MODULI OF CURVES ON ENRIQUES SURFACES

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Abstract. We compute the number of moduli of all irreducible components of the moduli space of smooth curves on Enriques surfaces. In most cases, the moduli maps to the moduli space of Prym curves are generically injective or dominant. Exceptional behaviour is related to existence of Enriques–Fano threefolds and to curves with nodal Prym-canonical model.

1. Introduction

Moduli of curves on projective surfaces have been the object of intensive study for a long time. In more recent times the so-called *Mukai map* c_g from the $(19 + g)$–dimensional moduli space of smooth $K3$ sections of genus g (that is, pairs (S, C), where S is a smooth $K3$ surface and $C \subset S$ is a smooth genus g curve) to $\mathcal{M}_g$ has been given much attention in relation to the birational geometry of $\mathcal{M}_g$ and of the moduli space of $K3$ surfaces of genus g. In particular c_g is dominant for $g \leqslant 11$ and $g \neq 10$, is birational onto its image for $g \geqslant 11$ and $g \neq 12$, and its image is a divisor in genus 10 and it has generically one-dimensional fibers in genus 12 [33, 34, 35, 32, 10]. Notable are the relations of pathologies of c_g with the existence of Fano and Mukai manifolds [9, 6]. Also recall that Mukai's program towards reconstructing a fiber of c_g is now proven [33, 1, 19], and that the image of c_g has been recently characterized, via the Gauss–Wahl map, for Brill-Noether-Petri general curves [2, 45].

In this paper we consider smooth curves on Enriques surfaces. The moduli of such curves have not been systematically investigated so far. Probably this is due to the fact that the Enriques case is much more complicated and rich compared to the $K3$ case due to the presence of many irreducible components of the moduli space of polarized such surfaces, whence also of the moduli space of smooth curves on Enriques surfaces, even when fixing the genus of the polarization. Remarkably enough our results give the number of moduli of *all* such components, equivalently, the dimension of the image (or of a general fiber) of the moduli map. It should be noted that there are some striking analogies with the $K3$ case, including behaviour induced by the existence of Enriques–Fano threefolds, as well as more exceptional behavior, e.g., related to curves with nodal Prym–canonical models.

We now present our results. Let $\mathcal{E}$ denote the smooth irreducible 10-dimensional moduli space parametrizing smooth, complex Enriques surfaces and $\mathcal{E}_{g,\phi}$ the (in general reducible) moduli space of pairs (S, H) such that S is a member of $\mathcal{E}$ and H is an ample line bundle on S satisfying $H^2 = 2g - 2$ and $\phi(H) = \phi$, where

$$(1) \qquad \phi(H) := \min\{E \cdot H \mid E \in \mathrm{NS}(S), E^2 = 0, E > 0\}.$$

Recall that $\phi^2 \leqslant 2g - 2$ by [15, Cor. 2.7.1].

Denote by $\mathcal{EC}_{g,\phi}$ the moduli space of triples (S, H, C) where (S, H) is a member of $\mathcal{E}_{g,\phi}$ and $C \in |H|$ is a smooth irreducible curve. Note that $\mathcal{EC}_{g,\phi}$ has as many irreducible components as $\mathcal{E}_{g,\phi}$. There are natural morphisms

(2)
$$\begin{array}{ccccc} & & \mathcal{EC}_{g,\phi} & & \\ & \swarrow^{p_{g,\phi}} & \downarrow{\scriptstyle \chi_{g,\phi}} & \searrow^{c_{g,\phi}} & \\ \mathcal{E}_{g,\phi} & & \mathcal{R}_g & \longrightarrow & \mathcal{M}_g, \end{array}$$

where $\mathcal{R}_g$ is the moduli space of *Prym curves*, that is, of pairs (C, η), with C a smooth, irreducible, genus g curve and η a non–zero 2-torsion element of $\mathrm{Pic}^0(C)$. The map $\chi_{g,\phi}$ sends (S, H, C) to the Prym curve $(C, \omega_S \otimes \mathcal{O}_C)$. The morphism $c_{g,\phi}$ is the composition of the latter with the forgetful covering map $\mathcal{R}_g \to \mathcal{M}_g$, which has degree $2^{2g} - 1$. By a dimension count, one could expect $\chi_{g,\phi}$ and $c_{g,\phi}$ to be dominant (on some or all components) for $g \leqslant 6$ and generically finite (on some or all components) for $g \geqslant 6$.

As far as we know, the only known result so far concerning the maps $\chi_{g,\phi}$ and $c_{g,\phi}$ is the one of Verra [43] stating that $\chi_{6,3}$ is dominant, equivalently generically finite (note that $\mathcal{E}_{6,3}$ is irreducible).

Our main results are the following. We present the cases $\phi \geqslant 3$, $\phi = 2$ and $\phi = 1$ separately. We refer to the tables in §2 and Notation 3.4 for the definition of the various components of $\mathcal{E}_{g,\phi}$ and $\mathcal{EC}_{g,\phi}$ showing up in the results below.

Theorem 1. *Assume that $\phi \geqslant 3$ (whence $g \geqslant 6$). The map $\chi_{g,\phi} : \mathcal{EC}_{g,\phi} \to \mathcal{R}_g$ is generically injective on any irreducible component of $\mathcal{EC}_{g,\phi}$ not appearing in the list below, for which the dimension of a general fiber is indicated:*

comp.	$\mathcal{EC}_{7,3}$	$\mathcal{EC}_{9,3}^{(II)}$	$\mathcal{EC}_{9,4}^{+}$	$\mathcal{EC}_{9,4}^{-}$	$\mathcal{EC}_{10,3}^{(II)}$	$\mathcal{EC}_{13,3}^{(II)}$	$\mathcal{EC}_{13,4}^{(II)+}$	$\mathcal{EC}_{13,4}^{(II)-}$	$\mathcal{EC}_{17,4}^{(IV)+}$	$\mathcal{EC}_{17,4}^{(IV)-}$
fib.dim.	1	1	3	0	2	1	1	0	1	0

In particular, we obtain that $\chi_{6,3} : \mathcal{EC}_{6,3} \longrightarrow \mathcal{R}_6$ is birational, improving the result of [43]. Moreover, in analogy with the $K3$ case, for any $g \geqslant 8$ there is a component of $\mathcal{E}_{g,\phi}$ on which $\chi_{g,\phi}$ is generically injective, whereas on $\mathcal{E}_{7,3}$ (which is irreducible), the map $\chi_{g,\phi}$ has generically one–dimensional fibers. However, in contrast to the $K3$ case, there are more components of $\mathcal{E}_{g,\phi}$ for $g \geqslant 8$ where $\chi_{g,\phi}$ is not generically finite. This phenomenon can be explained by the existence of Enriques–Fano threefolds, see §4.

For $\phi = 2$ we obtain:

Theorem 2. *The map $\chi_{g,2} : \mathcal{EC}_{g,2} \to \mathcal{R}_g$ is generically finite on all irreducible components of $\mathcal{EC}_{g,2}$ when $g \geqslant 10$. For $g \leqslant 9$ the dimension of a general fiber of $\chi_{g,2}$ on the various irreducible components of $\mathcal{EC}_{g,2}$ is as follows:*

comp.	$\mathcal{EC}_{9,2}^{(I)}$	$\mathcal{EC}_{9,2}^{(II)+}$	$\mathcal{EC}_{9,2}^{(II)-}$	$\mathcal{EC}_{8,2}$	$\mathcal{EC}_{7,2}^{(I)}$	$\mathcal{EC}_{7,2}^{(II)}$	$\mathcal{EC}_{6,2}$	$\mathcal{EC}_{5,2}^{(I)}$	$\mathcal{EC}_{5,2}^{(II)+}$	$\mathcal{EC}_{5,2}^{(II)-}$	$\mathcal{EC}_{4,2}$	$\mathcal{EC}_{3,2}$
fib.dim.	0	2	1	0	1	3	2	3	6	4	4	6

In particular, $\chi_{g,2}$ is dominant precisely on $\mathcal{EC}_{3,2}$ and $\mathcal{EC}_{4,2}$ and is generically finite on at least one component of $\mathcal{EC}_{g,2}$ precisely for $g \geqslant 8$. The positive-dimensional fibers of $\chi_{9,2}$ on $\mathcal{EC}_{9,2}^{(II)+}$ and $\mathcal{EC}_{9,2}^{(II)-}$ can again be explained by the existence of Enriques–Fano threefolds, see Corollary 4.3. The other positive-dimensional fibers are due to the fact that the image of $\chi_{g,2}$ lies in quite special loci, as we now explain. Define:

- $\mathcal{R}_g^0$ — the locally closed locus in $\mathcal{R}_g$ of pairs (C, η) for which the complete linear system $|\omega_C(\eta)|$ is base point free and the map $C \to \mathbb{P}^{g-2}$ it defines (the so–called *Prym–canonical map*) is not an embedding. This locus is irreducible (and unirational) of dimension $2g+1$ for $g \geqslant 5$ by [8, Thm. 1]. (Obviously, $\mathcal{R}_g^0$ is dense in $\mathcal{R}_g$ for $g \leqslant 4$.) Moreover, for the general element, the Prym–canonical map is birational onto its image, which has precisely two nodes, cf. [8, Prop. 1.2].

- $\mathcal{R}_g^{0,\mathrm{nb}}$ — the closed locus in $\mathcal{R}_g^0$ of pairs (C,η) for which the Prym–canonical map is not birational onto its image. This locus is irreducible of dimension $2g-2$ for $g \geqslant 4$ and dominates the bielliptic locus in $\mathcal{M}_g$ via the forgetful map $\mathcal{R}_g \to \mathcal{M}_g$ by [8, Cor. 2.2].
- $\mathcal{D}_5^0$ — the locally closed locus in $\mathcal{R}_5^0$ of pairs (C,η) with 4-nodal Prym-canonical model. By [8, Prop. 5.3] this locus is an irreducible (unirational) divisor in $\mathcal{R}_5^0$ whose closure in $\mathcal{R}_5$ coincides with closure of the locus of pairs (C,η) carrying a theta-characteristic θ such that $h^0(\theta) = h^0(\theta+\eta) = 2$.

The image of $\chi_{g,2}$ (on any component of $\mathcal{EC}_{g,2}$) always lies in $\mathcal{R}_g^0$, cf. Lemma 3.5(ii)-(v), and consequently, by counting dimensions one a priori sees that $\chi_{g,2}$ has expected fiber dimension $\max\{0, 8-g\}$. Furthermore, as a consequence of Theorem 2, the maps $\chi_{g,2}$ dominate some of the peculiar loci above in various cases. Indeed, it follows from Proposition 9.1(i)-(ii) and Corollary 8.7 that:

- $\chi_{5,2}$ on $\mathcal{EC}_{5,2}^{(I)}$ (respectively, $\chi_{6,2}$, $\chi_{7,2}$ on $\mathcal{EC}_{7,2}^{(I)}$, $\chi_{8,2}$) dominates $\mathcal{R}_5^0$ (resp., $\mathcal{R}_6^0$, $\mathcal{R}_7^0$, $\mathcal{R}_8^0$). In particular, the image of $\chi_{5,2}$ on $\mathcal{EC}_{5,2}^{(I)}$ is a divisor in $\mathcal{R}_5$; this parallells the situation of $\operatorname{im} c_{10}$ in the $K3$ case.
- $\chi_{5,2}$ on $\mathcal{EC}_{5,2}^{(II)^+}$ dominates $\mathcal{R}_5^{0,\mathrm{nb}}$.
- $\chi_{5,2}$ on $\mathcal{EC}_{5,2}^{(II)^-}$ dominates $\mathcal{D}_5^0$.

For $\phi = 1$ the moduli spaces $\mathcal{E}_{g,1}$ are irreducible for all g and the image of $\chi_{g,1}$ (and of $c_{g,1}$) lies in the hyperelliptic locus, cf. Lemma 3.5(i), hence the expected fiber dimension is $\max\{10-g, 0\}$. We prove that this is indeed the dimension of a general fiber:

Theorem 3. *The dimension of a general fiber of $\chi_{g,1}$ and of $c_{g,1}$ is $\max\{10-g, 0\}$. Hence, $c_{g,1}$ dominates the hyperelliptic locus if $g \leqslant 10$ and is generically finite if $g \geqslant 10$.*

An immediate consequence of the above results is:

Corollary 1.1. *A general curve of genus 2, 3, 4 and 6 lies on an Enriques surface, whereas a general curve of genus 5 or $\geqslant 7$ does not. A general hyperelliptic curve of genus g lies on an Enriques surface if and only if $g \leqslant 10$.*

The proof of Theorem 1 also has an application to the classification of projective varieties having Enriques surfaces as linear sections. We recall that a projective variety $V \subset \mathbb{P}^N$ is said to be *k-extendable* if there exists a projective variety $W \subset \mathbb{P}^{N+k}$, different from a cone, such that $V = W \cap \mathbb{P}^N$ (transversely). The question of k-extendability of Enriques surfaces is still open, although it is proved in [38, 27] that $N \leqslant 17$ is a necessary condition for 1–extendability, and *terminal threefolds* having Enriques surfaces as hyperplane sections have been classified in [4, 41, 30].

Corollary 1.2. *Let $S \subset \mathbb{P}^N$ be an Enriques surface not containing any smooth rational curve. If S is 1–extendable, then $(S, \mathcal{O}_S(1))$ belongs to the following list:*

$$\mathcal{E}_{17,4}^{(IV)^+},\ \mathcal{E}_{13,4}^{(II)^+},\ \mathcal{E}_{13,3}^{(II)},\ \mathcal{E}_{10,3}^{(II)},\ \mathcal{E}_{9,4}^{+},\ \mathcal{E}_{9,3}^{(II)},\ \mathcal{E}_{7,3}.$$

Furthermore, the members of this list are all at most 1–extendable, except for members of $\mathcal{E}_{10,3}^{(II)}$, which are at most 2–extendable, and of $\mathcal{E}_{9,4}^{+}$, which are at most 3–extendable.

This result is sharp in the case of 1-extendability: The general members of the moduli spaces of Corollary 1.2 indeed occur as hyperplane sections of threefolds different from cones, cf. Remark 4.9. Furthermore, one cannot remove the assumption about S not containing smooth rational curves, as there are threefolds different from cones enjoying the peculiar property that their Enriques hyperplane sections belong to $\mathcal{E}_{8,3}$ and $\mathcal{E}_{6,3}$ and contain a smooth rational curve, cf. Corollary 6.5. We remark that the proof of Corollary 1.2 is independent from, and much simpler than, the results in [38, 27], but it needs the technical assumption about rational curves, which can probably be avoided, at the expense of adding more cases, cf. Remark 6.1. We refer to Corollary 4.10 for another variant of Corollary 1.2.

Our general strategy is to compute the kernel of the differential of the map $c_{g,\phi}$, see §3; to this end we develop in §5 tools to compute the cohomology of twisted tangent bundles on Enriques surfaces. In some cases additional arguments are required, involving for instance extensions to Enriques–Fano threefolds (see §4), and specializations to Enriques surfaces containing smooth rational curves (see §§8–9). Theorem 1 and Corollary 1.2 are proved in §6; Theorem 2 is obtained by combining Propositions 6.6, 8.1 and 9.1; Theorem 3 is proved in §9.

In conclusion we remark that our work leaves several interesting open questions. For example: is it possible to characterize curves on Enriques surfaces in terms of the suitable Gauss–Prym map? In the cases of generic injectivity of $\chi_{g,\phi}$, is it possible to develop an analogue of Mukai's programme of explicit reconstruction of the Enriques surface from its Prym curve section? The latter question was proposed to us by Enrico Arbarello. Finally, in view of Corollary 1.2, are the general members of $\mathcal{E}^{(II)}_{10,3}$ (respectively, $\mathcal{E}^{+}_{9,4}$) 2–extendable (resp., 3–extendable)?

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2. Moduli spaces of Enriques surfaces

We first briefly recall some well-known properties of divisors on Enriques surfaces.

Any irreducible curve C on an Enriques surface satisfies $C^2 \geqslant -2$, with equality occurring if and only if $C \simeq \mathbb{P}^1$. The latter curves are called *nodal*, and Enriques surfaces containing (respectively, not containing) them are called *nodal* (resp., *unnodal*). It is well-known that the general Enriques surface is unnodal, cf. references in [14, p. 577].

A divisor E is said to be *isotropic* if $E^2 = 0$ and $E \not\equiv 0$ (where '$\equiv$' denotes numerical equivalence) and *primitive* if it is non-divisible in $\operatorname{Num} S$. If E is primitive, isotropic and nef, then $|2E|$ is a base point free pencil with general member a smooth elliptic curve, cf. [15, Prop. 3.1.2]. In this case, $\dim(|E|) = 0$ and E is called a *half-fiber*, cf. [15, p. 172]. Conversely, any elliptic pencil $|P|$ contains precisely two double fibers $2E$ and $2E'$, where E' is the only member of $|E + K_S|$. It is clear that, when H is big and nef, the invariant $\phi(H)$ in (1) is computed by a half-fiber.

Let $\mathcal{E}$ (resp. $\mathcal{K}^\iota$) denote the 10-dimensional smooth moduli space parametrizing smooth Enriques surfaces (respectively, smooth $K3$ surfaces with a fixed point free involution), and let $\mathcal{K}$ denote the 20-dimensional moduli space parametrizing smooth $K3$ surfaces, cf. [3, VIII.§12,§§19-21]. We have a natural bijective map sending a $K3$ surface with fixed point free involution to the quotient surface by the involution

$$\delta : \mathcal{K}^\iota \longrightarrow \mathcal{E}. \tag{3}$$

Let $\mathcal{E}_{g,\phi}$ (respectively, $\widehat{\mathcal{E}}_{g,\phi}$) denote the moduli space of *polarized* (resp., *numerically polarized*) *Enriques surfaces*, that is, pairs (S,H) (resp., $(S,[H])$) such that $[S] \in \mathcal{E}$ and $H \in \mathrm{Pic}(S)$ (resp., $[H] \in \mathrm{Num}(S)$) is ample with $H^2 = 2g-2 \geqslant 2$ and $\phi = \phi(H)$. There is an étale cover $\mathcal{E}_{g,\phi} \to \mathcal{E}$, and $\mathcal{E}_{g,\phi}$ is smooth. There is also an étale double cover $\rho : \mathcal{E}_{g,\phi} \to \widehat{\mathcal{E}}_{g,\phi}$ mapping (S,H) and $(S,H+K_S)$ to $(S,[H])$. We refer to [7, §2] for details and references and also recall that $\phi(H)^2 \leqslant H^2$ by [15, Cor. 2.7.1].

The spaces $\mathcal{E}_{g,\phi}$ need not be irreducible. In [7] various irreducible components were determined and their unirationality or uniruledness was proved. In particular, all components are determined and described for $\phi \leqslant 4$ and $g \leqslant 20$ respectively. The description is in terms of isotropic decompositions, as we now explain.

By [7, Cor. 4.6, Cor. 4.7, Rem. 4.11] any effective line bundle H with $H^2 \geqslant 0$ on an Enriques surface S can be written as (denoting linear equivalence by '$\sim$'):

$$H \sim a_1E_1 + \cdots + a_nE_n + \varepsilon K_S \tag{4}$$

where all E_i are effective, primitive and isotropic, all a_i are positive integers, $n \leqslant 10$,

$$\varepsilon = \begin{cases} 0, & \text{if } H+K_S \text{ is not 2-divisible in } \mathrm{Pic}(S),\\ 1, & \text{if } H+K_S \text{ is 2-divisible in } \mathrm{Pic}(S),\end{cases}$$

and moreover

$$\begin{cases} \text{either } n \neq 9,\ E_i \cdot E_j = 1 \text{ for all } i \neq j,\\ \text{or } n \neq 10,\ E_1 \cdot E_2 = 2 \text{ and otherwise } E_i \cdot E_j = 1 \text{ for all } i \neq j,\\ \text{or } E_1 \cdot E_2 = E_1 \cdot E_3 = 2 \text{ and otherwise } E_i \cdot E_j = 1 \text{ for all } i \neq j.\end{cases} \tag{5}$$

We call this a *simple isotropic decomposition* (up to reordering indices), cf. [7].

We say that two polarized Enriques surfaces (S,H) and (S',H') in $\mathcal{E}_{g,\phi}$ *admit the same simple decomposition type* if one has two simple isotropic decompositions

$$H \sim a_1E_1 + \cdots + a_nE_n + \varepsilon K_S \quad\text{and}\quad H' \sim a_1E'_1 + \cdots + a_nE'_n + \varepsilon K_{S'}$$

and $E_i \cdot E_j = E'_i \cdot E'_j$ for all $i \neq j$. This defines an equivalence relation on $\mathcal{E}_{g,\phi}$ by [7, Prop. 4.15].

By [7, Cor. 1.3 and 1.4] the irreducible components of $\mathcal{E}_{g,\phi}$ when $\phi \leqslant 4$ or $g \leqslant 20$ correspond precisely to the loci consisting of pairs (S,H) admitting the same decomposition type. Moreover, by [7, Cor. 1.5], in the same range, for $\mathcal{C} \subset \mathcal{E}_{g,\phi}$ any irreducible component, $\rho^{-1}(\rho(\mathcal{C}))$ is reducible if and only if $\mathcal{C}$ parametrizes pairs (S,H) such that H is 2-divisible in $\mathrm{Num}(S)$. The various irreducible components of $\widehat{\mathcal{E}}_{g,\phi}$ were labeled by roman numbers in the appendix of [7]. We will use the same labels for the irreducible components of $\mathcal{E}_{g,\phi}$, adding a superscript "+" and "−" in the cases there are two irreducible components lying above one irreducible component of $\widehat{\mathcal{E}}_{g,\phi}$. We also adopt the following from [7]:

Notation 2.1. When writing a simple isotropic decomposition (4) verifying (5) (up to permuting indices), we will adopt the convention that E_i, E_j, $E_{i,j}$ are primitive isotropic satisfying $E_i \cdot E_j = 1$ for $i \neq j$, $E_{i,j} \cdot E_i = E_{i,j} \cdot E_j = 2$ and $E_{i,j} \cdot E_k = 1$ for $k \neq i, j$.

In particular, we recall the following (cf. [7, Cor. 1.3 and Lemma 4.18]):

- $\mathcal{E}_{g,1}$ is irreducible and unirational, and $H \sim (g-1)E_1 + E_2$.
- If g is even (resp., $g = 3$), then $\mathcal{E}_{g,2}$ is irreducible and unirational, and $H \sim \frac{g-2}{2}E_1 + E_2 + E_3$ (resp., $H \sim E_1 + E_{1,2}$).
- If $g \geqslant 7$ and $g \equiv 3 \mod 4$, then $\mathcal{E}_{g,2}$ has two irreducible, unirational components $\mathcal{E}_{g,2}^{(I)}$ and $\mathcal{E}_{g,2}^{(II)}$ corresponding, respectively, to simple decomposition types:
  (I) $H \sim \frac{g-1}{2}E_1 + E_{1,2}$,
  (II) $H \sim \frac{g-1}{2}E_1 + 2E_2$.
- If $g \geqslant 5$ and $g \equiv 1 \mod 4$, then $\mathcal{E}_{g,2}$ has three irreducible, unirational components $\mathcal{E}_{g,2}^{(I)}$, $\mathcal{E}_{g,2}^{(II)^+}$ and $\mathcal{E}_{g,2}^{(II)^-}$, corresponding, respectively, to simple decomposition types
  (I) $H \sim \frac{g-1}{2}E_1 + E_{1,2}$,
  (II)$^+$ $H \sim \frac{g-1}{2}E_1 + 2E_2$,
  (II)$^-$ $H \sim \frac{g-1}{2}E_1 + 2E_2 + K_S$,

For later reference we list all irreducible components of $\mathcal{E}_{g,\phi}$ for $\phi \geqslant 2$ and $g \leqslant 10$, cf. [7, Appendix]:

g	ϕ	comp.	dec. type
3	2	$\mathcal{E}_{3,2}$	$H \sim E_1 + E_{1,2}$
4	2	$\mathcal{E}_{4,2}$	$H \sim E_1 + E_2 + E_3$
5	2	$\mathcal{E}_{5,2}^{(I)}$	$H \sim 2E_1 + E_{1,2}$
5	2	$\mathcal{E}_{5,2}^{(II)^+}$	$H \sim 2E_1 + 2E_2$
5	2	$\mathcal{E}_{5,2}^{(II)^-}$	$H \sim 2E_1 + 2E_2 + K_S$
6	2	$\mathcal{E}_{6,2}$	$H \sim 2E_1 + E_2 + E_3$
6	3	$\mathcal{E}_{6,3}$	$H \sim E_1 + E_2 + E_{1,2}$
7	2	$\mathcal{E}_{7,2}^{(I)}$	$H \sim 3E_1 + E_{1,2}$
7	2	$\mathcal{E}_{7,2}^{(II)}$	$H \sim 3E_1 + 2E_2$
7	3	$\mathcal{E}_{7,3}$	$H \sim E_1 + E_2 + E_3 + E_4$
8	2	$\mathcal{E}_{8,2}$	$H \sim 3E_1 + E_2 + E_3$
8	3	$\mathcal{E}_{8,3}$	$H \sim 2E_1 + E_3 + E_{1,2}$

g	ϕ	comp.	dec. type
9	2	$\mathcal{E}_{9,2}^{(I)}$	$H \sim 4E_1 + E_{1,2}$
9	2	$\mathcal{E}_{9,2}^{(II)^+}$	$H \sim 4E_1 + 2E_2$
9	2	$\mathcal{E}_{9,2}^{(II)^-}$	$H \sim 4E_1 + 2E_2 + K_S$
9	3	$\mathcal{E}_{9,3}^{(I)}$	$H \sim 2E_1 + E_2 + E_{1,2}$
9	3	$\mathcal{E}_{9,3}^{(II)}$	$H \sim 2E_1 + 2E_2 + E_3$
9	4	$\mathcal{E}_{9,4}^{+}$	$H \sim 2(E_1 + E_{1,2})$
9	4	$\mathcal{E}_{9,4}^{-}$	$H \sim 2(E_1 + E_{1,2}) + K_S$
10	2	$\mathcal{E}_{10,2}$	$H \sim 4E_1 + E_2 + E_3$
10	3	$\mathcal{E}_{10,3}^{(I)}$	$H \sim 2E_1 + E_2 + E_3 + E_4$
10	3	$\mathcal{E}_{10,3}^{(II)}$	$H \sim 3(E_1 + E_2)$
10	4	$\mathcal{E}_{10,4}$	$H \sim 2E_{1,2} + E_1 + E_2$

We also list all irreducible components of $\mathcal{E}_{13,3}$, $\mathcal{E}_{13,4}$ and $\mathcal{E}_{17,4}$:

g	ϕ	comp.	dec. type
13	3	$\mathcal{E}_{13,3}^{(I)}$	$H \sim 3E_1 + E_2 + E_3 + E_4$
13	3	$\mathcal{E}_{13,3}^{(II)}$	$H \sim 4E_1 + 3E_2$
13	4	$\mathcal{E}_{13,4}^{(I)}$	$H \sim 2E_1 + 2E_2 + E_{1,2}$
13	4	$\mathcal{E}_{13,4}^{(II)^+}$	$H \sim 2(E_1 + E_2 + E_3)$
13	4	$\mathcal{E}_{13,4}^{(II)^-}$	$H \sim 2(E_1 + E_2 + E_3) + K_S$
13	4	$\mathcal{E}_{13,4}^{(III)}$	$H \sim 3E_1 + 2E_{1,2}$

g	ϕ	comp.	dec. type
17	4	$\mathcal{E}_{17,4}^{(I)}$	$H \sim 3E_1 + 2E_2 + 2E_3$
17	4	$\mathcal{E}_{17,4}^{(II)}$	$H \sim 3E_1 + 2E_2 + E_{1,2}$
17	4	$\mathcal{E}_{17,4}^{(III)^+}$	$H \sim 4E_1 + 2E_{1,2}$
17	4	$\mathcal{E}_{17,4}^{(III)^-}$	$H \sim 4E_1 + 2E_{1,2} + K_S$
17	4	$\mathcal{E}_{17,4}^{(IV)^+}$	$H \sim 4E_1 + 4E_2$
17	4	$\mathcal{E}_{17,4}^{(IV)^-}$	$H \sim 4E_1 + 4E_2 + K_S$

3. Generalities on moduli maps

Recall that if a divisor H on an Enriques surface S is big and nef such that $H^2 = 2g-2$, then $\dim |H| = g-1$ and a general member C of $|H|$ is a smooth irreducible curve of genus g if either $g > 2$, or $g = 2$ and S is unnodal or H is ample, by [14, Prop. 2.4] and [12, Thm. 4.1 and Prop. 8.2]. As we explained in the introduction, one could expect $\chi_{g,\phi}$ and $c_{g,\phi}$ from diagram (2) to be dominant (on some or all irreducible components) for $g \leqslant 6$ and generically finite (on some or all irreducible components) for $g \geqslant 6$. This expectation fails in the cases $\phi = 1, 2$ for low genera as the curves in $|H|$ are all special from a Brill-Noether theoretical point of view, cf. Lemma 3.5 below. It also fails in case of existence of Enriques–Fano threefolds, as we will see in §4 below.

Recalling the map (3), set $\mathcal{K}^{\iota}_{g,\phi} = \delta^{-1}(\mathcal{E}_{g,\phi})$; thus $\mathcal{K}^{\iota}_{g,\phi}$ is a component of the moduli space of polarized $K3$ surfaces $(\widetilde{S}, \widetilde{H}, \iota)$ of genus $2g-1$ with a fixed point free involution ι, and we have a generically injective morphism $\alpha_{g,\phi} : \mathcal{K}^{\iota}_{g,\phi} \to \mathcal{K}_{2g-1}$ forgetting the involution, where $\mathcal{K}_{2g-1}$ denotes the moduli space of polarized $K3$ surfaces of genus $2g-1$ (as the general $K3$ surface with a fixed point free involution contains only one such). We have the commutative diagram

$$
\begin{array}{ccccc}
 & & \mathcal{M}_{2g-1} & & \\
 & \overset{c^{\iota}_{g,\phi}}{\nearrow} & & \overset{c_{2g-1}}{\nwarrow} & \\
\mathcal{KC}^{\iota}_{g,\phi} & & \xrightarrow{\widetilde{\alpha}_{g,\phi}} & & \mathcal{KC}_{2g-1} \\
{\scriptstyle p^{\iota}_{g,\phi}}\downarrow & & & & \downarrow{\scriptstyle q_{2g-1}} \\
\mathcal{K}^{\iota}_{g,\phi} & & \xrightarrow[\alpha_{g,\phi}]{} & & \mathcal{K}_{2g-1}
\end{array}
\tag{6}
$$

where $\mathcal{KC}^{\iota}_{g,\phi}$ is the moduli space of quadruples $(\widetilde{S}, \widetilde{H}, \iota, \widetilde{C})$, with $(\widetilde{S}, \widetilde{H}, \iota)$ in $\mathcal{K}^{\iota}_{g,\phi}$ and $\widetilde{C} \in |\widetilde{H}|$ is a smooth curve invariant under the involution ι, the map $\widetilde{\alpha}_{g,\phi}$ forgets ι, and $\mathcal{KC}_{2g-1}$ is the moduli space of triples (X, L, Y), with (X, L) in $\mathcal{K}_{2g-1}$ and $Y \in |L|$ is a smooth curve.

Recall now, for any smooth $C \in |H|$, the sheaf $\mathcal{T}_S\langle C\rangle$ defined by

$$0 \longrightarrow \mathcal{T}_S\langle C\rangle \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{N}_{C/S} \longrightarrow 0, \tag{7}$$

and fitting into the exact sequence

$$0 \longrightarrow \mathcal{T}_S(-C) \longrightarrow \mathcal{T}_S\langle C\rangle \longrightarrow \mathcal{T}_C \longrightarrow 0. \tag{8}$$

We have the following, cf. [42, §3.4.4] or [5]:

Lemma 3.1. *The differential of $c_{g,\phi}$ at (S, H, C) (resp., of c_{2g-1} at $(\widetilde{S}, \widetilde{H}, \widetilde{C})$) is the morphism $H^1(\mathcal{T}_S\langle C\rangle) \to H^1(\mathcal{T}_C)$ (resp., $H^1(\mathcal{T}_{\widetilde{S}}\langle \widetilde{C}\rangle) \to H^1(\mathcal{T}_{\widetilde{C}})$) induced by* (8). *Its kernel is $H^1(\mathcal{T}_S(-C))$ (resp., $H^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{C}))$).*

The spaces $H^1(\mathcal{T}_S(-C))$ and $H^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{C}))$ in the lemma are related in the following way. Let $\pi : \widetilde{S} \to S$ be the $K3$ double cover and set $\widetilde{H} := \pi^* H$. As π is étale, we have $\pi^*\mathcal{T}_S \simeq \mathcal{T}_{\widetilde{S}}$. Therefore,

$$H^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = H^1(\mathcal{T}_S(-H)) \oplus H^1(\mathcal{T}_S(-H + K_S)). \tag{9}$$

Lemma 3.2. *Assume that $\phi \geqslant 3$ (whence $g \geqslant 6$). Let $(C, K_S \otimes \mathcal{O}_C)$ be a general element of the image of $\chi_{g,\phi}$. Denote by $\widetilde{C} \to C$ its induced double cover. If $c^{-1}_{2g-1}(\widetilde{C})$ is finite, then it consists of only one point, and also $\chi^{-1}_{g,\phi}((C, K_S \otimes \mathcal{O}_C))$ consists of only one point.*

Proof. Thanks to the bijective map δ in (3) and $\alpha_{g,\phi}$ being generically injective, the fact that $\chi^{-1}_{g,\phi}((C, K_S \otimes \mathcal{O}_C))$ is a point is equivalent to the fact that $(c^{\iota}_{g,\phi})^{-1}(\widetilde{C})$ is a point, where $c^{\iota}_{g,\phi}$ is as in (6). The latter will follow if $c^{-1}_{2g-1}(\widetilde{C})$ is a point. By [10], this property follows if $\widetilde{C}$ has a corank one Gauss-Wahl map, cf. [28, Sketch of proof of Prop. 3.3]. Since $2g-2 = C^2 \geqslant \phi(C)^2 \geqslant 9$ (using [15, Cor. 2.7.1]), we have $g \geqslant 6$, hence $2g-1 \geqslant 11$. Therefore, if $\mathrm{Cliff}(\widetilde{C}) \geqslant 3$, the fiber $c_{2g-1}^{-1}(\widetilde{C})$ is positive dimensional as soon as the Gauss-Wahl map of $\widetilde{C}$ has corank > 1, cf. [6, Thm. 2.6]. Hence, $c_{2g-1}^{-1}(\widetilde{C})$ consists of exactly one point if it is finite.

We thus have left to prove that $\mathrm{Cliff}(\widetilde{C}) \geqslant 3$. As the Clifford index is constant among smooth curves in the linear system $|\widetilde{H}|$ (see [22]), we may assume that C is general in its linear system. Furthermore, $\mathrm{Cliff}(\widetilde{C}) \geqslant 3$ is equivalent to $\mathrm{gon}(\widetilde{C}) \geqslant 5$, which is satisfied if $\mathrm{gon}(C) \geqslant 5$. The cases with $\mathrm{gon}(C) < 2\phi(H)$ are classified in [26, Cor. 1.5] and a direct check shows that $\mathrm{gon}(C) \geqslant 5$ when $\phi \geqslant 3$ and $g \geqslant 7$. If $g = 6$, we use the assumption that S is general, so that $\mathrm{gon}(\widetilde{C}) = 2\phi(C) = 6$ by [39, Thm. 1.1]. □

Corollary 3.3. *Let (S,H) be a general element of an irreducible component $\mathcal{E}'_{g,\phi}$ of $\mathcal{E}_{g,\phi}$ and let $\chi'_{g,\phi}$ denote the restriction of $\chi_{g,\phi}$ to $p_{g,\phi}^{-1}(\mathcal{E}'_{g,\phi})$.*

(i) If $\phi \geqslant 3$ and $h^1(\mathcal{T}_S(-H)) = h^1(\mathcal{T}_S(-H+K_S)) = 0$, then $\chi'_{g,\phi}$ is generically injective.

(ii) If $h^1(\mathcal{T}_S(-H)) = 0$, then $\chi'_{g,\phi}$ is generically finite.

In any case, the dimension of a general fiber of $\chi'_{g,\phi}$ is $h^1(\mathcal{T}_S(-H))$.

Proof. This follows from Lemmas 3.2 and 3.1, as well as (9). □

In the rest of the paper we will adopt the following:

Notation 3.4. For any irreducible component $\mathcal{E}'_{g,\phi}$ of $\mathcal{E}_{g,\phi}$ we express the irreducible component $p_{g,\phi}^{-1}(\mathcal{E}'_{g,\phi})$, as well as the restrictions of the maps $\chi_{g,\phi}$ and $c_{g,\phi}$ to this irreducible component, by the same superscripts as the ones used to label $\mathcal{E}'_{g,\phi}$. For instance, we set $\mathcal{EC}_{5,2}^{(II)} := p_{5,2}^{-1}(\mathcal{E}_{5,2}^{(II)})$, $c_{5,2}^{(II)} := c_{5,2}|_{\mathcal{EC}_{5,2}^{(II)}}$ and $\chi_{5,2}^{(II)} := \chi_{5,2}|_{\mathcal{EC}_{5,2}^{(II)}}$.

We finish this section with a lemma that will be needed later. We refer to the introduction for the definitions of the loci $\mathcal{R}_g^0$, $\mathcal{R}_5^{0,\mathrm{nb}}$ and $\mathcal{D}_5^0$.

Lemma 3.5. *(i) For any $g \geqslant 2$ the image of $c_{g,1}$ lies in the hyperelliptic locus; in particular the fiber dimension is $\geqslant \max\{0, 10-g\}$.*

(ii) The image of $\chi_{5,2}^{(I)}$ lies in $\mathcal{R}_5^0$; in particular the fiber dimension is $\geqslant 3$.

(iii) The image of $\chi_{5,2}^{(II)+}$ lies in $\mathcal{R}_5^{0,\mathrm{nb}}$; in particular the fiber dimension is $\geqslant 6$.

(iv) The image of $\chi_{5,2}^{(II)-}$ lies in $\mathcal{D}_5^0$; in particular the fiber dimension is $\geqslant 4$.

(v) For any $g \geqslant 6$ the image of $\chi_{g,2}$ restricted to any component of $\mathcal{EC}_{g,2}$ lies in $\mathcal{R}_g^0$; in particular the fiber dimension is $\geqslant \max\{0, 8-g\}$.

Proof. Item (i) follows from [15, Prop. 4.5.1, Cor. 4.5.1], items (ii) and (v) from [8, Ex. 5.1], item (iii) from [8, Rem. 5.5] and (iv) from [8, Ex. 5.2]. □

4. Fibers of the moduli maps and Enriques–Fano threefolds

An *Enriques–Fano threefold of genus g* is a pair $(X, \mathcal{L})$ where X is a normal threefold and $\mathcal{L}$ is an ample line bundle on X with $\mathcal{L}^3 = 2g-2$ such that $|\mathcal{L}|$ contains a smooth Enriques surface S, and X is not a generalized cone over S, that is, X is not isomorphic to a variety obtained by contracting to a point a negative section of some $\mathbb{P}^1$-bundle over

S. Such threefolds with terminal singularities are classified in [4, 41, 30], and examples with canonical, nonterminal singularities are given in [27, 38], but a full classification of these threefolds is still missing, although it is proved in [27, 38] that $g \leqslant 17$. We say that a polarized Enriques surface (S, H) is *extendable* to an Enriques–Fano threefold $(X, \mathcal{L})$ if $S \in |\mathcal{L}|$ with $H = \mathcal{L}|_S$.

Lemma 4.1. *Let $(X, \mathcal{L})$ be an Enriques–Fano threefold of genus g, $\pi : \widetilde{X} \to X$ a desingularization and $S \in |\mathcal{L}|$ a smooth surface. Then*

(i) $h^0(X, \mathcal{L}) = g + 1$ and the restriction map $H^0(X, \mathcal{L}) \to H^0(S, \mathcal{L}|_S)$ is onto;
(ii) $h^1(\mathcal{O}_{\widetilde{X}}) = 0$ and $H^0(\widetilde{X}, \pi^*\mathcal{L}) \simeq H^0(X, \mathcal{L})$.

Proof. Since π is an isomorphism outside the singular locus of X, we may identify S with $\pi^{-1}(S)$. By the fact that $\pi^*\mathcal{L}$ is big and nef and

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(-\pi^*\mathcal{L}) \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}_S \longrightarrow 0, \tag{10}$$

we get $h^1(\mathcal{O}_{\widetilde{X}}) = 0$. The rest of (ii) follows from the normality of X. Tensoring (10) by $\pi^*\mathcal{L}$ and taking cohomology, we get (i). □

In particular, part (i) implies that $|\mathcal{L}|$ is base point free if and only if $|\mathcal{L}|_S|$ is, for any smooth Enriques surface $S \in |\mathcal{L}|$, which holds if and only if $\phi(\mathcal{L}|_S) \geqslant 2$ by [12, Thms. 4.1] or [15, Thm. 4.4.1]. Similarly, the morphism $\varphi_{\mathcal{L}}$ defined by $|\mathcal{L}|$ is an isomorphism on S if and only if $\phi(\mathcal{L}|_S) \geqslant 3$ by [12, Thms. 5.1] or [15, Thm. 4.6.1] (since $\mathcal{L}|_S$ is ample), in which case we get that $\varphi_{\mathcal{L}}(X) \subset \mathbb{P}^g$ is a (possibly non-normal) threefold whose general hyperplane section is a smooth Enriques surface.

The connection to the topic of this paper is given by:

Proposition 4.2. *Let $(X, \mathcal{L})$ be an Enriques–Fano threefold of genus $g \geqslant 6$. Let $S \in |\mathcal{L}|$ be general, and $C \in |\mathcal{L}_{|S}|$ be general, with $\phi = \phi(\mathcal{L}_{|S}) \geqslant 2$. Then the dimension of the fiber of $c_{g,\phi}$ at $(S, \mathcal{L}_{|S}, C)$ is at least 1.*

Proof. Consider the linear pencil $\mathfrak{l}$ in $|\mathcal{L}|$ with base locus C, so that $S \in \mathfrak{l}$. Consider the open subset U of $\mathfrak{l}$ whose points correspond to smooth sections of X. We claim that two general points of U correspond to non–isomorphic polarized Enriques surfaces $(S', \mathcal{L}_{|S'}), (S'', \mathcal{L}_{|S''})$. The assertion clearly follows from this claim.

To prove the claim, suppose, to the contrary, that all points of U correspond to isomorphic polarized Enriques surfaces. This implies that two general members in $|\mathcal{L}|$ are isomorphic as polarized Enriques surfaces. Since $g \geqslant 6$ and $\phi \geqslant 2$, Lemma 4.1 together with [12, Thms. 4.1 and 5.1] or [15, Thm. 4.4.1 and Prop. 4.7.1] yield that the map $\varphi_{\mathcal{L}}$ determined by $|\mathcal{L}|$ is a morphism that maps X birationally onto its image, which is not a cone. Hence, two general hyperplane sections of $Y = \varphi_{\mathcal{L}}(X)$ are projectively equivalent. By [37, Prop. 1.7] (which applies in fact to all varieties different from cones) this would imply that the general hyperplane section of Y is ruled, a contradiction. □

Corollary 4.3. *The maps $\chi^{(IV)^+}_{17,4}, \chi^{(II)^+}_{13,4}, \chi^{(II)^+}_{9,2}, \chi^{(II)^-}_{9,2}, \chi_{7,3}$ are not generically finite.*

Proof. This will follow from Lemmas 4.5, 4.6, 4.8 and Proposition 4.7 below, where we prove that the general members of $\mathcal{E}^{(IV)^+}_{17,4}$, $\mathcal{E}^{(II)^+}_{13,4}$, $\mathcal{E}^{(II)^+}_{9,2}$, $\mathcal{E}^{(II)^-}_{9,2}$, $\mathcal{E}_{7,3}$ are extendable. □

We will make use of the following auxiliary result:

Lemma 4.4. *Let (S,L) be a polarized Enriques surface of genus $g \geqslant 6$ with $\phi(L) \geqslant 2$. Assume that $(S, L+D)$ is extendable to an Enriques–Fano threefold $(Y,\mathcal{H})$ for an effective divisor D, and that Y is unirational. Then (S,L) is extendable to an Enriques–Fano threefold $(X,\mathcal{L})$ and the elements in $|\mathcal{L}|$ are in one-to one correspondence with the elements in $|\mathcal{H}\otimes\mathcal{J}_D|$.*

Proof. Let $\pi : \widetilde{Y} \to Y$ be a desingularization and identify S with $\pi^{-1}(S)$. Then $h^1(\mathcal{O}_{\widetilde{Y}}) = 0$ by Lemma 4.1(ii). Therefore, the exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{Y}} \longrightarrow \pi^*\mathcal{H}\otimes\mathcal{J}_D \longrightarrow \mathcal{O}_S(L) \longrightarrow 0, \tag{11}$$

shows, as $|L|$ is base point free and birational by [12, Thms. 4.1 and 5.1] or [15, Thm. 4.4.1 and Prop. 4.7.1], that the closure of the image of the rational map defined by the linear system $|\pi^*\mathcal{H}\otimes\mathcal{J}_D|$ is a threefold X' in $\mathbb{P}^g$, where $L^2 = 2g-2$, having the surfaces in $|\pi^*\mathcal{H}\otimes\mathcal{J}_D|$, including S, as hyperplane sections. Since Y is unirational, also X' is. If X' were a cone, then it would be birational to $H \times \mathbb{P}^1$, for a general hyperplane section H of X'. Thus, H would be unirational, a contradiction. Hence, X' is not a cone. Let $\nu : X \to X'$ be its normalization and $\mathcal{L} := \nu^*\mathcal{O}_{X'}(1)$. Then $(X,\mathcal{L})$ is an Enriques–Fano threefold extending (S,L). Identifying D with $\pi^{-1}(D)$, we get, as Y is normal,

$$H^0(Y,\mathcal{H}\otimes\mathcal{J}_D) \simeq H^0(\widetilde{Y},\pi^*\mathcal{H}\otimes\mathcal{J}_D) \simeq H^0(X',\mathcal{O}_{X'}(1)) \simeq \mathbb{C}^{g+1},$$

and the latter is contained in $H^0(X',\nu_*\mathcal{L}) \simeq H^0(X,\mathcal{L})$. Since $h^0(\mathcal{L}) = g+1$ by Lemma 4.1, we must have $H^0(Y,\mathcal{H}\otimes\mathcal{J}_D) \simeq H^0(X,\mathcal{L})$, proving the last assertion. □

The *classical Enriques–Fano threefold* Y of genus 13 is the image of $\mathbb{P}^3$ via the linear system of sextic surfaces that are double along the edges of a tetrahedron, cf. [11, 18]. Its smooth hyperplane sections are Enriques surfaces with polarization of the form $2(E_1 + E_2 + E_3)$ (cf. [27, Pf. of Prop. 13.1]), that is, they belong to $\mathcal{E}^{(II)^+}_{13,4}$.

Lemma 4.5. *Any $(S, H = 2(E_1+E_2+E_3)) \in \mathcal{E}^{(II)^+}_{13,4}$ such that E_1,E_2,E_3 are nef and $|E_1+E_2+E_3|$ is birational is extendable to the classical Enriques–Fano threefold.*

Proof. By assumption, $|E_1+E_2+E_3|$ maps S birationally onto a sextic surface in $\mathbb{P}^3$ singular along the edges of a tetrahedron, which are the images of all E_i and E_i', the only member of $|E_i + K_S|$, for $i = 1,2,3$, cf., e.g., [15, Thm. 4.9.3]. All such sextics are by construction hyperplane sections of the classical Enriques–Fano threefold. □

Lemma 4.6. *A general member of $\mathcal{E}_{7,3}$ is extendable.*

Proof. Let $(S,H) \in \mathcal{E}_{7,3}$ be general with $H \sim E_1+E_2+E_3+E_4$. In particular, S is unnodal, whence $|E_1+E_2+E_3|$ is birational by [12, Thm. 7.2]. Thus $(S, L := 2(E_1+E_2+E_3))$ is extendable to the classical Enriques–Fano threefold Y by Lemma 4.5. Note that $(E_1+E_2+E_3-E_4)^2 = 0$, so that $E_1+E_2+E_3 \sim E_4+F$, for an effective isotropic F. In particular, $L \sim H+F$, and the result follows from Lemma 4.4. □

Next we consider the only known Enriques–Fano threefold of genus 17, namely the one constructed by Prokhorov in [38, §3] with canonical nonterminal singularities in the following way: Let x and $y_{i,j}$, $0 \leqslant i,j \leqslant 2$ be homogeneous coordinates in $\mathbb{P}^9$ and consider the anticanonical embedding of $P := \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^8 = \{x = 0\} \subset \mathbb{P}^9$ given by

$$(u_0 : u_1) \times (v_0 : v_1) \mapsto (y_{0,0} : \cdots : y_{2,2}),\ y_{i,j} = u_0^i u_1^{2-i} v_0^j v_1^{2-j}.$$

Let V be the projective cone over P and $v = (0 : \cdots : 0 : 1)$ its vertex. Then V is a Gorenstein Fano threefold V with canonical singularities. Let $\pi : V \to W$ be the quotient map of the involution τ defined by $\tau(x) = -x$ and $\tau(y_{i,j}) = (-1)^{i+j} y_{i,j}$. Letting $\mathcal{M} := \mathcal{O}_V(1)$, we have $-K_V \sim 2\mathcal{M}$ by [38, Lemma 3.1] and every smooth member of $|-K_V|$ is a $K3$ surface. The τ-invariant ones are precisely the ones cut out on V by quadrics of the form $q_1(y_{0,0}, y_{0,2}, y_{2,0}, y_{2,2}, y_{1,1}) + q_2(y_{0,1}, y_{2,1}, y_{1,0}, y_{1,2}, x)$, where q_1 and q_2 are quadratic homogeneous forms, on which the action of τ is free. The quotient of any such τ-invariant $\widetilde{S}$ by τ is thus an Enriques surface S. Since $\pi^* S = \widetilde{S}$ we have $2g - 2 = S^3 = \frac{1}{2}\widetilde{S}^3 = \frac{1}{2}(2\mathcal{M})^3 = 32$, whence $g = 17$. Set $\mathcal{L} := \mathcal{O}_W(S)$. Then $(W, \mathcal{L})$ is an Enriques–Fano threefold of genus 17.

Proposition 4.7. *The threefold W is unirational and its polarized Enriques sections $(S, \mathcal{L}|_S)$ belong to $\mathcal{E}_{17,4}^{(IV)^+}$. Conversely, any $(S, H = 4(E_1 + E_2)) \in \mathcal{E}_{17,4}^{(IV)^+}$ with E_1 and E_2 nef is extendable to $(W, \mathcal{L})$.*

Proof. We keep the notation above. The unirationality of W follows from the rationality of V. Set $\mathcal{L}|_S = L$. As we have an induced double cover $\widetilde{S} \to \mathbb{P}^1 \times \mathbb{P}^1$, we have $\mathcal{M}|_{\widetilde{S}} \sim 2D$, with $D^2 = 4$. Thus, $\pi|_{\widetilde{S}}^* L = \pi^* \mathcal{L}|_{\widetilde{S}} \sim \mathcal{O}_{\widetilde{S}}(\widetilde{S}) \sim 2\mathcal{M}|_{\widetilde{S}} \sim 4D$, so that either L or $L + K_S$ is 4-divisible in $\operatorname{Pic}(S)$. By [27, Prop. 12.1], either L or $L - E$ with $E \cdot L = \phi(L)$ is 2-divisible in $\operatorname{Pic}(S)$, and this implies that $L + K_S$ is not 2-divisible. Hence, L is 4-divisible in $\operatorname{Pic}(S)$, and the only possibility is $L \sim 4(E_1 + E_2)$ as desired.

Now let $(S, H = 4(E_1 + E_2)) \in \mathcal{E}_{17,4}^{(IV)^+}$ with E_1 and E_2 nef, and denote by $p_i : S \to \mathbb{P}^1$ the morphism induced by the pencil $|2E_i|$, $i = 1, 2$. Let $\pi : \widetilde{S} \to S$ be the $K3$ double cover. Then each $|\pi^* E_i|$ is an elliptic pencil and we have a commutative diagram

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\ \pi\ } & S \\ {\scriptstyle \widetilde{p}_i}\downarrow & & \downarrow{\scriptstyle p_i} \\ \mathbb{P}^1 & \xrightarrow{\ \sigma_i\ } & \mathbb{P}^1, \end{array}$$

where $\widetilde{p}_i$ is the map induced by the pencil $|\pi^* E_i|$ and σ_i is a double cover branched at the two points corresponding to the double fibers of p_i.

The map $\widetilde{S} \to \mathbb{P}^1 \times \mathbb{P}^1$ given by $x \mapsto (\widetilde{p}_1(x), \widetilde{p}_2(x))$ is a double cover branched on a smooth curve in $|-2K_{\mathbb{P}^1 \times \mathbb{P}^1}|$. Equivalently, it is defined by the linear system $|\pi^*(E_1 + E_2)|$, as its image in $\mathbb{P}^3$ factors through $\mathbb{P}^1 \times \mathbb{P}^1$ by [40] (see also [3, VIII.§18]). Then $\widetilde{S}$ is embedded in the total space $T(-K_{\mathbb{P}^1 \times \mathbb{P}^1})$ of the line bundle $-K_{\mathbb{P}^1 \times \mathbb{P}^1}$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The variety $T(-K_{\mathbb{P}^1 \times \mathbb{P}^1})$ compactifies to $V' = \mathbb{P}(-K_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1})$ by adding a section at infinity Σ of V' corresponding to the surjection $-K_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$. Then V' identifies with the blow–up of the cone V at its vertex, the exceptional divisor being Σ. We have the inclusion $\widetilde{S} \subset T(-K_{\mathbb{P}^1 \times \mathbb{P}^1}) = V' - \Sigma \subset V'$, hence an inclusion $\widetilde{S} \subset V$, and it is easy to check that $\widetilde{S}$ identifies with a quadric section of V.

Let now $t_i : \mathbb{P}^1 \to \mathbb{P}^1$ be the involution corresponding to the double cover σ_i, for $i = 1, 2$. Consider the involution $t : (x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \to (t_1(x), t_2(y)) \in \mathbb{P}^1 \times \mathbb{P}^1$. By appropriately choosing coordinates we may assume that t coincides with the involution $t : (u_0 : u_1) \times (v_0 : v_1) \mapsto (-u_0 : u_1) \times (-v_0 : v_1)$. The involution t on $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to the involution τ of V defined above and one checks that $\widetilde{S}$ is τ-invariant. Indeed, τ lifts to an involution of $\widetilde{S}$ because t clearly fixes the branch divisor of the double cover

$\widetilde{S} \to \mathbb{P}^1 \times \mathbb{P}^1$, and $\widetilde{S}$ is also invariant by the involution of $T(-K_{\mathbb{P}^1\times\mathbb{P}^1})$ that sends a point z in a fibre $\mathbb{C}$ over a point of $\mathbb{P}^1 \times \mathbb{P}^1$ to $-z$. Thus, the quotient map $\pi : V \to W$ maps $\widetilde{S}$ back to S, and the last assertion follows. □

Lemma 4.8. *The general members of* $\mathcal{E}_{9,2}^{(II)^+}$ *and* $\mathcal{E}_{9,2}^{(II)^-}$ *are extendable.*

Proof. Let $(S,H) \in \mathcal{E}_{9,2}^{(II)^+}$ (respectively, $\mathcal{E}_{9,2}^{(II)^-}$) be general. We have $H \sim 4E_1 + 2E_2$ (resp., $4E_1+2E_2+K_S$). Set $D := 2E_2$ (resp., $2E_2+K_S$). Then $(S, H+D \sim 4(E_1+E_2))$ is extendable to $(W,\mathcal{L})$ by Proposition 4.7, and the result follows from Lemma 4.4. □

Remark 4.9. Similar reasonings yield that the general members of $\mathcal{E}_{13,3}^{(II)}$, $\mathcal{E}_{10,3}^{(II)}$, $\mathcal{E}_{9,3}^{(II)}$, $\mathcal{E}_{7,2}^{(II)}$, $\mathcal{E}_{6,2}$ are extendable, and the corresponding moduli maps are not generically finite, but we will not need this fact. Similarly, a thorough study of the only Enriques–Fano threefold of genus 9 in [4, 41] shows that the general member of $\mathcal{E}_{9,4}^{+}$ is extendable. In particular, the general members of all the moduli spaces occurring in Corollary 1.2 are extendable to an Enriques–Fano threefold $(X,\mathcal{L})$ such that the morphism $\varphi_{\mathcal{L}}$ defined by $|\mathcal{L}|$ is an isomorphism on the general member S of $|\mathcal{L}|$ (as $\phi(\mathcal{L}|_S) \geqslant 3$), whence $\varphi_{\mathcal{L}}(X) \subset \mathbb{P}^g$ is not a cone and has smooth Enriques surfaces as hyperplane sections.

We conclude the section by explaining how to use our results (without using [27, 38]) to bound the families of Enriques–Fano threefolds having the property that their Enriques sections are *general in moduli*, meaning that the family of polarized Enriques sections obtained from the family dominates the moduli space $\mathcal{E}$ of Enriques surfaces.

Corollary 4.10. *Consider a family of Enriques–Fano threefolds* $(X,\mathcal{L})$ *such that* $\mathcal{L}$ *is globally generated and whose Enriques sections are general in moduli. Then the general polarized Enriques sections of the family belong to one of the following moduli spaces:*

$$\mathcal{E}_{17,4}^{(IV)^+},\ \mathcal{E}_{13,4}^{(II)^+},\ \mathcal{E}_{13,3}^{(II)},\ \mathcal{E}_{10,3}^{(II)},\ \mathcal{E}_{9,4}^{+},\ \mathcal{E}_{9,3}^{(II)},\ \mathcal{E}_{7,3},\ \mathcal{E}_{9,2}^{(II)^+},\ \mathcal{E}_{9,2}^{(II)^-},$$
$$\mathcal{E}_{7,2}^{(I)},\ \mathcal{E}_{7,2}^{(II)},\ \mathcal{E}_{6,2},\ \mathcal{E}_{5,2}^{(I)},\ \mathcal{E}_{5,2}^{(II)^+},\ \mathcal{E}_{5,2}^{(II)^-},\ \mathcal{E}_{4,2},\ \mathcal{E}_{3,2}$$

Proof (granting Theorems 1-2). Let $(X,\mathcal{L})$ be general in the family and $S \in |\mathcal{L}|$ be general. The assumption that $\mathcal{L}$ is globally generated yields $\phi(\mathcal{L}|_S) \geqslant 2$ by [12, Thms. 4.1] or [15, Prop. 4.7.1]. If $g \leqslant 5$, then $\phi \leqslant 2$ and $(S,\mathcal{L}|_S)$ belongs to one of $\mathcal{E}_{5,2}^{(I)}, \mathcal{E}_{5,2}^{(II)^+}, \mathcal{E}_{5,2}^{(II)^-}, \mathcal{E}_{4,2}, \mathcal{E}_{3,2}$, as those are all irreducible components of $\mathcal{E}_{g,2}$. If $g \geqslant 6$, then Proposition 4.2 and the assumptions of the corollary yield that $(S,\mathcal{L}|_S)$ belongs to one of the components of $\mathcal{E}_{g,\phi}$ over which the moduli map $\chi_{g,\phi}$ is not generically finite. These are given in Theorems 1-2. □

5. Computing cohomology of twisted tangent bundles

In the rest of the paper we adopt the following:

Notation 5.1. For an Enriques surface S, we denote by $\pi : \widetilde{S} \to S$ the $K3$ double cover. For any divisor (or line bundle) D on S we write $\widetilde{D} := \pi^* D$.

In view of Corollary 3.3 and (9), in this section we will develop some tools for computing or bounding $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))$, where H is a big and nef line bundle on S.

Let F_1 and F_2 be two half-fibers such that $F_1 \cdot F_2 = 1$. Then $|\widetilde{F}_1 + \widetilde{F}_2|$ is base point free and (as in the proof of Proposition 4.7) it defines a double cover $g : \widetilde{S} \to \mathbb{P}^1 \times \mathbb{P}^1$, branched on a smooth curve $R \in |-2K_{\mathbb{P}^1\times\mathbb{P}^1}|$ (see also [40] and [3, VIII.§18]). Denote by $\widetilde{R} \in |2\widetilde{F}_1 + 2\widetilde{F}_2|$ the ramification divisor. Define, for any big and nef H on S

$$\alpha = \alpha(H, F_1, F_2) := h^1(H-2F_1)+h^1(H-2F_1+K_S)+h^1(H-2F_2)+h^1(H-2F_2+K_S)$$

and

$$\beta = \beta(H, F_1, F_2) := h^0(\mathcal{O}_{\widetilde{R}}(4\widetilde{F}_1 + 4\widetilde{F}_2 - \widetilde{H})).$$

Lemma 5.2. *Let H be a big and nef line bundle on S and F_1 and F_2 two half-fibers such that $F_1 \cdot F_2 = 1$. Then $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) \leqslant \alpha + \beta$, with equality if $\alpha = 0$.*

Proof. Dualizing the sequence of relative differentials and tensoring by $\mathcal{O}_{\widetilde{S}}(-\widetilde{H})$ we get

$$0 \to \mathcal{T}_{\widetilde{S}}(-\widetilde{H}) \to g^*\mathcal{T}_{\mathbb{P}^1\times\mathbb{P}^1}(-\widetilde{H}) \simeq \mathcal{O}_{\widetilde{S}}(2\widetilde{F}_1 - \widetilde{H}) \oplus \mathcal{O}_{\widetilde{S}}(2\widetilde{F}_2 - \widetilde{H}) \to \mathcal{O}_{\widetilde{R}}(2\widetilde{R} - \widetilde{H}) \to 0.$$

Since H is big and nef, we have $h^0(2\widetilde{F}_i - \widetilde{H}) = 0$, and the result follows. □

The following bounds on β will be useful later on:

$$\beta = 0 \quad \text{if} \quad (F_1 + F_2) \cdot H > 8, \tag{12}$$

and

$$\begin{aligned}\beta \leqslant\ & h^0(4F_1 + 4F_2 - H) + h^0(4F_1 + 4F_2 - H + K_S) \\ & + h^1(H - 2F_1 - 2F_2) + h^1(H - 2F_1 - 2F_2 + K_S).\end{aligned} \tag{13}$$

Indeed, (12) follows by reasons of degree, as

$$\begin{aligned}\deg(\mathcal{O}_{\widetilde{R}}(4\widetilde{F}_1 + 4\widetilde{F}_2 - \widetilde{H})) &= \pi^*(2F_1 + 2F_2) \cdot \pi^*(4F_1 + 4F_2 - H) \\ &= 4(F_1 + F_2)(4F_1 + 4F_2 - H) = 4\,(8 - (F_1 + F_2) \cdot H)\,,\end{aligned}$$

whereas (13) follows from the exact sequence

$$0 \longrightarrow \pi^*\mathcal{O}_S(2F_1 + 2F_2 - H) \longrightarrow \pi^*\mathcal{O}_S(4F_1 + 4F_2 - H) \longrightarrow \mathcal{O}_{\widetilde{R}}(2\widetilde{R} - \widetilde{H}) \longrightarrow 0. \tag{14}$$

Let next G_1 and G_2 be two effective primitive isotropic divisors such that $G_1 \cdot G_2 = 2$ and $G_1 + G_2$ is nef (e.g., G_1 and G_2 are half-fibers). Then $|\widetilde{G}_1 + \widetilde{G}_2|$ is base point free and embeds $\widetilde{S}$ into $\mathbb{P}^5$ as a complete intersection of three quadrics by [40]. Set

$$\begin{aligned}\gamma = \gamma(H, G_1, G_2) &:= h^1(H - G_1 - G_2) + h^1(H - G_1 - G_2 + K_S) \\ \delta = \delta(H, G_1, G_2) &:= h^0(2G_1 + 2G_2 - H) + h^0(2G_1 + 2G_2 - H + K_S) \\ \epsilon = \epsilon(H, G_1, G_2) &:= \operatorname{cork} \mu_{\widetilde{G}_1+\widetilde{G}_2, \widetilde{H}-\widetilde{G}_1-\widetilde{G}_2},\end{aligned}$$

where $\mu_{A,B} : H^0(A) \otimes H^0(B) \longrightarrow H^0(A + B)$ is the multiplication map of sections.

Lemma 5.3. *Let H be a big and nef line bundle on S with $H^2 \geqslant 4$ and let G_1 and G_2 be two effective primitive isotropic divisors such that $G_1 \cdot G_2 = 2$ and $G_1 + G_2$ is nef. If $H \not\equiv G_1 + G_2$, then $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) \leqslant \epsilon + 6\gamma + 3\delta$, with equality if $\epsilon = \gamma = 0$. If $H \equiv G_1 + G_2$, then $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 12$.*

Proof. The Euler sequence twisted by $\mathcal{O}_{\widetilde{S}}(-\widetilde{H})$ is

$$0 \longrightarrow \mathcal{O}_{\widetilde{S}}(-\widetilde{H}) \longrightarrow H^0(\widetilde{G}_1+\widetilde{G}_2)^\vee \otimes \mathcal{O}_{\widetilde{S}}(\widetilde{G}_1+\widetilde{G}_2-\widetilde{H}) \longrightarrow \mathcal{T}_{\mathbb{P}^5}|_{\widetilde{S}}(-\widetilde{H}) \longrightarrow 0.$$

The map on cohomology $H^2(\mathcal{O}_{\widetilde{S}}(-\widetilde{H})) \to H^0(\widetilde{G}_1+\widetilde{G}_2)^\vee \otimes H^2(\mathcal{O}_{\widetilde{S}}(\widetilde{G}_1+\widetilde{G}_2-\widetilde{H}))$ is the dual of $\mu_{\widetilde{G}_1+\widetilde{G}_2,\widetilde{H}-\widetilde{G}_1-\widetilde{G}_2}$. Thus, as $h^0(-\widetilde{H})=h^1(-\widetilde{H})=0$, we have

$$h^0(\mathcal{T}_{\mathbb{P}^5}|_{\widetilde{S}}(-\widetilde{H})) = 6h^0(\mathcal{O}_{\widetilde{S}}(\widetilde{G}_1+\widetilde{G}_2-\widetilde{H})) = \begin{cases} 6 & \text{if } H \equiv G_1+G_2, \\ 0 & \text{if } H \not\equiv G_1+G_2 \end{cases} \tag{15}$$

and

$$h^1(\mathcal{T}_{\mathbb{P}^5}|_{\widetilde{S}}(-\widetilde{H})) = \epsilon + 6\gamma. \tag{16}$$

The normal bundle sequence twisted by $\mathcal{O}_{\widetilde{S}}(-\widetilde{H})$ is

$$0 \longrightarrow \mathcal{T}_{\widetilde{S}}(-\widetilde{H}) \longrightarrow \mathcal{T}_{\mathbb{P}^5}|_{\widetilde{S}}(-\widetilde{H}) \longrightarrow \mathcal{O}_{\widetilde{S}}(2\widetilde{G}_1+2\widetilde{G}_2-\widetilde{H})^{\oplus 3} \longrightarrow 0. \tag{17}$$

Taking cohomology, and using (15) and (16), yields the desired result (using that $h^0(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))=0$ by the Mori-Sumihiro-Wahl Theorem [31, 44] when $H \equiv G_1+G_2$). □

Remark 5.4. Pushing forward (17) by π and using the fact that $\mathcal{T}_{\widetilde{S}} \simeq \pi^*\mathcal{T}_S$, we obtain a splitting of the coboundary map into the direct sum of $H^0(\mathcal{O}_S(2G_1+2G_2-H))^{\oplus 3} \to H^1(\mathcal{T}_S(-H))$ and $H^0(\mathcal{O}_S(2G_1+2G_2-H+K_S))^{\oplus 3} \to H^1(\mathcal{T}_S(-H+K_S))$.

We end this section with some results that will be useful to compute the corank of multiplication maps. They are similar to the generalization by Mumford of a theorem of Castelnuovo, cf. [36, Thm. 2, p. 41]:

Lemma 5.5. *Let F and G be divisors on a projective surface S and assume that $|G|$ is a base point free pencil. Then* $\operatorname{cork}(\mu_{F,G}) \leqslant h^1(F-G)$*, with equality if $h^1(F)=0$.*

Proof. This follows by tensoring the evaluation exact sequence

$$0 \longrightarrow \mathcal{O}_S(-G) \longrightarrow H^0(G)\otimes\mathcal{O}_S \longrightarrow \mathcal{O}_S(G) \longrightarrow 0$$

with $\mathcal{O}_S(F)$ and taking cohomology. □

Lemma 5.6. ([21, Obs. 1.4.1]) *Let F and $G=\sum_{i=1}^n G_i$ be divisors on a projective surface. If the multiplication maps $\mu_{F+G_1+\cdots+G_{i-1},G_i}$ are surjective for all $1 \leqslant i \leqslant n$, then also $\mu_{F,G}$ is surjective.*

Remark 5.7. When all $|G_i|$ are base point free pencils, the criterion in Lemma 5.6 is satisfied (by Lemma 5.5) if $h^1(F+G_1+\cdots+G_{i-1}-G_i)=0$ for all $i=1,\ldots,n$.

6. Fiber dimensions of moduli maps

In this section we will apply the results of the previous section, combined with Corollary 3.3 and (9), to prove Theorem 1 and part of Theorem 2.

We will make use of the following facts. Let S be an Enriques surface. By [25, Lemma 2.1], if A and B are effective divisors on S, then

$$A^2, B^2 \geqslant 0 \implies A\cdot B \geqslant 0, \text{ with equality iff } A^2=0 \text{ and } A \equiv kB \text{ for some } k\in\mathbb{Q}. \tag{18}$$

For any divisor D such that $D^2 \geqslant 0$ and $D \not\sim K_S$, either D or $-D$ is effective. If moreover S is unnodal, then any effective divisor D is nef, and it is ample if and only if $D^2 > 0$. Thus, for any divisor D on an unnodal S we have (by Riemann-Roch and Mumford vanishing)

$$(19) \qquad D^2 \leqslant -2 \implies \begin{cases} h^0(D) = h^0(D+K_S) = h^2(D) = h^2(D+K_S) = 0 \quad \text{and} \\ h^1(D) = h^1(D+K_S) = -\frac{1}{2}D^2 - 1, \end{cases}$$

$$(20) \ h^1(D) = h^1(D+K_S) = 0 \iff D^2 \geqslant -2 \text{ and } D \not\equiv \pm lE,\ l \geqslant 2, \text{ for a half-fiber } E.$$

Remark 6.1. Recall that the general Enriques surface is unnodal. The assumption in the results below that S be unnodal is not necessary in order to apply the results from §5. It is added to simplify the proofs of the vanishings of various cohomology groups. A more thorough study will yield bounds on $h^1(\mathcal{T}_S(-H))$ in terms of the existence of specific configurations of rational curves. We therefore expect that a result similar to Corollary 1.2 can also be obtained in the nodal cases (yielding for instance the two additional cases of Corollary 6.5 below), but it would require additional work that would bring us beyond the scope of this paper.

We use Notations 2.1 and 5.1. We say that a simple isotropic decomposition $H \sim \sum_{i=1}^n \alpha_i F_i + \varepsilon K_S$ contains $\sum_{i=1}^n a_i F_i$ if $\alpha_i \geqslant a_i$ for all $i \in \{1, \dots, n\}$.

Lemma 6.2. *Assume S is an unnodal Enriques surface and H a big and nef line bundle on S. We have $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 0$ if a simple isotropic decomposition of H contains*

(a) $E_1 + E_2 + E_3 + E_4 + E_5$,
(b) $2E_1 + E_2 + E_3 + E_4$,
(c) $3E_1 + E_2 + E_3$,
(d) $5E_1 + 3E_2$,
(e) $2E_1 + E_3 + E_{1,2}$,
(f) $E_1 + E_2 + E_{1,2}$,
(g) $3E_1 + 2E_{1,2}$,
(h) $2E_1 + 3E_{1,2}$.

Proof. **(a)** We have $H \sim E_1+E_2+E_3+E_4+E_5+D$, where D is nef. By [15, Cor. 2.5.6] there are primitive isotropic F_1, F_2 such that $F_1 \cdot F_2 = F_i \cdot E_j = 1$ for $i \in \{1,2\}$ and $j \in \{1,2,3,4,5\}$ and such that $F_j \not\equiv \frac{1}{2}(E_1 + \cdots + E_5) - \frac{1}{E_1 \cdot D} D$.

We apply Lemma 5.2. We have $(F_1 + F_2) \cdot H \geqslant 10$, whence $\beta = 0$ by (12). We have $(E_1 + \cdots + E_5 - 2F_1)^2 = 0$, whence

$$\begin{aligned} (H - 2F_1)^2 &= (E_1 + \cdots + E_5 - 2F_1)^2 + 2(E_1 + \cdots + E_5 - 2F_1) \cdot D + D^2 \\ &= 2(E_1 + \cdots + E_5 - 2F_1) \cdot D + D^2 \geqslant 0, \end{aligned}$$

with equality if and only if $D^2 = 0$ and $D \equiv k(E_1 + \cdots + E_5 - 2F_1)$ for some $k \in \mathbb{Q}$ by (18). In the latter case, intersecting with E_1 yields $k = \frac{1}{2} E_1 \cdot D$, whence $F_1 \equiv \frac{1}{2}(E_1 + \cdots + E_5) - \frac{1}{E_1 \cdot D} D$, a contradiction. Hence $h^1(H - 2F_1) = h^1(H - 2F_1 + K_S) = 0$ by (20). By symmetry, also $h^1(H - 2F_2) = h^1(H - 2F_2 + K_S) = 0$, so that $\alpha = 0$. The result then follows from Lemma 5.2.

(b) We have $H \sim 2E_1 + E_2 + E_3 + E_4 + D$, where D is nef. By symmetry, we may assume that

$$(21) \qquad D \cdot E_4 \geqslant D \cdot E_2.$$

We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$.

We have $(H - 2E_1)^2 = (E_2 + E_3 + E_4 + D)^2 > 0$, whence $h^1(H - 2E_1) = h^1(H - 2E_1 + K_S) = 0$ by (20). We have $(2E_1 + E_3 + E_4 - E_2)^2 = 2$, whence

$$(H - 2E_2)^2 = (2E_1 + E_3 + E_4 - E_2)^2 + 2(2E_1 + E_3 + E_4 - E_2) \cdot D + D^2 \geqslant 2,$$

since both $2E_1 + E_3 + E_4 - E_2$ and D are effective. It follows that $h^1(H - 2E_2) = h^1(H - 2E_2 + K_S) = 0$ again by (20). Hence $\alpha = 0$.

We next prove that $\beta = 0$. We will apply (13). We first note that, by (19),

$$h^0(4E_1 + 4E_2 - H) = h^0(2E_1 + 3E_2 - E_3 - E_4 - D) \leqslant h^0(2E_1 + 3E_2 - E_3 - E_4) = 0,$$

as $(2E_1 + 3E_2 - E_3 - E_4)^2 = -6$. Similarly, $h^0(4E_1 + 4E_2 - H + K_S) = 0$. We also have

$$\begin{aligned}(H - 2E_1 - 2E_2)^2 &= (E_3 + E_4 - E_2)^2 + 2(E_3 + E_4 - E_2) \cdot D + D^2 \\ &= -2 + 2(E_3 + E_4 - E_2) \cdot D + D^2 \geqslant -2,\end{aligned}$$

by (21), and $(H - 2E_1 - 2E_2)^2 = 0$ if and only if $(D^2, (E_3 + E_4 - E_2) \cdot D) \in \{(0, 1), (2, 0)\}$. But in the latter case we must have $D \cdot E_3 = 0$ by (21), contradicting (18). In the former case we have $D \cdot (H - 2E_1 - 2E_2) = 1$, implying that $H - 2E_1 - 2E_2$ is primitive. Hence, $h^1(H - 2E_1 - 2E_2) = h^1(H - 2E_1 - 2E_2 + K_S) = 0$ by (20). Thus, $\beta = 0$ by (13).

(c) We have $H \sim 3E_1 + E_2 + E_3 + D$, where D is nef. By symmetry, we may assume that

$$D \cdot E_3 \geqslant D \cdot E_2. \tag{22}$$

We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$.

We have $(H - 2E_1)^2 = (E_1 + E_2 + E_3 + D)^2 > 0$, whence $h^1(H - 2E_1) = h^1(H - 2E_1 + K_S) = 0$ by (20). We have

$$\begin{aligned}(H - 2E_2)^2 &= (3E_1 + E_3 - E_2)^2 + 2(3E_1 + E_3 - E_2) \cdot D + D^2 \\ &= -2 + 2(3E_1 + E_3 - E_2) \cdot D + D^2 \geqslant -2,\end{aligned}$$

by (22), and $(H - 2E_2)^2 = 0$ if and only if $(D^2, (3E_1 + E_3 - E_2) \cdot D) \in \{(0, 1), (2, 0)\}$. But in the latter case we must have $D \cdot E_1 = 0$ by (22), contradicting (18). In the first case we have $D \cdot (H - 2E_2) = 1$, implying that $H - 2E_2$ is primitive. It follows that $h^1(H - 2E_2) = h^1(H - 2E_2 + K_S) = 0$ again by (20). Hence $\alpha = 0$.

We next prove that $\beta = 0$. We will apply (13). We first note that, by (19),

$$h^0(4E_1 + 4E_2 - H) = h^0(E_1 + 3E_2 - E_3 - D) \leqslant h^0(E_1 + 3E_2 - E_3) = 0,$$

as $(E_1 + 3E_2 - E_3)^2 = -2$. Similarly, $h^0(4E_1 + 4E_2 - H + K_S) = 0$. We also have

$$\begin{aligned}(H - 2E_1 - 2E_2)^2 &= (E_1 + E_3 - E_2)^2 + 2(E_1 + E_3 - E_2) \cdot D + D^2 \\ &= -2 + 2(E_1 + E_3 - E_2) \cdot D + D^2 \geqslant -2,\end{aligned}$$

by (22), and $(H - 2E_1 - 2E_2)^2 = 0$ if and only if $(D^2, (E_1 + E_3 - E_2) \cdot D) \in \{(0, 1), (2, 0)\}$. But in the latter case we must have $D \cdot E_1 = 0$ by (22), contradicting (18). In the first we have $D \cdot (H - 2E_1 - 2E_2) = 1$, implying that $H - 2E_1 - 2E_2$ is primitive. Hence, $h^1(H - 2E_1 - 2E_2) = h^1(H - 2E_1 - 2E_2 + K_S) = 0$ by (20). Thus, $\beta = 0$ by (13).

(d) We have $H \sim 5E_1 + 3E_2 + D$, where D is nef. We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$ and argue as in (c).

(e) We have $H \sim 2E_1 + E_3 + E_{1,2} + D$, where D is nef. We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_3$.

We have $(H - 2E_1)^2 = (E_3 + E_{1,2} + D)^2 > 0$, whence $h^1(H - 2E_1) = h^1(H - 2E_1 + K_S) = 0$ by (20). We have $(2E_1 + E_{1,2} - E_3)^2 = 2$, whence

$$(H - 2E_3)^2 = (2E_1 + E_{1,2} - E_3)^2 + 2(2E_1 + E_{1,2} - E_3) \cdot D + D^2 > 0,$$

so that also $h^1(H - 2E_3) = h^1(H - 2E_3 + K_S) = 0$ by (20). It follows that $\alpha = 0$.

To prove that $\beta = 0$, we will use (12) and (13). We first note that

$$h^0(4E_1 + 4E_3 - H) = h^0(2E_1 + 3E_3 - E_{1,2} - D) \leqslant h^0(2E_1 + 3E_3 - E_{1,2}) = 0, \tag{23}$$

by (19), as $(2E_1 + 3E_3 - E_{1,2})^2 = -2$. Similarly, $h^0(4E_1 + 4E_3 - H + K_S) = 0$. To finish the proof that $\beta = 0$ we divide the treatment in different cases.

Assume that $E_{1,2}$ is present in the isotropic decomposition of D. Then $E_1 \cdot H \geqslant 5$ and $E_3 \cdot H \geqslant 4$, so that $\beta = 0$ by (12).

Assume that E_3 is present in the isotropic decomposition of D, whereas $E_{1,2}$ is not. Write $D' = D - E_3$. Then $H - 2E_1 - 2E_3 = E_{1,2} + D'$, so that $h^1(H - 2E_1 - 2E_3) = h^1(H - 2E_1 - 2E_3 + K_S) = 0$ by (20). Hence $\beta = 0$ by (23) and (13).

Assume that E_j is present in the isotropic decomposition of D, for $j = 1$ or 2, whereas $E_{1,2}$ is not. Then $D \cdot (E_{1,2} - E_3) \geqslant 1$ and

$$\begin{aligned}(H - 2E_1 - 2E_3)^2 &= (-E_3 + E_{1,2} + D)^2 = (E_{1,2} - E_3)^2 + 2(E_{1,2} - E_3) \cdot D + D^2 \\ &= -2 + 2(E_{1,2} - E_3) \cdot D + D^2 \geqslant 0,\end{aligned}$$

with equality if and only if $D = E_j$. In this case, $h^1(H - 2E_1 - 2E_3) = h^1(H - 2E_1 - 2E_3 + K_S) = 0$ by (20), as $E_j \cdot (H - 2E_1 - 2E_3) = 1$. Again, $\beta = 0$ by (23) and (13).

Finally, assume that neither E_1, E_2, E_3 nor $E_{1,2}$ are present in the isotropic decomposition of D. Then $D \cdot (E_{1,2} - E_3) = 0$, whence

$$(H-2E_1-2E_3)^2 = (-E_3+E_{1,2}+D)^2 = (E_{1,2}-E_3)^2+2(E_{1,2}-E_3)\cdot D+D^2 = D^2-2 \geqslant -2,$$

and is 0 if and only if $D^2 = 2$. In this case, we have $E_1 \cdot D \geqslant 2$ and $E_3 \cdot D \geqslant 2$, so that $E_1 \cdot H \geqslant 5$ and $E_3 \cdot H \geqslant 5$. Hence, $\beta = 0$ by (12).

(f) We have $H \sim E_1 + E_2 + E_{1,2} + D$, where D is nef. By symmetry between E_1 and E_2, we may assume that $D \not\equiv kE_2$ for any $k \geqslant 1$. We apply Lemma 5.3 with $G_1 = E_1$ and $G_2 = E_{1,2}$.

We have $H - E_1 - E_{1,2} = E_2 + D$, whence $h^1(H - E_1 - E_{1,2}) = h^1(H - E_1 - E_{1,2} + K_S) = 0$ by (20) and the fact that $D \not\equiv kE_2$. Therefore, $\gamma = 0$.

By (19) and the fact that $(E_1 + E_{1,2} - E_2)^2 = -2$, we have

$$h^0(2E_1 + 2E_{1,2} - H) = h^0(E_1 + E_{1,2} - E_2 - D) \leqslant h^0(E_1 + E_{1,2} - E_2) = 0.$$

Similarly, $h^0(2E_1 + 2E_{1,2} - H + K_S) = 0$. Hence, $\delta = 0$.

To check that the multiplication map $\mu_{\widetilde{E}_1+\widetilde{E}_{1,2},\widetilde{E}_2+\widetilde{D}}$ is surjective, we apply Lemmas 5.5 and 5.6, cf. Remark 5.7. Write $D \equiv \sum_{i=1}^n \alpha_i E_i + \alpha_0 E_{1,2}$ for some $n \leqslant 9$. The multiplication map

$$\mu_{\widetilde{E}_1+\widetilde{E}_{1,2},\widetilde{E}_2} : H^0(\widetilde{E}_1 + \widetilde{E}_{1,2}) \otimes H^0(\widetilde{E}_2) \longrightarrow H^0(\widetilde{E}_1 + \widetilde{E}_{1,2} + \widetilde{E}_2)$$

is surjective, since (20) and the fact that $(E_1 + E_{1,2} - E_2)^2 = -2$ imply that

$$h^1(\widetilde{E}_1 + \widetilde{E}_{1,2} - \widetilde{E}_2) = h^1(E_1 + E_{1,2} - E_2) + h^1(E_1 + E_{1,2} - E_2 + K_S) = 0.$$

Likewise, all multiplication maps $\mu_{\widetilde{E}_1+\widetilde{E}_{1,2}+j\widetilde{E}_2,\widetilde{E}_2}$ for $1 \leqslant j \leqslant \alpha_2$ are surjective, since all $((\widetilde{E}_1 + \widetilde{E}_{1,2} + j\widetilde{E}_2) - \widetilde{E}_2)^2 > 0$. For the same reason, all $\mu_{\widetilde{E}_1+\widetilde{E}_{1,2}+(\alpha_2+1)\widetilde{E}_2+j\widetilde{E}_1,\widetilde{E}_1}$, for $0 \leqslant j \leqslant \alpha_1 - 1$, are surjective, as well as all $\mu_{(\alpha_1+1)\widetilde{E}_1+\widetilde{E}_{1,2}+(\alpha_2+1)\widetilde{E}_2+j\widetilde{E}_{1,2},\widetilde{E}_2}$, for $0 \leqslant j \leqslant \alpha_0 - 1$. Finally, for any $i \in \{3, \dots, n\}$ and any $0 \leqslant j \leqslant \alpha_i - 1$, set

$$B_{ij} := (\alpha_1 + 1)E_1 + (\alpha_0 + 1)E_{1,2} + (\alpha_2 + 1)E_2 + \alpha_3 E_3 + \cdots + \alpha_{i-1}E_{i-1} + jE_i - E_i.$$

Then $\widetilde{B}_{ij} = \widetilde{E}_1 + \widetilde{E}_{1,2} + \widetilde{E}_2 - \widetilde{E}_i + \Delta$, with

$$\Delta := \alpha_1 \widetilde{E}_1 + \alpha_0 \widetilde{E}_{1,2} + \alpha_2 \widetilde{E}_2 + \alpha_3 \widetilde{E}_3 + \cdots + \alpha_{i-1} \widetilde{E}_{i-1} + j \widetilde{E}_i.$$

Since $\Delta^2 \geqslant 0$ and $(\widetilde{E}_1 + \widetilde{E}_{1,2} + \widetilde{E}_2 - \widetilde{E}_i)^2 = 8$, we have $\widetilde{B}_{ij}^2 > 0$, whence $h^1(\widetilde{B}_{ij}) = h^1(B_{ij}) + h^1(B_{ij} + K_S) = 0$ by (20). It follows by Lemma 5.6 and Remark 5.7 that $\mu_{\widetilde{E}_1+\widetilde{E}_{1,2},\widetilde{E}_2+\widetilde{D}}$ is surjective, whence $\epsilon = 0$.

(g) We have $H \sim 3E_1 + 2E_{1,2} + D$, where D is nef. If E_2 is present in D, we are done by (f). If any E_j, for $j \neq 1, 2$, is present in D, we are done by (e). We have therefore left to treat the case where $H \equiv a_1 E_1 + a_0 E_{1,2}$, with $a_1 \geqslant 3$ and $a_0 \geqslant 2$. By symmetry, we may assume that $a_1 \geqslant a_0$. As in the previous case, we apply Lemma 5.3 with $G_1 = E_1$ and $G_2 = E_{1,2}$.

We have $H - E_1 - E_{1,2} = (a_1 - 1)E_1 + (a_2 - 1)E_{1,2}$, whence $h^1(H - E_1 - E_{1,2}) = h^1(H - E_1 - E_{1,2} + K_S) = 0$ by (20). Therefore, $\gamma = 0$.

We have $2E_1 + 2E_{1,2} - H \equiv -(a_1 - 2)E_1 - (a_0 - 2)E_{1,2}$, whence $h^0(2E_1 + 2E_{1,2} - H) = h^0(2E_1 + 2E_{1,2} - H + K_S) = 0$. Thus, $\delta = 0$.

To check that the map $\mu_{\widetilde{E}_1+\widetilde{E}_{1,2},(a_1-1)\widetilde{E}_1+(a_0-1)\widetilde{E}_{1,2}}$ is surjective, we apply Lemma 5.6. The map $\mu_{\widetilde{E}_1+\widetilde{E}_{1,2},(a_0-1)(\widetilde{E}_1+\widetilde{E}_{1,2})}$ is surjective by [40, Thm. 6.1]. Finally, for $0 \leqslant j \leqslant a_1 - a_0 - 1$, set $B_j := (a_0 + j)E_1 + (a_0 - 1)E_{1,2} - E_1$. Then $B_j^2 > 0$, so that $h^1(\widetilde{B}_j) = h^1(B) + h^1(B + K_S) = 0$ by (20), whence all $\mu_{(a_0+j)\widetilde{E}_1+(a_0-1)\widetilde{E}_{1,2},\widetilde{E}_1}$ are surjective by Lemma 5.5. The map $\mu_{\widetilde{E}_1+\widetilde{E}_{1,2},(a_1-1)\widetilde{E}_1+(a_0-1)\widetilde{E}_{1,2}}$ is thus surjective by Lemma 5.6.

(h) This case is treated as the previous one, exchanging the roles of E_1 and $E_{1,2}$. □

Lemma 6.3. *Assume S is an unnodal Enriques surface and H a big and nef line bundle on S. If $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) \neq 0$, then we are in one of the following cases:*

component of moduli space	*simple isotr. decomp.*	$h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))$
$\mathcal{E}_{17,4}^{(IV)^+}$ and $\mathcal{E}_{17,4}^{(IV)^-}$	$H \equiv 4(E_1 + E_2)$	$= 1$
$\mathcal{E}_{13,3}^{(II)}$	$H \sim 4E_1 + 3E_2$	$= 2$
$\mathcal{E}_{13,4}^{(II)^+}$ and $\mathcal{E}_{13,4}^{(II)^-}$	$H \equiv 2(E_1 + E_2 + E_3)$	$\leqslant 1$
$\mathcal{E}_{10,3}^{(II)}$	$H \sim 3E_1 + 3E_2$	$= 4$
$\mathcal{E}_{9,3}^{(II)}$	$H \sim 2E_1 + 2E_2 + E_3$	$= 2$
$\mathcal{E}_{9,4}^{+}$ and $\mathcal{E}_{9,4}^{-}$	$H \equiv 2(E_1 + E_{1,2})$	$= 3$
$\mathcal{E}_{7,3}$	$H \sim E_1 + E_2 + E_3 + E_4$	$\leqslant 2$
$\mathcal{E}_{6,2}$	$H \sim 2E_1 + E_2 + E_3$	$= 4$
$\mathcal{E}_{4,2}$	$H \sim E_1 + E_2 + E_3$	$= 8$
$\mathcal{E}_{3,2}$	$H \sim E_1 + E_{1,2}$	$= 12$
$\mathcal{E}_{2k+1,2}^{(I)}$, $k \geqslant 2$	$H \sim kE_1 + E_{1,2}$, $k \geqslant 2$	
$\mathcal{E}_{2k+1,2}^{(II)}$ *if k is odd;* $\mathcal{E}_{2k+1,2}^{(II)+}$ *and* $\mathcal{E}_{2k+1,2}^{(II)-}$ *if k is even*	$H \equiv kE_1 + 2E_2$, $k \geqslant 2$	
$\mathcal{E}_{k+1,1}$	$H \equiv kE_1 + E_2$, $k \geqslant 1$	

Proof. Up to rearranging indices, the decompositions in the table are the only ones not covered by Lemma 6.2, except for $H \equiv E_1 + kE_3 + lE_{1,2}$, with $k, l \geqslant 1$. Set $F := E_1 + E_{1,2} - E_3$. Then $F^2 = 0$, $E_{1,2} \cdot F = 1$ and $E_3 \cdot F = 2$. Thus, $H \equiv (k+1)E_3 + F + (l-1)E_{1,2}$ is a simple isotropic decomposition, which can be rewritten, after renaming indices, $H \equiv (k+1)E_1 + E_{1,2} + (l-1)E_3$. This falls into case (e) of Lemma 6.2 if $l \geqslant 2$, and is present in the table of the lemma if $l = 1$. We now study $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))$.

• **Cases $\mathcal{E}_{17,4}^{(IV)^+}$ and $\mathcal{E}_{17,4}^{(IV)^-}$.** We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$. We have $(H-2E_1)^2 = (2E_1+4E_2)^2 > 0$, whence $h^1(H-2E_1) = h^1(H-2E_1+K_S) = 0$ by (20). Similarly, we have $h^1(H-2E_2) = h^1(H-2E_2+K_S) = 0$, so that $\alpha = 0$. We have $4\widetilde{F}_1 + 4\widetilde{F}_2 - \widetilde{H} = 0$, whence $\beta = 1$ by definition. Lemma 5.2 implies $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 1$.

• **Case $\mathcal{E}_{13,3}^{(II)}$.** We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$. We find $\alpha = 0$ as above. We have $4F_1 + 4F_2 - H = E_2$ and $2F_1 + 2F_2 - H = -2E_1 - E_2$. Hence $h^0(4\widetilde{F}_1 + 4\widetilde{F}_2 - \widetilde{H}) = h^0(E_2) + h^0(E_2+K_S) = 2$, and $h^i(2\widetilde{F}_1 + 2\widetilde{F}_2 - \widetilde{H}) = 0$ for $i = 0, 1$. Hence, $\beta = 2$ by the exact sequence (14). Thus, $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 2$ by Lemma 5.2.

• **Cases $\mathcal{E}_{13,4}^{(II)^+}$ and $\mathcal{E}_{13,4}^{(II)^-}$.** We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$. We find $\alpha = 0$ as above. We have $4F_1 + 4F_2 - H \equiv 2(E_1 + E_2 - E_3)$, which has square -8, whence $h^0(4F_1 + 4F_2 - H) = h^0(4F_1 + 4F_2 - H + K_S) = 0$ by (19). We have $2F_1 + 2F_2 - H \equiv -2E_3$, whence $h^1(2F_1 + 2F_2 - H) + h^1(2F_1 + 2F_2 - H + K_S) = 1$. Therefore, $\beta \leqslant 1$ by (13). It follows that $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) \leqslant 1$ by Lemma 5.2.

• **Case $\mathcal{E}_{10,3}^{(II)}$.** We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$. As above, $\alpha = 0$. We have $2F_1+2F_2-H = -E_1-E_2$, whence $h^i(2F_1+2F_2-H) = h^i(2F_1+2F_2-H+K_S) = 0$, $i = 0, 1$ by (20). We have $4F_1+4F_2-H = E_1+E_2$, whence $h^0(4F_1+4F_2-H) = h^0(4F_1+4F_2-H+K_S) = 2$. Thus, $h^i(2\widetilde{F}_1 + 2\widetilde{F}_2 - \widetilde{H}) = 0$, $i = 0, 1$ and $h^0(4\widetilde{F}_1 + 4\widetilde{F}_2 - \widetilde{H}) = 4$, so $\beta = 4$ by (14). Lemma 5.2 yields $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 4$.

• **Case $\mathcal{E}_{9,3}^{(II)}$.** We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$. We find $\alpha = 0$ as above. We have $2F_1 + 2F_2 - H = -E_3$, whence $h^i(2F_1 + 2F_2 - H) = h^i(2F_1 + 2F_2 - H + K_S) = 0$ for $i = 0, 1$ by (20). We have $4F_1+4F_2-H = 2E_1+2E_2-E_3$, which has square 0. Since $E_1\cdot(4F_1+4F_2-H) = 1$, we have $h^0(4F_1+4F_2-H) = h^0(4F_1+4F_2-H+K_S) = 1$ (using (20)). It follows that $h^i(2\widetilde{F}_1+2\widetilde{F}_2-\widetilde{H}) = 0$ for $i = 0, 1$ and $h^0(4\widetilde{F}_1+4\widetilde{F}_2-\widetilde{H}) = 2$, whence $\beta = 2$ by (14). Thus, $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 2$ by Lemma 5.2.

• **Cases $\mathcal{E}_{9,4}^+$ and $\mathcal{E}_{9,4}^-$.** We apply Lemma 5.3 with $G_1 = E_1$ and $G_2 = E_{1,2}$. We have $H - G_1 - G_2 \equiv E_1 + E_{1,2}$, whence $\gamma = 0$. We have $2G_1 + 2G_2 - H \equiv 0$, whence $\delta = 1$. Finally, the multiplication map $\mu_{\widetilde{E}_1+\widetilde{E}_{1,2},\widetilde{E}_1+\widetilde{E}_{1,2}}$ is surjective by [40, Thm. 6.1]. Hence, $\epsilon = 0$. Thus, $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 3$ by Lemma 5.3. (See also Remark 6.4 below.)

• **Case $\mathcal{E}_{7,3}$.** We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$. We have $(H - 2F_1)^2 = (E_2 + E_3 + E_4 - E_1)^2 = 0$ and $E_2 \cdot (H - 2F_1) = 1$, whence $h^1(H - 2F_1) = h^1(H - 2F_1 + K_S) = 0$ by (20). Similarly, $h^1(H - 2F_2) = h^1(H - 2F_2 + K_S) = 0$, whence $\alpha = 0$. We have $(4F_1 + 4F_2 - H)^2 = (3E_1 + 3E_2 - E_3 - E_4)^2 = -4$, whence $h^0(4F_1 + 4F_2 - H) = h^0(4F_1 + 4F_2 - H + K_S) = 0$ by (19). We have $(2F_1 + 2F_2 - H)^2 = (E_1 + E_2 - E_3 - E_4)^2 = -4$, whence $h^1(2F_1 + 2F_2 - H) = h^1(2F_1 + 2F_2 - H + K_S) = 1$ by (19). It follows that $h^1(2\widetilde{F}_1 + 2\widetilde{F}_2 - \widetilde{H}) = 2$ and $h^0(4\widetilde{F}_1 + 4\widetilde{F}_2 - \widetilde{H}) = 0$, whence $\beta \leqslant 2$ by (14). Thus, $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) \leqslant 2$ by Lemma 5.2.

• **Case $\mathcal{E}_{6,2}$.** We apply Lemma 5.2 with $F_1 = E_1$ and $F_2 = E_2$. We have $(H-2F_1)^2 = (E_2 + E_3)^2 = 2$, whence $h^1(H - 2F_1) = h^1(H - 2F_1 + K_S) = 0$ by (20). We have $(H - 2F_2)^2 = (2E_1 + E_3 - E_2)^2 = -2$, whence $h^1(H - 2F_2) = h^1(H - 2F_2 + K_S) = 0$ by (20). It follows that $\alpha = 0$. We have $(4F_1 + 4F_2 - H)^2 = (2E_1 + 3E_2 - E_3)^2 = 2$, whence $h^0(4F_1 + 4F_2 - H) = h^0(4F_1 + 4F_2 - H + K_S) = 2$ by (20) and Riemann-Roch. We have $(2F_1 + 2F_2 - H)^2 = (E_2 - E_3)^2 = -2$, whence $h^i(2F_1 + 2F_2 - H) =$

$h^i(2F_1+2F_2-H+K_S)=0$ for $i=0,1$ by (19). Thus, $h^i(2\widetilde{F}_1+2\widetilde{F}_2-\widetilde{H})=0$ for $i=0,1$ and $h^0(4\widetilde{F}_1+4\widetilde{F}_2-\widetilde{H})=4$, so $\beta=4$ by (14). Lemma 5.2 yields $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))=4$.

• **Case** $\mathcal{E}_{4,2}$**.** We apply Lemma 5.2 with $F_1=E_1$ and $F_2=E_2$. We have $(H-2F_1)^2=(E_2+E_3-E_1)^2=-2$, whence $h^1(H-2F_1)=h^1(H-2F_1+K_S)=0$ by (20). Similarly, $h^1(H-2F_2)=h^1(H-2F_2+K_S)=0$, whence $\alpha=0$. We have $(4F_1+4F_2-H)^2=(3E_1+3E_2-E_3)^2=6$, whence $h^0(4F_1+4F_2-H)=h^0(4F_1+4F_2-H+K_S)=4$. We have $(2F_1+2F_2-H)^2=(E_1+E_2-E_3)^2=-2$, whence $h^i(2F_1+2F_2-H)=h^i(2F_1+2F_2-H+K_S)=0$, $i=0,1$ by (19). Thus, $h^i(2\widetilde{F}_1+2\widetilde{F}_2-\widetilde{H})=0$ for $i=0,1$ and $h^0(4\widetilde{F}_1+4\widetilde{F}_2-\widetilde{H})=8$, so $\beta=8$ by (14). Lemma 5.2 yields $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))=8$.

• **Case** $\mathcal{E}_{3,2}$**.** Lemma 5.3 with $G_1=E_1$ and $G_2=E_{1,2}$ yields $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))=12$. □

Remark 6.4. In the cases $\mathcal{E}^+_{9,4}$ and $\mathcal{E}^-_{9,4}$, applying Remark 5.4, we obtain more precisely that $h^1(\mathcal{T}_S(-H))=3$ and $h^1(\mathcal{T}_S(-H+K_S))=0$ for $(S,H)\in\mathcal{E}^+_{9,4}$ and $h^1(\mathcal{T}_S(-H))=0$ and $h^1(\mathcal{T}_S(-H+K_S))=3$ for $(S,H)\in\mathcal{E}^-_{9,4}$.

We draw some consequences from the last two lemmas:

Proof of Theorem 1. The cases not in the table of Lemma 6.3 satisfy $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))=0$, where the result follows from Corollary 3.3 and (9). Let us consider the other cases.

• **Cases** $\mathcal{E}^{(IV)^+}_{17,4}$ **and** $\mathcal{E}^{(IV)^-}_{17,4}$**.** The moduli map $\chi^{(IV)^+}_{17,4}$ is not generically finite by Corollary 4.3, whence $h^1(\mathcal{T}_S(-H))>0$ for $(S,H)\in\mathcal{E}^{(IV)^+}_{17,4}$ by Corollary 3.3. Lemma 6.3 then implies that $h^1(\mathcal{T}_S(-H))=1$ and $h^1(\mathcal{T}_S(-H+K_S))=0$, so that $\chi^{(IV)^+}_{17,4}$ has generically one-dimensional fibers and $\chi^{(IV)^-}_{17,4}$ is generically finite by Corollary 3.3.

• **Cases** $\mathcal{E}^{(II)^+}_{13,4}$ **and** $\mathcal{E}^{(II)^-}_{13,4}$**.** These cases are treated exactly as the previous ones.

• **Cases** $\mathcal{E}^+_{9,4}$ **and** $\mathcal{E}^-_{9,4}$**.** Lemma 6.3 and Remark 6.4 imply that for $(S,H)\in\mathcal{E}^+_{9,4}$, we have $h^1(\mathcal{T}_S(-H))=3$ and $h^1(\mathcal{T}_S(-H+K_S))=0$. Thus $\chi^+_{9,4}$ has generically three-dimensional fibers by Corollary 3.3. It also follows that $h^1(\mathcal{T}_S(-H))=0$ for $(S,H)\in\mathcal{E}^-_{9,4}$, whence $\chi^-_{9,4}$ is generically finite.

• **Case** $\mathcal{E}_{7,3}$**.** By Corollary 4.3, the moduli map $\chi_{7,3}$ is not generically finite, whence $h^1(\mathcal{T}_S(-H))>0$ for $(S,H)\in\mathcal{E}_{7,3}$ by Corollary 3.3, and also $h^1(\mathcal{T}_S(-H+K_S))>0$, since $(S,H+K_S)\in\mathcal{E}_{7,3}$ as well. Lemma 6.3 then implies that $h^1(\mathcal{T}_S(-H))=h^1(\mathcal{T}_S(-H+K_S))=1$, in particular $\chi_{7,3}$ has generically one-dimensional fibers by Corollary 3.3.

• **Cases** $\mathcal{E}^{(II)}_{13,3}$**,** $\mathcal{E}^{(II)}_{10,3}$ **and** $\mathcal{E}^{(II)}_{9,3}$**.** Since these spaces are all irreducible and (S,H) and $(S,H+K_S)$ belong to the same spaces, we must have $h^1(\mathcal{T}_S(-H))=h^1(\mathcal{T}_S(-H+K_S))=\frac{1}{2}h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))$. Then Lemma 6.3 and Corollary 3.3 yield the rest. □

Proof of Corollary 1.2. Assume that $S\subset\mathbb{P}^N$ is k-extendable, for some $k\geqslant 1$. In the language of [29] this means that S can be nontrivially extended k steps. Since S is not a quadric, [29, Thm. 0.1] yields that

$$\alpha(S):=h^0(\mathcal{N}_{S/\mathbb{P}^N}(-1))-N-1\geqslant\min\{k,N\}. \tag{24}$$

The normal bundle and Euler sequences yield $h^0(\mathcal{N}_{S/\mathbb{P}^N}(-1))\leqslant N+1+h^1(\mathcal{T}_S(-1))$, whence $\alpha(S)\leqslant h^1(\mathcal{T}_S(-1))$. Hence, $h^1(\mathcal{T}_S(-1))\geqslant\min\{k,N\}$ by (24). In particular, we must have $h^1(\mathcal{T}_S(-1))>0$, which may also be deduced from Lemma 3.1 and Proposition 4.2. Since $\phi(\mathcal{O}_S(1))\geqslant 3$ by [12, Thm. 5.1] or [15, Thm. 4.6.1] and we assume S is unnodal, $(S, \mathcal{O}_S(1))$ must therefore be in one of the cases listed in Lemma 6.3. The proof of Theorem 1 shows that $h^1(\mathcal{T}_S(-1)) = 0$ for $(S, \mathcal{O}_S(1))$ in $\mathcal{E}_{17,4}^{(IV)^-}$, $\mathcal{E}_{13,4}^{(II)^-}$ and $\mathcal{E}_{9,4}^-$, leaving us with the list of the corollary. The same proof also shows that $h^1(\mathcal{T}_S(-1)) = 1$ in all cases, except for the cases $\mathcal{E}_{10,3}^{(II)}$ and $\mathcal{E}_{9,4}^+$, where $h^1(\mathcal{T}_S(-1)) = 2$ and 3, respectively. Since $N \geqslant 4$, we get from from (24) that $k \leqslant 2$, resp. 3, in these cases. □

Corollary 6.5. *The general Enriques surface sections of the Enriques–Fano threefolds (1) and (3) in the list of* [4, Thm. A] *are nodal Enriques surfaces.*

Proof. Let $(X, \mathcal{L})$ be one of the Enriques–Fano threefolds in question and $S \in |\mathcal{L}|$ be general. We have $(g, \phi) = (8, 3)$ and $(6, 3)$, which do not appear in the table of Lemma 6.3. By Proposition 4.2, the map $c_{g,\phi}$ has positive dimensional fiber at $(S, \mathcal{L}|_S, C)$ for general $C \in |\mathcal{L}|_S|$. The result thus follows from Lemma 6.3, Corollary 3.3 and (9). □

The next result proves part of Theorem 2.

Proposition 6.6. *(i) The moduli map $\chi_{g,2}$ is generically finite for even $g \geqslant 8$, dominant for $g = 3, 4$, and with image of codimension 2 for $g = 6$.*

(ii) A general fiber of $\chi_{5,2}^{(I)}$ is three-dimensional.

Proof. (i) By Lemma 6.3 we have $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 0$ for even $g \geqslant 8$, and the result follows from Corollary 3.3 and (9). In the remaining cases, as (S, H) and $(S, H+K_S)$ both belong to $\mathcal{E}_{g,2}$, which is irreducible by [7], we must have $h^1(\mathcal{T}_S(-H)) = h^1(\mathcal{T}_S(-H + K_S)) = \frac{1}{2} h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))$, whence Lemma 6.3 yields

$$h^1(\mathcal{T}_S(-H)) = \begin{cases} 2 & \text{if } (S, H) \in \mathcal{E}_{6,2} \\ 4 & \text{if } (S, H) \in \mathcal{E}_{4,2} \\ 6 & \text{if } (S, H) \in \mathcal{E}_{3,2}, \end{cases}$$

which is the dimension of a general fiber of $\chi_{g,2}$ by Corollary 3.3. Comparing dimensions of $\mathcal{EC}_{g,2}$ and $\mathcal{R}_g$ yields the rest.

(ii) Recalling that $H \sim 2E_1 + E_{1,2}$, we first apply Lemma 5.3 with $G_1 = E_1$ and $G_2 = E_{1,2}$ to compute $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))$. We have $H - G_1 - G_2 = E_1$, whence $\gamma = 0$. We have $2G_1 + 2G_2 - H = E_{1,2}$, whence $\delta = 2$. Finally, the multiplication map $\mu_{\widetilde{E}_1+\widetilde{E}_{1,2},\widetilde{E}_1}$ is surjective by Lemma 5.5, as

$$h^1(\widetilde{E}_1 + \widetilde{E}_{1,2} - \widetilde{E}_1) = h^1(\widetilde{E}_{1,2}) = h^1(E_{1,2}) + h^1(E_{1,2} + K_S) = 0.$$

Hence, $\epsilon = 0$. Thus, $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 6$ by Lemma 5.3 and the result follows as in (i). □

To finish the proof of Theorem 2 we will have to study the cases $\phi = 2$ of odd genus $g \geqslant 5$ apart from $\chi_{5,2}^{(I)}$. We will do this in Sections 8 and 9 after a technical result in the next section. Theorem 2 will follow from Propositions 6.6, 8.1 and 9.1.

7. A technical result

We here give a result that we will need in the next section, where we will bound the fiber dimension of a moduli map by specializing to a union $C \cup \Gamma$ of a smooth curve C and a rational curve Γ and using knowledge of the fiber dimension over C.

Although we will need the result in the case X is an Enriques–Fano threefold, we formulate it in a more general setting. Its proof is independent of the rest of the paper and its reading can be postponed.

Lemma 7.1. *Let X be a normal projective threefold and $\mathcal{L}$ a big and nef line bundle on X such that the general member of $|\mathcal{L}|$ is a smooth, regular surface. Assume that there is a smooth surface $S_0 \in |\mathcal{L}|$ containing a smooth irreducible rational curve Γ_0 such that:*

- *(i)* $\operatorname{kod}(S_0) \geqslant 0$ *(where* kod *denotes the Kodaira dimension);*
- *(ii) the general element in $|\mathcal{L}|$ does not contain any deformation of Γ_0;*
- *(iii) $\mathcal{L}|_{S_0} \sim M + N$ such that M and N are effective and nontrivial and M is globally generated; moreover, $\Gamma_0 \cdot M > 0$;*
- *(iv) there is a smooth, irreducible nonrational $D \in |N|$ such that $h^0(\mathcal{O}_D(\Gamma_0)) = 1$.*

Then, possibly up to substituting the pair (S_0, Γ_0) with a deformation of it keeping S_0 inside $|\mathcal{L} \otimes \mathcal{J}_D|$ (which automatically maintains $N \sim D$ and $M \sim \mathcal{L}|_{S_0} - N$), the following holds: For general $C \in |M|$, the linear system $|\mathcal{L} \otimes \mathcal{J}_{D \cup C}|$ is a pencil with base locus $D \cup C$ and either

- *(a) Γ_0 does not deform to a general member of $|\mathcal{L} \otimes \mathcal{J}_{D\cup C}|$, or*
- *(b) Γ_0 deforms to a general member of $|\mathcal{L} \otimes \mathcal{J}_{D\cup C}|$ in such a way that the intersection $\Gamma_t \cap C \neq \Gamma_0 \cap C$ for the general deformation Γ_t of Γ_0.*

Proof. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of X (which is an isomorphism on the smooth locus of X). Arguing precisely as in the proof of Lemma 4.1, one finds that $h^1(\mathcal{O}_{\widetilde{X}}) = 0$ and $H^0(\widetilde{X}, \pi^*\mathcal{L}) \simeq H^0(X, \mathcal{L})$. We can therefore without loss of generality assume that X is smooth and $h^1(\mathcal{O}_X) = 0$.

Let S_0, Γ_0, M, N and $D \in |N|$ be as in the statement. From

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L} \otimes \mathcal{J}_D \longrightarrow \mathcal{L}|_{S_0} - D \simeq M \longrightarrow 0,$$

and the fact that M is globally generated and $h^1(\mathcal{O}_X) = 0$, we see that $|\mathcal{L} \otimes \mathcal{J}_D|$ is base point free off D and $\dim |\mathcal{L} \otimes \mathcal{J}_D| = \dim |M| + 1 \geqslant 2$. For any $C \in |M|$ we see from

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L} \otimes \mathcal{J}_{D\cup C} \longrightarrow \mathcal{L}|_{S_0} - D - C \simeq \mathcal{O}_{S_0} \longrightarrow 0,$$

that $|\mathcal{L} \otimes \mathcal{J}_{D\cup C}|$ is a pencil (containing the element S_0) with base locus $D \cup C$. Conversely, for any pencil $\Lambda \subset |\mathcal{L} \otimes \mathcal{J}_D|$ containing S_0, the base locus is $C_\Lambda \cup D$ for some $C_\Lambda \in |M|$. Hence, giving a pencil $\Lambda \subset |\mathcal{L} \otimes \mathcal{J}_D|$ containing the element S_0 is equivalent to giving a curve $C \in |M|$ (which will be the base curve off D of Λ). Note that $\Gamma_0 \not\subseteq C_\Lambda \cup D$ for general Λ, since $|M|$ is base point free and $D \neq \Gamma_0$ by assumption (iv). A general pencil $\Lambda \subset |\mathcal{L} \otimes \mathcal{J}_D|$ containing S_0 therefore does not have Γ_0 in its base locus.

Denote by $\mathcal{R}$ the union of the components of the incidence variety

$$\{(\Gamma, S) \mid [\Gamma] \in \operatorname{Hilb} X, S \in |\mathcal{L} \otimes \mathcal{J}_D|, \Gamma \subset S\} \subset \operatorname{Hilb} X \times |\mathcal{L} \otimes \mathcal{J}_D|$$

containing (Γ_0, S_0), where we denote by $[\Gamma]$ the point corresponding to the curve Γ in the Hilbert scheme. We further denote by $p : \mathcal{R} \to |\mathcal{L} \otimes \mathcal{J}_D|$ the natural projection, which is generically finite as a general member in $|\mathcal{L} \otimes \mathcal{J}_D|$ has nonnegative Kodaira dimension by assumption (i), whence it contains at most finitely many curves that are deformations of Γ_0. If p is not dominant, we end up in case (a), taking the pencil generated by S_0 and a general element not in the image of p. We may thus henceforth assume p is dominant. In particular, we can assume that S_0 is general in $|\mathcal{L} \otimes \mathcal{J}_D|$.

Indeed, for general $S_t \in |\mathcal{L} \otimes \mathcal{J}_D|$, one may define $N_t := \mathcal{O}_{S_t}(D)$ and $M_t := \mathcal{L}|_{S_t} - D$, and properties (i)-(iv) are preserved passing from (S_0, Γ_0) to a general (S_t, Γ_t).

Let now $\Lambda \subset |\mathcal{L} \otimes \mathcal{J}_D|$ be a general pencil containing S_0. Its general member therefore does not contain Γ_0 and contains only finitely many rational curves deformations of Γ_0, as it has nonnegative Kodaira dimension by assumption (i). For $\lambda \in \Lambda$ we denote by S_λ the corresponding surface. Let $\mathcal{R}_\Lambda$ be any irreducible component of the incidence variety

$$\{(\Gamma, \lambda) \mid [\Gamma] \in \operatorname{Hilb} X, \lambda \in \Lambda, \Gamma \subset S_\lambda\} \subset \operatorname{Hilb} X \times \Lambda$$

containing $([\Gamma_0], 0 = S_0)$ and such that the second projection $p_\Lambda : \mathcal{R}_\Lambda \to \Lambda$ is dominant. Such a component exists since p is dominant. Moreover, p_Λ is generically finite by what we said above.

Assume p_Λ is not generically injective. Then the general $S \in \Lambda$ contains at least two curves that are deformations of Γ_0. As we assume that S_0 is general in $|\mathcal{L} \otimes \mathcal{J}_D|$, it is not a branch point of p, so that the limit curves on S_0 are all distinct. As we assume that Λ is general, the curve C_Λ (the base curve of Λ off D) is general in $|M|$ and therefore does not pass through the intersection points of the finitely many curves on S_0 in the component of $\operatorname{Hilb} X$ containing $[\Gamma_0]$. This forces the intersection points $\Gamma \cap C_\Lambda$ to vary as (Γ, S) varies in $\mathcal{R}_\Lambda$; indeed, Γ cannot specialize to a curve containing C_Λ, because the latter is not rational for general Λ, as it moves on S_0 and $\operatorname{kod}(S_0) \geqslant 0$. We thus end up in case (b).

Assume therefore that p_Λ is generically injective. In particular, there is a dense, open subset $\Lambda^\circ \subset \Lambda$ such that for all $\lambda \in \Lambda^\circ$, the surface S_λ is smooth and contains a distinguished curve $\Gamma_\lambda \neq \Gamma_0$ that is a deformation of Γ_0. Consider the irreducible closed surface $R_\Lambda := \overline{\cup_{\lambda \in \Lambda^\circ} \Gamma_\lambda} \subset X$. This surface can also be described as the image in X of the universal family over the image of $\Lambda \to \operatorname{Hilb} X$. Let us study the intersection $R_\Lambda \cap S_\lambda$ for general $\lambda \in \Lambda$. Clearly, $R_\Lambda \cap S_\lambda$ is a curve containing Γ_λ.

If $R_\Lambda \cap S_\lambda = \Gamma_\lambda$ (set-theoretically), then the intersection is transversal for general $\lambda \in \Lambda$, as Λ is base point free off $D \cup C_\Lambda$. Hence $\Gamma_\lambda = R_\Lambda \cdot S_\lambda = R_\Lambda \cdot \mathcal{L}$, and it would follow that a general member of $|\mathcal{L}|$ contains a deformation of Γ_0, contradicting (ii).

Therefore, $R_\Lambda \cap S_\lambda$ contains a curve F_λ in addition to Γ_λ. We claim that F_λ does not vary with λ, and therefore $F_\lambda = D$, C_Λ or $D \cup C_\Lambda$. Indeed, if F_λ varies, it cannot lie in $\overline{R_\Lambda \setminus \cup_{\lambda \in \Lambda^\circ} \Gamma_\lambda}$, as it consists of finitely many curves. But then F_λ, for general λ, must intersect $\cup_{\lambda \in \Lambda^\circ} \Gamma_\lambda$ in infinitely many points, and must therefore lie in the base locus $D \cup C_\Lambda$ of Λ, a contradiction. Thus, $R_\Lambda \cap S_\lambda = F_\lambda \cup \Gamma_\lambda$, with $F_\lambda = D$, C_Λ or $D \cup C_\Lambda$. Moreover, $R_\Lambda \cap S_\lambda$ contains D (respectively, C_Λ) for general $\lambda \in \Lambda^\circ$, only if $\Gamma_\lambda \cap D$ (resp., $\Gamma_\lambda \cap C_\Lambda$) varies with λ: indeed, the finitely many curves in $\overline{R_\Lambda \setminus \cup_{\lambda \in \Lambda^\circ} \Gamma_\lambda}$ are rational, being components of limit curves of the Γ_λ with $\lambda \in \Lambda^\circ$, whereas D is not rational by assumption (iv) and neither is C_Λ as it moves on S_0. Thus $R_\Lambda \cap S_\lambda$ cannot contain D, as $\{\Gamma_\lambda \cap D\}_{\lambda \in \Lambda^\circ}$ would then form a nonconstant family of rationally equivalent cycles on D, whence $h^0(\mathcal{O}_D(\Gamma_0)) \geqslant 2$, contradicting assumption (iv). Hence $R_\Lambda \cap S_\lambda$ contains C_Λ, and we end up in case (b). □

8. The moduli maps on $\mathcal{EC}_{g,2}^{(I)}$ for $g \geqslant 7$

The main result of this section is the following, which proves part of Theorem 2.

Proposition 8.1. *The map $\chi_{g,2}^{(I)}$ is generically finite if $g \geqslant 9$ and has generically one-dimensional fibers if $g = 7$.*

Recall that the irreducible component $\mathcal{E}^{(I)}_{g,2}$ occurs for all odd g, and corresponds to polarizations $\frac{g-1}{2}E_1 + E_{1,2}$. The proposition will be proved by semicontinuity, specializing the curves in $\mathcal{EC}^{(I)}_{g,2}$ to a union of a curve in $\mathcal{EC}_{g-1,2}$ and a smooth rational curve. We will therefore first develop some auxiliary results on $\mathcal{E}_{2k+2,2}$. Recall that by [7] these spaces are irreducible and that $H \sim kE_1 + E_2 + E_3$ for $(S,H) \in \mathcal{E}_{2k+2,2}$.

Let $k \geqslant 1$. We define $\mathcal{E}^{\circ}_{2k+2,2} \subset \mathcal{E}_{2k+2,2}$ to be the subset parametrizing pairs $(S, H = kE_1+E_2+E_3)$ such that E_1, E_2, E_3 are nef and both $|E_1+E_2+E_3|$ and $|E_1+E_2+E_3+K_S|$ map S birationally onto a sextic. Nonemptiness of this locus follows from [12, §7].

We set $\mathcal{EC}^{\circ}_{2k+2,2} := p^{-1}_{2k+2,2}(\mathcal{E}^{\circ}_{2k+2,2}) \subset \mathcal{EC}_{2k+2,2}$ and denote by $c^{\circ}_{2k+2,2} : \mathcal{EC}^{\circ}_{2k+2,2} \to \mathcal{M}_g$ the restriction of $c_{2k+2,2}$ to $\mathcal{EC}^{\circ}_{2k+2,2}$. A key result is the following stronger version of Proposition 6.6(i):

Proposition 8.2. *For any $C \in \operatorname{im} c^{\circ}_{2k+2,2}$, we have $\dim({c^{\circ}_{2k+2,2}}^{-1}(C)) = 0$ if $k \geqslant 3$, whereas ${c^{\circ}_{6,2}}^{-1}(C)$ is equidimensional of dimension two.*

To prove this, we first need an auxiliary result:

Lemma 8.3. *Let $(S, H \sim kE_1 + E_2 + E_3) \in \mathcal{E}^{\circ}_{2k+2,2}$. Then, for $\{\alpha,\beta,\gamma\} = \{1,2,3\}$, and any $l \in \mathbb{Z}$, we have $h^i(lE_\alpha + E_\beta - E_\gamma) = 0$, $i = 0,1,2$.*

Proof. Since $(lE_\alpha+E_\beta-E_\gamma)^2 = -2$, the statement is by Riemann-Roch and Serre duality equivalent to the fact that the divisor $lE_\alpha + E_\beta - E_\gamma$ is not numerically equivalent to an effective divisor for any $l \in \mathbb{Z}$ and $\{\alpha,\beta,\gamma\} = \{1,2,3\}$.

By [14, Def. 5.3.1 and (5.3.2)] (see also [12, §7]), neither $E_\alpha + E_\beta - E_\gamma$ nor $E_\alpha + E_\beta - E_\gamma + K_S$ is linearly equivalent to an effective divisor. It is clear that $lE_\alpha + E_\beta - E_\gamma$ cannot be numerically equivalent to an effective divisor if $l \leqslant 0$.

Assume therefore, to get a contradiction, that $lE_\alpha + E_\beta - E_\gamma$ is numerically equivalent to an effective divisor Δ for some $l \geqslant 2$. We claim that

$$2E_\alpha - \Delta > 0. \tag{25}$$

This yields the desired contradiction, as $E_\gamma \cdot (2E_\alpha - \Delta) = 1 - l < 0$.

Let us prove (25) by induction on l. We may assume that Δ does not contain any multiple of E_α or $E_\alpha + K_S$, as otherwise $(l-1)E_\alpha + E_\beta - E_\gamma$ would be numerically equivalent to an effective divisor. Since $\Delta \cdot E_\alpha = 0$, we have

$$\Delta'^2 \leqslant -2 \quad \text{for every effective subdivisor } \Delta' \text{ of } \Delta \tag{26}$$

by (18). Pick a (-2)-curve $R \leqslant \Delta$. Since $|2E_\alpha|$ is an elliptic pencil and $R \cdot E_\alpha = 0$, it follows that R must be part of a fiber of the elliptic fibration defined by $|2E_\alpha|$, whence $2E_\alpha - R > 0$. Set $\Delta' := \Delta - R$. If $\Delta' > 0$, then, using (26), we find

$$-2 = \Delta^2 = \Delta'^2 + R^2 + 2\Delta' \cdot R \leqslant -4 + 2\Delta' \cdot R,$$

whence $\Delta' \cdot R \geqslant 1$. Hence, there exists a (-2)-curve $R' \leqslant \Delta'$ such that $R' \cdot R \geqslant 1$; more precisely, we have $R' \cdot R = 1$, since otherwise $(R + R')^2 \geqslant 0$, contradicting (26). Since $R' \cdot (2E_\alpha - R) = -1$, we must have $2E_\alpha - R - R' > 0$. Repeating the procedure, if necessary, eventually yields (25). □

Proof of Proposition 8.2. By Lemmas 3.1 and 3.5(v), the result will follow if we prove that for any $(S,H) \in \mathcal{E}^{\circ}_{2k+2,2}$ and $k \geqslant 2$, we have

$$h^1(\mathcal{T}_S(-H)) + h^1(\mathcal{T}_S(-(H+K_S))) = \begin{cases} 0, & \text{if } \ k \geqslant 3, \\ 4, & \text{if } \ k = 2. \end{cases} \tag{27}$$

(Indeed, when $k=2$, since (S,H) and $(S,H+K_S)$ are both general in $\mathcal{E}^{\circ}_{6,2}$ we have that $h^1(\mathcal{T}_S(-H))$ and $h^1(\mathcal{T}_S(-(H+K_S))$ are equal, and (27) implies they are both equal to 2.)

We apply Lemma 5.2 with $H = kE_1 + E_2 + E_3$, $F_1 = E_1$ and $F_2 = E_2$, and (9).

As $H - 2F_1 \sim (k-2)E_1 + E_2 + E_3$ is big and nef, we have $h^1(H-2F_1) = h^1(H - 2F_1 + K_S) = 0$. We have $H - 2F_2 \sim kE_1 + E_3 - E_2$, so that $h^1(H-2F_2) = 0$ by Lemma 8.3 and $h^1(H - 2F_2 + K_S) = 0$ by the same lemma applied with E_3 replaced by $E_3 + K_S$. Hence $\alpha = 0$ by Lemma 8.3.

As $H - 2F_1 - 2F_2 \sim (k-2)E_1 + E_3 - E_2$, Lemma 8.3 (possibly applied again with E_3 replaced by $E_3 + K_S$) and (14) yield that $\beta = h^0(4F_1 + 4F_2 - H) + h^0(4F_1 + 4F_2 - H + K_S)$. We have $4F_1 + 4F_2 - H \sim (4-k)E_1 + 3E_2 - E_3$. As $E_2 \cdot (4F_1 + 4F_2 - H) = 3 - k$, we see that $\beta = 0$ if $k \geqslant 4$. Moreover, $\beta = 0$ if $k = 3$ by Lemma 8.3. If $k = 2$, we claim that $\beta = 4$. Indeed, as $(4F_1 + 4F_2 - H)^2 = 2$, the claim follows if we prove that $h^1(D) = 0$ for $D \equiv 2E_1 + 3E_2 - E_3$. If, by contradiction, $h^1(D) > 0$, there is by [25] an effective divisor Δ such that $\Delta^2 = -2$ and $\Delta \cdot D \leqslant -2$. Then $(D-\Delta)^2 \geqslant 4$ and $(D - \Delta)\cdot(E_1 + 2E_2) \leqslant D \cdot (E_1 + 2E_2) = 4$. Since $(E_1 + 2E_2)^2 = 4$, the Hodge index theorem yields $D - \Delta \equiv E_1 + 2E_2$, whence $\Delta \equiv E_1 + E_2 - E_3$, contradicting Lemma 8.3. We have therefore proved that $\beta = 4$ when $k = 2$.

By Lemma 5.2, we have $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 0$ if $k \geqslant 3$ and $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = 4$ if $k = 2$, and (27) follows from (9). □

The next key ingredient in the proof of Proposition 8.1 is the identification of a suitable sublocus of *nodal* Enriques surfaces.

Proposition 8.4. *The closed subset $\mathcal{E}'_{2k+2,2} \subset \mathcal{E}^{\circ}_{2k+2,2}$ parametrizing (S,H) such that S contains a smooth rational curve Γ with $\Gamma \cdot E_1 = 0$ and $\Gamma \cdot E_2 = \Gamma \cdot E_3 = 1$ (possibly after rearranging indices when $k=1$) is irreducible of codimension one. Moreover, for general (S,H) in $\mathcal{E}'_{2k+2,2}$, we have $\Gamma \cap E_2 \cap E_3 = \emptyset$.*

Again, to prove this we need an auxiliary result:

Lemma 8.5. *There exists an Enriques surface S containing three nef, primitive isotropic divisors E_1, E_2 and E_3 and smooth rational curves Γ and Γ' such that*
- *(i) $2E_1 \sim \Gamma + \Gamma'$, with $\Gamma \cdot \Gamma' = 2$, and the latter intersection is transversal;*
- *(ii) $E_2 \cdot E_3 = 1$, $\Gamma \cdot E_2 = \Gamma' \cdot E_2 = \Gamma \cdot E_3 = \Gamma' \cdot E_3 = 1$ (whence $E_1 \cdot E_2 = E_1 \cdot E_3 = 1$);*
- *(iii) the elliptic pencils $|2E_2|$ and $|2E_3|$ have no reducible fibers.*
- *(iv) $|E_1 + E_2 + E_3|$ is ample and maps S birationally onto a sextic surface.*

Proof. By [13, Lemma 3.2.1] there exists an Enriques surface S with ten elliptic pencils $|2F_i|$ and ten smooth rational curves D_i, with $1 \leqslant i \leqslant 10$, such that

$$D_i \cdot F_j = 1 \text{ for } i \neq j; \quad D_i \cdot F_i = 3; \quad F_i \cdot F_j = 1 \text{ for } i \neq j; \quad D_i \cdot D_j = 2 \text{ for } i \neq j.$$

Moreover, by [13, Rem. p. 747], the elliptic pencils $|2F_i|$ have no reducible fibers, and by [13, Prop. 3.2.6] the complete linear system $|D_i + D_j + D_k|$, for any distinct

i,j,k, defines a degree two morphism of S onto a Cayley cubic surface in $\mathbb{P}^3$. Thus, by [12, Thm. 7.2 and (7.7.1)], the surface S can equivalently be realized as the minimal desingularization of the double cover of $\mathbb{P}^2$ branched along a *Wirtinger sextic* (a sextic with six double points at the vertices of a complete quadrilateral) and the edges of its complete quadrilateral. The curves D_i, D_j, D_k are the inverse images of the three diagonals of the quadrilateral, whence they intersect pairwise transversely in two points.

By [14, Lemma 1.6.2] there exists a $B \in \mathrm{Pic}(S)$ such that $3B \sim F_1 + \cdots + F_{10}$. Set $F_{ij} := B - F_i - F_j$, for $i \neq j$. Then $F_{ij}^2 = 0$. We get $D_i \cdot B = 4$ for all i, whence $D_i \cdot F_{ij} = 0$ for $i \neq j$. As $(D_i + D_j)^2 = (D_i + D_j) \cdot F_{ij} = 0$, we must have $(D_i + D_j) \equiv qF_{ij}$ for some $q \in \mathbb{Q}$ by (18), and dotting both sides with F_i yields $q = 2$. In particular, F_{ij} is nef, so that $|2F_{i,j}|$ is an elliptic pencil. Hence, one necessarily has $2F_{i,j} \sim D_i + D_j$. The divisors $E_1 := F_{45}$, $E_2 := F_2$, $E_3 := F_3$, $\Gamma := D_4$ and $\Gamma' := D_5$ satisfy properties (i)–(iii). Moreover, (iii) implies that both E_2 and E_3 has positive intersection with any (-2)–curve, whence $E_1 + E_2 + E_3$ is ample. By [12, §7] or [14, (5.3.2)], either $|E_1 + E_2 + E_3|$ or $|E_1 + E_2 + E_3 + K_S|$ maps S birationally onto a sextic surface, whence (iv) follows possibly by replacing any of E_i with $E_i + K_S$. □

Proof of Proposition 8.4. We argue as in [7, §5]. Fix homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ on $\mathbb{P}^3$ and let $T = Z(x_0x_1x_2x_3)$ be the *coordinate tetrahedron*. We label by $\ell_1, \ell_2, \ell_3, \ell'_1, \ell'_2, \ell'_3$ the edges of T, in such a way that ℓ_1, ℓ_2, ℓ_3 are coplanar, and ℓ'_i is skew to ℓ_i for all $i = 1, 2, 3$.

Consider the linear system $\mathcal{S}$ of surfaces of degree 6 singular along the edges of T (called *Enriques sextics*). They have equations of the form

$$c_3(x_0x_1x_2)^2 + c_2(x_0x_1x_3)^2 + c_1(x_0x_2x_3)^2 + c_0(x_1x_2x_3)^2 + Qx_0x_1x_2x_3 = 0,$$

where $Q = \sum_{i\leqslant j} q_{ij}x_ix_j$. This shows that $\dim(\mathcal{S}) = 13$ and we may identify $\mathcal{S}$ with the $\mathbb{P}^{13}$ with homogeneous coordinates

$$q = (c_0 : c_1 : c_2 : c_3 : q_{00} : q_{01} : q_{02} : q_{03} : q_{11} : q_{12} : q_{13} : q_{22} : q_{23} : q_{33}).$$

As in [7, §5] we have a dominant rational map $\sigma_{2k+2,2} : \mathcal{S} \dashrightarrow \mathcal{E}_{2k+2,2}$, which assigns to a general $\Sigma \in \mathcal{S}$ the pair (S, H), where $\varphi : S \to \Sigma$ is the normalization and $H = k\varphi^*(\ell_1) + \varphi^*(\ell_2) + \varphi^*(\ell_3)$. Indeed, any $(S, H = kE_1 + E_2 + E_3) \in \mathcal{E}_{2k+2,2}$ such that $|E_1 + E_2 + E_3|$ is ample and birational lies in the image of $\sigma_{2k+2,2}$, because the image Σ of S via the map $\varphi := \varphi_{E_1+E_2+E_3}$ is singular precisely along the edges of T, cf. [15, Thm. 4.6.3], with $\ell_i = \varphi(E_i)$, after a suitable change of coordinates. Also note that the image of $\sigma_{2k+2,2}$ contains pairs $(S, H = kE_1 + E_2 + E_3)$ satisfying the conditions of Lemma 8.5, because of property (iv) therein.

The fiber $\sigma^{-1}_{2k+2,2}(S, H)$ consists of the orbit of $\Sigma = \varphi(S)$ via the 3–dimensional group of projective transformations fixing T.

We denote by $\mathcal{F}$ the family of smooth conics $F \subset \mathbb{P}^3$ such that F does not contain the vertex $\ell'_1 \cap \ell'_2 \cap \ell'_3$ of T and such that F meets the edges ℓ_2, ℓ'_2, ℓ_3 and ℓ'_3 exactly once and does not meet ℓ_1 and ℓ'_1.

Claim 8.6. *(a) The variety $\mathcal{F}$ is irreducible and 4-dimensional.*

(b) Each $F \in \mathcal{F}$ is contained in an 8-dimensional linear system of Enriques sextics.

Proof of the claim. (a) Each F in $\mathcal{F}$ spans a plane intersecting the edges of T in six points. The set of plane conics through four of these six points is a $\mathbb{P}^1$, proving (a).

(b) The linear system $\mathcal{S}$ of the Enriques sextics cuts out on each $F \in \mathcal{F}$ a linear system of divisors with base locus (containing) $T \cap F$ and a moving part of degree (at most) 4, whence of dimension at most 4. Hence F is contained in a linear system $\mathcal{S}_F$ of Enriques sextics of dimension at least 8.

We claim that for each $F \in \mathcal{F}$, one has $\dim(\mathcal{S}_F) = 8$. Consider the restriction rational map $\rho_F : \mathcal{S} \dashrightarrow \mathcal{S}|_F$, whose indeterminacy locus is $\mathcal{S}_F$. Pick any Enriques sextic Σ containing F and let S be its normalization. We consider by abuse of notation F as a curve in S. Then ρ_F factors through the restriction ρ_S to S and the restriction $\rho_{S,F}$ from S to F, i.e., $\rho_F : \mathcal{S} \overset{\rho_S}{\dashrightarrow} \mathcal{S}|_S \overset{\rho_{S,F}}{\dashrightarrow} \mathcal{S}|_F$. The indeterminacy locus of ρ_S is just the point $[\Sigma]$. Therefore, denoting by $\mathcal{S}_{S,F}$ the indeterminacy locus of $\rho_{S,F}$, we have $\dim(\mathcal{S}_F) = \dim(\mathcal{S}_{S,F}) + 1$. So we have to prove that $\dim(\mathcal{S}_{S,F}) = 7$.

The restricted linear system $\mathcal{S}|_S$ is $|2(E_1+E_2+E_3)|$; indeed, it is the sublinear system of $|6(E_1+E_2+E_3)|$ having base locus twice the sum of the pullback of the edges of the tetrahedron, which is

$$2(E_1 + E_2 + E_3 + (E_1 + K_S) + (E_2 + K_S) + (E_3 + K_S)) \sim 4(E_1 + E_2 + E_3).$$

Hence $\mathcal{S}_{S,F}$ is the projectivization of the kernel of the restriction map

$$H^0(\mathcal{O}_S(2(E_1 + E_2 + E_3)) \longrightarrow H^0(\mathcal{O}_F(2(E_1 + E_2 + E_3)),$$

which is $|2(E_1+E_2+E_3)-F|$. Set $D := 2(E_1+E_2+E_3)-F$. Then $D^2 = 14$. We want to prove that $\dim(\mathcal{S}_{S,F}) = \dim(|D|) = 7$, which amounts to proving that $h^1(D) = 0$. Assume $h^1(D) > 0$. Then, by [25], there exists an effective divisor Δ such that $\Delta^2 = -2$ and $\Delta \cdot D \leqslant -2$. In particular, $\Delta \cdot F \geqslant 2$. Since F is mapped by the morphism φ defined by $|E_1 + E_2 + E_3|$ to a smooth conic, Δ cannot be contracted by φ. Hence, $\Delta\cdot(E_1+E_2+E_3) > 0$. It follows that $(D-\Delta)\cdot(E_1+E_2+E_3) < D\cdot(E_1+E_2+E_3) = 10$. But this contradicts the Hodge index theorem, since $(D-\Delta)^2(E_1+E_2+E_3)^2 \geqslant 16\cdot 6 = 96$. □

By the claim the incidence variety $\mathcal{G} := \{(F, \Sigma) \in \mathcal{F} \times \mathcal{S} \mid F \subset \Sigma\}$ is irreducible of dimension 12. Denote by $\mathcal{E}''_{2k+2,2}$ the image of the projection $\mathcal{G} \to \mathcal{S}$ followed by $\sigma_{2k+2,2} : \mathcal{S} \dashrightarrow \mathcal{E}_{2k+2,2}$, which is nonempty by Lemma 8.5, as already remarked. The projection has finite fibers, since an Enriques surface contains only finitely many conics with respect to a given polarization, and $\sigma_{2k+2,2}$ has three-dimensional fibers, whence $\mathcal{E}''_{2k+2,2}$ is irreducible of dimension nine. It parametrizes by construction all pairs $(S, H = kE_1 + E_2 + E_3)$ such that E_1, E_2, E_3 are nef, $|E_1 + E_2 + E_3|$ is ample and birational and S contains a smooth rational curve Γ with $\Gamma \cdot E_1 = 0$ and $\Gamma \cdot E_2 = \Gamma \cdot E_3 = 1$ (possibly after rearranging indices when $k = 1$). Since it is irreducible, its general element has the property that also $|E_1 + E_2 + E_3 + K_S|$ is birational. Thus, $\mathcal{E}'_{2k+2,2} = \mathcal{E}''_{2k+2,2} \cap \mathcal{E}^{\circ}_{2k+2,2}$ is nonempty, whence irreducible of dimension nine, as stated. The last assertion of the proposition follows as $F \cap \ell_2 \cap \ell_3 = \emptyset$ for general $F \in \mathcal{F}$. □

We set $\mathcal{EC}'_{2k+2,2} := p^{-1}_{2k+2,2}(\mathcal{E}'_{2k+2,2}) \subset \mathcal{EC}^{\circ}_{2k+2,2}$, which is irreducible of codimension one in $\mathcal{EC}^{\circ}_{2k+2,2}$.

Proof of Proposition 8.1. Let $(S, kE_1 + E_2 + E_3) \in \mathcal{E}'_{2k+2,2}$ be general, $k \geqslant 2$. Set $H := kE_1 + E_2 + E_3 + \Gamma$. Then H is big and nef, but not ample, as $\Gamma \cdot H = 0$.

Consider $\overline{\mathcal{E}}^{(I)}_{2k+3,2}$, the closure of $\mathcal{E}^{(I)}_{2k+3,2}$ in the moduli space of pairs (X, L) where X is a smooth Enriques surface and L is a *big and nef* line bundle on X. (The existence of such a moduli space is indicated for instance in [24, §5.1.4] for $K3$ surfaces and the case of Enriques surfaces is analogous; see also [16].) We claim that (S,H) lies in $\overline{\mathcal{E}}^{(I)}_{2k+3,2}$. Indeed, set $B := E_2+E_3+\Gamma$. Then B is nef with $B^2=4$ and $\phi(B)=2$ (as $E_2\cdot B = E_3\cdot B = 2$). Since also $E_1\cdot B=2$, we may write $B\sim E_1+E_{1,2}$ for some effective isotropic primitive $E_{1,2}$ satisfying $E_1\cdot E_{1,2}=2$. Thus $H\sim kE_1+B\sim (k+1)E_1+E_{1,2}$, proving the claim.

Denote by $\overline{\mathcal{EC}}^{(I)}_{2k+3,2}$ the partial compactification of $\mathcal{EC}^{(I)}_{2k+3,2}$ parametrizing triples (S,H,C), where (S,H) lies in $\overline{\mathcal{E}}^{(I)}_{2k+3,2}$ and $C\in |H|$ has at most nodes as singularities. Denote by $\overline{c}^{(I)}_{2k+3,2} : \overline{\mathcal{EC}}^{(I)}_{2k+3,2} \to \overline{\mathcal{M}}_{2k+3}$ the extension of $c^{(I)}_{2k+3,2}$. Pick a general $(S,\mathcal{O}_S(C),C)\in \mathcal{EC}'_{2k+2,2}$ and consider $C' := C\cup\Gamma$. Then $(S,\mathcal{O}_S(C'),C')\in \overline{\mathcal{EC}}^{(I)}_{2k+3,2}$. By Proposition 8.2, the fiber $c^{-1}_{2k+2,2}(C)$ is finite for $k\geqslant 3$. Since Γ does not move on any Enriques surface, also the fiber $(\overline{c}^{(I)}_{2k+3,2})^{-1}(C')$ is finite. Hence, $c^{(I)}_{2k+3,2}$ is generically finite for $k\geqslant 3$, that is, $g\geqslant 9$, and so is $\chi^{(I)}_{g,2}$.

Assume now $k=2$, that is, $g=7$. Then $(S,2(E_1+E_2+E_3))$ is extendable to the classical Enriques–Fano threefold $(Y,\mathcal{O}_Y(1))$ in $\mathbb{P}^{13}$ by Lemma 4.5. Let $D\in|E_2+E_3|$ be general. Then Lemma 4.4 implies that $(S,2E_1+E_2+E_3)$ is extendable to an Enriques–Fano threefold $(Y',\mathcal{L})$ and the members in $|\mathcal{L}|$ are in one-to-one-correspondence to the members in $|\mathcal{O}_Y(1)\otimes\mathcal{J}_D|$. Since the hyperplane sections S' of Y such that $\mathcal{O}_{S'}(1)\sim 2(E'_1+E'_2+E'_3)$ with $(S',2E'_1+E'_2+E'_3)\in\mathcal{E}'_{6,2}$ (possibly after rearranging the E'_is) form a hypersurface in $|\mathcal{O}_Y(1)|$ by Proposition 8.4, the members in $|\mathcal{L}|$ yielding elements in $\mathcal{E}'_{6,2}$ form a subset $\mathcal{N}$ of codimension at most one in $|\mathcal{L}|$. Hence, a general pencil in $|\mathcal{L}|$ contains a general element in $\mathcal{N}$. By the proof of Proposition 4.2, two general members of the pencil are not isomorphic. This means that we may find a pencil $|\mathcal{O}_Y(1)\otimes\mathcal{J}_{C\cup D}|$ of hyperplane sections of Y containing S such that $(S,\mathcal{O}_S(C),C)$ is general in $\mathcal{EC}'_{6,2}$ and two general surfaces in the pencil are not isomorphic, that is, we have a finite rational map $\mathfrak{a} : |\mathcal{O}_Y(1)\otimes\mathcal{J}_{C\cup D}| \dashrightarrow {c^{\circ}_{6,2}}^{-1}(C)$. We also have a rational map $\mathfrak{b} : \left(\overline{c}^{(I)}_{7,2}\right)^{-1}(C') \dashrightarrow {c^{\circ}_{6,2}}^{-1}(C)$, forgetting Γ, which is finite, as Γ does not move on any Enriques surface. Accepting for a moment that the hypotheses of Lemma 7.1 are satisfied (for $X=Y$, $\mathcal{L}=\mathcal{O}_Y(1)$, $S_0=S$, $\Gamma_0=\Gamma$, $M\sim\mathcal{O}_S(2E_1+E_2+E_3)$ and $N\sim\mathcal{O}_S(E_2+E_3)$), we obtain that $\mathfrak{b}$ restricted to a neighborhood of $[(S,\mathcal{O}_S(C'),C')]$ is not dominant. Indeed, either (a) of Lemma 7.1 holds, in which case there are elements in the image of $\mathfrak{a}$ not containing any deformation of Γ, whence not lying in the image of $\mathfrak{b}$. Else, (b) of Lemma 7.1 holds, in which case Γ deforms to a general surface in the pencil $|\mathcal{O}_Y(1)\otimes\mathcal{J}_{C\cup D}|$, but in such a way that the moduli of $C'=C\cup\Gamma$ vary, thus again yielding an element in the image of $\mathfrak{a}$ outside the image of $\mathfrak{b}$. By Proposition 8.2, this implies that $\left(\overline{c}^{(I)}_{7,2}\right)^{-1}(C')$ has a component of dimension $\leqslant 1$. By semicontinuity of the dimension of the fibers of a morphism (see [23, Lemme (13.1.1)]), a general fiber of $c^{(I)}_{7,2}$ (and of $\chi^{(I)}_{7,2}$) has dimension $\leqslant 1$. Lemma 3.5(v) implies that equality holds, as desired.

Finally, we check the hypotheses in Lemma 7.1. The general hyperplane section of Y is unnodal by Lemma 4.5, whence (i) and (ii) are satisfied. We have that M is globally generated by [15, Prop. 3.1.6 and Thm. 4.4.1], since M is nef with $\phi(M)=E_1\cdot M=2$, and $|N|$ contains a smooth curve D by [15, Prop. 3.1.6] and [12, Prop. 8.2], as E_2 and E_3 are nef. The fact that $h^0(\mathcal{O}_D(\Gamma))=1$ can be verified, using semicontinuity, by specializing D to $E_2 + E_3$, considering

$$0 \longrightarrow \mathcal{O}_{E_3}(\Gamma - E_2) \longrightarrow \mathcal{O}_D(\Gamma) \longrightarrow \mathcal{O}_{E_2}(\Gamma) \longrightarrow 0$$

and using that $h^0(\mathcal{O}_{E_2}(\Gamma)) = 1$ and $h^0(\mathcal{O}_{E_3}(\Gamma - E_2)) = 0$ by the last assertion in Proposition 8.4. □

Corollary 8.7. *The maps $\chi_{8,2}$, $\chi^{(I)}_{7,2}$, $\chi_{6,2}$ and $\chi^{(I)}_{5,2}$ dominate $\mathcal{R}^0_g$.*

Proof. The result follows from Lemma 3.5(ii),(v), Propositions 6.6 and 8.1. □

Arguing similarly as above, we prove a result that we will need in the next section.

Lemma 8.8. *The map $c^{(II)^+}_{9,2}$ has some fibers of dimension $\geqslant 2$ whose general element $(S, H = 4E_1 + 2E_2, C)$ has the property that E_1 and E_2 are nef.*

Proof. To keep notation consistent with the rest of the section, we switch the roles of E_1 and E_2 and write $H = 2E_1 + 4E_2$ for pairs $(S, H) \in \mathcal{E}^{(II)^+}_{9,2}$. We define a dense, open subset $\mathcal{E}^\circ_{9,2} \subset \mathcal{E}^{(II)^+}_{9,2}$ parametrizing pairs $(S, H = 2E_1 + 4E_2)$ such that E_1 and E_2 are nef. In fact, $\mathcal{E}^\circ_{9,2}$ is non–empty because on the general Enriques surface S there are smooth irreducible elliptic curves E_1, E_2, which are therefore nef, with $E_1 \cdot E_2 = 1$. The openess of $\mathcal{E}^\circ_{9,2}$ follows from the fact that E_1, E_2 being nef on S is an open condition in the moduli space of Enriques surfaces.

We set $\mathcal{EC}^\circ_{9,2} := p^{-1}_{9,2}(\mathcal{E}^\circ_{9,2}) \subset \mathcal{EC}_{9,2}$ and $c^\circ_{9,2} := c^{(II)^+}_{9,2}|_{\mathcal{EC}^\circ_{9,2}}$. To prove the lemma, we want to find a curve C in $\operatorname{im} c^\circ_{9,2}$ with $\dim(c^\circ_{9,2})^{-1}(C) \geqslant 2$.

Claim 8.9. *There is an irreducible codimension-one sublocus $\mathcal{E}'_{9,2} \subset \mathcal{EC}^\circ_{9,2}$ parametrizing pairs $(S, H = 2E_1 + 4E_2)$ such that $2E_1 \sim \Gamma + \Gamma'$, where Γ and Γ' are smooth rational curves intersecting transversely in two points and such that $\Gamma \cdot E_2 = \Gamma' \cdot E_2 = 1$.*

Proof of the Claim. We argue as in the proof of Proposition 8.4, from where we keep the notation. Consider the map $\sigma^{(II)^+}_{9,2} : \mathcal{S} \dashrightarrow \mathcal{E}^{(II)^+}_{9,2}$ associating to a general $\Sigma \in \mathcal{S}$ the pair (S, H), where $\varphi : S \to \Sigma$ is the normalization and $H = 2E_1 + 4E_2$, with $E_i := \varphi^*(\ell_i)$, $i = 1, 2$. Let $\mathcal{E}''_{9,2}$ denote the image of the projection of the incidence variety $\mathcal{G}$ to $\mathcal{S}$ followed by $\sigma^{(II)^+}_{9,2}$, which has (as before) dimension nine. It parametrizes pairs $(S, H = 2E_1 + 4E_2) \in \mathcal{EC}^\circ_{9,2}$ such that S contains a smooth rational curve Γ satisfying $\Gamma \cdot E_1 = 0$ and $\Gamma \cdot E_2 = 1$ (and a nef, isotropic E_3 such that $E_1 \cdot E_3 = E_2 \cdot E_3 = \Gamma \cdot E_3 = 1$ and $|E_1 + E_2 + E_3|$ is birational). Since $|2E_1|$ is a pencil, we must have $\Gamma' := 2E_1 - \Gamma > 0$. By Lemma 8.5 there are elements in $\mathcal{E}''_{9,2}$ for which Γ' is a smooth irreducible rational curve intersecting Γ transversely in two points. (Also note property (iii) in Lemma 8.5 implies that E_2 has positive intersection with any (-2)–curve, so that $2E_1 + 4E_2$ is ample.) We let $\mathcal{E}'_{9,2}$ be the (open dense) locus of such pairs. □

We have $\dim(c^\circ_{9,2})^{-1}(C) \geqslant 1$ for any $C \in \operatorname{im} c^\circ_{9,2}$, as $\chi^{(II)^+}_{9,2}$ is not generically finite by Corollary 4.3. Assume, by contradiction, that

$$\dim(c^\circ_{9,2})^{-1}(C) = 1 \text{ for all } C \in \operatorname{im} c^\circ_{9,2}. \tag{28}$$

We now argue as in the last part of the proof of Proposition 8.1 (for $k = 2$). Denote by $\overline{\mathcal{EC}}^{(II)^+}_{17,4}$ the partial compactification of $\mathcal{EC}^{(II)^+}_{17,4}$ parametrizing curves with at most nodes as singularities and denote by $\overline{c}_{17,4}^{(II)^+} : \overline{\mathcal{EC}}_{17,4}^{(II)^+} \to \overline{\mathcal{M}}_{17}$ the extension of $c_{17,4}^{(II)^+}$. Pick a general $(S, \mathcal{O}_S(C) = 2E_1 + 4E_2, C) \in \mathcal{EC}'_{9,2}$ and consider $C' := C \cup \Gamma \cup \Gamma' \in |4E_1 + 4E_2|$. Then $(S, \mathcal{O}_S(C'), C') \in \overline{\mathcal{EC}}_{17,4}^{(II)^+}$.

Extend $(S, 4(E_1+E_2))$ to Prokhorov's Enriques–Fano threefold $(W, \mathcal{L})$ by Proposition 4.7. As in the last part of the proof of Proposition 8.1, we obtain for general $D \in |2E_1|$, a pencil $|\mathcal{L} \otimes \mathcal{J}_{C\cup D}|$ in W containing S such that $(S, \mathcal{O}_S(C), C)$ is general in $\mathcal{EC}'_{9,2}$ and two general surfaces in the pencil are not isomorphic, that is, we have a finite rational map $\mathfrak{a} : |\mathcal{L} \otimes \mathcal{J}_{C\cup D}| \dashrightarrow {c^{\circ}_{9,2}}^{-1}(C)$. We also have a finite rational map $\mathfrak{b} : \left(\overline{c}_{17,4}^{(II)^+}\right)^{-1}(C') \dashrightarrow {c^{\circ}_{9,2}}^{-1}(C)$, forgetting $\Gamma \cup \Gamma'$. By Lemma 7.1 (with $X = W$, $S_0 = S$, $\Gamma_0 = \Gamma$ or Γ', $M \sim \mathcal{O}_S(2E_1 + 4E_2)$ and $N \sim \mathcal{O}_S(2E_1)$) we obtain, arguing as in the last part of the proof of Proposition 8.1, that $\mathfrak{b}$ restricted to a neighborhood of $(S, \mathcal{O}_S(C'), C')$ is not dominant. By (28) this implies that $(\overline{c}_{17,4}^{(II)^+})^{-1}(C')$ has a zero-dimensional component. By semicontinuity of the dimension of the fibers of a morphism (see [23, Lemme (13.1.1)]), the general fiber of $c_{17,4}^{(II)^+}$ is zero-dimensional. Hence, also $\chi_{17,4}^{(II)^+}$ is generically finite, contradicting Corollary 4.3. □

9. The moduli maps on $\mathcal{EC}_{g,1}$, $\mathcal{EC}_{g,2}^{(II)}$, $\mathcal{EC}_{g,2}^{(II)^+}$ and $\mathcal{EC}_{g,2}^{(II)^-}$

The aim of this section is to prove Theorem 3 together with the following result, which concludes the proof of Theorem 2.

Proposition 9.1. *(i) A general fiber of $\chi_{5,2}^{(II)^+}$ has dimension six; in particular $\chi_{5,2}^{(II)^+}$ dominates $\mathcal{R}_5^{0,\mathrm{nb}}$.*

(ii) A general fiber of $\chi_{5,2}^{(II)^-}$ is four-dimensional; in particular $\chi_{5,2}^{(II)^-}$ dominates $\mathcal{D}_5^0$.

(iii) A general fiber of $\chi_{7,2}^{(II)}$ has dimension 3.

(iv) A general fiber of $\chi_{9,2}^{(II)^+}$ has dimension 2.

(v) A general fiber of $\chi_{9,2}^{(II)^-}$ has dimension 1.

(vi) The moduli maps $\chi_{g,2}$ are generically finite on all irreducible components of $\mathcal{EC}_{g,2}$ for all odd $g \geqslant 11$.

To prove the mentioned results, recall that for (S, H) in $\mathcal{E}_{g,2}^{(II)}$, $\mathcal{E}_{g,2}^{(II)^+}$ or $\mathcal{E}_{g,2}^{(II)^-}$ (note that $\mathcal{E}_{g,2}^{(II)}$ occurs for $g \equiv 3 \bmod 4$, and $\mathcal{E}_{g,2}^{(II)^+}$ and $\mathcal{E}_{g,2}^{(II)^-}$ occur for $g \equiv 1 \bmod 4$) we have $H \equiv kE_1 + 2E_2$, $g = 2k+1$, $k \geqslant 2$, whereas for $(S, H) \in \mathcal{E}_{g,1}$, we have $H \sim (g-1)E_1 + E_2$. Assume that E_1 and E_2 are nef and consider the double cover $g : \widetilde{S} \to P := \mathbb{P}^1 \times \mathbb{P}^1$ defined by $|\widetilde{E}_1 + \widetilde{E}_2|$, as in the beginning of §5.

We denote any line bundle on P by the obvious notation $\mathcal{O}_P(a, b)$, its restriction to any effective divisor $D \subset P$ by $\mathcal{O}_D(a, b)$, and for any sheaf $\mathcal{F}$ on P, we set $\mathcal{F}(a, b) := \mathcal{F} \otimes \mathcal{O}_P(a, b)$. Recall that the branch divisor of g is a smooth curve $R \in |\mathcal{O}_P(4, 4)|$.

Lemma 9.2. *For any $k, l \geqslant 1$ we have $h^1(\mathcal{T}_{\widetilde{S}}(-k\widetilde{E}_1 - l\widetilde{E}_2)) = h^1(\Omega_P(\log R)(k-2, l-2))$.*

Proof. By [17, Lemma 3.16] we have $g_*\mathcal{T}_{\widetilde{S}} \simeq \mathcal{T}_P(-2,-2) \oplus \mathcal{T}_P\langle R\rangle$, where $\mathcal{T}_P\langle R\rangle := \Omega_P(\log R)^\vee$ or is equivalently defined as in (7). We therefore have

$$\begin{aligned} h^1(\mathcal{T}_{\widetilde{S}}(-k\widetilde{E}_1 - l\widetilde{E}_2)) &= h^1(\mathcal{T}_{\widetilde{S}} \otimes g^*\mathcal{O}_P(-k,-l)) = h^1(\mathcal{O}_P(-k,-l) \otimes g_*\mathcal{T}_{\widetilde{S}}) \\ &= h^1(\mathcal{O}_P(-k,-l) \otimes \mathcal{T}_P(-2,-2)) + h^1(\mathcal{O}_P(-k,-l) \otimes \mathcal{T}_P\langle R\rangle) \\ &= h^1(\mathcal{O}_P(-k,-l-2)) + h^1(\mathcal{O}_P(-k-2,-l)) + h^1(\mathcal{T}_P\langle R\rangle(-k,-l)) \\ &= h^1(\Omega_P(\log R)(k-2,l-2)). \end{aligned}$$

□

The next lemma is the main ingredient in the proof of Proposition 9.1.

Lemma 9.3. *For any (S,H) such that $H \equiv kE_1+2E_2$ with E_1 and E_2 nef and $E_1\cdot E_2 = 1$, we have*

$$h^1(\mathcal{T}_S(-H)) + h^1(\mathcal{T}_S(-(H+K_S))) = \begin{cases} 10, & \text{if } k=2\ (g=5) \\ 6, & \text{if } k=3\ (g=7) \\ 3, & \text{if } k=4\ (g=9) \\ 0, & \text{if } k \geqslant 5\ (g \geqslant 11). \end{cases}$$

Proof. We will compute $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}))$ using Lemma 9.2 and use (9). We have

$$0 \longrightarrow \Omega_P(k-2,0) \longrightarrow \Omega_P(\log R)(k-2,0) \longrightarrow \mathcal{O}_R(k-2,0) \longrightarrow 0, \tag{29}$$

cf., e.g., [17, 2.3a]. Since $\Omega_P(k-2,0) \simeq \mathcal{O}_P(k-4,0) \oplus \mathcal{O}_P(k-2,-2)$, we get

$$h^2(\Omega_P(k-2,0)) = 0 \quad \text{and} \quad h^1(\Omega_P(k-2,0)) = \begin{cases} 2, & \text{if } k=2 \\ k-1, & \text{if } k>2. \end{cases}$$

We compute $h^1(\mathcal{O}_R(k-2,0)) = h^0(\mathcal{O}_R(4-k,2)) = h^0(\mathcal{O}_P(4-k,2)) = \max\{0, 15-3k\}$. Hence, from Lemma 9.2 and (29), we obtain

$$h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = h^1(\Omega_P(\log R)(k-2,0)) = \max\{0,15-3k\} + \operatorname{cork}(\partial), \tag{30}$$

with ∂ the coboundary map $H^0(\mathcal{O}_R(k-2,0)) \to H^1(\Omega_P(k-2,0))$ of (29).

When $k=2$, whence $g=5$, we have

$$\partial : \mathbb{C} \simeq H^0(\mathcal{O}_R) \longrightarrow H^1(\Omega_P) \simeq \mathbb{C}^2,$$

which is injective, its image being the 1-dimensional subspace of $H^{1,1}(P)$ generated by the class of R. Thus, $\operatorname{cork}(\partial) = 1$ and the lemma follows from (30) and (9).

Similarly, by (30) and (9) the lemma follows when $k>2$ if we prove the surjectivity of ∂. It suffices to prove that its restriction to the image of the multiplication map

$$H^0(\mathcal{O}_P(k-2,0)) \otimes H^0(\mathcal{O}_R) \longrightarrow H^0(\mathcal{O}_R(k-2,0))$$

is surjective. This restriction is the composed map

$$H^0(\mathcal{O}_P(k-2,0)) \otimes H^0(\mathcal{O}_R) \xrightarrow{\phi_1} H^0(\mathcal{O}_P(k-2,0)) \otimes H^1(\Omega_P) \xrightarrow{\phi_2} H^1(\Omega_P(k-2,0)), \tag{31}$$

where ϕ_1 is the tensor product of the identity with the same map $H^0(\mathcal{O}_R) \to H^1(\Omega_P) \simeq H^{1,1}(P)$ as above, and ϕ_2 is defined by cup-product.

As we saw, the map ϕ_1 is injective, and its image is $H^0(\mathcal{O}_P(k-2,0))\otimes\mathbb{C}\cdot[R]$, where $[R]$ is the class of R in $H^{1,1}(P)\simeq H^1(\Omega_P)$. By the Künneth formula we have

$$H^1(\Omega_P(k-2,0))\simeq \mathrm{pr}_1^*\left(H^0(\mathcal{O}_{\mathbb{P}^1}(k-2))\right)\otimes \mathrm{pr}_2^*\left(H^1(\Omega_{\mathbb{P}^1})\right)$$

where $\mathrm{pr}_i : P\to\mathbb{P}^1$, $1\leqslant i\leqslant 2$, are the two projections. Moreover $H^0(\mathcal{O}_P(k-2,0))\simeq \mathrm{pr}_1^*\left(H^0(\mathcal{O}_{\mathbb{P}^1}(k-2))\right)$. Hence the map

$$\phi_2 : \mathrm{pr}_1^*\left(H^0(\mathcal{O}_{\mathbb{P}^1}(k-2))\right)\otimes H^1(\Omega_P)\longrightarrow \mathrm{pr}_1^*\left(H^0(\mathcal{O}_{\mathbb{P}^1}(k-2))\right)\otimes \mathrm{pr}_2^*\left(H^1(\Omega_{\mathbb{P}^1})\right)$$

is the tensor product of the identity on the first factor and of the natural map $H^1(\Omega_P)\to \mathrm{pr}_2^*\left(H^1(\Omega_{\mathbb{P}^1})\right)$, which maps $\mathbb{C}\cdot[R]$ isomorphically to the target $\mathrm{pr}_2^*\left(H^1(\Omega_{\mathbb{P}^1})\right)\simeq\mathbb{C}$. Hence ϕ_2 maps the image of ϕ_1 isomorphically onto $H^1(\Omega_P(k-2,0))$, showing that the composed map (31) is surjective. Thus, ∂ is surjective, which ends the proof. □

Proof of Proposition 9.1. (i)–(ii) By Corollary 3.3 and Lemma 9.3, the sum of the dimensions of a general fiber of $\chi_{5,2}^{(II)^+}$ and a general fiber of $\chi_{5,2}^{(II)^-}$ is 10. Hence, assertions (i) and (ii) follow by Lemma 3.5(iii),(iv).

(iii) This is a consequence of Corollary 3.3 and Lemma 9.3, as both (S,H) and $(S,H+K_S)$ are general elements of $\mathcal{E}_{7,2}^{(II)}$.

(iv)–(v) By Lemma 3.1 and Lemma 8.8 there are pairs $(S,H=4E_1+2E_2)\in\mathcal{E}_{9,2}^{(II)^+}$ such that E_1 and E_2 are nef and $h^1(\mathcal{T}_S(-H))\geqslant 2$. Similarly, Lemma 3.1 and Corollary 4.3 imply that $h^1(\mathcal{T}_S(-(H+K_S)))\geqslant 1$. Hence, equality is attained in both cases by Lemma 9.3, whence also for general $(S,H)\in\mathcal{E}_{9,2}^{(II)^+}$. Corollary 3.3 yields the result.

(vi) This is an immediate consequence of Corollary 3.3 and Lemma 9.3. □

We next prove Theorem 3. We recall that the moduli spaces $\mathcal{E}_{g,1}$ are all irreducible (cf. [7]). By Corollary 3.3, the theorem is a consequence of the following lemma.

Lemma 9.4. *For general $(S,H)\in\mathcal{E}_{g,1}$, $g\geqslant 2$, we have $h^1(\mathcal{T}_S(-H))=\max\{0,10-g\}$.*

Proof. By (9) and the fact that $\mathcal{E}_{g,1}$ is irreducible, it suffices to prove that $h^1(T_{\widetilde{S}}(-\widetilde{H}))=\max\{0,20-2g\}$.

Consider $\sigma_1\in H^0(\mathcal{O}_R(4,2))$ and $\sigma_2\in H^0(\mathcal{O}_R(2,4))$ two sections (uniquely defined up to constants) whose zero schemes $Z(\sigma_1)=Z_1$ and $Z(\sigma_2)=Z_2$ are the ramification divisors of the $4:1$ maps $R\to\mathbb{P}^1$ defined by the two projections of P to $\mathbb{P}^1$. Note that $Z_1\cap Z_2=\emptyset$. Indeed a point in $Z_1\cap Z_2$ would be singular for R, a contradiction.

We remark for later use that the scheme $Z_1\in|\mathcal{O}_R(4,2)|=|\omega_R(2,0)|$ has length 24 and consists of the ramification points of the first projection $R\to\mathbb{P}^1$, thus of the points where the fibers in $|\mathcal{O}_P(1,0)|$ are tangent to R. On $\widetilde{S}$ these fibers become singular members of $|\widetilde{E}_1|$, that are mapped pairwise onto singular members of $|2E_1|$ on S. Thus, if S is general, Z_1 consists of 24 points on distinct elements of $|\mathcal{O}_P(1,0)|$, as it is well-known that an elliptic pencil on a general Enriques surface has precisely 12 singular reduced fibres, all nodal, cf., e.g., [20, Thm. 4.8 and Rem. 4.9.1].

For any integer $k\geqslant 1$, consider $H_k\sim kE_1+E_2$. Note that $H=H_{g-1}$.

Claim 9.5. *For every $k\geqslant 1$, one has*

$$h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}_k))=18-2k+h^0(\mathcal{O}_P(k+2,1)\otimes\mathcal{J}_{Z_1}). \tag{32}$$

Proof of the Claim. We have an exact sequence (cf., e.g. [17, 2.3c])

$$0 \longrightarrow \Omega_P(\log R)(k-2,-1) \longrightarrow \Omega_P(k+2,3) \xrightarrow{\gamma} \omega_R(k+2,3) \longrightarrow 0.$$

Since $\Omega_P(k+2,3) \simeq \mathcal{O}_P(k,3) \oplus \mathcal{O}_P(k+2,1)$, we have $h^1(\Omega_P(k+2,3)) = 0$, whence

$$h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H}_k)) = h^1(\Omega_P(\log R)(k-2,-1)) = \operatorname{cork} H^0(\gamma) \quad (33)$$

(where the left equality follows from Lemma 9.2). Using the fact that $\omega_R \simeq \mathcal{O}_R(2,2)$, we may write $H^0(\gamma)$ as

$$H^0(\gamma) : H^0(\mathcal{O}_P(k,3)) \oplus H^0(\mathcal{O}_P(k+2,1)) \longrightarrow H^0(\mathcal{O}_R(k+4,5)).$$

Moreover, computing dimensions yields that the domain has dimension $6k+10$ and the target has dimension $4k+28$, whence

$$\operatorname{cork}(H^0(\gamma)) = 18 - 2k + \dim(\ker H^0(\gamma)). \quad (34)$$

We have $H^0(\gamma) = \gamma_1 + \gamma_2$, where

$$H^0(\mathcal{O}_P(k,3)) \xrightarrow{\gamma_1} H^0(\mathcal{O}_R(k+4,5)), \quad s \xmapsto{\gamma_1} s|_R \cdot \sigma_1$$

$$H^0(\mathcal{O}_P(k+2,1)) \xrightarrow{\gamma_2} H^0(\mathcal{O}_R(k+4,5)), \quad t \xmapsto{\gamma_2} t|_R \cdot \sigma_2.$$

The restriction $H^0(\mathcal{O}_P(k,3)) \to H^0(\mathcal{O}_R(k,3))$ is an isomorphism, as $h^0(\mathcal{O}_P(k-4,-1)) = h^1(\mathcal{O}_P(k-4,-1)) = 0$. Hence, γ_1 is injective and $\operatorname{im}\gamma_1 = H^0(\mathcal{O}_R(k+4,5) \otimes \mathcal{J}_{Z_1})$. Since $h^0(\mathcal{O}_P(k-2,-3)) = 0$, the restriction map $H^0(\mathcal{O}_P(k+2,1)) \to H^0(\mathcal{O}_R(k+2,1))$ is injective (but not surjective). It follows that γ_2 is injective and

$$\ker H^0(\gamma) \simeq (\operatorname{im}\gamma_1 \cap \operatorname{im}\gamma_2) = H^0(\mathcal{O}_P(k+2,1) \otimes \mathcal{J}_{Z_1}). \quad (35)$$

Thus, (32) follows from (33), (34) and (35). □

Claim 9.6. *If $1 \leqslant k \leqslant g-1$ and (S,H) is general, then $h^0(\mathcal{O}_P(k+2,1) \otimes \mathcal{J}_{Z_1})$ is even.*

Proof of the Claim. By (9) written for H_k, the fact that (S,H) is general (whence also all (S,H_k) are general) and the fact that $\mathcal{E}_{k+1,1}$ is irreducible, we have $h^1(T_S(-H_k)) = h^1(T_S(-H_k+K_S))$, so that $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H_k}))$ is even. Hence the claim follows from (32). □

Claim 9.7. *One has $h^0(\mathcal{O}_P(k+2,1) \otimes \mathcal{J}_{Z_1}) = 0$ for $1 \leqslant k \leqslant 9$ and (S,H) general.*

Proof of the Claim. Assume $h^0(\mathcal{O}_P(k+2,1) \otimes \mathcal{J}_{Z_1}) > 0$. Then $h^0(\mathcal{O}_P(k+2,1) \otimes \mathcal{J}_{Z_1}) \geqslant 2$ by Claim 9.6. Write $|\mathcal{O}_P(k+2,1) \otimes \mathcal{J}_{Z_1}| = M + \Delta$, where M is the moving part and Δ the fixed part.

Assume first that Δ contains an irreducible curve $B \in |\mathcal{O}_P(\beta,1)|$, for some $\beta \leqslant k+2$. Then $\Delta = B + F_1 + \cdots + F_\alpha$, where $F_i \in |\mathcal{O}_P(1,0)|$ and $0 \leqslant \alpha \leqslant k+2-\beta$. Hence M consists of divisors in $|\mathcal{O}_P(k+2-\alpha-\beta,0)|$. Since M has no fixed part, then $Z_1 \subset \Delta$. Therefore $M = |\mathcal{O}_P(k+2-\alpha-\beta,0)|$ and Δ is the unique curve in $|\mathcal{O}_P(\alpha+\beta,1) \otimes \mathcal{J}_{Z_1}|$. In particular $h^0(\mathcal{O}_P(\alpha+\beta,1) \otimes \mathcal{J}_{Z_1}) = 1$. So Claim 9.6 implies that $\alpha+\beta \leqslant 2$. As $Z_1 \subset R \in |\mathcal{O}_P(4,4)|$, then we must have $24 = \deg(Z_1) \leqslant \mathcal{O}_P(\alpha+\beta,1) \cdot \mathcal{O}_P(4,4) = 4(\alpha+\beta+1) \leqslant 12$, a contradiction.

The remaining case is $\Delta = F_1 + \cdots + F_\alpha$ where $F_i \in |\mathcal{O}_P(1,0)|$ and $0 \leqslant \alpha \leqslant k+2$. Let Z'' be the largest subset of Z_1 contained in Δ and set $Z' = Z_1 - Z''$. We thus have $M = |\mathcal{O}_P(k+2-\alpha,1) \otimes \mathcal{J}_{Z'}|$ and $\dim(M) = h^0(\mathcal{O}_P(k+2,1) \otimes \mathcal{J}_{Z_1}) - 1 \geqslant 1$ by Claim 9.6. As M is base component free, it contains irreducible members. Hence $\deg(Z') \leqslant \mathcal{O}_P(k+2-\alpha,1)^2 = 2(k+2-\alpha)$. Since $\deg(Z'') \leqslant \alpha$, because the points of

Z_1 lie in different elements of $|\mathcal{O}_P(1,0)|$, we have $2(k+2) \geqslant 2\alpha + \deg(Z') \geqslant 2\deg(Z'') + \deg(Z') \geqslant \deg(Z_1) = 24$. Hence $k \geqslant 10$, which proves the claim. □

We can now finish the proof of the lemma. By (32) written for $k = g-1$, we have

$$h^1(T_{\widetilde{S}}(-\widetilde{H})) = 20 - 2g + h^0(\mathcal{O}_P(g+1,1) \otimes \mathcal{J}_{Z_1}). \tag{36}$$

Assume $g \leqslant 10$. By Claim 9.7 we have $h^0(\mathcal{O}_P(g+1,1) \otimes \mathcal{J}_{Z_1}) = 0$, so $h^1(T_{\widetilde{S}}(-\widetilde{H})) = 20 - 2g$ by (36), as wanted.

Assume $g \geqslant 11$. For any $n \geqslant 0$ and $F \in |\mathcal{O}_P(1,0)|$ such that $F \cap Z_1 = \emptyset$, we have

$$0 \longrightarrow \mathcal{O}_P(n,1) \otimes \mathcal{J}_{Z_1} \longrightarrow \mathcal{O}_P(n+1,1) \otimes \mathcal{J}_{Z_1} \longrightarrow \mathcal{O}_P(n+1,1) \otimes \mathcal{O}_F \simeq \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0,$$

whence $h^0(\mathcal{O}_P(n+1,1) \otimes \mathcal{J}_{Z_1}) \leqslant h^0(\mathcal{O}_P(n,1) \otimes \mathcal{J}_{Z_1}) + 2$. Arguing inductively, we have

$$h^0(\mathcal{O}_P(g+1,1) \otimes \mathcal{J}_{Z_1}) \leqslant h^0(\mathcal{O}_P(g-i,1) \otimes \mathcal{J}_{Z_1}) + 2(i+1)$$

for every $i \in \{0, \ldots, g\}$. Setting $i = g - 11$ and applying Claim 9.7 we get

$$h^0(\mathcal{O}_P(g+1,1) \otimes \mathcal{J}_{Z_1}) \leqslant h^0(\mathcal{O}_P(11,1) \otimes \mathcal{J}_{Z_1}) + 2(g-11+1) = 2g - 20.$$

Inserting in (36) we get $h^1(T_{\widetilde{S}}(-\widetilde{H})) = 0$, as wanted. □

References

[1] E. Arbarello, A. Bruno, E. Sernesi, *Mukai's program for curves on a $K3$ surface*, Alg. Geom. **1** (2014), 532–557.

[2] E. Arbarello, A. Bruno, E. Sernesi, *On hyperplane sections of $K3$ surfaces*, Alg. Geom. **4** (2017), 562–596.

[3] W. Barth, K. Hulek, C. Peters, A. van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Volume 4 (2004), Springer-Verlag Berlin Heidelberg.

[4] L. Bayle, *Classification des variétés complexes projectives de dimension trois dont une section hyperplane générale est une surface d'Enriques*, J. Reine Angew. Math. **449** (1994), 9–63.

[5] A. Beauville, *Fano Threefolds and $K3$ Surfaces*, The Fano Conference. Univ. Torino, Turin, 175–184.

[6] C. Ciliberto, T. Dedieu, E. Sernesi, *Wahl maps and extensions of canonical curves and K3 surfaces*, J. Reine Angew. Math., `DOI: https://doi.org/10.1515/crelle-2018-0016`.

[7] C. Ciliberto, T. Dedieu, C. Galati, A. L. Knutsen, *Irreducible unirational and uniruled components of moduli spaces of polarized Enriques surfaces*, `arxiv:1809.10569`

[8] C. Ciliberto, T. Dedieu, C. Galati, A. L. Knutsen, *On the locus of Prym curves where the Prym–canonical map is not an embedding*, `arXiv:1903.05702`, to appear in Arkiv för Matematik.

[9] C. Ciliberto, A. F. Lopez, R. Miranda, *Classification of varieties with canonical curve section via Gaussian maps on canonical curves*, Am. J. Math. **120** (1998), 1–21.

[10] C. Ciliberto, A. F. Lopez, R. Miranda, *Projective degenerations of $K3$ surfaces, Gaussian maps and Fano threefolds*, Invent. Math. **114** (1993), 641–667.

[11] A. Conte, J. P. Murre, *Algebraic varieties of dimension three whose hyperplane sections are Enriques surfaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **12** (1985), 43–80.

[12] F. R. Cossec, *Projective models of Enriques surfaces*, Math. Ann. **265** (1983), 283–334.

[13] F. R. Cossec, *Reye congruences*, Trans. Amer. Math. Soc. **280** (1983), 737–751.

[14] F. R. Cossec, *On the Picard group of Enriques surfaces*, Math. Ann. **271** (1985), 577–600.

[15] F. R. Cossec, I. V. Dolgachev, *Enriques surfaces. I.* Progress in Mathematics, **76**. Birkhäuser Boston, Inc., Boston, MA, 1989.

[16] I. V. Dolgachev, *A brief introduction to Enriques surfaces*, in: Development of Moduli Theory–Kyoto 2013, Advanced Studies in Pure Mathematics **69** (2016), 1–32.

[17] H. Esnault, E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992.

[18] G. Fano, *Sulle varietà algebriche a tre dimensioni le cui sezioni iperpiane sono superficie di genere zero e bigenere uno*, Mem. Soc. It. d. Scienze, Serie 3, **24** (1938), 41–66.

[19] S. Feyzbakhsh, *Mukai's program (reconstructing a K3 surface from a curve) via wall-crossing*, J. Reine Angew. Math., doi:10.1515/crelle-2019-0025.

[20] R. Friedman, J. W. Morgan, *Smooth four-manifolds and complex surfaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete **27**. Springer-Verlag, Berlin, 1994.

[21] F. J. Gallego, B. P. Purnaprajna, *Projective normality and syzygies of algebraic surfaces*, J. Reine und angew. Math. **506** (1999), 145–180.

[22] M. Green, R. Lazarsfeld, *Special divisors on curves on a $K3$ surface*, Invent. Math. **89** (1987), 357–370.

[23] A. Grothendieck, *Éléments de géométrie algébrique: IV*, Publ. math. I.H.É.S. **28** (1966), 5–255.

[24] D. Huybrechts, *Lectures on $K3$ Surfaces*, Cambridge Studies in Advanced Mathematics **158** (2016), Cambridge University Press.

[25] A. L. Knutsen, A. F. Lopez, *A sharp vanishing theorem for line bundles on $K3$ or Enriques surfaces*, Proc. Amer. Math. Soc. **135** (2007) 3495–3498.

[26] A. L. Knutsen, A. F. Lopez, *Brill-Noether theory for curves on Enriques surfaces, I: the positive cone and gonality*, Math. Zeit. **261** (2009), 659–690.

[27] A. L. Knutsen, A. F. Lopez, R. Muñoz, *On the extendability of projective surfaces and a genus bound for Enriques–Fano threefolds*, J. Diff. Geom. **88** (2011), 483–518.

[28] M. Lelli-Chiesa, *Curves on surfaces with trivial canonical bundle*, Boll. Unione Mat. Ital. **11** (2018), 93–105.

[29] S. L'vovsky, *Extensions of projective varieties and deformations, II*, Michigan Math. J. **39** (1992), 65–70.

[30] T. Minagawa, *Deformations of $\mathbf{Q}$-Calabi-Yau 3-Folds and $\mathbf{Q}$-Fano 3-Folds of Fano Index 1*, J. Math. Sci. Univ. Tokyo **6** (1999), 397–414.

[31] S. Mori, H. Sumihiro, *On Hartshorne's conjecture*, J. Math. Kyoto Univ. **18** (1978), 523–533.

[32] S. Mori, S. Mukai, *The uniruledness of the moduli space of curves of genus 11*, Algebraic Geometry, Proc. Tokyo/Kyoto, 334–353, Lecture Notes in Math. **1016**, Springer, Berlin, 1983.

[33] S. Mukai, *Curves and $K3$ surfaces of genus eleven*, Moduli of vector bundles, Lecture Notes in Pure and Applied Math. **179**, Dekker (1996), 189–197.

[34] S. Mukai, *Curves, $K3$ surfaces and Fano 3-folds of genus $\leqslant 10$.* Algebraic geometry and commutative algebra, Vol. I, 357–377, Kinokuniya, Tokyo (1988).

[35] S. Mukai, *Fano 3–folds*, in "Complex Projective Geometry", LMS Lecture Notes Series **179** (1992), 255–261.

[36] D. Mumford, *Varieties Defined by Quadratic Equations.* In: Questions on Algebraic Varieties. C.I.M.E. Summer Schools, vol **51**. Springer, Berlin, Heidelberg.

[37] R. Pardini, *Some remarks on varieties with projectively isomorphic hyperplane sections*, Geometriae Dedicata **52** (1994), 15–32.

[38] Yu. G. Prokhorov, *On Fano-Enriques varieties*, Mat. Sb. **198** (2007), 117–134.

[39] M. Ramponi, *Special divisors on curves on $K3$ surfaces carrying an Enriques involution*, Manuscripta Math. **153** (2017), 315–322.

[40] B. Saint-Donat, *Projective models of K–3 surfaces*, Amer. J. Math. **96** (1974), 602–639.

[41] T. Sano, *On classification of non-Gorenstein $\mathbb{Q}$-Fano 3-folds of Fano index 1*, J. Math. Soc. Japan **47** (1995), 369–380.

[42] E. Sernesi, *Deformations of Algebraic Schemes*, Grundlehren der mathematischen Wissenschaften **334**. Springer-Verlag, Berlin, Heidelberg, New York, 2006.

[43] A. Verra, *A short proof of the unirationality of $\mathcal{A}_5$*, Indagationes Math. **46** (1984), 339–355.

[44] J. Wahl, *A cohomological characterization of $\mathbb{P}^n$*, Invent. Math. **72** (1983), 315–322.

[45] J. Wahl, *The Jacobian algebra of a graded Gorenstein singularity*, Duke Math. J. **55** (1987), 843–871.

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