ON SURFACES OF HIGH DEGREE WITH RESPECT TO THE SECTIONAL GENUS

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Abstract. We study and classify linearly normal surfaces in $\mathbf{P}^n$, of degree d and sectional genus g, such that $d \geqslant 2g-1$.

Contents

1. Introduction

Let S be a complex, projective, linearly normal, irreducible and non-degenerate surface in $\mathbf{P}^n$, arbitrarily singular, of degree d and sectional (geometric) genus g. This paper is devoted to the study and classification of such surfaces under the hypothesis that $d \geqslant 2g-1$, which implies that the Kodaira dimension of S is $-\infty$.

One of the main ideas in this article is very classical, namely it is to use the adjoint and pluriadjoint systems $|mK_{S'}+H|$ of the pull back $|H|$ of the hyperplane system on a minimal desingularization S' of S. First of all, we recall (and elaborate on) the classical fact that $|K_{S'} + H|$ is empty if and only if S is either a scroll or the Veronese surface of degree 4 in $\mathbf{P}^5$ (see Section 4). Next we consider the case in which $|K_{S'} + H|$ is non–empty. In this case we prove that $K_{S'} + H$ is nef, hence $(K_{S'} + H)^2 \geqslant 0$; then we have separate discussions depending on whether the latter inequality is strict or not. On the one hand, we classify the cases in which $(K_{S'} + H)^2 = 0$: then S is either a weak Del Pezzo surface and $g = 1$, or $g \geqslant 2$ and S is ruled by conics (see Section 5.1).

On the other hand, one of the main results of the article is a sufficient condition (always verified if the surface is regular) for $|K_{S'}+H|$ to be base point free, under the assumption that it is non–empty and $(K_{S'}+H)^2 > 0$ (see Proposition 5.4). In Section 5.3 we give examples showing the sharpness of this result. We further prove that if $|K_{S'} + H|$ is non–empty, $(K_{S'} + H)^2 > 0$, and $d \geqslant 10$, then $|K_{S'} + H|$ determines a birational map (see Proposition 5.11).

In Section 6 we focus on the case when the surface is irregular. The main result here is that if $q > 0$, $h^0(S', K_{S'} + H) > 0$, and $(K_{S'} + H)^2 > 0$, then $g - q \geqslant 3$, which we show implies that $|K_{S'} + H|$ is not composed with a pencil (see Theorem 6.5). In Section 7 we go on studying the irrational case and we prove an extension of a classical result by C. Segre; the latter says that if a scroll has linearly normal hyperplane sections of positive genus, then it is a cone (see [7, Thm. 2.3]), while our result says that if $d > 2g + 5$, and S is ruled by conics and has linearly normal hyperplane sections, then it is essentially the 2–Veronese re-embedding of a cone (see Theorem 7.1). Moreover we show that if $d > 3g - 3$ and $q > 0$, then S is ruled by lines or conics, and therefore it is either a cone or a 2–Veronese thereof if it has linearly normal hyperplane sections (see Lemma 6.2 and Corollary 7.6).

Finally we focus on rational surfaces. In this case, we give a complete classification of surfaces for which the biadjoint system $|2K_{S'} + H|$ (see Theorem 9.1) or the triadjoint system $|3K_{S'} + H|$ is empty (see Theorems 10.4 and 10.6). This, plus the aforementioned extension of Segre's theorem, gives a complete classification of surfaces of degree $d > 3g - 3$ with linearly normal hyperplane sections. The classification for $d \geqslant 4g - 4$ had been given previously by the two first-named authors in [7]. As a brief account, the classification gives the following list: Veronese surfaces, represented by plane curves of degree at most 8, Del Pezzo surfaces or 2–Veronese images thereof, and surfaces represented by linear systems of k–gonal curves, with $k \leqslant 5$.

One of our important tools, besides projective and birational techniques, is an analogue, or rather a slight strengthening, of Reider's method, which is based on a detailed study of properties of m–connected effective divisors on a smooth surface (see in particular Lemma 3.5, due to the third-named author).

Finally we point out that, on the way, we provide many auxiliary results, too many to be described here, that are not strictly necessary for us, but will be useful in future work, we believe.

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2. Setup and notation

2.1. **Setup and notation.** In this paper we consider irreducible complex projective surfaces $S \subseteq \mathbf{P}^n$ that are linearly normal and non-degenerate; we let d be the degree of S, and assume $n \geqslant 3$. We set $d = n + a$, with $a \geqslant -1$. Let $\pi : S' \longrightarrow S$ be the minimal desingularization of S and set $H := \pi^*(\mathcal{O}_S(1))$; it is a nef and big line bundle on S', generated by its global sections. The general curve $C \in |H|$ is smooth and irreducible, and we denote its genus by g, that will be called the *sectional genus* of S. Note that $H^2 = d$. By the minimality assumption, there is no (-1)-curve θ on S' such that $H \cdot \theta = 0$.

We assume $d \geqslant 2g - 1$. Since $2g - 2 = K_{S'} \cdot H + H^2$, this assumption is equivalent to the assumption $K_{S'} \cdot H < 0$. Since H is nef, $K_{S'} \cdot H < 0$ implies that $h^0(S', mK_{S'}) = 0$ for all $m \in \mathbf{N}$, and so the Kodaira dimension of S' is $-\infty$.

We set $q := q(S')$ the *irregularity* of S'. If $q > 0$, since $\kappa(S') = -\infty$, the *Albanese morphism* of S' is a morphism alb : $S' \longrightarrow \Gamma$, where Γ is a smooth, projective curve of genus q, and the general fibre G of alb is a smooth rational curve. The fibres of alb form a pencil of genus q, called the *Albanese pencil* of S'.

For all positive integers m, we call the linear system $|mK_{S'} + H|$ the *m–adjoint system* to H.

2.2. **Further notation.** For all $e \in \mathbf{N}$, we let $\mathbf{F}_e$ be the rational ruled surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e))$, and we denote by E a section with self-intersection $-e$ (in case $e > 0$ this section is unique), and by F the class of the fibres of the structure map $\mathbf{F}_e \longrightarrow \mathbf{P}^1$.

For all $d \in \mathbf{N}$, we will denote by v_d the *d–Veronese map* and by $V_d \subseteq \mathbf{P}^{\frac{d(d+3)}{2}}$ the *d–Veronese image of* $\mathbf{P}^2$.

A complete linear system of plane curves of degree d with n base points of multiplicities $m_1, \ldots, m_n$ will be denoted by $(d; m_1, \ldots, m_n)$. We will use the exponential notation $(d; m_1^{i_1}, \ldots, m_n^{i_n})$ for repeated multiplicities.

We will denote by $\sim$ the linear equivalence and by $\equiv$ the numerical equivalence of divisors.

For all $m \in \mathbf{N}$, an effective divisor D on a smooth, irreducible, projective surface T is *m-connected* if, for every decomposition $D = D_1 + D_2$ with D_1 and D_2 effective and non-zero, one has $D_1 \cdot D_2 \geqslant m$.

We call *(-1)-divisor* a 1-connected divisor D on a smooth, irreducible, projective surface T such that $D^2 = K_T \cdot D = -1$ and the intersection form of D is negative definite. For all such divisor, there exists a morphism $f : T \to T'$ onto a smooth, projective surface T', which contracts D to a point and is an isomorphism on the complement of D.

2.3. **Simple internal projections.** Let $S \subseteq \mathbf{P}^N$ be a degree d surface, of sectional genus g. We call *simple internal projection* of S a surface $T \subseteq \mathbf{P}^{N'}$ obtained by projecting S from a curvilinear subscheme Z of length b supported on the smooth locus of S, with $N' = N - \dim(\langle Z \rangle) - 1$, such that the intersection scheme of $\langle Z \rangle$ with S is Z and such that the projection map is birational. We recall that a scheme Z is *curvilinear* if for every point p in the support of Z, the Zariski tangent space of Z at p has dimension at most one.

For T a simple internal projection of S as above, one has $\deg(T) = d - b$ and T has the same sectional genus g as S. Note that, if $d - b \geqslant 2g + 1$ and S is regular and linearly normal, then any projection from a curvilinear subscheme Z of length b supported on the smooth locus of S is a simple internal projection, and $N' = N - b$, because in this case the linear system of hyperplane sections of S passing through Z restricts on its general member to a complete, non-special, very ample linear system.

3. Some preliminary results

We keep the setup and notation as in Section 2.1.

Lemma 3.1. *Let S be a degree d, irreducible and non-degenerate surface in $\mathbf{P}^n$ with sectional genus g. If $d > 3g-3$ and S has linearly normal hyperplane sections, then $d < \frac{3}{2}n$.*

Proof. One has $3g-2 \geqslant 2g-1$ if $g \geqslant 1$, hence the divisor $H|_C$ is non-special for all g. Since, by the hypotheses, the hyperplane sections of S are linearly normal, one has $n = d-g+1$ by Riemann–Roch. Then

$$d > 3g-3 \iff d > 3(d-n),$$

which proves the assertion. □

The rays generated by H and $K_{S'}$ in the cone of divisors of S' lie respectively inside and outside of the cone of effective divisors, since H is effective and no multiple of $K_{S'}$ is effective, because in our hypotheses, S' has Kodaira dimension $-\infty$. When we let the integer m move to infinity, the ray generated by $mK_{S'}+H$ travels in straight line between the two rays generated by H and $K_{S'}$ respectively; therefore, if m is large enough, then no multiple of $mK_{S'}+H$ has sections. More precisley we have:

Lemma 3.2. *Suppose that $d > \frac{2m_0}{m_0-1}(g-1)$ for some integer m_0. Then for all $m \geqslant m_0$, no multiple of $mK_{S'}+H$ has sections. In particular, if $d > 3g-3$, then for all $m \geqslant 3$, no multiple of $mK_{S'}+H$ has sections.*

Proof. Let C be a general curve in $|H|$. It is an irreducible curve, and $C^2 = d \geqslant 0$. Therefore, if $(mK_{S'}+H)\cdot C < 0$, then no multiple of $mK_{S'}+H$ has sections, see [2, Useful Remark III.5] for instance. Now,

$$(mK_{S'}+H)\cdot C = m(K_{S'}+H)\cdot H - (m-1)H^2 = m(2g-2) - d(m-1),$$

which is negative if $m \geqslant m_0$. □

In Section 10 below, we classify rational surfaces under the assumption that the tri-adjoint linear system $|3K_{S'}+H|$ is empty. By Lemma 3.2, this is certainly the case if $d > 3g-3$. The general idea is to use that this implies that, since S is rational, any irreducible curve in $|2K_{S'}+H|$ is rational.

If $d = 3g-3$, it is possible that $3K_{S'}+H$ has sections. An example is provided by the Veronese surface $V_9 \subseteq \mathbf{P}^{54}$: then $S' = \mathbf{P}^2$ is the projective plane, and $H = 9L$ where L is the class of lines, so that $d = 81$ and $g = 28$.

Proposition 3.3. *Let S, S', H be as in Section 2.1. The following facts hold:*
(i) $h^0(S', K_{S'}+H) = g-q$; in particular, $q \leqslant g$, and $h^0(S', K_{S'}+H) = 0$ if and only if $g = q$;
(ii) $h^1(S', H) \leqslant q$;
(iii) $g = a+1-q+h^1(S', H) \leqslant a+1$;
(iv) if $g = q$ then $a+1 = 2q - h^1(S', H)$, and in particular $q \leqslant a+1 \leqslant 2q$;
(v) if $q = a+1$ then $g = q = h^1(S', H)$;
(vi) the hyperplane sections of S are linearly normal if and only if $q = h^1(S', H)$, and this

happens if and only if $g = a + 1$;
(vii) if $q = 0$, then $g = a + 1$ and the hyperplane sections of S are linearly normal; if moreover $d \geqslant 2g + 1$, then S has only isolated singularities;
(viii) $(K_{S'} + H)^2 = K_{S'}^2 + 4g - 4 - d$;
(ix) $(K_{S'} + H)^2 \leqslant K_{S'}^2 - n + 3a$, and if $g = a + 1$ (i.e., if the hyperplane sections of S are linearly normal), in particular if $q = 0$, then equality holds.

Remark 3.4. In the above proposition, the assumption that $d \geqslant 2g - 1$ may be substituted with the assumptions that S has Kodaira dimension $-\infty$ and $h^1(C, H|_C) = 0$.

Proof. Let $C \in |H|$ be a smooth curve. Since H is big and nef, one has $h^1(S', K_{S'} + H) = 0$ by Kawamata-Viehweg vanishing. Then the long exact sequence obtained from

$$0 \to \mathcal{O}_{S'}(K_{S'}) \to \mathcal{O}_{S'}(K_{S'} + H) \to \mathcal{O}_C(K_C) \to 0.$$

gives $g = h^0(S', K_{S'} + H) + q$, which proves (i).

One has $h^1(C, H|_C) = 0$ hence, from the long exact sequence obtained from

$$0 \to \mathcal{O}_{S'} \to \mathcal{O}_{S'}(H) \to \mathcal{O}_C(H|_C) \to 0, \tag{3.1}$$

we have $h^1(S', H) \leqslant q$, proving (ii).

Since S is linearly normal, one has $h^0(S', H) = n + 1$, and so, again by (3.1), one has $h^0(C, H|_C) = n + q - h^1(S', H) \leqslant n + q$. On the other hand, since $h^1(C, H|_C) = 0$, we have $h^0(C, H|_C) = n + a - g + 1$. Comparing both expressions of $h^0(C, H|_C)$, we obtain $g = a + 1 - q + h^1(S', H) \leqslant a + 1$, proving (iii) (cf. [12, Prop. 1.1]). Then (iv) follows right away. As for (v), if $q = a+1$, then from (i), (ii), and (iii), one has $h^1(S', H) \leqslant q \leqslant g = h^1(S', H)$.

The hyperplane sections of S are linearly normal if and only if the restriction map $H^0(S', H) \to H^0(C, H|_C)$ is surjective. By (3.1), and since $h^1(C, H|_C) = 0$, this happens if and only if $h^1(S', H) = q$, which is equivalent to $g = a + 1$ by (iii), proving (vi).

In particular, if $q = 0$, then $h^1(S', H) = 0$ by (ii), and by (vi) the hyperplane sections of S are linearly normal, and $g = a + 1$. If $d \geqslant 2g + 1$, then $H|_C$ is very ample, hence the general hyperplane section of S is smooth, and so S has only isolated singularities, which proves (vii).

Then $(K_{S'} + H)^2 = K_{S'}^2 + 4g - 4 - 2n - 2a + n + a = K_{S'}^2 + 4g - 4 - n - a$, proving (viii). Finally, since $g - 1 \leqslant a$, also (ix) follows. □

Lemma 3.5 ([13, Lemma 2.6])**.** *Let T be a smooth, irreducible, projective surface, and let L be a big and nef divisor on T. Then all divisors in $|L|$ are 1-connected, and thus have non-negative arithmetic genus.*

If $L = A + B$ is a decomposition of L with A and B effective divisors such that $A \cdot B = 1$ and $A^2 \leqslant B^2$, then A and B are 1-connected, and only the following possibilities can occur:
(a) $A^2 = -1$ (and so $L \cdot A = 0$);
(b) $A^2 = 0$ (and so $L \cdot A = 1$);
(c) $A^2 = B^2 = 1$, $A \equiv B$, and $L^2 = 4$.

Lemma 3.6 ([8, Lemma (A.4)])**.** *Let T be a smooth, irreducible, projective surface. Let D be an m-connected effective divisor on T. Let D_1, D_2 be effective divisors such that $D = D_1 + D_2$.*
(i) If $D_1 \cdot D_2 = m$, then D_1 and D_2 are $\lfloor (m + 1)/2 \rfloor$-connected.
(ii) If D_1 is minimal with respect to the condition $D_1 \cdot (D - D_1) = m$, then D_1 is $\lfloor (m + 3)/2 \rfloor$-connected.

Lemma 3.7. *Let T be a smooth, irreducible, projective surface, equipped with a big and nef divisor L satisfying $L^2 \geqslant 3$. Let D be a non-zero, effective, divisor, such that $D^2 = 0$ and $L \cdot D = 1$. Then D is 1-connected.*

Moreover, if D decomposes as $D = A+B$ with A and B effective divisors such that $A \cdot B = 1$, then $A^2 = B^2 = -1$ and, possibly up to exchanging A and B, $L \cdot A = 0$ and $L \cdot B = 1$.

Assume in addition that $L = K_T + L_0$ with L_0 a big and nef divisor, and D is not 2-connected. Then for every decomposition $D = A+B$ as above with $L \cdot A = 0$, one has, $K_T \cdot A = -1$, hence A is a (-1)-divisor, and $L_0 \cdot A = 1$.

Proof. First note that for any decomposition of $D = A+B$ with A, B effective divisors, we may assume without loss of generality that $L \cdot A = 0$ and $L \cdot B = 1$, because L is nef and $L \cdot D = 1$. Then the index theorem implies that $A^2 < 0$ and $B^2 \leqslant 0$.

Thus, the equality $D^2 = A^2 + 2A \cdot B + B^2 = 0$ implies that $A \cdot B \geqslant 1$, hence D is 1-connected.

Suppose now in addition that $A \cdot B = 1$. Then again from $D^2 = A^2 + 2A \cdot B + B^2 = 0$, we have either $A^2 = -1$ and $B^2 = -1$, and then the decomposition is as asserted in the lemma, or $A^2 = -2$ and $B^2 = 0$. This second possibility, however, cannot occur, for otherwise we would have $(D + B)^2 = 2$, $L \cdot (D + B) = 2$, and $L^2 \geqslant 3$, in contradiction with the index theorem. This proves the second assertion of the lemma.

Assume, finally, that $L = K_T + L_0$ with L_0 a big and nef divisor. Since A is 1-connected, by Lemma 3.6, and $A^2 = -1$, one has $K_T \cdot A = 2p_a(A) - 1 \geqslant -1$. Then $(K_T + L_0) \cdot A = 0$ gives $L_0 \cdot A = -2p_a(A) + 1$. Since L_0 is nef, this implies that $p_a(A) = 0$, and $K_T \cdot A = -1$ and $L_0 \cdot A = 1$. □

Proposition 3.8. *Let T be a smooth, irreducible, projective surface, and D be a non-zero, effective divisor on T, such that D is nef and $D^2 = 0$. Then:*
(i) for all effective A such that $A \leqslant D$, one has $D \cdot A = 0$ and $A^2 \leqslant 0$;
(ii) D is 0-connected.

If moreover $D = K_T + L$ with L nef and big, and T has negative Kodaira dimension, then there exists a (possibly non-linear) base-point-free pencil $\{G\}$ of rational curves, such that D consists of members of $\{G\}$.

Proof. Let A be an effective divisor such that $A \leqslant D$. Then there exists an effective B such that $D = A + B$. One has $D^2 = D \cdot A + D \cdot B = 0$, and $D \cdot A$ and $D \cdot B$ are non-negative; therefore $D \cdot A = D \cdot B = 0$. Then $A^2 > 0$ would contradict the index theorem. This proves (i).

Consider a decomposition $D = A + B$ with A and B non-zero and effective. Since $D^2 = A^2 + B^2 + 2A \cdot B = 0$, if $A \cdot B < 0$, then either $A^2 > 0$ or $B^2 > 0$, which is impossible by (i). Thus D is 0-connected.

Assume now that $D = K_T + L$ with L nef and big, and T has negative Kodaira dimension. Assume first that D is 1-connected. One has $D^2 = D \cdot K_T + D \cdot L = 0$, and $D \cdot L \geqslant 0$, hence $D \cdot K_T \leqslant 0$. Since $D^2 = 0$ and $p_a(D) \geqslant 0$, one must have $D \cdot K_T = -2$ and $p_a(D) = 0$. Then either $q = 0$, and then D moves in a linear base-point-free pencil of rational curves, as required, or $q > 0$, and then D is a fiber of the Albanese pencil.

If D is not 1-connected, since it is 0-connected, there exists a decomposition $D = A+B$ with A and B non-zero and effective, and $A \cdot B = 0$. Assume A is minimal with respect to this condition. Then by Lemma 3.6, A is 1-connected and $p_a(A) \geqslant 0$. One has $A \cdot D = A^2 + A \cdot B = A^2$. Since $A \cdot D = 0$ by (i), one has $A^2 = 0$. By the index theorem, this implies that $A \cdot L > 0$. Now,

$A \cdot K_T + A \cdot L = 0$, and therefore $A \cdot K_T < 0$, which implies that $p_a(A) = 0$. Then, as above, either $q = 0$ and A moves in a base-point-free pencil, or $q > 0$ and A is a fiber of the Albanese pencil. Finally, since $A \cdot B = 0$ and $B^2 = 0$, B is a union of members of the pencil $\{A\}$. □

4. Vanishing of the first adjoint

Proposition 4.1. *Let S, S', H, C be as in Section 2.1, and assume that $h^0(S', K_{S'} + H) = 0$ (equivalenty, $g = q$, by Proposition 3.3,* (i)*). Then:*
(i) $h^0(S', K_{S'} + 2H) = d + q - 1 > 0$;
(ii) $K_{S'} + 2H$ is nef;
(iii) $K_{S'}^2 = 8 - 8q$ and $(K_{S'} + 2H)^2 = 0$, unless S is the Veronese surface V_2 of degree 4 in $\mathbf{P}^5$, in which case $K_{S'}^2 = 9$ and $(K_{S'} + 2H)^2 = 1$;
(iv) either $S = V_2$, or S is a scroll.

Proof. Since $h^0(S', K_{S'} + H) = 0$ by the hypotheses, and $h^1(S', K_{S'} + H) = 0$ by the Kawamata–Viehweg theorem, one has $h^0(S', K_{S'} + 2H) = h^0(C, (K_{S'} + 2H)|_C)$. Note that $(K_{S'} + 2H) \cdot C = 2g - 2 + d$, so that $h^1(C, (K_{S'} + 2H)|_C) = 0$. Applying Riemann–Roch we get

$$h^0(S', K_{S'} + 2H) = (K_{S'} + 2H) \cdot C - g + 1 = 2g - 2 + d - g + 1 = d + g - 1 = d + q - 1.$$

Moreover $d \geqslant n - 1 \geqslant 2$, so $d + q - 1 > 0$, and (i) follows.

(ii) Assume that $K_{S'} + 2H$ is not nef. If θ is an irreducible curve such that $(K_{S'} + 2H) \cdot \theta < 0$, then $\theta^2 < 0$ and, since $2H$ is nef, we must have $K_{S'} \cdot \theta < 0$. So the only possibility is that θ is a (-1)-curve contracted by H, contradicting the assumption that S' is a minimal desingularization of S.

(iii) We have $(K_{S'} + 2H)^2 \geqslant 0$. Since $H^2 = d$ and $K_{S'} \cdot H = 2g - 2 - d = 2q - 2 - d$, we have $(K_{S'} + 2H)^2 = K_{S'}^2 + 8q - 8$, and so $K_{S'}^2 \geqslant 8 - 8q$. Since the Kodaira dimension of S' is $-\infty$, we get $K_{S'}^2 = 8 - 8q$, unless S' is isomorphic to $\mathbf{P}^2$. In this case we have $(K_{S'} + 2H)^2 = 1$, which implies that $K_{S'} + 2H \sim L$ where L is a line, i.e., $H \sim 2L$ and so $S = V_2$.

(iv) Assume $S \neq V_2$. Then $K_{S'}^2 = 8 - 8q$ and $(K_{S'} + 2H)^2 = 0$. Then, by Proposition 3.8 applied to $L = 2H$, $|K_{S'} + 2H|$ is composed with a (possibly non-linear) pencil $\{G\}$ of rational curves with $G^2 = 0$. Thus, $K_{S'} \cdot G = -2$. Since $(K_{S'} + 2H) \cdot G = 0$, we must have $H \cdot G = 1$, so that S is a scroll. □

Remark 4.2. In the situation of Proposition 4.1, suppose that $S \neq V_2$, so that $K_{S'}^2 = 8 - 8q$ and $(K_{S'} + 2H)^2 = 0$. If $q = 0$, one has $K_{S'}^2 = 8$, and $S' \cong \mathbf{F}_d$ is a Hirzebruch surface. If $q > 0$, then S' is a minimal ruled surface of genus q.

If $q = 0$, one has $a = -1$ (as $q \leqslant a + 1 \leqslant 2q$, see Proposition 3.3, (iv)) hence $d = n - 1$, so $S \neq V_2$ is a surface of minimal degree that is a scroll. Precisely the pencil $\{G\}$ is the ruling $|F|$ of $S' = \mathbf{F}_e$ and $K_{\mathbf{F}_e} + 2H \sim (d - 2)F$. Since $K_{\mathbf{F}_e} \sim -2E - (e + 2)F$ we obtain

$$2H \sim 2E + (d + e)F, \qquad \text{i.e.,} \quad H \sim E + \frac{d + e}{2} F,$$

in particular $d + e$ must be even.

Note that we must have $H \cdot E \geqslant 0$, which is equivalent to $d \geqslant e$. If $d = e$ then S is the cone over a rational normal curve, while if $d > e$, H is very ample and $S = S'$.

If $q > 0$ then the pencil $\{G\}$ is the Albanese pencil, and from $h^0(S, K_{S'} + 2H) = d + q - 1 > q$ we obtain $K_{S'} + 2H \equiv (n + a + 2q - 2)G$. Indeed, let $\mathrm{alb} : S' \to \Gamma$ be the Albanese fibration;

$K_{S'} + 2H$ is the pull-back by alb of a divisor D on Γ such that $H^0(S, K_{S'} + 2H) = H^0(\Gamma, D)$. Having $h^0(\Gamma, D) > q$, D is non-special, hence by Riemann–Roch it has degree $d + 2q - 2$.

The following lemma essentially gives the converse to Proposition 4.1, (iv):

Lemma 4.3. *If S is a scroll, then $h^0(S', K_{S'} + H) = 0$ and $g = q$.*

Proof. Suppose that S is a scroll and that $h^0(S', K_{S'} + H) > 0$. Denote by L the pull back to S' of a general line of the ruling of S. Then $H \cdot L = 1$ and $L^2 = 0$. Since L is a rational curve, the adjunction formula implies $K_{S'} \cdot L = -2$. Then $(K_{S'} + H) \cdot L < 0$, a contradiction because L is nef. This proves the first assertion. The second assertion follows from Proposition 3.3, (i). □

Remark 4.4. Let $S \subseteq \mathbf{P}^n$ be a linearly normal, non–degenerate rational surface of degree d. If S is a scroll, then by Remark 4.2, S is a surface of minimal degree $d = n - 1$. In particular, if $d \geqslant n$, then S is not a scroll. Conversely, if $d = n - 1$, then S is a surface of minimal degree and therefore it is either the Veronese surface V_2 or a scroll.

Corollary 4.5. *Let S, S', H, C be as in Section 2.1, and assume that $q > 0$ and $h^0(S', K_{S'} + H) = 0$. Then S is a scroll, and:*
(i) one has $d \geqslant n + q - 1$; if equality holds, then S is a cone over a curve of geometric genus q;
(ii) one has $d < n + 2q$.

Proof. That S is a scroll is contained in Proposition 4.1.

(i) By Proposition 3.3, (iii), one has $q = g \leqslant a + 1$, hence $d = n + a \geqslant n + q - 1$. If equality holds, then $h^1(S', H) = q = g$ (see Proposition 3.3, (v)), and this occurs if and only if the hyperplane sections of S are linearly normal (see Proposition 3.3, (vi)). Then the assertion follows by C. Segre's theorem [7, Thm 2.3].

(ii) By Proposition 3.3, (iv), we have $a = 2q - 1 - h^1(S', H)$, thus $d = n + a = n + 2q - 1 - h^1(S', H) \leqslant n + 2q - 1$, proving the assertion. □

5. General properties of the first adjoint system

In this section we want to discuss the first adjoint system when $h^0(S', K_{S'} + H) > 0$. By Proposition 3.3, (i), this assumption is equivalent to $g > q$.

5.1. **Nefness of the adjoint system.**

Proposition 5.1. *Let S, S' and H be as in Section 2.1, and assume that $h^0(S', K_{S'} + H) > 0$. Then the following assertions hold:*
(i) $K_{S'} + H$ is nef, hence $(K_{S'} + H)^2 \geqslant 0$;
(ii) $K_{S'}^2 - n + 3a \geqslant 0$;
(iii) $d \leqslant 4g - 8q + 4$, unless S is the Veronese surface V_3 of degree 9 in $\mathbf{P}^9$ ($d = 9$, $g = 1$);
(iv) $3a \geqslant n + 8q - 8$, unless S is the Veronese surface V_3 of degree 9.

Proof. (i) Assume that $K_{S'} + H$ is not nef, and let θ be an irreducible curve such that $(K_{S'} + H) \cdot \theta < 0$. Since H is nef, one has $K_{S'} \cdot \theta < 0$. On the other hand, since $K_{S'} + H$ is effective, $(K_{S'} + H) \cdot \theta < 0$ yields $\theta^2 < 0$. This implies that θ is a (-1)-curve and $H \cdot \theta = 0$, which is impossible because S' is a minimal desingularization of S.

(ii) By Proposition 3.3, (ix), we have $0 \leqslant (K_{S'} + H)^2 \leqslant K_{S'}^2 - n + 3a$, as required.

(iii)–(iv) By Proposition 3.3, (viii) and (ix), we have

$$d = 4g - 4 + K_{S'}^2 - (K_{S'} + H)^2 \quad \text{and} \quad 3a - n \geqslant (K_{S'} + H)^2 - K_{S'}^2. \tag{5.1}$$

Since S' has negative Kodaira dimension, either $K_{S'}^2 \leqslant 8-8q$, or $K_{S'}^2 = 9$ and S' is isomorphic to $\mathbf{P}^2$. In the former case, we obtain the required inequalities at once, using that $(K_{S'} + H)^2 \geqslant 0$. In the latter case, $q = 0$, and the inequality in (5.1) is an equality by Proposition 3.3, (ix); thus the two inequalities in (iii) and (iv) hold unless $(K_{S'} + H)^2 = 0$. If $(K_{S'} + H)^2 = 0$, then $K_{S'} + H \sim 0$, because $\mathbf{P}^2$ does not contain effective non-zero divisors with self-intersection 0, so $H \sim -K_{S'}$, and S must be the Veronese surface of degree 9 in $\mathbf{P}^9$. □

Proposition 5.2. *Let S, S' and H be as in Section 2.1 and suppose that $h^0(S', K_{S'} + H) > 0$ and $(K_{S'} + H)^2 = 0$. Then either*
(a) $K_{S'} + H \sim 0$ and S' is a Del Pezzo or weak Del Pezzo surface (and $g = 1$), or
(b) $g \geqslant 2$, and $|K_{S'} + H| = |(g-1)G|$ where $\{G\}$ is a pencil of rational curves (and so, the Albanese pencil if $q > 0$), satisfying $G \cdot H = 2$; thus S is ruled by conics.

Proof. If $K_{S'} + H \sim 0$, then S' is a Del Pezzo or weak Del Pezzo surface.

Assume $K_{S'} + H \not\sim 0$. Then, since $K_{S'} + H$ is nef and $(K_{S'} + H)^2 = 0$, it follows from Proposition 3.8 that $|K_{S'} + H|$ is composed with a base point free pencil $\{G\}$ of rational curves. Then $G \cdot H = 2$ because $G \cdot (K_{S'} + H) = 0$. □

In the remainder of this section we will consider the situation when $h^0(S', K_{S'} + H) > 0$ and $(K_{S'} + H)^2 > 0$. The following lemma implies in particular that in this situation S is not ruled by conics.

Lemma 5.3. *Let S, S' and H be as in Section 2.1. If $h^0(S', mK_{S'} + H) > 0$ for some integer m, then any pencil of rational curves $\{G\}$ with self-intersection 0 (that always exists if $S' \neq \mathbb{P}^2$) satisfies $H \cdot G \geqslant 2m$. Furthermore, if equality holds, then either $mK_{S'} + H \sim 0$, or all the components of any effective divisor in $|mK_{S'} + H|$ are contained in fibers of $\{G\}$.*

Proof. Since G is nef, $G \cdot (mK_{S'} + H) \geqslant 0$. From $K_{S'} \cdot G = -2$, we obtain $H \cdot G \geqslant 2m$.

If $H \cdot G = 2m$, then $G \cdot (mK_{S'} + H) = 0$, hence either $mK_{S'} + H \sim 0$ or, since G is nef, any component of an effective divisor in $|mK_{S'} + H|$ is contained in a fiber of $\{G\}$. □

5.2. **Base points of the adjoint system.**

Proposition 5.4. *Let S, S' and H be as in Section 2.1, and suppose that $h^0(S', K_{S'} + H) > 0$ and $(K_{S'} + H)^2 > 0$. If $|K_{S'} + H|$ has base points, then $q > 0$, and there exists a smooth irreducible curve θ of genus q such that $\theta^2 = -1$ and $H \cdot \theta = 0$.*

Proof. Note first that the hypotheses imply $g \geqslant 2$, and thus $H^2 \geqslant 4$. In fact $(K_{S'} + H)^2 > 0$ implies, by the index theorem, that $(K_{S'} + H) \cdot H > 0$, and so $g \geqslant 2$.

Let x be a base point of $|K_{S'} + H|$. Since $h^0(S', H) \geqslant 4$, x is a multiple point of some curve C in $|H|$ and so, by Theorem 3.1 of [13], C is not 2-connected. Then C has a decomposition $C = A + B$ with $A, B > 0$, $A \cdot B = 1$, and $A^2 \leqslant B^2$; by Lemma 3.5, A and B are 1-connected and only the following possibilities can occur:
(a) $A^2 = B^2 = 1$, $A \equiv B$, $H^2 = 4$ and $H \equiv 2A$ or
(b) $A^2 = -1$, $H \cdot A = 0$, or
(c) $A^2 = 0$, $H \cdot A = 1$.

In case (a), from $K_{S'} \cdot H < 0$ and $p_a(H) \geqslant 2$, one has $K_{S'} \cdot H = -2$. Then the hypothesis $(K_{S'} + H)^2 > 0$ implies $K_{S'}^2 \geqslant 1$, and so S' is a rational surface, hence $H \sim 2A$. On the other hand $A \cdot (A + K_{S'}) = 0$ and $(A + K_{S'})^2 \geqslant 0$ yield by the index theorem that $A \sim -K_{S'}$. By the Riemann-Roch theorem, $h^0(S', A) \geqslant 2$. Since $p_a(A) = 1$ and $H \cdot A = 2$, the map defined by $|H|$ restricted to a general member of $|A|$ is not birational, contradicting our assumptions on H. So case (a) does not occur.

In case (b), we claim that $p_a(A) \geqslant 1$. It suffices to prove that $K_{S'} \cdot A \geqslant 1$. By the adjunction formula, $K_{S'} \cdot A$ is odd, and so, since A is 1-connected, one has $K_{S'} \cdot A \geqslant -1$. But $K_{S'} \cdot A = -1$ and $H \cdot A = 0$ cannot occur, because in that case $A \cdot (K_{S'} + H) = -1$, in contradiction with the fact that $K_{S'} + H$ is nef. This proves the claim.

If $q = 0$, then $h^0(K_{S'}+A) > 0$, which leads to a contradiction, since $(K_{S'}+A)\cdot H = K_{S'}\cdot H < 0$ and H is nef and effective.

The upshot is that $q > 0$ and $p_a(A) \geqslant 1$. Then A is not contained in the fibres of the Albanese map of S', and so $p_a(A) \geqslant q$. If $p_a(A) > q$, then $h^0(K_{S'} + A) > 0$, as follows from the exact sequence

$$0 \to \mathcal{O}_{S'}(K_{S'}) \to \mathcal{O}_{S'}(K_{S'} + A) \to \omega_A \to 0.$$

This leads to a contradiction as above, since $H \cdot (K_{S'} + A) < 0$.

Thus $p_a(A) = q > 0$. Since $A^2 = -1$, one has $K_{S'} \cdot A = 2q - 1$. Now A, having arithmetic genus $q > 0$, cannot be contained in a fibre of the Albanese map of S', and so it has at least one component θ with geometric genus $\gamma \geqslant q$. Since $q \leqslant \gamma \leqslant p_a(\theta) \leqslant q$, the curve θ is smooth of genus q.

Since H is nef and $H \cdot A = 0$, one has $H \cdot \theta = 0$, hence also $\theta^2 < 0$. On the other hand $K_{S'} + H$ being nef and $(K_{S'} + H) \cdot A = 2q - 1$ imply that $(K_{S'} + H) \cdot \theta \leqslant 2q - 1$. Hence, from the adjunction formula, we obtain $\theta^2 = -1$, and the property asserted in the statement holds.

Lastly, assume we are in case (c). Then $H \cdot A = 1$ so, since $K_{S'} + H$ is nef, $K_{S'} \cdot A \geqslant -1$ and $p_a(A) \geqslant 1$. Since $H \cdot A = 1$ and H is base point free, $h^0(A, H) \geqslant 2$. Then, by [8, Prop. A.5, (ii)], A is not 2-connected. So A has a decomposition $A = A_1 + A_2$ with $A_1 \cdot A_2 = 1$ and, by Lemma 3.7, we have $A_1^2 = A_2^2 = -1$ and $H \cdot A_1 = 0$ and $H \cdot A_2 = 1$, possibly up to relabelling A_1 and A_2. Moreover, A_1 is 1-connected by Lemma 3.6, so we are back in case (b) with A_1 instead of A, and the existence of the curve θ follows by the argument given in that case. □

Remark 5.5. Conversely, the existence of a curve θ as in Proposition 5.4 tends to impose that the adjoint system has base points. Indeed, in the situation of the proposition, consider $\hat{H} = H - \theta$; it is an effective divisor, and

$$(K_{S'} + H)|_\theta = (K_{S'} + \theta)|_\theta + \hat{H}|_\theta = K_\theta + \hat{H}|_\theta\,.$$

Since $\hat{H} \cdot \theta = H \cdot \theta - \theta^2 = 1$, the linear series $|K_\theta + \hat{H}|_\theta|$ on θ has a base point if $\hat{H}|_\theta$ is effective, and in that case the adjoint system $|K_{S'} + H|$ has a base point as well. Note that $\hat{H}|_\theta$ is certainly effective if $q = 1$, or if $|\hat{H}|$ doesn't have θ as a fixed component.

Proposition 5.6. *Let S, S' and H be as in Section 2.1, and suppose that $h^0(S', K_{S'} + H) > 0$, and $q = 0$. Then $|K_{S'} + H|$ is base-point-free.*

Proof. If $(K_{S'}+H)^2 > 0$, the conclusion follows from Proposition 5.4. Otherwise, since $K_{S'}+H$ is nef, one has $(K_{S'} + H)^2 = 0$. Then, by Proposition 3.8, there exists a base-point-free pencil $\{G\}$ of rational curves, such that D consists of members of $\{G\}$. Since S' is rational, $h^0(S',G) > 1$, and the conclusion follows. □

5.3. **Examples.** In this section we present some examples illustrating Proposition 5.4 above. They show that it is indeed possible that the adjoint system has base points, and then these base points are located on the curve θ as in the proposition. Example 5.7 satisfies all the assumptions of the proposition, in particular $d \geqslant 2g-1$. In Examples 5.8 and 5.9 however, one has $d = 2g-2$; still, the two latter examples have many interesting features. All our examples live in elliptic ruled surfaces. It is possible to cook up similar examples on ruled surfaces of irregularity $q > 1$, but then d becomes smaller than $2g-1$.

Let Γ be a smooth, irreducible, elliptic curve. We shall consider elliptic ruled surfaces $R = \mathbf{P}(\mathcal{E})$ for various rank two vector bundles $\mathcal{E}$ on Γ, which we assume to be normalized as in [11, V, Notation 2.8.1]. In each case, we fix a section C_0 such that $\mathcal{O}_R(C_0) \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. For each divisor $\mathfrak{d}$ on Γ, we denote by $F(\mathfrak{d})$ the pull-back of $\mathfrak{d}$ to R; we denote by F the numerical equivalence class of $F(\mathfrak{d})$ for all degree 1 divisors $\mathfrak{d}$ on Γ. One has $K_R \sim -2C_0 + F(\det \mathcal{E})$.

Example 5.7. Fix a point $p \in \Gamma$ and consider $R = \mathbf{P}(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-p))$, with the notation introduced above for all of Section 5.3. For all $k \in \mathbf{N}$, let

$$H_k \sim k\big(C_0 + F(p)\big).$$

The curve C_0 has genus 1 and self-intersection -1, and intersects H_k with degree 0. We shall prove the following.

Claim 5.7.1. For all $k \geqslant 3$, the linear system $|H_k|$ defines a birational morphism which does not contract any (-1)-curve, onto a surface $S \subseteq \mathbf{P}^{\frac{1}{2}k(k+1)}$ of degree $d = k^2$ and sectional genus $g = \frac{1}{2}k(k-1)+1$, and with linearly normal hyperplane sections. The adjoint system $|K_R + H_k|$ has a base point on C_0.

Before we prove the claim, let us observe that

$$d = H_k^2 = 2g - 2 + k.$$

Thus, we get examples fitting in the assumptions of Proposition 5.4, with degree d arbitrarily large with respect to the genus g. Since S has linearly normal hyperplane sections results by C. Segre (mentioned in the introduction) and Hartshorne (see [10]) imply that the $d \leqslant 4g+4$, see [7, Corollary 2.6]. As a sanity check we note that

$$\frac{d}{2g-2} = 1 + \frac{1}{k-1},$$

so that indeed the degree does not exceed $4g+4$.

These examples are Veronese re-embeddings of cones as studied in Section 7 below. After we prove the claim, we briefly discuss the variant when we consider

$$H_k^{(p')} = k\big(C_0 + F(p')\big)$$

with $p' \neq p$.

Proof of the claim. One has $d = H_k^2 = k^2$ and $H_k \cdot C_0 = 0$. Moreover, $-K_R \sim 2C_0 + F(p)$, hence

$$(K_R + H_k)|_{C_0} = (H_{k-2} + F(p))|_{C_0} = p,$$

and thus the adjoint system $|K_R + H_k|$ has a base point at $p \in C_0$. A direct computation gives

$$(K_R + H_k) \cdot H_k = k(k-1),$$

and therefore the members of $|H_k|$ have genus $g = \frac{1}{2}k(k-1)+1$. Then it is straightforward that $d = 2g - 2 + k$.

One has

$$\begin{aligned} h^0(R, H_k) &= h^0(\Gamma, \mathrm{Sym}^k(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-p)) \otimes \mathcal{O}_\Gamma(kp)) \\ &= h^0(\Gamma, \mathcal{O}_\Gamma(kp) \oplus \mathcal{O}_\Gamma((k-1)p) \oplus \cdots \oplus \mathcal{O}_\Gamma(p) \oplus \mathcal{O}_\Gamma) \\ &= k + \cdots + 2 + 1 + 1 \\ &= 1 + \frac{1}{2}k(k+1), \end{aligned}$$

and similarly

$$\begin{aligned} h^1(R, H_k) &= h^1(\Gamma, \mathcal{O}_\Gamma(kp) \oplus \mathcal{O}_\Gamma((k-1)p) \oplus \cdots \oplus \mathcal{O}_\Gamma(p) \oplus \mathcal{O}_\Gamma) \\ &= h^1(\mathcal{O}_\Gamma) = 1. \end{aligned}$$

This, by Proposition 3.3, (vi), implies that S has linearly normal hyperplane sections. □

If, on the other hand, one considers $H_k^{(p')} = k\big(C_0 + F(p')\big)$ for $p' \in C_0$ such that $kp' \not\sim kp$, then

$$(K_R + H_k^{(p')})|_{C_0} = ((k-2)C_0 + F(kp'-p))|_{C_0} \sim p' + (k-1)(p'-p),$$

and thus the adjoint system $|K_R + H_k|$ has a base point at the unique point $p^{(k)} \in C_0$ linearly equivalent to $p' + (k-1)(p'-p)$. The numerical characters computed above remain unchanged, but

$$\begin{aligned} h^0(R, H_k^{(p')}) &= h^0(\Gamma, \mathrm{Sym}^k(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-p)) \otimes \mathcal{O}_\Gamma(kp')) \\ &= h^0(\Gamma, \mathcal{O}(kp') \oplus \mathcal{O}((kp'-p) \oplus \cdots \oplus \mathcal{O}(kp'-(k-1)p) \oplus \mathcal{O}(kp'-kp)) \\ &= k + \cdots + 2 + 1 + 0 \\ &= \frac{1}{2}k(k+1), \end{aligned}$$

and

$$\begin{aligned} h^1(R, H_k^{(p')}) &= h^1(\Gamma, \mathcal{O}(kp') \oplus \mathcal{O}((kp'-p) \oplus \cdots \oplus \mathcal{O}(kp'-(k-1)p) \oplus \mathcal{O}(kp'-kp)) \\ &= h^1(\mathcal{O}(p'-p)) = 0. \end{aligned}$$

This implies that for all $C \in |H_k^{(p')}|$, the linear system $|H_k^{(p')}|$ cuts out a codimension one linear subsystem of $|\ H_k^{(p')}|_C\ |$ on C. For $k \geqslant 3$, there are then two possibilities: either this subsystem is base-point-free, and then the surface gotten from $|H_k^{(p')}|$ has degree k^2 and hyperplane sections that are not linearly normal, or this subsystem has a unique base point, and then the surface gotten from $|H_k^{(p')}|$ has degree k^2-1 and linearly normal hyperplane sections.

Example 5.8. Let $\mathfrak{e}$ be a non-torsion degree 0 divisor on Γ, and $\mathcal{E} = \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(\mathfrak{e})$. We consider $R = \mathbf{P}(\mathcal{E})$, with the notation introduced above, at the beginning of Section 5.3. The curve C_0 is the unique effective divisor in its linear equivalence class, and there is a unique, irreducible, curve $C_{\mathfrak{e}} \sim C_0 - F(\mathfrak{e})$. One has $-K_R \sim C_0 + C_{\mathfrak{e}}$. Let p be a general point on Γ. For all $g \in \mathbf{N}$, let $H_g \sim gC_0 + F(p)$.

The following properties hold. For all $g \geqslant 3$, the linear system $|H_g|$ has dimension g; it has two base points on C_0 and $C_{\mathfrak{e}}$ respectively, and its proper transform $|H'_g|$ on the blow-up $R' \to R$ at these two base points defines a birational morphism $R' \to S \subseteq \mathbf{P}^g$ onto a surface with canonical hyperplane sections. The proper transforms C'_0 and $C'_{\mathfrak{e}}$ of C_0 and $C_{\mathfrak{e}}$ have self-intersection -1 and intersect H'_g trivially; in particular, the surface $S \subseteq \mathbf{P}^g$ has two elliptic double points. The adjoint system $|K_{R'} + H'_g|$ has two base points, lying on C'_0 and $C'_{\mathfrak{e}}$ respectively.

We omit the proof of these properties, because it is similar to the proof of the analogous properties in the next example, which is thoroughly treated in [14].

The next example is kind of a degenerate version of the previous one: the anticanonical divisor becomes a section with multiplicity two, and the two base points of the linear system $|H_g|$ become infinitely near. The adjoint system $|K_{R'} + H'_g|$ has a fixed part, and the surface image of $|H'_g|$ has an interesting singularity.

Example 5.9. Let $\mathcal{E}$ be the unique indecomposable vector bundle with trivial determinant on Γ, see [11, Thm. 2.15, p. 377], and consider $R = \mathbf{P}(\mathcal{E})$ with the notation introduced at the beginning of Section 5.3. The curve C_0 is the unique effective divisor in its linear equivalence class, and one has $-K_R \sim 2C_0$. Let p be a general point on Γ. For all $g \in \mathbf{N}$, let $H_g \sim gC_0 + F(p)$.

The following properties hold, see [14]. For all $g \geqslant 3$, the linear system $|H_g|$ has dimension g; it has two infinitely near base points, with the proper base point lying on C_0. The proper transform $|H'_g|$ on the blow-up $R' \to R$ at these two base points defines a birational morphism $R' \to S \subseteq \mathbf{P}^g$ onto a surface with canonical hyperplane sections. The proper transform C'_0 of C_0 has self-intersection -1 and intersects H'_g trivially. The surface $S \subseteq \mathbf{P}^g$ has a genus two singularity,[1] consisting of a double point with an infinitely near double line. The adjoint system $|K_{R'} + H'_g|$ has the exceptional locus of $R' \to R$ as a fixed part.

5.4. **Further properties of the adjoint system.**

Proposition 5.10. *Let S, S' and H be as in Section 2.1. Assume that $h^0(S', K_{S'} + H) > 0$ and $(K_{S'} + H)^2 > 0$. Let D be a divisor in $|K_{S'} + H|$. Then:*
(i) $h^1(S', K_{S'} + H) = h^1(S', 2K_{S'} + H) = 0$;
(ii) $h^0(D, K_{S'} + H) = g - 1$ *and* $h^1(D, K_{S'} + H) = 0$;
(iii) $h^0(S', 2K_{S'} + H) = h^0(D, \omega_D) - q$;
(iv) $p_a(K_{S'} + H) \geqslant q$, *and equality holds if and only if* $h^0(S', 2K_{S'} + H) = 0$;
(v) $p_a(K_{S'} + H) - 1 = K_{S'}^2 + 3g - 3 - d$;
(vi) $(K_{S'} + H)^2 = g - 2 + p_a(K_{S'} + H) \geqslant g - 2 + q$.

Proof. (i) comes from the Kawamata-Viehweg theorem, because both H and $K_{S'} + H$ are nef and big, see Proposition 5.1, (i).

For (ii), consider the long exact sequence coming from

$$0 \to \mathcal{O}_{S'} \to \mathcal{O}_{S'}(K_{S'} + H) \to \mathcal{O}_D(K_{S'} + H) \to 0,$$

[1]This means that, denoting by $f : S' \to S$ of the singularity, the stalk of $R^1 f_* \mathcal{O}_{S'}$ at the singular point has dimension 2, see [9, Chap. I, Definition 5.1].

namely

$$0 \to H^0(S', \mathcal{O}_{S'}) \to H^0(S', K_{S'} + H) \to H^0(D, K_{S'} + H) \\ \to H^1(S', \mathcal{O}_{S'}) \to H^1(S', K_{S'} + H) \to H^1(D, K_{S'} + H) \to H^2(S', \mathcal{O}_{S'}) = 0.$$

Then, (ii) follows from $h^0(S', K_{S'} + H) = g - q$ (Proposition 3.3, (i)), $h^1(S', K_{S'} + H) = 0$, and $h^2(S', \mathcal{O}_{S'}) = h^0(S', K_{S'}) = 0$.

(iii) follows from the long exact sequence coming from

$$0 \to \mathcal{O}_{S'}(K_{S'}) \to \mathcal{O}_{S'}(2K_{S'} + H) \to \omega_D \to 0$$

and the fact that $h^1(S', 2K_{S'} + H) = 0$.

For (iv), note that $K_{S'} + H$ is 1-connected because it is nef and big (Lemma 3.5), therefore $p_a(K_{S'} + H) = h^0(D, \omega_D)$. Then, (iv) follows from (iii).

For (v), just use the adjunction formula.

For (vi), note that by the Riemann–Roch theorem and assertion (ii), $g-1 = h^0(D, K_{S'}+H) = (K_{S'}+H)^2+1-p_a(K_{S'}+H)$, and so $(K_{S'}+H)^2 = g-2+p_a(K_{S'}+H)$. The inequality follows from (iv). □

Proposition 5.11. *Let S, S' and H be as in Section 2.1, and assume that $h^0(S', K_{S'} + H) > 0$ and $(K_{S'} + H)^2 > 0$. If $H^2 \geqslant 10$, then the map defined by $|K_{S'} + H|$ is birational.*

Proof. Suppose by contradiction that the map defined by $|K_{S'}+H|$ is not birational. Let $x \in S'$ be a general point, and $y \in S'$ be such that $x \neq y$ and x and y have the same image. Then, by the main theorem 2.1 of [3], there is an effective divisor D, containing x and y, such that

$$D \cdot H - 2 \leqslant D^2 \leqslant 1.$$

Since $h^0(S', K_{S'} + H) > 0$, S is not a scroll, hence $D \cdot H > 1$, and $D \cdot H$ equals 2 or 3. If $D \cdot H = 3$, then $D^2 = 1$, and this contradicts the index theorem because $H^2 \geqslant 10$. So we must have $D \cdot H = 2$, and S is swept out by irreducible conics. Let A be the preimage in S' of a general member of this family of conics. Being A rational, if $A^2 > 0$, then A moves in S' in a family of dimension at least two, and therefore S contains a family of dimension at least two of conics; this implies that S is the Veronese surface V_2, which contradicts the assumption that $h^0(S', K_{S'} + H) > 0$. Therefore $A^2 = 0$, and A moves in a (possibly non-linear) pencil. Now, $H \cdot A = 2$ and $K_{S'} \cdot A = -2$, hence $A \cdot (K_{S'} + H) = 0$, and therefore every member of $|K_{S'} + H|$ consists of curves contained in members of the pencil $\{A\}$; this implies that $(K_{S'} + H)^2 \leqslant 0$, in contradiction with our assumptions. This ends the proof. □

6. The irrational case

Here we consider surfaces with $q > 0$ and $h^0(S', K_{S'} + H) > 0$; the latter assumption is equivalent to $g > q$ by Proposition 3.3, (i). We set $\mu = H \cdot G$, where G denotes the general fibre of the Albanese pencil. Since $g > q$, we have $\mu \geqslant 2$; on the other hand, $g - 1 \geqslant \mu(q - 1)$ by Riemann–Hurwitz.

6.1. **The case** $(K_{S'}+H)^2=0$.

Lemma 6.1. *Let* S, S' *and* H *be as in Section 2.1, and assume* $q>0$ *and* $h^0(S', K_{S'}+H)>0$.
If $\mu=2$ *(hence* S *is ruled by conics), then* $(K_{S'}+H)^2=0$ *(and so* $K_{S'}^2+4g-4-d=0$*), and* $|K_{S'}+H|$ *is composed with the Albanese pencil.*
Conversely, if $(K_{S'}+H)^2=0$*, then* $|K_{S'}+H|$ *is composed with the Albanese pencil and* $\mu=2$.

Proof. The first assertion follows from Lemma 5.3. The converse is Proposition 5.2. □

Lemma 6.2. *Let* S, S' *and* H *be as in Section 2.1, and assume* $q>0$ *and* $h^0(S', K_{S'}+H)>0$. *If* $d>3g-3$*, then* $\mu=2$ *and* $g>8q-7$.

Proof. By [10, Thm. (2.3)], one has

$$d \leqslant \frac{2\mu}{\mu-1}(g-1).$$

If $\mu \geqslant 3$, this yields $d \leqslant 3(g-1)$, a contradiction. Hence we have $\mu=2$.
So, by Lemma 6.1, $(K_{S'}+H)^2=0$, i.e., $K_{S'}^2+4g-4-d=0$. Then the hypothesis $d>3g-3$ gives $K_{S'}^2+g-1>0$, and so $g>-K_{S'}^2+1 \geqslant 8q-7$. □

6.2. **The case** $(K_{S'}+H)^2>0$.

Lemma 6.3. *Let* S, S' *and* H *be as in Section 2.1, and assume that* $q>0$*,* $h^0(S', K_{S'}+H)>0$*, and* $(K_{S'}+H)^2>0$*. Then* $g \geqslant 9q-7$*. Furthermore, if equality holds, then* S' *is minimal,* $d=2g-1$*,* $p_a(K_{S'}+H)=q$*, and* $h^0(S', 2K_{S'}+H)=0$.

Proof. From $q>0$, we have $K_{S'}^2 \leqslant 8-8q$. Moreover, we assume $d \geqslant 2g-1$. Then:

$$\begin{aligned}(K_{S'}+H)^2 &= K_{S'}^2+4g-4-d\\ &\leqslant 8-8q+4g-4+1-2g = 5-8q+2g.\end{aligned}$$

On the other hand, by Proposition 5.10, (vi), $(K_{S'}+H)^2 \geqslant g-2+q$. Thus,

$$5-8q+2g \geqslant g-2+q,$$

i.e., $g \geqslant 9q-7$. If equality holds, then all the above inequalities are equalities, in particular $K_{S'}^2=8-8q$, $d=2g-1$, and equality holds in Proposition 5.10, (vi), which means $p_a(K_{S'}+H)=q$, and then $h^0(S', 2K_{S'}+H)=0$ by Proposition, 5.10, (iv). □

Proposition 6.4. *Let* S, S' *and* H *be as in Section 2.1, and assume that* $q>0$*,* $h^0(S', K_{S'}+H)>0$*, and* $(K_{S'}+H)^2>0$*. If* $g-q \geqslant 3$*, then* $|K_{S'}+H|$ *is not composed with a pencil.*

Proof. Suppose by contradiction that $|K_{S'}+H|$ is composed with a pencil $\{P\}$. Since $h^0(K_{S'}+H)=g-q$, we can then write

$$K_{S'}+H \equiv \alpha P+Z$$

where Z is the fixed part (possibly zero), and $\alpha \geqslant g-q-1$, with equality holding if and only if the pencil is rational. Being $K_{S'}+H$ nef and big, it is 1-connected (see Lemma 3.5), so if $Z \neq 0$ then $P \cdot Z>0$.
Since $h^0(S', K_{S'}+H)>0$ and $(K_{S'}+H)^2>0$, if $\{P\}$ is the Albanese pencil, then $H \cdot P \geqslant 3$, by Lemma 6.1. If $\{P\}$ is not the Albanese pencil, then the general curve in $\{P\}$ has genus at

least q and so, since the map defined by $|H|$ is birational, we must have $H \cdot P \geqslant 3$ in this case as well. The upshot is that $H \cdot P \geqslant 3$ in any event.

Since H is nef and $H \cdot (K_{S'} + H) = 2g - 2$, we obtain $\alpha H \cdot P \leqslant 2g - 2$, and thus

$$3\alpha \leqslant \alpha H \cdot P \leqslant 2g - 2. \tag{6.1}$$

If $\alpha \geqslant g - q$, we obtain $3g - 3q \leqslant 2g - 2$, i.e., $g \leqslant 3q - 2$. Since, by Lemma 6.3, $g \geqslant 9q - 7$, we see that $\alpha \geqslant g - q$ is impossible.

Therefore, one has $\alpha = g - q - 1 \geqslant 2$, and the pencil $\{P\}$ is rational; in particular, $\{P\}$ is not the Albanese pencil, and therefore $p_a(P) > 0$. From (6.1), we obtain

$$3 \leqslant \frac{2g-2}{g-q-1} = 2 + \frac{2q}{g-q-1},$$

hence $2q \geqslant g - q - 1$, i.e., $g \leqslant 3q + 1$. Since, by Lemma 6.3, $g \geqslant 9q - 7$, we obtain $6q \leqslant 8$, hence $q = 1$. Then, since $g - q \geqslant 3$ and $g \leqslant 3q + 1$, we have $g = 4$, and $\alpha = 2$.

Suppose this case occurs. By the index theorem, we have $H^2(K_{S'} + H)^2 \leqslant 4(g-1)^2 = 36$. Since $H^2 = d \geqslant 7$, this implies that $(K_{S'} + H)^2 \leqslant 5$. Moreover, being composed with a pencil, $|K_{S'} + H|$ has base points. So, by Proposition 5.4, there is a curve θ such that $\theta^2 = -1$, $\theta \cdot K_{S'} = 1$, and $\theta \cdot H = 0$. So,

$$1 = \theta \cdot (K_{S'} + H) = \theta \cdot (2P + Z) = 2\theta \cdot P + \theta \cdot Z,$$

which implies that $Z \neq 0$. Now notice that, since

$$4P^2 + 2P \cdot Z = 2P \cdot (K_{S'} + H) \leqslant (2P + Z) \cdot (K_{S'} + H) \leqslant 5,$$

one must have $P^2 = 0$. But then $p_a(P) \geqslant 1$ implies that $P \cdot K_{S'} \geqslant 0$, and so $2P \cdot (K_{S'} + H) \geqslant 2P \cdot H \geqslant 6$: this contradicts $(K_{S'} + H)^2 \leqslant 5$.

We conclude that $|K_{S'} + H|$ is not composed with a pencil. □

In fact, as we will now prove, the condition $g - q \geqslant 3$ always holds.

Theorem 6.5. *Let S, S' and H be as in Section 2.1 and assume that $q > 0$, $h^0(S', K_{S'} + H) > 0$ and $(K_{S'} + H)^2 > 0$. Then $g - q \geqslant 3$, and therefore $|K_{S'} + H|$ is not composed with a pencil.*

The proof starts by the following lemma, which shows that there is only one numerical possibility.

Lemma 6.6. *Let S, S' and H be as in Section 2.1 and assume that $q > 0$, $h^0(S', K_{S'} + H) > 0$ and $(K_{S'} + H)^2 > 0$. Then $g - q \geqslant 3$ unless, possibly, if $q = 1$, $g = 3$, $d = 5$, $h^0(S', H) = 4$, $(K_{S'} + H)^2 = 2$, and $K_{S'}^2 = -1$.*

Proof. From $h^0(S', K_{S'} + H) > 0$, we know that $g > q$ by Proposition 3.3, (i).

Assume that $g - q \leqslant 2$. Then, by Lemma 6.3, one has $g - q \geqslant 8q - 7$, and so we get $q = 1$, implying that either $g = 2$ or $g = 3$.

Let $g = 2$. We have $0 < (K_{S'} + H)^2 = K_{S'}^2 + 4g - 4 - d = K_{S'}^2 + 4 - d$, see Proposition 3.3, (viii). Since $q = 1$ implies $K_{S'}^2 \leqslant 0$, we have $4 - d > 0$, and since $d \geqslant 2g - 1 = 3$, we conclude that $d = 3$. But if $d = 3$, the map defined by $|H|$ cannot be birational because curves of degree 3 have genus at most 1. So $g = 2$ does not occur.

If $g = 3$, by Proposition 5.10, (vi), we have $(K_{S'} + H)^2 \geqslant 2$, yielding, as above, that d can only be 5 or 6, hence also $(K_{S'} + H)^2 \leqslant 3$. Since $h^0(S', K_{S'} + H) = g - q = 2$, $|K_{S'} + H|$ has base points and so there is a curve θ as in Proposition 5.4.

Now, $(K_{S'}+H+\theta)^2=(K_{S'}+H)^2+1$. Moreover, $4=(K_{S'}+H)\cdot H=(K_{S'}+H+\theta)\cdot H$. Then, by the index theorem, $H^2(K_{S'}+H+\theta)^2-16\leqslant 0$, and we are left with the only possibility $d=5$, $(K_{S'}+H)^2=2$. In this case, from $2=(K_{S'}+H)^2=K_{S'}^2+4g-4-d=K_{S'}^2+3$, we get $K_{S'}^2=-1$. Finally we have $h^0(S',H)=4$, because curves of degree 5 in $\mathbf{P}^n$ with $n>2$ have genus smaller than 3 by Castelnuovo's bound. □

Therefore, it only remains to prove that the sole possibility left open by Lemma 6.6 in fact does not occur.

Proof of Theorem 6.5. By Proposition 6.4 and Lemma 6.6, it is enough to show that there is no surface S with $q=1$, $g=3$, $d=5$, $h^0(S',H)=4$, $(K_{S'}+H)^2=2$, and $K_{S'}^2=-1$. So let us assume from now on, by contradiction, that $q=1$, $g=3$, $d=5$, $h^0(S',H)=4$, $(K_{S'}+H)^2=2$, $K_{S'}^2=-1$. As we have seen in the proof of Proposition 6.6, $|K_{S'}+H|$ has base points and so there is an irreducible curve θ of genus 1 with $\theta^2=-1$ and $H\cdot\theta=0$, as in Proposition 5.4.

Consider the linear system $|K_{S'}+H+\theta|$. Look at the exact sequence

$$0\to\mathcal{O}_{S'}(K_{S'}+H)\longrightarrow\mathcal{O}_{S'}(K_{S'}+H+\theta)\longrightarrow\mathcal{O}_\theta(K_{S'}+H+\theta)\longrightarrow 0.$$

We have $h^0(S',K_{S'}+H)=g-q=2$ and $h^1(S',K_{S'}+H)=0$. Moreover, $(K_{S'}+H+\theta)|_\theta$ is trivial on θ (it is $K_\theta+H|_\theta$), hence $h^0(\theta,\mathcal{O}_\theta(K_{S'}+H+\theta))=1$. So we have $h^0(S',K_{S'}+H+\theta)=3$, and θ is not a component of the fixed part of $|K_{S'}+H+\theta|$ (if any). Note that $(K_{S'}+H+\theta)^2=3$ and $K_{S'}\cdot(K_{S'}+H+\theta)=-1$.

We are going to see that the general curve in $|K_{S'}+H+\theta|$ is irreducible.

Claim 1) *$|K_{S'}+H+\theta|$ is not composed with a pencil.*

Suppose otherwise. Then we can write $|K_{S'}+H+\theta|=\alpha P+Z$ where $\{P\}$ is a pencil, $\alpha\geqslant 2$ (because $h^0(S',K_{S'}+H+\theta)=3$) and Z (possibly zero) is the fixed part.

From $(K_{S'}+H+\theta)\cdot H=4$, we have $H\cdot P\leqslant 2$. Since $(K_{S'}+H)^2>0$, $\mu\geqslant 3$ by Lemma 6.1, and so $\{P\}$ is not the Albanese pencil.

But then the general curve $P\in\{P\}$ has arithmetic genus $\geqslant 1$, and so $H\cdot P\leqslant 2$ gives a contradiction to the fact that the map defined by $|H|$ is birational.

So $|K_{S'}+H+\theta|$ is not composed with a pencil and the claim is proven.

Claim 2) *The general curve in $|K_{S'}+H+\theta|$ is irreducible.*

Suppose otherwise. Then there is a fixed divisorial part Z and we can write $|K_{S'}+H+\theta|=|M|+Z$ where M is the moving part. Since, by Claim 1), $|K_{S'}+H+\theta|$ is not composed with a pencil, the general curve in $|M|$ is irreducible. Note that, since S' is not birational to $\mathbb{P}^2$, one has $M^2\geqslant 2$, since $\dim(|M|)=\dim(|K_{S'}+H+\theta|)=2$.

Remark now that, since $K_{S'}+H$ is nef and $(K_{S'}+H+\theta)\cdot\theta=0$, also $K_{S'}+H+\theta$ is nef. Moreover, $(K_{S'}+H+\theta)^2=3$, hence $K_{S'}+H+\theta=M+Z$ is also big, and therefore 1-connected by Lemma 3.5. Thus, $M\cdot Z\geqslant 1$, and $(M+Z)\cdot Z\geqslant 0$. Since $(M+Z)^2=(K_{S'}+H+\theta)^2=3$, the only possibility is $M^2=2, M\cdot Z=1, Z^2=-1$, and $(K_{S'}+H+\theta)\cdot Z=0$.

Since, as we saw above, θ is not a component of Z (the restriction map $H^0(S',K_{S'}+H+\theta)\to H^0(\theta,\mathcal{O}_\theta)$ is surjective), one has $\theta\cdot Z\geqslant 0$. Then from $H\cdot Z\geqslant 0$ and $(K_{S'}+H+\theta)\cdot Z=0$, we conclude that $K_{S'}\cdot Z\leqslant 0$. From $M\cdot Z=1$ we know that Z is 1-connected hence, from the adjunction formula, we obtain $K_{S'}\cdot Z=-1$. So Z is a (-1)–divisor, and in fact it is a single (-1)-curve because $K_{S'}^2=-1$. Hence, because $|H|$ does not contract (-1)-curves, one has $H\cdot Z>0$. Therefore, from $H\cdot(M+Z)=H\cdot(K_{S'}+H+\theta)=4$, we have $H\cdot M\leqslant 3$.

On the other hand, from $K_{S'} \cdot (M + Z) = K_{S'} \cdot (K_{S'} + H + \theta) = -1$ and $K_{S'} \cdot Z = -1$, we obtain $K_{S'} \cdot M = 0$, implying that $p_a(M) = 2$. This is in contradiction with $H \cdot M \leqslant 3$ because $|H|$ defines a birational map. Hence $Z = 0$, and Claim 2) is proven.

From Claims 1) and 2), the general curve M in $|K_{S'} + H + \theta|$ is irreducible; moreover, as we have seen, $M^2 = 3$ and $K_{S'} \cdot M = -1$, hence $p_a(M) = 2$.

Since $H \cdot M = H \cdot (K_{S'} + H + \theta) = 4$, the images in S of the curves in $|K_{S'} + H + \theta|$ must be plane quartics of genus 2, and the residual with respect to $|H|$ are lines. So S would contain infinitely many lines, which is not possible. ☐

Examples 5.8 and 5.9 with $g = 3$ show that there exist surfaces S, S' with H and H as in Section 2.1 except that $d = 2g - 2$, with $q = 1$, $g = 3$, $d = 4$, $h^0(S', H) = 4$, $(K_{S'} + H)^2 = 2$, $K_{S'}^2 = -1$.

6.3. **Empty biadjoint system.** In this section, we consider the case when $q > 0$, and $h^0(S', 2K_{S'} + H) = 0$. Observe that if $(K_{S'} + H)^2 = 0$ then, by Lemma 6.1, $|K_{S'} + H|$ is composed with the Albanese pencil and thus $h^0(S', 2K_{S'} + H) = 0$. We will concentrate on the case when $(K_{S'} + H)^2 > 0$.

Proposition 6.7. *Let S, S', H be as in Section 2.1, and assume that $q > 0$, $h^0(S', K_{S'}+H) > 0$, $h^0(S', 2K_{S'} + H) = 0$, and $(K_{S'} + H)^2 \geqslant 5$. Then $\mu = 3$. Moreover, any curve in $|K_{S'} + H|$ consists of a smooth, irreducible, curve of genus q, plus possibly curves contained in the fibres of the Albanese map.*

Proof. First note that $K_{S'} + H$ is nef by Proposition 5.1, (i), hence also big since we assume $(K_{S'}+H)^2 \geqslant 5$. By the main theorem 2.1 of [3], since $h^0(S', 2K_{S'}+H) = 0$ and $(K_{S'}+H)^2 \geqslant 5$, if x is a general point of S', there is an effective divisor D containing x such that

$$-1 \leqslant (K_{S'} + H) \cdot D - 1 \leqslant D^2 \leqslant 0.$$

— If $D^2 = -1$, then $(K_{S'} + H) \cdot D = 0$. We claim that D is 1-connected: let A and B be non-zero, effective divisors, such that $D = A+B$. Since $K_{S'}+H$ is nef, we have $(K_{S'}+H)\cdot A = (K_{S'} + H) \cdot B = 0$ hence, by the index theorem, $A^2 < 0$ and $B^2 < 0$. Then $(A + B)^2 = -1$ implies $A \cdot B > 0$, which proves the claim. Now, since $H \cdot D \geqslant 0$, one has $K_{S'} \cdot D \leqslant 0$, and thus D is a (-1)-divisor. Since D contains a general point of S', this is impossible.
— If $D^2 = 0$, then $(K_{S'} + H) \cdot D \leqslant 1$; In fact, $(K_{S'} + H) \cdot D = 1$, since $(K_{S'} + H) \cdot D > 0$ by the index theorem. Besides, since D passes through a general point, one has $H \cdot D \geqslant 1$, hence $K_{S'} \cdot D \leqslant 0$. If $K_{S'} \cdot D = 0$, then $H \cdot D = 1$. Let D_0 be an irreducible component of D that is movable. Then $H \cdot D_0 \geqslant 1$ and, since H is nef, in fact $H \cdot D_0 = 1$, and thus S is a scroll, in contradiction with $h^0(K_{S'} + H) > 0$. Otherwise, $K_{S'} \cdot D < 0$. Since D is 1-connected by Lemma 3.7, $K_{S'} \cdot D = -2$, and D is rational, hence it moves in the Albanese pencil $\{G\}$. Then, $H \cdot D = 1 - K_{S'} \cdot D = 3$ implies that $\mu = H \cdot G = 3$, as we wanted to show.

Let A be a member of $|K_{S'} + H|$. One has $A \cdot G = H \cdot G + K \cdot G = 1$. Since G is nef, there is one irreducible component A_0 of A such that $A_0 \cdot G = 1$, and thus A_0 is smooth, irreducible, of genus q, whereas $A_i \cdot G = 0$ for any other component of A. This proves the assertion. ☐

Remark 6.8. In the situation of Proposition 6.7, one can have $(K_{S'} + H)^2 < 5$ only if $q = 1$, and either $g = 5$ and $(K_{S'} + H)^2 = 4$, or $g = 4$ and $(K_{S'} + H)^2 = 3$.

Indeed, assume $(K_{S'}+H)^2>0, q>0$, $h^0(S',K_{S'}+H)>0$ and $h^0(S',2K_{S'}+H)=0$ and $(K_{S'}+H)^2<5$. By Proposition 5.10, (iv) and (vi), we have $(K_{S'}+H)^2=g-2+q$ and by Lemma 6.3, $g\geqslant 9q-7$, i.e., $g+q\geqslant 10q-7$. Now $(K_{S'}+H)^2<5$ yields $g+q<7$, so $(K_{S'}+H)^2<5$ can only occur if $q=1$. Since $g-q\geqslant 3$ and $g<7-q=6$ we have only the possibilities $g=5$, $(K_{S'}+H)^2=4$ or $g=4$, $(K_{S'}+H)^2=3$.

We shall use the following lemma to give an application to Proposition 6.7 above.

Lemma 6.9. *Let S,S',H be as in Section 2.1, and assume that $q>0$, and $\mu=3$. If $H^2\geqslant 10$ and $h^0(S',K_{S'}+H)>0$, then $|K_{S'}+H|$ has no base points.*

Proof. We prove the contrapositive. By Proposition 5.4, if $K+H$ has base points, then there is a genus q curve θ such that $\theta^2=-1$, $\theta\cdot H=0$. Let $\{G\}$ be the Albanese pencil, and let $\alpha=\theta\cdot G$. Since θ has genus q, one has $\alpha\geqslant 1$. Then the divisor $B:=G+\theta$ satisfies $B^2=2\alpha-1\geqslant 1$ and $H\cdot B=\mu$. By the index theorem, one has $H^2B^2\leqslant(H\cdot B)^2=\mu^2$, hence $H^2\leqslant 9$. □

Corollary 6.10. *Let S,S',H be as in Section 2.1, and assume that $q>0$, $h^0(S',K_{S'}+H)>0$, and $h^0(S',2K_{S'}+H)=0$. If $(K_{S'}+H)^2>0$ and $H^2\geqslant 10$, then the image of S' by the adjoint series $|K_{S'}+H|$ has degree $g-2+q$ in $\mathbf{P}^{g-q-1}$.*

Let us point out that $g-2+q$ is the maximal possible degree for a surface in $\mathbf{P}^{g-q-1}$ with sectional genus q equal to the irregularity, see Proposition 3.3, (iv).

Proof. By Proposition 5.11, the linear system $|K_{S'}+H|$ defines a birational map; moreover, by Proposition 6.7 and Lemma 6.9, it is base-point-free. Therefore, the image of S' by this map is a degree $(K_{S'}+H)^2$ surface in $\mathbf{P}^{g-q-1}$ (see Proposition 3.3, (i)). By Proposition 5.10, (iii), it has sectional genus q, and by Proposition 5.10, (vi), it has degree $g-2+q$. □

7. An extension of a theorem of C. Segre

In this section we continue our study of irrational surfaces as in Section 2.1 (in fact we consider slightly more restrictive hypotheses). The main result of this section is the following generalization of C. Segre's classical theorem [7, Thm. 2.3] mentioned in the introduction. We shall apply it in particular to get Corollary 7.6.

Theorem 7.1. *Let $C\subseteq\mathbf{P}^r$ be a (smooth) linearly normal, non–degenerate, projective curve of genus g and degree $d\geqslant 2g+5$ (so $r=d-g$). If C is a hyperplane section of an irregular surface $S\subseteq\mathbf{P}^n$, with $n=r+1$, ruled in conics, then S is either the 2-Veronese re-embedding of a cone or a simple internal projection thereof.*

Let $S\subseteq\mathbf{P}^n$ be a surface having C as a hyperplane section as in the theorem. It satisfies the assumptions of Section 2.1, and we will use the notation introduced there. In particular, we let $\pi:S'\to S$ be the minimal desingularization of S. By abuse of notation we will denote by C the proper transform of C on S', that is isomorphic to C. By assumption, the fibres of the Albanese map $S'\to\Gamma$ are mapped by π to conics sweeping out S.

We let $\gamma:S''\to\Gamma$ be a relative minimal model of S' so that there is a birational morphism $h:S'\longrightarrow S''$ such that $\mathrm{alb}=\gamma\circ h$. Let $\bar{C}$ be the image of C via h.

Lemma 7.2. *The curve $\bar{C}$ is smooth.*

Proof. To prove this it suffices to show that C, which intersects positively any (-1)–curve, intersects any (-1)–curve in exactly one point. Let E be a (-1)–curve. It is contained in a fibre of alb. On the other hand any reducible fibre of alb either contains only one (-1)–curve with multiplicity 2 met by C in one point, or it contains exactly two distinct (-1)–curves that are met by C in one point (remember that the intersection number of C with the fibres of alb is 2). Indeed, since C intersects any (-1)–curve positively, and since the intersection number of C with the fibres of alb is 2, any such fibre cannot contain more than two distinct (-1)–curves, and if it contains a (-1)–curve with multiplicity, it contains only that (-1)–curve with multiplicity 2. To finish the proof it suffices to show that a fibre of alb cannot contain only one (-1)–curve with multiplicity 1. In fact if E is such a (-1)–curve, then its residual with respect to the fibre of alb in which it sits is a (-1)–divisor by Zariski's lemma, and therefore it must contain another (-1)–curve. □

Next, S is the image of S'' via a linear subsystem of $|\bar{C}|$ which may have some simple base points. In that case, we can replace S with the image of S'' via $\varphi_{|\bar{C}|}$, of which S will be an internal projection. Note that the sectional genus g is not affected by this operation, whereas the degree may increase, but the hyperplane sections stay linearly normal. So from now on we may and will assume that S is the image of S'' via the complete linear system $|\bar{C}|$.

Write $S'' = \mathbf{P}(\mathcal{E})$, where $\mathcal{E}$ is a normalized rank 2 vector bundle of degree $-e$. There exists an invertible sheaf $\mathcal{L}$ on Γ and an exact sequence

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{L} \to 0$$

with $\deg(\mathcal{L}) = \deg(\mathcal{E}) = -e$ (see [11, p. 372, proof of Prop. 2.8]). Since C is a hyperplane section of S, and the latter is swept out by conics, there exists an invertible sheaf $\mathcal{A}$ on Γ such that

$$\bar{C} \sim 2E + \gamma^*(A),$$

where $E \in |\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$ and $A \in |\mathcal{A}|$. Let α be the degree of $\mathcal{A}$.

Lemma 7.3. *The following two relations hold,*

$$g = 2q - 1 + \alpha - e$$
$$d = 4g + 4 - 8q.$$

Proof. We have

$$K_{S''} \sim -2E + \gamma^*(K_\Gamma + \mathcal{L})$$

(see [11, p. 373, Lemma 2.10]), hence

$$K_{S''} \equiv -2E + (2q - 2 - e)G \quad \text{and} \quad \bar{C} \equiv 2E + \alpha G,$$

where G denotes the numerical equivalence class of the fibres of γ. Then one computes

$$2g - 2 = (K_{S''} + \bar{C}) \cdot \bar{C} = 2(2q - 2 - e + \alpha)$$

and

$$d = \bar{C}^2 = 4(\alpha - e),$$

as wanted. □

The following observation is the keystone of our proof of Theorem 7.1.

Lemma 7.4. *The following relation holds:*

$$d - 2(g-1) = 2\big[\alpha - e - 2(q-1)\big].$$

Thus,

$$d \geqslant 2g+5 \quad \Longleftrightarrow \quad \alpha - e \geqslant 2q+2.$$

Proof. This can be proved by using the two relations stated in the previous lemma. Yet we find it more satisfactory to observe that the quantity on the left-hand-side is $-K_{S''} \cdot \bar{C}$, and then

$$\begin{aligned} d - 2(g-1) = -K_{S''} \cdot \bar{C} &= \big(2E - (2q-2-e)F\big) \cdot \big(2E + \alpha F\big) \\ &= 2(\alpha - e) - 2(2q-2). \end{aligned}$$

□

Proof of Theorem 7.1. Since $C \subseteq \mathbf{P}^r$ is linearly normal, one has

$$\begin{aligned} h^0(S'', \mathcal{O}_{S''}(\bar{C})) - 1 &= d - g + 1 \\ &= 3(\alpha - e) - 2(q-1). \end{aligned}$$

For $i = 0, 1$, one has

$$H^i(S'', \mathcal{O}_{S''}(\bar{C})) \cong H^i(\Gamma, \gamma_*(\mathcal{O}_{S''}(\bar{C})) = H^i(\Gamma, (\mathrm{Sym}^2(\mathcal{E})) \otimes \mathcal{A}).$$

Moreover, there exists a locally free sheaf $\mathcal{Q}$ such that we have two exact sequences

$$0 \to \mathcal{O} \to \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{Q} \to 0 \qquad \text{and} \qquad 0 \to \mathcal{L} \to \mathcal{Q} \to \mathcal{L}^{\otimes 2} \to 0,$$

hence also

$$0 \to \mathcal{A} \to \mathrm{Sym}^2(\mathcal{E}) \otimes \mathcal{A} \to \mathcal{Q} \otimes \mathcal{A} \to 0 \qquad \text{and} \qquad 0 \to \mathcal{L} \otimes \mathcal{A} \to \mathcal{Q} \otimes \mathcal{A} \to \mathcal{L}^{\otimes 2} \otimes \mathcal{A} \to 0.$$

One has

$$\begin{aligned} \deg(\mathcal{A}) &= \alpha \\ \deg(\mathcal{A} \otimes \mathcal{L}) &= \alpha - e \\ \deg(\mathcal{A} \otimes \mathcal{L}^{\otimes 2}) &= \alpha - 2e. \end{aligned}$$

Since we are assuming $d \geqslant 2g+5$, we have $\alpha - e \geqslant 2q+2$ by Lemma 7.4, hence $\mathcal{A} \otimes \mathcal{L}$ is non-special. Taking this into account, we have the two exact sequences

$$0 \to H^0(\Gamma, \mathcal{A}) \to H^0(\Gamma, \mathrm{Sym}^2(\mathcal{E}) \otimes \mathcal{A}) \to H^0(\Gamma, \mathcal{Q} \otimes \mathcal{A}) \to H^1(\Gamma, \mathcal{A})$$

and

$$0 \to H^0(\Gamma, \mathcal{L} \otimes \mathcal{A}) \to H^0(\Gamma, \mathcal{Q} \otimes \mathcal{A}) \to H^0(\Gamma, \mathcal{L}^{\otimes 2} \otimes \mathcal{A}) \to 0,$$

so that

$$h^0(S'', \mathcal{O}_{S''}(\bar{C})) = h^0(\Gamma, \mathrm{Sym}^2(\mathcal{E}) \otimes \mathcal{A}) \leqslant h^0(\Gamma, \mathcal{A}) + h^0(\Gamma, \mathcal{A} \otimes \mathcal{L}) + h^0(\Gamma, \mathcal{A} \otimes \mathcal{L}^{\otimes 2}),$$

with equality holding if $H^1(\Gamma, \mathcal{A}) = 0$.

Let us first assume that $e \geqslant 0$. Then $\alpha = \deg(\mathcal{A})$ is larger than $\deg(\mathcal{A} \otimes \mathcal{L})$, hence $\mathcal{A}$ is non-special as well, and thus

$$h^0(\Gamma, \mathcal{A}) + h^0(\Gamma, \mathcal{A} \otimes \mathcal{L}) + h^0(\Gamma, \mathcal{A} \otimes \mathcal{L}^{\otimes 2}) = 3(\alpha - e) - 3(q-1) + i,$$

with

$$i = h^1(\Gamma, \mathcal{A} \otimes \mathcal{L}^{\otimes 2}).$$

Therefore the condition that C is linearly normal implies $i \geqslant q$, hence $i = q$ and $\mathcal{A} \otimes \mathcal{L}^{\otimes 2} = \mathcal{O}_\Gamma$, equivalently $\mathcal{A} = \mathcal{L}^{\otimes -2}$, and in particular $\alpha = 2e$. Then the curve $\bar{C}$ is a member of the linear system $|\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(2)|$, where $\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1)$ is defined relatively to the vector bundle $\mathcal{E}' = \mathcal{E} \otimes \mathcal{L}^{-1}$ and, of course, $S'' = \mathbf{P}(\mathcal{E}')$. One has

$$\deg(\mathcal{E}') = -\deg(\mathcal{L}) = e = \alpha - e \geqslant 2q + 2$$

by Lemma 7.4. Then, by [6, Lemma 3.5], $\mathcal{E}'$ splits as $\mathcal{L}^{-1} \oplus \mathcal{O}_\Gamma$ if $h^1(S'', \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1)) \geqslant q$.

To prove this inequality, we will relate $H^1(S'', \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1))$ to $H^1(S'', \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(2))$. Since $\bar{C} \in |\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(2)|$ is linearly normal and $\mathcal{O}_{S''}(\bar{C})\big|_{\bar{C}}$ is non-special, the restriction exact sequence of $\bar{C} \subseteq S''$ gives an isomorphism

$$H^1(S'', \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(2)) = H^1(S'', \mathcal{O}_{S''}(\bar{C})) \cong H^1(\mathcal{O}_{S''}) = q.$$

Now, consider a general member D of the linear system $|\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1)|$; since the vector bundle $\mathcal{E}'$ is positive enough, D is a smooth curve isomorphic to Γ, and $\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(2)\big|_D$ is non-special. Therefore, it follows from the restriction exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1) \to \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(2) \to \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(2)\big|_D \to 0$$

that

$$h^1(S'', \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1)) \geqslant h^1(S'', \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(2)).$$

We thus conclude by [6, Lemma 3.5] that $\mathcal{E}' = \mathcal{L}^{-1} \oplus \mathcal{O}_\Gamma$, and S'' is mapped by the linear system $|\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1)|$ to the cone over Γ in its embedding defined by $|\mathcal{L}^{-1}|$. The conclusion follows, since $\bar{C}$ is a member of $|\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(2)|$.

It remains to explain how to adapt these arguments when $e < 0$. In this case $\mathcal{A} \otimes \mathcal{L}$ and $\mathcal{A} \otimes \mathcal{L}^{\otimes 2}$ are non-special, and then the linear normality of C implies in the same way as above that $h^1(\Gamma, \mathcal{A}) = q$, hence $\mathcal{A} = \mathcal{O}_\Gamma$ and $\alpha = 0$. Thus, $\bar{C}$ is a member of $|\mathcal{O}_{\mathbf{P}(\mathcal{E})}(2)|$. One has

$$\deg(\mathcal{E}) = -e = \alpha - e \geqslant 2q + 2,$$

and $h^1(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) \geqslant q$ by the exact same argument as above. It therefore follows yet again from [6, Lemma 3.5] that $\mathcal{E}$ is split, which contradicts the assumptions that $\mathcal{E}$ is normalized and $e < 0$. □

Conversely, virtually every curve which is a double cover has a linearly normal projective model which is a hyperplane section of a surface ruled in conics, as the following example shows.

Example 7.5. Let Γ be a smooth curve of genus q, and $\pi : C \to \Gamma$ a smooth double cover of genus g, with branch divisor $B \subseteq \Gamma$. Correspondingly, there exists a line bundle $\mathcal{G}$ on Γ such that $\mathcal{G}^{\otimes 2} = \mathcal{O}_\Gamma(B)$. Then C is a divisor in the surface $S = \mathbf{P}(\mathcal{O}_\Gamma \oplus \mathcal{G})$, member of the linear system $|\mathcal{O}_{\mathbf{P}(\mathcal{O}_\Gamma \oplus \mathcal{G})}(2)|$, with normal bundle $\mathcal{O}_S(C)|_C = \pi^*(\mathcal{G})^{\otimes 2} = \pi^*(\mathcal{O}_\Gamma(B))$. If $\deg(\mathcal{G}) > 2q - 2$, the linear system $|\mathcal{O}_{\mathbf{P}(\mathcal{O}_\Gamma \oplus \mathcal{G})}(1)|$ maps S to a cone over Γ, and C in its embedding defined by $|\pi^*(\mathcal{O}_\Gamma(B))|$ is linearly normal, and a hyperplane section of the 2-Veronese re-embedding of this cone; this is a particular case of Example 7.7 below.

Using Theorem 7.1 and Lemma 6.2 together, one obtains the following.

Corollary 7.6. *Let S be an irreducible and non-degenerate, linearly normal, irregular surface of degree d in $\mathbf{P}^n$. Assume that the hyperplane sections of S are smooth, linearly normal, of genus $g \geqslant 7$. If $d > 3g-3$, then S is either the 2-Veronese re-embedding of a cone or a simple internal projection thereof.*

(Note that $3g-2 \geqslant 2g+5$ if and only if $g \geqslant 7$).

If we drop the assumption that $d > 3g-3$, then there are examples with $\mu > 2$ of irregular surfaces with linearly normal hyperplane sections (compare with Lemma 6.2), where μ is as defined at the beginning of Section 6. Moreover, some of these examples have empty bi-adjoint system (compare with Section 6.3). We check below that certain 3-Veronese re-embeddings of cones provide such examples.

Example 7.7. Let us consider μ-Veronese re-embeddings of cones, for arbitrary $\mu > 1$. Let Γ be a smooth curve of genus $q > 0$, and let $\mathcal{G}$ be a line bundle on Γ of degree $e > 2q-2$. Let $S = \mathbf{P}(\mathcal{O}_\Gamma \oplus \mathcal{G})$, and consider a smooth curve $C \subseteq S$ defined by a section of $\mathcal{O}_{\mathbf{P}(\mathcal{O}_\Gamma \oplus \mathcal{G})}(\mu)$. Beware that in general a μ-tuple cover of Γ may not be realized in this way.

We shall see the following:
(i) the projective curve image of C by the linear system $|\mathcal{O}_{\mathbf{P}(\mathcal{O}_\Gamma \oplus \mathcal{G})}(\mu)|$ is linearly normal; in other words, the hyperplane sections of the μ-Veronese re-embedding of the cone over $(\Gamma, \mathcal{G})$ (i.e., the projective curve image of Γ by $|\mathcal{G}|$) are linearly normal;
(ii) for all m such that $2m > \mu$, the m-adjoint linear system $|mK_{S'} + C|$ is empty;
(iii) one has $d > 2g-2$, where $d = C^2$ and g is the genus of C.

Let us first compute the genus g of C. For numerical equivalence, one has

$$C \equiv \mu D \quad \text{and} \quad K_S \equiv -2D + (2q-2+e)F,$$

where D is the divisor class of $\mathcal{O}_{\mathbf{P}(\mathcal{O}_\Gamma \oplus \mathcal{G})}(1)$ (hence $D^2 = e$), and F is the numerical class of the fibres of $S \to \Gamma$. Thus,

$$\begin{aligned} 2g-2 &= (K_S + C) \cdot C \\ &= \mu D \cdot \big[(\mu-2)D + (2q-2+e)F\big] \\ &= \mu\,\big[2(q-1) + (\mu-1)e\big]. \end{aligned} \tag{7.1}$$

On the other hand the degree d of C in the embedding defined by $|\mathcal{O}_S(C)|$ is $C^2 = e\mu^2$, and thus

$$\begin{aligned} d > 2g-2 &\iff \mu e > 2(q-1) + (\mu-1)e \\ &\iff e > 2q-2, \end{aligned}$$

which proves (iii). As a remark, note that

$$\begin{aligned} d > 3g-3 &\iff 2e\mu > 3\,\big[2(q-1) + (\mu-1)e\big] \\ &\iff (\mu-3)e < -6(q-1), \end{aligned}$$

so, since $q > 0$, one has $\mu \leqslant 2$ if $d > 3g-3$, as predicted by Lemma 6.2.

One has

$$(C + mK_S) \cdot F = \mu - 2m,$$

which proves (ii).

Finally, let us prove (i). Since $e > 2q-2$, all positive multiples of $\mathcal{G}$ are non-special, hence

$$\begin{aligned} h^0(S, \mathcal{O}_{\mathbf{P}(\mathcal{O}_\Gamma \oplus \mathcal{G})}(\mu)) &= \sum_{0 \leqslant k \leqslant \mu} h^0(\Gamma, \mathcal{G}^{\otimes k}) \\ &= 1 + \frac{1}{2}\mu(\mu+1)e + \mu(1-q). \end{aligned}$$

On the other hand, if $\pi : C \to \Gamma$ is the natural morphism, $\pi^*(\mathcal{G})^{\otimes \mu}$ is non-special as well, because

$$\deg(\pi^*(\mathcal{G})^{\otimes \mu}) = e\mu^2 = d > 2g-2.$$

Therefore

$$h^0(C, \pi^*(\mathcal{G})^{\otimes \mu}) = \mu^2 e - g + 1,$$

and

$$\begin{aligned} h^0(C, \pi^*(\mathcal{G})^{\otimes \mu}) = h^0(S, \mathcal{O}_{\mathbf{P}(\mathcal{O}_\Gamma \otimes \mathcal{G})}(\mu)) - 1 &\iff \mu(\mu+1)e + 2\mu(1-q) = 2\mu^2 e - (2g-2) \\ &\iff 2g-2 = \mu(\mu-1)e + 2\mu(q-1). \end{aligned}$$

Since the last equality indeed holds, see (7.1), C is linearly normal as we wanted.

8. The rational case: first results

In this section we consider the case when S is rational and $d \geqslant n$.

Proposition 8.1. *Let S, S', H be as in Section 2.1, and assume that S is rational, and $d \geqslant n$ (i.e., $a \geqslant 0$). One has:*
(i) the hyperplane sections of S are linearly normal, $g = a+1 = h^0(K_{S'}+H)$, and S is not a scroll;
(ii) if $a = 0$ (equivalently $g = 1$), then $K_{S'} + H \sim 0$;
(iii) if $a = 1$ (equivalently $g = 2$), then $|K_{S'}+H|$ is a base-point-free pencil of rational curves, hence $(K_{S'}+H)^2 = 0$ and $h^0(S', 2K_{S'}+H) = 0$;
(iv) if $|K_{S'}+H|$ is composed with a pencil $|G|$, then $G^2 = 0$ and $K_{S'}+H \sim (g-1)G$, hence $(K_{S'}+H)^2 = 0$, $K_{S'} \cdot G = -2$, and $H \cdot G = 2$; thus S is ruled by conics, and the curves in $|H|$ are hyperelliptic; moreover, $h^0(S', 2K_{S'}+H) = 0$;
(v) if $(K_{S'}+H)^2 > 0$, then $|K_{S'}+H|$ is not composed with a pencil, $a \geqslant 2$ (equivalently $g \geqslant 3$), and the morphism $\varphi_{|K_{S'}+H|}$ maps S' onto a non-degenerate surface in $\mathbf{P}^{g-1}$, and $K_{S'}^2 \geqslant n-2g+1$ (or equivalently $(K_{S'}+H)^2 \geqslant g-2 = a-1$); moreover, if equality holds, then the morphism $\varphi_{|K_{S'}+H|}$ is birational onto its image, which is a surface S'' of minimal degree $g-2 = a-1$ in $\mathbf{P}^{g-1}$, and $h^0(S', 2K_{S'}+H) = 0$. In addition, in this case, if S'' is neither $\mathbf{P}^2$ nor the Veronese surface $V_2 \subseteq \mathbf{P}^5$, then S has a 1-dimensional family of rational cubic curves.

Proof. (i) follows from Proposition 3.3, (i), Proposition 3.3, (ii) and (vi), and Remark 4.4.

(ii) follows from the fact that $h^0(K_{S'}+H) = g = 1$, together with Proposition 5.6.

If $a = 1$, then $h^0(K_{S'}+H) = g = 2$ hence $|K_{S'}+H|$ is a pencil. It is base-point-free by Proposition 5.6, so $(K_{S'}+H)^2 = 0$. Similarly, if $|K_{S'}+H|$ is composed with a pencil $|G|$, since it is base-point-free, we have $G^2 = 0$; moreover, $K_{S'}+H \sim (g-1)G$, because $h^0(K_{S'}+H) = g$. In all cases (setting $G = K_{S'}+H$ if $g = 2$), $0 = G \cdot (K_{S'}+H) = G \cdot K_{S'} + G \cdot H > G \cdot K_{S'}$. Then $G \cdot K_{S'} = -2$, which implies that the curves in $|G|$ are rational, and $H \cdot G = 2$. Finally, $(2K_{S'}+H) \cdot G = -2$, so $2K_{S'}+H$ is not effective. This proves both (iii) and (iv).

(v) The image of $\varphi_{|K_{S'}+H|}$ is a non–degenerate surface in $\mathbf{P}^{g-1}$. By Proposition 5.10, (vi), $(K_{S'}+H)^2 \geqslant g-2$. If equality holds, then $\varphi_{|K_{S'}+H|}$ is birational onto its image, which is a surface S'' of minimal degree $g-2$, $p_a(K_{S'}+H) = 0$ and, by (iv) of Proposition 5.10, $h^0(S', 2K_{S'}+H) = 0$. If S'' is neither $\mathbf{P}^2$ nor the Veronese surface $V_2 \subseteq \mathbf{P}^5$, then it is a scroll, and the family of rational cubics on S corresponds to the ruling of S''. □

Proposition 8.2. *Let S, S', H be as in Section 2.1, and assume that S is rational, $g \geqslant 1$, and $(K_{S'}+H)^2 > 0$. Then:*
(i) S is not ruled by conics;
(ii) $0 \leqslant p_a(K_{S'}+H) = K_{S'}^2 - n + 2a + 1$, and $h^0(2K_{S'}+H) = p_a(K_{S'}+H) = K_{S'}^2 - n + 2a + 1$;
(iii) If $(K_{S'}+H)^2 > a-1$ (i.e., $K_{S'}^2 \geqslant n-2a$), then S has no 1-dimensional family of rational curves of degree $\delta < 4$.

Proof. The assumption $g \geqslant 1$ is equivalent to $h^0(S', K_{S'}+H) > 0$ (see Proposition 3.3, (i)).

Assertion (i) follows Lemma 5.3.

Assertion (ii) follows from Proposition 5.10, (iii) and (v).

For assertion (iii): If $(K_{S'}+H)^2 > a-1 = g-2$ then, by Proposition 5.10, (vi), $p_a(K_{S'}+H) = h^0(2K_{S'}+H) > 0$. Suppose that S has a 1-dimensional family of rational curves of degree $\delta < 4$. Let L be the pull back to S' of a general member of the family. Then $H \cdot L \leqslant 3$, and $L^2 \geqslant 0$. Since L is a rational curve, $K_{S'} \cdot L \leqslant -2$ by adjunction. Hence $(2K_{S'}+H) \cdot L < 0$, which is impossible because $h^0(2K_{S'}+H) > 0$ and L is nef. □

9. The rational case: empty biadjoint system

In this section we go on considering the same situation as in Section 8, with some additional assumptions.

Theorem 9.1. *Let S, S', H be as in Section 2.1, and assume that S is rational and $d \geqslant n$, i.e., $a \geqslant 0$. If $h^0(S', 2K_{S'}+H) = 0$, then one of the following cases occurs:*
(a) $g = 1$ and S is a (weak) Del Pezzo surface;
(b) $g = 3$ and S is the Veronese surface V_4, or a simple internal projection thereof;
(c) $g = 6$ and S is the Veronese surface V_5, or a simple internal projection thereof;
(d) S is a surface with hyperelliptic sections, which is either the degree $4g+4$ surface image of $\mathbf{F}_e$ by the linear system $|2E+(g+1+e)F|$, where $0 \leqslant e \leqslant g+1$, or a simple internal projection thereof;
(e) S is a surface with trigonal sections, which is either the degree $3g+6$ surface image of $\mathbf{F}_e$ by a linear system of the form $|3E+(h+e+2)F|$, where $e \geqslant 0$ and $h \geqslant \max\{2e-2, e\}$ are integers such that $g = 2h-e+2$, or a simple internal projection thereof.

Conversely, in all these cases, $h^0(S', 2K_{S'}+H) = 0$.

Proof. By Proposition 5.6, $|K_{S'}+H|$ is base-point-free of dimension $g-1 \geqslant 0$. By Proposition 8.1, (ii) and (iv), if $g = 1$ then we are in case (a), and if $|K_{S'}+H|$ is composed with a pencil then we are in case (d). Else, $(K_{S'}+H)^2 > 0$, and the general curve in $|K_{S'}+H|$ is smooth and irreducible. Since $|2K_{S'}+H|$ is empty, the curves in $|K_{S'}+H|$ have genus 0, hence $(K_{S'}+H)^2 = \dim(|K_{S'}+H|) - 1 = g-2 = a-1$. Consider the morphism

$$\varphi_{|K_{S'}+H|} : S' \longrightarrow S'' \subseteq \mathbf{P}^{g-1}$$

which, by Proposition 8.1, (v), is birational onto its image, which is a surface of minimal degree $g-2$ in $\mathbf{P}^{g-1}$. Hence S'' is either the plane (then $g=3$), or the Veronese surface V_2 in $\mathbf{P}^5$ (then $g=6$), or a rational normal scroll. In any event, we denote by $\phi : X \longrightarrow S''$ the minimal desingularization of S'' and we set $\mathcal{L} = \phi^*(|\mathcal{O}_{S''}(1)|)$. Then we have a birational map $\varphi : S' \dashrightarrow X$, which is a morphism if S'' is not a cone, and $|K_{S'}+H|$ is the pull–back via φ of the linear system $\mathcal{L}$. We have the following cases:
(i) $X = \mathbf{P}^2$ and $\mathcal{L} = |\mathcal{O}_{\mathbf{P}^2}(1)|$ (then $g=3$);
(ii) $X = \mathbf{P}^2$ and $\mathcal{L} = |\mathcal{O}_{\mathbf{P}^2}(2)|$ (then $g=6$);
(iii) $X = \mathbf{F}_e$ and $\mathcal{L} = |\mathcal{O}_{\mathbf{F}_e}(E+hF)|$, with $h \geqslant e$, but not $e=h=1$. Note that $\phi : X \longrightarrow S''$ is an isomorphism and $\varphi = \varphi_{|K_{S'}+H|}$ unless we are in case (iii) and $h=e$ (i.e., if S' is a cone).

In case (i) the general curve $C \in |H|$, that has genus 3, is mapped via φ to a smooth plane curve of degree $2g-2=4$. Hence the surface $S \subseteq \mathbf{P}^n$ is the image of $\mathbf{P}^2$ via a linear system of generically smooth plane quartics, possibly with simple base points, and the system has to be complete under the condition of containing these base points. Thus we are in case (b). In case (ii), by the same argument we end up in case (c).

Suppose we are in case (iii). Then the surface S'' is a rational normal scroll of degree $2h-e$ in $\mathbf{P}^{2h-e+1}$, hence $g = 2h-e+2$ and the general curve $C \in |H|$ is mapped via $\varphi_{|K_{S'}+H|}$ to a canonical curve of degree $2g-2 = 4h-2e+2$. We abuse notation and denote by C the image of C on X. One has $C \cdot F = 3$, thus $C \sim 3E + kF$, with k a suitable integer. Since

$$4h-2e+2 = 2g-2 = C\cdot(E+hF) = (3E+kF)\cdot(E+hF) = k+3h-3e,$$

one has $k = h+e+2$, hence $C \sim 3E+(h+e+2)F$. Since C is nef, one has $C \cdot E \geqslant 0$, which gives $h \geqslant 2e-2$. Moreover

$$C^2 = (3E+(h+e+2)F)^2 = 6h-3e+12.$$

Note that we can be in the cone case only if $e=h=2$ (hence $g=4$). In any event, we are in case (e).

Lastly, the fact that $|2K_{S'}+H|$ is empty in all these cases follows by a direct examination. □

Remark 9.2. In case (b) of Theorem 9.1, let b be the length of the 0–dimensional curvilinear scheme Z from which we make the internal projection of $V_4 \subseteq \mathbf{P}^{14}$. Since $14-b = n \geqslant 3$, one must have $b \leqslant 11$. On the other hand, any curvilinear scheme Z of length $b \leqslant 11$ lying on a smooth irreducible plane quartic gives independent conditions to plane curves of degree 4, so every such scheme is allowed.

In case (c), let again b be the length of the 0–dimensional curvilinear scheme Z from which we make the internal projection of $V_5 \subseteq \mathbf{P}^{20}$. Since $20-b = n \geqslant 3$, one must have $b \leqslant 17$. Any curvilinear scheme Z of length $b \leqslant 14$ lying on a smooth irreducible plane quintic D gives independent conditions to plane curves of degree 5, so every such scheme is allowed. If $15 \leqslant b \leqslant 17$, Z is allowed if and only if there exists a non-special divisor Z' on D such that $Z+Z' \in |\mathcal{O}_D(5)|$, i.e., Z' is not contained in a conic. In this case Z still gives independent conditions to plane curves of degree 5.

10. The rational case: empty triadjoint system

In this section we again consider the situation of Section 8 with some additional assumptions, but the latter are different from the additional assumptions we considered in Section 9.

Setup 10.1. Let S, S', H be as in Section 2.1, and assume that S is rational, and $d \geqslant n$, i.e., $a \geqslant 0$. This time we suppose that the bi-adjoint system $|2K_{S'} + H|$ is non-empty, whereas the tri-adjoint system $|3K_{S'} + H|$ is empty. This latter condition is verified if $d > 3g - 3$ (see Lemma 3.2).

As usual, $|K_{S'} + H|$ is base-point-free by Proposition 5.6; moreover, by Proposition 8.1, since $|2K_{S'} + H|$ is non-empty, $|K_{S'} + H|$ is not composed with a pencil and $(K_{S'} + H)^2 > 0$. In particular, $K_{S'} + H$ is big and nef, and the general curve $M \in |K_{S'} + H|$ is smooth and irreducible. We set

$$|2K_{S'} + H| = \Phi_2 + |M_2|$$

where $|M_2|$ is the (possibly empty) movable part, and Φ_2 is the fixed part. One has $M \cdot \Phi_2 = 0$.

Lemma 10.2. *There exist a (-1)-curve E on S' such that $H \cdot E = 1$ if and only if S is an internal projection.*

Proof. The "if" part is clear, so let us prove the other implication. Let E be a (-1)-curve as in the statement, and set $\tilde{H} = H + E$. The goal is to prove that $h^0(S', \tilde{H}) = h^0(S', H) + 1$. By considering the exact sequence

$$0 \to \mathcal{O}_{S'}(H) \to \mathcal{O}_{S'}(\tilde{H}) \to \mathcal{O}_E(\tilde{H}) \simeq \mathcal{O}_E \to 0,$$

one sees that it suffices to prove that $h^1(S', H) = 0$. Now, this follows from the cohomology sequence associated to the exact sequence

$$0 \to \mathcal{O}_{S'} \to \mathcal{O}_{S'}(H) \to \mathcal{O}_H(H) \to 0,$$

taking into account that $h^1(\mathcal{O}_{S'}) = h^1(\mathcal{O}_H(H)) = 0$. □

We will use the following:

Lemma 10.3. *Let T, X be smooth, irreducible, projective surfaces, and let $\varphi : T \dashrightarrow X$ be a birational map. Let us consider the minimal resolution of the indeterminacies of φ by the following diagram*

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & & \searrow^{f'} & \\ T & & \overset{\varphi}{\dashrightarrow} & & X. \end{array}$$

Let D be a smooth curve on T. Then one has

$$\varphi_*(D + K_T) = \varphi_*(D) + K_X + A, \qquad (10.1)$$

where A is the image via f'_* of some f-exceptional divisor. If $|D|$ is base-point-free, one has $\varphi_*(D) \cdot A = 0$.

Proof. This follows from [5, Formula (13)]. □

Theorem 10.4. *Consider Setup 10.1, and assume that $|M_2|$ is empty. Then:*
(a) if $g \geqslant 4$, then S is the image by the Veronese map v_2 of a (weak) Del Pezzo surface of degree $g - 1$ in $\mathbf{P}^{g-1}$, or a simple internal projection thereof, as in [7, Ex. 3.3(a)]. *In this case one has $g \leqslant 10$;*
(b) if $g = 3$, then S can be obtained as the image of the plane via the map determined by a linear system of the form $(6; 2^7)$, possibly with some further simple base points.

Proof. We assume without loss of generality that S is not an internal projection. By Lemma 10.2, this tells us that for every (-1)-curve E on S', $H \cdot E > 1$, hence $M \cdot E > 0$. Since $\dim(|2K_{S'} + H|) = 0$, the curves in $|M|$ have arithmetic genus 1, and $\dim(|M|) = g - 1 = M^2$.

(a) Assume that $g \geqslant 4$. Then the map defined by $|M|$ is birational. Since moreover there are no (-1)-curves E such that $M \cdot E = 0$, we can apply Proposition 5.6 to $|M|$, and conclude that $|K_{S'} + M| = |2K_{S'} + H|$ has no base points, so that $\Phi_2 = 0$, and therefore $2K_{S'} + H \sim 0$, i.e., $H \sim -2K_{S'}$. Then $2g - 2 = 2K_{S'}^2$, so that $g = K_{S'}^2 + 1 \leqslant 10$ and, since $g \geqslant 4$, one has $K_{S'}^2 \geqslant 3$. Moreover, $h^1(S', -K_{S'}) = h^1(S', K_{S'} + H) = 0$, and therefore, by the Riemann–Roch theorem, $\dim(| - K_{S'}|) = K_{S'}^2 \geqslant 3$. Then $| - K_{S'}|$ is a linear system of curves of genus 1, and $\varphi_{|-K_{S'}|}$ maps S' to a (weak) Del Pezzo surface of degree $K_{S'}^2$. The assertion follows.

(b) Assume that $g = 3$. In this case, $|M|$ is a net of elliptic curves. We cannot apply Proposition 5.6, but we will verify by hand that an analogous conclusion holds. By the classification theorem of linear systems of elliptic curves on a rational surface, see [4, p. 97], there is a birational map $\varphi : S' \dashrightarrow X$, where X is the blow–up of $\mathbf{P}^2$ at 7 suitable (proper or infinitely near) points $p_1, \ldots, p_7$ forming a curvilinear scheme, and $|M|$ is the pull back via φ of the anticanonical system of X.

Let $\psi : X \longrightarrow \mathbf{P}^2$ be the blow-up map, and consider the composite birational map $\gamma = \psi \circ \varphi : S' \dashrightarrow \mathbf{P}^2$. Then γ maps $|M|$ to the linear system $\mathcal{M} = (3; 1^7)$ of curves of degree 3 passing through the points $p_1, \ldots, p_7$. Set $\Psi = \psi_*(A)$, in the notation of Lemma 10.3. The map γ may contract some (-1)–curves to (proper or infinitely near) further points $p_8, \ldots, p_k$, different from $p_1, \ldots, p_7$, which are not base points for $\mathcal{M}$. The divisor Ψ belongs to some 0–dimensional linear system $(\ell; m_1, \ldots, m_k)$. The linear system $|H|$ is mapped by γ to a linear system $\mathcal{H}$ of the form $(\ell'; m_1', \ldots, m_k')$.

The system $\Psi + \mathcal{M}$, that is $(\ell + 3; m_1 + 1, \ldots, m_7 + 1, m_8, \ldots, m_k)$, is the adjoint system to $\mathcal{H}$ by (10.1). This implies that

$$\ell' = \ell + 6, \;\; m_i' = m_i + 2, \;\; \text{for } \; 1 \leqslant i \leqslant 7, \;\; m_i' = m_i + 1, \;\; \text{for } \; 8 \leqslant i \leqslant k.$$

We have the following two important pieces of information: (1) the curves in $\mathcal{M}$ cut out on the general curve in $\mathcal{H}$, off the base points, the complete canonical series; (2) the curves in $\mathcal{H}$ intersect Ψ only at the base points (by Lemma 10.3). Property (1) implies

$$3(\ell + 6) - \sum_{i=1}^{7} (m_i + 2) = 4, \quad \text{hence} \quad 3\ell - \sum_{i=1}^{7} m_i = 0, \tag{10.2}$$

which means that the curves in $\mathcal{M}$ intersect Ψ only at the base points. Property (2) reads

$$\ell(\ell + 6) - \sum_{i=1}^{7} m_i(m_i + 2) - \sum_{i=8}^{k} m_i(m_i + 1) = 0$$

which, taking into account (10.2), implies

$$\ell^2 - \sum_{i=1}^{k} m_i^2 = \sum_{i=8}^{k} m_i \geqslant 0. \tag{10.3}$$

But, since the curves in $\mathcal{H}$ intersect Ψ only at the base points, $\Psi \neq 0$ implies that

$$\ell^2 - \sum_{i=1}^{k} m_i^2 < 0;$$

thus, (10.3) yields that $\Psi = 0$, which implies that φ is regular. In the upshot, $\ell = 0$ and $m_i = 0$ for all $i = 1, \ldots, k$, therefore $\mathcal{H}$ is the system $(6; 2^7, 1^{k-7})$, which proves the assertion. $\square$

Remark 10.5. Theorem 10.4, (i), tells us that the surface S can be obtained as the image of the plane via the map determined by a linear system of the form $(6; 2^h)$, with $h \leqslant 6$, possibly with some further simple base points, or as the 2–Veronese image of a quadric in $\mathbf{P}^3$ or a simple internal projection thereof. This shows the similarity of part (i) of Theorem 10.4 with part (ii).

Next we consider the case in which $|M_2|$ is non-empty.

Theorem 10.6. *Consider Setup 10.1. If $|M_2|$ is non-empty, then one of the following cases occurs:*
(a) S is the surface image of $\mathbf{P}^2$ by a linear subsystem of $|\mathcal{O}_{\mathbf{P}^2}(7)|$ determined by s double base points, or an internal projection thereof from a scheme Z of length t, where s and t are non-negative integers such that

$$d = 49 - 4s - t \quad \text{and} \quad g = 15 - s;$$

(b) S is the surface image of $\mathbf{P}^2$ by a linear subsystem of $|\mathcal{O}_{\mathbf{P}^2}(8)|$ determined by s double base points, or an internal projection thereof from a scheme Z of length t, where s and t are non-negative integers such that

$$d = 64 - 4s - t \quad \text{and} \quad g = 21 - s;$$

(c) S is the surface image of $\mathbf{F}_e$ by a linear subsystem of $|4E + (h + 2e + 4)F|$ determined by s double base points, or an internal projection thereof from a scheme Z of length t, with integers $e \geqslant 0$, $h \geqslant \max\{2e - 4, e - 2\}$, $s \geqslant 0$, and $t \geqslant 0$ such that

$$d = 8h + 32 - 4s - t \quad \text{and} \quad g = 3h + 9 - s;$$

(d) S is the surface image of $\mathbf{F}_e$ by a linear subsystem of $|5E + (h + 2e + 4)F|$ determined by s double base points, or an internal projection thereof from a scheme Z of length t, with integers $e \geqslant 0$, $h \geqslant \max\{3e - 4, 2e - 2, e\}$, $s \geqslant 0$, and $t \geqslant 0$ such that

$$d = 10h - 5e + 40 - 4s - t \quad \text{and} \quad g = 4h - 2e + 12 - s.$$

Proof. As before, we assume without loss of generality that S is not an internal projection. By Lemma 10.2, this tells us that for every (-1)-curve E on S', $H \cdot E > 1$, hence $M \cdot E > 0$. As stated in Setup 10.1, $|M| = |K_{S'} + H|$ is base-point-free, $M^2 > 0$, and the general curve in $|M|$ is smooth. Since $|3K_{S'} + H|$ is empty, any irreducible curve contained in a curve of $|2K_{S'} + H| = \Phi_2 + |M_2|$ is smooth and rational. We start by proving various claims.

Claim 10.6.1. The divisor $2K_{S'} + H = K_{S'} + M$ is nef.

Proof of the claim. Suppose there exists an irreducible curve θ such that $\theta \cdot (K_{S'} + M) < 0$. Then $\theta^2 < 0$. Moreover, $K_{S'} \cdot \theta \leqslant \theta \cdot (K_{S'} + M) < 0$. This implies that θ is a (-1)-curve and $M \cdot \theta \leqslant 0$, which is excluded by our assumption that S is not an internal projection. $\square$

Claim 10.6.2. One has $M^2 \geqslant 3$.

Proof of the claim. One has $M^2 > 0$, so it suffices to prove that $M^2 \neq 1, 2$.

Let us first prove that $K_{S'} \cdot M < 0$. One has $K_{S'} \cdot M = K_{S'} \cdot (K_{S'} + H) < K_{S'}^2$. Thus, if $K_{S'}^2 \leqslant 0$, the wanted inequality holds. Else, $K_{S'}^2 > 0$. Then, by Riemann–Roch, $-K_{S'}$ is effective and, by the index theorem, $-K_{S'} \cdot (K_{S'} + H) > 0$, which is the required inequality.

Assume by contradiction that $M^2 = 1$ or 2. Since $K_{S'} \cdot M < 0$ and the curves in $|M|$ are not rational, we must have $(K_{S'} + M) \cdot M = 0$. Besides, $(K_{S'} + M)^2 \geqslant 0$ because $K_{S'} + M$ is nef. Therefore, by the index theorem, $K_{S'} + M \sim 0$, i.e., $2K_{S'} + H \sim 0$. Then $|M_2|$ is empty, contrary to the assumption. □

Claim 10.6.3. The divisor $2K_{S'} + H = K_{S'} + M$ is base-point-free. In particular, $\Phi_2 = 0$.

Proof of the claim. Let us first consider the case in which $(2K_{S'} + H)^2 = 0$. Then, by Proposition 3.8, there exists a base-point-free pencil $|G|$ of rational curves such that $|2K_{S'} + H| = |hG|$ for some integer $h > 0$, and the assertion follows.

Else, $(2K_{S'} + H)^2 > 0$. Then we argue in a way similar to the proof of Proposition 5.4. Assume by contradiction that there exists a base point x of $|K_{S'} + M|$. There is a curve $M \in |M|$ passing through x. Then the canonical series of M has a base point, and therefore M is not 2-connected, by [8, Proposition (A.7)]. However, by Lemma 3.5, M is 1-connected because it is big and nef. Therefore, by Lemma 3.5, there exists a decomposition $M = A + B$, with A and B effective and 1-connected, $A \cdot B = 1$, and $A^2 \leqslant B^2$, and one of the following holds:

(α) $A^2 = -1$, $M \cdot A = 0$, or

(β) $A^2 = 0$, $M \cdot A = 1$, or

(γ) $A^2 = B^2 = 1$, $A \equiv B$, $M^2 = 4$ and $M \equiv 2A$.

In case (α), $A \cdot K_{S'} \leqslant A \cdot M = 0$. Since $A^2 = -1$ and A is 1-connected, $A \cdot K_{S'} = -1$, and A is a (-1)-divisor. Since $A \cdot H = 1$, this contradicts our assumption that S is not an internal projection.

In case (β), $A \cdot K_{S'} \leqslant A \cdot M = 1$. Thus, $A \cdot K_{S'}$ equals either -2, and then $p_a(A) = 0$, or 0, and then $p_a(A) = 1$. The first case cannot happen: indeed, in that case, A would move in a pencil of rational curves intersecting M in one point, and the curves in $|M|$ would be rational, in contradiction with our assumptions.

Therefore, we must have $A \cdot K_{S'} = 0$. Since $A \cdot M = 1$, and $|M|$ is base-point-free, $h^0(A, M|_A) \geqslant 2$. Then, by [8, Prop. A.5, (ii)], A is not 2-connected. So, by Lemma 3.7, A has a decomposition $A = A_1 + A_2$ in which A_1 is a (-1)-divisor such that $A_1 \cdot H = 1$, in contradiction with our assumption that S is not an internal projection.

In case (γ), $K_{S'} \cdot A = \frac{1}{2} K_{S'} \cdot M < 0$. Since $K_{S'} \cdot A$ is odd, it equals either -1, and then $p_a(A) = 1$, or -3, and then $p_a(A) = 0$. The latter case cannot happen: if A is rational, since $A^2 = 1$ and $M \sim 2A$, M must be rational as well, which is excluded. Thus, we must have $K_{S'} \cdot A = -1$. Then, by the index theorem, $(K_{S'} + M)^2 \leqslant 1$, and in fact equality holds because we are assuming $(K_{S'} + M)^2 > 0$, so $K_{S'} + M \sim A$, hence $2(K_{S'} + M) \sim M$. Finally $H \sim -3K_{S'}$ and therefore $|H + 3K_{S'}|$ is non-empty, in contradiction with our assumptions. □

We may now proceed with the proof of the theorem. Suppose first that the curves in $|M_2|$ are reducible, hence there is a base-point-free pencil $|G|$ of rational curves, such that $|M_2| = |hG|$ for some positive integer h. We have a birational morphism $\varphi : S' \longrightarrow \mathbf{F}_e$, for some non–negative integer e, such that $|G|$ is the pull back via φ of the ruling $|F|$. Since $M \sim M_2 - K_{S'}$, we have $M \cdot G = 2$. Then the image $\mathcal{M}$ of $|M|$ via φ is a fixed-component-free linear subsystem of a system of the form $|2E + kF|$, and $\mathcal{M}$ may have some base points that are simple or double. However, up to performing elementary transformations, we can get rid of the double base points, and thus we assume without loss of generality that $\mathcal{M}$ has only $s \geqslant 0$ simple base points.

Note that, since $\mathcal{M}$ is fixed-component-free, one has $0 \leqslant E \cdot (2E + kF) = k - 2e$, i.e., $k \geqslant 2e$. Since $K_{\mathbf{F}_e} \sim -2E - (e+2)F$, we have $\mathcal{M} + K_{\mathbf{F}_e} = (k - e - 2)F$, and therefore we must have $k - e - 2 = h$, hence $k = h + e + 2$, and so $h + e + 2 \geqslant 2e$, i.e., $h \geqslant e - 2$.

Therefore, $\mathcal{M}$ is a linear subsystem of $|2E + (h + e + 2)F|$, with $s \geqslant 0$ simple base points. Thus, the image $\mathcal{H}$ of $|H|$ via φ, is a linear subsystem of

$$|\mathcal{M} - K_{\mathbf{F}_e}| = |4E + (h + 2e + 4)F|$$

with s double base points and maybe t simple base points. Again we must have $0 \leqslant E \cdot (4E + (h + 2e + 4)F) = h - 2e + 4$, so $h \geqslant 2e - 4$. We are then in case (c). This ends the analysis in the case when $|M_2|$ is composed with a pencil.

Assume now that $|M_2|$ is not composed with a pencil, thus the general curve in $|M_2|$ is smooth and irreducible. Consider the morphism $\varphi_{|M_2|} : S' \longrightarrow S'' \subseteq \mathbf{P}^r$ determined by the complete linear system $|M_2|$ of rational curves. Then, as in the proof of Theorem 9.1, the surface S'' is a surface of minimal degree $r-1$ in $\mathbf{P}^r$. We denote by $\phi : X \longrightarrow S''$ the minimal desingularization of S'', and set $\mathcal{L} = \phi^*(|\mathcal{O}_{S''}(1)|)$. Then we have a birational map $\varphi : S' \dashrightarrow X$, and $|M'|$ is the pull-back via φ of the linear system $\mathcal{L}$. We have the following cases:
(i) $X = \mathbf{P}^2$ and $\mathcal{L} = |\mathcal{O}_{\mathbf{P}^2}(1)|$;
(ii) $X = \mathbf{P}^2$ and $\mathcal{L} = |\mathcal{O}_{\mathbf{P}^2}(2)|$;
(iii) $X = \mathbf{F}_e$ and $\mathcal{L} = |\mathcal{O}_{\mathbf{F}_e}(E + hF)|$, with $h \geqslant e$, but not $e = h = 1$.
The map $\phi : X \longrightarrow S''$ is an isomorphism and $\varphi = \varphi_{|M'|}$, unless we are in case (iii) and $h = e$ (i.e., if S' is a cone).

Assume we are in case (i). Then the linear system $|M|$ is mapped by φ to a linear subsystem of $|\mathcal{O}_{\mathbf{P}^2}(4)|$ with $s \geqslant 0$ simple base points and therefore $|H|$ is mapped to a linear subsystem $\mathcal{H}$ of $|\mathcal{O}_{\mathbf{P}^2}(7)|$ with $s \geqslant 0$ double base points and t simple base points, and we are in case (a). Case (ii) is analogous and leads to case (b).

Finally, assume we are in case (iii). One has $\varphi_*(|M_2|) = |E + hF|$, so $\dim(|M_2|) = \dim(|E + hF|) = 2h - e + 1$, and the genus of the curves in $|M|$ is $p = 2h - e + 2$.

Let $\mathcal{M} = \varphi_*(|M|)$. The image of the general curve of $\mathcal{M}$ via ϕ is a canonical curve on S'', that is therefore trigonal. Hence $\mathcal{M}$ is a linear subsystem of a system of the form $|3E+kF|$, that may have a certain number s of simple base points. We must have $0 \leqslant E \cdot (3E + kF) = k - 3e$, i.e., $k \geqslant 3e$. Note that, if S'' is a cone, the images of the general curves in $\mathcal{M}$ via ϕ do not contain the vertex of the cone (because they are trigonal), and therefore $E \cdot (3E + kF) = 0$, i.e., $k = 3e$. We have

$$4h - 2e + 2 = 2p - 2 = (3E + kF) \cdot (E + hF) = 3h + k - 3e,$$

hence

$$k = e + h + 2. \tag{10.4}$$

Since $k \geqslant 3e$, one has $h \geqslant 2e - 2$. Then $\mathcal{M}$ is a linear subsystem of $|3E + (e + h + 2)F|$ with s simple base points. So $|H|$ is mapped by φ to a linear subsystem of $|5E + (2e + h + 4)F|$ with

s double base points, and possibly t simple base points, and we are in case (d). Notice that $E \cdot (5E + (2e + h + 4)F) \geqslant 0$, which reads $h \geqslant 3e - 4$. Notice also that S'' can be a cone only if $e = 2$, because in that case $h = e$, $k = 3e$, and therefore $e = 2$ follows from (10.4). ☐

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