NUMERICAL CHARACTERISATION OF QUADRICS

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Abstract. Let X be a Fano manifold such that $-K_X \cdot C \geq \dim X$ for every rational curve $C \subset X$. We prove that X is a projective space or a quadric.

1. Introduction

Let X be a Fano manifold, i.e. a complex projective manifold with ample anticanonical divisor $-K_X$. If the Picard number of X is at least two, Mori theory shows the existence of at least two non-trivial morphisms $\varphi_i : X \to Y_i$ which encode some interesting information on the geometry of X. On the contrary, when the Picard number equals one Mori theory does not yield any information, and one is thus led to studying X in terms of the positivity of the anticanonical bundle. A well-known example of such a characterisation is the following theorem of Kobayashi–Ochiai.

1.1. Theorem [KO73]**.** *Let X be a projective manifold of dimension n. Suppose that $-K_X \sim dH$ with H an ample divisor on X.*

a) Then one has $d \leq n+1$ and equality holds if and only if $X \simeq \mathbb{P}^n$.
b) If $d = n$, then $X \simeq \mathbb{Q}^n$.

The divisibility of $-K_X$ in the Picard group is a rather restrictive condition, so it is natural to ask for similar characterisations under (a priori) weaker assumptions. Based on Kebekus' study of singular rational curves [Keb02b], Cho, Miyaoka and Shepherd-Barron proved a generalisation of the first part of Theorem 1.1:

1.2. Theorem [CMSB02, Keb02a]**.** *Let X be a Fano manifold of dimension n. Suppose that*

$$-K_X \cdot C \geq n+1 \quad \text{for all rational curves } C \subset X.$$

Then $X \simeq \mathbb{P}^n$.

The aim of this paper is to prove the following, which is a similar generalisation for the second part of Theorem 1.1:

1.3. Theorem. *Let X be a Fano manifold of dimension n. Suppose that*

$$-K_X \cdot C \geq n \quad \text{for all rational curves } C \subset X.$$

Then $X \simeq \mathbb{P}^n$ or $X \simeq \mathbb{Q}^n$.

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This statement already appeared in a paper of Miyaoka [Miy04, Thm.0.1], but the proof there is incomplete (cf. Remark 5.2 for instance). In this paper we borrow some ideas and tools from Miyaoka's, yet give a proof based on a completely different strategy. Note also that Hwang gave a proof under the additional assumption that the general VMRT (see below) is smooth [Hwa13, Thm.1.11], a property that does not hold for every Fano manifold [CD15, Thm.1.10].

In the proof of Theorem 1.3, we have to assume $n \geq 4$; for $n \leq 3$ the statement follows directly from classification results.

The assumption that X is Fano assures that $\rho(X) = 1$ because of the Ionescu–Wiśniewski inequality [Ion86, Thm.0.4], [Wiś91, Thm.1.1] (see §4.1). It is possible to remove this assumption: the Ionescu–Wiśniewski inequality together with [HN13, Thm.1.3] enable one to deal with the case $\rho(X) > 1$, and one gets the following.

1.4. Corollary. *Let X be a projective manifold of dimension n containing a rational curve. If*

$$-K_X \cdot C \geq n \quad \textit{for all rational curves } C \subset X,$$

then X is a projective space, a hyperquadric, or a projective bundle over a curve.

(Note that under the assumptions of Corollary 1.4, if $\rho(X) = 1$ then X is Fano.)

Outline of the proof. In the situation of Theorem 1.3 let $\mathcal{K}$ be a family of minimal rational curves on X. By Mori's bend-and-break lemma a minimal curve $[l] \in \mathcal{K}$ satisfies $-K_X \cdot l \leq n+1$ and if equality holds then $X \simeq \mathbb{P}^n$ by [CMSB02]. By our assumption we are thus left to deal with the case $-K_X \cdot l = n$. Then, for a general point $x \in X$ the normalisation $\mathcal{K}_x$ of the space parametrising curves in $\mathcal{K}$ passing through x has dimension $n-2$, and by [Keb02b, Thm.3.4] there exists a morphism

$$\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega_{X,x})$$

which maps a general curve $[l] \in \mathcal{K}_x$ to its tangent direction $T_{l,x}^{\perp}$ at the point x. By [HM04, Thm.1] this map is birational onto its image $\mathcal{V}_x$, the *variety of minimal rational tangents* (VMRT) at x. We denote by $\mathcal{V} \subset \mathbb{P}(\Omega_X)$ the total VMRT, i.e. the closure of the locus covered by the VMRTs $\mathcal{V}_x$ for $x \in X$ general. To prove Theorem 1.3, we compute the cohomology class of the total VMRT $\mathcal{V} \subset \mathbb{P}(\Omega_X)$ in terms of the tautological class ζ and $\pi^* K_X$, where $\pi : \mathbb{P}(\Omega_X) \to X$ is the projection map. This computation is based on the construction, on the manifold X, of a family $\mathcal{W}^{\circ}$ of smooth rational curves such that for every $[C] \in \mathcal{W}^{\circ}$ one has

$$T_X|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus n};$$

it lifts to a family of curves on $\mathbb{P}(\Omega_X)$ by associating to a curve $C \subset X$ the image $\tilde{C}$ of the morphism $C \to \mathbb{P}(\Omega_X)$ defined by the invertible quotient

$$\Omega_X|_C \to \Omega_C.$$

The main technical statement of this paper is:

1.5. Proposition. *Let $X \not\simeq \mathbb{P}^n$ be a Fano manifold of dimension $n \geq 4$, and suppose that*

$$-K_X \cdot C \geq n \quad \textit{for all rational curves } C \subset X.$$

Then, in the above notation, one has $\mathcal{V} \cdot \tilde{C} = 0$ for all $[C] \in \mathcal{W}^{\circ}$.

Once we have shown this statement a similar intersection computation involving a general minimal rational curve l yields that the VMRT $\mathcal{V}_x \subset \mathbb{P}(\Omega_{X,x})$ is a hypersurface of degree at most two. We then conclude with some earlier results of Araujo, Hwang, and Mok [Ara06, Hwa07, Mok08].

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2. Notation and conventions

We work over the field $\mathbb{C}$ of complex numbers. Throughout the paper, $\mathbb{Q}^n$ designates a smooth quadric hypersurface in $\mathbb{P}^{n+1}$ for any positive integer n. Topological notions refer to the Zariski topology.

We use the modern notation for projective spaces, as introduced by Grothendieck: if $\mathcal{E}$ is a locally free sheaf on a scheme X, we let $\mathbb{P}(\mathcal{E})$ be $\mathbf{Proj}\,(\mathrm{Sym}\,\mathcal{E})$. If L is a line in a vector space V, $L^\perp$ designates the corresponding point in $\mathbb{P}(V^\vee)$. The symbols $\equiv$ and $\sim_{\mathbb{Q}}$ refer to numerical and $\mathbb{Q}$-linear equivalence respectively.

A variety is an integral scheme of finite type over $\mathbb{C}$, a manifold is a smooth variety. A fibration is a proper surjective morphism with connected fibres $\varphi : X \to Y$ such that X and Y are normal and $\dim X > \dim Y > 0$.

We use the standard terminology and results on rational curves, as explained in [Kol96, Ch.II], [Deb01, Ch.2,3,4], and [Hwa01]. Let X be a projective variety. We remind the reader that following [Kol96, II, Def.2.11], the notation $\mathrm{RatCurves}^{\mathrm{n}}\,X$ refers to the union of the normalisations of those locally closed subsets of the Chow variety of X parametrising irreducible rational curves (the superscript $^{\mathrm{n}}$ is a reminder that we normalised, and has nothing to do with the dimension).

For technical reasons, we have to consider families of rational curves on X as living alternately in $\mathrm{RatCurves}^{\mathrm{n}}\,X$ and in $\mathrm{Hom}(\mathbb{P}^1, X)$. Our general policy is to call $\mathrm{Hom}_{\mathcal{R}} \subset \mathrm{Hom}(\mathbb{P}^1, X)$ the family corresponding to a normal variety $\mathcal{R} \subset \mathrm{RatCurves}^{\mathrm{n}}\,X$.

3. Preliminaries on conic bundles

In this section, we establish some basic facts about conic bundles over a curve and compute some intersection numbers which will turn out to be crucial for the proof of Proposition 1.5. All these statements appear in one form or another in [Miy04, §2], but we recall them and their proofs for the clarity of exposition.

3.1. Definition. *A* conic bundle *is an equidimensional projective fibration $\varphi : X \to Y$ such that there exists a rank three vector bundle $V \to Y$ and an embedding $X \hookrightarrow \mathbb{P}(V)$ that maps every φ-fibre $\varphi^{-1}(y)$ onto a conic (i.e. the zero scheme of a degree 2 form) in $\mathbb{P}(V_y)$. The set*

$$\Delta := \{y \in Y \mid \varphi^{-1}(y) \text{ is not smooth}\}$$

is called the discriminant locus *of the conic bundle.*

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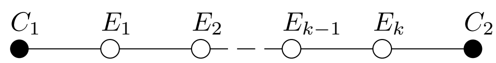
3.2. Lemma. *Let S be a smooth surface admitting a projective fibration $\varphi : S \to T$ onto a smooth curve such that the general fibre is $\mathbb{P}^1$, and such that $-K_S$ is φ-nef. Let F be a reducible φ-fibre and suppose that*

$$F = C_1 + C_2 + F',$$

where the C_i are (-1)-curves and $C_i \not\subset Supp(F')$. Then $F' = \sum E_j$ is a reduced chain of (-2)-curves and the dual graph of F is as depicted in Figure 1.

Figure 1



Proof. Write $F' = \sum_{j=1}^k a_j E_j$, $a_j \in \mathbb{N}$, where $E_1, \ldots, E_k$ are the irreducible components of F'. First note that since $-K_S \cdot F = 2$ and $-K_S \cdot C_i = 1$, the fact that $-K_S$ is φ-nef implies $-K_S \cdot E_j = 0$ for all j. Since E_j is an irreducible component of a reducible fibre, we have $E_j^2 < 0$. Thus we see that each E_j is a (-2)-curve.

We will now proceed by induction on the number of irreducible components of F', the case $F' = 0$ being trivial. Let $\mu : S \to S'$ be the blow-down of the (-1)-curve C_2; then by the rigidity lemma [Deb01, Lemma 1.15], there is a morphism $\varphi' : S' \to T$ such that $\varphi = \varphi' \circ \mu$. Note that S' is smooth and $-K_{S'}$ is φ'-nef. We also have

$$0 = C_2 \cdot F = -1 + C_2 \cdot (C_1 + \sum_{i=1}^k a_i E_i),$$

so C_2 meets $C_1 + \sum_{i=1}^k a_i E_i$ transversally in exactly one point. If $C_2 \cdot C_1 > 0$, then $\mu_*(C_1)$ has self-intersection 0, yet it is also an irreducible component of the reducible fibre $\mu_*(C_1 + \sum_{i=1}^k a_i E_i)$, a contradiction. Thus (up to renumbering) we can suppose that $C_2 \cdot E_1 = 1$ and $a_1 = 1$. In particular $\mu_*(E_1)$ is a (-1)-curve, so

$$\mu_*(C_1 + \sum_{i=1}^k a_i E_i) = \mu_*(C_1) + \mu_*(E_1) + \mu_*(\sum_{i=2}^k a_i E_i)$$

satisfies the induction hypothesis. $\square$

In the following we use that for every normal surface one can define an intersection theory using the Mumford pull-back to the minimal resolution, cf. [Sak84].

3.3. Lemma. *Let S be a normal surface admitting a projective fibration $\varphi : S \to T$ onto a smooth curve such that the general fibre is $\mathbb{P}^1$ and such that every fibre is reduced and has at most two irreducible components. Then*

a) φ is a conic bundle;
b) S has at most A_k-singularities; and
c) if $s \in S_{\text{sing}}$, then $s = F_{\varphi(s),1} \cap F_{\varphi(s),2}$ where $F_{\varphi(s)} = F_{\varphi(s),1} + F_{\varphi(s),2}$ is the decomposition of the fibre over $\varphi(s)$ in its irreducible components. In particular $F_{\varphi(s)}$ is a reducible conic.

Proof. If a fibre $\varphi^{-1}(t)$ is irreducible, then φ is a $\mathbb{P}^1$-bundle over a neighbourhood of t [Kol96, II, Thm.2.8]. Thus we only have to consider points $t \in T$ such that $S_t := \varphi^{-1}(t)$ is reducible. Since $p_a(S_t) = 0$ and $S_t = C_1 + C_2$ is reduced, we see that S_t is a union of two $\mathbb{P}^1$'s meeting transversally in a point. Since $S_t = \varphi^* t$ is a Cartier divisor, this already implies c).

Let $\varepsilon : \hat{S} \to S$ be the canonical modification [Kol13, Thm.1.31] of the singular points lying on S_t. Then we have

$$K_{\hat{S}} \equiv \varepsilon^* K_S - E,$$

with E an effective ε-exceptional $\mathbb{Q}$-divisor whose support is equal to the ε-exceptional locus. Denote by $\hat{C}_i$ the proper transform of C_i. If $K_{\hat{S}} \cdot \hat{C}_i < -1$, then $\hat{C}_i$ deforms in $\hat{S}$ [Kol96, II, Thm.1.15]. Yet $\hat{C}_i$ is an irreducible component of a reducible $\varphi \circ \varepsilon$-fibre, so this is impossible. So we have

$$K_S \cdot C_i \geq K_{\hat{S}} \cdot \hat{C}_i \geq -1$$

for $i = 1, 2$. Since $K_S \cdot (C_1 + C_2) = -2$, this implies that $K_S \cdot C_i = -1$ and $E = 0$. Thus S has canonical singularities. Since canonical surface singularities are Gorenstein we see that $-K_S$ is Cartier, φ-ample and defines an embedding

$$S \subset \mathbb{P}\big(V := \varphi_*(\mathcal{O}_S(-K_S))\big)$$

into a $\mathbb{P}^2$-bundle mapping each fibre onto a conic. This proves a).

Let now $\tilde{\varepsilon} : \tilde{S} \to S$ be the minimal resolution. It is crepant, so the divisor $-K_{\tilde{S}}$ is $\varphi \circ \tilde{\varepsilon}$-nef. Moreover the proper transforms $\tilde{C}_i$ of the curves C_i are (-1)-curves in $\tilde{S}$. By Lemma 3.2 this proves b). $\square$

The following fundamental lemma should be seen as an analogue of the basic fact that a projective bundle over a curve contains at most one curve with negative self-intersection.

3.4. Lemma [Miy04, Prop.2.4]**.** *Let S be a normal projective surface that is a conic bundle $\varphi : S \to T$ over a smooth curve T, and denote by Δ the discriminant locus. Suppose that φ has two disjoint sections σ_1 and σ_2, both contained in the smooth locus of S. Suppose moreover that for every $t \in \Delta$, the fibre F_t has a decomposition $F_t = F_{t,1} + F_{t,2}$ such that*

(C1) $$\sigma_i \cdot F_{t,j} = \delta_{i,j}$$

(Kronecker's delta). Assume also that we have

(C2) $$\sigma_1^2 < 0 \text{ and } \sigma_2^2 < 0.$$

Let $\varepsilon : \hat{S} \to S$ be the minimal resolution. Let σ be a φ-section, and $\hat{\sigma} \subset \hat{S}$ its proper transform. Then the following holds:

a) If $(\hat{\sigma})^2 < 0$, then $\sigma = \sigma_1$ or $\sigma = \sigma_2$.
b) If $(\hat{\sigma})^2 = 0$ then σ is disjoint from $\sigma_1 \cup \sigma_2$.

3.5. Remarks. 1. In the situation above all the fibres are reduced, since there exists a section that is contained in the smooth locus.

2. The two inequalities (C2) are satisfied if there exists a birational morphism $S \to S'$ onto a projective surface S' that contracts σ_1 and σ_2. More generally, the

Hodge index theorem implies that (C2) holds if there exists a nef and big divisor H on S such that $H \cdot \sigma_1 = H \cdot \sigma_2 = 0$.

Proof. Preparation: contraction to a smooth ruled surface. Lemma 3.3 applies to the surface S. It follows that S has an A_{k_t}-singularity ($k_t \geq 0$) in $F_{t,1} \cap F_{t,2}$ for every $t \in \Delta$, and no further singularity. In particular, the dual graph of $(\varphi \circ \varepsilon)^{-1}(t)$ is as described in Figure 1 for every $t \in \Delta$.

We consider the birational morphism

$$\hat{\mu} : \hat{S} \to S^{\flat}$$

defined as the composition, for every $t \in \Delta$, of the blow-down of the proper transform $\hat{F}_{t,1}$ of $F_{t,1}$ and of all the k_t (-2)-curves contained in $(\varphi \circ \varepsilon)^{-1}(t)$. Since $\hat{\mu}$ is a composition of blow-down of (-1)-curves, the surface $S^{\flat}$ is smooth. By the rigidity lemma [Deb01, Lemma 1.15], there is a morphism $\varphi^{\flat} : S^{\flat} \to T$. All its fibres are irreducible rational curves, so it is a $\mathbb{P}^1$-bundle by [Kol96, II, Thm.2.8]. Again by the rigidity lemma, $\hat{\mu}$ factors through ε, i.e. there is a birational morphism $\mu : S \to S^{\flat}$ such that $\hat{\mu} = \mu \circ \varepsilon$; it is the contraction of all the curves $F_{t,1}$, $t \in \Delta$.

Since σ_1 meets $F_{t,1}$ in a smooth point of S, the proper transforms $\hat{\sigma}_1$ and $\hat{F}_{t,1}$ meet in the same point. Thus (the successive images of) $\hat{\sigma}_1$ meets the exceptional divisor of all the blow-downs of (-1)-curves composing $\hat{\mu}$, and since the section $\sigma_1^{\flat} := \hat{\mu}(\hat{\sigma}_1)$ is smooth, all the intersections are transversal. Vice versa we can say that $\hat{S}$ is obtained from $S^{\flat}$ by blowing up points on (the successive proper transforms of) $\sigma_1^{\flat}$.

By the symmetry condition (C1) the curve σ_2 is disjoint from the μ-exceptional locus, so if we set $\sigma_2^{\flat} := \mu(\sigma_2)$, then we have $(\sigma_2^{\flat})^2 = (\sigma_2)^2 < 0$. Thus, in the notation of [Har77, V,Ch.2], $\varphi^{\flat} : S^{\flat} \to T$ is a ruled surface with invariant $-e := (\sigma_2^{\flat})^2 > 0$. In particular the Mori cone $\overline{\mathrm{NE}}(S^{\flat})$ is generated by a general $\varphi^{\flat}$-fibre F and $\sigma_2^{\flat}$. Since $\sigma_1^{\flat} \cdot \sigma_2^{\flat} = 0$ and $\sigma_1^{\flat} \cdot F = 1$, we have

$$\sigma_1^{\flat} \equiv \sigma_2^{\flat} + eF. \tag{3.5.1}$$

Conclusion. Let now $\sigma \subset S$ be a section that is distinct from both σ_1 and σ_2. Then $\sigma^{\flat} := \mu(\sigma)$ is distinct from both $\sigma_1^{\flat}$ and $\sigma_2^{\flat}$. Since $\sigma^{\flat} \neq \sigma_2^{\flat}$ we have

$$\sigma^{\flat} \equiv \sigma_2^{\flat} + cF \tag{3.5.2}$$

for some $c \geq e$ [Har77, V, Prop.2.20]. Since $\sigma^{\flat} \neq \sigma_1^{\flat}$ we have

$$\sigma^{\flat} \cdot \sigma_1^{\flat} \geq \sum_{t \in \Delta} \tau_t, \tag{3.5.3}$$

where τ_t is the intersection multiplicity of $\sigma^{\flat}$ and $\sigma_1^{\flat}$ at the point $F_t \cap \sigma_1^{\flat}$. Denote by $\hat{\sigma} \subset \hat{S}$ the proper transform of $\sigma \subset S$, which is also the proper transform of $\sigma^{\flat} \subset S^{\flat}$. By our description of $\hat{\mu}$ as a sequence of blow-ups in $\sigma_1^{\flat}$ we obtain

$$(\hat{\sigma})^2 = (\sigma^{\flat})^2 - \sum_{t \in \Delta} \min(\tau_t, k_t + 1) \geq (\sigma^{\flat})^2 - \sum_{t \in \Delta} \tau_t.$$

By (3.5.3) this implies

$$(\hat{\sigma})^2 \geq (\sigma^{\flat})^2 - \sigma^{\flat} \cdot \sigma_1^{\flat} = \sigma^{\flat} \cdot (\sigma^{\flat} - \sigma_1^{\flat}).$$

Plugging in (3.5.1) and (3.5.2) we obtain

$$(\hat{\sigma})^2 \geq c - e \geq 0. \tag{3.5.4}$$

This shows statement a).

Suppose now that $(\hat{\sigma})^2 = 0$. Then by (3.5.4) we have $c = e$, hence $\sigma^\flat \cdot \sigma_2^\flat = 0$. Being distinct, the two curves $\sigma^\flat$ and $\sigma_2^\flat$ are therefore disjoint, and so are their proper transforms $\hat{\sigma}$ and $\hat{\sigma}_2$. Note now that ε is an isomorphism in a neighbourhood of $\hat{\sigma}_2$, so $\sigma = \varepsilon(\hat{\sigma})$ is disjoint from $\sigma_2 = \varepsilon(\hat{\sigma}_2)$. In order to see that σ and σ_1 are disjoint, we repeat the same argument but contract those fibre components which meet σ_2. This proves statement b). □

4. The main construction

4.1. Set-up. For the whole section, we let $X \not\simeq \mathbb{P}^n$ be a Fano manifold of dimension $n \geq 4$, and suppose that

$$-K_X \cdot C \geq n \quad \text{for all rational curves } C \subset X; \tag{4.1.1}$$

this is the situation of Proposition 1.5. It then follows from the Ionescu–Wiśniewski inequality that the Picard number $\rho(X)$ equals 1, see [Miy04, Lemma 4.1].

Recall that a family of *minimal rational curves* is an irreducible component $\mathcal{K}$ of $\mathrm{RatCurves}^{\mathrm{n}}(X)$ such that the curves in $\mathcal{K}$ dominate X, and for $x \in X$ general the algebraic set $\mathcal{K}_x^\flat \subset \mathcal{K}$ parametrising curves passing through x is proper. We will use the following simple observation:

4.2. Lemma. *In the situation of Proposition 1.5, let $l \subset X$ be a rational curve such that $-K_X \cdot l = n$. Then any irreducible component $\mathcal{K}$ of* $\mathrm{RatCurves}^{\mathrm{n}} X$ *containing $[l]$ is a family of minimal rational curves.*

Proof. Condition (4.1.1) implies the properness of $\mathcal{K}$ [Kol96, II, (2.14)]. On the other hand, we know by [Kol96, IV, Cor.2.6.2] that the curves parametrised by $\mathcal{K}$ dominate X. □

4.3. Minimal rational curves and VMRTs. Since X is Fano, it contains a rational curve l [Mor79, Thm.6]. Since $X \not\simeq \mathbb{P}^n$, there exists a rational curve with $-K_X \cdot l = n$ by [CMSB02], and by Lemma 4.2 there exists a family of minimal rational curves containing the point $[l] \in \mathrm{RatCurves}^{\mathrm{n}}(X)$. We fix once and for all such a family, which we call $\mathcal{K}$.

For $x \in X$ general, denote by $\mathcal{K}_x$ the normalisation of the algebraic set $\mathcal{K}_x^\flat \subset \mathcal{K}$ parametrising curves passing through x. Every member of $\mathcal{K}_x^\flat$ is a free curve (this follows from the argument of [Kol96, II, proof of Thm.3.11]), so $\mathcal{K}_x$ is smooth and has dimension $n - 2 \geq 2$ [Kol96, II, (1.7) and (2.16)].

By results of Kebekus, a general curve $[l] \in \mathcal{K}_x^\flat$ is smooth [Keb02b, Thm.3.3], and the *tangent map*

$$\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega_{X,x})$$

which to a general curve $[l]$ associates its tangent direction $T_{l,x}^\perp$ at the point x is a finite morphism [Keb02b, Thm.3.4]. Its image $\mathcal{V}_x$ is called the *variety of minimal rational tangents* (VMRT) at x. The map τ_x is birational by [HM04, Thm.1], so the normalisation of $\mathcal{V}_x$ is $\mathcal{K}_x$, which is smooth (this is [HM04, Cor.1]). Also, one can associate to a general point $v \in \mathcal{V}_x$ a unique minimal curve $[l] \in \mathcal{K}_x$. We denote by $\mathcal{V} \subset \mathbb{P}(\Omega_X)$ the *total VMRT*, i.e. the closure of the locus covered by the VMRTs $\mathcal{V}_x$ for $x \in X$ general. Since $\mathcal{K}_x$ has dimension $n-2$, the total VMRT $\mathcal{V}$ is a divisor in $\mathbb{P}(\Omega_X)$.

For a general $[l] \in \mathcal{K}$, one has

$$T_X|_l \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^1} \tag{4.3.1}$$

[Kol96, IV, Cor.2.9]. We call a minimal rational curve $[l] \in \mathcal{K}$ *standard* if l is smooth and the bundle $T_X|_l$ has the same splitting type as in (4.3.1).

4.4. Smoothing pairs of minimal curves. For a general point $x_1 \in X$ the curves parametrised by $\mathcal{K}_{x_1}$ cover a divisor $D_{x_1} \subset X$ [Kol96, IV, Prop.2.5]. This divisor is ample because $\rho(X) = 1$, so for $x_2 \in X$ and $[l_2] \in \mathcal{K}_{x_2}$ the curve l_2 intersects D_{x_1}. Thus for a general point $x_2 \in X$ we can find a chain of two standard minimal curves $l_1 \cup l_2$ connecting the points x_1 and x_2. By [Kol96, II, Ex.7.6.4.1] the union $l_1 \cup l_2$ is dominated by a transverse union $\mathbb{P}^1 \cup \mathbb{P}^1$. Since both rational curves are free we can smooth the tree $\mathbb{P}^1 \cup \mathbb{P}^1$ keeping the point x_1 fixed [Kol96, II, Thm.7.6.1]. Since x_1 is general in X this defines a family of rational curves dominating X, and we denote by $\mathcal{W}$ the normalisation of the irreducible component of $\mathrm{Chow}(X)$ containing these rational curves.

4.5. Since a general member $[C]$ of the family $\mathcal{W}$ is free and $-K_X \cdot C = 2n$, we have $\dim \mathcal{W} = 3n-3$. We pick an arbitrary irreducible component of the subset of $\mathcal{W}$ parametrising cycles containing x_1, and let $\mathcal{W}_{x_1}$ be its normalisation; then we have $\dim \mathcal{W}_{x_1} = 2n-2$. Let $\mathcal{U}_{x_1}$ be the normalisation of the universal family of cycles over $\mathcal{W}_{x_1}$. The evaluation map $\mathrm{ev}_{x_1} : \mathcal{U}_{x_1} \to X$ is surjective: its image is irreducible, and it contains both the divisor D_{x_1} (because it is contained in the image of the restriction of ev_{x_1} to those members of $\mathcal{W}_{x_1}$ that contain a minimal curve through x_1) and the point x_2 which is *general* in X (in particular $x_2 \notin D_{x_1}$).

Next, we choose an arbitrary irreducible component of the subset of $\mathcal{W}$ parametrising cycles passing through x_1 and x_2, and let $\mathcal{W}_{x_1,x_2}$ be its normalisation, $\mathcal{U}_{x_1,x_2}$ the normalisation of the universal family over $\mathcal{W}_{x_1,x_2}$. We denote by

$$q : \mathcal{U}_{x_1,x_2} \to \mathcal{W}_{x_1,x_2}, \qquad \mathrm{ev} : \mathcal{U}_{x_1,x_2} \to X$$

the natural maps. It follows from the considerations above that $\mathcal{W}_{x_1,x_2}$ is non-empty of dimension $n-1$.

By construction, a general curve $[C] \in \mathcal{W}_{x_1,x_2}$ is smooth at x_i, $i \in \{1,2\}$, so the preimage $\mathrm{ev}^{-1}(x_i)$ contains a unique divisor σ_i that surjects onto $\mathcal{W}_{x_1,x_2}$. Since ev is finite on the q-fibres and $\mathcal{W}_{x_1,x_2}$ is normal, we obtain that the degree one map $\sigma_i \to \mathcal{W}_{x_1,x_2}$ is an isomorphism. We call the divisors σ_i the distinguished sections of q. We denote by $\Delta \subset \mathcal{W}_{x_1,x_2}$ the locus parametrising non-integral cycles.

Let $\mathrm{loc}^1_{x_1}$ be the locus covered by *all* the minimal rational curves of X passing through x_1. It is itself a divisor, but may be bigger than D_{x_1} since in general there are finitely many families of minimal curves. From now on we choose a general point $x_2 \in X$ such that $x_2 \notin \mathrm{loc}^1_{x_1}$ (which implies $x_1 \notin \mathrm{loc}^1_{x_2}$).

4.6. Lemma. *In the situation of Proposition 1.5 and using the notation introduced above, let C be a non-integral cycle corresponding to a point $[C] \in \Delta$. Then $C = l_1 + l_2$, with the l_i minimal rational curves such that $x_i \in l_j$ if and only if $i = j$.*

Remark. Note that we do not claim that the curves l_i belong to the family $\mathcal{K}$. However by construction of the family $\mathcal{W}$ as smoothings of pairs $l_1 \cup l_2$ in $\mathcal{K}$ there exists an irreducible component $\Delta_{\mathcal{K}} \subset \Delta$ such that $l_i \in \mathcal{K}$ when $[l_1 + l_2] \in \Delta_{\mathcal{K}}$.

Proof. We can write $C = \sum a_i l_i$ where the a_i are positive integers and l_i integral curves. By [Kol96, II, Prop.2.2] all the irreducible components l_i are rational curves. We can suppose that up to renumbering one has $x_1 \in l_1$. If $a_1 \geq 2$, then $-K_X \cdot C = 2n$ and $-K_X \cdot l_1 \geq n$ implies that $C = 2l_1$ and l_1 is a minimal rational curve. Yet this contradicts the assumption $x_2 \notin \operatorname{loc}^1_{x_1}$. Thus we have $a_1 = 1$ and since C is not integral there exists a second irreducible component l_2. Again $-K_X \cdot C = 2n$ and $-K_X \cdot l_i \geq n$ implies $C = l_1 + l_2$ and the l_i are minimal rational curves by Lemma 4.2. The last property now follows by observing that $x_2 \notin \operatorname{loc}^1_{x_1}$ implies that $x_1 \notin \operatorname{loc}^1_{x_2}$. □

By [Kol96, II, Thm.2.8], the fibration $q : \mathcal{U}_{x_1,x_2} \to \mathcal{W}_{x_1,x_2}$ is a $\mathbb{P}^1$-bundle over the open set $\mathcal{W}_{x_1,x_2} \setminus \Delta$. Although Lemma 4.6 essentially says that the singular fibres are reducible conics, it is a priori not clear that q is a conic bundle (cf. Definition 3.1). This becomes true after we make a base change to a smooth curve.

4.7. Lemma. *In the situation of Proposition 1.5 and using the notation introduced above, let $Z \subset \mathcal{W}_{x_1,x_2}$ be a curve such that a general point of Z parametrises an irreducible curve. Then there exists a finite morphism $T \to Z$ such that the normalisation S of the fibre product $\mathcal{U}_{x_1,x_2} \times_{\mathcal{W}_{x_1,x_2}} T$ has a conic bundle structure $\varphi : S \to T$ that satisfies the conditions of Lemma 3.4.*

Proof. Let $\nu : \tilde{Z} \to Z$ be the normalisation, and let N be the normalisation of $\mathcal{U}_{x_1,x_2} \times_{\mathcal{W}_{x_1,x_2}} \tilde{Z}$, $f_N : N \to X$ the morphism induced by $\operatorname{ev} : \mathcal{U}_{x_1,x_2} \to X$. Since all the curves pass through x_1 and x_2 there exists a curve $Z_1 \subset N$ (resp. $Z_2 \subset N$) that is contracted by f_N onto the point x_1 (resp. x_2). Since ev is finite on the q-fibres, the curves Z_1 and Z_2 are multisections of $N \to \tilde{Z}$. If $\tilde{Z}_i$ is the normalisation of Z_i, then the fibration $(N \times_{\tilde{Z}} \tilde{Z}_i) \to \tilde{Z}_i$ has a section given by $c \mapsto (c, c)$. Thus there exists a finite base change $T \to \tilde{Z}$ such that the normalisation $\varphi : S \to T$ of the fibre product $(\mathcal{U}_{x_1,x_2} \times_{\mathcal{W}_{x_1,x_2}} T) \to T$ has a natural morphism $f : S \to X$ induced by $\operatorname{ev} : \mathcal{U}_{x_1,x_2} \to X$ and contracts two φ-sections σ_1 and σ_2 on x_1 and x_2 respectively.

Since $Z \not\subset \Delta$, the general φ-fibre is $\mathbb{P}^1$. Moreover by Lemma 4.6 all the φ-fibres are reduced and have at most two irreducible components. By Lemma 3.3 this implies that φ is a conic bundle and if $s \in S_{\text{sing}}$, then $F_{\varphi(s)}$ is a reducible conic and the two irreducible components meet in s. Thus we have $\sigma_i \subset S_{sm}$, where S_{sm} denotes the smooth locus, since otherwise both irreducible components would pass through x_i, thereby contradicting the property that $x_2 \notin \operatorname{loc}^1_{x_1}$. For the same reason we can decompose any reducible φ-fibre F_t by defining $F_{t,i}$ as the unique component meeting the section σ_i. Since $\sigma_i \cdot F = 1$ for a general φ-fibre we see that (C1) holds. Condition (C2) holds with H the pull-back of an ample divisor on X. □

From this one deduces with Lemma 3.4 the following statement, in the spirit of the bend-and-break lemma [Deb01, Prop.3.2].

4.8. Lemma. *The restriction of the evaluation map* ev : $\mathcal{U}_{x_1,x_2} \to X$ *to the complement of* $\sigma_1 \cup \sigma_2$ *is quasi-finite. In particular* ev *is generically finite onto its image.*

Proof. We argue by contradiction. Since ev is finite on the q-fibres there exists a curve $Z \subset \mathcal{W}_{x_1,x_2}$ such that the natural map from the surface $q^{-1}(Z)$ onto $\mathrm{ev}(q^{-1}(Z))$ contracts three disjoint curves σ_1, σ_2 and σ onto the points x_1, x_2 and $x := \mathrm{ev}(\sigma)$.

If $Z \not\subset \Delta$, then by Lemma 4.7 we can suppose, possibly up to a finite base change, that $q^{-1}(Z) \to Z$ satisfies the conditions (C1) of Lemma 3.4. After a further base change we can assume that σ is a section. Since σ is contracted by ev we have $\sigma^2 < 0$. By Lemma 3.4,a), this implies $\sigma = \sigma_1$ or $\sigma = \sigma_2$, a contradiction.

If $Z \subset \Delta$, then all the fibres over Z are unions of two minimal rational curves. Thus the normalisation of $q^{-1}(Z)$ is a union of two $\mathbb{P}^1$-bundles mapping onto Z and by construction they contain three curves which are mapped onto points. However a ruled surface contains at most one contractible curve, a contradiction. ☐

4.9. Since $\dim \mathcal{U}_{x_1,x_2} = \dim X$, one deduces from Lemma 4.8 above that the cycles $[C] \in \mathcal{W}$ passing through x_1, x_2 cover the manifold X. By [Deb01, 4.10] this implies that a general member $[C] \in \mathcal{W}_{x_1,x_2}$ is a 2-free rational curve [Deb01, Defn.4.5]. Since $-K_X \cdot C = 2n$, this forces

$$f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus n}, \tag{4.9.1}$$

where $f : \mathbb{P}^1 \to C \subset X$ is the normalisation of C. As a consequence, one sees from [Kol96, II, Thm.3.14.3] that a general member $[C] \in \mathcal{W}$ is a *smooth* rational curve in X.

Let $\mathrm{Hom}^\circ_{\mathcal{W}} \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the irreducible open set parametrising morphisms $f : \mathbb{P}^1 \to X$ such that the image $C := f(\mathbb{P}^1)$ is smooth, the associated cycle $[C] \in \mathrm{Chow}(X)$ is a point in $\mathcal{W}$, and f^*T_X has the splitting type (4.9.1). By what precedes, the image of $\mathrm{Hom}^\circ_{\mathcal{W}}$ in $\mathcal{W}$ under the natural map $\mathrm{Hom}(\mathbb{P}^1, X) \to \mathrm{Chow}(X)$ is a dense open set $\mathcal{W}^\circ \subset \mathcal{W}$.

4.10. Denote by $\pi : \mathbb{P}(\Omega_X) \to X$ the projection map. We define an injective map

$$i : \mathrm{Hom}^\circ_{\mathcal{W}} \hookrightarrow \mathrm{Hom}(\mathbb{P}^1, \mathbb{P}(\Omega_X))$$

by mapping $f : \mathbb{P}^1 \to X$ to the morphism $\tilde{f} : \mathbb{P}^1 \to \mathbb{P}(\Omega_X)$ corresponding to the invertible quotient $f^*\Omega_X \to \Omega_{\mathbb{P}^1}$. Correspondingly, for $[C] \in \mathcal{W}^\circ$ with normalisation f, we call $[\tilde{C}]$ the member of $\mathrm{Chow}(\mathbb{P}(\Omega_X))$ corresponding to the lifting $\tilde{f}$.

We let $\mathrm{Hom}^{\sim}_{\mathcal{W}}$ be the image of i. Note that it parametrises a family of rational curves that dominates $\mathbb{P}(\Omega_X)$, but it is not an irreducible component of $\mathrm{Hom}(\mathbb{P}^1, \mathbb{P}(\Omega_X))$. Indeed, $\mathrm{Hom}^{\sim}_{\mathcal{W}}$ is contained in a (much bigger) irreducible component defined by morphisms corresponding to arbitrary quotients $f^*\Omega_X \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-2)$.

The following property is well-known to experts. Since $\mathrm{Hom}^{\sim}_{\mathcal{W}}$ is not an open set of the space $\mathrm{Hom}(\mathbb{P}^1, \mathbb{P}(\Omega_X))$, we have to adapt the proof of [Kol96, II,Prop.3.7].

4.11. Lemma. *In the situation of Proposition 1.5, let* $\mathcal{V}_0 \subset \mathcal{V}$ *be a dense, Zariski open set in the total VMRT* $\mathcal{V}$*, and let* $\tilde{C} := \tilde{f}(\mathbb{P}^1)$ *be a rational curve parametrised by a general point of* $\mathrm{Hom}^{\sim}_{\mathcal{W}}$*. Then one has*

$$(\mathcal{V} \cap \tilde{C}) \subset (\mathcal{V}_0 \cap \tilde{C}).$$

Proof. Set $Z := \mathcal{V} \setminus \mathcal{V}_0$. A point $z \in \mathbb{P}(\Omega_X)$ is $z = (v_z^\perp, x)$, where $\mathbb{C}v_z \subset T_{X,x}$ is a tangent direction in X at $x = \pi(z)$. So for all $p \in \mathbb{P}^1$, $z = (v_z^\perp, x) \in \mathbb{P}(\Omega_X)$, the morphisms $[\tilde{f}] \in \mathrm{Hom}^{\sim}_{\mathcal{W}}$ mapping p to z correspond to morphisms $f : \mathbb{P}^1 \to X$ in $\mathrm{Hom}^{\circ}_{\mathcal{W}}$ mapping p to x with tangent direction $\mathbb{C}v_z$. Since f has the splitting type (4.9.1), the set of these morphisms has dimension exactly n. It follows that

$$\mathrm{Hom}^{\sim}_{\mathcal{W},Z} := \left\{[\tilde{f}] \in \mathrm{Hom}^{\sim}_{\mathcal{W}} \mid \tilde{f}(\mathbb{P}^1) \cap Z \neq \emptyset\right\} = \bigcup_{z \in Z} \bigcup_{p \in \mathbb{P}^1} \left\{[\tilde{f}] \in \mathrm{Hom}^{\sim}_{\mathcal{W}} \mid \tilde{f}(p) = z\right\}$$

has dimension at most $\dim Z + 1 + n$.

Now $\mathcal{V} \subset \mathbb{P}(\Omega_X)$ is a divisor, and Z has codimension at least one in $\mathcal{V}$, so Z has dimension at most $2n - 3$, and the set $\mathrm{Hom}^{\sim}_{\mathcal{W},Z}$ above has dimension at most $3n - 2$. Since $\mathrm{Hom}^{\circ}_{\mathcal{W}}$ has dimension $3n$ and $\mathrm{Hom}^{\circ}_{\mathcal{W}} \to \mathrm{Hom}^{\sim}_{\mathcal{W}}$ is injective, a general point $[\tilde{f}] \in \mathrm{Hom}^{\sim}_{\mathcal{W}}$ is not in $\mathrm{Hom}^{\sim}_{\mathcal{W},Z}$. ☐

We need one more technical statement:

4.12. Lemma. *In the situation of Proposition 1.5 and using the notation introduced above, let $[f] \in \mathrm{Hom}^{\circ}_{\mathcal{W}}$ be a general point. Then for* every $x \in f(\mathbb{P}^1)$ *we have* $f(\mathbb{P}^1) \not\subset \mathrm{loc}^1_x$.

Proof. Fix two general points $x_1, x_2 \in X$. A general morphism $[f] \in \mathrm{Hom}^{\circ}_{\mathcal{W}}$ passing through x_1 and x_2 is 2-free and up to reparametrisation we have $f(0) = x_1, f(\infty) = x_2$. Set $g := f|_{\{0,\infty\}}$, then f is free over g [Kol96, II, Defn.3.1]. Suppose now that such a curve has the property $f(\mathbb{P}^1) \subset \mathrm{loc}^1_{x_0}$ for some $x_0 \in f(\mathbb{P}^1)$. Thus $x_1, x_2 \in \mathrm{loc}^1_{x_0}$, hence by symmetry $x_0 \in (\mathrm{loc}^1_{x_1} \cap \mathrm{loc}^1_{x_2})$. Yet the intersection

$$\mathrm{loc}^1_{x_1} \cap \mathrm{loc}^1_{x_2}$$

has codimension two in X. By [Kol96, II, Prop.3.7] a general deformation of f over g is disjoint from this set. ☐

4.13. Proof of Proposition 1.5. Arguing by contradiction, we suppose that $\mathcal{V} \cdot \tilde{C} > 0$ ($\tilde{C}$ is not contained in $\mathcal{V}$ for the general $[C] \in \mathcal{W}^\circ$). Applying Lemma 4.11 with

$$\mathcal{V}_0 := \{v^\perp \in \mathcal{V} \mid \mathbb{C}v = T_{l,\pi(v)} \text{ where } [l] \in \mathcal{K} \text{ is standard}\},$$

we see that for a general point $[C] \in \mathcal{W}$ there exists a point $x_1 \in C$ and a standard curve $[l] \in \mathcal{K}_{x_1}$ such that

$$T_{C,x_1} = T_{l,x_1}. \tag{4.13.1}$$

We shall now reformulate the property (4.13.1) in terms of the universal family $\mathcal{U}_{x_1,x_2}$, with x_2 a point chosen in $C \setminus \mathrm{loc}^1_{x_1}$ thanks to Lemma 4.12. Consider the blow-up $\varepsilon : \tilde{X} \to X$ at the point x_1, with exceptional divisor E_1. There is a rational map $\tilde{\mathrm{ev}} : \mathcal{U}_{x_1,x_2} \dashrightarrow \tilde{X}$ such that $\varepsilon \circ \tilde{\mathrm{ev}} = \mathrm{ev}$ (on the locus where $\tilde{\mathrm{ev}}$ is defined); since the general member of $\mathcal{W}_{x_1,x_2}$ is smooth at x_1, this map $\tilde{\mathrm{ev}}$ is well-defined in a general point of σ_1, and restricts to a rational map $\sigma_1 \dashrightarrow E_1$. The latter is dominant and therefore generically finite, because the general member of $\mathcal{W}_{x_1,x_2}$ is 2-free. In particular we may assume it is finite in a neighbourhood of the point $C \cap \sigma_1$.

We then consider the proper transform $\tilde{l}$ of l under ε, and let Γ be an irreducible component of $\tilde{\mathrm{ev}}^{-1}(\tilde{l})$ passing through $C \cap \sigma_1$. It is a curve that is mapped to a curve in $\mathcal{W}_{x_1,x_2}$ by q. Also, applying the same construction to the divisor $D_{x_1} \subset X$, one gets a prime divisor $G \subset \mathcal{U}_{x_1,x_2}$ mapping surjectively onto D_{x_1} and $\mathcal{W}_{x_1,x_2}$ respectively.

In general the curve Γ could be contained in the locus where $q|_G$ or $\mathrm{ev}|_G$ are not étale. However the standard rational curves $[l] \in \mathcal{K}$ such that a corresponding curve Γ is not contained in these ramification loci form a non-empty Zariski open set in $\mathcal{K}$. Hence their tangent directions define a non-empty Zariski open set in $\mathcal{V}$. Applying Lemma 4.11 a second time we can thus replace C by a general curve C' such that $[C'] \in \mathcal{W}^\circ \cap \mathcal{W}_{x_1,x_2}$ and hence l by a general $[l'] \in \mathcal{K}_{x_1}$ such that there exists a curve $\Gamma' \subset G$ such that $q(\Gamma')$ is a curve, $\mathrm{ev}(\Gamma') = l'$, and both maps $q|_G$ and $\mathrm{ev}|_G$ are étale at the general point $x \in \Gamma'$. By construction the point $C' \cap \sigma_1$ lies on Γ'. This is a contradiction to Proposition 4.14 below. □

4.14. Proposition [Miy04, Lemma 3.9]**.** *In the situation of Proposition 1.5, let $x_1, x_2 \in X$ be general points, and $[l]$ a general member of $\mathcal{K}_{x_1}$. Consider an irreducible curve $\Gamma \subset \mathcal{U}_{x_1,x_2}$ such that $\mathrm{ev}(\Gamma) = l$ and $q(\Gamma)$ is a curve, and assume there exists a prime divisor $G \subset \mathcal{U}_{x_1,x_2}$ mapped onto D_{x_1} by ev and containing Γ, such that both maps $q|_G$ and $\mathrm{ev}|_G$ are étale at a general point of Γ. Then $\Gamma \cap \sigma_1$ does not contain any point $C \cap \sigma_1$ with $[C] \in \mathcal{W}^\circ \cap \mathcal{W}_{x_1,x_2}$.*

We give the proof for the sake of completeness.

Proof. Since $[l]$ is general in $\mathcal{K}_{x_1}$, we have

$$T_X|_l \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n-2} \oplus \mathcal{O}_{\mathbb{P}^1},$$

and $\mathcal{K}_{x_1}$ is smooth with tangent space $H^0(l, N^+_{l/X} \otimes \mathcal{O}_l(-x_1))$ at $[l]$, where $\mathcal{E}^+$ denotes the ample part of a vector bundle $\mathcal{E} \to \mathbb{P}^1$, i.e. its ample subbundle of maximal rank.

Let $x \in \Gamma$ be a general point, and set $y = \mathrm{ev}(x) \in l$. For some analytic neighbourhood $V \subset \mathcal{K}_{x_1}$ of $[l]$, we have an evaluation map

$$\mathbb{P}^1 \times V \longrightarrow D_{x_1}$$

which is étale at $(y, [l])$, and the tangent space to D_{x_1} at y is thus

$$T_{D_{x_1},y} = T_{l,y} \oplus \left(N^+_{l/X} \otimes \mathcal{O}_l(-x_1)\right)_y = T_X|^+_{l,y}.$$

Since $\mathrm{ev}|_G$ is étale in x, we obtain that the tangent map

$$d_x\mathrm{ev} : T_{\mathcal{U}_{x_1,x_2},x} \to \mathrm{ev}^*(T_{X,\mathrm{ev}(x)})$$

maps $T_{G,x}$ isomorphically into the ample part i.e. we have

$$d_x\mathrm{ev}(T_{G,x}) \simeq \mathrm{ev}^*(T_X|^+_{l,\mathrm{ev}(x)}). \tag{4.14.1}$$

We argue by contradiction and suppose that there exists $[C] \in \mathcal{W}^\circ \cap \mathcal{W}_{x_1,x_2}$ such that $(C \cap \sigma_1) \in (\Gamma \cap \sigma_1)$. Since Γ maps onto l it is not contained in the divisor σ_1. Since the smooth rational curve C is 2-free, there exists by semicontinuity a neighbourhood U of $[C] \in \mathcal{W}_{x_1,x_2}$ parametrising 2-free smooth rational curves. For a 2-free rational curve, the evaluation morphism ev is smooth in the complement of the distinguished divisors σ_i [Kol96, II, Prop.3.5.1]. Thus if we denote by $R \subset \mathcal{U}_{x_1,x_2}$ the ramification divisor of ev, σ_1 is the unique irreducible component of R

containing the point $C \cap \sigma_1$. Thus Γ is not contained in the ramification divisor of ev.

Since $q(\Gamma)$ is a curve, there exists by Lemma 4.7 a finite base change $T \to q(\Gamma)$ with T a smooth curve, such that the normalisation S of the fibre product $T \times_{\mathcal{W}_{x_1,x_2}} \mathcal{U}_{x_1,x_2}$ is a surface with a conic bundle structure $\varphi : S \to T$ satisfying the conditions of Lemma 3.4. After a further base change we may suppose that there exists a φ-section Γ_1 that maps onto Γ. Note that since we obtained S by a base change from $\mathcal{U}_{x_1,x_2}$, the ramification divisor of the map $\mu : S \to \mathcal{U}_{x_1,x_2}$ is contained in the φ-fibres, i.e. its image by φ has dimension 0. In particular Γ_1 is not contained in this ramification locus.

Since the rational curve C is smooth and 2-free, the universal family $\mathcal{U}_{x_1,x_2}$ is smooth in a neighbourhood of $C \cap \sigma_1$. Thus σ_1 is a Cartier divisor in a neighbourhood of $C \cap \sigma_1$, and we can use the projection formula to see that

$$\Gamma_1 \cdot \mu^* \sigma_1 = \mu_*(\Gamma_1) \cdot \sigma_1 > 0.$$

In particular Γ_1 is not disjoint from the distinguished sections in the conic bundle $S \to T$. Let now $\varepsilon : \hat{S} \to S$ be the minimal resolution of singularities, and $\hat{\Gamma}_1$ the proper transform of Γ_1. Since the distinguished sections are in the smooth locus of S, the section $\hat{\Gamma}_1$ is not disjoint from the distinguished sections of $\hat{S} \to T$. We shall now show that

$$(\hat{\Gamma}_1)^2 \leq 0,$$

which is a contradiction to Lemma 3.4.

Denote by $f : \hat{\Gamma}_1 \to l$ the restriction of $\mathrm{ev} \circ \mu \circ \varepsilon : \hat{S} \to X$. Since $\hat{\Gamma}_1$ is not in the ramification locus of $\mu \circ \varepsilon$ and Γ is not in the ramification divisor of ev, the tangent map

$$T_{\hat{S}}|_{\hat{\Gamma}_1} \to f^* T_X|_l$$

is generically injective. Since $\hat{\Gamma}_1$ is a $\varphi \circ \varepsilon$-section, we have an isomorphism

$$T_{\hat{S}/T}|_{\hat{\Gamma}_1} \simeq N_{\hat{\Gamma}_1/\hat{S}}. \tag{4.14.2}$$

Since l has the standard splitting type (4.3.1) we have a (unique) trivial quotient $f^* T_X|_l \twoheadrightarrow \mathcal{O}_{\hat{\Gamma}_1}$, and thanks to (4.14.2) we are done if we prove that the natural map

$$T_{\hat{S}/T}|_{\hat{\Gamma}_1} \hookrightarrow T_{\hat{S}}|_{\hat{\Gamma}_1} \to f^* T_X|_l \twoheadrightarrow \mathcal{O}_{\hat{\Gamma}_1}$$

is not zero. It is sufficient to check this property for a general point in $\hat{\Gamma}_1$, and since $\hat{\Gamma}_1 \to \Gamma$ is generically étale, it is sufficient to check that for a general $x \in \Gamma$, the natural map

$$T_{\mathcal{U}_{x_1,x_2}/\mathcal{W}_{x_1,x_2},x} \to \mathrm{ev}^*(T_{X,\mathrm{ev}(x)})$$

does not have its image into the ample part $\mathrm{ev}^*(T_X|^+_{l,\mathrm{ev}(x)})$. Yet if $T_{\mathcal{U}_{x_1,x_2}/\mathcal{W}_{x_1,x_2},x}$ maps into the ample part, the decomposition $T_{\mathcal{U}_{x_1,x_2},x} = T_{\mathcal{U}_{x_1,x_2}/\mathcal{W}_{x_1,x_2},x} \oplus T_{G,x}$ (given by the fact that $q|_G$ is étale in x) combined with (4.14.1) implies that the tangent map

$$d_x \mathrm{ev} : T_{\mathcal{U}_{x_1,x_2},x} \to \mathrm{ev}^*(T_{X,\mathrm{ev}(x)})$$

cannot be surjective. Since Γ is not contained in the ramification locus of ev this is impossible. □

5. Proof of the main theorem

5.1. Proof of Theorem 1.3. If $X \simeq \mathbb{P}^n$ we are done, so suppose that this is not the case. Then consider the family of minimal rational curves $\mathcal{K}$ constructed in Section 4 and the associated total VMRT $\mathcal{V}$. Denote by $d \in \mathbb{N}$ the degree of a general VMRT $\mathcal{V}_x \subset \mathbb{P}(\Omega_{X,x})$.

Step 1. Using the family $\mathcal{W}^\circ$. In this step we prove that

$$\mathcal{V} \sim_{\mathbb{Q}} d(\zeta - \frac{1}{n}\pi^* K_X), \tag{5.1.1}$$

where ζ is the tautological divisor class on $\mathbb{P}(\Omega_X)$. Note that $\mathbb{P}(\Omega_X)$ has Picard number two, so we can always write

$$\mathcal{V} \sim_{\mathbb{Q}} a\zeta + b\frac{-1}{n}\pi^* K_X$$

with $a, b \in \mathbb{Q}$. Let now $\mathcal{W}^\circ$ be the family of rational curves constructed in Section 4, and let $\tilde{C}$ be the lifting of a curve $C \in \mathcal{W}^\circ$. By Proposition 1.5 we have $\mathcal{V} \cdot \tilde{C} = 0$. Since by the definition of $\tilde{C}$ one has $\zeta \cdot \tilde{C} = -2$ and $-\frac{1}{n}\pi^* K_X \cdot \tilde{C} = 2$, it follows that $a = b$. Since $\mathcal{V}_x = \mathcal{V}|_{\mathbb{P}(\Omega_{X,x})} \sim_{\mathbb{Q}} d\zeta|_{\mathbb{P}(\Omega_{X,x})}$, we have $a = b = d$. This proves (5.1.1).

Step 2. Bounding the degree d. Denote by $\mathcal{K}^\circ \subset \mathcal{K}$ the open set parametrising smooth standard rational curves in $\mathcal{K}$. We define an injective map

$$j : \mathcal{K}^\circ \hookrightarrow \mathrm{RatCurves}^{\mathrm{n}}(\mathbb{P}(\Omega_X))$$

by mapping a curve l to the image $\tilde{l}$ of the morphism $s : l \to \mathbb{P}(\Omega_X)$ defined by the invertible quotient $\Omega_X|_l \to \Omega_l$. We denote by $\tilde{\mathcal{K}}^\circ$ the image of j. Let us start by showing that $\tilde{\mathcal{K}}^\circ$ is dense in an irreducible component of $\mathrm{RatCurves}^{\mathrm{n}}(\mathbb{P}(\Omega_X))$. Since l is standard, the relative Euler sequence restricted to $\tilde{l}$ implies that $H^0(\tilde{l}, T_{\mathbb{P}(\Omega_X)/X}|_{\tilde{l}}) = 0$. Then, using the exact sequence

$$0 \to T_{\mathbb{P}(\Omega_X)/X}|_{\tilde{l}} \to T_{\mathbb{P}(\Omega_X)}|_{\tilde{l}} \to (\pi^* T_X)|_{\tilde{l}} \simeq T_X|_l \to 0$$

we obtain that the Zariski tangent space of $\mathrm{Hom}(\mathbb{P}^1, \mathbb{P}(\Omega_X))$ at a point corresponding to the rational curve $\tilde{l}$ has dimension at most $h^0(l, T_X|_l) = 2n$. Thus we can use [Kol96, II, Thm.2.15] to see that $\mathrm{RatCurves}^{\mathrm{n}}(\mathbb{P}(\Omega_X))$ has dimension at most $2n-3$ at the point $[\tilde{l}]$, which is exactly the dimension of $\tilde{\mathcal{K}}^\circ$.

By construction the lifted curves $\tilde{l}$ are contained in $\mathcal{V}$. Thus the open set $\tilde{\mathcal{K}}_0 \subset \mathrm{RatCurves}^{\mathrm{n}}(\mathbb{P}(\Omega_X))$ is actually an open set in $\mathrm{RatCurves}^{\mathrm{n}}(\mathcal{V})$. Since $\mathcal{V} \subset \mathbb{P}(\Omega_X)$ is a hypersurface, the algebraic set $\mathcal{V}$ has lci singularities. Thus we can apply [Kol96, II, Thm.1.3, Thm.2.15] and obtain

$$2n - 3 = \dim \tilde{\mathcal{K}}_0 \geq \deg \omega_{\mathcal{V}}^{-1}|_{\tilde{l}} + (2n-2) - 3.$$

We thus have $\deg \omega_{\mathcal{V}}^{-1}|_{\tilde{l}} \leq 2$.

Now by construction we have $-\frac{1}{n}\pi^* K_X \cdot \tilde{l} = 1$ and $\zeta \cdot \tilde{l} = -2$. Since $K_{\mathbb{P}(\Omega_X)} = 2\pi^* K_X - n\zeta$, the adjunction formula and (5.1.1) yield

$$2 \geq \deg \omega_{\mathcal{V}}^{-1}|_{\tilde{l}} = -(K_{\mathbb{P}(\Omega_X)} + \mathcal{V}) \cdot \tilde{l} = d.$$

Step 3. Conclusion. If $d = 1$ or $d = 2$ but $\mathcal{V}_x$ is reducible, we obtain a contradiction to [Hwa07, Thm.1.5] (cf. also [Ara06, Thm.3.1]). If $d = 2$ and $\mathcal{V}_x$ is irreducible, $\mathcal{V}_x$ is normal [Har77, II,Ex.6.5(a)], and therefore isomorphic to its normalisation $\mathcal{K}_x$

which is smooth (see §4.3). It is thus a smooth quadric and we conclude by [Mok08, Main Thm.]. ☐

5.2. Remark. Let us explain the difference of our proof with Miyaoka's approach: in the notation of Section 4, he considers the family $\mathcal{W}_{x_1,x_2}$. As we have seen above the evaluation map $\mathrm{ev} : \mathcal{U}_{x_1,x_2} \to X$ is generically finite and his goal is to prove that ev is birational. He therefore analyses the preimage $\mathrm{ev}^{-1}(l_1 \cup l_2)$, where the $l_i \subset X$ are general minimal curves passing through x_i respectively such that $[l_1 \cup l_2] \in \mathcal{W}_{x_1,x_2}$. If $\Gamma \subset \mathrm{ev}^{-1}(l_1 \cup l_2)$ is an irreducible curve mapping onto l_1 one can make a case distinction: if $q(\Gamma)$ is a curve that is not contained in the discriminant locus $\Delta \subset \mathcal{W}_{x_1,x_2}$ (Case **C** in [Miy04, p.227]) Miyaoka makes a very interesting observation which we stated as Proposition 4.14. However the analysis of the 'trivial' case (Case **A** in [Miy04, p.227]) where $q(\Gamma)$ is a point is not correct: it is not clear that $q(\Gamma) = [l_1 \cup l_2]$, because there might be another curve in $\mathcal{W}_{x_1,x_2}$ which is of the form $l_1 \cup l'_2$ with $l_2 \neq l'_2$. This possibility is an obvious obstruction to the birationality of ev and invalidates [Miy04, Cor.3.11(2), Cor.3.13(1)]. The following example shows that this possibility does indeed occur in certain cases.

5.3. Example. Let $H \subset \mathbb{P}^n$ be a hyperplane and $A \subset H \subset \mathbb{P}^n$ a projective manifold A of dimension $n-2$ and degree $3 \leq a \leq n$. Let $\mu : X \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along A. Then X is a Fano manifold [Miy04, Rem.4.2] and $-K_X \cdot C \geq n$ for every rational curve $C \subset X$ passing through a *general* point (the μ-fibres are however rational curves with $-K_X \cdot C = 1$). The general member of a family of minimal rational curves $\mathcal{K}$ is the proper transform of a line that intersects A. Consider the family $\mathcal{W}$ whose general member is the strict transform of a reduced, connected degree two curve C such that $A \cap C$ is a finite scheme of length two. For general points $x_1, x_2 \in X$ the (normalised) universal family $\mathcal{U}_{x_1,x_2} \to \mathcal{W}_{x_1,x_2}$ is a conic bundle and the evaluation map $\mathrm{ev} : \mathcal{U}_{x_1,x_2} \to X$ is generically finite. We claim that ev is not birational.

Proof of the claim. For simplicity of notation we denote by x_1, x_2 also the corresponding points in $\mathbb{P}^n$. Let $l_1 \subset \mathbb{P}^n$ be a general line through x_1 that intersects A. Since $x_2 \in \mathbb{P}^n$ is general there exists a unique plane Π containing l_1 and x_2. Moreover the intersection $\Pi \cap A$ consists of exactly a points, one of them the point $A \cap l_1$. For every point $x \in \Pi \cap A$ other than $A \cap l_1$, there exists a unique line $l_{2,x}$ through x and x_2. By Bezout's theorem $l_1 \cup l_2$ is connected, so its proper transform belongs to $\mathcal{W}_{x_1,x_2}$. Yet this shows that $\mathrm{ev}^{-1}(l_1)$ contains $a-1 > 1$ copies of l_1, one for each point $x \in \Pi \cap A \setminus l_1 \cap A$. This proves the claim. ☐

Let us conclude this example by mentioning that the conic bundle $\mathcal{U}_{x_1,x_2} \to \mathcal{W}_{x_1,x_2}$ does not satisfy the symmetry conditions of Lemma 3.4.

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