains O. A natural question arises: does there exist a $\mathbb{T}^r$-action in a neighborhood of O or N, which preserves the system and is transitive on O ?

The above question has been answered positively in [89], under a weak condition called the *finite type condition.* To formulate this condition, denote by $M_{\mathbb{C}}$ a small open complexification of M^m on which the complexification $\mathbf{X}_{\mathbb{C}}, \mathbf{F}_{\mathbb{C}}$ of $\mathbf{X}$ and $\mathbf{F}$ exists. Denote by $N_{\mathbb{C}}$ a connected component of $\mathbf{F}_{\mathbb{C}}^{-1}(\mathbf{F}(O))$ which contains N.

Definition 3.6. With the above notations, the singular orbit O is called of *finite type* if there is only a finite number of orbits of the infinitesimal action of $\mathbb{C}^p$ in $N_{\mathbb{C}}$, and $N_{\mathbb{C}}$ contains a regular point of the map $\mathbf{F}$.

For example, all nondegenerate singular orbits are of finite type (see Section 4). It is conjectured that every singular orbit of an algebraically integrable system is of finite type.

Theorem 3.7 ([89])**.** *With the above notations, if O is a compact finite type singular orbit of dimension r, then there is a real analytic torus action of $\mathbb{T}^r$ in a neighborhood of O which preserves the integrable system* $(\mathbf{F}, \mathbf{X})$ *and which is transitive on O. If moreover N is compact, then this torus action exists in a neighborhood of N. In the Hamiltonian case this torus action also preserves the Poisson structure.*

Notice that Theorem 3.7, together with Theorem 3.3 and the toric characterization of Poincaré-Birkhoff normalization, provides an analytic Poincaré-Birkhoff normal form in the neighborhood a singular invariant torus of an integrable system.

Denote by $\mathcal{A}_O$ the local automorphism group of the integrable system $(\mathbf{F}, \mathbf{X})$ at O, i.e. the group of germs of local analytic automorphisms of $(\mathbf{F}, \mathbf{X})$ in vicinity of O (which preserve the Poisson structure in the Hamiltonian case). Denote by $\mathcal{A}_O^0$ the subgroup of $\mathcal{A}_O$ consisting of elements of the type g_Z^1, where Z is a analytic vector field in a neighborhood of O which preserves the system and g_Z^1 is the time-1 flow of Z. The torus in the previous theorem is of course a Abelian subgroup of

$\mathcal{A}_O^0$. Actually, the automorphism group $\mathcal{A}_O$ itself is essentially Abelian in the finite type case:

Theorem 3.8 ([89]). *If O is a compact finite type singular orbit as above, then $\mathcal{A}_O^0$ is an Abelian normal subgroup of $\mathcal{A}_O$, and $\mathcal{A}_O/\mathcal{A}_O^0$ is a finite group.*

The above two theorems are very closely related: their proofs are almost the same. Let us indicate here the main ingredients of the proof of Theorem 3.7:

For simplicity, we will assume that $r = 1$, i.e. O is a circle (the case $r > 1$ is absolutely similar). Since O is of finite type, there is a regular complex orbit Q in $N_{\mathbb{C}}$ of dimension p whose closure contains O. Q is a flat affine manifold (the affine structure is given by the $\mathbb{C}^p$-action, so we can talk about geodesics on Q. If we can find a closed geodesic γ_Q on Q, then it is a periodic orbit of period 1 of a vector field of the type $\sum a_j X_j$ on Q (with a_j being constants) on Q. Since the points of Q are regular for the map $\mathbf{F}$, using implicit function theorem, we can construct a vector field of the type $\sum a_j X_j$, with a_j now being holomorphic functions which are functionally dependent on $\mathbf{F}$ (so that this vector fields preserves the system), and which is periodic of period 1 near γ_Q. With some luck, we will be able to extend this vector field holomorphically to a vector field in a neighborhood of O so that O becomes a periodic orbit of it, and we are almost done: if the vector field is not real-analytic, then its image under a complex involution will be another periodic vector field which preserves the system; the two vector fields commute (because the system is integrable) and we can fabricate from them a real-analytic periodic vector field, i.e. a real-analytic $\mathbb{T}^1$-action in a neighborhood of O, for which O is a periodic orbit.

The difficulties lies in finding the closed geodesic γ_Q (which satisfies some additional conditions). We will do it inductively: let $O_1 = O_{\mathbb{C}}$ $(O \subset O_C), O_2, \ldots, O_k = Q$ be a maximal chain of complex orbits of the system in $N_{\mathbb{C}}$ such that O_i lies in the closure of O_{i+1} and $O_i \neq O_{i+1}$. Then on each O_i, we will find a closed geodesic γ_i, such that each γ_{i+1} is homotopic to a multiple of γ_i in $O_i \cup O_{i+1}$, starting with $\gamma_1 = O$. We will show how to go from $O = \gamma_1$ to γ_2 (the other steps are similar). Without loss of generality, we may assume that O is a closed orbit for X_1. Take a small section D to O in M, and consider the Poincare map ϕ of X_1 on D. Let $Y = O_2 \cap D_{\mathbb{C}}$. Then Y is a affine manifold (whose affine structure is projected from O_2 by X_1). Let y be a point in Y. We want to connect y to $\phi(y)$ by a geodesic in Y. If we can do it, then the sum of this geodesic segment with the orbit of X_1 going from y to $\phi(Y)$ can be modified into a closed geodesic γ_2 on O_2. Unfortunately, in general, we cannot connect y to $\phi(y)$ by a geodesic in Y, because Y is not "convex". But an interesting lemma says that Y can be cut into a finite number of convex pieces, and as a consequences y can be connected geodesically to $\phi^N(y)$ for some power ϕ^N (N-time iteration) of ϕ. See [89] for the details. □

Theorem 3.7 reduces the study of compact singular orbits to the study of fixed points with a finite Abelian group of symmetry (this group arises from the fact that the torus action is not free in general, only locally free). In the case of *corank-1* singularities of Liouville-integrable systems, the local reduced system is a family of functions on a 2-dimensional symplectic disk which are invariant under the rotation action of a finite cyclic group $\mathbb{Z}/\mathbb{Z}_k$, see [83].

3.6. **Explicit Birkhoff coordinates.**

It is an important problem to find explicit *Birkhoff coordinates* for a integrable Hamiltonian system near a singular point or singular orbit (i.e. a system of coordinates in which the Hamiltonian has Birkhoff normal form), or explicit action-angle variables near regular tori. (Note that, in the infinite-dimensional case, usually every finite-dimensional invariant torus is singular). These coordinates are used, for example, in K.A.M. theory (for perturbations of finite and infinite-dimensional integrable systems), see e.g. [48].

Among the first results in this direction, one may mention the work of Flaschka–McLaughlin on action-angle variables for finite-zone solutions of the periodic KdV equation, [31]. Based on this and other works, Kappeler et al. constructed Birkhoff coordinates for various integrable systems in both finite and infinite dimension, including the Toda lattice, the periodic KdV equation, and the integrable defocusing nonlinear Schrödinger equation, see e.g. [8, 48] and references therein.

An analog of Theorem 3.3 is not yet available for infinite-dimensional integrable systems, so we don't know if all analytic infinite-dimensional integrable systems admit local analytic Birkhoff normal form. But even in the finite dimensional case, when we know that analytic Birkhoff coordinates exist, to find them explicitly is a non-trivial task: in general, the action functions for the local Hamiltonian torus action can be written down explicitly by Mineur-Arnold formula, but Guillemin–Sternberg–Marle linearization theorem does not give an explicit linearization of this action. (The proof of this linearization theorem relies on Moser's path method, which does not give a practical formula for computing the coordinate transformation).

In a joint work in progress [49], Kappeler and I are trying to characterize infinite-dimensional Birkhoff normal forms by infinite-dimensional torus actions, and generalize Theorem 3.3 to the infinite-dimensional case. We also made the following observation, useful for the construction of explicit Birkhoff coordinates: if we have an *anti-symplectic involution*, then the se of fixed points of this involution can be used to define Birkhoff coordinates, by the following theorem.

Theorem 3.9 (Kappeler–Zung [49]). *If z is a nondegenerate elliptic fixed point of a real analytic Liouville-integrable Hamiltonian system on a symplectic $2n$-dimensional manifold, and ρ is an anti-symplectic involution which preserves the momentum map and fixes z, then there is a local analytic system of Birkhoff coordinates (x_i, y_i) in a neighborhood of 0, such that the Lagrangian subspace $\{x_1 = \cdots = x_n = 0\}$ consists of fixed points of the involution ρ. This local Birkhoff coordinate system is uniquely defined up to an action of the Weyl group of $sp(2n)$.*

The above theorem works well in the case of periodic Toda lattice, and can probably be applied to periodic KdV and other systems of physical interest too, see [49]. Normal forms for Hamiltonian torus actions together with an anti-symplectic involution were first studied by Duistermaat [25], and the above theorem is in fact a simple consequence of Duistermaat's results (plus the existence of a torus action).

4. Nondegenerate singularities

In this section, we will consider only smooth Liouville-integrable Hamiltonian systems, though many ideas and results can probably be extended to other kinds of integrable systems.

4.1. **Nondegenerate singular points.**

Consider the momentum map $\mathbf{F} = (F_1, ..., F_n) : (M^{2n}, \omega) \to \mathbb{R}^n$ of a smooth integrable Hamiltonian system on a symplectic manifold (M^{2n}, ω). In this Section, we will forget about the original Hamiltonian function, and study the momentum map instead.

For a point $z \in M$, denote rank z = rank $d\mathbf{F}(z)$, where $d\mathbf{F}$ denotes the differential of $\mathbf{F}$. This number is equal to the dimension of the orbit of the system (i.e. the infinitesimal Poisson $\mathbb{R}^n$-action generated by $X_{F_1}, ..., X_{F_n}$) which goes through z. If rank $z < n$ then z is called a *singular point*. If rank $z = 0$ then z is a *fixed point* of the system.

If z is a fixed point, then the quadratic parts $F_1^{(2)}, ..., F_n^{(2)}$ of the components $F_1, ..., F_n$ of the momentum map at z are Poisson-commuting and they form an Abelian subalgebra, A_z, of the Lie algebra $Q(2n, \mathbb{R})$ of homogeneous quadratic functions of $2n$ variables under the standard Poisson bracket. Observe that the algebra $Q(2n, \mathbb{R})$ is isomorphic to the symplectic algebra $sp(2n, \mathbb{R})$.

A fixed point z will be called *nondegenerate* if A_z is a Cartan subalgebra of $Q(2n, \mathbb{R})$. In this case, according to Williamson [79], there is a triple of nonnegative integers (k_e, k_h, k_f) such that $k_e + k_h + 2k_f = n$, and a canonical coordinate system (x_i, y_i) in $\mathbb{R}^{2n}$, such that A_z is spanned by the following quadratic functions $h_1, ..., h_n$:

$$\begin{aligned}
&h_i = x_i^2 + y_i^2 \ \text{ for } \ 1 \le i \le k_e \ ; \\
&h_i = x_i y_i \ \text{ for } \ k_e + 1 \le i \le k_e + k_h \ ; \\
&h_i = x_i y_{i+1} - x_{i+1} y_i \ \text{ and} \\
&h_{i+1} = x_i y_i + x_{i+1} y_{i+1} \ \text{ for } \ i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f \ .
\end{aligned} \tag{4.1}$$

The triple (k_e, k_h, k_f) is called the *Williamson type* of (the system at) z. k_e is the number of *elliptic components* (and $h_1, ..., h_{k_e}$ are elliptic components), k_h is the number of *hyperbolic components*, and k_f is the number of *focus-focus components*. If $k_h = k_f = 0$ then z is called an *elliptic singular point*.

The local structure of nondegenerate singular point is given by the following theorem.

Theorem 4.1 (Eliasson [28, 29])**.** *If z is a nondegenerate fixed point of a smooth Liouville-integrable Hamiltonian system then there is a smooth Birkhoff normalization. In other words, the singular Lagrangian foliation given by the momentum map $\mathbf{F}$ in a neighborhood of z is locally smoothly symplectomorphic to the "linear" singular Lagrangian fibration given by the quadratic map $(h_1, ..., h_n) : \mathbb{R}^{2n} \to \mathbb{R}^n$ with the standard symplectic structure on $\mathbb{R}^{2n}$.*

The elliptic case of the above theorem is also obtained independently by Dufour and Molino [23]. The case is one degree of freedom is due to Colin de Verdière and Vey [16]. The analytic case of the above theorem is due to Vey [71], and is superceded by Theorem 3.3. Vu Ngoc San [74] obtained the semiclassical version of the above theorem (quantum Birkhoff normal form).

A direct consequence of Eliasson's theorem is that, near a nondegenerate fixed point of Williamson type (k_e, k_h, k_f), there is a local smooth Hamiltonian $\mathbb{T}^{k_e+k_f}$-action which preserves the system: each elliptic or focus-focus component provides one $\mathbb{T}^1$-action. In the analytic case, Birkhoff normalization gives us a $\mathbb{T}^n$-action, but it acts in the complex space, and in the real space we only see a $\mathbb{T}^{k_e+k_f}$-action.

The proof of Eliasson's theorem [28, 29] is quite long and highly technical: The first step is to use division lemmas in singularity theory to show that the local singular fibration given by the momentum map is diffeomorphic (without the symplectic structure) to the linear model. Then one uses a combination of averaging, Moser's path method, and technics similar to the ones used in the proof of Sternberg's smooth linearization theorem for vector fields, to show that the symplectic form can also be normalized smoothly.

4.2. **Nondegenerate singular orbits.**

Let $x \in M$ be a singular point of rank $x = m \geq 0$. We may assume without loss of generality that $dF_1 \wedge ... \wedge dF_m(x) \neq 0$, and a local symplectic reduction near x with respect to the local free $\mathbb{R}^m$-action generated by the Hamiltonian vector fields $X_{F_1}, ..., X_{F_m}$ will give us an m-dimensional family of local integrable Hamiltonian systems with $n - m$ degrees of freedom. Under this reduction, x will be mapped to a fixed point in the reduced system, and if this fixed point is nondegenerate according to the above definition, then x is called a *nondegenerate singular point of rank m and corank $(n - m)$*. In this case, we can speak about the Williamson type (k_e, k_h, k_f) of x, and we have $k_e + k_h + 2k_f = m$.

A *nondegenerate singular orbit* of the system is an orbit (of the infinitesimal Poisson $\mathbb{R}^n$-action) which goes through a nondegenerate singular point. Since all points on a singular orbit have the same Williamson type, we can speak about the Williamson type and the corank of a nondegenerate singular orbit. We have the following generalization of Theorem 4.1 to the case of compact nondegenerate singular orbits:

Theorem 4.2 (Miranda–Zung [58])**.** *If O is a compact nondenenerate singular orbit of a smooth Liouville-integrable Hamiltonian system, then the singular Lagrangian fibration given by the momentum map in a neighborhood of O is smoothly symplectomorphic to a linear model. Moreover, if the system is invariant under a symplectic action of a compact Lie group G in a neighborhood of O, then the above smooth symplectomorphism to the linear model can be chosen to be G-equivariant.*

The linear model in the above theorem can be constructed as follows: Denote by $(p_1, ..., p_m)$ a linear coordinate system of a small ball D^m of dimension m, $(q_1(mod\ 1), ..., q_m(mod\ 1))$ a standard periodic coordinate system of the torus $\mathbb{T}^m$, and $(x_1, y_1, ..., x_{n-m}, y_{n-m})$ a linear coordinate system of a small ball $D^{2(n-m)}$ of dimension $2(n - m)$. Consider the manifold

$$V = D^m \times \mathbb{T}^m \times D^{2(n-m)} \tag{4.2}$$

with the standard symplectic form $\sum dp_i \wedge dq_i + \sum dx_j \wedge dy_j$, and the following momentum map: $(\mathbf{p}, \mathbf{h}) = (p_1, ..., p_m, h_1, ..., h_{n-m}) : V \to \mathbb{R}^n$, where $(h_1, ..., h_{n-m})$ are quadratic functions given by Equation (4.1). A symplectic group action on V which preserves the above momentum map is called *linear* if it on the product $V = D^m \times \mathbb{T}^m \times D^{2(n-m)}$ componentwise, the action on D^m is trivial, the action on $\mathbb{T}^m$ is by translations with respect to the coordinate system $(q_1, ..., q_m)$, and the action on $D^{2(n-m)}$ is linear.

Let Γ be a finite group with a free linear symplectic action $\rho(\Gamma)$ on V which preserves the momentum map. Then we can form the quotient integrable system with the momentum map

$$(\mathbf{p}, \mathbf{h}) = (p_1, ..., p_m, h_1, ..., h_{n-m}) : V/\Gamma \to \mathbb{R}^n\ . \tag{4.3}$$

The set $\{p_i = x_i = y_i = 0\} \subset V/\Gamma$ is a compact orbit of Williamson type (k_e, k_f, k_h) of the above system. The above system on V/Γ is called the *linear model* of Williamson type (k_e, k_f, k_h) and twisting group Γ, or more precisely, twisting action $\rho(\Gamma)$. (It is called a direct model if Γ is trivial, and a twisted model if Γ is nontrivial). A symplectic action of a compact group G on V/Γ which preserves the momentum map $(p_1, ..., p_m, h_1, ..., h_{n-m})$ is called linear if it comes from a linear symplectic action of G on V which commutes with the action of Γ.

The case with G trivial and $n = 2, k_h = 1, k_e = k_f = 0$ of Theorem 4.2 is due to Colin de Verdière and Vu Ngoc San [17], and independently Currás-Bosch and Miranda [19]. A direct consequence of Theorem 4.2 is that the group of local smooth symplectic automorphisms of a smooth Liouville-integrable system near a compact nondegenerate singular orbit is Abelian, see [58].

4.3. **Nondegenerate singular fibers.**

In this subsection, we will assume that the momentum map $\mathbf{F} : M^{2n} \to \mathbb{R}^n$ is proper. A singular connected component of a level set of the momentum map will be called a *singular fiber* of the system. A singular fiber may contain one orbit (e.g. in the elliptic nondegenerate case), or many orbits, some of them singular and some of them regular. A singular fiber N_c is called *nondegenerate* if $\forall z \in N_c$ is either regular or nondegenerate singular. Of course, if a singular fiber of the system is nondegenerate then nearby singular fibers are also nondegenerate.

By a *singularity* of a Liouville-integrable system, we mean the germ of the system near a singular fiber, together with the symplectic form and the Lagrangian fibration. We will denote a singularity by $(\mathcal{U}(N_c), \omega, \mathcal{L})$, where $\mathcal{U}(N_c)$ denotes a small "tubular" neighborhood of N_c, and $\mathcal{L}$ denotes the Lagrangian fibration. If N_c is nondegenerate then $(\mathcal{U}(N_c), \omega, \mathcal{L})$ is also called nondegenerate.

A simple lemma [82] says that if N_c is a nondegenerate singular fiber, then all singular points of maximal corank in N_c have the same Williamson type. We define the rank and the Williamson type of a nondegenerate singularity $(\mathcal{U}(N_c), \omega, \mathcal{L})$ to be the rank and the Williamson type of a singular point of maximal corank in N_c.

The following theorem may be viewed as the generalization of Liouville–Mineur–Arnold theorem to the case of nondegenerate singular fibers:

Theorem 4.3 ([82]). *Let $(\mathcal{U}(N_c), \omega, \mathcal{L})$ be a nondegenerate smooth singularity of rank m and Williamson type (k_e, k_h, k_f). Then*
a) There is effective Hamiltonian $\mathbb{T}^{m+k_e+k_f}$-action in $(\mathcal{U}(N_c), \omega, \mathcal{L})$ which preserves the system. The dimension $m + k_e + k_f$ is maximal possible. There is a locally free $\mathbb{T}^m$-subaction of this action.
b) There is a partial action-angle coordinate system.
c) Under a mild additional condition, $(\mathcal{U}(N_c), \mathcal{L})$ is topologically equivalent to an almost direct product of simplest (corank 1 elliptic or hyperbolic and corank 2 focus-focus) singularities.

Assertion b) of the above theorem means that we can write $(\mathcal{U}(N_c), \omega)$ as $(D^m \times \mathbb{T}^m \times P^{2k})/\Gamma$ with

$$\omega = \sum_1^m dp_i \wedge dq_i + \omega_1 \tag{4.4}$$

where ω_1 is a symplectic form on P^{2k}, the finite group Γ acts on the product component-wise, its action is linear on $\mathbb{T}^m$, and the momentum map $\mathbf{F}$ does not depend on the variables $q_1, ..., q_m$.

The additional condition in Assertion c) prohibits the bifurcation diagram (i.e. the set of singular values of the momentum map) from having "pathologies", see [82], and it's satisfied for all nondegenerate singularities of physical integrable systems met in practice. The almost direct product means a product of the type

$$(\mathcal{T}^{2m} \times \mathcal{E}_1^2 \times ... \times \mathcal{E}_{k_e}^2 \times \mathcal{H}_1^2 \times ... \times \mathcal{H}_{k_h}^2 \times \mathcal{F}_1^4 \times ... \times \mathcal{F}_{k_f}^4)/\Gamma \tag{4.5}$$

where $\mathcal{T}^{2m}$ is the germ of $(D^m \times \mathbb{T}^m, \sum_1^m dp_i \wedge dq_i)$ with the standard Lagrangian torus fibration; $\mathcal{E}_i^2, \mathcal{F}_i^2$ and $\mathcal{H}_i^4$ are elliptic, hyperbolic and focus-focus singularities of integrable systems on symplectic manifolds of dimension 2, 2 and 4 respectively; the finite group Γ acts freely and component-wise. Remark that, in general, a nondegenerate singularity is only topologically equivalent, but not symplectically equivalent, to an almost direct product singularity.

The above almost direct product may remind one of the decomposition of algebraic reductive groups into almost direct products of simple groups and tori: though the two objects are very different, there are some common ideas behind them, namely infinitesimal direct decomposition, and twisting by a finite group.

4.4. **Focus-focus singularities.**

The singularities $\mathcal{E}_i^2, \mathcal{H}_i^2, \mathcal{F}_i^4$ in (4.5) may be called *elementary nondegenerate singularities*; they are characterized by the fact that $k_e + k_h + k_f = 1$ and rank $= 0$. Among them, elliptic singularities $\mathcal{E}_i^2$ are very simple: each elementary elliptic singularity is isomorphic to a standard linear model (a harmonic oscillator). Elementary hyperbolic singularities $\mathcal{H}_i^2$ are also relatively simple because they are given by hyperbolic singular level sets of Morse functions on 2-dimensional symplectic surfaces. On the other hand, *focus-focus singularities* $\mathcal{F}_i^4$ live in 4-dimensional symplectic manifolds, so their topological structure is somewhat more interesting. Let us mention here some results about the structure of these 4-dimensional focus-focus singularities, see [82, 87] and references therein for more details.

- One of the most important facts about focus-focus singularities is the existence of a $\mathbb{T}^1$-action (this is a special case of Assertion a) of Theorem 4.3); many other important properties are consequences of this $\mathbb{T}^1$-action. In fact, in many integrable systems with a focus-focus singularity, e.g. the spherical pendulum and the Lagrangian top, this $\mathbb{T}^1$-action is the obvious rotational symmetry, though in some systems, e.g. the Manakov integrable system on $so(4)$, this local $\mathbb{T}^1$-action is "hidden". Dynamically speaking, a focus-focus point is roughly an unstable equilibrium point with a $\mathbb{T}^1$- symmetry.

- Each focus-focus singularity has only one singular fiber: the focus- focus fiber, which is homeomorphic to a *pinched torus* (take a torus, and ℓ parallel homotopically non-trivial simple closed curves on it, $\ell \geq 1$, then collapse each of these curves into one point). This fact was known to Lerman and Umanskij [52]

- From the topological point of view, we have a singular torus fibration in a four-dimensional manifold with one singular fiber. These torus fibrations have been studied by Matsumoto and other people, see e.g. [55] and references therein, and of course the case with a singular fiber of focus-focus type is included in their topological classification. In particular, the number of pinches ℓ is the only topological invariant. The monodromy of the torus fibration (over a punched 2-dimensional disk) around the focus-focus fiber is given by the matrix $\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$. By the way, the case with $\ell > 1$ is topologically an ℓ-sheet covering of the case with $\ell = 1$, and a concrete example with $\ell = 1$ is the unstable equilibrium of the usual spherical pendulum.

- The above phenomenon of *nontrivial monodromy* (of the foliation by Liouville tori) was first observed by Duistermaat and Cushman [24], and then by some other people for various concrete integrable systems. The general formula for monodromy around focus-focus singularities was observed in my thesis (see [82]). Now we have many different ways to look at this monodromy: from the purely topological point of view (using Matsumoto's theory [55]), from the point of view of Picard-Lefschetz theory (see Audin [6] and references therein), or as a consequence of Duistermaat-Heckman formula with respect to the above-mentioned $\mathbb{T}^1$-action (see [87]).

- Quantization of focus-focus singularities leads to *quantum monodromy*, see Vu Ngoc San [73] and Subsection 6.1.

Similar results, including the existence of a $\mathbb{T}^1$-action, for focus-focus singularities of *non-Hamiltonian* systems, have been obtained by Cushman an Duistermaat [20], see also [87].

5. Action functions and convexity

5.1. **Convexity properties of momentum maps.**

Probably the most famous theorem about convexity properties of momentum maps in symplectic geometry is the following

Theorem 5.1 (Atiyah–Guillemin–Sternberg–Kirwan [3, 39, 50])**.** *Let $\mu : M \to \mathfrak{g}^*$ be an equivariant momentum map of a Hamiltonian action of a compact Lie group G on a connected compact sympletic manifold (M, ω). Let $\mathfrak{t}^*_+$ denote a positive Weyl chamber, $\mathfrak{t}^*_+ \subset \mathfrak{t}^* \subset \mathfrak{g}^*$ where $\mathfrak{t}^*$ is the dual of a Cartan subalgebra. Then $\mu(M) \cap \mathfrak{t}^*_+$ is a convex polytope.*

The Abelian case of the above theorem ($G = \mathbb{T}^m$ is a torus, and $\mathfrak{t}^*_+ = \mathfrak{t}^* = \mathfrak{g}^*$) is due to Atiyah [3] and Guillemin–Sternberg [39]. The non-Abelian case (G is not a torus) is proved by Kirwan [50], using Morse theory and the results of [39].

There are many other convexity theorems. Let us mention just two of them:

- Nonlinear convexity theorem of Flaschka–Ratiu [32]. G is now a compact Poisson-Lie group, the action of G on the connected compact symplectic manifold (M, ω) is a Poisson action (i.e. the action map $G \times M \to M$ is a Poisson map), the momentum map μ now goes from M to the dual Poisson-Lie group G^* of G (momentum map in the sense of Lu). Flaschka–Ratiu theorem says that in this case, the momentum map μ also has a convexity property similar to the one given by Theorem 5.1, see [32] for a precise formulation.

- Noncompact convexity theorem of Weinstein [77]. G is now a noncompact real semisimple Lie group, the Hamiltonian action of G on M is proper, and the image of the momentum map $\mu : M \to \mathfrak{g}^*$ lies in the "stable region" of $\mathfrak{g}^*$ (the region in which the coadjoint action of G is proper). Then the momentum map μ also enjoys convexity properties in this case. A special case of Weinstein's noncompact convexity theorem (with $G = Sp(2n, \mathbb{R})$) is the following theorem, directly related to Hamiltonian dynamics:

Theorem 5.2 (Weinstein [77]). *For any positive-definite quadratic Hamiltonian function H on $\mathbb{R}^{2n}$, let $F(H)$ be the n-tuple $(\lambda_1, \dots \lambda_n)$, where $\lambda_1 \leq \dots \leq \lambda_n$ are the frequencies of the normal modes of oscillation for the linear hamiltonian system generated by H; i.e. $F(H)$ are the coefficients of the normal form $\sum_{j=1}^{n} \frac{\lambda_j}{2}(q_j^2 + p_j^2)$ for H in suitably chosen canonical coordinates. If λ and μ are nondecreasing n-tuples of positive real numbers, then*

$$\{F(H_1 + H_2) | F(H_1) = \lambda \text{ and } F(H_2) = \mu\} \tag{5.1}$$

is a closed, convex, locally polyhedral set.

In [76, 77, 78], and in private communication, Weinstein presented an unified approach for looking at the convexity theorems, from an intrinsic point of view, using the language of Poisson geometry. Let us recall some of his ideas here:

- In Kirwan's non-Abelian convexity theorem (and other non-Abelian convexity theorems), there is something not very intuitive: the image $\mu(M)$ is *not* convex in general, only its intersection with $\mathfrak{t}^*_+$ convex. Weinstein's idea is that $\mu(M)$ has some intrinsic convexity property, and the convexity of $\mu(M) \cap \mathfrak{t}^*_+$ is just an appearance of that.

- Another idea of Weinstein is that *proper symplectic groupoid actions* generalize at the same time Hamiltonian compact group actions, Hamiltonian proper noncompact group actions, and Poisson compact group actions, so one can use the language of proper symplectic groupoid actions to unify the above-mentioned theorems and other theorems. This idea led Weinstein to his *Poisson convexity conjecture*, see [76] and Subsection 5.4, and his study of proper groupoids [78].

In [90, 91], we studied proper groupoids and intrinsic convexity of momentum maps, developing Weinstein's ideas. To speak about intrinsic convexity, we need intrinsic transverse affine structures. It turns out that these affine structures are given by action functions of integrable systems (in generalized Liouville sense) that arise naturally. Their local intrinsic convexity is provided by local normal form theorems, due to the properness (of the groupoids in question). Finally, one can go from local convexity to global convexity by the local-global principle (used by Condevaux–Dazord–Molino [18] in their different proof of Theorem 5.1), which says that, under some hypothese, something locally convex is also globally convex. See [91] for the details. In the next three Subsections, we will briefly discuss transverse affine structures, proper groupoids, and intrinsic convexity.

5.2. Action functions and transverse affine structures.

Let $\mathcal{V}$ be a (singular) foliation in a symplectic manfiold (M, ω), which is *symplectically complete*, in the sense that there is another (singular) foliation $\mathcal{V}^\perp$ in M, such that the tangent space $V_x^\perp$ of $\mathcal{V}^\perp$ at a generic point $x \in M$ is the symplectically orthogonal to the tangent space V_x of $\mathcal{V}$. Such a $\mathcal{V}$ is also called a (singular) *Libermann foliation*, and the pair $(\mathcal{V}, \mathcal{V}^\perp)$ is called a *dual pair*. For example, the orbits of a Hamiltonian group action on M form a symplectically complete singular foliation.

Given $\mathcal{V}$, we can create two other (singular) foliations: the *coisotropic hull* $\mathcal{W}$, and the *isotropy* $\mathcal{W}^\perp$ of $\mathcal{V}$: at a generic point x we have

$$W_x = V_x + V_x^\perp \ \text{ and } \ W_x^\perp = V_x \cap V_x^\perp . \tag{5.2}$$

We will say that $\mathcal{V}$ has *compact isotropy*, if the leaves of $\mathcal{W}^\perp$ are compact. For example, the foliation given by the orbits of a Hamiltonian compact group action on a symplectic manifold has compact isotropy.

If $\mathcal{V}$ has compact isotropy, then its coisotropic hull $\mathcal{W}$ has a unique natural *transverse affine structure*. In fact, in this case, $\mathcal{W}^\perp$ is the geometric version of an integrable Hamiltonian system in generalized Liouville sense, i.e. its regular leaves are invariant manifolds of an integrable system. In particular, regular leaves of $\mathcal{W}^\perp$ are isotropic tori, say of dimension d. Near each such torus, say T^d, we can define d independent action functions $I_1, \dots, I_d$, using Mineur-Liouville formula (2.5), by fixing a basis of $H_1(T^d, \mathbb{Z})$ and a primitive of the symplectic form near T^d. These action functions are local first integrals of $\mathcal{W}$. They are not defined globally in general: by going around, we may have to change the primitive form, or the basis of $H_1(T^d, \mathbb{Z})$. But these change lead to integral affine transformations of $(I_1, \dots, I_d)$ ("integral" means that the coefficients of the linear part of the transformations are integers). So even though the functions $(I_1, \dots, I_d)$ are not defined globally, they define a transverse affine structure to $\mathcal{W}$ (the codimension of $\mathcal{W}$ is exactly d). This is the natural transverse affine structure that we were talking about.

For example, consider an effective Hamiltonian torus action of $\mathbb{T}^d$ on (M, ω). In this case, the foliation $\mathcal{V}$ by the orbits of this action coincides with its isotropy, and its generic orbits are d- dimensional tori. The generic leaves of the coisotropy hull are just regular level sets of the momentum map, the action functions are actually components of the momentum map, and the transverse affine structure is the pullback of the standard affine structure on $\mathbb{R}^d$ by the momentum map.

Another typical example: let G be a compact Lie group. The action of G on itself by conjugacy lifts to a Hamiltonian action of G on its cotangent bundle T^*G, with the momentum map $\nu : T^*G \to \mathfrak{g}^*$. The leaves of the coisotropy hull of the corresponding foliation in T^*G project to coadjoint orbits in $\mathfrak{g}^*$ by the map ν. So the transverse affine structure in T^*G projects to the transverse affine structure in $\mathfrak{g}^*$ with respect to the foliation given by the coadjoint action (i.e. the symplectic foliation of $\mathfrak{g}^*$). Factoring $\mathfrak{g}^*$ by coadjoint action, we get an affine structure on $\mathfrak{g}^*/Ad^*$. This quotient space is naturally isomorphic to a Weyl chamber $\mathfrak{t}^*_+$ (because every coadjoint orbit intersects with a fixed Weyl chamber at exactly one point). Under this identification, the affine structure that we just defined on $\mathfrak{g}^*/Ad^*$ happens to be the same as the affine structure on $\mathfrak{t}^*_+$ induced from $\mathfrak{t}^*$. That's why there is another formulation of Kirwan's theorem, in which one can use the intrinsic transverse affine structure in $\mathfrak{g}^*$ instead of taking intersection with $\mathfrak{t}^*_+$.

5.3. **Proper groupoids.**

A *groupoid* is a (small) category in which each morphism is invertible: it consists of a space Γ called the *arrow space* (the space of morphisms), a subspace B of Γ called the *base space* (the space of objects), two maps $s : \Gamma \to B$ and $t : \Gamma \to B$ called the *source map* and the *target map*, a map $m : \Gamma_2 \to \Gamma$ called the *multiplication map* (composition of two morphisms), where $\Gamma_2 = \{(g, h) \in \Gamma \times \Gamma, s(g) = t(h)\}$, and a map from Γ to itself called the inversion (each morphism has its inverse), such that the usual axioms of a category are satisfied. A groupoid is usually denoted by a double map $\Gamma \rightrightarrows B$. For a point $x \in B$, the set $G_x = s^{-1}(x) \cap t^{-1}(x)$ is a group and is called the *isotropy group* of x. The set $\mathcal{O}(x) = t(s^{-1}(x)) = s(t^{-1}(x))$ is called the *orbit* of x. If $\mathcal{O}(x) = \{x\}$ then x is called a *fixed point*.

For example, each group is a groupoid (the base space is just one point). If a group G acts on a space B, then it gives rise to an *action groupoid* $G \times B \rightrightarrows B$: $s(g,x) = x, t(g,x) = g.x$.

By a *proper groupoid*, we mean a groupoid $\Gamma \rightrightarrows B$ which satisfies the following conditions (see [78]): a) Γ and B are smooth paracompact manifolds and the maps s, t are smooth submersions; b) The map s is a locally trivial fibration; c) The map $(s,t) : \Gamma \to B \times B$ is a proper map.

Let $\Gamma \rightrightarrows B$ be a proper groupoid and x a point in B. Then the isotropy group G_x is a compact Lie group, and the orbit $\mathcal{O}(x)$ of x is an embedded submanifold in B. Let D be a small disk such that D cuts $\mathcal{O}(x)$ transversally at x, and define $\Gamma_D = \{g \in \Gamma \mid \ s(g) \in D, t(g) \in D\}$. Then D can be chosen (arbitrarily small) so that $\Gamma_D \rightrightarrows D$ is again a proper groupoid. This groupoid is called the *slice* of $\Gamma \rightrightarrows B$ at x; it has x as a fixed point. Weinstein's slice theorem (Theorem 9.1 of [78]), together with the following local structure (i.e. local linearization near a fixed point) theorem, give a normal form (i.e. linearization) for a proper groupoid in the neighborhood of an orbit.

Theorem 5.3 ([90])**.** *A proper groupoid $\Gamma_D \rightrightarrows D$ with a fixed point x is locally isomorphic to the action groupoid $G_x \times D \rightrightarrows D$ of a linear action of G_x on D.*

The proof of Theorem 5.3 is based on the averaging method. The main point is to find a homomorphism from Γ_D to G_x, whose restriction to $G_x = s^{-1}(x) \cap t^{-1}(x)$ is identity. One starts with a near-homomorphism from Γ_D to G_x (which always exists after shrinking D if necessary), then averages it with respect to a Haar system on Γ to get a new map from Γ_D to G_x which is closer to a homomorphism than the original near-homomorphism. By repeating the process and taking the limit, one finds a true homomorphism from Γ_D to G_x. This proof is similar to the proof of Grove–Karcher–Ruh theorem [37], which says that a near-homomorphism from a compact Lie group to another compact Lie group can be approximated by a homomorphism.

Symplectic groupoids have been introduced independently by Karasev, Weinstein, and Zakrzewski, in relation with symplectic realization and quantization of Poisson manifolds, see e.g. [75] and references therein. A groupoid $\Gamma \rightrightarrows B$ is called a *symplectic groupoid* if Γ is equipped with a symplectic form σ, and if we denote by $\overline{\Gamma}$ the manifold Γ with the opposite symplectic form $-\sigma$, then the graph of the multiplication map

$$\{(g, h, g.h) \in \Gamma \times \Gamma \times \overline{\Gamma} \mid \ (g,h) \in \Gamma_2\} \tag{5.3}$$

is a Lagrangian submanfiold of $\Gamma \times \Gamma \times \overline{\Gamma}$. For example, if G is a group, then we have the *standard symplectic groupoid* $T^*G \rightrightarrows \mathfrak{g}^*$ $(= T^*_e G)$, where the two maps are left and right translations.

If $(\Gamma, \omega) \rightrightarrows B$ is a symplectic groupoid, then B has a unique natural Poisson structure, for which the source map is a Poisson map, and the target map is anti-Poisson. In the case of $T^*G \rightrightarrows \mathfrak{g}^*$, the induced Poisson structure on $\mathfrak{g}^*$ is the standard linear Poisson structure.

A symplectic groupoid $(\Gamma, \sigma) \rightrightarrows B$ is called proper if it is a proper groupoid. A slice of a proper symplectic groupoid is again a proper symplectic groupoid, and its local structure is given by the following theorem, whose proof follows from Theorem 5.3 and Moser's path method.

Theorem 5.4 ([91])**.** *A proper symplectic groupoid $\Gamma_D \rightrightarrows D$ with a fixed point x is locally isomorphic to the standard symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$, where $G = G_x$ is a compact Lie group.*

A groupoid $\Gamma \rightrightarrows B$ can act on a space M equipped with a map $\mu : M \to B$, called the *momentum map* of the action. By definition, the action is a map $(g, y) \mapsto g.y$ from $\Gamma * M = \{(g, y)|s(g) = \mu(y)\}$ to M satisfying the condition $t(g.y) = t(g)$ and the usual laws for an action. A *symplectic groupoid action* of a symplectic groupoid $(\Gamma, \sigma) \rightrightarrows B$ on a symplectic manifold (M, ω) is a groupoid action such that the graph $\{(g, y, z) \mid z = g.y\}$ is a coisotropic submanifold of $\Gamma \times M \times \overline{M}$. The momentum map $\mu : M \to B$ is then a Poisson map. For example, actions of the standard symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ correspond to Hamiltonian actions of G.

5.4. **Intrinsic convexity.**

Suppose now that a proper symplectic groupoid $(\Gamma, \sigma) \rightrightarrows B$ acts on a symplectic manifold (M, ω). Weinstein's Poisson convexity conjecture [76] says that, in this case, the corresponding momentum map $\mu : M \to B$ must have convexity properties. The original formulation of this conjecture is a little bit vague, so let us spell it out, and sketch an intrinsic proof of it.

The orbits of the action of $(\Gamma, \sigma) \rightrightarrows B$ on M form a singular symplectically complete foliation on M, which we will denote by $\mathcal{V}$. The coisotropic hull of $\mathcal{V}$ is denoted by $\mathcal{W}_M$.

The singular foliation of B by the symplectic leaves is denoted by $\mathcal{W}_B$. The foliation on Γ which is the pullback of $\mathcal{W}_B$ by the source map is denoted by $\mathcal{W}_\Gamma$. $\mathcal{W}_\Gamma$ is a coisotropic singular foliation. Since $(\Gamma, \sigma) \rightrightarrows B$ is proper, $\mathcal{W}_\Gamma$ has compact isotropy, so it admits a transverse integral affine structure, which projects to a transverse integral affine structure for $\mathcal{W}_B$ in B. Similarly, $\mathcal{W}_M$ also has a natural transverse affine structure in M.

The momentum map $\mu : M \to B$ sends $\mathcal{W}_M$ to $\mathcal{W}_B$, i.e. it sends each regular leaf of $\mathcal{W}_M$ onto exactly one leaf of $\mathcal{W}_B$. An important point is that μ is a *transversally affine map*: due to the properness, we can form the quotient map $\hat{\mu} : M/\mathcal{W}_M \to B/\mathcal{W}_B$, and it's an affine map with respect to the corresponding induced affine structures. ($M/\mathcal{W}_M$ and $B/\mathcal{W}_B$ are orbifolds with boundary and corners in general).

Due to Theorem 5.4, the transverse affine structure of $(\Gamma, \mathcal{W}_\Gamma)$ near each point is locally isomorphic to a positive Weyl chamber of a compact Lie group; in particular we have the local intrinsic convexity of this affine structure.

Similarly, due to Theorem 5.4, proper symplectic groupoid actions are locally just like Hamiltonian actions of compact Lie groups. In particular, we can apply Guillemin–Sternberg-Marle local linearization theorem of Hamiltonian actions and Guillemin–Sternberg local convexity theorem (see [39]) to conclude that the transverse affine structure of $(M, \mathcal{W}_M)$ is also locally convex. Moreover, the momentum map μ is locally transversally injective.

So we have the image of something (transversally) locally convex affine under a (transversally) locally injective affine map, into a (transversally) affine space which is also locally convex. This is what we call *intrinsic convexity.* Under some mild conditions, one can conclude that the image must be convex too.

6. Global aspects of local torus actions

6.1. Sheaf of local $\mathbb{T}^1$-actions.

Consider a smooth proper integrable system on a manifold M with a given p-tuple of commuting vector fields $\mathbf{X} = (X_1, ..., X_q)$ and q-tuple of common first integrals $\mathbf{F} = (F_1, ..., F_q)$.

We will call the space of connected components of the level sets of the map $\mathbf{F}$ the *base space* of the integrable system, and denote it by $\mathcal{B}$. Since the system is proper, the space $\mathcal{B}$ with the induced topology from M is a Hausdorff space. We will denote by $\mathbf{P} : M \to \mathcal{B}$ the projection map from M to $\mathcal{B}$.

For each open set U of $\mathcal{B}$, denote by $\mathcal{R}(U)$ the set of all $\mathbb{T}^1$-actions in $\mathbf{P}^{-1}(U)$ which preserve the integrable system $(\mathbf{F}, \mathbf{X})$ (in the Hamiltonian case, due to generalized Liouville-Mineur-Arnold theorem, elements of $\mathcal{R}(U)$ will automatically preserve the Poisson structure). $\mathcal{R}(U)$ is an Abelian group: if two elements ρ_1, ρ_2 of $\mathcal{R}(U)$ are generated by two periodic vector fields Y_1, Y_2 respectively, then Y_1 will automatically commute with Y_2, and the sum $Y_1 + Y_2$ generates another $\mathbb{T}^1$-action which can be called the sum of ρ_1 and ρ_2. Actually, $\mathcal{R}(U)$ is a free Abelian group, and its dimension can vary from 0 to p (the dimension of a regular invariant torus of the system), depending on U and on the system. If U is a small disk in the regular region of $\mathcal{B}$ then $\dim_{\mathbb{Z}} \mathcal{R}(U) = p$.

The association $U \mapsto \mathcal{R}(U)$ forms a free Abelian sheaf $\mathcal{R}$ over $\mathcal{B}$, which we will call the *toric monodromy sheaf* of the system. This sheaf was first introduced in [84] for the case of Liouville-integrable systems, but its generalization to the cases of non- Hamiltonian integrable systems and integrable systems in generalized Liouville sense is obvious.

If we restrict $\mathcal{R}$ to the regular region $\mathcal{B}_0$ of $\mathcal{B}$ (the set of regular invariant tori of the system), then $\mathcal{B}$ is a locally trivial free Abelian sheaf of dimension m (one may view it as a $\mathbb{Z}^p$-bundle over $\mathcal{B}_0$), and its monodromy (which is a homomorphism from the fundamental group $\pi_1(\mathcal{B}_0)$ of $\mathcal{B}_0$ to $GL(p, \mathbb{Z})$) is nothing but the topological monodromy of the torus fibration of the regular part of the system. This topological monodromy, in the case of Liouville-integrable system, is known as the monodromy in the sense of Duistermaat [24], and it is a topological obstruction to the existence of global action-angle variables. In the case of Liouville-integrable systems with only nondegenerate elliptic singularities, studied by Boucetta and Molino [13], $\mathcal{R}$ is still a locally free Abelian sheaf of dimension $p = \frac{1}{2} \dim M$.

When the system has non-elliptic singularities, the structure of $\mathcal{R}$ can be quite complicated, even locally, and it contains a lot more information than the monodromy in the sense Duistermaat. For example, in the case of 2-degree-of-freedom Liouville-integrable systems restricted to isoenergy 3-manifolds, $\mathcal{R}$ contains information on the "marks" of the *Fomenko-Zieschang invariant*, which is a complete topological invariant for such systems, see e.g. [34, 11]. In fact, as found out by Fomenko, these isoenergy 3-manifolds are graph-manifolds, so the classical theory of graph-manifolds can be applied to the topological study of these 2-degree-of-freedom Liouville-integrable systems. A simple explanation of the fact that these manifolds are graph-manifolds is that they admit (natural) local $\mathbb{T}^1$-actions.

In the case of Liouville-integrable systems, the base space $\mathcal{B}$ has a natural stratified integral affine structure (rational affine functions on $\mathcal{B}$ are action functions for the system), and the structure of $\mathcal{R}$ can be read off the affine structure of $\mathcal{B}$, see

[84]. One can think of Bohr-Sommerfeld quantization as a disretization of the integral affine structure of $\mathcal{B}$ (see e.g. [87] ad references therein): after quantization, in place of a stratified integral affine manifold, we get a "stratified nonlinear lattice" (of joint spectrum of the system). The monodromy of this joint spectrum stratified lattice (of the quantized system) is called *quantum monodromy*, and it naturally resembles the monodromy of the classical system.

The second cohomology group $H^2(\mathcal{B}, \mathcal{R})$ plays an important role in the global topological study of integrable systems: In fact, if two integrable systems have the same base space, the same singularities, and the same toric monodromy sheaf, then their remaining topological difference can be characterized by an element in $H^2(\mathcal{B}, \mathcal{R})$, called the (relative) *Chern class*. We refer to [84] for a precise definition of this Chern class for Hamiltonian integrable systems (the definition is quite technical when the system has non-elliptic singularities), and the corresponding topological classification theorem. In the case of systems without singularities or with only elliptic singularities, this Chern class was first defined and studied by Duistermaat [24], and then by Dazord–Delzant [22] and Boucetta–Molino [13].

6.2. **Integrable surgery.**

The idea of *integrable surgery*, introduced in [84], is as follows: if we look at integrable systems from differential topology point of view (singular torus foliations), instead of dynamical point of view (quasi-periodic flows), then we can perform surgery on them, in order to modify them, and obtain new integrable systems from old ones. Integrable surgery may be useful also for the study of topology an symplectic geometry of ambient manifolds (which arise as phase spaces of integrable systems). Some simple examples of integrable surgery can be found in [84], including a construction of *exotic* symplectic spaces, and a construction of integrable systems on symplectic manifolds diffeomorphic to K3 surfaces.

Symington [68, 69] recently obtained several interesting results concerning symplectic 4-manifolds, using integrable surgery.

6.3. **Localization formulas.**

A general idea in analysis and geometry is to express global invariants in terms of local invariants, via *localization formulas*.

Various global topological invariants, including the Chern classes (of the tangent bundle), of the symplectic ambient manifold of a Liouville-integrable system, can be localized at singularities of the system. Some results in this direction can be found in recent papers of Gross [36] and Smith [66], though much still waits to be worked out for general integrable systems. For example, consider a 4-dimensional symplectic manfiold with a proper integrable system whose fixed points are nondegenerate. Then to find c_2 (the Euler class) of the manifold, one simply needs to count the number of fixed points with signs: the plus sign for elliptic-elliptic ($k_e = 2, k_h = k_f = 0$ in Williamson type), hyperbolic-hyperbolic ($k_h = 2$) and focus-focus points, and the minus sign for elliptic-hyperbolic ($k_e = k_h = 1$) points.

In symplectic geometry, there is a famous localization formula for Hamiltonian torus actions, due to Duistermaat and Heckman [27]. There is a topological version of this formula, in terms of equivariant cohomology, due to Atiyah–Bott [4] and Berline–Vergne [9], and a non-Abelian version due to Witten [80] and Jeffrey–Kirwan [45]. We refer to [5, 26, 38] for an introduction to these formulas. It would be nice to have analogs of these formulas for proper groupoid actions and general integrable systems.

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