

Intrinsic pseudo-volume forms for logarithmic pairs

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Abstract

We study an adaptation to the logarithmic case of the Kobayashi-Eisenman pseudo-volume form, or rather an adaptation of its variant defined by Claire Voisin, for which she replaces holomorphic maps by holomorphic K -correspondences. We define an intrinsic logarithmic pseudo-volume form $\Phi_{X,D}$ for every pair (X, D) consisting of a complex manifold X and a normal crossing Weil divisor D on X , the positive part of which is reduced. We then prove that $\Phi_{X,D}$ is generically non-degenerate when X is projective and $K_X + D$ is ample. This result is analogous to the classical Kobayashi-Ochiai theorem. We also show the vanishing of $\Phi_{X,D}$ for a large class of log- K -trivial pairs, which is an important step in the direction of the Kobayashi conjecture about infinitesimal measure hyperbolicity in the logarithmic case.

Introduction

In the standard non logarithmic case, Kobayashi and Eisenman have defined an intrinsic pseudo-volume form Ψ_X on every complex manifold X ([13]). The definition involves all holomorphic maps from the unit polydisk $\mathbf{D}^n \subset \mathbf{C}^n$ to X . Ψ_X coincides with the Poincaré hyperbolic volume form on X when X is a quotient (by a group acting freely and properly discontinuously) of the unit polydisk \mathbf{D}^n . In fact, if X is a smooth curve of genus g , then we have the following dichotomy as a consequence of the Klein-Poincaré uniformization theorem : if $g = 0$ or $g = 1$, then the universal covering of X is \mathbf{P}^1 or \mathbf{C} , and Ψ_X vanishes ; if $g \geq 2$, then the universal covering of X is the unit disk \mathbf{D} , and Ψ_X is induced by the Poincaré volume form on \mathbf{D} . For an n -dimensional manifold X , one expects the situation to follow the same outline. This is in part proved by the Kobayashi-Ochiai theorem ([15]), which states that if X is of general type, then Ψ_X is non degenerate outside a proper closed algebraic subset of X . A variety X such that $\Psi_X > 0$ almost everywhere is said to be infinitesimal measure hyperbolic. On the other hand, Kobayashi conjectured that if X is not of general type, then Ψ_X vanishes on a Zariski open subset of X . The Kobayashi conjecture is proved in the 2-dimensional case for algebraic varieties, using the classification of surfaces (see [9]) : Green and Griffiths show that $\Psi_X = 0$ on a dense Zariski open set when X is covered by abelian varieties, and use the fact that algebraic $K3$ surfaces are swept out by elliptic curves.

It is indeed an important step towards the Kobayashi conjecture to show that if X is a Calabi-Yau variety, then Ψ_X vanishes generically. In [24], Claire Voisin defines a new intrinsic pseudo-volume form $\Phi_{X,an}$, which is a variant of Ψ_X , and for which she is able to show that a very wide range of Calabi-Yau varieties satisfy the Kobayashi conjecture (in fact, she shows that the pseudo-volume form $\Phi_{X,an}$ vanishes on these varieties). She also proves a theorem relative to $\Phi_{X,an}$, which is exactly analogous to the Kobayashi-Ochiai theorem. The definition of $\Phi_{X,an}$ is obtained from the definition of Ψ_X by replacing the holomorphic maps from \mathbf{D}^n to X by K -correspondences. A K -correspondence between two complex manifolds X and Y of the same dimension is a closed analytic subset $\Sigma \subset X \times Y$ satisfying the following properties :

- (i) the projections $\Sigma \rightarrow X$ and $\Sigma \rightarrow Y$ are generically of maximal rank on each irreducible component of Σ ,
- (ii) the first projection $\Sigma \rightarrow X$ is proper,
- (iii) for every desingularization $\tau : \tilde{\Sigma} \rightarrow \Sigma$, letting $f := \text{pr}_1 \circ \tau$ and $g := \text{pr}_2 \circ \tau$, one has the inequality $R_f \leq R_g$ between the ramification divisors of f and g .

$$\begin{array}{ccc}
 & \tilde{\Sigma} & \\
 & \downarrow \tau & \\
 f \swarrow & \Sigma & \searrow g \\
 \text{pr}_1 \swarrow & & \searrow \text{pr}_2 \\
 X & & Y
 \end{array} \tag{1}$$

A K -correspondence Σ has to be seen as the graph of a multivalued map between X and Y . The last condition (iii) ensures the existence of a generalized Jacobian map $(J_{\tilde{\Sigma}})^T : g^*K_Y \rightarrow f^*K_X$. This definition of a K -correspondence, which was introduced in [24], derives from the notion of K -equivalence, for which both projections $\Sigma \rightarrow X$ and $\Sigma \rightarrow Y$ are birational.

It is nowadays understood that for certain problems, it is more relevant to consider logarithmic pairs (X, D) rather than simply considering varieties. In this situation, one replaces the canonical bundle K_X of X by the log-canonical bundle $K_X(D)$. A first very classical example of this is given by the study of open varieties. If U is a complex manifold, such that there exist a compact variety X and a normal crossing divisor $D \subset X$, such that $U = X \setminus D$, then the study of the pair (X, D) provides a lot of information about U . For example, the Betti cohomology with complex coefficients of U can be computed as the hypercohomology of the logarithmic de Rham complex $\Omega_X^\bullet(\log D)$, see *e.g.* [22]. It has also been made clear, that the minimal model program has to be worked out for pairs, rather than simply for varieties. But the best clue, showing that it is indeed necessary to define an intrinsic pseudo-volume form for logarithmic pairs (analogous to Ψ_X), is perhaps the following.

In [3], Campana shows that to decompose a compact Kähler variety into components of special and hyperbolic types, one necessarily has to consider fibrations with orbifold bases. By definition, a complex manifold X is of special type if there does not exist any non trivial meromorphic fibration $X \dashrightarrow Y$ with orbifold base of general type. Fano and K -trivial manifolds are special, but for every $n > 0$ and $\kappa \in \{-\infty, 0, 1, \dots, n-1\}$, there exist n -dimensional manifolds X with $\kappa(X) = \kappa$ that are special. If Y is a complex manifold, an orbifold structure on Y is the data of a \mathbf{Q} -divisor $\Delta = \sum_j a_j D_j$, where $0 < a_j \leq 1$, $a_j \in \mathbf{Q}$, and the D_j are distinct irreducible divisors on Y . The canonical bundle of the orbifold (Y/Δ) is the \mathbf{Q} -divisor $K_Y + \Delta$ on Y . If X and Y are smooth complex varieties, and if $f : X \rightarrow Y$ is a holomorphic fibration, then the orbifold base of f is $(Y/\Delta(f))$, where

$$\Delta(f) := \sum_{D \subset Y} \left(1 - \frac{1}{m(f, D)} \right) \cdot D,$$

$m(f, D)$ being the multiplicity of the fiber of f above the generic point of D . Campana constructs for every variety X (or rather for every orbifold (X/D)) a functorial fibration $c_X : X \rightarrow C(X)$, the core of X , which is characterized by the fact that the generic fibers are special, and that the orbifold base is hyperbolic. In addition, he conjectures that the Kobayashi pseudo-metric d_X on X (note that this is not the same as the Kobayashi-Eisenman pseudo-volume form) is the pull-back *via* c_X of a pseudo-metric δ_X on the orbifold base of the core.

In this paper, we seek the definition of a pseudo-volume form $\Phi_{X,D}$ on a logarithmic pair (X, D) . Let X be a complex manifold of dimension n , and D be a normal crossing Weil divisor

on X , the positive part of which is reduced (we say that D is a normal crossing divisor if its support has normal crossings). Note that we do not require that D has a non zero positive part.

Theorem 1 (i) *There exists a logarithmic pseudo-volume form $\Phi_{X,D}$ on the pair (X,D) , i.e. a pseudo-metric on the line bundle $\bigwedge^n T_X(-D)$, satisfying the following functoriality property. Let Y be a complex manifold, and $\nu : Y \rightarrow X$ be a proper morphism with ramification divisor R , such that $\nu^*D - R$ is a normal crossing divisor, the positive part of which is reduced. Then we have*

$$\nu^*\Phi_{X,D} = \Phi_{Y,\nu^*D-R} \quad (2)$$

(when ν is not proper, we only get the inequality $\nu^*\Phi_{X,D} \leq \Phi_{Y,\nu^*D-R}$).

(ii) *Let D and D' be two normal crossing Weil divisors on X , the respective positive part of which are reduced. If $D \leq D'$, then $\Phi_{X,D} \leq \Phi_{X,D'}$.*

(iii) *If $D = 0$, then $\Phi_{X,0} = \Phi_{X,an}$.*

This is obtained by following the definition of $\Phi_{X,an}$ in [24]. One replaces the holomorphic maps between \mathbf{D}^n and X in the definition of Ψ_X by log- K -correspondences between (\mathbf{D}^n, Δ_k) and (X, D) , where Δ_k is the divisor given in \mathbf{D}^n by the equation $z_{n-k+1} \cdots z_n = 0$. They are closed analytic subsets $\Sigma \subset \mathbf{D}^n \times X$, satisfying the following three properties : (i) the projections to X and Y are generically of maximal rank on each irreducible component of Σ , (ii) the first projection $\Sigma \rightarrow \mathbf{D}^n$ is proper, and (iii) with the same notations as in (1) above ($\tau : \widetilde{\Sigma} \rightarrow \Sigma$ is a desingularization, $f = \text{pr}_1 \circ \tau$, and $g = \text{pr}_2 \circ \tau$)

$$R_f - f^*\Delta_k \leq R_g - g^*D. \quad (3)$$

The ramification divisor R_f (resp. R_g) is the zero divisor of the section of $K_{\widetilde{\Sigma}} \otimes (f^*K_{\mathbf{D}^n})^{-1}$ (resp. $K_{\widetilde{\Sigma}} \otimes (g^*K_X)^{-1}$) given by the Jacobian map of f (resp. g). Condition (iii) ensures the existence of a generalized Jacobian morphism $(J_{\widetilde{\Sigma}})^T : g^*K_X(D) \rightarrow f^*K_{\mathbf{D}^n}(\Delta_k)$.

In [24], Claire Voisin uses local K -isocorrespondences to transport the Poincaré volume form of \mathbf{D}^n to small open sets on X . Here we use local log- K -isocorrespondences to transport the logarithmic Poincaré volume form of (\mathbf{D}^n, Δ_k) . This is done in section 2. Log- K -correspondences are defined and studied in section 1.

Section 3 is devoted to the proof of the following result, which generalizes the Kobayashi-Ochiai theorem to our case.

Theorem 2 *Let (X, D) be a pair consisting of a projective n -dimensional complex manifold X and a normal crossing Weil divisor D on X , the effective part of which is reduced and has global normal crossings. If $K_X + D$ is ample, then $\Phi_{X,D} > 0$ (instead of Ψ_X , and indeed another motivation for our construction) away from a proper closed algebraic subset of X .*

As in [24], this is proved by a standard curvature argument, namely an adaptation to our case of the Ahlfors-Schwarz lemma, which is more or less an incarnation of the maximum principle (see e.g. [6] or [23]). We also use a result of Carlson and Griffiths, about the existence of metrics with negative Ricci curvature on the complement of hypersurfaces of projective algebraic manifolds, enjoying some further properties of Kähler-Einstein type (see [4] and [11]).

Let X and Y be two projective complex manifolds of the same dimension, and assume that X is of general type. Let $\nu : Y \rightarrow X$ be a dominant morphism. Using the volume decreasing property $\nu^*\Psi_X \leq \Psi_Y$, it is well known how to obtain an upper bound on $\deg \nu$ from the classical Kobayashi-Ochiai theorem. However, when X is not of general type, then we cannot say much about the degree of ν .

One of the major benefits we get by considering $\Phi_{X,D}$ instead of Ψ_X , is that we get much more precise decreasing volume properties (cf. theorem 1 and section 2.3). Indeed, by formula (2), where $D = 0$, the standard volume decreasing inequality is even replaced by an equality

for proper morphisms, the ramification divisors of which have normal crossings. Taking the ramification of a morphism $\nu : Y \rightarrow X$ into consideration gives a greater accuracy in the comparison between the intrinsic (logarithmic) pseudo-volume forms on X and Y respectively.

Combining this with theorem 2, we obtain a way to control the degree of a dominant morphism $\nu : Y \rightarrow X$ even if X is not of general type (X and Y are again of the same dimension). Assume there exists a normal crossing Weil divisor D on X , with positive part reduced and with global normal crossings, and such that $K_X + D$ is ample. One decomposes the ramification divisor R of ν into $R = \nu^*D + E$, where E is a Weil divisor of Y , which has a non zero negative part as soon as D has a non zero positive part. Assume that E has normal crossings, and that its negative part is reduced. Then, by integrating on Y the decreasing volume inequality

$$\nu^*\Phi_{X,D} \leq \Phi_{Y,-E},$$

which is an equality if ν is proper, one gets the inequality

$$\left(\int_X \Phi_{X,D} \right) (\deg \nu) \leq \int_Y \Phi_{Y,-E}. \quad (4)$$

The pseudo-volume forms $\Phi_{X,D}$ and $\Phi_{Y,-E}$ are allowed to have poles along the positive part of D , and the negative part of E respectively. They are however integrable at the neighbourhood of these poles, as in the same way the Poincaré volume form on the punctured disk $\mathbf{D} \setminus \{0\}$ is integrable at the neighbourhood of 0. On the other hand, we have $\int_X \Phi_{X,D} > 0$ by theorem 2, so inequality (4) yields an upper bound on $\deg \nu$.

Eventually, we prove our main result in section 4, that for many log- K -trivial pairs, the pseudo-volume form $\Phi_{X,D}$ vanishes. This, of course, has to be seen as a step in the direction of the logarithmic version of the Kobayashi conjecture. This is stated as follows.

Theorem 3 *Let Y be a smooth rationally connected variety, and (X, D) be a pair such that $X \subset Y$ is a smooth hypersurface, D is reduced and has normal crossings, and $D = X \cap X'$, where $X' \subset Y$ is a reduced hypersurface such that*

$$X + X' \in |-K_Y|.$$

Then the pseudo-volume form $\Phi_{X,D}$ vanishes.

Note that by adjunction, the variety $\overline{X} := X \cup X'$ has trivial canonical bundle. Using the equality of line bundles $K_X(D) = (K_{\overline{X}})|_X$, we see that the hypotheses impose that $K_X(D)$ is trivial. As in [24], the key point in the proof is the production of log- K -autocorrespondences of the pair (X, D) , *i.e.* correspondences $\Sigma \subset X \times X$ satisfying the equality $R_f - f^*D = R_g - g^*D$, and with the additional dilating property

$$f^*\eta_X = \lambda g^*\eta_X,$$

where $|\lambda| \neq 1$, and η_X is a generator of $H^0(X, K_X(D))$. This construction is in some way analogous to the definition of the multiplication by a complex number on an elliptic curve, realized as a cubic plane curve. Another important feature in the proof is the use of pull-backs on differential forms induced by correspondences, following original ideas of Mumford ([18], see also chapter 22 in [22]). One has to be slightly more careful with them than with pull-backs on cohomology, which are used in [24].

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1 Log- K -correspondences

1.1 Definition and basic properties

In this section, we define and study the notion of log- K -correspondence. This will be used in the next section to define and study properly our variant of the Kobayashi-Eisenman pseudo-volume form.

Definition 1.1 *Let (X, D) and (Y, D') be pairs of the same dimension, i.e. X and Y are complex manifolds of dimension n , and D and D' are (not necessarily effective) Weil divisors of X and Y respectively. A log- K -correspondence from (X, D) to (Y, D') is a reduced n -dimensional closed analytic subset $\Sigma \subset X \times Y$, satisfying the three following properties.*

(i) *The projections to X and Y are generically of maximal rank on each irreducible component of Σ .*

(ii) *The first projection $\Sigma \rightarrow X$ is proper.*

(iii) *Let $\tau : \tilde{\Sigma} \rightarrow \Sigma$ be a desingularization, $f = \text{pr}_1 \circ \tau : \tilde{\Sigma} \rightarrow X$, and $g = \text{pr}_2 \circ \tau : \tilde{\Sigma} \rightarrow Y$. The ramification divisors R_f and R_g (of f and g respectively) satisfy the inequality*

$$R_f - f^*D \leq R_g - g^*D'.$$

The above notations are summed up in the following commutative diagram. They will be used very often without further notice in the end of this text.

$$\begin{array}{ccc} & \tilde{\Sigma} & \\ & \downarrow \tau & \\ f \swarrow & \Sigma & \searrow g \\ \text{pr}_1 \swarrow & & \searrow \text{pr}_2 \\ X & & Y \end{array} \tag{5}$$

Note that if condition (iii) is true for one desingularization of Σ , then it is true for all desingularizations. Let us now explain the meaning of this last condition. The two Jacobian maps $\bigwedge^n df$ and $\bigwedge^n dg$ (or rather their transpose) give isomorphisms of line bundles on $\tilde{\Sigma}$

$$f^*(K_X(D)) \cong K_{\tilde{\Sigma}}(-R_f + f^*D) \quad \text{and} \quad g^*(K_Y(D')) \cong K_{\tilde{\Sigma}}(-R_g + g^*D').$$

So condition (iii) ensures the existence of a holomorphic (rather than just meromorphic) map

$$(J_{\tilde{\Sigma}})^T : g^*(K_Y(D')) \rightarrow f^*(K_X(D)),$$

given by the transpose of the holomorphic map

$$J_{\tilde{\Sigma}} := \bigwedge^n dg \circ (\bigwedge^n df)^{-1} : f^*(\bigwedge^n T_X(-D)) \rightarrow g^*(\bigwedge^n T_Y(-D')),$$

which we call the generalized logarithmic Jacobian map. When $D = 0$ and $D' = 0$, the notion of log- K -correspondence coincides with the notion of K -correspondence introduced in [24].

It is also important to note the following inequality of divisors on $\tilde{\Sigma}$. We write $D = D_1 - D_2$, with both D_1 and D_2 non negative, and similarly $D' = D'_1 - D'_2$. If D_1 is reduced, then the negative part of $R_f - f^*D$ is the sum of the reduced divisor $(f^*D_1)_{\text{red}}$ and possibly of some f -exceptional components contained in f^*D_1 . We write this f -exceptional sum E_1 . In the same way, if D'_1 is reduced, the negative part of $R_g - g^*D'$ writes $(g^*D'_1)_{\text{red}} + E'_1$, where E'_1 is a sum of

g -exceptional components contained in $g^*D'_1$. So if both D_1 and D'_1 are reduced, then condition (iii) implies the inequality

$$(f^*D_1)_{\text{red}} + E_1 \geq (g^*D'_1)_{\text{red}} + E'_1.$$

In particular, if D' has a positive part (i.e. if D'_1 does not vanish), then D necessarily has a positive part as well.

We shall now describe some enlightening examples.

Example 1.2 *Let X and Y be complex manifolds of dimension n , and $D \subset X$ be an effective divisor. A morphism $\varphi : X \rightarrow Y$ such that the ramification divisor R_φ contains D (with multiplicities) yields a morphism of line bundles*

$$\varphi^*K_Y \rightarrow K_X(-D). \quad (6)$$

The graph $\Gamma_\varphi \subset X \times Y$ is isomorphic to X . It satisfies properties (i) and (ii) of definition 1.1, and with the notations of (5), one has $R_g - R_f = R_\varphi \geq f^*D$. So Γ_φ is a (smooth) log- K -correspondence between $(X, -D)$ and $(Y, 0)$, and the generalized logarithmic Jacobian map

$$g^*K_Y \rightarrow f^*(K_X(-D))$$

identifies to (6).

Example 1.3 *Let (X, D) and (X', D') be smooth logarithmic pairs, where X and X' are complex manifolds of the same dimension, and $D \subset X$ and $D' \subset X'$ are effective divisors. One also usually assumes D and D' to be normal crossing divisors, but this is not necessary for this example. A morphism of pairs $\varphi : (X, D) \rightarrow (X', D')$ is a morphism of complex manifolds $X \rightarrow X'$, such that the ramification divisor R_φ contains $\varphi^*D' - D$. In other words, we require that there exists an effective divisor $R \subset X$, such that $K_X + D = \varphi^*(K_{X'} + D') + R$ as divisors on X . Such a morphism yields a morphism of line bundles on X*

$$\varphi^*(K_{X'}(D')) \rightarrow K_X(D). \quad (7)$$

Again, the graph $\Gamma_\varphi \subset X \times Y$ is isomorphic to X , and satisfies both properties (i) and (ii) of definition 1.1, and we have $R_g - R_f = R_\varphi \geq g^*D' - f^*D$. So Γ_φ is a log- K -correspondence between (X, D) and (X', D') , and the generalized logarithmic Jacobian map

$$g^*(K_{X'}(D')) \rightarrow f^*(K_X(D))$$

identifies to (7).

Finally, the following notion of log- K -isocorrespondence will be useful later.

Definition 1.4 *Let (X, D) and (Y, D') be logarithmic pairs of the same dimension n , and let $\Sigma \subset X \times Y$ be a reduced closed analytic subset, generically finite over X and Y . We let $\tau : \tilde{\Sigma} \rightarrow \Sigma$ be a desingularization, and use the notations (5). If both projections pr_1 and pr_2 are proper, and if*

$$R_f - f^*D = R_g - g^*D',$$

then Σ is a log- K -isocorrespondence between (X, D) and (Y, D') .

Note that under these hypotheses, Σ is a log- K -correspondence between (X, D) and (Y, D') , and its transpose $\Sigma^T \subset Y \times X$ is a log- K -correspondence between (Y, D') and (X, D) . The generalized logarithmic Jacobian map then induces an isomorphism of line bundles on $\tilde{\Sigma}$

$$g^*(K_Y(D')) \cong f^*(K_X(D)).$$

Example 1.5 We consider the unit disk \mathbf{D} , with maps $f : z \in \mathbf{D} \mapsto z^p \in \mathbf{D}$ and $g : z \in \mathbf{D} \mapsto z^q \in \mathbf{D}$, where p and q are two relatively prime integers. Then the diagram

$$\begin{array}{ccc} & \mathbf{D} & \\ z \mapsto z^p \swarrow & & \searrow z \mapsto z^q \\ (\mathbf{D}, \{0\}) & & (\mathbf{D}, \{0\}) \end{array}$$

yields a log- K -autocorrespondence of the pair $(\mathbf{D}, \{0\})$. Indeed, we have the equality of divisors on \mathbf{D}

$$R_f - f^*\{0\} = (p-1)\{0\} - p\{0\} = -\{0\} = R_g - g^*\{0\}.$$

1.2 Composition of log- K -correspondences

We shall now study carefully the composition of two log- K -correspondences. This will allow us in the next section to prove some properties of volume decreasing type for our logarithmic pseudo-volume form.

We first need to define the weaker notion of 0-correspondence, and to study the composition of two of them. Let X and Y be two n -dimensional complex manifolds.

Definition 1.6 A 0-correspondence between X and Y is a reduced closed analytic subset $\Sigma \subset X \times Y$, which is generically finite over X and Y , and such that the first projection $\Sigma \rightarrow X$ is proper.

In other words, $\Sigma \subset X \times Y$ has only to satisfy conditions (i) and (ii) of definition 1.1 to be a 0-correspondence.

Let Z be a third n -dimensional complex manifold. We denote by p_{lq} the projection of $Z \times X \times Y$ to the l -th and q -th factors.

Proposition-Definition 1.7 Let $\Sigma \subset X \times Y$ and $\Sigma' \subset Z \times X$ be two 0-correspondences. We define $\Sigma \circ \Sigma' \subset Z \times Y$ as the union of the components of $p_{13}(p_{12}^{-1}(\Sigma') \cap p_{23}^{-1}(\Sigma))$ on which the projections to Z and Y are generically of maximal rank. Then $\Sigma \circ \Sigma' \subset Z \times Y$ is a 0-correspondence.

Before stating the proof of this, let us see on a simple example why it may be necessary to remove certain irreducible components of $p_{13}(p_{12}^{-1}(\Sigma') \cap p_{23}^{-1}(\Sigma))$. Assume for simplicity that Z , X and Y are surfaces. Suppose we are given two 0-correspondences $\Sigma \subset X \times Y$ and $\Sigma' \subset Z \times X$, and that there exist two irreducible curves $C_Z \subset Z$ and $C_Y \subset Y$, and a point $x_0 \in X$, such that Σ' contains $C_Z \times \{x_0\}$ and Σ contains $\{x_0\} \times C_Y$. In other words, Σ' contains a contraction of the curve C_Z to the point x_0 , and Σ contains a blow-up of the point x_0 onto the curve C_Y . Then $C_Z \times C_Y$ is an irreducible component of $p_{13}(p_{12}^{-1}(\Sigma') \cap p_{23}^{-1}(\Sigma))$ of dimension 2. It is obvious that this component does not satisfy condition (i) of definition 1.1. Note that it would correspond to a blow-up of every point of C_X onto the curve C_Z .

Proof. We have a natural identification between $p_{12}^{-1}(\Sigma') \cap p_{23}^{-1}(\Sigma)$ and $\Sigma' \times_X \Sigma$. The first projection $\Sigma' \times_X \Sigma \rightarrow \Sigma'$ is proper by the stability of properness under base change ($\Sigma \rightarrow X$ is proper by the definition of a 0-correspondence). Since the projection $\Sigma' \rightarrow Z$ is proper as well, the composition $\Sigma' \times_X \Sigma \rightarrow \Sigma' \rightarrow Z$ is proper. *A fortiori* $\Sigma \circ \Sigma'$ satisfies condition (ii) of definition 1.1. Now condition (i) of definition 1.1 is clearly satisfied, and one sees that a component of $p_{13}(p_{12}^{-1}(\Sigma') \cap p_{23}^{-1}(\Sigma))$ which is generically of maximal rank over both Z and Y is necessarily of dimension n . □

We now specify definition 1.7 to the case of log- K -correspondences.

Proposition 1.8 *Let D_Z and D_Y be Weil divisors of Z and Y respectively, and D_X and D'_X be Weil divisors of X . Assume Σ' is a log- K -correspondence between (Z, D_Z) and (X, D'_X) , and Σ is a log- K -correspondence between (X, D_X) and (Y, D_Y) . If $D'_X \geq D_X$, then $\Sigma \circ \Sigma'$ is a log- K -correspondence between (Z, D_Z) and (Y, D_Y) .*

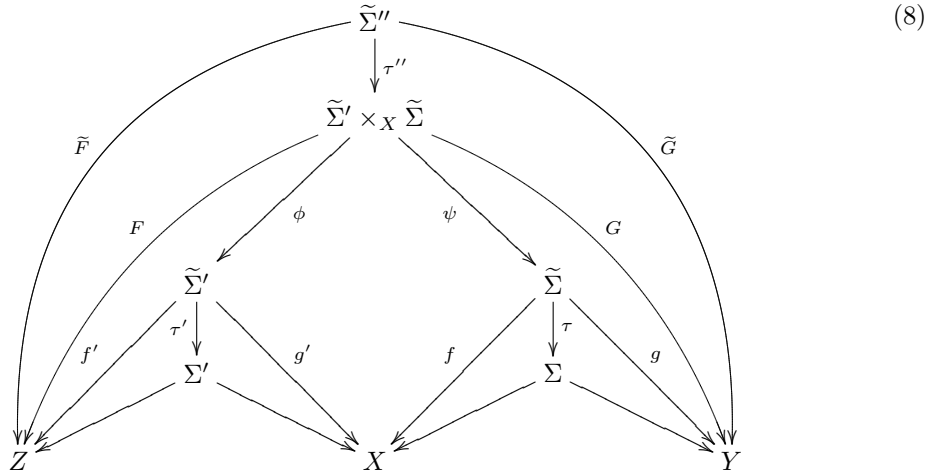
We have to prove that the two generalized logarithmic Jacobian maps $J_{\Sigma}^T : g^*(K_Y(D_Y)) \rightarrow f^*(K_X(D_X))$ and $J_{\Sigma'}^T : g'^*(K_X(D'_X)) \rightarrow f'^*(K_Z(D_Z))$ can be composed on $\tilde{\Sigma}' \times_X \tilde{\Sigma}$, to obtain a generalized logarithmic Jacobian map for $\Sigma \circ \Sigma'$. This composition is of course well defined, since we have a morphism $K_X(D_X) \rightarrow K_X(D'_X)$ of line bundles on X , because of the inequality $D_X \leq D'_X$. We just have to show that all this lifts to a desingularization $\tilde{\Sigma}'' \rightarrow \Sigma \circ \Sigma'$, and that the map we obtain in this way is actually the generalized logarithmic Jacobian map of $\Sigma \circ \Sigma'$. This is a consequence of the following lemma 1.9.

We need some further notations to state the lemma properly. We let $\tau' : \tilde{\Sigma}' \rightarrow \Sigma'$ and $\tau : \tilde{\Sigma} \rightarrow \Sigma$ be desingularizations of Σ' and Σ respectively. We call f' and g' (resp. f and g) the maps from $\tilde{\Sigma}'$ (resp. $\tilde{\Sigma}$) to Z and X (resp. X and Y). Let ϕ and ψ be the natural projections from $\tilde{\Sigma}' \times_X \tilde{\Sigma}$ to $\tilde{\Sigma}'$ and $\tilde{\Sigma}$. We call Σ'' the union of the components of $\tilde{\Sigma}' \times_X \tilde{\Sigma}$ on which the maps $F := f' \circ \phi$ and $G := g \circ \psi$ to Z and Y are generically of maximal rank. We consider a desingularization $\tau'' : \tilde{\Sigma}'' \rightarrow \Sigma'' \subset \tilde{\Sigma}' \times_X \tilde{\Sigma}$, and call \tilde{F} and \tilde{G} the natural maps from $\tilde{\Sigma}''$ to Z and Y .

Lemma 1.9 *With the above notations (see also the diagram below), we have*

$$R_{\tilde{G}} - R_{\tilde{F}} = (\phi \circ \tau'')^*(R_{g'} - R_{f'}) + (\psi \circ \tau'')^*(R_g - R_f)$$

as an equality of divisors on $\tilde{\Sigma}''$.



Proof. Consider $\sigma \in \tilde{\Sigma}''$ and let $z = \tilde{F}(\sigma)$, $x = g' \circ \phi \circ \tau''(\sigma) = f \circ \psi \circ \tau''(\sigma)$, and $y = \tilde{G}(\sigma)$. Let ω_z , ω_x and ω_y be holomorphic n -forms, which generate respectively K_Z near z , K_X near x and K_Y near y . We then have

$$g'^*\omega_x = \chi' \cdot f'^*\omega_z \quad \text{and} \quad g^*\omega_y = \chi \cdot f^*\omega_x,$$

where χ' is a meromorphic function defined on the inverse image $U' \subset \tilde{\Sigma}'$ of a neighbourhood of $(z, x) \in Z \times X$, with divisor $(\chi') = (R_{g'} - R_{f'}) \cap U'$, and χ is similarly a meromorphic function

defined on $U \subset \widetilde{\Sigma}$ with divisor $(\chi) = (R_g - R_f) \cap U$. Pulling back these equalities on $\widetilde{\Sigma}''$ via $\phi \circ \tau''$ and $\psi \circ \tau''$, one gets

$$\tau''^* \phi^* g'^* \omega_x = \chi' \circ \phi \circ \tau'' \cdot \tau''^* \phi^* f'^* \omega_z \quad \text{and} \quad \tau''^* \psi^* g^* \omega_y = \chi \circ \psi \circ \tau'' \cdot \tau''^* \psi^* f^* \omega_x.$$

Now since $g' \circ \phi = f \circ \psi$, we have $\phi^* g'^* \omega_x = \psi^* f^* \omega_x$, and therefore

$$\widetilde{G}^* \omega_y = \chi \circ \psi \circ \tau'' \cdot \chi' \circ \phi \circ \tau'' \cdot \widetilde{F}^* \omega_z \quad (9)$$

on the inverse image $U'' \subset \widetilde{\Sigma}''$ of a neighbourhood of $(z, y) \in Z \times Y$. Note that U'' contains a neighbourhood of $\sigma \in \widetilde{\Sigma}''$. The meromorphic function $\chi \circ \psi \circ \tau'' \cdot \chi' \circ \phi \circ \tau''$ has divisor

$$(\tau''^* \phi^* (R_{g'} - R_{f'}) + \tau''^* \psi^* (R_g - R_f)) \cap U''.$$

□

Proof of proposition 1.8. Because of proposition 1.7, to show that $\Sigma \circ \Sigma'$ is a log- K -correspondence, it only remains to prove the inequality

$$R_{\widetilde{F}} - \widetilde{F}^* D_Z \leq R_{\widetilde{G}} - \widetilde{G}^* D_Y.$$

Since Σ' and Σ are log- K -correspondences, we have

$$R_{g'} - R_{f'} \geq g'^* D'_X - f'^* D_Z \quad \text{and} \quad R_g - R_f \geq g^* D_Y - f^* D_X.$$

By lemma 1.9, this yields

$$R_{\widetilde{G}} - R_{\widetilde{F}} \geq \tau''^* \phi^* (g'^* D'_X - f'^* D_Z) + \tau''^* \psi^* (g^* D_Y - f^* D_X).$$

On the other hand, since $D'_X \geq D_X$, and $g' \circ \phi = f \circ \psi$, we have

$$\phi^* g'^* D'_X - \psi^* f^* D_X \geq 0,$$

and therefore

$$R_{\widetilde{G}} - R_{\widetilde{F}} \geq \tau''^* \psi^* g^* D_Y - \tau''^* \phi^* f'^* D_Z = \widetilde{G}^* D_Y - \widetilde{F}^* D_Z,$$

which is the desired inequality. Eventually, we see from equality (9) that the morphism $J_{\widetilde{\Sigma}''}^T$ of line bundles on $\widetilde{\Sigma}''$ given by the log- K -correspondence $\Sigma \circ \Sigma'$ is obtained as the composition

$$\begin{array}{ccc} \widetilde{G}^* (K_Y(D_Y)) & \xrightarrow{J_{\widetilde{\Sigma}''}^T} & \widetilde{F}^* (K_Z(D_Z)) \\ & \searrow (\tau'' \circ \psi)^* J_{\widetilde{\Sigma}}^T & \nearrow (\tau'' \circ \phi)^* J_{\widetilde{\Sigma}'}^T \\ & (\tau'' \circ \psi)^* f^* (K_X(D_X)) \subset (\tau'' \circ \phi)^* g'^* (K_X(D'_X)) & \end{array}$$

of the generalized logarithmic Jacobian maps given by Σ and Σ' .

□

2 Intrinsic logarithmic pseudo-volume forms

2.1 The standard Kobayashi-Eisenman pseudo-volume form

We first recall the classical definition of the Kobayashi-Eisenman pseudo-volume form and its fundamental properties.

Let X be an n -dimensional complex manifold. The Kobayashi-Eisenman pseudo-volume form Ψ_X is defined by its associated Hermitian pseudo-norm on $\bigwedge^n T_X$

$$\|\xi\|_{\Psi_{X,x}} = \inf \{ \lambda > 0 \text{ s.t. } \exists \phi : \mathbf{D}^n \rightarrow X \text{ with } \phi(0) = x \text{ and } \lambda \cdot J_\phi(\partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_n) = \xi \}$$

for $x \in X$, $\xi \in \bigwedge^n T_{X,x}$, where ϕ denotes a holomorphic map from the unit polydisk in \mathbf{C}^n . Note that if ϕ is ramified at the origin, then $J_\phi(\partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_n) = 0$, and there does not exist any $\lambda > 0$ such that $\lambda \cdot J_\phi(\partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_n) = \xi$ for $\xi \neq 0$.

Ψ_X is closely related to the Poincaré hyperbolic volume form on the polydisk \mathbf{D}^n

$$\kappa_n = \frac{i^n}{2^n} \bigwedge_{1 \leq j \leq n} \frac{dz_j \wedge d\bar{z}_j}{(1 - |z_j|^2)^2}.$$

Indeed, $\text{Aut}(\mathbf{D}^n)$ acts transitively on the polydisk and leaves the Poincaré volume form invariant, and since the latter coincides with the standard Euclidean volume form at 0, we find that

$$\Psi_{X,x} = \inf \{ (\phi_b^{-1})^* \kappa_n, \phi : \mathbf{D}^n \rightarrow X \text{ s.t. } \phi(b) = x \text{ and } \phi \text{ unramified at } b \},$$

where again ϕ runs through all holomorphic maps $\mathbf{D}^n \rightarrow X$ (ϕ_b^{-1} is the local inverse of ϕ near b). One also has the following result, which is a consequence of Ahlfors-Schwarz lemma (see section 3).

Theorem 2.1 (Kobayashi) *If X is isomorphic to the unit polydisk $\mathbf{D}^n \subset \mathbf{C}^n$ (resp. to the quotient of \mathbf{D}^n by a group acting freely and properly discontinuously), then the Kobayashi pseudo-volume form Ψ_X is equal to the Poincaré hyperbolic volume form κ_n (resp. to the hyperbolic volume form on the quotient induced by κ_n).*

Finally, the following decreasing volume property is a straightforward consequence of the definition. If Y is another smooth manifold of dimension n , and if $\phi : X \rightarrow Y$ is a holomorphic map, then we have the inequality between pseudo-volume forms on X

$$\phi^* \Psi_Y \leq \Psi_X. \tag{10}$$

There is also a meromorphic version $\tilde{\Psi}_X$ of Ψ_X , introduced by Yau in [25]. It is obtained by considering all meromorphic maps $\phi : \mathbf{D}^n \dashrightarrow X$ defined near the origin, instead of considering holomorphic maps as in the definition of Ψ_X . It is invariant under birational maps.

2.2 The Poincaré volume form on the punctured disk

In this paragraph, we describe the Poincaré volume form on the punctured disk $\mathbf{D} \setminus \{0\}$. We will use it later on as a local model to define intrinsic pseudo-volume forms for logarithmic pairs.

The punctured disk $\mathbf{D} \setminus \{0\}$ is the quotient of \mathbf{D} under the action of the subgroup of $\text{Aut}(\mathbf{D})$ generated by a parabolic transformation $g \in \text{Aut}(\mathbf{D})$. In order to compute the Poincaré volume form of $\mathbf{D} \setminus \{0\}$, it is however more convenient to see it as a quotient of the Poincaré upper half plane \mathbf{H} .

Proposition 2.2 *The Poincaré volume form of the punctured disk $\mathbf{D} \setminus \{0\}$ is*

$$\frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2 (\log |z|^2)^2}.$$

Proof. The punctured disk $\mathbf{D} \setminus \{0\}$ is the quotient of \mathbf{H} by $\langle Z \mapsto Z + 1 \rangle$. The projection map is

$$\pi : Z \in \mathbf{H} \mapsto z = \exp(2\pi i Z) \in \mathbf{D} \setminus \{0\}.$$

In particular, we have $Z = \log(z)/2\pi i$, and

$$dZ = \frac{1}{2\pi i} \frac{dz}{z}.$$

Now, we get the Poincaré volume form on $\mathbf{D} \setminus \{0\}$ by its expression on \mathbf{H}

$$\frac{i}{2} \frac{dZ \wedge d\bar{Z}}{|Z - \bar{Z}|^2} = \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2 (\log |z|^2)^2}. \quad (11)$$

□

Proposition 2.2 is in fact a particular case of a more general fact. Let X be a punctured Riemann surface, which is universally covered by \mathbf{D} . Then every point x in the puncture corresponds to a subgroup of $\text{Aut}(\mathbf{D})$ generated by a parabolic transformation, and there is a neighbourhood of x in X which is isomorphic to the quotient of the circle $\{\Im(z) > 1\} \subset \mathbf{H}$ by $\langle Z \mapsto Z + 1 \rangle$ (see *e.g.* [8]). Therefore, the Poincaré volume form on X is given by (11) around x .

The Poincaré volume form on $\mathbf{D} \setminus \{0\}$ yields a logarithmic volume form on the pair $(\mathbf{D}, \{0\})$, *i.e.* a volume form with a pole at $z = 0$. It is left invariant by the log- K -autocorrespondences

$$\begin{array}{ccc} & \mathbf{D} & \\ z \mapsto z^p \swarrow & & \searrow z \mapsto z^q \\ (\mathbf{D}, \{0\}) & & (\mathbf{D}, \{0\}) \end{array} \quad (12)$$

described in example 1.5. Of course, it is also left invariant by the log- K -autocorrespondences of $(\mathbf{D}, \{0\})$ given by the rotations $z \in \mathbf{D} \mapsto e^{i\alpha} \cdot z \in \mathbf{D}$, $\alpha \in \mathbf{R}$.

Lemma 2.3 *Up to multiplication by a constant, the Poincaré volume form on $\mathbf{D} \setminus \{0\}$ is the only logarithmic pseudo-volume form on $(\mathbf{D}, \{0\})$ that is left invariant by the rotations centred at 0 and by the log- K -autocorrespondences (12).*

Proof. Indeed, let

$$\frac{i}{2} \alpha(z) \frac{dz \wedge d\bar{z}}{|z|^2}$$

be such a logarithmic volume form on the pair $(\mathbf{D}, \{0\})$. Since it is invariant under the action of the rotations, $\alpha(z)$ depends only on $|z|$. On the other hand, letting $z = z'^p$, we find

$$\frac{dz}{z} = p \frac{z'^{p-1}}{z'^p} dz' = p \frac{dz'}{z'}.$$

By invariance under the action of the log- K -autocorrespondences (12), we find that for every $r \in]0, 1[$, and for every relatively prime integers p and q , one has

$$p^2 \alpha(r^p) = q^2 \alpha(r^q).$$

This implies that there exists $\lambda > 0$ such that

$$\frac{i}{2} \alpha(z) \frac{dz \wedge d\bar{z}}{|z|^2} = \lambda \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2 (\log |z|^2)^2}.$$

□

This shows in particular that the Poincaré logarithmic volume form on $(\mathbf{D}, \{0\})$ is essentially characterized by its invariance under the action of log- K -autocorrespondences of $(\mathbf{D}, \{0\})$. Note that it is locally integrable around $0 \in \mathbf{D}$. We will use its n -dimensional version given below, to define intrinsic logarithmic pseudo-volume forms on general pairs in the next subsection.

We let \mathbf{D}^n be the unit polydisk, with coordinates (z_1, \dots, z_n) , and Δ_k be the divisor given by the equation $z_{n-k+1} \cdots z_n = 0$. Then the pair (\mathbf{D}^n, Δ_k) is equipped with the Poincaré logarithmic volume form

$$\kappa_{n,k} = \left(\frac{i}{2}\right)^n \left(\bigwedge_{1 \leq j \leq n-k} \frac{dz_j \wedge d\bar{z}_j}{(1-|z_j|^2)^2} \right) \wedge \left(\bigwedge_{n-k+1 \leq j \leq n} \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2 (\log |z_j|^2)^2} \right). \quad (13)$$

This is a C^∞ logarithmic volume form on $\mathbf{D}^n \setminus \Delta_k$, and it is singular along Δ_k .

2.3 Log- K -correspondences and intrinsic logarithmic pseudo-volume forms

In this paragraph, we define the central object of this paper, the intrinsic pseudo-volume form $\Phi_{X,D}$ of a logarithmic pair (X, D) .

We first need to introduce the notion of logarithmic pseudo-volume form.

Definition 2.4 *Let (X, D) be a pair composed by a complex manifold X of dimension n , and a Weil divisor D of X . A logarithmic pseudo-volume form on (X, D) is a pseudo-metric on the line bundle $\bigwedge^n T_X(-D)$.*

Let μ be a logarithmic pseudo-volume form on (X, D) . In case it is C^∞ , it writes locally

$$\mu = \frac{1}{|h|^2} \mu',$$

where μ' is a C^∞ pseudo-volume form, and h is a meromorphic function with divisor D : if $D = D_1 - D_2$ with D_1 and D_2 non negative, then h has zeroes exactly along D_1 , and poles exactly along D_2 . It will often be useful to allow μ' to have singularities along the positive part D_1 of D (cf. definition 3.3). This is already clear from the expression of the logarithmic Poincaré volume form on $(\mathbf{D}, \{0\})$, given in proposition 2.2 above, and more generally from the expression of $\kappa_{n,k}$ in (13) above.

Now $\Phi_{X,D}$ is defined as follows (we use the notations introduced in the diagram (5)).

Definition 2.5 *Let (X, D) be a pair composed by an n -dimensional complex manifold X and a normal crossing Weil divisor D of X , such that the positive part of D is reduced. For every $x \in X$, we let*

$$\Phi_{X,D,x} = \inf_{0 \leq k \leq n} (\inf \{ (f^* \kappa_{n,k})_\sigma, \text{ where } \sigma \in \widetilde{\Sigma}, \Sigma \text{ log-}K\text{-correspondence between } (\mathbf{D}^n, \Delta_k) \text{ and } (X, D), \text{ unramified at } \sigma, \text{ with } g(\sigma) = x \}).$$

A log- K -correspondence Σ between (\mathbf{D}^n, Δ_k) and (X, D) is said to be unramified at σ if the inequality of divisors

$$R_f - f^* \Delta_k \leq R_g - g^* D$$

is an equality locally around σ . In this case, the transpose of the generalized logarithmic Jacobian yields a morphism of line bundles on $\widetilde{\Sigma}$

$$g^* (K_X(D)) \rightarrow K_{\mathbf{D}^n}(\Delta_k),$$

which is an isomorphism locally around σ . This authorizes the identification of $f^*\kappa_{n,k}$ with a Hermitian metric on $g^*(K_X(D))^\vee$ locally around σ , and hence with a logarithmic pseudo-volume element at $x = g(\sigma)$.

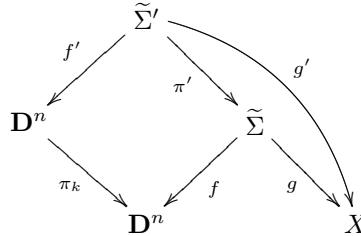
Under the hypotheses of definition 2.5, there exists an unramified log- K -correspondence between (\mathbf{D}^n, Δ_k) and (X, D) around every point $x \in X$ (the integer k depends on x). To see this, we write $D = D_1 - D_2$, with D_1 and D_2 non negative, and we choose a local holomorphic system of coordinates (z_1, \dots, z_n) centred at x , and defined on an open set $U \subset X$, such that D_2 is given in U by $z_1^{l_1} \cdots z_r^{l_r} = 0$, and D_1 is given in U by $z_{n-k+1} \cdots z_n = 0$ ($r+k \leq n$). Then D_2 is the ramification divisor of the morphism $U \rightarrow V$ defined by $(z_1, \dots, z_n) \mapsto (z_1^{l_1+1}, \dots, z_r^{l_r+1}, z_{r+1}, \dots, z_n)$, where V is an open subset of \mathbf{C}^n . In particular, the graph Γ of this morphism is an unramified log- K -correspondence between $(X, D_1 - D_2)$ and (V, D_1') . Since D_1' is given in V by the equation $z_{n-k+1} \cdots z_n = 0$, there exists an unramified log- K -correspondence between (\mathbf{D}^n, Δ_k) and (V, D_1') . Then $\Gamma^T \circ \Sigma$ is an unramified log- K -correspondence between (\mathbf{D}^n, Δ_k) and (X, D) as we wanted. Note that k is the number of branches at the point x of the positive part D_1 of D .

Proposition 2.6 *When $D = 0$, one has $\Phi_{X,0} = \Phi_{X,an}$. More generally, when D is effective, we have*

$$\Phi_{X,D}|_{X \setminus D} = \Phi_{X \setminus D, an}.$$

Proof. The reason for this is simply that the logarithmic volume form $\kappa_{n,k}$ on (\mathbf{D}^n, Δ_k) comes from the Poincaré volume form $\kappa_{n,0}$ on $(\mathbf{D}^n, 0)$, due to the fact that $\mathbf{D}^n \setminus \Delta_k$ is a quotient of \mathbf{D}^n by a group acting freely and properly discontinuously (see paragraph 2.2). Let $\pi_k : \mathbf{D}^n \rightarrow \mathbf{D}^n$ be the projection corresponding to this quotient. One has by definition $\pi_k^*\kappa_{n,k} = \kappa_{n,0}$.

Let x be a point in $X \setminus D$, and Σ be a log- K -correspondence between (\mathbf{D}^n, Δ_k) and (X, D) , with a point $\sigma \in \tilde{\Sigma}$ above x , where Σ is unramified. It yields by base change $\pi_k : \mathbf{D}^n \rightarrow \mathbf{D}^n$ a log- K -correspondence Σ' between $(\mathbf{D}^n, 0)$ and $(X \setminus D, 0)$. By the same base change, we get a desingularization $\tilde{\Sigma}'$ of Σ' .



At every point $\sigma' \in \tilde{\Sigma}'$ above σ , we have $(f'^*\kappa_{n,0})_{\sigma'} = (f'^*\pi_k^*\kappa_{n,k})_{\sigma'} = \pi'^*(f^*\kappa_{n,k})_{\sigma}$. In particular, $(f'^*\kappa_{n,0})_{\sigma'}$ and $(f^*\kappa_{n,k})_{\sigma}$ yield the same logarithmic volume element at x . At x , it is therefore sufficient to take the infimum of all $f'^*\kappa_{n,0}$ in the expression of $\Phi_{X,D}$ in definition 2.5 (*i.e.* we only consider $k = 0$), with Σ' log- K -correspondence between $(\mathbf{D}^n, 0)$ and $(X \setminus D, 0)$. By definition, this gives $\Phi_{X \setminus D, an}$ at the point x . □

When $D = -D_2$ is a negative divisor, a similar argument shows that $\Phi_{X, -D_2}$ can be computed only by looking at log- K -correspondences between $(\mathbf{D}^n, 0)$ and $(X, -D_2)$. In other words, at a point $x \in X$, $\Phi_{X, -D_2}$ is the infimum of all $(f^*\kappa_{n,0})_{\sigma}$ where $\sigma \in \tilde{\Sigma}$, and Σ is a log- K -correspondence between $(\mathbf{D}^n, 0)$ and $(X, -D_2)$ as in definition 2.5.

One has the following obvious comparison results.

Proposition 2.7 *Let X be a complex manifold. Let D and D' be normal crossing Weil divisors of X , such that their respective positive parts are reduced. If $D \leq D'$, then one has*

$$\Phi_{X,D} \leq \Phi_{X,D'}.$$

When $D = 0$, one has

$$\Phi_{X,0} = \Phi_{X,an} \leq \Psi_X.$$

The inequality $\Phi_{X,an} \leq \Psi_X$ is already contained in [24]. It is a simple consequence of example 1.2. The inequality $\Phi_{X,D} \leq \Phi_{X,D'}$ when $D \leq D'$ comes from the fact that a log- K -correspondence between (\mathbf{D}^n, Δ_k) and (X, D') is in particular a log- K -correspondence between (\mathbf{D}^n, Δ_k) and (X, D) .

In [24], Claire Voisin shows that if X is a quotient of \mathbf{D}^n by a group acting freely and properly discontinuously, then $\Phi_{X,an}$ is simply the Poincaré volume form on X . Then by proposition 2.6, we have

$$\Phi_{\mathbf{D}^n, \Delta_k} = \kappa_{n,k}. \quad (14)$$

The proof involves curvature arguments, which are generalized in section 3. When this is done, we will be able to show equality (14) directly (*cf.* theorem 3.5).

Lemma 2.8 *For any pair (X, D) as in definition 2.5, the logarithmic pseudo-volume form $\Phi_{X,D}$ is locally integrable.*

Proof. As we already saw, the Poincaré logarithmic pseudo-volume form

$$\frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2 (\log |z|^2)^2}$$

on $(\mathbf{D}, \{0\})$ is integrable at the neighbourhood of 0. As a consequence, the Poincaré pseudo-volume form $\kappa_{n,k}$ on (\mathbf{D}^n, Δ_k) is locally integrable as well.

Now for any pair (X, D) as in definition 2.5, and for any point $x \in X$, there exists an integer k , and a log- K -isocorrespondence Σ between (\mathbf{D}^n, Δ_k) and (X, D) , which is unramified at the neighbourhood of a point $\sigma \in \tilde{\Sigma}$ over x . By the expression of $\Phi_{X,D}$ in definition 2.5, one then has $(g^* \Phi_{X,D})_\sigma \leq (f^* \kappa_{n,k})_\sigma$. In particular, the growth of $\Phi_{X,D}$ at x is bounded from above by the growth of the Poincaré logarithmic volume form on (\mathbf{D}^n, Δ_k) , and hence $\Phi_{X,D}$ is locally integrable at x . □

An important feature of these intrinsic pseudo-volume forms is their decreasing volume properties. In the standard case, this is inequality (10). For $\Phi_{X,D}$, we even obtain an equality in the case of a proper morphism. These properties are obtained by using the study of the composition of log- K -correspondences, which was carried out in paragraph 1.2. The main result is the following.

Theorem 2.9 *Let (X, D) and (Y, D') be two pairs composed of a complex manifold and a normal crossing Weil divisor, the positive part of which is reduced. Assume there exists a log- K -correspondence $\Sigma \subset X \times Y$ between (X, D) and (Y, D') . We consider a desingularization $\tau : \tilde{\Sigma} \rightarrow \Sigma$, and use the notations of (5). Then we have the inequality of logarithmic pseudo-volume forms on $\tilde{\Sigma}$*

$$g^* \Phi_{Y,D'} \leq f^* \Phi_{X,D}.$$

In the case when $D = D' = 0$, this is proved in [24]. As an immediate consequence of theorem 2.9, we have the following.

Corollary 2.10 *If Σ is a log- K -isocorrespondence between (X, D) and (Y, D') , then we have the equality of logarithmic pseudo-volume forms on $\tilde{\Sigma}$*

$$g^* \Phi_{Y, D'} = f^* \Phi_{X, D}.$$

Proof. Applying theorem 2.9 to the log- K -correspondence Σ on the one hand, and to the log- K -correspondence Σ^T on the other hand gives both inequalities

$$g^* \Phi_{Y, D'} \leq f^* \Phi_{X, D} \quad \text{and} \quad f^* \Phi_{X, D} \leq g^* \Phi_{Y, D'}.$$

One then obviously has the required equality. □

Applying the former corollary to examples 1.2 and 1.3, we get the following.

Corollary 2.11 *(i) Let $\pi : X \rightarrow Y$ be a proper morphism, with ramification divisor R with normal crossings. Then we have*

$$\Phi_{X, -R} = \pi^* \Phi_{Y, 0}.$$

If π is not proper, then we only have $\Phi_{X, -R} \geq \pi^ \Phi_{Y, 0}$.*

(ii) Let $\nu : Y \rightarrow X$ be a proper morphism, and D be an effective, reduced, normal crossing divisor on X . We call D' the proper transform of D with a reduced scheme structure. Then D' has normal crossings, and there exist an effective divisor $R \subset Y$, and an exceptional divisor $E \subset Y$, such that ν ramifies exactly along $(\nu^ D - D') - E + R$. If $D' + E - R$ has normal crossings, and if E is reduced, then*

$$\nu^* \Phi_{X, D} = \Phi_{Y, D' + E - R}.$$

If ν is not proper, then we only have $\nu^ \Phi_{X, D} \leq \Phi_{Y, D' + E - R}$.*

Having the result of proposition 1.8, it is fairly easy to prove the decreasing volume property. The argument is the same as in the standard case of Ψ_X .

Proof of theorem 2.9. Let $\sigma \in \tilde{\Sigma}$, $x = f(\sigma)$, and $y = g(\sigma)$. We have to show that for every $\xi \in f^*(\bigwedge^n T_X(-D))_\sigma \subset g^*(\bigwedge^n T_Y(-D'))_\sigma$, one has the inequality $\Phi_{Y, D'}(g_* \xi) \leq \Phi_{X, D}(f_* \xi)$.

Let $\Sigma' \subset \mathbf{D}^n \times X$ be a log- K -correspondence between (\mathbf{D}^n, Δ_k) and (X, D) , and $\sigma' \in \tilde{\Sigma}'$ such that $g'(\sigma') = x$, and Σ' is not ramified at σ' . Then there exists $\xi' \in g'^*(\bigwedge^n T_X(-D))_{\sigma'}$ satisfying $g'_* \xi' = f_* \xi$. By definition, $\Phi_{X, D}(f_* \xi)$ is the infimum of the $\kappa_{n, k}(f'_* \xi')$, taken on all such Σ' and ξ' .

Now $\Sigma'' := \Sigma \circ \Sigma' \subset \mathbf{D}^n \times Y$ is a log- K -correspondence between (\mathbf{D}^n, Δ_k) and (Y, D') . It has a desingularization $\tilde{\Sigma}''$ obtained as in the diagram (8). We use the same notations here. Since $g'(\sigma') = f(\sigma) = x$, there is a point $\sigma'' \in \tilde{\Sigma}''$ above both σ and σ' , and there exists $\xi'' \in (\psi \circ \tau'')^* f^*(\bigwedge^n T_X(-D))_{\sigma''} = (\phi \circ \tau'')^* g'^*(\bigwedge^n T_X(-D))_{\sigma''}$, such that $(\psi \circ \tau'')_*(\xi'') = \xi$ and $(\phi \circ \tau'')_*(\xi'') = \xi'$. We thus have $\tilde{G}_* \xi'' = g_* \xi$ on the one hand, and $\tilde{F}_* \xi'' = f'_* \xi'$, which implies that $\kappa_{n, k}(\tilde{F}_* \xi'') = \kappa_{n, k}(f'_* \xi')$ on the other hand.

Eventually, we have shown the inclusion of sets

$$\begin{aligned} & \{\kappa_{n, k}(f'_* \xi'), \text{ with } \Sigma' \subset \mathbf{D}^n \times X, \text{ and } g'_* \xi' = f_* \xi\} \\ & \subset \{\kappa_{n, k}(f''_* \xi''), \text{ with } \Sigma'' \subset \mathbf{D}^n \times Y, \text{ and } g''_* \xi'' = g_* \xi\}, \end{aligned}$$

where Σ' (resp. Σ'') runs through all log- K -correspondences between (\mathbf{D}^n, Δ_k) and (X, D) (resp. (Y, D')). Letting k take any value between 0 and n , and taking the infima, we get the desired inequality $\Phi_{Y, D'}(g_* \xi) \leq \Phi_{X, D}(f_* \xi)$. □

3 Curvature arguments

Let X be a complex manifold. In the case of the standard Kobayashi-Eisenman pseudo-volume form, one has the following result, connecting the positivity (or rather the negativity) of the curvature of the canonical line bundle of X , and the infinitesimal measure hyperbolicity of X (i.e. positivity of Ψ_X , see [6], or [23]).

Theorem 3.1 (Kobayashi-Ochiai) *If X is a projective complex manifold which is of general type, then Ψ_X is non-degenerate outside a proper closed algebraic subset of X .*

It is proved in [15], [10] and for $\tilde{\Psi}_X$ in [25]. The main ingredient in the proof is Ahlfors-Schwarz lemma, which we generalize in this section. The converse to theorem 3.1 is conjectured to be true in [13].

Conjecture 3.2 (Kobayashi) *If X is a projective complex manifold which is not of general type, then $\Psi_X = 0$ on a dense Zariski open subset of X .*

We study a logarithmic variant of this conjecture in the logarithmic Calabi-Yau case in section 4. In this section, we first establish a generalized Ahlfors-Schwarz lemma in paragraph 3.1. Then in paragraph 3.2, we use it together with a result of Carlson and Griffiths to prove a version of the Kobayashi-Ochiai theorem relative to the logarithmic pseudo-volume forms $\Phi_{X,D}$. Note that for $D = 0$, this is done in [24].

3.1 Metrics with negative curvature on \mathbf{D}^n

We first need to recall a few facts about the logarithmic hyperbolic volume form $\kappa_{n,k}$ on (\mathbf{D}^n, Δ_k) and its curvature. It is obtained from the Poincaré metric, which has Kähler form

$$\omega_{n,k} = \frac{i}{2} \left(\sum_{1 \leq j \leq n-k} \frac{dz_j \wedge d\bar{z}_j}{(1 - |z_j|^2)^2} + \sum_{n-k+1 \leq j \leq n} \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2 (\log |z_j|^2)^2} \right)$$

($\kappa_{n,k} = \omega_{n,k}^n / n!$, cf. (13)). It has Ricci form

$$\begin{aligned} & -i\partial\bar{\partial} \log \left(\prod_{1 \leq j \leq n-k} \frac{1}{(1 - z_j \bar{z}_j)^2} \prod_{n-k+1 \leq j \leq n} \frac{1}{z_j \bar{z}_j (\log(z_j \bar{z}_j))^2} \right) \\ &= 2i \left(\sum_{1 \leq j \leq n-k} \frac{\partial}{\partial z_j} \frac{-z_j}{1 - z_j \bar{z}_j} d\bar{z}_j + \sum_{n-k+1 \leq j \leq n} \frac{\partial}{\partial z_j} \frac{1}{z_j \log(z_j \bar{z}_j)} d\bar{z}_j \right) = -4\omega_{n,k}. \end{aligned}$$

This gives the Kähler-Einstein equation

$$(i\partial\bar{\partial} \log \kappa_{n,k})^n = 4^n \omega_{n,k}^n = 4^n n! \kappa_{n,k}. \quad (15)$$

Eventually, we need the following definition to state properly the generalization of Ahlfors-Schwarz lemma in the logarithmic case.

Definition 3.3 *Let (X, D) be a pair composed by a complex manifold X , and a Weil divisor $D = D' - D''$, with D' and D'' non negative, and D' reduced and normal crossing. A logarithmic pseudo-volume form μ on (X, D) is said to have singularities of Poincaré type if it is C^∞ on $X \setminus D'$, and if it is equivalent to a non zero multiple of*

$$\prod_{n-k+1 \leq j \leq n} \frac{1}{|z_j|^2 (\log |z_j|^2)^2}$$

at the neighbourhood of D' , assuming it is given by the equation $z_{n-k+1} \cdots z_n = 0$.

Of course, the logarithmic Poincaré volume form $\kappa_{n,k}$ on (\mathbf{D}^n, Δ_k) has singularities of Poincaré type.

Proposition 3.4 *Let (X, D) be a pair composed by an n -dimensional complex manifold X and a normal crossing Weil divisor D of X , the positive part of which is reduced. Let $\Sigma \subset \mathbf{D}^n \times X$ be a log- K -correspondence between (\mathbf{D}^n, Δ_k) and (X, D) , with desingularization $\tau : \tilde{\Sigma} \rightarrow \Sigma$. Let μ be a logarithmic pseudo-volume form on (X, D) , satisfying the three following properties.*

- (a) $i\partial\bar{\partial} \log \mu > 0$.
- (b) μ has singularities of Poincaré type.
- (c) $(i\partial\bar{\partial} \log \mu)^n \geq 4^n n! \mu$.

Then one has the inequality of logarithmic pseudo-volume forms on $\tilde{\Sigma}$

$$g^* \mu \leq f^* \kappa_{n,k}.$$

Note that if D has a positive part, then μ has poles, and therefore inequality (c) cannot be true if μ has no singularity, *i.e.* if it writes locally $\mu'/|h|^2$, where h is a meromorphic function with divisor D , and μ' is a C^∞ pseudo-volume form on X . In this case, $i\partial\bar{\partial} \log \mu = i\partial\bar{\partial} \log \mu'$ is a C^∞ (1,1)-form. In particular, it has no pole on X .

Proof of proposition 3.4. We begin by restricting Σ to $\mathbf{D}_{1-\varepsilon}^n := \mathbf{D}(0, 1-\varepsilon)^n$ as follows. We let Σ_ε be the inverse image of Σ *via* the map $((1-\varepsilon)\text{id}, \text{id}) : \mathbf{D}^n \times X \rightarrow \mathbf{D}^n \times X$. This would correspond in the morphism case to the transformation of $\varphi : \mathbf{D}^n \rightarrow X$ into $\tilde{\varphi} := \varphi|_{\mathbf{D}_{1-\varepsilon}^n}((1-\varepsilon) \times \cdot) : \mathbf{D}^n \rightarrow X$. Of course one gets from $\tilde{\Sigma}$ a desingularization $\tilde{\Sigma}_\varepsilon$ of Σ_ε in a natural way, with maps f_ε and g_ε to \mathbf{D}^n and X .

Next, one considers the ratio

$$\psi_\varepsilon := \frac{g_\varepsilon^* \mu}{f_\varepsilon^* \kappa_n}.$$

It is a non-negative C^∞ -function on $\tilde{\Sigma}_\varepsilon$. To see why, we first write locally $\mu = \mu'/|h|^2$, where h is a meromorphic function with divisor D (*i.e.* it has zeroes along the positive part D_1 of D , and poles along the negative part D_2 of D), and $\mu' = \chi_{\mu'}(i/2)^n \bigwedge_{1 \leq j \leq n} dz_j \wedge d\bar{z}_j$ is a pseudo-volume form on X , which is C^∞ on $X \setminus D_1$, and singular along D_1 , so that μ has singularities of Poincaré type. On the other hand, we let similarly

$$\chi_{n,k} = \left(\prod_{1 \leq j \leq n-k} (1 - |t_j|^2)^{-2} \right) \left(\prod_{n-k+1 \leq j \leq n} (\log |t_j|^2)^{-2} \right),$$

so that $\kappa_{n,k} = \chi_{n,k}(i/2)^n \bigwedge_{1 \leq j \leq n} dt_j \wedge d\bar{t}_j / |t_{n-k+1} \cdots t_n|^2$. We let $h_k(t) = t_{n-k+1} \cdots t_n$ be a holomorphic equation of $\Delta_k \subset \mathbf{D}^n$. Then ψ_ε writes locally

$$\psi_\varepsilon = \frac{|s_g|^2 g_\varepsilon^* \chi_{\mu'}}{|g_\varepsilon^* h|^2} \cdot \frac{|f_\varepsilon^* h_k|^2}{|s_f|^2 f_\varepsilon^* \chi_{n,k}},$$

where s_f and s_g are local analytic equations of the ramification divisors R_f and R_g respectively. Now since Σ is a log- K -correspondence between (\mathbf{D}^n, Δ_k) and (X, D) , one has $R_g - g^* D \geq R_f - f^* \Delta_k$, and therefore $s_g \cdot f_\varepsilon^* h_k / s_f \cdot g_\varepsilon^* h$ is C^∞ on X . In addition, since μ has singularities of Poincaré type, the ratio $g_\varepsilon^* \chi_{\mu'} / f_\varepsilon^* \chi_{n,k}$ is C^∞ on X as well.

Now, $1/\chi_{n,k}$ tends to 0 on the boundary of \mathbf{D}^n , while $\psi_\varepsilon \cdot f_\varepsilon^* \chi_{n,k}$ stays bounded near the boundary of $\tilde{\Sigma}_\varepsilon$, since $\overline{\mathbf{D}_{1-\varepsilon}^n}$ is compact and f is proper. This gives

$$\lim_{f_\varepsilon(x) \rightarrow \partial \mathbf{D}^n} \psi_\varepsilon(x) = 0,$$

and so by properness of f_ε , the ratio ψ_ε has a maximum on $\tilde{\Sigma}_\varepsilon$. Let $x_0 \in \tilde{\Sigma}_\varepsilon$ be a point where this maximum is reached, and write $c := \psi_\varepsilon(x_0)$. Then we just have to show that $c \leq 1$ (we will then get the proposition by letting ε tend to 0).

We now suppose that $c > 1$, and show as in the standard proof that this contradicts hypotheses (a), (b) and (c). For $\alpha \in]1, c[$, define

$$\tilde{\Sigma}_{\varepsilon, \alpha} = \left\{ x \in \tilde{\Sigma}_{\varepsilon} \text{ s.t. } \psi_{\varepsilon}(x) \geq \alpha \right\}.$$

$\psi_{\varepsilon}(x)$ tends to 0 near the boundary of $\tilde{\Sigma}_{\varepsilon}$, so $\tilde{\Sigma}_{\varepsilon, \alpha}$ is compact, and with smooth boundary for α generic. Since $i\partial\bar{\partial} \log \chi_{\mu'} = i\partial\bar{\partial} \log \mu > 0$, and $i\partial\bar{\partial} \log \chi_{n,k} = 4\omega_{n,k} > 0$,

$$\theta := g_{\varepsilon}^* (i\partial\bar{\partial} \log \chi_{\mu'})^{n-1} + g_{\varepsilon}^* (i\partial\bar{\partial} \log \chi_{\mu'})^{n-2} f_{\varepsilon}^* (i\partial\bar{\partial} \log \chi_{n,k}) + \cdots + f_{\varepsilon}^* (i\partial\bar{\partial} \log \chi_{n,k})^{n-1}$$

is a semi-positive $(n-1, n-1)$ -form, positive away from the positive part of $R_f - f^* \Delta_k$. Note that it is $+\infty$ along the negative part of $R_g - g^* D$, i.e. along the positive part of the reduced divisor $(g^* D)_{\text{red}}$, where μ' is singular. Now ψ_{ε} has Laplacian $i\partial\bar{\partial} \log \psi_{\varepsilon} = i\partial\bar{\partial} \log(g_{\varepsilon}^* \chi_{\mu'}) - i\partial\bar{\partial} \log(f_{\varepsilon}^* \chi_{n,k})$, and we have

$$\begin{aligned} (i\partial\bar{\partial} \log \psi_{\varepsilon}) \theta &= g_{\varepsilon}^* (i\partial\bar{\partial} \log \chi_{\mu'})^n - f_{\varepsilon}^* (i\partial\bar{\partial} \log \chi_{n,k})^n \\ &\geq 4^n n! (g_{\varepsilon}^* \mu - f_{\varepsilon}^* \kappa_{n,k}), \end{aligned}$$

where the inequality is given by condition (c) on μ , and by the Kähler-Einstein equation (15) for the hyperbolic volume form. Now in $\tilde{\Sigma}_{\varepsilon, \alpha}$ we have $\psi_{\varepsilon} > 1$, and therefore

$$g_{\varepsilon}^* \mu - f_{\varepsilon}^* \kappa_n \geq 0,$$

with strict inequality away from the positive part of $R_f - f^* \Delta_k$ (again, this is $+\infty$ along the positive part of $(g^* D)_{\text{red}}$). Finally the Laplacian $i\partial\bar{\partial} \log \psi_{\varepsilon}$ is semi-positive, and positive away from the positive part of $R_f - f^* \Delta_k$.

When x_0 does not belong to the positive part of $R_f - f^* \Delta_k$, one can conclude from the maximum principle for pluri-subharmonic functions that ψ_{ε} cannot have a maximum at x_0 , which is a contradiction, and proves $c \leq 1$ as we wanted. Otherwise, one has to apply the following standard argument. One chooses m satisfying $\log \alpha < m < \log c$, and then defines a function

$$\mu^+(x) = \max(0, \log \psi_{\varepsilon}(x) - m).$$

It is non-negative, positive at x_0 , and vanishes identically near the boundary $\partial\tilde{\Sigma}_{\varepsilon, \alpha}$. Therefore we have

$$\int_{\tilde{\Sigma}_{\varepsilon, \alpha}} \mu^+ (i\partial\bar{\partial} \log \psi_{\varepsilon}) \theta > 0.$$

Note that the form $(i\partial\bar{\partial} \log \psi_{\varepsilon}) \theta$ is indeed integrable in $\tilde{\Sigma}_{\varepsilon, \alpha}$, because both μ and $\kappa_{n,k}$ only have singularities of Poincaré type. The derivatives of μ^+ are integrable, so we can integrate by parts the previous integral. This gives

$$- \int_{\tilde{\Sigma}_{\varepsilon, \alpha}} \partial \mu^+ \wedge (i\bar{\partial} \log \psi_{\varepsilon}) \theta > 0.$$

Since $\mu^+ = \log \psi_{\varepsilon} - m$ when it is non-zero, the former inequality gives

$$\int_{\tilde{\Sigma}_{\varepsilon, \alpha}} \partial \mu^+ \wedge (i\bar{\partial} \log \psi_{\varepsilon}) \theta = \int_{\{\log \psi_{\varepsilon}(x) \geq m\}} i(\partial \log \psi_{\varepsilon}) \wedge (\bar{\partial} \log \psi_{\varepsilon}) \theta < 0,$$

which is a contradiction, since the right-hand side integral is obviously positive. \square

As a first application of this result, we can show directly that $\Phi_{\mathbf{D}^n, \Delta_k}$ is indeed given by the hyperbolic logarithmic Poincaré volume form (see proposition 2.6 as well).

Theorem 3.5 For $0 \leq k \leq n$, we have the equality of logarithmic volume forms on (\mathbf{D}^n, Δ_k)

$$\Phi_{\mathbf{D}^n, \Delta_k} = \kappa_{n,k}.$$

Proof. The diagonal in $\mathbf{D}^n \times \mathbf{D}^n$ is a log- K -correspondence between (\mathbf{D}^n, Δ_k) and itself. By definition 2.5, we thus have $\Phi_{\mathbf{D}^n, \Delta_k} \leq \kappa_{n,k}$.

On the other hand, let Σ be a log- K -correspondence between (\mathbf{D}^n, Δ_p) and (\mathbf{D}^n, Δ_k) , with $0 \leq p \leq n$. Then by proposition 3.4, and using the standard notations of (5), we have $g^* \kappa_{n,k} \leq f^* \kappa_{n,p}$. This implies that $\kappa_{n,k} \leq \Phi_{\mathbf{D}^n, \Delta_k}$. \square

3.2 Mappings onto pairs with positive logarithmic canonical bundle

Before we state our generalization of the Kobayashi-Ochiai theorem 3.1, let us first recall that a variety X is said to be of general type if the canonical bundle K_X is big. A line bundle L on X is said to be big if it has maximal Iitaka-Kodaira dimension $\kappa(X, L) = \dim X$. This is equivalent to the fact, that the image of the rational map associated to the linear system $|mL|$ is of maximal dimension $\dim X$, for m big enough and divisible enough (see *e.g.* [20]). Given a Weil divisor D on X , the condition corresponding to the fact that X is of general type is in our case the bigness of $K_X + D$. However, we shall need some slightly stronger hypotheses to prove our result.

Theorem 3.6 Let (X, D) be a pair composed by a projective n -dimensional complex manifold X , and a normal crossing Weil divisor D of X , the positive part of which is reduced and has global normal crossings. If $K_X + D$ is ample, then $\Phi_{X,D} > 0$ away from a proper closed algebraic subset of X .

As an important consequence of this result, we can bound from above the degree of a morphism of logarithmic pairs, which is onto a pair with positive logarithmic canonical bundle.

Corollary 3.7 Let (X, D) be a pair composed by a projective manifold X , and a normal crossing Weil divisor D , the positive part of which is reduced, and has global normal crossings. We assume that $K_X + D$ is ample. Let (Y, D') be another logarithmic pair. We assume that D' has normal crossings, and that its positive part is reduced. For every dominant morphism $\phi : (Y, D') \rightarrow (X, D)$, we have

$$\deg \phi \leq \frac{\int_Y \Phi_{Y,D'}}{\int_X \Phi_{X,D}}.$$

Proof. The morphism ϕ induces a log- K -correspondence between (Y, D') and (X, D) (*cf.* example 1.3). By decreasing volume property, we thus have

$$\phi^* \Phi_{X,D} \leq \Phi_{Y,D'}. \tag{16}$$

Since for $0 \leq k \leq n$, the volume form $\kappa_{n,k}$ is integrable at the neighbourhood of Δ_k , and since X and Y are compact, definition 2.5 implies that the pseudo-volume forms $\Phi_{X,D}$ and $\Phi_{Y,D'}$ are integrable on X and Y respectively. In particular, (16) yields

$$(\deg \phi) \int_X \Phi_{X,D} \leq \int_Y \Phi_{Y,D'}.$$

Now, by theorem 3.6, the integral $\int_X \Phi_{X,D}$ is positive, so $\deg \phi$ is bounded from above. \square

Along with Ahlfors-Schwarz lemma 3.4, the main ingredient in the proof of theorem 3.6 above is the following result.

Lemma 3.8 (Carlson-Griffiths, [4]) *Let (X, D) be a pair as in theorem 3.6. Then there exists a logarithmic volume form with Poincaré singularities μ on (X, D) , such that $i\partial\bar{\partial}\log\mu > 0$ and $(i\partial\bar{\partial}\log\mu)^n \geq \mu$.*

For a complete proof of this, we refer to [11], proposition 2.17. Let us still explain how this pseudo-volume form is constructed. We choose a Hermitian metric h_0 on K_X . Equivalently, h_0^{-1} is a C^∞ volume form μ_X on X . We assume for simplicity that D is effective, the general case follows easily. Then D is assumed to have global normal crossings, so it writes $D = D_1 + \dots + D_p$, where the D_j are smooth divisors on X , meeting transversally. We choose a Hermitian metric h_j for every line bundle $\mathcal{O}_X(D_j)$, and we let s_j be a global section of this line bundle, with zero divisor D_j . Then for $\alpha > 0$ sufficiently small, the pseudo-volume form

$$\mu_\alpha := \frac{\mu_X}{\prod_{j=1}^p h_j(s_j) (\log(\alpha h_j(s_j)))^2}$$

satisfies the required properties.

Before we continue, we want to underline that this result is used as a first step in the construction of complete Kähler-Einstein metrics with negative Ricci curvature on the complement of hypersurfaces of projective algebraic manifolds (see [12] and [19]).

Proof of theorem 3.6. By lemma 3.8, there exists a logarithmic pseudo-volume form μ on (X, D) , with Poincaré singularities, and such that $i\partial\bar{\partial}\log\mu > 0$, and $(i\partial\bar{\partial}\log\mu)^n \geq \mu$. Up to a rescaling, one can assume that μ satisfies hypothesis (c) of lemma 3.4. Then if $0 \leq k \leq n$, and for every log- K -correspondence Σ between (\mathbf{D}^n, Δ_k) and (X, D) , we have the inequality of pseudo-volume forms on a desingularization $\tilde{\Sigma}$

$$g^*\mu \leq f^*\kappa_{n,k}.$$

It thus follows from the definition 2.5 of $\Phi_{X,D}$ that $\Phi_{X,D} \geq g^*\mu$, which implies that $\Phi_{X,D}$ is positive on a Zariski dense open subset of X , since μ is a true logarithmic volume form on (X, D) . □

4 Log- K -autocorrespondences on log- K -trivial pairs

In this section, we prove that for many pairs (X, D) , where X is a complex manifold, and D an effective divisor on X , which is reduced and has normal crossings, such that the line bundle $K_X(D)$ is trivial, the pseudo-volume form $\Phi_{X,D}$ vanishes. This can be interpreted as a special case of the Kobayashi conjecture 3.2 in the logarithmic case. Again, the case $D = 0$ is handled in [24]. However, one has to be slightly more careful in our case for the proof. This is due to the fact that we need to pull-back differential forms *via* correspondences, instead of cohomology classes.

4.1 Log- K -autocorrespondences and the Kobayashi conjecture

In view of theorem 3.6, which has to be seen as a logarithmic version of the Kobayashi-Ochiai theorem for $\Phi_{X,D}$, the Kobayashi conjecture generalizes as follows in the logarithmic case.

Conjecture 4.1 *Let (X, D) be a pair composed by a projective manifold X and a normal crossing Weil divisor $D \subset X$, the positive part of which is reduced. If (X, D) is not of log-general type (i.e. if $K_X + D$ is not big), then $\Phi_{X,D}$ vanishes on a dense Zariski open subset of X .*

This section is devoted to the proof of the following result, which goes in the direction of this conjecture.

Theorem 4.2 *Let (X, D) be a pair consisting of a smooth projective variety X , and an effective divisor $D \subset X$, which is reduced and has normal crossings, such that $K_X(D)$ is trivial. Assume that there exists a smooth, rationally connected variety Y , such that X can be realized as a hypersurface $X \subset Y$, $D = X \cap X'$, where $X' \subset Y$ is a reduced hypersurface such that $X + X' \in |-K_Y|$. Then $\Phi_{X,D} = 0$.*

Note that this shows the log-Kobayashi conjecture for a very wide class of log- K -trivial pairs.

The next proposition shows why this theorem is an easy consequence of the existence on such pairs of log- K -autocorrespondences (i.e. log- K -isocorrespondences between a pair (X, D) and itself, see definition 1.4) satisfying a certain dilation property.

Proposition 4.3 *Let (X, D) be a pair composed by a smooth projective variety X , and an effective divisor $D \subset X$, which is reduced and has normal crossings, such that $K_X(D)$ is trivial. Let η_X be a generator of $H^0(X, K_X(D))$. If there exists a log- K -autocorrespondence Σ of the pair (X, D) , such that for a desingularization $\tau : \tilde{\Sigma} \rightarrow \Sigma$, and with the notations (5) of definition 1.1, one has*

$$f^*\eta_X = \lambda g^*\eta_X,$$

where λ is a complex number with $|\lambda| \neq 1$, then $\Phi_{X,D} = 0$.

Proof. Let $\Omega_{X,D}$ be defined as

$$\Omega_{X,D} = (-1)^{\frac{n(n-1)}{2}} i^n \eta_X \wedge \bar{\eta}_X.$$

The dilation property satisfied by Σ shows that

$$f^*\Omega_{X,D} = |\lambda|^2 g^*\Omega_{X,D},$$

while corollary 2.10 gives the equality of pseudo-volume forms

$$f^*\Phi_{X,D} = g^*\Phi_{X,D}. \tag{17}$$

Now there exists a bounded, upper semi-continuous function χ on X , such that $\Phi_{X,D} = \chi \Omega_{X,D}$. χ has a maximum on X . Let x be a point on X where $\chi(x)$ is this maximum. Then take $\sigma \in \tilde{\Sigma}$ such that $f(\sigma) = x$, and let $y = g(\sigma)$. Equality (17) eventually gives $\chi(y) = |\lambda|^2 \chi(x)$. Since of course we can assume $|\lambda| > 1$ by symmetry, this shows that $\chi(x) = 0$, and hence that $\chi = 0$ as we wanted. □

Now the proof of 4.2 amounts to show the following theorem. This is done in the remainder of this section.

Theorem 4.4 *If (X, D) is a pair satisfying the condition of theorem 4.2, then there exists a log- K -autocorrespondence of (X, D) , such that (with the notations of proposition 4.3)*

$$mf^*\eta_X = -m'g^*\eta_X, \tag{18}$$

where m and m' are distinct positive integers.

In fact, by the following remark, it is enough to prove that there exists a proper 0-correspondence $\Sigma \subset X \times X$ (cf. definition 1.6) satisfying relation (18).

Remark 4.5 Let $\Sigma \subset X \times X$ be a reduced closed analytic subset, with both projections proper and generically finite. If $mf^*\eta_X = -m'g^*\eta_X$ (with the notations of proposition 4.3), then Σ is a log- K -autocorrespondence of (X, D) .

Indeed, η_X is an everywhere non zero section of $K_X(D)$, so the meromorphic n -form $f^*\eta_X$ (resp. $g^*\eta_X$) on $\tilde{\Sigma}$ has divisor $R_f - f^*D$ (resp. $R_g - g^*D$), i.e. it has zeroes along R_f , and poles along f^*D . Equality (18) thus yields the equality of divisors

$$R_f - f^*D = R_g - g^*D$$

on the desingularization $\tilde{\Sigma}$, proving that Σ is a log- K -autocorrespondence of (X, D) .

4.2 Geometric construction of self-correspondences on K -trivial pairs

From now on, we let Y be a smooth, rationally connected variety of dimension $n+1$, and $\bar{X} \subset Y$ be a reduced hypersurface in the anticanonical linear class of Y (i.e. $\bar{X} \in |-K_Y|$), with smooth locus $\text{sm}(\bar{X}) \subset \bar{X}$. By adjunction, \bar{X} has a canonical sheaf $K_{\bar{X}}$, and it is trivial. We construct in this section a 0-correspondence $\Sigma \subset \bar{X} \times \bar{X}$. In [24], such correspondences were constructed in the case of smooth \bar{X} , and proved to be K -autocorrespondences. The process is in some way analogous to the geometric definition of addition on an elliptic curve in \mathbf{P}^2 . Of course, we have in mind the case when $\bar{X} = X \cup X'$ is the reunion of two reduced hypersurfaces, X smooth, and $D := X \cap X'$ is reduced and has normal crossings, which will give theorem 4.4 for the pair (X, D) , where D is seen as a divisor of X .

Since Y is rationally connected, there exists a rational curve $C_0 \subset Y$ satisfying the two following conditions.

- (i) C_0 does not meet the singular part of \bar{X} , and $C_0 \cap \bar{X} = mx_0 + m'y_0 + z_0$ as a divisor on C_0 , where x_0 and y_0 are distinct points of $\text{sm}(\bar{X})$, m and m' are (fixed) distinct positive integers, and z_0 is a reduced 0-cycle on C_0 , disjoint from x_0 and y_0 .
- (ii) The deformations of the subscheme $C_0 \subset Y$ induce arbitrary deformations of the M -jet of C_0 at two points of intersection with \bar{X} , where $M := \max(m, m')$.

This is given by the fact that C_0 can be chosen with arbitrarily ample normal bundle, because Y is rationally connected (see [16]).

In addition, one chooses a hypersurface $W \subset \bar{X}$ containing the 0-cycle z_0 . We denote by $|C_0|$ the space parametrizing all deformations of the subscheme $C_0 \subset Y$. Now define $\Sigma_{tot.} \subset \text{sm}(\bar{X}) \times \text{sm}(\bar{X}) \times |C_0|$ by

$$\begin{aligned} \Sigma_{tot.} := \{ & (x, y, C) \text{ s.t. } C \text{ deformation of } C_0, \\ & C \cap \bar{X} = mx + m'y + z, \text{ supp}(z) \subset W \}. \end{aligned} \tag{19}$$

Proposition 4.6 For a generic choice of W , the Zariski closure of $\Sigma_{tot.}$ in $\bar{X} \times \bar{X} \times |C_0|$ has a unique n -dimensional irreducible component through the point (x_0, y_0, C_0) . We call this component Σ' . The Zariski closure of the projection of Σ' in $\bar{X} \times \bar{X}$ is irreducible of dimension n . Let us denote it Σ . Then $\Sigma \subset \bar{X} \times \bar{X}$ is a 0-correspondence.

We call f' and g' the two morphisms $\Sigma' \rightarrow \bar{X}$ given by the projections of $\bar{X} \times \bar{X} \times |C_0|$ on its first and second factors respectively. Σ is an n -dimensional component of the closure of $(f', g')(\Sigma')$.

Proof. The proposition essentially follows from the fact that, for a generic choice of W , $\Sigma_{tot.}$ is smooth and of dimension $n = \dim \bar{X}$ at the point (x_0, y_0, C_0) . This gives the existence of a unique component Σ' at once. On the other hand, since the deformations of $C_0 \subset Y$ induce arbitrary deformations of the M -jet of C_0 at x_0 and y_0 , this implies that the image $(f', g')(\Sigma') \subset \bar{X} \times \bar{X}$ has a component of dimension n .

So let us show that $\Sigma_{tot.}$ is smooth and of dimension n near (x_0, y_0, C_0) . We first study the Hilbert scheme of $C_0 \subset Y$ at the infinitesimal neighbourhood of C_0 . Since C_0 is a rational curve, and its normal bundle $N_{C_0/Y}$ is ample, we have $h^1(C_0, N_{C_0/Y}) = 0$, and the Hilbert scheme of $C_0 \subset Y$ is smooth, and of dimension

$$h^0(C_0, N_{C_0/Y}) = \chi(C_0, N_{C_0/Y}).$$

By the Riemann-Roch formula,

$$\begin{aligned} \chi(C_0, N_{C_0/Y}) &= \deg(N_{C_0/Y}) + \text{rg}(N_{C_0/Y})(1 - g) \\ &= -K_Y \cdot C_0 + (2g - 2) + n(1 - g) = n - 2 - K_Y \cdot C_0 \end{aligned}$$

($g = 0$ is the geometric genus of C_0). Now to compute the dimension of $\Sigma_{tot.}$, we note that we impose to the deformations C of C_0 to meet \bar{X} properly and in the smooth locus $\text{sm}(\bar{X})$ (this is open), and to have intersection $C \cap \bar{X} = mx + m'y + z$, with $\text{supp}(z) \subset W$. This imposes at most $(m - 1) + (m' - 1) + \deg z$ conditions. In fact, for a generic choice of W , these conditions are infinitesimally independent at the starting point (x_0, y_0, C_0) . We deduce from this that $\Sigma_{tot.}$ is smooth at the neighbourhood of (x_0, y_0, C_0) , and, since $\bar{X} \in |-K_Y|$, of dimension

$$(-K_Y \cdot C_0 + n - 2) - (m + m' - 2 + \deg z) = n - K_Y \cdot C_0 - \bar{X} \cdot C_0 = n.$$

□

4.3 Realization as log- K -autocorrespondences

When $\bar{X} = X \cup X'$ is the union of two reduced hypersurfaces, and if X is smooth, then a section of $K_{\bar{X}}$ gives a meromorphic n -form on the component X by restriction to the smooth locus of \bar{X} . In this subsection, we show the following.

Theorem 4.7 *Let $\Sigma \subset \bar{X} \times \bar{X}$ be as before (see proposition 4.6). We consider a desingularization $\tau : \tilde{\Sigma} \rightarrow \Sigma$ and use the notations (5) of definition 1.1. Let $\eta_{\bar{X}}$ be a generator of $H^0(\bar{X}, K_{\bar{X}})$ (recall that $K_{\bar{X}}$ is trivial). We have the equality of meromorphic differential n -forms on $\tilde{\Sigma}$*

$$mf^*\eta_{\bar{X}} + m'g^*\eta_{\bar{X}} = 0. \quad (20)$$

Specializing this result to the case $\bar{X} = X \cup X'$ already mentioned, we get the following corollary. Of course, it proves theorem 4.4, and because of proposition 4.3, we get our main theorem 4.2 (cf. subsection 4.1).

Corollary 4.8 *If $\bar{X} = X \cup X'$ is the union of two reduced hypersurfaces, X smooth, and $D := X \cap X'$ reduced, then Σ induces a 0-correspondence $\Sigma_1 := \Sigma \cap (X \times X) \subset X \times X$. If η_X is a generator of $H^0(X, K_X(D))$, then we have the equality of meromorphic differential n -forms*

$$mf_1^*\eta_X + m'g_1^*\eta_X = 0$$

on a desingularization $\tilde{\Sigma}_1 \rightarrow \Sigma_1$, where f_1 and g_1 are the natural morphisms $\Sigma_1 \rightarrow X$. In particular, Σ_1 is a log- K -autocorrespondence of the pair (X, D) .

Proof. In this case, $\Sigma \subset \bar{X} \times \bar{X}$ splits into four components, contained in $X \times X$, $X \times X'$, $X' \times X$ and $X' \times X'$ respectively. Of course, the desingularization $\tau : \tilde{\Sigma} \rightarrow \Sigma$ splits as well. The first component gives $\Sigma_1 \subset X \times X$, equipped with a desingularization $\tilde{\Sigma}_1 \rightarrow \Sigma_1$ induced by τ .

On the other hand, we have by adjunction

$$(K_{\bar{X}})|_X = K_Y(\bar{X})|_X = K_Y(X)|_X \otimes \mathcal{O}_X(X') = K_X(D).$$

In particular, since $K_{\overline{X}}$ is trivial, $K_X(D)$ is trivial as well, and if $\eta_{\overline{X}}$ is a generator of $H^0(\overline{X}, K_{\overline{X}})$, then its restriction $\eta_X := \eta_{\overline{X}}|_X$ is a generator of $H^0(X, K_X(D))$. Since X is smooth, η_X is a meromorphic differential n -form, with polar divisor D . It is now clear from theorem 4.7 that we have the equality of meromorphic n -forms

$$mf_1^*\eta_X + m'g_1^*\eta_X = 0$$

on the desingularization $\widetilde{\Sigma}_1 \rightarrow \Sigma_1$. By remark 4.5, this implies that Σ_1 is a log- K -autocorrespondence of (X, D) . □

Proof of theorem 4.7. We consider a desingularization $\tau' : \widetilde{\Sigma}' \rightarrow \Sigma'$, with maps $\widetilde{f} : \widetilde{\Sigma}' \rightarrow \overline{X}$ and $\widetilde{g} : \widetilde{\Sigma}' \rightarrow \overline{X}$. It is enough to show (20) on $\widetilde{\Sigma}'$, that is

$$m\widetilde{f}^*\eta_{\overline{X}} = -m'\widetilde{g}^*\eta_{\overline{X}}. \quad (21)$$

The reason for this is that $(\Sigma, \text{pr}_1, \text{pr}_2)$ is the Stein factorization of (Σ', f', g') (and also of $(\widetilde{\Sigma}', \widetilde{f}, \widetilde{g})$). In fact, since (21) is an equality of meromorphic differential forms, it is enough to prove it locally, and on a dense open subset of $\widetilde{\Sigma}'$.

We construct three 0-correspondences $\Gamma_{x,y}$, Γ_z and Γ_C between Σ' and \overline{X} . Each of them is defined by its fiber over a generic point $\sigma = (x, y, C) \in \Sigma'$. We describe these generic fibers as subschemes of \overline{X} .

-The fiber of $\Gamma_{x,y}$ over σ is the 0-cycle $m.x + m'.y$.

-The fiber of Γ_z over σ is the 0-cycle $C \cdot X - m.x - m'.y$. For σ generic, this is the part of the intersection $C \cap \overline{X}$ that lies on the hypersurface $W \subset \overline{X}$.

-The fiber of Γ_C over σ is the 0-cycle $C \cap \overline{X}$. Γ_C can also be defined globally as $\mathcal{C} \cap (\Sigma' \times \overline{X}) \subset \Sigma' \times Y$, where \mathcal{C} is the universal curve over Σ' .

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \Sigma' \times Y \\ & \searrow & \downarrow \\ & & \Sigma' \end{array}$$

Now, following an idea of Mumford ([18]), we define the pull-back of $\eta_{\overline{X}}$ by a 0-correspondence $\Gamma \subset \Sigma' \times \overline{X}$. In fact, since it is enough to prove (21) locally and on a dense open subset of $\widetilde{\Sigma}'$, we only need to define the pull-back of the restriction of $\eta_{\overline{X}}$ to a dense open subset of \overline{X} . We thus define a smooth, dense open subset $\Sigma^\circ \subset \Sigma'$ in the following way (since it is smooth, Σ° can also be seen as a dense open subset of $\widetilde{\Sigma}'$). We let pr_{12} be the projection $\overline{X} \times \overline{X} \times |\mathcal{C}_0| \rightarrow \overline{X} \times \overline{X}$. Then Σ° is the subset of $\Sigma' \cap \text{pr}_{12}^{-1}(\text{sm}(\overline{X}) \times \text{sm}(\overline{X}))$, above which the projection $\Gamma_C \rightarrow \Sigma'$ is finite. By restricting Σ° , one can assume that it is smooth, and that it is a self-0-correspondence of a smooth, dense open subset $X^\circ \subset \overline{X}$. We call Γ° the restriction of Γ to $\Sigma^\circ \times X^\circ$, and φ and ψ the two projections of Γ° on Σ° and X° respectively. The key point here is the fact that φ is proper. Of course, one can also assume Γ° to be smooth.

$$\begin{array}{ccc} & \Gamma^\circ & \\ \varphi \swarrow & & \searrow \psi \\ \Sigma^\circ & & X^\circ \end{array}$$

We now define the pull-back by Γ° of the restriction $\eta^\circ := \eta_{\overline{X}}|_{X^\circ}$. Since $X^\circ \subset \text{sm}(\overline{X})$, η° is a holomorphic differential n -form on X° . Its pull-back $\psi^*\eta^\circ$ is then a holomorphic n -form on Γ° . Eventually, we let

$$(\Gamma^\circ)^*\eta^\circ := \varphi_*\psi^*\eta^\circ,$$

where φ_* is the trace map relative to φ . $(\Gamma^\circ)^*\eta^\circ$ is a holomorphic n -form on Σ° . It extends to a meromorphic n -form on the smooth variety $\widetilde{\Sigma}'$; we call it $\Gamma^*\eta_{\overline{X}}$. The definition of the trace map φ_* is classical. It goes as follows. Let ω be a holomorphic n -form on Γ° . If $U \subset \Sigma^\circ$ is an open subset above which φ is not ramified, then $\varphi^{-1}U$ is the disjoint union of d open subsets $U_1, \dots, U_d \subset \Gamma^\circ$, which are all isomorphic to U . On U , $\varphi_*\omega$ is defined as the sum

$$\omega|_{U_1} + \dots + \omega|_{U_d}.$$

Then all these local definitions glue together to give a well defined holomorphic n -form $\varphi_*\omega$ on Σ° , even if φ ramifies on Γ° .

We now have all the definitions we need to end the proof. The equality (21) is a simple consequence of the following proposition.

Proposition 4.9 *The holomorphic n -form $(\Gamma_{x,y}^\circ)^*\eta^\circ$ on Σ° vanishes.*

To see why this implies (21), we let $\Gamma_{f'}^\circ \subset \Sigma^\circ \times X^\circ$ and $\Gamma_{g'}^\circ \subset \Sigma^\circ \times X^\circ$ denote the restrictions to $\Sigma^\circ \times X^\circ$ of the graphs of $f' : \Sigma' \rightarrow \overline{X}$ and $g' : \Sigma' \rightarrow \overline{X}$ respectively. By construction, we have the equality of n -cycles in $\Sigma^\circ \times X^\circ$

$$\Gamma_{x,y}^\circ = m\Gamma_{f'}^\circ + m'\Gamma_{g'}^\circ.$$

This implies that $(\Gamma_{x,y}^\circ)^*\eta^\circ = mf'^*\eta^\circ + m'g'^*\eta^\circ$. So proposition 4.9 yields the equality (21) by continuity. □

Proof of proposition 4.9. By construction, we have $\Gamma_{x,y} + \Gamma_z = \Gamma_C$ as n -cycles in $\Sigma' \times \overline{X}$, and therefore $\Gamma_{x,y}^\circ + \Gamma_z^\circ = \Gamma_C^\circ$ in $\Sigma^\circ \times X^\circ$. The vanishing of proposition 4.9 is a consequence of the two vanishings $(\Gamma_z^\circ)^*\eta^\circ = (\Gamma_C^\circ)^*\eta^\circ = 0$ of holomorphic n -forms on Σ° .

The first one is given by the fact that Γ_z is contained in $\Sigma' \times W \subset \Sigma' \times X$. Let $\nu : \widetilde{W} \rightarrow W^\circ \subset X^\circ$ be a desingularization of $W^\circ := W \cap X^\circ$, and consider the following 0-correspondence obtained by base change.

$$\begin{array}{ccc} & (\Gamma_z^\circ)' & \\ \tilde{\varphi} \swarrow & & \searrow \psi \\ \Sigma^\circ & & \widetilde{W} \xrightarrow{\nu} X^\circ \end{array}$$

One has $(\Gamma_z^\circ)^*\eta^\circ = (\Gamma_z^\circ)'^*\nu^*\eta^\circ = 0$, because $\dim W < n$ and therefore $\nu^*\eta^\circ = 0$.

As for the second one, it is a consequence of proposition 4.10 below. This proposition says that we have an equality of meromorphic forms on $\widetilde{\Sigma}'$

$$\Gamma_C^*\eta_{\overline{X}} = \mathcal{C}^*l_*\eta_{\overline{X}}. \quad (22)$$

Here, $l_* : H^0(\overline{X}, K_{\overline{X}}) \rightarrow H^1(Y, K_Y)$ is a push-forward map induced by the inclusion $l : \overline{X} \subset Y$, and $\mathcal{C}^* : H^1(Y, K_Y) \rightarrow H^0(\widetilde{\Sigma}', K_{\widetilde{\Sigma}'})$ is given by the correspondence between smooth varieties $\widetilde{\mathcal{C}} \subset \widetilde{\Sigma}' \times Y$, obtained by base change from the universal family $\mathcal{C} \subset \Sigma' \times Y$. Now, since Y is rationally connected, we have $H^1(Y, K_Y) = 0$. We deduce from this that $l_*\eta_{\overline{X}} = 0$, and therefore that $\Gamma_C^*\eta_{\overline{X}} = 0$. *A fortiori*, we have $(\Gamma_C^\circ)^*\eta^\circ = 0$ by restriction to Σ° . □

Let us now define precisely the various maps involved in formula (22). Let l be the inclusion $\overline{X} \subset Y$. We define a push-forward morphism l_* , as the boundary morphism $H^0(\overline{X}, K_{\overline{X}}) \rightarrow H^1(Y, K_Y)$ of the long exact sequence associated to the Poincaré residue exact sequence

$$0 \rightarrow K_Y \rightarrow K_Y(\overline{X}) \rightarrow K_{\overline{X}} \rightarrow 0.$$

On the other hand, we consider $\mathcal{C} \subset \Sigma' \times Y$, which is the universal curve relative to the parametrizing variety Σ' . We call Φ and Ψ the two projections on Σ' and Y respectively. By base change, \mathcal{C} gives a correspondence in $\tilde{\Sigma}' \times Y$. Desingularizing this new correspondence, we obtain the following diagram, where $\tilde{\mathcal{C}}$ is smooth, as well as $\tilde{\Sigma}'$ and Y .

$$\begin{array}{ccc} & \tilde{\mathcal{C}} & \\ \tilde{\Phi} \swarrow & & \searrow \tilde{\Psi} \\ \tilde{\Sigma}' & & Y \end{array}$$

This yields a Mumford pull-back $\mathcal{C}^* : \mathbb{H}^1(Y, K_Y) \rightarrow \mathbb{H}^0(\tilde{\Sigma}', K_{\tilde{\Sigma}'}),$ defined as the composition

$$\mathbb{H}^1(Y, K_Y) \xrightarrow{\tilde{\Psi}^*} \mathbb{H}^1(\tilde{\mathcal{C}}, K_{\tilde{\mathcal{C}}}) \xrightarrow{(\tilde{\Phi}^*)^T} \mathbb{H}^0(\tilde{\Sigma}', K_{\tilde{\Sigma}'}),$$

where the last morphism is obtained by Serre duality as the transpose of the pull-back map $\Phi^* : \mathbb{H}^n(\tilde{\Sigma}', \mathcal{O}_{\tilde{\Sigma}'}) \rightarrow \mathbb{H}^n(\tilde{\mathcal{C}}, \mathcal{O}_{\tilde{\mathcal{C}}})$.

Proposition 4.10 *We have $\Gamma_{\mathcal{C}}^* \eta_{\bar{X}} = \mathcal{C}^* l_{*} \eta_{\bar{X}}$, as an equality of meromorphic forms on $\tilde{\Sigma}'$.*

Again, this is an equality of meromorphic forms, so we can prove it locally in the dense open subset $\Sigma^\circ \subset \tilde{\Sigma}'$, where it is an equality of holomorphic differential forms.

Now the situation is the following. \mathcal{C} is a family of curves over Σ' . These curves are embedded in Y . $\Gamma_{\mathcal{C}}$ is cut out on $\mathcal{C} \subset \Sigma' \times Y$ by $\Sigma' \times \bar{X}$, and since $\bar{X} \subset Y$ is a divisor, $\Gamma_{\mathcal{C}}$ can be seen as a family of divisors on the curves of the family \mathcal{C} . So basically, proposition 4.10 is just a generalization of the following result on curves, which is an application of the residue theorem.

Lemma 4.11 *Let C be a smooth curve, and $j : Z \subset C$ be an effective divisor. We have a push-forward morphism $j_* : \mathbb{H}^0(Z, K_Z) \rightarrow \mathbb{H}^1(C, K_C)$, taken from the long exact sequence associated to the Poincaré residue exact sequence*

$$0 \rightarrow K_C \rightarrow K_C(Z) \rightarrow K_Z \rightarrow 0.$$

The following diagram is commutative.

$$\begin{array}{ccc} \mathbb{H}^0(Z, K_Z) & \xrightarrow{j_*} & \mathbb{H}^1(C, K_C) \\ & \searrow j_Z & \downarrow j_C \\ & & \mathbf{C} \end{array}$$

Proof. We first study the case when Z is a single point $x \in C$ with multiplicity μ . It will then easily extend to the general case, as will be seen from the proof. We begin by defining a C^∞ function

$$\rho : z \in \mathbf{D} \mapsto \exp\left(\frac{|z|^2}{|z|^2 - 1}\right) \in \mathbf{R}.$$

It extends by 0 on the complementary of the unit disk in \mathbf{C} to a C^∞ function $\mathbf{C} \rightarrow \mathbf{R}$. It also satisfies $\rho(0) = 1$.

We then consider a neighbourhood $U \subset C$ of $x \in C$, equipped with a holomorphic coordinate z centered at x , and such that $U \cong \mathbf{D}$ via z , and $Z \subset C$ is given by the equation $z^\mu = 0$. Let $\omega_Z \in \mathbb{H}^0(Z, K_Z)$. It writes

$$\frac{a_{-\mu+1} + a_{-\mu+2}z + \dots + a_0 z^{\mu-1}}{z^{\mu-1}} \in \mathbf{C}[z]/(z^\mu) \cdot \frac{1}{z^{\mu-1}}, \quad (23)$$

and we have $\int_Z \omega_Z = a_0$. The differential form

$$\frac{\rho(z^\mu)}{2\pi i} (a_{-\mu+1} + a_{-\mu+2}z + \cdots + a_0 z^{\mu-1}) \frac{dz}{z^\mu}$$

is sent to ω_Z by the residue map. Its $\bar{\partial}$ -differential

$$\frac{1}{2\pi i} \frac{-\mu |z^{\mu-1}|^2}{(|z^\mu|^2 - 1)^2} \exp\left(\frac{|z^\mu|^2}{|z^\mu|^2 - 1}\right) \left(\frac{a_{-\mu+1}}{z^{\mu-1}} + \cdots + a_0\right) d\bar{z} \wedge dz \quad (24)$$

thus represents $j_*\omega_Z$. Note that in (24), the pole $z^{\mu-1}$ is eliminated by the $|z^{\mu-1}|^2$ from the numerator. The differential form $j_*\omega_Z$ is therefore a C^∞ section of $\Omega_C^{1,1}$. It is supported on $U \subset C$. The integral of this form on \mathbf{D} is

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\mathbf{D}} \frac{-\mu(r^{\mu-1})^2}{(r^{2\mu} - 1)^2} \exp\left(\frac{r^{2\mu}}{r^{2\mu} - 1}\right) \left(\frac{a_{-\mu+1}e^{-i(\mu-1)\theta}}{r^{\mu-1}} + \cdots + a_0\right) (2irdr \wedge d\theta) \\ &= a_0 \int_{0 \leq r \leq 1} \frac{-2r^\mu}{(r^{2\mu} - 1)^2} \exp\left(\frac{r^{2\mu}}{r^{2\mu} - 1}\right) (\mu r^{\mu-1} dr) = a_0. \end{aligned}$$

This proves lemma 4.11. □

The next step of the proof of proposition 4.10 is to extend lemma 4.11 to the family of curves \mathcal{C} . In fact, we have seen that it is enough to consider the smaller family $\mathcal{C}^\circ \rightarrow \Sigma^\circ$.

$$\begin{array}{ccc} \Gamma_C^\circ & \xrightarrow{(\text{id}_{\Sigma^\circ}, l)} & \mathcal{C}^\circ \\ & \searrow \varphi & \downarrow \Phi \\ & & \Sigma^\circ \end{array}$$

We recall that φ and ψ are the natural morphisms from Γ_C to Σ' and X respectively, that Φ and Ψ are the natural morphisms from \mathcal{C} to Σ' and Y , and that l is the inclusion $\bar{X} \subset Y$. The following lemma is proved by using lemma 4.11, and the fact that the push-forward map Φ_* is given locally above Σ° by the integration along the fibers of Φ .

Lemma 4.12 *The following diagram is commutative.*

$$\begin{array}{ccc} H^0(\Gamma_C^\circ, K_{\Gamma_C^\circ}) & \xrightarrow{(\text{id}_{\Sigma^\circ}, l)_*} & H^1(\mathcal{C}^\circ, K_{\mathcal{C}^\circ}) \\ & \searrow \varphi_* & \downarrow \Phi_* \\ & & H^0(\Sigma^\circ, K_{\Sigma^\circ}) \end{array}$$

To be more precise, the vertical arrows of the preceding diagram are given by φ_*° and Φ_*° respectively, where $\varphi^\circ := \varphi|_{\Gamma_C^\circ}$ and $\Phi^\circ := \Phi|_{\mathcal{C}^\circ}$.

Proof. Again, we want to show an equality of holomorphic differential forms on Σ° . This can be done locally. We thus choose an open subset $V \subset \Sigma^\circ$ over which the map $\Gamma_C^\circ \rightarrow \Sigma^\circ$ is étale (or rather the map $(\Gamma_C^\circ)_{red} \rightarrow \Sigma^\circ$ is étale). Then $\varphi^{-1}(V)$ is a disjoint union of open subsets $V_j \subset \Gamma_C^\circ$, such that for every j one has $(V_j)_{red} \cong V$.

One can assume that V is equipped with a holomorphic system of coordinates $v = (v_1, \dots, v_n)$, and that there exists a neighbourhood U_j of every $V_j \subset \Phi^{-1}(V)$ equipped with a holomorphic system of coordinates (v, t_j) , such that Φ is simply given locally by $(v, t_j) \mapsto v$, and that

V_j is given by the local equation $t_j^{\mu_j} = 0$ (μ_j is equal to 1, m or m'). The coordinates t_j give local parameters of the curves C_σ ($\sigma \in V$) at their points of intersection with \overline{X} .

Let $\eta \in H^0(\Gamma_C^\circ, K_{\Gamma_C^\circ})$. Its restriction to $\varphi^{-1}(V)$ is a collection of meromorphic forms

$$\eta_j = \left(a_{j,-\mu_j+1}(v) + a_{j,-\mu_j+2}(v)t_j + \cdots + a_{j,0}(v)t_j^{\mu_j-1} \right) \frac{dv_1 \wedge \cdots \wedge dv_n}{t_j^{\mu_j-1}}$$

on the open subsets V_j , exactly as in (23). The map φ_* is locally defined as the trace map, so $\varphi_*\eta$ is given in V by

$$\varphi_*\eta = \left(\sum_j a_{j,0}(v) \right) dv_1 \wedge \cdots \wedge dv_n.$$

On the other hand, $(\text{id}, l)_*\eta$ is given in $\Phi^{-1}(V)$ by a $\bar{\partial}$ -closed, C^∞ differential form of type $(n+1, 1)$, which vanishes outside from the neighbourhood $\bigcup_j U_j$ of $\Gamma_C^\circ \subset \mathcal{C}^\circ$. In the neighbourhood U_j of $V_j \subset \Phi^{-1}(V)$, it is given by

$$\bar{\partial} \left(\frac{\rho(t_j^{\mu_j})}{2\pi i} \left(a_{j,-\mu_j+1}(v) + a_{j,-\mu_j+2}(v)t_j + \cdots + a_{j,0}(v)t_j^{\mu_j-1} \right) \frac{dt_j}{t_j^{\mu_j}} \wedge dv_1 \wedge \cdots \wedge dv_n \right),$$

exactly as in the proof of lemma 4.11. Eventually, Φ_* is given by the integration along the fibers of Φ . To compute this, we let ζ be a holomorphic n -vector field over V (i.e. a holomorphic section of $\bigwedge^n T_{\Sigma^\circ}$ over V), given in coordinates by

$$h(v) \frac{\partial}{\partial v_1} \wedge \cdots \wedge \frac{\partial}{\partial v_n}. \quad (25)$$

We lift it to a C^∞ n -vector field $\tilde{\zeta}$ on $\Phi^{-1}(V)$. Using a partition of unity, we can assume that in every U_j , $\tilde{\zeta}$ is simply given by the expression (25). At a point $\sigma \in V$ of coordinates v , the inner product $(\Phi_*(\text{id}, l)_*\eta)(\zeta)_\sigma$ is by definition

$$\int_{C_\sigma} ((\text{id}, l)_*\eta)(\tilde{\zeta}),$$

where C_σ is the fiber of Φ over σ . We let $Z_\sigma := (\Gamma_C^\circ \cap C_\sigma)$, and denote by j_σ the inclusion $Z_\sigma \subset C_\sigma$. The inner product $\eta(\tilde{\zeta})_{Z_\sigma} := \sum_{z \in \text{Supp}(Z_\sigma)} \eta(\tilde{\zeta})_z$ sits naturally in $H^0(Z_\sigma, K_{Z_\sigma})$. Now the $(1, 1)$ -form $((\text{id}, l)_*\eta)(\tilde{\zeta})$ restricted to C_σ is precisely $(j_\sigma)_*(\eta(\tilde{\zeta})_{Z_\sigma})$, so by lemma 4.11, we have

$$(\Phi_*(\text{id}, l)_*\eta)(\zeta)_\sigma = \int_{C_\sigma} (j_\sigma)_*(\eta(\tilde{\zeta})_{Z_\sigma}) = \int_{Z_\sigma} \eta(\tilde{\zeta})_{Z_\sigma} = (\varphi_*\eta)(\zeta)_\sigma.$$

This shows the equality of holomorphic differential forms $\Phi_*(\text{id}, l)_*\eta = \varphi_*\eta$ on V . \square

To conclude the proof of proposition 4.10, it only remains to show the following commutativity result.

Lemma 4.13 *The following diagram is commutative.*

$$\begin{array}{ccc} H^0(\overline{X}, K_{\overline{X}}) & \xrightarrow{l_*} & H^1(Y, K_Y) \\ \psi^* \downarrow & & \downarrow \Psi^* \\ H^0(\Gamma_C^\circ, K_{\Gamma_C^\circ}) & \xrightarrow{(\text{id}_{\Sigma^\circ}, l)_*} & H^1(\mathcal{C}^\circ, K_{\mathcal{C}^\circ}) \end{array}$$

In fact, the vertical arrows of this diagram are rather $(\psi^\circ)^*$ and $(\Psi^\circ)^*$, where $\psi^\circ : \Gamma_C^\circ \rightarrow \bar{X}$ and $\Psi^\circ : \mathcal{C}^\circ \rightarrow Y$ are the restrictions of ψ and Ψ to Γ_C° and \mathcal{C}° respectively.

Proof. It is clear from the definitions that $\psi^\circ = \Psi^\circ|_{\Gamma_C^\circ}$. On the other hand, we have $\Psi^*\bar{X} = \Gamma_C^\circ$ as an equality of divisors on \mathcal{C}° . This shows that we have a morphism of short exact sequences as follows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Psi^*K_Y & \longrightarrow & \Psi^*K_Y(\bar{X}) & \longrightarrow & \psi^*K_{\bar{X}} \longrightarrow 0 \\ & & \downarrow \Psi^* & & \downarrow \Psi^* & & \downarrow \psi^* \\ 0 & \longrightarrow & K_{\mathcal{C}^\circ} & \longrightarrow & K_{\mathcal{C}^\circ}(\Gamma_C^\circ) & \longrightarrow & K_{\Gamma_C^\circ} \longrightarrow 0 \end{array}$$

It yields a morphism between the associated long exact sequences in cohomology. In particular, the following diagram is commutative.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^0(\Gamma_C^\circ, \psi^*K_{\bar{X}}) & \longrightarrow & H^1(\mathcal{C}^\circ, \Psi^*K_Y) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^0(\Gamma_C^\circ, K_{\Gamma_C^\circ}) & \longrightarrow & H^1(\mathcal{C}^\circ, K_{\mathcal{C}^\circ}) & \longrightarrow & \cdots \end{array}$$

Up to the shrinking of $\Sigma^\circ \subset \Sigma$, we can assume that both ψ° and Ψ° are smooth, so we have $H^0(\Gamma_C^\circ, \psi^*K_{\bar{X}}) \cong H^0(\bar{X}, K_{\bar{X}})$ and $H^1(\mathcal{C}^\circ, \Psi^*K_Y) \cong H^1(Y, K_Y)$, and the lemma is proved. \square

Eventually, lemmas 4.12 and 4.13 give the commutativity of the left-hand side square and of the right-hand side triangle in the following diagram.

$$\begin{array}{ccccc} & & (\Gamma_C^\circ)^* & & \\ & \searrow & \text{---} & \searrow & \\ H^0(\bar{X}, K_{\bar{X}}) & \xrightarrow{\psi^*} & H^0(\Gamma_C^\circ, K_{\Gamma_C^\circ}) & \xrightarrow{\varphi_*} & H^0(\Sigma^\circ, K_{\Sigma^\circ}) \\ \downarrow l_* & & \downarrow (\text{id}_{\Sigma^\circ}, l)_* & \nearrow \Phi_* & \\ H^1(Y, K_Y) & \xrightarrow{\psi^*} & H^1(\mathcal{C}^\circ, K_{\mathcal{C}^\circ}) & & \\ & \searrow & \text{---} & \searrow & \\ & & (\mathcal{C}^\circ)^* & & \end{array}$$

On the other hand, it follows from the definitions of $(\Gamma_C^\circ)^*$ and $(\mathcal{C}^\circ)^*$ that the bended arrows commute with the rest of the diagram. We thus have the equality of holomorphic forms on Σ°

$$(\Gamma_C^\circ)^*\eta^\circ = (\mathcal{C}^\circ)^*l_*\eta_{\bar{X}},$$

which yields by continuity the desired equality of meromorphic differential forms on $\tilde{\Sigma}'$. This ends the proof of proposition 4.10. Theorem 4.7 is thus completely proved.

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