Wahl maps and extensions of canonical curves and $K3$ surfaces

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Abstract. Let C be a smooth projective curve (resp. (S, L) a polarized $K3$ surface) of genus $g \geqslant 11$, non-tetragonal, considered in its canonical embedding in $\mathbf{P}^{g-1}$ (resp. in its embedding in $|L|^\vee \cong \mathbf{P}^g$). We prove that C (resp. S) is a linear section of an arithmetically Gorenstein normal variety Y in $\mathbf{P}^{g+r}$, not a cone, with $\dim(Y) = r+2$ and $\omega_Y = \mathcal{O}_Y(-r)$, if the cokernel of the Gauss–Wahl map of C (resp. $\mathrm{H}^1(T_S \otimes L^\vee)$) has dimension larger or equal than $r+1$ (resp. r). This relies on previous work of Wahl and Arbarello–Bruno–Sernesi. We provide various applications.

A central theme of this text is the *extendability problem*: Given a projective (irreducible) variety $X \subset \mathbf{P}^n$, when does there exist a projective variety $Y \subset \mathbf{P}^{n+1}$, not a cone, of which X is a hyperplane section? Given a positive integer r, an *r-extension* of $X \subset \mathbf{P}^n$ is a variety $Y \subset \mathbf{P}^{n+r}$ having X as a section by a linear space. The variety X is *r-extendable* if it has an r-extension that is not a cone, and *extendable* if it is at least 1-extendable. The following result provides a necessary condition for extendability.

(0.1) Theorem (Lvovski [26]). *Let $X \subset \mathbf{P}^n$ be a smooth, projective, irreducible, non-degenerate variety, not a quadric. Set*

$$\alpha(X) = h^0(N_{X/\mathbf{P}^n}(-1)) - n - 1.$$

If X is r-extendable and $\alpha(X) < n$, then $r \leqslant \alpha(X)$.

In particular, if X is extendable then $\alpha(X) > 0$. The condition $\alpha(X) < n$ is necessary in Lvovski's proof, and implies that X is not a complete intersection. The so-called Babylonian tower theorem, due to Barth, Van de Ven, and Tyurin (see, e.g., [15]), asserts that complete intersections are the only infinitely extendable varieties among local complete intersection varieties. As far as we know, it is an open question whether the assumption $\alpha(X) < n$ in Theorem (0.1) can be replaced by the a priori weaker condition that X is not a complete intersection.

One of the objectives of this article is to establish that conversely, the condition $\alpha(X) \geqslant r$ is sufficient for the r-extendabilty of canonical curves (Theorem (2.1)) and $K3$ surfaces (Theorem (2.18)).

Let $C \subset \mathbf{P}^{g-1}$ be a canonical curve of genus g. We consider its *Wahl map*

$$\Phi_C : \ \textstyle\sum_i s_i \wedge t_i \in \bigwedge^2 \mathrm{H}^0(C, \omega_C) \ \longmapsto \ \sum_i (s_i \cdot dt_i - t_i \cdot ds_i) \in \mathrm{H}^0(C, \omega_C^{\otimes 3}),$$

see, e.g., [10]. The invariant $\alpha(C)$ in Theorem (0.1) equals the corank $\mathrm{cork}(\Phi_C)$ of the Wahl map, see Lemma (3.2). Thus, as a particular case of Theorem (0.1), one has that if a smooth curve C sits on a $K3$ surface then Φ_C is non-surjective. This was originally proved by Wahl [40], using the deformation theory of cones. Beauville and Mérindol [5] gave another proof, based on the observation that for a smooth and irreducible curve C sitting on a $K3$ surface S, the surjectivity of Φ_C implies the splitting of the normal bundle exact sequence,

$$0 \to T_C \to T_S|_C \to N_{C/S} \to 0.$$

This introduced the idea, explicit in Voisin's article [38], that the elements of $(\operatorname{coker}(\Phi_C))^\vee$ (or rather of $\ker({}^{\mathsf{T}}\Phi_C)$) should be interpreted as *ribbons*, or infinitesimal surfaces, embedded in $\mathbf{P}^g$ and extending C: see Section 4.

The following statement is a first converse to Theorem (0.1), and a central element of the proofs of our Theorems (2.1) and (2.18).

(0.2) Theorem (Wahl [43], Arbarello–Bruno–Sernesi [3]). *Let C be a smooth curve of genus $g \geqslant 11$, and Clifford index $\operatorname{Cliff}(C) > 2$. Every ribbon $v \in \ker({}^{\mathsf{T}}\Phi_C)$ may be integrated to (i.e. is contained in) a surface S in $\mathbf{P}^g$ having the canonical model of C as a hyperplane section.*

Note that if $v \neq 0$, then the surface S is not a cone as only the trivial ribbon may be integrated to a cone. Conversely, we observe that actually unicity holds in Theorem (0.2) (see (2.2.2) and Remark (4.8)): up to isomorphisms, given a ribbon $v \in \ker({}^{\mathsf{T}}\Phi_C)$, the surface S integrating it in $\mathbf{P}^g$ is unique. For $v = 0$, this is the content of the aforementioned theorem of Wahl and Beauville–Mérindol, see (2.3).

We prove a statement for $K3$ surfaces analogous to Theorem (0.2) (Theorem (2.17)).

Theorem (0.2) provides a characterization of those curves having non-surjective Wahl map in the range $g \geqslant 11$ and $\operatorname{Cliff} > 2$. Wahl [42, p. 80] suggested to study the stratification of the moduli space of curves by the corank of the map Φ_C: This is done in our Theorem (2.1) to the effect that, in the same range, the curves with $\operatorname{cork}(\Phi_C) \geqslant r$ are those which are r-extendable.

We give various applications of our results, in particular to the smoothness of the fibres of the forgetful map which to a pair (S, C) associates the modulus of C, where $S \subset \mathbf{P}^g$ is a $K3$ surface and C is a canonical curve hyperplane section of S (Theorem (2.6)). The same result is proven for the analogous map on pairs (V, S) where V is a Fano threefold and S a smooth anticanonical section of V (Theorem (2.19)); in this case, this is closely related to Beauville's main result in [6]. We also answer a question asked in that article, see Proposition (2.21).

We study the Wahl maps and extensions of (the smooth models of) plane curves with up to nine ordinary singularities, and apply this to solve a conjecture of Wahl [42, p. 80] in the particular case of Del Pezzo surfaces (Proposition (9.5)).

We give a detailed account of our results in § 2. The substance of the proofs, together with the technical material needed for them, is contained in § 3–9. More information on the organization of the paper is given along § 2.

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1 – Notation and conventions

We work over the field $\mathbf{C}$ of complex numbers. All varieties, e.g., curves, surfaces, threefolds, etc., are assumed to be integral and projective.

A $K3$ *surface* is a smooth complete surface S such that $\omega_S \cong \mathcal{O}_S$ and $\mathrm{H}^1(S, \mathcal{O}_S) = 0$; a $K3$ *surface with canonical singularities*, or $K3$ *surface possibly with ADE singularities*, or *possibly singular* $K3$ *surface*, is a surface with canonical singularities whose minimal desingularisation is a $K3$ surface. A *fake* $K3$ *surface* is a non-degenerate, projective surface in $\mathbf{P}^g$, not a possibly singular $K3$ surface, having as a hyperplane section a smooth, canonical curve $C \subset \mathbf{P}^{g-1}$ of genus $g \geqslant 3$.

The *Clifford index* $\mathrm{Cliff}(S, L)$ of a polarized $K3$ surface (S, L) is the Clifford index of any smooth curve $C \in |L|$; by [20], this does not depend on the choice of C.

We denote by:

- $\mathcal{M}_g$ the moduli stack of smooth curves of genus g;
- $\mathcal{K}_g$ (resp. $\mathcal{K}_g^{\mathrm{prim}}$) the moduli stack of polarised (resp. primitively polarised) $K3$ surfaces of genus g, i.e. pairs (S, L) with S a smooth $K3$ surface, and L an ample, globally generated (resp. and primitive) line bundle on S with $L^2 = 2g - 2$;
- $\mathcal{K}_g^{\mathrm{can}}$ the moduli stack of polarised, possibly singular, $K3$ surfaces of genus g, i.e. pairs (S, L) with S a $K3$ surface with canonical singularities, and L an ample, globally generated line bundle on S with $L^2 = 2g - 2$, see [21, 5.1.4];
- $\mathcal{KC}_g$ (resp. $\mathcal{KC}_g^{\mathrm{prim}}$, $\mathcal{KC}_g^{\mathrm{can}}$) the moduli stack of pairs (S, C) with C a smooth curve on S and $(S, \mathcal{O}_S(C)) \in \mathcal{K}_g$ (resp. $\mathcal{K}_g^{\mathrm{prim}}$, $\mathcal{K}_g^{\mathrm{can}}$);
- $\mathcal{F}_g$ the moduli stack of Fano threefolds V of genus g (not necessarily of index 1), i.e. smooth varieties V with $-K_V$ ample, and $K_V^3 = 2 - 2g$;
- $\mathcal{FS}_g$ the moduli stack of pairs (V, S) with $V \in \mathcal{F}_g$ and $S \in |-K_V|$ a smooth surface, so that $(S, -K_V|_S) \in \mathcal{K}_g$;
- $\mathcal{K}_g^{\mathrm{R}}$, where $\mathrm{R} = (R, \lambda)$ consists of a lattice R and a distinguished element $\lambda \in R$ with $\lambda^2 = 2g - 2$, the moduli stack of R-*polarised* $K3$ surfaces, i.e. polarised $K3$ surfaces (S, L) together with a fixed embedding of R as a primitive sublattice of $\mathrm{Pic}(S)$, sending λ to the class of L;
- $\mathcal{KC}_g^{\mathrm{R}}$, with R as above, the moduli stack of pairs (S, C) with S an R-polarised $K3$ surface and C a smooth curve on S in the class λ;
- $\mathcal{F}_g^{\mathrm{R}}$ and $\mathcal{FS}_g^{\mathrm{R}}$ the stacks of Fano varieties analogous to $\mathcal{K}_g^{\mathrm{R}}$ and $\mathcal{KC}_g^{\mathrm{R}}$;
- $c_g : \mathcal{KC}_g \to \mathcal{M}_g$, $c_g^{\mathrm{prim}} : \mathcal{KC}_g^{\mathrm{prim}} \to \mathcal{M}_g$, and $c_g^{\mathrm{R}} : \mathcal{KC}_g^{\mathrm{R}} \to \mathcal{M}_g$ the forgetful maps;
- $s_g : \mathcal{FS}_g \to \mathcal{K}_g$ and $s_g^{\mathrm{R}} : \mathcal{FS}_g^{\mathrm{R}} \to \mathcal{K}_g^{\mathrm{R}}$ the forgetful maps.

2 – Main results

Canonical curves. Our first result is the following converse to Lvovski's Theorem (0.1) for canonical curves.

(2.1) Theorem. *Let C be a smooth genus g curve with Clifford index $\mathrm{Cliff}(C) > 2$, and r a non-negative integer. We consider the following two propositions:*
(i) $\mathrm{cork}(\Phi_C) \geqslant r + 1$*;*
(ii) there is an arithmetically Gorenstein normal variety Y in $\mathbf{P}^{g+r}$, not a cone, with $\dim(Y) = r + 2$, $\omega_Y = \mathcal{O}_Y(-r)$, which has a canonical image of C as a section with a $(g-1)$-dimensional linear subspace of $\mathbf{P}^{g+r}$ (in particular, $C \subset \mathbf{P}^{g-1}$ is $(r+1)$-extendable).

If $g \geqslant 11$, then (i) *implies* (ii)*. Conversely, if $g \geqslant 22$ and the canonical image of C is a hyperplane section of some smooth $K3$ surface in $\mathbf{P}^g$, then* (ii) *implies* (i).

(2.2) Actually, we prove more than Theorem (2.1), see Corollary (5.5). Let C be a smooth curve of genus $g \geqslant 11$ with $\mathrm{Cliff}(C) > 2$, and let $r := \mathrm{cork}(\Phi_C) - 1$.

(2.2.1) There is an arithmetically Gorenstein normal variety X of dimension $r + 2$ in $\mathbf{P}^{g+r}$, not a cone, containing a canonical image C_0 of C as a section by a linear $(g-1)$-space, and satisfying the following property: for all $[v] \in \mathbf{P}(\ker({}^{\mathsf{T}}\Phi_C))$, there is a unique section of X by a linear g-space containing a ribbon over C_0 belonging to the isomorphism class $[v]$.

(See Section 4 for background on ribbons and their relation with the Wahl map Φ_C).

(2.2.2) For all $[v] \in \mathbf{P}(\ker({}^{\mathsf{T}}\Phi_C))$*, there is a* unique *(up to projectivities pointwise fixing* C*, see Remark (4.8)) surface* $S \subset \mathbf{P}^g$ *containing a ribbon over a canonical model of* C *in the isomorphism class* $[v]$*.*

(2.2.3) Definition. We say that an extension X of C_0 as in (2.2.1) is *universal.* By (2.2.2), a universal extension of C_0 has as linear sections all possible surface extensions of C_0 but cones.

No matter the genus, Lvovski's Theorem (0.1) tells us that (ii) of Theorem (2.1) implies the inequality

$$\operatorname{cork}(\Phi_C) \geqslant \min(g-1, r+1).$$

When $r = 0$, '(ii) $\Rightarrow$ (i)' in Theorem (2.1) was proved by Wahl [40] and later independently by Beauville and Mérindol [5], and '(i) $\Rightarrow$ (ii)' is Theorem (0.2) by Wahl and Arbarello–Bruno–Sernesi. To prove (i) $\Rightarrow$ (ii) for arbitrary r, we show that Wahl's extension construction [43, Theorem 7.1] (the requirements of which are met thanks to [3, Theorem 3]) works in families, see § 5.

Statement (2.2.2) is implicitly contained in the proof of [43, Theorem 7.1] as we observe in Remark (4.8), although it apparently remained unnoticed so far.

(2.3) If $g \geqslant 11$ and $\operatorname{Cliff}(C) > 2$, the unicity of the extension of the ribbon $v = 0$ in Theorem (0.2) (see Remark (4.8)) tells us that the cone over a canonical model of C is the only surface in $\mathbf{P}^g$ containing the trivial ribbon over C. Thus, if C sits on a $K3$ surface, the ribbon over C defined by S is non-trivial, hence Φ_C is not surjective: this is the theorem of Wahl and Beauville–Mérindol, though a priori only for curves of genus $g \geqslant 11$ and Clifford index $\operatorname{Cliff}(C) > 2$; the remaining cases can be dealt with directly: curves with $g < 10$ or $\operatorname{Cliff}(C) \leqslant 2$ all have non-surjective Wahl map [42, 13, 14, 7], and curves of genus 10 sitting on a $K3$ have non-surjective Wahl map by [17].

Proof of Theorem (2.1). The fact that (i) implies (ii) provided $g \geqslant 11$ is the content of Corollary (5.6.1). The converse implication is given by Lvovski's Theorem (0.1) as follows. Identify C with its canonical model in $\mathbf{P}^{g-1}$. Then $\alpha(C) = h^0(N_{C/\mathbf{P}^{g-1}}(-1)) - g$ equals $\operatorname{cork}(\Phi_C)$ by Lemma (3.2), so one has $\alpha(C) \leqslant 20$ by Corollary (8.5). It follows that the assumption $\alpha(C) < g-1$ holds if $g \geqslant 22$; in this case, Theorem (0.1) says that (ii) implies (i). □

We obtain the bound $\alpha(C) \leqslant 20$ used in the above proof of Theorem (2.1) as a corollary of Proposition (8.4). The latter Proposition also has the following consequence, of independent interest.

(2.4) Proposition (Corollary (8.6)). *Let* $(S, C) \in \mathcal{KC}_g^{\mathrm{can}}$ *with* $g \geqslant 11$ *and* $\operatorname{Cliff}(C) > 2$*. There are only finitely many members* C' *of* $|\mathcal{O}_S(C)|$ *that are isomorphic to* C*.*

In fact it follows from the arguments in the proof of Corollary (8.6) that if $\operatorname{cork}(\Phi_C) = 1$ (which happens for instance if $g > 37$, see Corollary (2.10) below), then for all $C' \in |\mathcal{O}_S(C)|$ isomorphic to C there exists an automorphism of the polarized surface $(S, \mathcal{O}_S(C))$ taking C to C' (because in this case the two curves C and C' have the same ribbon in S). In particular, if the automorphism group of S is trivial (which happens for instance if S has Picard number 1), then the smooth members of $|\mathcal{O}_S(C)|$ are pairwise non-isomorphic.

The following is a consequence of (2.2):

(2.5) Corollary. *Let C be a smooth curve of genus $g \geqslant 11$ with* $\mathrm{Cliff}(C) > 2$. *The curve C cannot sit on two $K3$ surfaces S and S' such that its respective classes in* $\mathrm{Pic}(S)$ *and* $\mathrm{Pic}(S')$ *have different divisibilities.*

Proof. By (2.2), all extensions of the canonical model of C are packaged together in an irreducible family. The Corollary thus follows from the fact that the divisibility of $[C]$ in $\mathrm{Pic}(S)$ is a topological character, hence constant under deformations of the pair (C, S). □

Next, we study the ramification of the forgetful map $c_g : \mathcal{KC}_g \to \mathcal{M}_g$. To put our results in perspective, recall that

$$\dim(\mathcal{KC}_g) - \dim(\mathcal{M}_g) = (19+g) - (3g-3) = 2(11-g).$$

The primitively polarised case has been classically studied: for $g \geqslant 11$, the map c_g^{prim} is birational onto its image if $g \neq 12$, whereas its generic fibre is irreducible of dimension 1 if $g = 12$ ([12, § 5.3] and [29]); for $g \leqslant 11$, the map c_g^{prim} is dominant if $g \neq 10$ [28], and onto a hypersurface of $\mathcal{M}_{10}$ if $g = 10$ [17], with irreducible general fibre in any case [13, 12]. The non-primitively polarised cases have been studied in [9, 24] where it is shown that, if $g \geqslant 11$ then c_g is generically finite in all but possibly finitely many cases.

It turns out that in the range $g \geqslant 11$, the map c_g has smooth fibres over the locus of curves with Clifford index greater than 2.

(2.6) Theorem. *Let $(S, C) \in \mathcal{KC}_g$ with $g \geqslant 11$ and* $\mathrm{Cliff}(C) > 2$. *Then*

$$\dim(\ker(dc_g)_{(S,C)}) = \dim(c_g^{-1}(C)) = \mathrm{cork}(\Phi_C) - 1.$$

Over curves with $\mathrm{Cliff}(C) \leqslant 2$ the situation is more complicated, if only because then the spaces $\mathrm{H}^0(N_{C/\mathbf{P}^{g-1}}(-k))$, $k \geqslant 2$, don't necessarily vanish (equivalently the higher Gaussian maps $\Phi_{\omega_C^{\otimes k}, \omega_C}$, $k \geqslant 2$, are not necessarily surjective [41]), contrary to what happens when $\mathrm{Cliff}(C) > 2$, compare Lemma (3.6). See [13, Cor. 4.4] for the situation over the general curve of genus $g \leqslant 6$.

(2.7) Remark. Beauville [6, Sec. 5] observed that the map c_g is not everywhere unramified, as it has positive dimensional fibres at those points (S, C) such that S is an anticanonical divisor of some smooth Fano threefold V. Theorems (2.6) and (2.1) together say that, in the range of application of Theorem (2.6), all the ramification of c_g is accounted for by this phenomenon.

This reflects the fact that for $g \leqslant 12$, $g \neq 11$, the positive dimensionality of the generic fibre of c_g^{prim} is explained by the existence of Fano varieties with coindex 3 and Picard number 1 (see, e.g., [22, Chap. 5]).

Proof of Theorem (2.6). We have the following chain of (in)equalities:

$$\begin{aligned}
\mathrm{cork}(\Phi_C) - 1 &\leqslant \dim(c_g^{-1}(C)) && \text{by Corollary (7.1)} \\
&\leqslant \dim(\ker(dc_g)_{(S,C)}) && \text{obviously} \\
&= h^1(T_S(-1)) && \text{by Lemma (7.2)} \\
&\leqslant \mathrm{cork}(\Phi_C) - 1 && \text{by Proposition (7.3).}
\end{aligned}$$

□

The following result is a straightforward but noteworthy consequence of the proof of Theorem (2.6). It says in particular that the corank of the Wahl map is the same for all smooth hyperplane sections of a given $K3$ surface.

(2.8) Corollary. *Let $(S, L) \in \mathcal{K}_g$, and assume that $g \geqslant 11$ and $\mathrm{Cliff}(S, L) > 2$. For every smooth member $C \in |L|$, one has*

$$\mathrm{cork}(\Phi_C) = h^1(T_S(-1)) + 1.$$

(2.9) It is a known fact that a threefold V in $\mathbf{P}^{g+1}$ having as hyperplane section a $K3$ surface S, possibly with ADE singularities, is an arithmetically Gorenstein normal Fano threefold with canonical singularities, see Corollary (5.6). Consequently, in the setting of Theorem (2.1), if we assume in addition that there exists an extension of C to a surface S with at worst ADE singularities (so that S is a $K3$ surface, possibly singular), then the sections of Y with linear subspaces of dimension $g+1$ containing S are Fano threefolds of genus g, with canonical singularities. We may thus use the boundedness of Fano varieties to derive the following corollary from our previous results.

(2.10) Corollary. *Let C be a smooth curve of genus $g > 37$, and Clifford index* $\mathrm{Cliff}(C) > 2$. *If the canonical model of C is a hyperplane section of a $K3$ surface S, possibly with ADE singularities, then* $\mathrm{cork}(\Phi_C) = 1$.

Proof. If C is a hyperplane section of a $K3$ surface S and $\mathrm{cork}(\Phi_C) > 1$, then by Corollary (5.6) there is an arithmetically Gorenstein Fano threefold of genus g, with canonical singularities, and having C as a curve section. By [30, Thm. 1.5] all such threefolds have genus $g \leqslant 37$. □

(2.11) Remark. Based on the above statement, one may be tempted to speculate that all smooth curves C of genus $g > 37$ with $\mathrm{Cliff}(C) > 2$ have $\mathrm{cork}(\Phi_C) \leqslant 1$; this is not true.

If one drops the assumption that the curve C lies on a $K3$ surface in Corollary (2.10), one has to deal with the possibility that all surface extensions of the curve C may have singularities worse than ADE singularities. In such a situation, threefolds extending C are no longer Fano, and there is no boundedness result in this case.

As a matter of fact, plane curves provide examples of curves of arbitrarily large genus, having Clifford index greater than 2, and for which the Wahl map has corank 10 [42, Thm. 4.8].

(2.12) In Section 9 we study the extensions of the canonical models of the normalizations of plane curves, continuing a long story contributed to by numerous authors (see at least [18, 3] and the references therein). Such surface extensions are rational, hence not $K3$, and have indeed an elliptic (in general non-smoothable) singularity.

We give an explicit construction of the universal extensions of such curves. These extensions are not Fano, and provide an unbounded family of "fake Fano" varieties, i.e. irreducible varieties X of dimension $r+2$ in $\mathbf{P}^{g+r}$ ($r > 0$), with non-canonical singularities, and with curve sections canonical curves of genus g. Whereas fake $K3$ surfaces are fairly well understood (for instance, there is a classification [18]), understanding fake Fano varieties is a wide open problem.

We use the precise relation between extensions and cokernel of the Gauss map to prove a conjecture of Wahl [42, p. 80] in the case of Del Pezzo surfaces, see Proposition (9.5); the case of the projective plane was already handled in [42].

(2.13) Remark. *All canonical curves in smooth Fano threefolds V with Picard number $\rho(V) \geqslant 2$ are Brill–Noether special.*

More precisely, we claim that if V is a smooth Fano threefold V with $\rho(V) \geqslant 2$, then the smooth curves in V complete intersections of two elements of $|-K_V|$ are Brill–Noether special.

To see this, one has to consider one by one all the elements in the list [22, Chap. 12], and check that in each case there is a line bundle on V that gives Brill–Noether special linear series on the canonical curves contained in V. We do not dwell on this here.

Since all smooth Fano threefolds with Picard number 1 have genus $g \leqslant 12$, $g \neq 11$, Remark (2.13) together with (2.9) leads to the following question.

(2.14) Question. *Does there exist any Brill–Noether general curve of genus $g \geqslant 11$, $g \neq 12$, such that* $\operatorname{cork}(\Phi_C) > 1$*?*

We cannot answer this question so far, for the following two reasons: (i) as far as we know, no classification of Fano threefolds with Gorenstein canonical singularities is available, and (ii) there is the possibility that all surface extensions of a given curve have singularities worse than ADE singularities.

Note however that the singularities of a surface extension of a Brill–Noether–Petri general curve cannot be too bad: it is proven in [3] that such an extension is smoothable to a $K3$ surface if $g \geqslant 12$.

The so-called *du Val curves* (a particular instance of the curves we study in § 9, see [2]) are an interesting example with regard to this problematic. Under suitable generality assumptions, a du Val curve is Brill–Noether–Petri general [36, 2]; its Wahl map has corank 1 if g is odd [1], and is unknown otherwise; this leaves Question (2.14) open. Note that when g is odd, the canonical model of a general du Val curve has a unique surface extension, which is a rational surface with a smoothable elliptic singularity (see Proposition (9.1)).

(2.15) In Corollary (8.5), we prove as a consequence of Proposition (2.4) (Proposition (8.4)) that $\operatorname{cork}(\Phi_C) \leqslant 20$ for any smooth curve of genus $g \geqslant 11$ and $\operatorname{Cliff}(C) > 2$ lying on a smooth $K3$ surface. We suspect that this bound is far from being sharp.

The corank of the Wahl map of a general tetragonal curve of genus $g \geqslant 7$ equals 9 [7]. On the other hand, the corank of the Wahl map of a hyperelliptic (resp. trigonal) curve of genus g is $3g-2$ [42, 14] (resp. $g+5$ [7, 14]). Note that [42, 14] also assert that $3g-2$ is the maximal possible value for the corank of the Gauss map of a curve of genus g, and is attained only for hyperelliptic curves.

(2.16) Question. *Does there exist a universal, genus independent, bound on* $\operatorname{cork}(\Phi_C)$ *for curves C with Clifford index* $\operatorname{Cliff}(C) > 2$*?*

$K3$ surfaces. Generally speaking, the results about canonical curves discussed above pass to their smooth extensions in $\mathbf{P}^g$, namely $K3$ surfaces. First of all, we prove the following result for $K3$ surfaces, analogous to Theorem (0.2). Given $(S, L) \in \mathcal{K}_g$, we consider S in its embedding in $\mathbf{P}^g = |L|^\vee$.

(2.17) Theorem. *Let $(S, L) \in \mathcal{K}_g$ be a polarized $K3$ surface of genus $g \geqslant 11$, such that* $\operatorname{Cliff}(S, L) > 2$*. Every ribbon $v \in \mathrm{H}^1(T_S \otimes L^\vee)$ may be integrated to a unique threefold V in $\mathbf{P}^{g+1}$, up to projectivities.*

As in Theorem (0.2), if $v \neq 0$ in the above statement, then V is not a cone. In particular, a polarized $K3$ surface (S, L) with $g \geqslant 11$ and $\operatorname{Cliff}(S, L) > 2$ lies on a Fano threefold (with canonical Gorenstein singularities, see (2.9)) if and only if $\mathrm{H}^1(T_S \otimes L^\vee) \neq 0$.

The necessary background on ribbons is given in § 4, and the proof of Theorem (2.17) in § 6; it relies on the existence of a universal extension for canonical curves (see (2.2)) and on Corollary (2.8). Next, Theorems (2.18) and (2.19) are the exact analogues for $K3$ surfaces of Theorems (2.1) and (2.6).

(2.18) Theorem. *Let $(S, L) \in \mathcal{K}_g$ be a polarized $K3$ surface with Clifford index* $\mathrm{Cliff}(S, L) > 2$. *We consider the following two propositions:*
(i) $h^1\big(T_S \otimes L^\vee\big) \geqslant r$;
(ii) there is an arithmetically Gorenstein normal variety X in $\mathbf{P}^{g+r}$, with $\dim(X) = r + 2$, *$\omega_X = \mathcal{O}_X(-r)$, X not a cone, having the image of S by the linear system $|L|$ as a section with a linear subspace of dimension g.*

If $g \geqslant 11$, then (i) *implies* (ii). *Conversely, if $g \geqslant 22$ then* (ii) *implies* (i).

By Lemma (3.5), one has

$$\alpha(S) = h^1\big(T_S(-1)\big) = h^1\big(T_S \otimes L^\vee\big)$$

Similar to the curve case, if (ii) holds then $h^1\big(T_S(-1)\big) \geqslant \min(g, r)$ by Lvovski's Theorem (0.1); in particular, S is extendable if and only if $h^1\big(T_S(-1)\big) > 0$.

Proof of Theorem (2.18). Let C be a smooth hyperplane section of $S \subset \mathbf{P}^g$: it is a canonical curve of genus g and Clifford index $\mathrm{Cliff}(C) = \mathrm{Cliff}(S, L)$, and one has $\mathrm{cork}(\Phi_C) = h^1(T_S(-1))+1$ by Corollary (2.8). Then Theorem (2.18) follows at once from Theorems (2.1) and (2.2). $\square$

(2.19) Theorem. *Let $(V, S) \in \mathcal{FS}_g$ with $g \geqslant 11$ and* $\mathrm{Cliff}(C) > 2$. *Then*

$$\dim(\ker(ds_g)_{(V,S)}) = \dim(s_g^{-1}(S)) = h^1\big(T_S(-1)\big) - 1.$$

The proof of Theorem (2.19) is exactly the same as that of Theorem (2.6). In analogy with Corollary (2.8), it gives

$$h^1(T_S(-1)) - 1 = h^1(T_V(-S)) = h^{2,1}(V) = b_3(V)/2. \tag{2.19.1}$$

Theorem (2.19) is closely related to the following result.

(2.20) Theorem (Beauville, [6]). *The morphism $s_g^{\mathrm{R}} : \mathcal{FS}_g^{\mathrm{R}} \to \mathcal{K}_g^{\mathrm{R}}$ is smooth and dominant. Its relative dimension at the point (V, S) is $b_3(V)/2$.*

Beauville [6, (4.4)] asked: *For those families of Fano threefolds for which $b_3 = 0$, the map s_g^{R} is étale; is it an isomorphism onto an open substack of $\mathcal{K}_g^{\mathrm{R}}$?* We give the following answer.

(2.21) Proposition. *Let $(V, S) \in \mathcal{FS}_g$ be such that $g \geqslant 11$,* $\mathrm{Cliff}(S, -K_V|_S) > 2$, *and $b_3(V) = 0$. The fibre $(s_g^{\mathrm{R}})^{-1}(S)$ is reduced to a point if and only if there is no non-trivial automorphism of V induced by a projectivity of $|-K_V|$.*

Proof. One has $h^1(T_S(-1)) = 1$ by (2.19.1). From this we deduce by Theorem (2.17) that up to isomorphism V is the only Fano threefold that may contain S as an anticanonical divisor. As a consequence, the fibre $s_g^{-1}(S)$ has cardinality greater than 1 if and only if there are several anticanonical divisors in V isomorphic to S. The latter property implies the existence of an automorphism of V, induced by a projectivity of $\mathbf{P}^{g+1}$, that transforms one copy of S as an anticanonical divisor into another. $\square$

3 – Gaussian maps and twisted normal bundles

(3.1) Let X be a smooth variety and L a line bundle on X. We consider the multiplication map

$$\mu_{L,\omega_X} : \mathrm{H}^0(L) \otimes \mathrm{H}^0(\omega_X) \to \mathrm{H}^0(L \otimes \omega_X), \tag{3.1.1}$$

whose kernel we denote by $R(L, \omega_X)$. If X is a curve C, one defines the (L, ω_C)-*Gaussian map*

$$\Phi_{L,\omega_C} : R(L, \omega_C) \to \mathrm{H}^0(L \otimes \omega_C^{\otimes 2}) \quad \text{by} \quad \textstyle\sum_i s_i \otimes t_i \mapsto \sum_i (s_i \cdot dt_i - t_i \cdot ds_i), \tag{3.1.2}$$

see [10] for more details. The map Φ_{ω_C,ω_C} restricted to $\bigwedge^2 \mathrm{H}^0(\omega_C)$ identifies with the Wahl map Φ_C.

(3.2) Lemma ([41], see also [13, Prop. 1.2]). *Let C be a smooth curve of positive genus and L a very ample line bundle on C. We consider $C \subset \mathbf{P}^r := \mathbf{P}(\mathrm{H}^0(L)^\vee)$. Then one has the exact sequence*

$$0 \to \mathrm{H}^0(L)^\vee \to \mathrm{H}^0\big(N_{C/\mathbf{P}^r} \otimes L^\vee\big) \to \mathrm{coker}(\Phi_{L,\omega_C})^\vee \to 0. \tag{3.2.1}$$

(3.3) In order to state some identifications worth keeping in mind, valid for a curve of arbitrary genus, we sketch the proof of Lemma (3.2). The Euler exact sequence twisted by $L^\vee$, together with Serre duality, gives the exact sequence

$$0 \to \mathrm{H}^0(L)^\vee \to \mathrm{H}^0\big(T_{\mathbf{P}^r} \otimes L^\vee\big) \to \mathrm{H}^0(L \otimes \omega_C)^\vee \xrightarrow{{}^\mathsf{T}\mu_{L,\omega_C}} \mathrm{H}^0(L)^\vee \otimes \mathrm{H}^0(\omega_C)^\vee \to \mathrm{H}^1\big(T_{\mathbf{P}^r} \otimes L^\vee\big) \to 0,$$

from which it follows that:

$$0 \to \mathrm{H}^0(L)^\vee \to \mathrm{H}^0\big(T_{\mathbf{P}^r}|_C \otimes L^\vee\big) \to \ker\big({}^\mathsf{T}\mu_{L,\omega_C}\big) \to 0 \quad \text{is an exact sequence;} \tag{3.3.1}$$

$$\mathrm{H}^1\big(T_{\mathbf{P}^r}|_C \otimes L^\vee\big) \cong \mathrm{coker}\big({}^\mathsf{T}\mu_{L,\omega_C}\big) \cong R(L, \omega_C)^\vee. \tag{3.3.2}$$

Then, ${}^\mathsf{T}\Phi_{L,\omega_C}$ identifies with the map

$$\mathrm{H}^1\big(T_C \otimes L^\vee\big) \to \mathrm{H}^1\big(T_{\mathbf{P}^r}|_C \otimes L^\vee\big) \tag{3.3.3}$$

induced by the inclusion $T_C \subset T_{\mathbf{P}^r}|_C$. Eventually, if $C \subset \mathbf{P}^r$ is neither a line nor a conic, the normal bundle exact sequence twisted by $L^\vee$ gives the exact sequence

$$0 \to \mathrm{H}^0\big(T_{\mathbf{P}^r}|_C \otimes L^\vee\big) \to \mathrm{H}^0\big(N_{C/\mathbf{P}^r} \otimes L^\vee\big) \to \ker({}^\mathsf{T}\Phi_{L,\omega_C}) \to 0. \tag{3.3.4}$$

When C has positive genus, the map μ_{L,ω_C} is surjective [4, 8, 19], and Lemma (3.2) follows from (3.3.1) and (3.3.4).

(3.4) The space $\mathrm{H}^0(N_{C/\mathbf{P}^r} \otimes L^\vee)$ is the Zariski tangent space to the space of deformations of C in $\mathbf{P}^r$ fixing a given hyperplane section $H \cap C$. The inclusion of $\mathrm{H}^0(L)^\vee$ in this space, at the left-hand-side of (3.2.1), comes from the isomorphism $\mathrm{H}^0(L)^\vee \cong \mathrm{H}^0(T_{\mathbf{P}^r} \otimes L^\vee)$ given by (3.3.1), which identifies $\mathrm{H}^0(L)^\vee$ with a space of infinitesimal automorphisms inside $\mathrm{H}^0(N_{C/\mathbf{P}^r} \otimes L^\vee)$.

It is useful in our setup to express this identification in coordinates. Fix homogeneous coordinates $(x_0 : \ldots : x_r)$ such that H has equation $x_0 = 0$. The space $\mathrm{H}^0(T_{\mathbf{P}^r})$ is the tangent space at the origin to PGL_{r+1}, and $H^0(T_{\mathbf{P}^r} \otimes L^\vee)$ is the tangent space at the origin to the subgroup

$$\mathbf{C}^r \rtimes \mathbf{C}^* < \mathrm{PGL}_{r+1}$$

of projectivities fixing H point by point. The latter are given in the affine chart $x_0 = 1$ by $x_i \mapsto \lambda x_i + a_i$ $(1 \leqslant i \leqslant r)$, with $(a_1, \ldots, a_r) \in \mathbf{C}^r$ and $\lambda \in \mathbf{C}^*$. The elements of $H^0(T_{\mathbf{P}^r} \otimes L^\vee)$ thus identify with the infinitesimal automorphisms given in affine coordinates by $x_i \mapsto (1+t\varepsilon_0)x_i + t\varepsilon_i$ $(1 \leqslant i \leqslant r)$, where $t^2 = 0$ and $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r) \in \mathbf{C}^{r+1}$. The isomorphism $\mathrm{H}^0(L)^\vee \cong \mathrm{H}^0(T_{\mathbf{P}^r} \otimes L^\vee)$ maps the linear form $\mathbf{x} \mapsto \boldsymbol{\varepsilon} \cdot \mathbf{x}$ to the latter infinitesimal automorphism.

The following statement is a higher dimensional version of Lemma (3.2), and the proofs of the two go along the same lines.

(3.5) Lemma. *Let X be a smooth variety of dimension $n \geqslant 2$ with $h^1(\mathcal{O}_X) = 0$, and L a very ample line bundle on X, with (X, L) different from $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$. If X is a surface, we assume in addition that the multiplication map μ_{L,ω_X} is surjective. We consider $X \subset \mathbf{P}^r := \mathbf{P}(\mathrm{H}^0(L)^\vee)$. Then one has the exact sequence*

$$0 \to \mathrm{H}^0(L)^\vee \to \mathrm{H}^0\big(N_{X/\mathbf{P}^r} \otimes L^\vee\big) \to \mathrm{H}^1(T_X \otimes L^\vee) \to 0. \tag{3.5.1}$$

If $n = 2$ and X is a K3 surface, then μ_{L,ω_X} is surjective, and moreover

$$\mathrm{H}^1\big(N_{X/\mathbf{P}^r} \otimes L^\vee\big) \cong \mathrm{H}^2(T_X \otimes L^\vee).$$

Proof. The Euler exact sequence twisted by $L^\vee$, together with the the Kodaira Vanishing Theorem imply that

$$\mathrm{H}^0(T_{\mathbf{P}^r}|_X \otimes L^\vee) \cong \mathrm{H}^0(L)^\vee. \tag{3.5.2}$$

Moreover

$$\mathrm{H}^1(T_{\mathbf{P}^r}|_X \otimes L^\vee) = 0 : \tag{3.5.3}$$

this follows from the vanishing of $\mathrm{H}^1(\mathcal{O}_X)$ and, if $n \geqslant 3$ the vanishing of $\mathrm{H}^2(L^\vee)$, or if $n = 2$ the surjectivity of μ_{L,ω_X}.

Next we consider the twisted normal bundle exact sequence

$$0 \to T_X \otimes L^\vee \to T_{\mathbf{P}^r}|_X \otimes L^\vee \to N_{X/\mathbf{P}^r} \otimes L^\vee \to 0. \tag{3.5.4}$$

By the Mori–Sumihiro–Wahl Theorem [27, 39], one has $\mathrm{H}^0(T_X \otimes L^\vee) = 0$. Then (3.5.1) follows from (3.5.2), (3.5.3), and the long exact sequence of cohomology of (3.5.4).

If ω_X is trivial then μ_{L,ω_X} is an isomorphism hence, if $n = 2$, $\mathrm{H}^2(T_{\mathbf{P}^r}|_X \otimes L^\vee)$ vanishes, and the final assertion follows from the cohomology sequence of (3.5.4). □

(3.6) Lemma (see, e.g., [40, § 2] and [25, Lemma 2.7 (ii)]). *Let $X \subset \mathbf{P}^n$ be a local complete intersection variety such that the homogeneous ideal of X is generated by quadrics and the first syzygy module is generated by linear syzygies. Then $\mathrm{H}^0(N_X(-k)) = 0$ for all $k \geqslant 2$.*

Note that this applies to any canonical curve C with $\mathrm{Cliff}(C) > 2$ by [37, 32], resp. to any $K3$ surface $S \subset \mathbf{P}^g$ with $\mathrm{Cliff}(S, \mathcal{O}_S(1)) > 2$ [31]. In the latter case, Andreas Knutsen kindly indicated to us how to prove that $\mathrm{H}^0(N_S(-2)) = 0$ if $g \geqslant 11$ without any assumption on the Clifford index. We don't dwell on this here.

4 – Ribbons and extensions

In this Section we recall the required background on ribbons, and their relation with Wahl maps in the case of canonical curves. We review [43, Proof of Thm 7.1] in some details, as we will need this later. We make our observation that unicity holds in Theorem (0.2) (Remark (4.8)).

(4.1) Let Y be a reduced connected scheme and L an invertible sheaf on Y. A *ribbon* over Y with *normal bundle* L (or *conormal bundle* $L^\vee$) is a scheme $\tilde{Y}$ such that $\tilde{Y}_{\mathrm{red}} = Y$, $\mathcal{I}^2_{Y/\tilde{Y}} = 0$ and $L^\vee \cong \mathcal{I}_{Y/\tilde{Y}} = \ker\big(\mathcal{O}_{\tilde{Y}} \to \mathcal{O}_Y\big)$. To each ribbon one associates the extension class $e_{\tilde{Y}} \in \mathrm{Ext}^1_Y(\Omega^1_Y, L^\vee)$ determined by the conormal sequence of $Y \subset \tilde{Y}$:

$$\begin{array}{cc} & 0 \to L^\vee \longrightarrow \mathcal{O}_{\tilde{Y}} \longrightarrow \mathcal{O}_Y \to 0 \\ & \quad\ \big\| \qquad\quad \downarrow \qquad\quad \downarrow \\ e_{\tilde{Y}}: & 0 \to L^\vee \to \Omega^1_{\tilde{Y}}\big|_Y \to \Omega^1_Y \to 0 \end{array}$$

Note that the upper row is an extension of sheaves of algebras, while the lower one is an extension of $\mathcal{O}_Y$-modules; the middle and right vertical arrows are differentials and therefore are not $\mathcal{O}_{\tilde{Y}}$-linear. Conversely, to each element of $\mathrm{Ext}^1_Y(\Omega^1_Y, L^\vee)$ there is associated a unique ribbon constructed in a standard way (see, e.g., [33, Thm. 1.1.10]).

(4.1.1) The trivial extension corresponds to the *split ribbon*, the unique one such that the inclusion $Y \subset \tilde{Y}$ admits a retraction $\tilde{Y} \to Y$. Two extensions define isomorphic ribbons if and only if they are proportional. Therefore the set of isomorphism classes of non-split ribbons is in $1:1$ correspondence with $\mathbf{P}\big(\mathrm{Ext}^1_Y(\Omega^1_Y, L^\vee)\big)$.

(4.2) Let $Y \subset X$ be a nonsingular hypersurface in a variety X smooth along Y. The conormal sequence of $Y \subset X$ yields an element $\kappa_{Y/X} \in \mathrm{Ext}^1_Y(\Omega^1_Y, N^\vee_{Y/X})$, defining a ribbon $\tilde{Y}$ over Y with normal bundle $N_{Y/X}$. A priori we have another ribbon $\bar{Y}$ over Y, defined by $\mathcal{O}_{\bar{Y}} = \mathcal{O}_X/\mathcal{I}^2_Y = \mathcal{O}_X/\mathcal{O}_X(-2Y)$; by definition, one has $\bar{Y} \subset X$. On the other hand it follows from the conormal sequence of $\bar{Y} \subset X$ that $\Omega^1_X\big|_Y = \Omega^1_{\bar{Y}}\big|_Y$. Therefore $\tilde{Y} = \bar{Y}$. We call $\tilde{Y}$ a *double hypersurface* in X and we denote it by $2Y_X$.

(4.2.1) If $H \subset \mathbf{P}^{r+1}$ is a hyperplane, then $2H_{\mathbf{P}^{r+1}}$ is a split ribbon. This can be seen in two ways. Firstly, projecting from a point $p \notin H$ we obtain a retraction $2H_{\mathbf{P}^{r+1}} \to H$. Alternatively, the extension defining $2H_{\mathbf{P}^{r+1}}$ belongs to $\mathrm{Ext}^1_H(\omega_H, \mathcal{O}_H(-1)) = \mathrm{H}^1(H, \mathcal{O}_H(r)) = 0$, and therefore splits.

(4.3) Consider a smooth variety $X \subset \mathbf{P}^n$, and identify this $\mathbf{P}^n$ with a hyperplane $H \subset \mathbf{P}^{n+1}$. Let $L = \mathcal{O}_X(1)$. The restriction map $r : \Omega^1_H\big|_X \to \Omega^1_X$ induces a map

$$\eta : \mathrm{Ext}^1_X(\Omega^1_X, L^\vee) \to \mathrm{Ext}^1_X(\Omega^1_H\big|_X, L^\vee).$$

The following result characterizes in terms of this map η those abstract ribbons $\tilde{X}$ over X with normal bundle L, which can be embedded in the embedded ribbon $2H_{\mathbf{P}^{n+1}} \subset \mathbf{P}^{n+1}$ in a way compatible with the embedding $X \subset \mathbf{P}^n = H$.

(4.4) Lemma (see [38, § 0]). *In the situation of (4.3), consider an element $e \in \mathrm{Ext}^1_X(\Omega^1_X, L^\vee)$, and let $\tilde{X}$ be the ribbon over X defined by e. There exists an inclusion $\tilde{X} \subset 2H_{\mathbf{P}^{n+1}}$ such that $X = \tilde{X} \cap H$ if and only if $\eta(e) = 0$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
e_{2H_{\mathbf{P}^{n+1}}}\big|_X : & 0 \arrow[r] & L^\vee \arrow[r] \arrow[d, equal] & \Omega^1_{\mathbf{P}^{n+1}}\big|_X \arrow[r] \arrow[d, dashed] & \Omega^1_H\big|_X \arrow[r] \arrow[d, equal] & 0 \\
\eta(e) : & 0 \arrow[r] & L^\vee \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \Omega^1_H\big|_X \arrow[r] \arrow[d, "r"] & 0 \\
e : & 0 \arrow[r] & L^\vee \arrow[r] & \Omega^1_{\tilde{X}}\big|_X \arrow[r] & \Omega^1_X \arrow[r] & 0
\end{tikzcd}$$

There exists an inclusion $\tilde{X} \subset 2H_{\mathbf{P}^{n+1}}$ such that $X = \tilde{X} \cap H$ if and only if there exists a dashed arrow such that the diagram commutes, and the latter condition is equivalent to $\eta(e) \cong e_{2H_{\mathbf{P}^{n+1}}}\big|_X$. The result follows by the fact that the ribbon $2H_{\mathbf{P}^{n+1}}$ is split, whence $e_{2H_{\mathbf{P}^{n+1}}} = 0$. $\square$

In particular, Lemma (4.4) tells us that if η is injective, then every ribbon $\tilde{X} \subset 2H_{\mathbf{P}^{n+1}}$ such that $X = \tilde{X} \cap H$ is split.

(4.5) Lemma. *When $X = C$ is a canonical curve (resp. $X = S$ is a $K3$ surface), the restriction to $\bigwedge^2 \mathrm{H}^0(\omega_C)$ of the map η is ${}^\mathsf{T}\Phi_C$ (resp. the map η is 0).*

Proof. In the case of a canonical curve, the stated identification is merely the definition of the Wahl map Φ_C, see (3.3). In the case of a $K3$ surface, the target of the map η is $\mathrm{H}^1(T_{\mathbf{P}^g}|_S(-1))$, which vanishes by (3.5.3). $\square$

(4.6) Let $C \subset \mathbf{P}^{g-1}$ be a canonical curve. The upshot of the previous paragraphs is that the space $\mathbf{P}(\ker({}^\mathsf{T}\Phi_C))$ canonically identifies with the space of isomorphism classes of ribbons $\tilde{C}$ over C for which there may be a surface $S \subset \mathbf{P}^g$ not a cone, such that $\tilde{C} = 2C_S$.

Similarly, for a $K3$ surface $S \subset \mathbf{P}^g$, the space $\mathbf{P}(\mathrm{H}^1(T_S(-1)))$ parametrizes isomorphism classes of ribbons $\tilde{S}$ over S that may come from a threefold $V \subset \mathbf{P}^{g+1}$ not a cone, having S as a hyperplane section. The vanishing of η for $K3$ surfaces tells us that, quite surprisingly, if a ribbon over S has the appropriate normal bundle $\mathcal{O}_S(1)$, there is no obstruction to embed it as an infinitesimal threefold in $\mathbf{P}^{g+1}$ with hyperplane section S.

(4.7) Let us now recall some detail of Wahl's extension construction [43, Proof of Thm 7.1]; it follows Stevens' approach, see, e.g., [35]. Let $C \subset \mathbf{P}^{g-1}$ be a canonical curve with $\mathrm{Cliff}(C) > 2$, hence of genus $g \geqslant 7$. Let $\mathbf{x} = (x_0 : \ldots : x_{g-1})$ be homogeneous coordinates in $\mathbf{P}^{g-1}$ and let $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ be the homogeneous quadratic equations of C in the form of a vector of length m. Since $\mathrm{Cliff}(C) > 2$, we know by [37, 32] that the homogeneous ideal of C has a minimal presentation

$$\mathcal{O}_{\mathbf{P}^{g-1}}(-3)^{\oplus m_1} \xrightarrow{\mathbf{r}} \mathcal{O}_{\mathbf{P}^{g-1}}(-2)^{\oplus m} \xrightarrow{\mathbf{f}} \mathcal{I}_{C/\mathbf{P}^{g-1}} \longrightarrow 0. \tag{4.7.1}$$

Assume now that $g \geqslant 11$. By [3, Theorem 3] one has $\mathrm{H}^1(\mathbf{P}^{g-1}, \mathcal{I}^2_{C/\mathbf{P}^{g-1}}(k)) = 0$ for all $k \geqslant 3$, so that [43, Theorem 7.1] can be applied: Consider a non-zero $v \in \mathrm{coker}(\Phi_C)^\vee$, and let C_v be the ribbon over C corresponding to v; this ribbon C_v lies in $\mathbf{P}^g$ by Lemmas (4.4) and (4.5); the construction in the proof of [43, Theorem 7.1] provides a surface S_v in $\mathbf{P}^g$ such that $C_v = 2C_{S_v}$.

We shall now outline this construction. Because of (3.2.1), $\mathrm{coker}(\Phi_C)^\vee$ is a quotient of $\mathrm{H}^0(C, N_{C/\mathbf{P}^{g-1}}(-1))$; we choose a lift of v with respect to this quotient. The inclusion of

$\mathrm{H}^0(C, N_{C/\mathbf{P}^{g-1}}(-1))$ in $\mathrm{H}^0(C, \mathcal{O}_C(1))^{\oplus m}$ coming from (4.7.1) represents this lift of v as a length m vector $\mathbf{f}_v$ of linear forms on $\mathbf{P}^{g-1}$. The scheme C_v is defined by the equations

$$\mathbf{f}(\mathbf{x}) + t\mathbf{f}_v(\mathbf{x}) = \mathbf{0}, \quad t^2 = 0 \tag{4.7.2}$$

in the g-dimensional projective space with homogeneous coordinates $(\mathbf{x} : t) = (x_0 : \ldots : x_{g-1} : t)$. Wahl proves that there is a vector $\mathbf{h}_v$ of constants such that S_v is defined by the equations

$$\mathbf{f}(\mathbf{x}) + t\mathbf{f}_v(\mathbf{x}) + t^2\mathbf{h}_v = \mathbf{0}. \tag{4.7.3}$$

(4.8) Remark. *Let $C \subset \mathbf{P}^{g-1}$ be a canonical curve of genus $g \geqslant 11$ and Clifford index $\mathrm{Cliff}(C) > 2$. Given a ribbon $v \in \ker({}^{\mathsf{T}}\Phi_C)$ over C, there is a surface $S_v \subset \mathbf{P}^g$ extending it; it is uniquely determined up to the action of a group of projective transformations of $\mathbf{P}^g$ pointwise fixing C, whose tangent space identifies with $\mathrm{H}^0(\omega_C)^\vee$ (see (3.4)).*

This is a mere consequence of [43, Proof of Thm. 7.1]. In a nutshell, the idea is that any 1-extension of C is given by equations as in (4.7.3), where, by sequence (3.2.1), $\mathbf{f}_v$ is determined by v up to an element of $\mathrm{H}^0(\omega_C)^\vee$, i.e. up to an infinitesimal automorphism as in (3.4), which does not change the isomorphism class of the ribbon C_v. Then the extension S_v depends only on the choice of $\mathbf{h}_v$. Now any two such choices differ by an element of $\mathrm{H}^0(C, N_{C/\mathbf{P}^{g-1}}(-2))$ as we recall in (4.9) below, and this space is zero by Lemma (3.6) because $\mathrm{Cliff}(C) > 2$.

(4.9) To justify our affirmations above, let us briefly recall how the vector of constants $\mathbf{h}_v$ may be chosen in [43, Proof of Thm. 7.1]. Set $S = \mathrm{Sym}^\bullet \mathrm{H}^0(C, \omega_C)$, $I_C \subset S$ the homogeneous ideal of C in $\mathbf{P}^{g-1}$, and $S_C = S/I_C$. In terms of graded S-modules, the presentation (4.7.1) writes

$$S(-3)^{\oplus m_1} \xrightarrow{\mathbf{r}} S(-2)^{\oplus m} \xrightarrow{\mathbf{f}} S \longrightarrow S_C \longrightarrow 0. \tag{4.9.1}$$

We need to recall the definition of $T^2_{S_C}$ from [33, § 3.1.2]. Denote by

$$R_C := \ker(\mathbf{f}) = \mathrm{im}(\mathbf{r}) \subset S(-2)^{\oplus m}$$

the *graded module of relations*. It contains the graded submodule R_0 of *trivial* (or *Koszul*) *relations*. An elementary remark shows that R_C/R_0 is killed by I_C and therefore it is an S_C-module. Thus the presentation (4.9.1) induces an exact sequence:

$$R_C/R_0 \longrightarrow S_C(-2)^{\oplus m} \longrightarrow I_C/I_C^2 \longrightarrow 0.$$

The following exact sequence,

$$\begin{array}{c} 0 \to \mathrm{Hom}(I_C/I_C^2, S_C) \to \mathrm{Hom}(S_C^{\oplus m}(-2), S_C) \xrightarrow{\tilde{\mathbf{r}}} \mathrm{Hom}(R_C/R_0, S_C) \to T^2_{S_C} \to 0 \\ \| \\ \mathrm{Hom}(S_C^{\oplus m}, S_C(2)) \end{array} \tag{4.9.2}$$

defines $T^2_{S_C}$.

By flatness of the family of affine schemes over $\mathrm{Spec}(\mathbf{C}[t]/(t^2))$ defined by (4.7.2), the relations $\mathbf{r}$ lift, i.e. there is an $m \times m_1$ matrix of constants $\mathbf{r}_v$ such that

$$(\mathbf{f} + t\mathbf{f}_v)(\mathbf{r} + t\mathbf{r}_v) = 0 \bmod t^2, \quad \text{i.e.} \quad t(\mathbf{f}_v\mathbf{r} + \mathbf{f}\mathbf{r}_v) = 0.$$

Now the vector of constants $\mathbf{h}_v$ is only subject to the condition that the equations (4.7.3) define a flat family of affine schemes over $\operatorname{Spec}(\mathbf{C}[t]/(t^3))$ (this, as in the proof of [43, Proof of Thm. 7.1], eventually ensures flatness over $\operatorname{Spec}(\mathbf{C}[t])$), which in turn boils down to

$$t^2(\mathbf{f}_v\mathbf{r}_v + \mathbf{h}_v\mathbf{r}) = 0. \tag{4.9.3}$$

The map $\mathbf{f}_v\mathbf{r}_v : S^{\oplus m_1} \to S(1) \to S_C(1)$ induces a map belonging to $\operatorname{Hom}(R_C/R_0, S_C)_{-2}$, and condition (4.9.3) is equivalent to

$$\mathbf{h}_v \in \operatorname{Hom}(S^{\oplus m}, S)_0 = \operatorname{Hom}(S_C^{\oplus m}, S_C)_0 = \operatorname{Hom}(S_C^{\oplus m}, S_C(2))_{-2}$$

being a lift of $-\mathbf{f}_v\mathbf{r}_v \in \operatorname{Hom}(R_C/R_0, S_C)_{-2}$ with respect to the map $\tilde{\mathbf{r}}$ in the exact sequence (4.9.2); two such lifts differ by an element of

$$\operatorname{Hom}(I_C/I_C^2, S_C)_{-2} = \mathrm{H}^0(C, N_{C/\mathbf{P}^{g-1}}(-2)) = (0).$$

□

The *existence* of a lift $\mathbf{h}_v$, on the other hand, comes from the identification $(T^2_{S_C})^\vee_{-2} \cong \mathrm{H}^1(\mathbf{P}^{g-1}, \mathcal{I}^2_{C/\mathbf{P}^{g-1}}(-3))$ [43, Cor. 1.6] and the vanishing of the latter cohomology group [3, Thm. 3].

(4.9.4) Observation. In the above proof, as both $\mathbf{f}_v$ and $\mathbf{r}_v$ depend linearly on v, and $\mathbf{h}_v$ is a lift of $-\mathbf{f}_v\mathbf{r}_v$, the vector of constants $\mathbf{h}_v$ depends quadratically on v.

(4.10) Remark. It is not always true that the extension of a ribbon over a canonical curve is unique. Beauville and Mérindol [5, Proposition 3 et Remarque 4] classify the curves C for which there is a $K3$ surface extending the trivial ribbon over the canonical model of C (this indeed contradicts the unicity, as in any event the cone over C extends the trivial ribbon). They show that such a curve is either the normalization of a plane sextic, or a complete intersection of bidegree $(2,4)$ in $\mathbf{P}^3$. In both cases one has $\mathrm{H}^0(C, N_{C/\mathbf{P}^{g-1}}(-2)) \neq 0$; we leave this to the reader.

(4.11) The line of argument of (4.9) may be applied to the more general situation in which C is a curve with Clifford index greater than 2, embedded by the complete linear system of an arbitrary very ample line bundle L, with the proviso that the multiplication map μ_{L,ω_C} is surjective, which is equivalent to the condition that C has positive genus. If the multiplication map is not surjective, then the relation between $\mathrm{H}^0(N_C \otimes L^\vee)$ and $\ker({}^\mathsf{T}\Phi_{L,\omega_C})$ is more complicated (see (3.3)), and indeed for rational normal curves of degree $d > 3$, there exist ribbons with several extensions, see [44, p. 276]. For non-linearly normal curves, the same problem may also appear in positive genus, e.g., for hyperplane sections of irregular scrolls.

5 – Wahl maps and extensions of canonical curves

This Section is devoted to the proof of our main extension result, Theorem (2.1), and its variant (2.2). We start by recalling the following auxiliary result.

(5.1) Theorem. *Let $X \subset \mathbf{P}^m$ be a variety of dimension n having a linear section which is a canonical curve. Then X is arithmetically Gorenstein, normal, and has canonical sheaf $\omega_X \cong \mathcal{O}_X(2-n)$.*

This theorem is clear for $n = 1$ and follows in general by the hyperplane principle; proofs in the cases $n = 2, 3$ may be found in [18], [16] respectively.

(5.2) For the rest of the Section, we let $C \subset \mathbf{P}^{g-1}$ be a canonical curve of genus $g \geqslant 11$ and Clifford index $\mathrm{Cliff}(C) > 2$. We choose a section

$$v \in \ker({}^{\mathsf{T}}\Phi_C) \longmapsto \mathbf{f}_v \in \mathrm{H}^0(C, N_{C/\mathbf{P}^{g-1}}(-1)) \tag{5.2.1}$$

of the extension (3.2.1) (for $L = \omega_C$) of vector spaces, and fix homogeneous coordinates $(\mathbf{x} : t) = (x_0 : \ldots : x_{g-1} : t)$ on $\mathbf{P}^g$, so that for all $v \in \ker({}^{\mathsf{T}}\Phi_C)$ there is a uniquely determined extension S_v of the ribbon C_v in $\mathbf{P}^g$, given by equations (4.7.3).

(5.3) Lemma. *Let $v \in \ker({}^{\mathsf{T}}\Phi_C)$, $v \neq 0$, and $\lambda \in \mathbf{C}^*$. The surface $S_{\lambda v}$ is obtained by applying to S_v the projective transformation $\omega_{\lambda^{-1}} : (\mathbf{x} : t) \mapsto (\mathbf{x} : \lambda^{-1} t)$.*

Proof. By linearity of the map (5.2.1), the equations of $S_{\lambda v}$ are

$$\mathbf{f}(\mathbf{x}) + \lambda t \mathbf{f}_v(\mathbf{x}) + t^2 \mathbf{h}_{\lambda v} = \mathbf{0}.$$

Then the equations of the surface $\omega_\lambda(S_{\lambda v})$ are

$$\mathbf{f}(\mathbf{x}) + t \mathbf{f}_v(\mathbf{x}) + \frac{t^2}{\lambda^2} \mathbf{h}_{\lambda v} = \mathbf{0}.$$

The surface $\omega_\lambda(S_{\lambda v})$ thus contains C_v, and therefore coincides with S_v. $\square$

We remark that the above proof shows that $\mathbf{h}_v$ depends quadratically on v, thus giving another justification to our Observation (4.9.4).

(5.4) Proposition. *Set* $\mathrm{cork}(\Phi_C) = r + 1$ *and* $\mathbf{P}^r = \mathbf{P}(\ker({}^{\mathsf{T}}\Phi_C))$*. There is a diagram*[1]

$$\begin{tikzcd} \mathcal{S} \arrow[dr, "p"'] & \subset \mathbf{P}(\mathcal{O}_{\mathbf{P}^r}^{\oplus g} \oplus \mathcal{O}_{\mathbf{P}^r}(1)) \arrow[d, "\pi"] \\ & \mathbf{P}^r \end{tikzcd} \tag{5.4.1}$$

where $p : \mathcal{S} \to \mathbf{P}^r$ is a flat family of surfaces such that:
(i) the intersection of $\mathcal{S}$ with $\mathbf{P}(\mathcal{O}_{\mathbf{P}^r}^{\oplus g}) \cong \mathbf{P}^{g-1} \times \mathbf{P}^r$ is equal to $C \times \mathbf{P}^r$;
(ii) for any $\xi = [v] \in \mathbf{P}^r$, the inclusion $\mathcal{S}_\xi \subset \mathbf{P}(\mathcal{O}_{\mathbf{P}^r}^{\oplus g} \oplus \mathcal{O}_{\mathbf{P}^r}(1))_\xi \cong \mathbf{P}^g$ of fibres of p and π, is the extension S_v of $C = \mathcal{S} \cap \mathbf{P}(\mathcal{O}_{\mathbf{P}^r}^{\oplus g})_\xi$.

Proof. For simplicity we will do the case $r = 1$, the general case being similar. Let v_0, v_1 be a basis of $\mathrm{coker}(\Phi_C)^\vee$. For $i = 0, 1$, consider the diagrams

$$\begin{tikzcd} \mathcal{S}_i \arrow[dr, "p_i"'] & \subset \mathbf{P}^g \times \mathbf{A}^1 \arrow[d, "\pi_i"] \\ & \mathbf{A}^1 \end{tikzcd} \tag{5.4.2}$$

given by the equations

$$\mathbf{f}(\mathbf{x}) + t\mathbf{f}_{v_0 + a_1 v_1}(\mathbf{x}) + t^2 \mathbf{h}_{v_0 + a_1 v_1} = \mathbf{0}, \quad \text{resp.} \quad \mathbf{f}(\mathbf{x}) + t\mathbf{f}_{a_0 v_0 + v_1}(\mathbf{x}) + t^2 \mathbf{h}_{a_0 v_0 + v_1} = \mathbf{0},$$

where a_1, resp. a_0, are affine coordinates on $\mathbf{A}^1$. By Lemma (5.3), these two diagrams are isomorphic over $\mathbf{A}^1 - \{0\}$ via the map $([\mathbf{x} : t], a_1) \in \mathcal{S}_0 \mapsto ([\mathbf{x} : a_1 t], 1/a_1) \in \mathcal{S}_1$. Diagram (5.4.1) is obtained by glueing the diagrams (5.4.2) via this map. $\square$

[1] Beware that here $\mathbf{P}(\mathcal{O}_{\mathbf{P}^r}^{\oplus g} \oplus \mathcal{O}_{\mathbf{P}^r}(1))$ denotes the projective bundle of one-dimensional quotients, whereas everywhere else in the text we use the classical notation for projective spaces.

(5.5) Corollary. *Under the assumptions of Proposition (5.4), there is an arithmetically Gorenstein normal variety X of dimension $r+2$ in $\mathbf{P}^{g+r}$ with $\omega_X \cong \mathcal{O}_X(-r)$, not a cone, having C as a linear section, and satisfying the following property: the surface linear sections of X containing C are in $1:1$ correspondence with the surface extensions of C in $\mathbf{P}^g$ that are not cones.*

Proof. We keep the notation of Proposition (5.4). The $\mathcal{O}(1)$ bundle of $\mathbf{P}(\mathcal{O}_{\mathbf{P}^r}^{\oplus g} \oplus \mathcal{O}_{\mathbf{P}^r}(1))$ defines a morphism ϕ to $\mathbf{P}^{g+r}$ which is the blow-up of $\mathbf{P}^{g+r}$ along the $\mathbf{P}^{g-1}$ image of the trivial subbundle $\mathbf{P}(\mathcal{O}_{\mathbf{P}^r}^{\oplus g})$. Let $X = \phi(\mathcal{S})$. The map $\phi_{|\mathcal{S}}$ is the contraction of $C \times \mathbf{P}^r = \mathcal{S} \cap \mathbf{P}(\mathcal{O}_{\mathbf{P}^r}^{\oplus g})$ to $C \subset \mathbf{P}^{g-1} \subset \mathbf{P}^{g+r}$. The fibres of p are isomorphically mapped to the sections of X with the $\mathbf{P}^g$'s containing the $\mathbf{P}^{g-1}$. None of these surfaces is a cone, because the corresponding first order extensions of C on them are non-trivial. Therefore X is not a cone. The property that the surface linear sections of X containing C are in $1:1$ correspondence with the surface extension of C in $\mathbf{P}^g$ other than cones follows from assertion (ii) in Proposition (5.4) and Remark (4.8). The rest of the assertions follows by Theorem (5.1). □

(5.6) Corollary. *Consider a non-negative integer r such that* $\operatorname{cork}(\Phi_C) \geqslant r+1$.
(5.6.1) There is an arithmetically Gorenstein normal variety $Y \subset \mathbf{P}^{g+r}$ of dimension $r+2$ with $\omega_Y \cong \mathcal{O}_Y(-r)$, not a cone, having C as a curve section with $\mathbf{P}^{g-1}$.
(5.6.2) Assume $r > 0$. If there is a surface section of Y with at worst ADE singularities, then the general threefold section V of Y has canonical singularities.

Note that (5.6.2) applies to a universal extension of C as soon as C sits on a $K3$ surface with at worst ADE singularities.

Proof. Assertion (5.6.1) follows directly from the previous Corollary (5.5): take Y a linear section of X containing C of the appropriate dimension. To prove (5.6.2), we note that by the argument in [30, Introduction], if V had non-canonical singularities, it would be a cone, a contradiction. □

6 – Integration of ribbons over $K3$ surfaces

In this Section we prove Theorem (2.17), to the effect that any ribbon on a $K3$ surface in $\mathbf{P}^g$ may be integrated to a unique threefold in $\mathbf{P}^{g+1}$ (under suitable assumptions). It will be deduced from the integrability of ribbons on canonical curves by a hyperplane principle. Key to this principle is the relation between ribbons over a variety in projective space and over its hyperplane sections, as explained in the following paragraph.

(6.1) In this paragraph, we use without further reference the notions and results recalled in Section 4. Let S be a smooth $K3$ surface in $\mathbf{P}^g$, and C a smooth hyperplane section of S. Recall that we denote by $[2C_S] \in \mathbf{P}(\ker({}^{\mathsf{T}}\Phi_C)) \subseteq \mathbf{P}(\mathrm{H}^1(T_C(-1))$ the ribbon over C, considered up to isomorphism, given by its being a hypersurface in S.

Let $[\tilde{S}] \in \mathbf{P}(\mathrm{H}^1(T_S(-1)))$. This is the isomorphism class of the ribbon $\tilde{S} \subset \mathbf{P}^{g+1}$ over S, contained in the ribbon $2(H_S)_{\mathbf{P}^{g+1}}$ over the hyperplane $H_S = \langle S \rangle$, and such that $\tilde{S} \cap H_S = S$.

Now for any hyperplane $H \subset \mathbf{P}^{g+1}$ containing C, $H \neq H_S$, the intersection $H \cap \tilde{S}$ is a ribbon C_H over C in $H \cong \mathbf{P}^g$, contained in the ribbon $2\langle C \rangle_H$ over $\langle C \rangle \cong \mathbf{P}^{g-1}$, and such that $C_H \cap \langle C \rangle = C$. As such, it determines a point of $\mathbf{P}(\ker({}^{\mathsf{T}}\Phi_C))$.

Thus, the pencil of hyperplanes of $\mathbf{P}^{g+1}$ containing C defines a line in $\mathbf{P}(\ker({}^{\mathsf{T}}\Phi_C))$ passing through the point $[2C_S]$; in other words, $\tilde{S}$ defines a point $\mathfrak{l}_{\tilde{S}} \in \mathbf{P}\big(\ker({}^{\mathsf{T}}\Phi_C)\big)/[2C_S]$.[2] We are abusing terminology here, as this "line" may actually be reduced to the sole point $[2C_S]$ if all H containing C cut out the same ribbon over C on $\tilde{S}$, and in this case $\mathfrak{l}_{\tilde{S}}$ is not well-defined; it will be a consequence of (6.2) below that this does not happen.

(6.2) *Proof of the existence part of Theorem (2.17).* We identify S with its image in $\mathbf{P}^g = |L|^\vee$. Choose any smooth hyperplane section C of S. It satisfies the same assumptions as S on the genus and Clifford index, so we may consider its universal extension $X \subset \mathbf{P}^{g+r}$ constructed in Corollary (5.5), with

$$r = \operatorname{cork}(\Phi_C) - 1 = h^1(T_S(-1)),$$

the second equality in this equation coming from Corollary (2.8). By Corollary (5.5), we may consider S as a linear section of X.

Now, every linear $(g+1)$-subspace Λ of $\mathbf{P}^{g+r}$ containing S cuts out a threefold X_Λ on X having S as a hyperplane section, hence determines a ribbon $2S_\Lambda := 2S_{X\cap\Lambda} \in \mathrm{H}^1(T_S(-1))$, which in turn determines a point of $\mathbf{P}\big(\ker({}^{\mathsf{T}}\Phi_C)\big)/[2C_S]$ via the mechanism described in (6.1). We thus have a composed map

$$\text{(6.2.1)} \qquad \psi_S : \Lambda \in \mathbf{P}^{g+r}/\langle S\rangle \longmapsto [2S_\Lambda] \in \mathbf{P}(\mathrm{H}^1(T_S(-1))) \longmapsto \mathfrak{l}_{2S_\Lambda} \in \mathbf{P}\big(\ker({}^{\mathsf{T}}\Phi_C)\big)/[2C_S],$$

albeit maybe only defined so far on a (possibly empty!) Zariski open subset of $\mathbf{P}^{g+r}/\langle S\rangle$ because of the abuse of terminology mentioned in (6.1).

We claim that the universality of X implies the surjectivity of ψ_S. Consider a point of $\mathbf{P}\big(\ker({}^{\mathsf{T}}\Phi_C)\big)/[2C_S]$, and represent it as a point $[\tilde{C}]$ of $\mathbf{P}(\ker({}^{\mathsf{T}}\Phi_C))$ distinct from $[2C_S]$. The universality of X tells us that there exists a linear g-subspace Γ of $\mathbf{P}^{g+r}$ such that $\tilde{C} = 2C_{X\cap\Gamma}$. Then $\Lambda := \langle\Gamma, S\rangle$ is a $(g+1)$-subspace of $\mathbf{P}^{g+r}$ such that, by construction, $\psi_S(\Lambda) = [\tilde{C}]$. This proves our claim.

Now note that in diagram (6.2.1), all three projective spaces have the same dimension $r-1$, and the two maps whose composition is ψ_S are linear. Therefore, the map ψ_S may be surjective only if it is an isomorphism, and the two maps in (6.2.1) are isomorphisms as well. We conclude by observing that the surjectivity of the first map in (6.2.1) tells us that for every isomorphism class of ribbons $[\tilde{S}] \in \mathbf{P}(\mathrm{H}^1(T_S(-1)))$ there is a threefold $X \cap \Lambda$ such that $[\tilde{S}] = [2S_{X\cap\Lambda}]$. $\square$

(6.3) *Proof of the unicity part of Theorem (2.17).* Consider two threefold extensions V and V' of S such that the two corresponding ribbons $2S_V$ and $2S_{V'}$ are proportional. It follows from the considerations in (6.1) and the unicity of integrals of ribbons over canonical curves (Remark (4.8)), that the two threefolds V and V' respectively contain two isomorphic pencils of hyperplane sections, and this implies that they are isomorphic. $\square$

(6.4) Remark. In the case of the trivial ribbon, the conclusion of (6.3) is that if a $K3$ surface as in Theorem (2.17) sits on a threefold $V \subset \mathbf{P}^{g+1}$, not a cone, then the conormal exact sequence of S in V is not split. By reproducing the argument of [5, Proposition 3], this implies that there does not exist any automorphism of V of order 2 and with S as fix locus.

[2] Here and in the rest of this Section, we use the following non-standard but convenient notation: if W is a vector subspace of V, we write $\mathbf{P}(V)/\mathbf{P}(W)$ for $\mathbf{P}(V/W)$.

7 – Study of the moduli maps

This Section contains the building blocks of the proofs of Theorems (2.6) and (2.19). The following Corollary of Theorems (0.2) and (2.17) comes in a straightforward manner once one understands the latter Theorems as integration results for ribbons.

(7.1) Corollary. *Let* $(S, C) \in \mathcal{KC}_g$ *(resp.* $(V, S) \in \mathcal{FS}_g$*) be such that* $\mathrm{Cliff}(C) > 2$ *(resp.* $\mathrm{Cliff}(S, -K_V|_S) > 2$*). Then*

$$\dim(c_g^{-1}(C)) \geqslant \mathrm{cork}(\Phi_C) - 1 \qquad \textit{(resp. } \dim(s_g^{-1}(S)) \geqslant h^1\big(T_S(-1)\big) - 1\textit{)}.$$

Proof. Consider the family $p : \mathcal{S} \to \mathbf{P}(\ker({}^\mathsf{T}\Phi_C))$ constructed in Proposition (5.4). The $K3$ surface S is a fibre of p, so the fibre $S_{[v]}$ of p over the general $[v] \in \mathbf{P}(\ker({}^\mathsf{T}\Phi_C))$ is a $K3$ surface as well, hence gives rise to a point $(S_{[v]}, C) \in c_g^{-1}(C)$. We claim that these points are pairwise distinct, from which the assertion follows at once.

Let $[v], [v']$ be two distinct points of $\mathbf{P}(\ker({}^\mathsf{T}\Phi_C))$. If $S_{[v]}$ and $S_{[v']}$ are not isomorphic, then the claim is trivial; else, we may assume $S_{[v]} = S_{[v']}$, and call this surface S_0. There are two copies $C_{[v]}$ and $C_{[v']}$ of C in S_0, and since $[v] \neq [v']$, the respective infinitesimal neighbourhoods of $C_{[v]}$ and $C_{[v']}$ in S_0 are not isomorphic, which implies that $C_{[v]}$ and $C_{[v']}$ correspond to two distinct points of the linear system $|\mathcal{O}_{S_0}(C)|$ and there is no automorphism of S_0 sending one of the two curves to the other. This proves the first instance of the statement.

The proof of the second instance is exactly the same, after one notes that there exists a family $p : \mathcal{V} \to \mathbf{P}(H^1(T_S(-1)))$ with properties analogous to those of the previous family $p : \mathcal{S} \to \mathbf{P}(\ker({}^\mathsf{T}\Phi_C))$, as follows from the arguments in Section 6: this is Theorem (2.18)! □

The two following results bound from above the dimensions of the kernels of the differentials of c_g and s_g.

(7.2) Lemma (see [33], § 3.4.4). *Let* $(S, C) \in \mathcal{KC}_g$ *(resp.* $(V, S) \in \mathcal{FS}_g$*). The kernel of the differential of* c_g *at* (S, C) *(resp. of* s_g *at* (V, S)*) is* $\mathrm{H}^1\big(T_S(-C)\big)$ *(resp.* $\mathrm{H}^1\big(T_V(-S)\big)$*).*

(7.3) Proposition. *Let* $(S, C) \in \mathcal{KC}_g$ *(resp.* $(V, S) \in \mathcal{FS}_g$*) be such that* $\mathrm{Cliff}(C) > 2$ *(resp.* $\mathrm{Cliff}(S, -K_V|_S) > 2$*). Then*

$$h^1(T_S(-1)) + 1 \leqslant \mathrm{cork}(\Phi_C) \qquad \textit{(resp. } h^1(T_V(-1)) + 1 \leqslant h^1(T_S(-1))\textit{)}.$$

Proof. We prove only the first instance of the statement, the other one being entirely similar. The curve C is the complete intersection of $S \subset \mathbf{P}^g$ with a hyperplane $H \cong \mathbf{P}^{g-1}$, so one has

$$N_{C/\mathbf{P}^{g-1}} \cong N_{S/\mathbf{P}^g}\big|_C . \tag{7.3.1}$$

By Lemma (3.6) one has $\mathrm{H}^0(N_{S/\mathbf{P}^g}(-2)) = 0$, so one deduces from the twisted restriction exact sequence that

$$h^0(N_{S/\mathbf{P}^g}(-1)) \leqslant h^0(N_{S/\mathbf{P}^g}\big|_C(-1)). \tag{7.3.2}$$

We may now conclude:

$$\begin{aligned}
\operatorname{cork}(\Phi_C) + g &= h^0(N_{C/\mathbf{P}^{g-1}}(-1)) && \text{by Lemma (3.2)}\\
&= h^0(N_{S/\mathbf{P}^g}(-1)\big|_C) && \text{by (7.3.1)}\\
&\geqslant h^0(N_{S/\mathbf{P}^g}(-1)) && \text{by (7.3.2)}\\
&= h^1(T_S(-1)) + g + 1 && \text{by Lemma (3.5).}
\end{aligned}$$

□

8 – A general bound on the corank of the Wahl map

In this Section we prove that under our usual assumptions, a given canonical curve can be integrated to a given $K3$ surface in only finitely many ways, and use this to bound the corank of the Wahl maps of the curves that sit on a $K3$ surface. We first recall the two following results from [18].

(8.1) Theorem [18, p. iii]. *Let S be a non-degenerate projective surface in $\mathbf{P}^g$, having as a hyperplane section a smooth canonical curve $C \subset \mathbf{P}^{g-1}$ of genus $g \geqslant 3$. Then only the following cases are possible:*
(i) S is a $K3$ surface with canonical singularities;
(ii) S is a rational surface with a minimally elliptic singularity, plus perhaps canonical singularities;
(iii) S is a ruled surface over a curve of genus $q \geqslant 1$ with only one singularity of genus $q+1$, *plus perhaps canonical singularities;*
(iv) S is a ruled surface over a curve of genus $q = 1$, with two simple elliptic singularities, plus perhaps canonical singularities.

Surfaces of type (ii)–(iv) are fake K3 surfaces; their Kodaira dimension is $-\infty$.

(8.2) Proposition [18, Theorem 2.1, p. 38]. *Assume we are in one of the cases (iii)–(iv) of Theorem (8.1). Let $\mu : S \to \Sigma$ be a minimal model of S; it has a structure of $\mathbf{P}^1$-bundle $f : \Sigma \to D$, where D is a smooth curve of genus q. If the image of C in Σ is a section of f, then S is a cone over C.*

(8.3) Let $C \subset \mathbf{P}^{g-1}$ be a smooth canonical curve of genus $g \geqslant 11$ with $\operatorname{Cliff}(C) > 2$, set $r + 1 = \operatorname{cork}(\Phi_C)$ and $\mathbf{P}^r = \mathbf{P}(\operatorname{coker}(\Phi_C)^\vee)$, and assume $r \geqslant 0$.

Consider the flat family $p : \mathcal{S} \to \mathbf{P}^r$ constructed in Proposition (5.4). Note that no surface of this family is a cone over C. Suppose that the general member of this family is a $K3$ surface, possibly with ADE singularities. Then we have the rational modular map

$$s : \mathbf{P}^r \dashrightarrow \mathcal{K}_g^{\mathrm{can}},$$

whose indeterminacy locus Z consists of the points $[v] \in \mathbf{P}^r$ such that the corresponding extension S_v of C is a fake $K3$ surface. So s is defined on the dense Zariski open subset $U = \mathbf{P}^r - Z$.

(8.4) Proposition. *The morphism $s|_U : U \to \mathcal{K}_g^{\mathrm{can}}$ has finite fibres.*

Proof. We argue by contradiction, and suppose there is an irreducible curve $\gamma \subset U$ such that $s(\gamma)$ is a point. Let us first rule out the possibility that s be defined everywhere on

$\mathbf{P}^r = \mathbf{P}(\mathrm{coker}(\Phi_C)^\vee)$, i.e. $Z = \varnothing$. In this case, since s contracts a curve it must be constant. Then there would exist a $K3$ surface S containing a complete 1-dimensional family of curves all pairwise isomorphic. Since $\mathrm{Pic}(S)$ is discrete, all these curves belong to the same linear equivalence class $[C] \in \mathrm{Pic}(S)$. But since the discriminant locus in $|C|$ is a hypersurface, it is impossible to have a complete positive-dimensional family of smooth curves in $|C|$, and we have a contradiction.

We may thus assume that the locus of indeterminacy Z of s is non-empty. Let Γ be the Zariski closure of γ in $\mathbf{P}^r$. The rational map s is defined by a linear system on $\mathbf{P}^r$ with base locus Z. The curve Γ necessarily intersects the divisors in this linear system, and since it is contracted by s the intersection must be contained in Z. We may thus consider a point $\xi \in \Gamma \cap Z$. Looking at the normalization of Γ at ξ, we see that there is an analytic morphism $\nu : \mathbf{D} \to \Gamma$, where $\mathbf{D}$ is the complex unit disc and $\nu(0) = \xi$. By pulling back $p : \mathcal{S} \to \mathbf{P}^r$ to $\mathbf{D}$, we find a flat family $p' : \mathcal{S}' \to \mathbf{D}$ which is isotrivial over $\mathbf{D} - \{0\}$, with general fibre a $K3$ surface S_1, and central fibre a fake $K3$ surface S_0. Since the automorphisms of S_1 as a polarised surface have finite order we may assume, up to performing a finite base change, that p' is actually trivial over $\mathbf{D} - \{0\}$. Moreover there is an inclusion:

$$\begin{tikzcd} \mathbf{D} \times C \arrow[dr, "\mathrm{pr}_1"'] & \subset & \mathcal{S}' \arrow[d, "p'"] \\ & & \mathbf{D} \end{tikzcd}$$

Now consider a semistable reduction $\tilde{p} : \tilde{S} \to \mathbf{D}$ of $p' : \mathcal{S}' \to \mathbf{D}$; we still have the inclusion:

$$\begin{tikzcd} \mathbf{D} \times C \arrow[dr, "\mathrm{pr}_1"'] & \subset & \tilde{\mathcal{S}} \arrow[d, "\tilde{p}"] \\ & & \mathbf{D} \end{tikzcd} \tag{8.4.1}$$

The central fibre of $\tilde{p}$ consists of the proper transform $\tilde{S}_0$ of the central fibre S_0 of p, plus possibly other components. The central fibre C of the trivial family $\mathbf{D} \times C \subset \tilde{\mathcal{S}}$ sits on $\tilde{S}_0$, and is entirely contained in the smooth locus of the central fibre of $\tilde{p}$.

Since p' is trivial over $\mathbf{D} - \{0\}$, so is $\tilde{p}$. This implies that there is a diagram:

$$\begin{tikzcd} \tilde{\mathcal{S}} \arrow[r, dashed, "\psi"] \arrow[dr, "\tilde{p}"'] & \mathbf{D} \times S_1 \arrow[d, "\mathrm{pr}_1"] \\ & \mathbf{D} \end{tikzcd}$$

where ψ is a birational map contracting all components of the central fibre of $\tilde{p}$ but one, and S_1 is the general fibre of p'. By composing the inclusion (8.4.1) with ψ, we still have an inclusion:

$$\begin{tikzcd} \mathbf{D} \times C \arrow[dr, "\mathrm{pr}_1"'] & \subset & \mathbf{D} \times S_1 \arrow[d, "\mathrm{pr}_1"] \\ & & \mathbf{D} \end{tikzcd} \tag{8.4.2}$$

We claim that $\tilde{S}_0$ has to be contracted by ψ. Indeed, being birational to S_0 which is a fake $K3$ surface, $\tilde{S}_0$ has Kodaira dimension $-\infty$, whereas S_1 is a genuine $K3$ surface. On the other hand, because of the inclusion (8.4.2), $\tilde{S}_0$ has to be contracted to a curve isomorphic to C. This implies that $\tilde{S}_0$ is a ruled surface over C, and so is S_0. Consider a minimal model $\mu : S_0 \to \Sigma$ of S_0 (and of $\tilde{S}_0$ as well). Then Σ is a $\mathbf{P}^1$-bundle $f : \Sigma \to C$, and the image of C to Σ via μ is a section of $f : \Sigma \to C$. By Proposition (8.2), S_0 must be a cone over C, a contradiction. □

(8.5) Corollary. *Let C be a canonical curve of genus $g \geqslant 11$ in $\mathbf{P}^{g-1}$, with Clifford index* $\mathrm{Cliff}(C) > 2$. *If C is a hyperplane section of a $K3$ surface S (possibly with ADE singularities) in $\mathbf{P}^g$, then* $\mathrm{cork}(\Phi_C) \leqslant 20$.

Proof. By Proposition (8.4), there is a rational map $s : \mathbf{P}(\ker({}^\mathsf{T}\Phi_C)) \dashrightarrow \mathcal{K}_g^{\mathrm{can}}$ which is generically finite on its image. Therefore $\mathrm{cork}(\Phi_C) - 1 \leqslant \dim(\mathcal{K}_g^{\mathrm{can}}) = 19$. □

(8.6) Corollary (Proposition (2.4)). *Let $(S, C) \in \mathcal{KC}_g^{\mathrm{can}}$ with $g \geqslant 11$ and* $\mathrm{Cliff}(C) > 2$. *There are only finitely many members C' of $|\mathcal{O}_S(C)|$ that are isomorphic to C.*

Proof. Assume by contradiction that there is an infinite family (C_i) of curves isomorphic to C in $|\mathcal{O}_S(C)|$. By Proposition (8.4), we may furthermore assume that the curves C_i all have the same ribbon in S. We consider the pair (S, C) embedded in $\mathbf{P}^g$. Taking C as a common canonical model for all the curves C_i, we obtain a family (S_i) of surfaces in $\mathbf{P}^g$, such that each S_i is the image of S by a projectivity of $\mathbf{P}^g$ fixing C. By unicity of the integration of ribbons, see Remark (4.8), we must have $S_i = S$ for all i, and it follows that S has infinitely many projective automorphisms, a contradiction. □

(8.7) Remark. It is claimed in [11, Proposition 1.2 and Corrigendum] that for a smooth curve C sitting on a $K3$ surface S, one has $\mathrm{H}^0(C, T_S|_C) = 0$; this would imply the injectivity of the coboundary map

$$\partial : \mathrm{H}^0(C, N_{C/S}) \to \mathrm{H}^1(C, T_C)$$

induced by the conormal exact sequence of C in S.

Let $|\mathcal{O}_S(C)|^\circ$ be the Zariski open subset of $|\mathcal{O}_S(C)|$ parametrizing smooth members of the linear system, and $c : |\mathcal{O}_S(C)|^\circ \to \mathcal{M}_g$ be the morphism mapping a smooth member C' of $|\mathcal{O}_S(C)|$ to its modulus in $\mathcal{M}_g$. Since ∂ is the differential of c, the injectivity of ∂ would be a stronger result than Corollary (8.6).

However, there exist smooth curves C sitting on $K3$ surfaces S for which the conormal exact sequence is split [5, Proposition 3 et Remarque 4], see also Remark (4.10). For such pairs (C, S), the boundary map ∂ is downright zero. This shows that there is a problem with the claim of [11, Proposition 1.2 and Corrigendum]. This problem does not affect the results of [ibid.].

Note that for a pair (C, S) with split conormal sequence as above, the assumptions of Proposition (8.4), described in (8.3), are not verified: one has $\mathrm{Cliff}(C) = 2$, as C carries either a g^1_4 or a g^2_6 [5, Remarque 4], see also Remark (4.10).

9 – Plane curves with ordinary singularities

In this Section we construct an extension of plane curves with $a \leqslant 9$ ordinary singularities to an $(11 - a)$-dimensional variety, and thus give a lower bound on the coranks of their respective Gauss maps. The construction proposed in the following proposition is not new, see, e.g., [18].

(9.1) Proposition. *Let $C \subset \mathbf{P}^2$ be an integral curve with $a \leqslant 9$ singular points in general position, such that a simple blow-up of $\mathbf{P}^2$ at these a points resolves the singularities of C (e.g., C has a ordinary singular points in general position and no other singularities). Assume moreover that C has genus $g \geqslant 3$. There is a family of dimension $9 - a$ of mutually non-isomorphic surfaces in $\mathbf{P}^g$ having the canonical image of the resolution of C as a hyperplane section.*

Proof. Let $C \subset \mathbf{P}^2$ be an integral curve of degree d satisfying the assumptions of the Proposition; we call $p_1, \ldots, p_a$ its singular points, and $m_1, \ldots, m_a > 1$ the respective multiplicities of C at these points. Let T be a smooth cubic passing through $p_1, \ldots, p_a$. We call $p_{a+1}, \ldots, p_h$ the intersection points, possibly infinitely near, of T and C off $p_1, \ldots, p_a$, and set $m_{a+1} = \cdots = m_h = 1$, so that $\sum_{i=1}^h m_i = 3d$.

Consider the blow-up $\sigma_T : \tilde{\mathbf{P}}_T \to \mathbf{P}^2$ at all the intersection points $p_1, \ldots, p_h$ of T and C, and call E_i the exceptional divisor over the point p_i (note that it is a chain of reduced rational curves, such that $E_i^2 = -1$). The proper transform C_T of C is smooth by assumption, and disjoint from the proper transform $\hat{T}$ of T. The curve $\hat{T}$ is an anticanonical divisor on $\tilde{\mathbf{P}}_T$. Let H be the line class on $\mathbf{P}^2$, and consider the linear system

(9.1.1) $\big|C_T\big| = \big|d \cdot \sigma_T^* H - \sum_{i=1}^h m_i E_i\big| = \big|(d-3)\sigma_T^* H - \sum_{i=1}^h (m_i - 1) E_i + \hat{T}\big| = \big|K_{\tilde{\mathbf{P}}_T} + C_T + \hat{T}\big|.$

It restricts to the complete canonical series on C_T, and defines a birational map $\phi_T : \tilde{\mathbf{P}}_T \dashrightarrow \mathbf{P}^g$. It follows that the image surface $S_T = \phi_T(\tilde{\mathbf{P}}_T)$ is an extension of the canonical model of the resolution of C. The curve $\hat{T}$ is contracted to an elliptic singularity by ϕ_T.

Now the cubic curves passing through $p_1, \ldots, p_a$ form a linear system of dimension $9 - a$, and the generic such cubic is smooth. The Proposition therefore follows from the fact that two different choices of T give two non-isomorphic surfaces S_T, which is the content of Lemma (9.2) below. □

(9.2) Lemma. *Maintain the notation of the proof of Proposition (9.1), and let T' be another cubic satisfying the same assumptions as T. The surfaces S_T and $S_{T'}$ are not isomorphic.*

Proof. The cubics through $p_1, \ldots, p_a$ cut a base-point-free g_{h-a}^{9-a} on the normalisation of C. On the other hand, S_T and $S_{T'}$ each have $h - a$ lines, corresponding respectively to the simple base points of the linear systems $\sigma_{T*}|C_T|$ and $\sigma_{T'*}|C_{T'}|$ on the plane.

If S_T and $S_{T'}$ are isomorphic, the two elliptic curves T and T' are isomorphic as well. Also, the isomorphism $S_T \cong S_{T'}$ sends the aforementioned lines on S_T to their counterparts on $S_{T'}$. This implies that T and T' cut out the same member of the g_{h-a}^{9-a} on the normalisation of C, hence coincide. □

Note that if $h \geqslant 19$ (recall that h is the number of points in the set-theoretic intersection $C \cap T$), the surfaces S_T are neither $K3$ surfaces nor limits of such, because in this case the curve $\hat{T}$ is contracted by ϕ_T to an elliptic singularity which is not smoothable; see [3] and the references therein for more details.

(9.3) Corollary. *Let $C \subset \mathbf{P}^2$ be an integral curve of geometric genus $g \geqslant 11$, with $a \leqslant 9$ singular points in general position, such that a simple blow-up of $\mathbf{P}^2$ at these a points resolves the singularities of C. Let $\bar{C}$ be the normalization of C. One has*

(9.3.1) $$\operatorname{cork}(\Phi_{\bar{C}}) \geqslant 10 - a.$$

Proof. We first prove (9.3.1) under the assumption that $\operatorname{Cliff}(C) > 2$. By Proposition (9.1), there is a family of dimension $9 - a$ of mutually non-isomorphic extensions of the canonical model of $\bar{C}$. By Remark (4.8), these correspond to mutually non-isomorphic ribbons $\tilde{C}$ over $\bar{C} \subset \mathbf{P}^{g-1}$, and it follows that $\mathbf{P}(\ker({}^{\mathsf{T}}\Phi_C))$ has dimension at least $9 - a$, see (4.6).

On the other hand, if $\operatorname{Cliff}(C) \leqslant 2$, then $\bar{C}$ is either hyperelliptic, trigonal, or tetragonal, since $g \geqslant 11$. In all these cases, it is known that $\operatorname{cork}(\Phi_{\bar{C}}) \geqslant 10 - a$ by the results quoted in (2.15), except possibly if $a = 0$ and $\bar{C}$ is tetragonal; in the latter case C is necessarily a smooth plane quintic, in contradiction with the assumption $g \geqslant 11$. □

(9.4) Conjecture [42, p. 80]. *Let S be a regular surface. There should exist an integer g_0 such that for every non-singular curve $C \subset S$ of genus $g \geqslant g_0$, one has*

$$\operatorname{cork}(\Phi_C) \geqslant \mathrm{h}^0(S, \omega_S^{-1}).$$

In the same article, this conjecture is proved for $S = \mathbf{P}^2$ [42, Thm. 4.8].

(9.5) Proposition. *The Wahl Conjecture (9.4) holds for any blow-up of the projective plane having an anticanonical curve, e.g., when S is the projective plane blown-up at $a \leqslant 9$ points in general position.*

Proof. Let $\varepsilon : S \to \mathbf{P}^2$ be the blow-up of $a \leqslant 9$ points in general position, and set $g_0 = 11$. One has $\mathrm{h}^0(S, \omega_S^{-1}) = 10 - a$. Let C be a smooth curve in S of genus $g \geqslant g_0$; it is the normalization of $\varepsilon(C)$, and the latter has at most a singular points in general position resolved in one single blow-up, namely $\varepsilon|_C$. It thus follows from Corollary (9.3) that $\operatorname{cork}(\Phi_C) \geqslant 10 - a$ as required.

The same argument works for any blow-up of the plane having an anticanonical curve $\hat{T}$ (which is easily seen to have $h^0(\mathcal{O}_{\hat{T}}) = 1$); we leave this to the reader. □

(9.6) Remark. Corollary (9.3) and Conjecture (9.4) contradict [23, Theorem B, (ii)], which asserts that the Wahl map of the normalization of a plane curve of degree d with one node and one ordinary $(d-5)$-fold point, and no other singularity, has corank 7. We double-checked, using cohomological methods, that the corank is indeed greater or equal than 8 in this case.

(9.7) Example. For curves C as in Proposition (9.1), it is possible to construct a $(9-a)$-extension containing as linear sections all the surface extensions constructed in the proof of Proposition (9.1). This is the universal extension of C whenever one has equality in (9.3.1), which happens when C is smooth [42, Thm. 4.8], or has up to two nodes [23], or in various other cases [23, 34].

Assume for simplicity that C is smooth, and consider the product $\mathbf{P}^2 \times \mathbf{P}^9$. It contains $\mathcal{C} = C \times \mathbf{P}^9$ and the universal family of plane cubics $\mathcal{T}$ over $\mathbf{P}^9 \cong |\mathcal{O}_{\mathbf{P}^2}(3)|$. We let $\mathfrak{L}$ be the linear system of hypersurfaces of bidegree $(d,1)$ in $\mathbf{P}^2 \times \mathbf{P}^9$ containing the intersection scheme $\mathcal{C} \cap \mathcal{T}$; we claim that it defines a birational map, the image of which is the extension $X \subset \mathbf{P}^{g+9}$ of C we are looking for.

We first observe that the linear system $\mathfrak{L}$ restricts on the fibres of the second projection to the linear systems (9.1.1) defining the various extensions of the canonical model of C. It follows that it defines a birational map, and that its image has as linear sections the various surfaces images of the linear systems (9.1.1). Moreover, it maps $\mathcal{T}$ to a $\mathbf{P}^9$; for each surface extension S_T of the canonical model of C, this $\mathbf{P}^9$ image of $\mathcal{T}$ intersects $\langle S_T \rangle \cong \mathbf{P}^g$ at one point, which is the elliptic singularity of S_T.

On the other hand, the members of $\mathfrak{L}$ restrict to hyperplanes on the fibres of the first projection; over a point $p \in C \subset \mathbf{P}^2$, they all restrict to the same hyperplane of $\mathbf{P}^9 \cong |\mathcal{O}_{\mathbf{P}^2}(3)|$, namely the one parametrizing plane cubics passing through p. It follows that the birational map defined by $\mathfrak{L}$ contracts $\mathcal{C} = C \times \mathbf{P}^9$ to C. We leave the remaining details to the reader.

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