Extensions of curves with high degree with respect to the genus

Ciro Ciliberto and Thomas Dedieu

Dedicated to Claire Voisin on the occasion of her birthday, with admiration and heartful gratitude.

Abstract. We classify linearly normal surfaces $S \subseteq \mathbf{P}^{r+1}$ of degree d such that $4g-4 \leqslant d \leqslant 4g+4$, where $g > 1$ is the sectional genus (it is a classical result that for larger d there are only cones). We apply this to the study of the extension theory of pluricanonical curves and genus 3 curves, whenever they verify Property N_2, using and slightly expanding the theory of integration of ribbons of the authors and E. Sernesi. We compute the corank of the relevant Gaussian maps, and we show that all ribbons over such curves are integrable, and thus there exists a universal extension.

We carry out a similar program for linearly normal hyperelliptic curves of degree $d \geqslant 2g+3$. We classify surfaces having such a curve C as a hyperplane section, compute the corank of the relevant Gaussian maps, and prove that all ribbons over C are integrable if and only if $d = 2g+3$. In the latter case we obtain the existence of a universal extension.

Contents

1 – Introduction

In this article we study the extensions of certain non-special curves of genus $g \geqslant 2$. We shall consider smooth, irreducible, linearly normal projective curves C of genus $g \geqslant 2$ in $\mathbf{P}^r$, and of degree d; it will always be the case that $d > 2g-2$, hence $r = d-g$. The *extensions* we want to study are surfaces $S \subseteq \mathbf{P}^{r+1}$ having C as a hyperplane section, and more generally $(c+1)$-dimensional varieties $X \subseteq \mathbf{P}^{r+c}$ having the curve C as a section by a linear space. An extension of C is *non-trivial* if it is not a cone. In this article we will be interested in non-trivial extensions.

A classical theorem by C. Segre says that if a surface extension of C is a scroll, then it is actually a cone over C. We shall give this theorem a modern proof in the present text, see [39, §2] for the original proof. On the other hand, a theorem by Hartshorne [25, Thm. (4.1)] says that if $d > 4g+4$, then S is a scroll; together with the previous result by C. Segre, this implies that it is a cone. The upshot is that for our study, we can assume without loss of generality that $d \leqslant 4g+4$. Our first result is the classification of surface extensions of a curve as above in the range $4g-4 \leqslant d \leqslant 4g+4$.

The following notation will be used throughout the text.

Notation 1.1. For all $e \in \mathbf{N}$, we let $\mathbf{F}_e$ be the rational ruled surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e))$, and we denote by E the section with self-intersection $-e$ (in case $e = 0$, this has to be taken with a grain of salt), by F the class of the fibres, and set $H = E + eF$.

Theorem 1.2. *Let $S \subseteq \mathbf{P}^{r+1}$ be a nondegenerate irreducible, projective surface of degree $d \geqslant 4g-4$, and whose general hyperplane section C is smooth, of genus $g \geqslant 2$, and linearly normal. If S is not a cone, then one of the following holds:*

(a) *S is the image by the Veronese map v_2 of a cone over an elliptic normal curve of degree $g-1$, and the hyperplane sections of S are bielliptic bicanonical curves, as in Example 3.1;*
(b) *S is a rational surface represented by a linear system of smooth plane δ-ics, $4 \leqslant \delta \leqslant 6$, as in Example 3.2;*
(c) *S is the image by the Veronese map v_2 of a Del Pezzo surface, as in Example 3.3;*
(d) *S is a rational surface with hyperelliptic sections, represented by a linear subsystem of $|2H + (g+1-e)F|$ on $\mathbf{F}_e$, as in Example 3.4;*
(e) *S is a rational surface with trigonal sections, represented by a linear subsystem of $|3H + \frac{1}{2}(g-3e+2)F|$ on $\mathbf{F}_e$, and $g \leqslant 10$, as in Example 3.5.*

In the above statement, linear (sub)systems (in Cases (b), (d), and (e)) are defined by simple base points, possibly infinitely near but along a curvilinear scheme. Our proof of Theorem 1.2 is based on a careful analysis of the adjoint system of C on S.

On the other hand, if $C \subseteq \mathbf{P}^r$ enjoys Property N_2 (see Theorem 2.2), then its extension theory is governed by ribbons. The latter are nonreduced schemes supported on C, that are potential first-infinitesimal neighbourhoods of C in an extension $S \subseteq \mathbf{P}^{r+1}$. The salient points of the theory, which we recall from [16] and slightly expand in Section 6, are the following: (i) isomorphism classes of non-trivial ribbons are parametrized by the projective space $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$, where $L = \mathcal{O}_{\mathbf{P}^r}(1)|_C$, and $\gamma_{C,L}$ is a Gaussian map, the definition of which is recalled in (5.1); (ii) each ribbon may be the first-infinitesimal neighbourhood of C in at most one extension, in particular C may have a non-trivial extension only if $\gamma_{C,L}$ is not surjective; (iii) if all ribbons in $\mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$ can be realized as first-infinitesimal neighbourhoods of C in an extension, then there exists a *universal extension* of C, i.e., a $(c+1)$-dimensional variety $X \subseteq \mathbf{P}^{r+c}$, $c = \operatorname{cork}(\gamma_{C,L})$, having C as a curve section, and such that all surface extensions of C are realized in a unique way as a section of X by some $(r+1)$-dimensional linear space containing C.

We shall apply our Classification Theorem 1.2 to study the existence of non-trivial extensions of polarized curves and, in favourable cases, prove the existence of universal extensions, as follows. First, we have found some situations in which we can compute the dimension of $\mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$, which in general is a very difficult task. Then our idea is to consider the family of all possible extensions of the curve C, using our Classification Theorem, which gives the dimension of the locus in $\mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$ of those ribbons corresponding to an actual extension of (C, L) by the Unicity Property (ii) above. When the two dimensions match, we can conclude that there exists a universal extension by the general Theorem 6.8.

The dimension of the space $\mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$ is $\operatorname{cork}(\gamma_{C,L}) - 1$, where $\operatorname{cork}(\gamma_{C,L})$ denotes the corank of the map $\gamma_{C,L}$, i.e., the codimension of its image in $H^0(C, 2K_C + L)$, see (5.1). We compute it in the following situations.

Theorem 1.3. *Let C be a smooth projective curve of genus $g \geqslant 2$, and L a line bundle on C of degree d.*
- (a) *Assume C is hyperelliptic; if either $d \geqslant 2g+3$ or $d \geqslant g+4$ and L is general, then* $\operatorname{cork}(\gamma_{C,L}) = 2g+2$.
- (b) *Assume $g = 3$ and C is non-hyperelliptic; if either $d \geqslant 2g = 6$, or $d \geqslant g+1 = 4$ and L is general, then* $\operatorname{cork}(\gamma_{C,L}) = h^0(C, 4K_C - L)$.

In fact the above result is an application of Proposition 5.5, which enables one to compute the corank of $\gamma_{C,L}$ in virtually any situation, provided C either is hyperelliptic or has genus 3. For the genus 3 case, we essentially give another proof to an earlier result by Knutsen and Lopez [29, Prop.2.9(a)]. The following on the other hand is essentially a compilation of previously known results.

Theorem 1.4. *Let C be a smooth projective curve of genus $g \geqslant 5$, non-hyperelliptic, and $L = mK_C$ for some integer $m > 1$. Then* $\operatorname{cork}(\gamma_{C,mK_C}) = 0$ *if $m > 2$ or* $\operatorname{Cliff}(C) > 2$.
If $\operatorname{Cliff}(C) = 1$*, then either*
- (a) *C is trigonal, and then* $\operatorname{cork}(\gamma_{C,2K_C}) = h^0(K_C - (g-4)\mathfrak{g})$*, with $\mathfrak{g}$ the class of the g^1_3; or*
- (b) *C is a plane quintic, and then* $\operatorname{cork}(\gamma_{C,2K_C}) = h^0(\mathbf{P}^2, -2K_{\mathbf{P}^2} - C) = 3$.

If $\operatorname{Cliff}(C) = 2$*, then* $\operatorname{cork}(\gamma_{C,mK_C}) = 0$ *except in the following cases:*
- (a) *$g = 5$, and then* $\operatorname{cork}(\gamma_{C,2K_C}) = 3$*;*
- (b) *C is a bi-anticanonical divisor in a Del Pezzo surface X, and then* $\operatorname{cork}(\gamma_{C,2K_C}) = h^0(X, -2K_X - C) = 1$*;*
- (c) *C is bielliptic, and then* $\operatorname{cork}(\gamma_{C,2K_C}) = 1$.

The case of curves of genus $g \leqslant 4$ is elementary, see (9.4.1).

The next stage of our program is to examine in all three cases above (genus 3 curves, hyperelliptic curves, and pluricanonical curves) the families of all surface extensions of a given polarized curve (C, L). For pluricanonical curves, Theorem 1.2 above tells us all the surfaces we need to consider. In general we will assume that $\deg(L) \geqslant 2g+3$, in order for Property N_2 for (C, L) (which is needed to apply the theory of ribbons and extensions) to be granted by the Green Theorem 2.2. For pluricanonical curves, this condition is automatic except for a few sporadic cases in genus $g \leqslant 3$. For an arbitrary polarized curve of genus 3, this condition is stronger than $\deg(L) \geqslant 4g-4$, so that again Theorem 1.2 tells us all the surfaces we need to consider.

For a polarized hyperelliptic curve (C,L) of arbitrary genus however, we need a stronger classification result. We prove the following, which extends classical results by Castelnuovo [8], and more recent ones by Serrano [41] and Sommese–Van de Ven [42] (see (4.8) for more comments on these results).

Theorem 1.5. *Let $C \subseteq \mathbf{P}^{d-g}$ be a linearly normal hyperelliptic curve of genus $g \geqslant 2$ and degree $d \geqslant 10$, unless $g = 2$ or 3 in which case we only make the looser assumption that $d \geqslant 2g+3$. For all surfaces $S \subseteq \mathbf{P}^{d-g+1}$ having C as a hyperplane section, if S is not a cone then it is rational, and ruled by conics. In particular, its general hyperplane section is hyperelliptic.*

Corollary 1.6. *In the setting of Theorem 1.5, the surface S is represented by a linear subsystem of $|2H + (g+1-e)F|$ on $\mathbf{F}_e$, as in case* (d) *of Theorem 1.2.*

Our main tool in proving Theorem 1.5 is the Reider and Beltrametti–Sommese Theorem 4.1.

Eventually, we can complete our program, to the effect that we obtain the following results.

Theorem 1.7. *Let (C,L) be a non-hyperelliptic polarized curve of genus $g = 3$ and degree $d \geqslant 2g+3 = 9$. Then the following hold.*
(1.7.1) *There exists a non-trivial extension of the polarized curve (C,L) if and only if there exist points $p_1,\dots,p_{16-d} \in C$ such that $L = 4K_C - \sum_{i=1}^{16-d} p_i$ (in particular, $d \leqslant 16$).*
(1.7.2) *Every ribbon in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$ is the first infinitesimal neighbourhood of C in some extension of (C,L), hence there exists a universal extension of (C,L).*

We refer to (7.8) for a discussion of the universal extensions of polarized genus 3 curves. We also include an analysis of what happens for degrees below $2g+3$, in which case Property N_2 is no longer granted by Green's Theorem, and many results about ribbons and extensions are no longer available; in particuliar a given ribbon may a priori be the first infinitesimal neighbourhood of C in several different extensions. Notably we give examples of polarized curves having two distinct families of extensions, one of the expected dimension $\operatorname{cork}(\gamma_{C,L}) - 1$ and one superabundant, which would be impossible if Property N_2 holded.

Theorem 1.8. *Let (C,L) be a polarized hyperelliptic curve of genus g and degree $d \geqslant 2g+3$.*
(1.8.1) *If $d \leqslant 4g+4$, then there exists a non-trivial extension of (C,L).*
(1.8.2) *If $d = 2g+3$, then every ribbon in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$ is the first infinitesimal neighbourhood of C in some extension of (C,L), hence there exists a universal extension of (C,L), of degree $2g+3$ and dimension $2g+3$ in $\mathbf{P}^{3g+5}$.*
(1.8.3) *If $d > 2g+3$, for general (C,L) there exist ribbons in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$ which may not be realized as first infinitesimal neighbourhoods of C in some extension of (C,L).*

When $d = 4g+4$, (C,L) has in general only finitely many extensions, but more than one; thus there cannot exist a universal extension of (C,L). Also in this case we analyze briefly the situation for degrees below $2g+3$, and we find that in this case the extensions form a superabundant family, i.e., this family has dimension greater than $\operatorname{cork}(\gamma_{C,L}) - 1$. This would be impossible if Property N_2 holded.

Theorem 1.9. *Let (C, mK_C) be a pluricanonical curve, and assume that either $g \geqslant 4$ and $m \geqslant 2$, or $g = 3$ and $m \geqslant 3$. Then there exists a non-trivial extension of the polarized curve*

(C, mK_C) *if and only if* (C, mK_C) *falls into one of the cases of Theorem 1.4 or* (9.4) *in which* $\mathrm{cork}(\gamma_{C,mK_C}) \neq 0$ *(in particular, either* $g \leqslant 10$*, or* C *is bielliptic).*

Every ribbon in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,mK_C}))$ *is the first infinitesimal neighbourhood of* C *in some extension of* (C, mK_C)*, hence there exists a universal extension of* (C, mK_C)*.*

We refer to Subsection 9.3 for a discussion of the universal extensions of pluricanonical curves. Except in the trigonal case, we provide an explicit construction.

The organization of the text is as follows. In Section 2 we recall general results on projective curves, revisit the C. Segre Theorem and recall the Hartshorne Bound. Section 3 is devoted to the proof of the Classification Theorem 1.2, and Section 4 to that of the Classification Theorem 1.5 for hyperelliptic curves. In Section 5 we recall the definition of the Gaussian map $\gamma_{C,L}$, and compute its corank in a number of cases, thus proving Theorem 1.3. In Section 6 we recall the theory of ribbons and extensions, and provide all the necessary material for its application as described in the introduction. These applications are to polarized genus three curves (in Section 7), to polarized hyperelliptic curves (in Section 8), and to pluricanonical curves (in Section 9). All references in this text are invisible but clickable hyperlinks.

We work over the field $\mathbf{C}$ of complex numbers throughout. We use the symbols '$\equiv$' and '$\sim$' to denote numerical and linear equivalence, respectively.

Acknowledgements. We thank Andreas Knutsen and Angelo Lopez for useful comments.

2 – General preliminary results

2.1 – Results on projective curves

We will need the following results. The first one is an improvement on a theorem by Castelnuovo, see [10] and [13, Thm. (1.11)].

Theorem 2.1 (Castelnuovo)**.** *Let* C *be a smooth curve and* $L \to C$ *be a globally generated line bundle. If the image of the map associated to the complete linear series* $|L|$ *is not a rational curve, or if* $|L|$ *is a pencil, then the multiplication map*

$$H^0(L) \otimes H^0(K_C) \to H^0(K_C + L)$$

is surjective.

The other one is due to Green.

Theorem 2.2 ([24, Thm. 4.a.1])**.** *Let* C *be a smooth curve of any genus* g*, and* $L \to C$ *a line bundle of degree* d*. For all* $k \geqslant 0$*, if*

$$d \geqslant 2g + 1 + k,$$

then L *enjoys Property* N_k*, i.e.,*

(i) L *defines a projectively normal embedding of* C*, and*

(ii) if $k \geqslant 1$*, the ideal of* C *in this embedding is generated by quadrics, and all syzygies are generated by linear syzygies up to the* k*-th step.*

2.2 – Scrolls as extensions of linearly normal curves

The main object of this subsection is to discuss Theorem 2.3 below. We distinguish between *ruled surfaces*, by which me mean "abstract ruled surfaces", i.e., surfaces S equipped with a locally trivial morphism $S \to C$ onto a smooth curve whose fibres are $\mathbf{P}^1$, and *scrolls*, by which we mean a ruled surface embedded in some projective space in such a way that the fibres are lines.

Theorem 2.3. *Let $C \subseteq \mathbf{P}^{n-1}$ be a smooth linearly normal and non-degenerate curve of genus $g > 0$. Assume there exists a scroll $\Sigma \subseteq \mathbf{P}^n$ such that C is a hyperplane section of Σ. Then the scroll Σ is necessarily a cone.*

This first appeared in [39, §2] by C. Segre, and also later in [40, §14] by the same author, under the additional assumption that $\mathcal{O}_C(1)$ is non-special. In the particular case when C is a canonical curve, this is [20, Thm. III.2.1, p. 38].

We shall need the following lemma for the proof.

Lemma 2.4 ([3, Lemme 1])**.** *Let C be a smooth curve, and let*

$$0 \to E' \to E \to E'' \to 0 \tag{2.4.1}$$

be an exact sequence of vector bundles on C. Assume that
(i) the boundary map $\partial : H^0(E'') \to H^1(E')$ is zero, and
(ii) the multiplication map $\alpha : H^0(E'') \otimes H^0((E')^\vee \otimes \omega_C) \to H^0(E'' \otimes (E')^\vee \otimes \omega_C)$ is surjective.
Then the exact sequence (2.4.1) *is split.*

Proof of Theorem 2.3. Let $f : S \to \Sigma$ be the minimal resolution of singularities. The surface S is ruled with C as a section of the ruling, and Σ is a scroll, hence there exists a rank 2 vector bundle $\mathcal{E}$ on C such that $S \cong \mathbf{P}(\mathcal{E})$ and the map $S \to \Sigma \subseteq \mathbf{P}^n$ is given by a linear subsystem of $|H^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))|$. Moreover, there is the following exact sequence of locally free sheaves on C:

$$0 \to \underset{\mathcal{O}_C}{\underset{\|}{\mathcal{O}_C(1) \otimes N_{C/S}^{-1}}} \to \mathcal{E} \to \mathcal{O}_C(1) \to 0, \tag{2.4.2}$$

see [26, Prop. V.2.6]. We shall use Lemma 2.4 to show that this exact sequence is split. Then $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_C(1)$ and the linear system $|H^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))|$ contracts the section corresponding to the trivial quotient $\mathcal{E} \to \mathcal{O}_C$, and Σ is a cone, which is the result we wanted to prove.

It thus only remains to apply Lemma 2.4 to the exact sequence (2.4.2). Condition (ii) is satisfied by the Castelnuovo Theorem 2.1, i.e., the multiplication map $H^0(L) \otimes H^0(K_C) \to H^0(K_C + L)$ is surjective, where $L = \mathcal{O}_C(1)$. To see that condition (i) of the lemma holds as well, we write the long exact sequence associated to (2.4.2):

$$0 \to H^0(\mathcal{O}_C) \to \underset{H^0(\mathcal{O}_S(1))}{\underset{\|}{H^0(\mathcal{E})}} \to H^0(\mathcal{O}_C(1)) \to \underset{H^1(\mathcal{O}_S)}{\underset{\|}{H^1(\mathcal{O}_C)}} \to \underset{H^1(\mathcal{O}_S(1))}{\underset{\|}{H^1(\mathcal{E})}} \to H^1(\mathcal{O}_C(1)) \to 0,$$

for the vertical identifications, see, e.g., [26, Lem. V.2.4]; $\mathcal{O}_S(1)$ stands for $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. It follows from the fact that $C \subseteq \mathbf{P}^{n-1}$ is linearly normal that $H^0(\mathcal{O}_\Sigma(1)) \to H^0(\mathcal{O}_C(1))$ is surjective, hence $H^0(\mathcal{O}_S(1)) \to H^0(\mathcal{O}_C(1))$ is surjective, and thus the boundary map $H^0(\mathcal{O}_C(1)) \to H^1(\mathcal{O}_C)$ is zero, i.e., condition (i) holds. We may thus apply Lemma 2.4 and, as explained above, this concludes the proof of the theorem. □

2.3 – The Hartshorne Bound

Theorem 2.5 ([25])**.** *Let C be a smooth curve of genus g, sitting in a smooth surface S. If $C^2 > 4g+5$, then there exists a ruled surface Σ having C as a section, and a birational map $S \dashrightarrow \Sigma$ which is an isomorphism on an open subset containing C.*

If $C^2 = 4g+5$, the only other possibility is that there is a birational map $S \dashrightarrow \mathbf{P}^2$ which is an isomorphism on an open subset containing C, and identifies C with a cubic curve.

This result is also a particular case of [27, Thm. A].

Corollary 2.6. *Let $C \subseteq \mathbf{P}^n$ be a smooth curve of genus $g > 1$ and degree d, non-degenerate and linearly normal. If $d > 4g+4$, then every extension of C is trivial.*

If $g = 1$ and $d = 4g+5$, the only possibility for C to have a non-trivial extension is that it is a hyperplane section of the Veronese surface $v_3(\mathbf{P}^2) \subseteq \mathbf{P}^9$.

Proof. Let $S \subseteq \mathbf{P}^{n+1}$ be a surface having C as a hyperplane section. Then $\deg(S) = \deg(C) = d$, hence $C^2 = d$ as a divisor in S. We consider a minimal desingularization $\pi : S' \to S$ and, abusing notation, we still denote by C its proper transform on S'. Note that S may have at most isolated singularities and, by the minimality of the resolution, there is no irreducible (-1)-curve Γ on S' such that $C \cdot \Gamma = 0$.

By Hartshorne's Theorem 2.5 above, there are two cases to be considered. Assume first that there exists a birational map $S' \dashrightarrow \mathbf{P}(E)$ which is an isomorphism on an open subset $U \subseteq S'$ containing C, where E is a rank two vector bundle over C. The pull-back to S' of the linear system $|C|$ on S is the complete linear system $|C|$ on S' because $C \subseteq \mathbf{P}^n$ is linearly normal, and it is base-point-free. In turn, the image on $\mathbf{P}(E)$ of this system is again the complete linear system $|C|$, because $S' \dashrightarrow \mathbf{P}(E)$ is an isomorphism on U. The upshot is that S is the image of $\mathbf{P}(E)$ defined by this linear system, and by Theorem 2.3 this is a cone.

In the remaining case of Hartshorne's Theorem, which may occur only if $g = 1$ and $d = 4g+5 = 9$, similar arguments show that S is the image of $\mathbf{P}^2$ by the complete linear system of plane cubics. $\square$

3 – Classification of extensions with high degree

This section is devoted to Theorem 1.2. We first expand on the description of the items in the classification, and then give the proof.

3.1 – Detailed description of the items in the classification

Let $S \subseteq \mathbf{P}^N$ be a degree d surface, of sectional (geometric) genus g. We call *simple internal projection* of S a surface $S' \subseteq \mathbf{P}^{N'}$ obtained by projecting S from a curvilinear subscheme Z of length b supported on the smooth locus of S, $N' = N - \dim(\langle Z \rangle)$, such that the projection map is birational. We recall that a scheme Z is *curvilinear* if for all point p in the support of Z, the Zariski tangent space of Z at p has dimension at most one.

For S' a simple internal projection of S as above, one has $\deg(S') = d - b$ and S' has the same sectional genus g as S. Note that, if $d - b \geqslant 2g+1$ and S is regular and linearly normal, then any projection from a curvilinear subscheme Z of length b supported on the smooth locus of S is a simple internal projection, and $N' = N - b$, for in this case the linear system of hyperplane sections of S passing through Z restricts on its general member to a complete, non-special, very ample linear system.

Example 3.1. Let C be a bielliptic curve of genus $g \geqslant 4$. Then the canonical model of C in $\mathbf{P}^{g-1}$ sits on a cone X with vertex a point p over a normal elliptic curve E of degree $g-1$ in a hyperplane Π of $\mathbf{P}^{g-1}$ not containing p, and C is the complete intersection of X with a quadric in $\mathbf{P}^{g-1}$. The bielliptic involution is the restriction to C of the projection from p to Π.

Note that the minimal resolution of X is the projective bundle $\mathbf{P}(\mathcal{O}_E \oplus \mathcal{L})$ where $\mathcal{L}$ is the hyperplane bundle of E in $\Pi \cong \mathbf{P}^{g-2}$. The map $\mathbf{P}(\mathcal{O}_E \oplus \mathcal{L}) \to X$ is induced by the $\mathcal{O}(1)$ bundle on $\mathbf{P}(\mathcal{O}_E \oplus \mathcal{L})$.

Consider the 2–Veronese image S of X. Since

$$h^0(E, \mathrm{Sym}^2(\mathcal{O}_E \oplus \mathcal{L})) = h^0(E, \mathcal{O}_E) + h^0(E, \mathcal{L}) + h^0(E, \mathcal{L}^{\otimes 2}) = 3g-2,$$

the surface S is linearly normally embedded in $\mathbf{P}^{3(g-1)}$. The bicanonical image of C is a hyperplane section of S, and it is linearly normally embedded with degree $d = 4(g-1)$. In this case we will say that S presents the *bicanonical bielliptic case*.

Example 3.2. Consider the linear system $|\mathcal{O}_{\mathbf{P}^2}(\delta)|$ of plane curves of degree δ, whose self–intersection is δ^2 and whose genus is

$$g = \frac{(\delta-1)(\delta-2)}{2}.$$

We assume $\delta \geqslant 4$ so that $g > 1$. One has $\delta^2 \geqslant 4g-4$ if and only if $\delta \leqslant 6$. This means that for $4 \leqslant \delta \leqslant 6$, the degree of the δ–Veronese image of $\mathbf{P}^2$ (and suitable simple internal projections of it) is in the range $[4g-4, 4g+4]$.

More precisely, in case $\delta = 4$ one has $\delta^2 = 4g+4$, which is the maximum possible degree with respect to the sectional genus. We still have degree in the above range if we make simple internal projections of the 4–Veronese of $\mathbf{P}^2$ from $b \leqslant 8$ points.

If $\delta = 5$, we have $\delta^2 = 4g+1$, and the degree is in the above range if we make simple internal projections of the 5–Veronese of $\mathbf{P}^2$ from $b \leqslant 5$ points.

Finally, if $\delta = 6$ we have $\delta^2 = 4g-4$.

In all these cases we will say that the surfaces present the *planar case*.

Example 3.3. a) Let X be the plane blown up at $h \leqslant 7$ (proper or infinitely near) points such that there is an irreducible cubic curve passing simply through these points. Let $E_1, \ldots, E_h$ be the exceptional (-1)-divisors over the blown–up points, set $E = E_1 + \cdots + E_h$ and let H be the pull back on X of a general line of $\mathbf{P}^2$. Note that the anticanonical system on X is $|3H - E|$. We will say that we are here in a *Del Pezzo situation* (even though X is a genuine *Del Pezzo surface* only if the anticanonical system is ample).

Consider the linear system $|6H - 2E|$. This linear system is base point free and its general curve is irreducible of genus $g = 10-h$ and self–intersection $4(9-h) = 4g-4$. Moreover it is not difficult to see that $\phi_{|6H-2E|}$ is a birational morphism to the image S, that is non-degenerate in $\mathbf{P}^{27-3h} = \mathbf{P}^{3g-3}$. Note that the hyperplane sections of these surfaces are bicanonically embedded. For $h = 0$ we get again the planar case for $\delta = 6$.

b) Similarly, let X be an irreducible quadric in $\mathbf{P}^3$, and consider the linear system $|-2K_X|$, which is the linear system of quadric sections of X. Thus, either X is the image of $\mathbf{F}_0$ by the linear system $|H+F|$ and $|-2K_X| = |4H+4F|$ (curves of bidegree $(4,4)$ on $\mathbf{P}^1 \times \mathbf{P}^1$), or X is the image of $\mathbf{F}_2$ by the linear system $|2H|$ and we may identify X with $\mathbf{F}_2$ and $|-2K_X|$ with $|4H|$. As in case a), the linear system $|-2K_X|$ is base point free, its general member is irreducible of genus $g = 9$ and self–intersection $4g-4 = 32$, and the associated map $\phi_{|-2K_X|}$ is a birational morphism to the image S, which is a non-degenerate surface in $\mathbf{P}^{3g-3} = \mathbf{P}^{24}$ with bicanonical hyperplane sections.

We will say that the surfaces in cases a) and b) above present the *bicanonical Del Pezzo case*.

Example 3.4. Consider, in the Notation 1.1, the linear system

$$|2H + kF| = |2E + (k + 2e)F| = |H + E + (k + e)F|$$

on a rational ruled surface $\mathbf{F}_e$, with $k \geqslant \max(0, 3 - e)$. It is base point free, and very ample unless $k = 0$, in which case the morphism $\phi_{|2H|}$ maps $\mathbf{F}_e$ birationally onto its image, which is the 2-Veronese image of the cone in $\mathbf{P}^{e+1}$ over a rational normal curve in $\mathbf{P}^e$. In any event the general curve in $|2H + kF|$ is smooth and irreducible.

Since

$$K_{\mathbf{F}_e} \sim -2E - (e + 2)F \sim -2H + (e - 2)F, \tag{3.4.1}$$

the adjoint system to $|2H + kF|$ is $|(k + e - 2)F|$, and therefore the curves in $|2H + kF|$ are hyperelliptic of genus $g = k+e-1$. The assumption that $k \geqslant 3-e$ grants that $g \geqslant 2$. Moreover,

$$(2H + kF)^2 = 4e + 4k = 4g + 4.$$

If C is a smooth curve in $|2H + kF|$, from the exact sequence

$$0 \to \mathcal{O}_{\mathbf{F}_e} \to \mathcal{O}_{\mathbf{F}_e}(2H + kF) \to \mathcal{O}_C(2H + kF) \to 0$$

and from the fact that

$$h^1(\mathbf{F}_e, \mathcal{O}_{\mathbf{F}_e}) = 0, \quad h^0(C, \mathcal{O}_C(2H + kF)) = 3g + 5$$

we deduce that $h^0(\mathbf{F}_e, \mathcal{O}_{\mathbf{F}_e}(2H + kF)) = 3g + 6$. If we set $S = \phi_{|2H+kF|}(\mathbf{F}_e)$, then S is non–degenerate of degree $4g + 4$ in $\mathbf{P}^{3g+5}$, and its general curve section is hyperelliptic of genus $g \geqslant 2$. Any surface which is a simple internal projection of S from $b \leqslant 8$ points on S still has degree in the range $[4g - 4, 4g + 4]$, and sectional genus g. We will say that surfaces of this type present the *hyperelliptic case*.

Example 3.5. Let C be a trigonal canonical curve of genus $g \geqslant 4$ in $\mathbf{P}^{g-1}$. Then C sits on a smooth rational normal scroll Y of degree $g - 2$ in $\mathbf{P}^{g-1}$. We denote by $\mathcal{H}$ the hyperplane section class of $Y \subseteq \mathbf{P}^{g-1}$, and by F a line of the ruling of Y (be careful not to mistake $\mathcal{H}$ with the class H in our Notation 1.1; the minimal resolution of singularities of Y is isomorphic to some rational ruled surface $\mathbf{F}_e$, and one has $\mathcal{H} = H + lF$ for some $l \geqslant 0$).

It is easy to check that $C \in |3\mathcal{H} - (g - 4)F|$. Conversely, if Y is a rational normal scroll of degree $g - 2$ in $\mathbf{P}^{g-1}$, and if a smooth curve C sits in $|3\mathcal{H} - (g - 4)F|$, then C is a trigonal canonical curve. One has $(3\mathcal{H} - (g-4)F)^2 = 3g+6$, hence the linear system $|\mathcal{O}_C(3\mathcal{H} - (g-4)F)|$ is very ample of dimension $2g + 6$. This shows that $\phi_{|3\mathcal{H}-(g-4)F|}$ is a morphism that maps Y birationally to a non-degenerate surface $S \subseteq \mathbf{P}^{2g+6}$. If $g \leqslant 10$, one has $3g + 6 \geqslant 4g - 4$. We will say that surfaces of this sort, as well as their simple internal projections of degree at least $4g - 4$, present the *trigonal case*.

To connect with the notation in Theorem 1.2, note that if $\mathcal{H} = H + lF$ on $\mathbf{F}_e$, then

$$(H + lF)^2 = e + 2l = g - 2,$$

and

$$3\mathcal{H} - (g - 4)F = 3H + (3l - g + 4)F = 3H + \frac{1}{2}(g - 3e + 2)F.$$

Note that none of the surfaces in the above Examples is a cone. Indeed, they have irregularity $q \leqslant 1$, and sectional genus $g \geqslant 2$.

3.2 – Previously known results

We quote the following from [32, §7], which we do not use in our proof.

Theorem 3.6 ([29, Cor. 2.10], [5, Thm. 2])**.** *Let $C \subseteq \mathbf{P}^r$ be a smooth irreducible nondegenerate linearly normal curve of genus $g \geqslant 4$ and degree d. Then C is not extendable if*
(i) C is trigonal, $g \geqslant 5$ and $d \geqslant \max\{4g-6, 3g+7\}$;
(ii) C is a plane quintic and $d \geqslant 26$;
(iii) $\mathrm{Cliff}(C) = 2$ and $d \geqslant 4g-3$;
(iv) $\mathrm{Cliff}(C) \geqslant 3$ and $d \geqslant 4g+1-3\mathrm{Cliff}(C)$.

Part (iv) tells us in particular that no curve C with $\mathrm{Cliff}(C) > 2$ is extendable in the range of degree under consideration in the present text, namely $d \geqslant 4g-4$. If $\mathrm{Cliff}(C) \leqslant 2$, our Theorem 1.2 classifies those extensions that indeed exist in the possibilities left open by the above statement.

If $\mathrm{Cliff}(C) = 2$, the only possibility in our range left by the above theorem is $d = 4g-4$. We find that there indeed exist extensions in this degree, and they are all extensions of bicanonical curves, for some special curves.

Items (i) and (ii) deal with curves of Clifford index one. For plane quintics the maximal degree is 25, and it is indeed realized in our classification, by rational surfaces representend by a linear system of plane quintics. For smooth quintics, $g = 6$ hence $4g-4 = 20$, and $4g+4 = 28$.

For trigonal curves, the above theorem says that extensions may have degree at most

$$\max(4g-7, 3g+6) = \begin{cases} 4g-7 & \text{if } g \geqslant 13 \\ 3g+6 & \text{if } g \leqslant 13. \end{cases}$$

If $g \geqslant 13$, this implies that there is nothing in the range $[4g-4, 4g+4]$. For $g \leqslant 13$ the bound is sharp and, by our classification, realized exclusively by trisecant scrolls (cf. Example 3.5) as soon as $3g+6 \geqslant 4g-4$, i.e., $g \geqslant 10$.

For curves with Clifford index zero, i.e., hyperelliptic curves, there exist extensions in all degrees $4g-4, \dots, 4g+4$, and they are all bisecant scrolls (cf. Example 3.4).

3.3 – Setup of the Proof of Theorem 1.2

We now start proving Theorem 1.2. Let $S \subseteq \mathbf{P}^{r+1}$ be a surface as in the theorem. It may have at most isolated singular points. We consider its minimal desingularization $\pi : S' \to S$ and, abusing notation, we still denote by C its proper transform on S'. By the minimality of the resolution, there is no irreducible (-1)-curve E such that $C \cdot E = 0$ on S'.

Since $K_{S'} \cdot C = 2g-2-d < 0$, the Kodaira dimension of S' is $\kappa(S') = -\infty$. We let q be the irregularity of S', and consider a minimal model $f : S' \to \Sigma$ of S'. If $q > 0$, then Σ is a $\mathbf{P}^1$-bundle over a smooth curve Γ of genus q, while if S' is rational, then Σ is either $\mathbf{P}^2$ or a rational ruled surface $\mathbf{F}_e$, with $e \geqslant 0$ and $e \neq 1$.

3.4 – The irregular case

In this section we will prove Theorem 1.2 in case $q > 0$, which amounts to proving the following proposition.

Proposition 3.7. *Let S be as in Theorem 1.2 with $q > 0$. Then S presents the bicanonical bielliptic case as in Example 3.1, hence we are in case* (a) *of Theorem 1.2.*

In this case the minimal desingularization S' of S has a surjective morphism $\varphi : S' \to \Gamma$ to a smooth curve Γ of genus q with connected, rational fibres. We denote by θ the class in the Néron–Severi group of S' of a general fibre of φ, and set

$$m = C \cdot \theta.$$

This is the degree of the images of the fibres of φ on S. If $m = 1$ then, by Theorem 2.3, S is a cone, which is excluded. Thus we have $m \geqslant 2$. In the case $m = 2$, the images of the fibres of φ on S are conics, and we will say we are in the *conic case*.

The proof of Proposition 3.7 consists of a few steps.

3.4.1 Reduction to the conic case

The first step consists in the following:

Proposition 3.8. *Let S be as in Proposition 3.7. Then $m = 2$ and $d = 4(g-1)$.*

Proof. We apply [25, Thm. (2.3)] to the effect that if C is a smooth, irreducible curve of genus g on an irregular ruled surface S' with $m > 1$, then

$$C^2 \leqslant \frac{2m}{m-1}(g-1).$$

Hence we have

$$4(g-1) \leqslant d = C^2 \leqslant \frac{2m}{m-1}(g-1),$$

and the assertion immediately follows. □

3.4.2 Passing to a minimal model

Consider the image $\bar{C}$ of C in Σ via the map $f : S' \to \Sigma$. This is an irreducible curve and, since $m = 2$ by Proposition 3.8, $\bar{C}$ may have at most double points that can be proper or infinitely near. Let h be the number of double points of $\bar{C}$. One has

$$\bar{C}^2 = C^2 + 4h + \nu, \quad p_a(\bar{C}) = g + h,$$

with ν the number of (-1)-curves E contracted by f and such that $C \cdot E = 1$.

By performing a sequence of elementary transformations on Σ based at the double points of $\bar{C}$, we produce a birational map

$$\alpha : \Sigma \dashrightarrow \Sigma'$$

where Σ' is still a $\mathbf{P}^1$–bundle over Γ, the image C' of $\bar{C}$ via α is smooth and $C'^2 = C^2 + \nu = 4(g-1) + \nu$. Applying Proposition 3.8 to $C' \subseteq \Sigma'$, one finds that $\nu = 0$.

Abusing notation, we will still denote by θ the class of a fibre of the structure morphism $\varphi' : \Sigma' \to \Gamma$.

The pair (S', C) is birational to the pair (Σ', C'), i.e., there is a birational map $S' \dashrightarrow \Sigma'$ that maps C to C'. Hence the image of Σ' via the map $\phi_{|C'|}$ determined by the linear system $|C'|$ is the original surface S. So, rather than studying the linear system $|C|$ on S', we may study the linear system $|C'|$ on Σ'.

3.4.3 The adjoint system

Next we consider the *adjoint linear system* $|K_{\Sigma'} + C'|$.

Let $\varphi'|_{C'} : C' \to \Gamma$ be the double cover, with branch divisor B. By the Riemann–Hurwitz Formula we have

$$g = 2q - 1 + b, \quad \text{with} \quad \deg(B) = 2b.$$

From the cohomology sequence of the exact sequence

$$0 \to \mathcal{O}_{\Sigma'}(K_{\Sigma'}) \to \mathcal{O}_{\Sigma'}(K_{\Sigma'} + C') \to \mathcal{O}_{C'}(K_{C'}) \to 0,$$

since

$$\begin{aligned}
&h^0(\Sigma', \mathcal{O}_{\Sigma'}(K_{\Sigma'})) = h^1(\Sigma', \mathcal{O}_{\Sigma'}(K_{\Sigma'} + C')) = 0,\\
&h^1(\Sigma', \mathcal{O}_{\Sigma'}(K_{\Sigma'})) = q,\\
&h^0(C', \mathcal{O}_{C'}(K_{C'})) = g,
\end{aligned}$$

we get

$$\dim(|K_{\Sigma'} + C'|) = g - 1 - q = q - 2 + b.$$

Moreover, $q - 2 + b \geqslant 0$ because we are assuming $q > 0$ and $g \geqslant 2$.

Note that $K_{\Sigma'} \cdot \theta = -2$, hence $(K_{\Sigma'} + C') \cdot \theta = 0$, so, since $K_{\Sigma'} + C'$ is effective, we have the numerical equivalence $K_{\Sigma'} + C' \equiv k\theta$, for some integer k.

Lemma 3.9. *In the above setting, one has $q = 1$.*

Proof. We have

$$0 = (K_{\Sigma'} + C')^2 = K^2_{\Sigma'} + 2(K_{\Sigma'} + C') \cdot C' - (C')^2 = K^2_{\Sigma'} + 4(g-1) - C'^2 = K^2_{\Sigma'}.$$

On the other hand, $K^2_{\Sigma'} = 8(1-q)$, and the assertion follows. □

3.4.4 The classification

We may now finish the proof of Proposition 3.7. To do so, we first identify the surface Σ'. We have $\Sigma' = \mathbf{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank two vector bundle on a curve Γ of genus 1. We can suppose that $\mathcal{E}$ is *normalized* (see [26, p. 373]), with invariant e. We denote by E the section such that $E^2 = -e$. We have

$$K_{\Sigma'} \equiv -2E - e\theta$$

by [26, Cor. V.2.11], and

$$C' \equiv 2E + a\theta \tag{3.9.1}$$

for some integer a.

One has

$$2g - 2 = (K_{\Sigma'} + C') \cdot C' = (a - e)\theta \cdot C' = 2(a - e),$$

hence

$$a = g - 1 + e. \tag{3.9.2}$$

Lemma 3.10. *The vector bundle $\mathcal{E}$ is decomposable.*

Proof. Suppose by contradiction that $\mathcal{E}$ is indecomposable. Then one has $-1 \leqslant e \leqslant 0$ by [26, Thm. V.2.15]. If $e = 0$, then by (3.9.2) we have

$$C' - K_{\Sigma'} \equiv 4E + (g-1)\theta, \tag{3.10.1}$$

which is big and nef. Then, by Kawamata–Viehweg Vanishing, one has $h^1(\Sigma', \mathcal{O}_{\Sigma'}(C')) = 0$. But then from the cohomology sequence of the exact sequence

$$0 \to \mathcal{O}_{\Sigma'} \to \mathcal{O}_{\Sigma'}(C') \to \mathcal{O}_{C'}(C') \to 0,$$

we see that the restriction map

$$H^0(\Sigma', \mathcal{O}_{\Sigma'}(C')) \to H^0(\Sigma', \mathcal{O}_{C'}(C'))$$

has corank $h^1(\Sigma', \mathcal{O}_{\Sigma'}) = 1$, so it is not surjective, which in turn implies that C is not linearly normal, a contradiction.

In case $e = -1$, one has $h^1(\Sigma', \mathcal{O}_{\Sigma'}(C')) = 0$ by [11, Thm. 1.17], and then one concludes as in the previous case. $\square$

We can now finish the:

(3.11) Proof of Proposition 3.7. From Lemma 3.10 we have that

$$\mathcal{E} = \mathcal{O}_\Gamma \oplus \mathcal{L}$$

where $\mathcal{L}$ is a line bundle of degree $-e \leqslant 0$, so that $e \geqslant 0$ (see [26, Thm. V.2.12]). We want to compute $h^1(\Sigma', \mathcal{O}_{\Sigma'}(C'))$. To do so, consider again the structure morphism $\varphi' : \Sigma' = \mathbf{P}(\mathcal{E}) \to \Gamma$, and note that

$$h^1(\Sigma', \mathcal{O}_{\Sigma'}(C')) = h^1(\Gamma, \varphi'_* \mathcal{O}_{\Sigma'}(C')).$$

From (3.9.1) and (3.9.2), we have

$$\varphi'_* \mathcal{O}_{\Sigma'}(C') = \mathrm{Sym}^2(\mathcal{E}) \otimes \mathcal{D} \tag{3.11.1}$$

where $\mathcal{D}$ is a line bundle of degree $g - 1 + e$ on Γ. One has

$$\mathrm{Sym}^2(\mathcal{E}) = \mathcal{O}_\Gamma \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2},$$

hence

$$\mathrm{Sym}^2(\mathcal{E}) \otimes \mathcal{D} = \mathcal{D} \oplus (\mathcal{D} \otimes \mathcal{L}) \oplus (\mathcal{D} \otimes \mathcal{L}^{\otimes 2}),$$

with

$$\deg(\mathcal{D} \otimes \mathcal{L}) = g - 1, \quad \deg(\mathcal{D} \otimes \mathcal{L}^{\otimes 2}) = g - 1 - e.$$

Therefore,

$$h^1(\Sigma', \mathcal{O}_{\Sigma'}(C')) = h^1(\Gamma, \mathcal{D}) + h^1(\Gamma, \mathcal{D} \otimes \mathcal{L}) + h^1(\Gamma, \mathcal{D} \otimes \mathcal{L}^{\otimes 2}) = h^1(\Gamma, \mathcal{D} \otimes \mathcal{L}^{\otimes 2}).$$

By the same argument we made in the proof of Lemma 3.10, we must have $h^1(\Sigma', \mathcal{O}_{\Sigma'}(C')) > 0$. So we must have $h^1(\Gamma, \mathcal{D} \otimes \mathcal{L}^{\otimes 2}) > 0$, hence $g - 1 - e = \deg(\mathcal{D} \otimes \mathcal{L}^{\otimes 2}) \leqslant 0$, and $e \geqslant g - 1$. On the other hand, by (3.9.1) and (3.9.2) we have

$$C' \cdot E = (2E + (g - 1 + e)\theta) \cdot E = g - 1 - e,$$

hence $e \leqslant g-1$, thus $e = g-1$, and from $h^1(\Gamma, \mathcal{D} \otimes \mathcal{L}^{\otimes 2}) > 0$ we deduce that

$$\mathcal{D} \otimes \mathcal{L}^{\otimes 2} \cong \mathcal{O}_\Gamma, \quad \text{i.e.,} \quad \mathcal{D} \cong (\mathcal{L}^\vee)^{\otimes 2}.$$

On the other hand, let us compute the linear equivalence class of C'. By (3.9.1),

$$\mathcal{O}_{\Sigma'}(C') = \mathcal{O}_{\Sigma'}(2E) \otimes \varphi'^* \mathcal{M}$$

for some line bundle $\mathcal{M}$ on Γ, hence $\varphi'_* \mathcal{O}_{\Sigma'}(C') = \operatorname{Sym}^2 \mathcal{E} \otimes \mathcal{M}$. Then (3.11.1) implies that $\mathcal{M} = \mathcal{D}$. The upshot is that

$$\mathcal{O}_{\Sigma'}(C') = \mathcal{O}_{\Sigma'}(2E) \otimes \varphi'^*((\mathcal{L}^\vee)^{\otimes 2}) = \mathcal{A}^{\otimes 2}$$

where we set

$$\mathcal{A} = \mathcal{O}_{\Sigma'}(E) \otimes \varphi'^*(\mathcal{L}^\vee).$$

The map $\phi_{\mathcal{A}}$ determined by the line bundle $\mathcal{A}$ maps $\Sigma' = \mathbf{P}(\mathcal{O} \oplus \mathcal{L})$ to a cone $X \subseteq \mathbf{P}^{g-1}$ over the elliptic normal curve of degree $g-1$ in $\mathbf{P}^{g-2}$ which is the image of Γ via the map $\phi_{\mathcal{L}^\vee}$. In this map the curve C' is mapped to a quadratic section of X. This implies that we are in the bicanonical bielliptic case. $\square$

3.5 – The rational case

In this subsection we finish the proof of Theorem 1.2 by considering the case in which the surface S is rational. We will thus prove the following:

Proposition 3.12. *Let S be as in Theorem 1.2 with $q = 0$, i.e., S is rational. Then S presents either the planar case (see Example 3.2), or the bicanonical Del Pezzo case (see Example 3.3), or the hyperelliptic case (see Example 3.4), or the trigonal case (see Example 3.5).*

The proof will consist of various steps that we will perform in the next subsections. We consider the adjoint system $|C + K_{S'}|$, which we write

$$|C + K_{S'}| = F + |M|,$$

where F is the fixed part and $|M|$ the movable part, with $\dim(|M|) = g-1$. Since $|C + K_{S'}|$ cuts the complete canonical linear series on C, we have $C \cdot F = 0$.

There are two cases to be considered:

(3.12.a) $|M|$ is composed with a pencil $|\Phi|$, including the case $g = 2$ in which $|M|$ itself is a pencil;

(3.12.b) the general curve in $|M|$ is irreducible, and $\dim(|M|) \geqslant 2$, hence $g \geqslant 3$.

In case (3.12.a) the curves in $|C|$ are hyperelliptic and $C \cdot \Phi = 2$. So the curves in $|\Phi|$ are mapped to conics on $S \subseteq \mathbf{P}^{r+1}$ and therefore are rational.

3.5.1 The hyperelliptic case

Proof of Proposition 3.12 in case (3.12.a). There is a birational morphism $\xi : S' \to \mathbf{F}_e$ such that the pencil $|\Phi|$ of rational curves is mapped to the system $|F|$ of fibres of the structure morphism $\mathbf{F}_e \to \mathbf{P}^1$. Then the linear system $|C|$ is mapped to a linear system of curves of the type $|2H + kF|$ and, by acting if necessary with elementary transformations, we may assume that the general curve in this system is smooth of genus g. The adjoint system to $|2H + kF|$ is $|(k+e-2)F|$, and therefore $g = k+e-1$. Since $g \geqslant 2$ we must have $k \geqslant 3-e$. Moreover, since $k = (2H + kF) \cdot E \geqslant 0$, we must also have $k \geqslant 0$. Thus we are in the hyperelliptic case. $\square$

3.5.2 The non–hyperelliptic case

Next we consider case (3.12.b). In particular, $g \geqslant 3$.

Lemma 3.13. *In case (3.12.b) the general curve in $|C|$ is not hyperelliptic.*

Proof. Suppose by contradiction that the general curve in $|C|$ is hyperelliptic. Let p be a general point of S'. Let us consider a general pencil $\mathcal{P}$ of curves in $|C|$ having p as base point. If C is a general curve in $\mathcal{P}$ and q is the point conjugate to p in the g^1_2 on C, then, by the generality of $\mathcal{P}$ we can assume that q is not a base point of $\mathcal{P}$. Consider the Zariski closure D in S' of the set of points $q \in S'$, such that $p+q$ is a divisor of the g^1_2 on the curves of $\mathcal{P}$, i.e.,

$$D = \overline{\bigcup_{C \in \mathcal{P}^0} \{q : p+q \in g^1_2(C)\}},$$

with $\mathcal{P}^0 \subseteq \mathcal{P}$ the Zariski dense open subset parametrizing smooth members of $\mathcal{P}$. Then D is a (rational) curve on S'.

Any curve of $|M|$ passing through p (which, by the generality of p, is a general curve of $|M|$), contains D, and therefore coincides with D. On the other hand, we claim that D does not cut out a canonical divisor on C, hence it cannot be a member of $|M|$, a contradiction.

It thus only remains to prove the claim. Let m be the multiplicity of p in D. Then

$$D \cdot C = mp + q + R,$$

where R is contained in $\mathrm{Bs}(\mathcal{P}) - p$, with $\mathrm{Bs}(\mathcal{P})$ the base locus of $\mathcal{P}$. If $R = 0$, the claim holds (recall that $g \geqslant 3$). Otherwise, R must contain all $d-1$ points of $\mathrm{Bs}(\mathcal{P}) - p$, since by the generality of $\mathcal{P}$ there is a monodromy action on $\mathrm{Bs}(\mathcal{P}) - p$, and it acts as the full symmetric group by [1, p.111–113]. Then the claim follows, as $d \geqslant 4g-4$. □

One has $C \cdot M = 2g-2$. Hence for an irreducible curve M to be contained in a curve in $|C|$ is at most $2g-1$ conditions, and equality holds if and only if M is smooth and rational, and the restriction of $|C|$ to M is a complete linear series. We thus have

$$\dim(|C-M|) \geqslant 3g-3+\varepsilon-(2g-1) = g-2+\varepsilon \geqslant 1,$$

where ε is the nonnegative integer such that $C^2 = 4g-4+\varepsilon$.

(3.14) Proof of Proposition 3.12 in case $\dim(|C-M|) = g-2+\varepsilon$**.** In this case the general curve in $|M|$ is smooth and rational. Since $\dim(|M|) = g-1$, we have $M^2 = g-2$, $|M|$ is base point free, and $\phi_{|M|}$ is a morphism mapping S' to a surface $Y \subseteq \mathbf{P}^{g-1}$. In this map the curves C are mapped to canonical curves of degree $2g-2$. Since Y has rational hyperplane sections, we have only the following possibilities:
(a) $g = 3$ and Y is $\mathbf{P}^2$;
(b) $g = 6$ and Y is the 2–Veronese image of $\mathbf{P}^2$;
(c) Y is a rational normal scroll.

In cases (a) and (b) we are in the planar case with $\delta = 4$ and $\delta = 5$ respectively. In case (c) we are in the trigonal case. □

Next we assume $s := \dim(|C-M|) \geqslant g-1+\varepsilon$. Recall that

$$C - M \sim -K_{S'} + F.$$

Lemma 3.15. *In the above setting, F is in the fixed part of $|C-M|$.*

Proof. Let D be an irreducible component of F. One has $C \cdot D = 0$, hence $D^2 < 0$. Then $K_{S'} \cdot D \geqslant 0$, otherwise D would be a (-1)–curve contracted by $|C|$, a contradiction. So we have $K_{S'} \cdot F \geqslant 0$ and therefore $(-K_{S'} + F) \cdot F < 0$. Indeed we also have $F^2 < 0$ by the Hodge Index Theorem, because $C \cdot F = 0$. Since $-K_{S'} + F \sim C - M$ is effective, there is a non–zero divisor $G \leqslant F$ that is in the fixed part of $|-K_{S'} + F|$. If $G = F$ we are done. Otherwise set $F_1 = F - G$ and consider the linear system $|-K_{S'} + F_1|$. By the same argument as above, we have $(-K_{S'} + F_1) \cdot F_1 < 0$, so there is a non–zero divisor $G_1 \leqslant F_1$ that is in the fixed part of $|-K_{S'} + F_1|$. If $G_1 = F_1$ we are done. Otherwise we repeat this argument till we eliminate all of F from the fixed part of $|-K_{S'} + F|$. □

(3.16) Conclusion of the proof of Proposition 3.12. By Lemma 3.15 we can assume that $s = \dim(|-K_{S'}|) \geqslant g - 1 + \varepsilon \geqslant 2$. Consider the map

$$\phi_{|-K_{S'}|} : S' \dashrightarrow Y \subseteq \mathbf{P}^s.$$

One has

$$(-K_{S'}) \cdot C = (C - M - F) \cdot C = 4g - 4 + \varepsilon - (2g - 2) = 2g - 2 + \varepsilon.$$

Hence the curves in $|C|$ are mapped via $\phi_{|-K_{S'}|}$ to curves of degree $\delta \leqslant 2g - 2 + \varepsilon$. Let γ be the number of conditions that containing the curve C imposes to the members of $|-K_{S'}|$. One has

$$\begin{cases} \gamma = g & \text{if } \ \varepsilon = 0 \ \text{ and } \ C \ \text{ is mapped via } \ \phi_{|-K_{S'}|} \ \text{ to a canonical curve;} \\ \gamma \leqslant g - 1 + \varepsilon & \text{otherwise.} \end{cases}$$

In any event, unless $\varepsilon = 0$, $s = g - 1$ and $\gamma = g$, one has

$$\dim(|-K_{S'} - C|) = s - \gamma \geqslant 0$$

so that $-K_{S'} - C$ is effective. Hence we have $-K_{S'} \sim C + T$, with T effective. Then $0 \sim K_{S'} + C + T \sim F + M + T$, which is not possible.

So the only possibility is that $\varepsilon = 0$, $s = g - 1$, $\gamma = g$, and $|-K_{S'}|$ cuts out the complete canonical series on C. Since $\dim(|-K_{S'}|) \leqslant 9$, we have $g \leqslant 10$.

From $-K_{S'}|_C = K_C$ we deduce that $\mathcal{O}_C(C) = \mathcal{O}_C(2K_C) = \mathcal{O}_C(-2K_{S'})$.

Let us set $|-K_{S'}| = A + |B|$, where A is the fixed part and $|B|$ the movable part of $|-K_{S'}|$. Since $|-K_{S'}|$ cuts out the complete canonical series on C, and C is not hyperelliptic by Lemma 3.13, we have that $C \cdot A = 0$ and $|B|$ is not composed with a pencil, hence the general curve in $|B|$ is irreducible.

Suppose first that $A = 0$. Then, for any irreducible curve $B \in |B|$, one has $p_a(B) = 1$. Moreover, from the exact sequence

$$0 \to \mathcal{O}_{S'} \to \mathcal{O}_{S'}(B) \to \mathcal{O}_B(B) \to 0,$$

and from $h^0(S', \mathcal{O}_{S'}(B)) = g$, we deduce that $h^0(B, \mathcal{O}_B(B)) = g - 1$, which implies that $B^2 = g - 1$. From the Index Theorem applied to C and B, we have that $C \sim 2B$. Moreover, there is a birational map $\eta : S' \dashrightarrow \mathbf{P}^2$, that maps $|B|$ to the linear system of cubics with $10 - g$ simple base points. From this we see we are in the bicanonical Del Pezzo case.

Suppose next that A is non–zero. By Lemma 3.17 below, $A \cdot B = 2$ and the general member of $|B|$ is rational. By the same argument we made above, we have $B^2 = g - 2$ and $\phi_{|B|} = \phi_{|-K_{S'}|}$ is a birational map of S' to its image Y that is a surface of minimal degree in $\mathbf{P}^{g-1}$.

If $Y = \mathbf{P}^2$, then $\phi_{|B|}$ maps $|B|$ to the linear system of lines and A to a conic Γ. The curves in $|C|$ are mapped to plane quartics. Since $C \cdot A = 0$, the linear system $|C|$ is mapped to a linear system of quartics with 8 (proper or infinitely near) base points on the conic Γ. This shows that we are in the planar case with $\delta = 4$ and $b = 8$ (see notation in Example 3.2).

If Y is the 2–Veronese image of $\mathbf{P}^2$, then we may identify Y with $\mathbf{P}^2$ and $\phi_{|B|}$ maps $|B|$ to the linear system of conics and A to a line R. The curves in $|C|$ are mapped to plane quintics. Since $C \cdot A = 0$, the linear system $|C|$ is mapped to a linear system of quintics with 5 (proper or infinitely near) base points along the line R. Hence we are in the planar case with $\delta = 5$ and $b = 5$ (see Example 3.2 again).

Finally, if $g \geqslant 4$, Y can be a rational normal scroll in $\mathbf{P}^{g-1}$ and $\phi_{|B|}$ maps $|B|$ to the linear system $|\mathcal{H}|$, where $\mathcal{H}$ is the hyperplane section class on Y. The curves in $|C|$ are mapped to canonical trigonal curves C' on Y. To see that C' is trigonal, consider a divisor D cut out on C' by a line of the ruling of Y, and let d be its degree; by the Geometric Riemann–Roch Theorem, the linear series $|D|$ has dimension $d-2$, and then it follows from Clifford's Theorem that it is a g^1_3 as C' is non-hyperelliptic. Let F be a line of the ruling of Y. As in Example 3.5, one sees that $C' \in |3\mathcal{H} - (g-4)F|$, so that Y is smooth, unless maybe $g = 4$. In the latter case, Y can be a quadric cone, so its minimal desingularization is $\mathbf{F}_2$ and in this case we will work on $\mathbf{F}_2$ rather than on Y.

Suppose that $Y = \mathbf{F}_e$ and let, as usual, E be the section such that $E^2 = -e$. Then g and e have the same parity and

$$\mathcal{H} \sim E + \big(\frac{e+g}{2} - 1\big)F$$

and accordingly

$$C' \sim 3\mathcal{H} - (g-4)F \sim 3E + \big(\frac{3e+g}{2} + 1\big)F.$$

The morphism $\phi_{|B|} : S' \to Y$ consists in blowing down a number of (-1)–exceptional divisors. Let D be the total such exceptional divisor. Then

$$-K_{S'} = -\phi^*_{|B|}(K_{\mathbf{F}_e}) - D.$$

Since $\phi^*_{|B|}(\mathcal{H}) = B$, we have

$$A \sim \phi^*_{|B|}(-K_{\mathbf{F}_e} - \mathcal{H}) - D.$$

By (3.4.1), we have

$$-K_{\mathbf{F}_e} - \mathcal{H} \sim E + \big(\frac{e-g}{2} + 3\big)F,$$

thus

$$A \sim \phi^*_{|B|}\Big(E + \big(\frac{e-g}{2} + 3\big)F\Big) - D.$$

One has

$$C' \cdot \Big(E + \Big(\frac{e-g}{2} + 3\Big)F\Big) = 10 - g$$

and recall that $C \cdot A = 0$. Moreover

$$-K_{\mathbf{F}_e} \cdot \Big(E + \Big(\frac{e-g}{2} + 3\Big)F\Big) = 8 - g < 10 - g.$$

In conclusion:

(i) S' is obtained from $\mathbf{F}_e$ by blowing up the (curvilinear) scheme Z of length $10-g$ that is the complete intersection of a smooth curve C' with a curve N of $\big|E + (\frac{e-g}{2} + 3)F\big|$;

(ii) the linear system $|C|$ on S' is the strict transform of the linear systems of the curves of $|C'|$ on $\mathbf{F}_e$ containing Z and

$$C^2 = C'^2 - (10-g) = 3g + 6 - (10-g) = 4g - 4,$$

as we wanted;

(iii) the strict transform A of N on S' splits off the anticanonical system on S'.
Thus we are here in the trigonal case. $\square$

This concludes the proof of Proposition 3.12, hence also that of Theorem 1.2. We end this section by an elementary lemma that has been used above.

Lemma 3.17. *Let S be a smooth rational surface, and write*

$$|-K_S| = A + |B|$$

with A the fixed part and $|B|$ the movable part. We assume that A is effective and nonzero, and B big and nef. Then $A \cdot B = 2$, and the general member of $|B|$ is rational if it is irreducible.

Proof. First, $h^0(-A) = 0$. Next, by Kawamata–Viehweg vanishing, $h^1(-A) = h^1(B + K_S) = 0$. Eventually,

$$h^2(-A) = h^0(K_S + A) = h^0(-B) = 0.$$

Thus $\chi(-A) = 0$, hence by Riemann–Roch $A \cdot B = 2$. Therefore, $(B + K_S) \cdot B = -A \cdot B = -2$, and the result follows. $\square$

4 – Classification of surfaces with a hyperelliptic section

This section is dedicated to Theorem 1.5. We prove it in Subsections 4.1 and 4.2, and formulate it in useful alternative ways in Subsection 4.3.

We consider a minimal resolution of singularities $S' \to S$ and work on S'. Being a hyperplane section of S, C must be contained in the smooth locus of S, so that considered in S' it cannot intersect any curve contracted by $S' \to S$. It follows that, considered in S', C is a big and nef divisor such that $C^2 = d$.

We want to apply the Reider–Beltrametti–Sommese Theorem below to C for $k = 1$, which requires $C^2 \geqslant 9$. We thus split the proof of Theorem 1.5 in two: the *general case* in which we assume that $d \geqslant 9$, equivalently either $g \geqslant 3$ or $g = 2$ and $d \neq 7, 8$, and the *sporadic cases* in which $g = 2$ and $d = 7$ or 8.

4.1 – Proof of Theorem 1.5 in the general case

Theorem 4.1 (Reider [37], Beltrametti–Sommese [4])**.** *Let L be a nef line bundle on a smooth surface S, and k be a positive integer. Assume that $L^2 \geqslant 4k+5$, and there exists a 0-dimensional subscheme $Z \subseteq S$ of length $k+1$ such that the restriction*

$$H^0(S, K_S + L) \to H^0(Z, (K_S + L)|_Z)$$

is not surjective. Then there exists an effective divisor D containing Z, and such that

$$L \cdot D - k - 1 \leqslant D^2 < k + 1.$$

(4.2) Proof of Theorem 1.5 in the general case. The general case means that we assume $d \geqslant 9$, see above.

For all divisor $x_1 + x_2$ in the g^1_2 of C, the adjoint system $|K_{S'} + C|$ does not separate x_1 and x_2, so Theorem 4.1 tells us that there exists a divisor D on S' such that

$$x_1, x_2 \in D \quad \text{and} \quad D \cdot C - 2 \leqslant D^2 \leqslant 1.$$

The inequality $D \cdot C \leqslant 3$ forbids that D contains C: indeed if $D = C + D'$ with D' effective, then $D \cdot C = C^2 + C \cdot D' \geqslant C^2 \geqslant 9$, a contradiction with $D \cdot C \leqslant 3$. This implies that D cannot be fixed as $x_1 + x_2$ moves in the g^1_2, for otherwise D would necessarily have C as an irreducible component.

Let M be the part of D that contains x_1 and x_2. By the previous observation, the family of these M has no fixed part. Thus we have

$$2 \leqslant M \cdot C \leqslant D \cdot C \leqslant 3.$$

We first claim that M is irreducible. Indeed, otherwise at least one component M' of M would verify $M' \cdot C = 1$, hence it would be mapped to a line by the map $S' \to S$, and give a ruling of S by lines, which implies that S is a cone by Theorem 2.3. So M is irreducible, and we have the following possibilities 1) and 2) to consider.

1) $M \cdot C = 2$. Then M is mapped to a conic by $S' \to S$, in particular it is a rational curve. Since it moves in a family parametrized by the g^1_2 on C, the surface S is then unirational hence rational, so the conclusion of our theorem holds in this case.

2) $M \cdot C = 3$. Then M is mapped to a cubic by $S' \to S$, hence it spans either a) a plane or b) a 3-space in $\mathbf{P}^{d-g+1}$.

This case may happen only if $d \leqslant 9$, for the following reason. If $D \cdot C = 3$, then $D^2 = 1$, and thus by the Hodge Index Theorem,

$$D^2 \cdot C^2 \leqslant (D \cdot C)^2 \Longleftrightarrow C^2 \leqslant 9.$$

The upshot is that $d = 9$ and $g = 2$ or 3, hence C is an intersection of quadrics by Green's Theorem 2.2.

In case a), the three points of $M \cap C$ are located on the line $\langle M \rangle \cap \langle C \rangle$: to see that the latter is indeed a line, note that $\langle C \rangle$ is a hyperplane in $\langle S \rangle$, hence it is impossible that the plane $\langle M \rangle$ be contained in $\langle C \rangle$ for general C. On the other hand, C is an intersection of quadrics, so this situation is impossible. In conclusion, case a) cannot happen.

In case b) the curve M is rational. Hence if $M^2 = 0$, then the curves M are parametrized by a curve $\mathcal{H}_M$. The curve $\mathcal{H}_M$ is rational, because there exists a morphism $\mathbf{P}^1 \to \mathcal{H}_M$ mapping the element $x_1 + x_2$ of the g^1_2 to the corresponding curve M. The upshot is that in this case $|M|$ is a base point free pencil. The restriction of such a pencil to C would be a base point free g^1_3 (remember that C viewed on S' does not intersect any curve contracted by $S' \to S$) containing all divisors of the g^1_2: by Lemma 4.3 below this is impossible. So we must have $M^2 > 0$. But in this case $|M|$ cuts out a g^r_3 on C with $r \geqslant 2$, in contradiction with $g \geqslant 2$.

The conclusion is that only case 1) may happen, in which the conclusion of our theorem holds. $\square$

Lemma 4.3. *Let C be a curve of genus g, and assume it has a g^1_2, $\mathfrak{l}$, and a g^1_3, $\mathfrak{m}$. We consider the following condition:*

$$\forall x_1 + x_2 \in \mathfrak{l},\ \exists z \in C: \quad x_1 + x_2 + z \in \mathfrak{m}. \tag{4.3.1}$$

If (4.3.1) *holds, then either z is a base point of $\mathfrak{m}$, or $g = 0$. If* (4.3.1) *does not hold, then C is birational to a curve of type $(2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^1$, in particular $g \leqslant 2$.*

Proof. First assume that (4.3.1) holds. Consider $z, z' \in C$ such that $x_1 + x_2 + z \in \mathfrak{m}$ and $x'_1 + x'_2 + z' \in \mathfrak{m}$ for some members $x_1 + x_2$ and $x'_1 + x'_2$ of $\mathfrak{l}$. Then

$$x_1 + x_2 + z \sim x'_1 + x'_2 + z', \quad \text{hence} \quad z \sim z',$$

and either C is rational, or $z = z'$. In the latter case z is a base point of $\mathfrak{m}$. This proves the first part of the lemma.

For the second part, consider the map

$$\phi = (\phi_{\mathfrak{l}}, \phi_{\mathfrak{m}}) : C \to \mathbf{P}^1 \times \mathbf{P}^1$$

defined by the two pencils $\mathfrak{l}$ and $\mathfrak{m}$. It is straightforward to verify that (4.3.1) holds if and only if ϕ is not birational, and this proves the second part of the statement. □

4.2 – Proof of Theorem 1.5 in the sporadic cases

(4.4) Proof of Theorem 1.5 in the sporadic cases. The sporadic cases are those when $g = 2$ and $d = 7$ or 8. Let $S \subseteq \mathbf{P}^{d-1}$, $d = 7, 8$, be a degree d surface with one hyperplane section a genus 2 curve C, such that S is not a cone, hence not a scroll either by Theorem 2.3. Since $d > 2g - 2$, we have $\kappa(S) = -\infty$ as in Subsection 3.3.

Assume by contradiction that it is not rational. Then S is abstractly a ruled surface, necessarily elliptic as it is not a scroll. Letting S' be a minimal resolution of the singularities of S, and Σ a minimal model of S'; Σ is a ruled surface over an elliptic curve B. By [12, Thm 1.4], since $d \geqslant 7$, if $S \subseteq \mathbf{P}^{d-1}$ is not 1-weakly defective then for general $p_1, p_2 \in S$, the general section of S by a hyperplane tangent at both p_1 and p_2 is a curve C_0 with two nodes at p_1 and p_2 and no other singularity.

If C_0 is irreducible, then it is a rational curve because S has sectional genus 2. Moreover, C_0 dominates the base B since $C_0^2 > 0$, a contradiction to the non-rationality of S. Otherwise, there are the two following possibilities:
(i) $C_0 = \bar{C} + F_1 + F_2$, with $F_i \cap \bar{C} = p_i$ for $i = 1, 2$ and $F_1 \cap F_2 = \varnothing$, all three $\bar{C}, F_1, F_2$ smooth;
(ii) $C_0 = \bar{C} + F$, $\bar{C} \cap F = \{p_1, p_2\}$, $\bar{C}$ and F smooth.

In case (i), by generality the situation must be symmetric with respect to p_1 and p_2, hence F_1 and F_2 are algebraically equivalent. Let F be their algebraic equivalence class. Necessarily, $F^2 = 0$, and F moves in a 1-dimensional family. Moreover,

$$C \cdot F = \bar{C} \cdot F = 1$$

hence F is mapped to a line, and S is a scroll, a contradiction.

In case (ii), at least one among $\bar{C}$ and F dominates the base B; we assume it is $\bar{C}$, thus $\bar{C}$ has genus at least one. Since C has arithmetic genus 2, the only possibility is that $\bar{C}$ has genus 1 and F has genus 0. Therefore F is in the class of the ruling of Σ. This leads to a contradiction, as the two general points p_1 and p_2 may not sit on a common ruling.

It thus only remains to examine the finitely many possibilities in which $S \subseteq \mathbf{P}^{d-1}$ is 1-weakly defective, listed in Proposition 4.5 below.

1) If S were on the cone over the Veronese surface, then its hyperplane section C would be on the Veronese surface itself. But then necessarily $S \subseteq \mathbf{P}^6$, i.e., $d = 7$, and since all curves on the Veronese surface have even degree this case cannot be realized.

2) In this case S has sectional genus 3 (the general hyperplane section of S is birational to a plane quartic everywhere tangent to the branch curve of the double cover $S \xrightarrow{2:1} \mathbf{P}^2$), and this is contrary to our assumptions.

3) In this case S is contained in a cone $K(\Lambda, B) \subseteq \mathbf{P}^{d-1}$. Then B is an irreducible curve in $\mathbf{P}^{d-3}$, hence it has degree $\delta \geqslant d - 3$. The cone $K(\Lambda, B)$ is swept out by a 1-dimensional family $(\Pi_b)_{b \in B}$ of planes containing Λ. Let m be the number of points cut out on C off Λ by a general Π_b. If $m = 1$, then S is ruled by lines which is excluded. Otherwise, let us consider the section

of C by a general hyperplane H containing the line Λ: this is the sum of the points $\Pi_b \cap C$ for $b \in H \cap B$, plus possibly some points on the vertex line Λ. The upshot is that

$$d = \deg(H \cap S) \geqslant m\delta \geqslant m(d-3),$$

which is possible only if $d \leqslant 6$, in contradiction with our assumptions. Thus this case cannot happen either.

This ends the proof by contradiction that S is rational. Now, let us write

$$|K_{S'} + C| = F + |M|,$$

where as usual F is the fixed part, and $|M|$ the mobile part. Since S is rational, it cuts out the complete canonical series on C, hence $|M|$ has dimension 1, and $F \cdot C = 0$ and $M \cdot C = 2$. Thus $|M|$ is mapped to a pencil of conics on S, and the proof is over. □

Proposition 4.5 (Chiantini–Ciliberto [12]). *Let $S \subseteq \mathbf{P}^r$, $r \geqslant 6$, be a 1-weakly defective surface with isolated singularities. Then S falls into one of the following four cases.*
1) $S \subseteq \mathbf{P}^6$ is contained in the cone over the Veronese surface and vertex a point, i.e.,

$$S \subseteq K\big(p, v_2(\mathbf{P}^2)\big).$$

2) $S \subseteq \mathbf{P}^6$ is a quartic double plane $\pi : S \xrightarrow{2:1} \mathbf{P}^2$ embedded by the complete linear system $|\pi^* 2H|$, where H is the line class in $\mathbf{P}^2$.
3) S is contained in the cone with vertex a line over a curve B,

$$S \subseteq K\big(\Lambda, B\big) \subseteq \mathbf{P}^r.$$

The proof is a direct application of [12, Thm 1.3] and is left to the reader.

4.3 – The Castelnuovo Classification

As a consequence of Theorem 1.5, we can prove the following classification result, from which Corollary 1.6 will follow.

Corollary 4.6. *Let S be as in Theorem 1.5. If S is not a cone, then S is represented by the complete linear system*

$$|2E + (g+1+e)F|$$

on the rational ruled surface $\mathbf{F}_e$, $0 \leqslant e \leqslant g+1$, or possibly by a linear subsystem defined by simple base points along a curvilinear scheme.

Proof. By Theorem 1.5, we know that S' is rational and contains a pencil $|M|$ of rational curves such that $C \cdot M = 2$. Then there is a birational morphism $S' \to \Sigma$, with Σ a minimal rational ruled surface on which $|M|$ is mapped to the ruling. Since $C \cdot M = 2$ on S', the linear system $|C|$ is mapped in Σ to a linear system with only simple and double base points. One may get rid of all double base points by performing elementary transformations. Thus we may assume that $\Sigma \cong \mathbf{F}_e$ and the general member of $|C|$ (in Σ) is smooth.

In our usual notation, $C \sim 2E + kF$, and one finds $k = g+1+e$ with the adjunction formula. Then,

$$0 \leqslant C \cdot E = g + 1 - e.$$

□

Corollary 4.7. *Let S be as in Theorem 1.5. If S is not a cone, then it is rational and represented by one of the following linear systems, or one obtained from those by adding simple base points along a curvilinear subscheme:*

a) $\mathcal{L}_{g+3}(g+1,2)$, the linear system of plane curves of degree $g+3$ with one base point of multiplicity $g+1$ and one double point;

b) $\mathcal{L}_{2g+2-\mu}([2g-\mu,2^{g-\mu}])$, $\mu=0,\dots,g$, the linear system of plane curves of degree $2g+2-\mu$, with one base point p of multiplicity $2g-\mu$, and $g-\mu$ double base points infinitely near to p, pairwise distinct on the exceptional divisor of the blow-up of p.

Proof. Take into account Corollary 4.6. If $e=0$, we are in case a). If $e=1$, we are in case b) with $\mu=g$. If $e>1$, we perform $e-1$ elementary transformations at general points, thus ending up on $\mathbf{F}_1$, after what we arrive at $\mathbf{P}^2$ by contracting the (-1)-curve E on $\mathbf{F}_1$, with a linear system as in case b) with $\mu=g+1-e$. ☐

(4.8) The classification of rational surfaces such that the general hyperplane section is hyperelliptic had been classically worked out by Castelnuovo [8]. In more recent times, the classification of arbitrary smooth surfaces having at least one smooth hyperelliptic hyperplane section has been worked out by Serrano [41] and Sommese–Van de Ven [42]. Our classification takes into account singular surfaces as well, with the same condition that they have one hyperelliptic section, albeit with the restriction that the degree d be at least 10, or at least $2g+3$ if $g=2$ or 3. Note that without this assumption there are other possibilities, including rational surfaces such that the general hyperplane section is not hyperelliptic, if the pair (g,d) equals $(3,8)$ or $(4,9)$, and an elliptic ruled surface, if $(g,d)=(3,8)$, cf. [41].

5 – Gaussian maps and their cokernels

We consider $C\subseteq\mathbf{P}^r$ a linearly normal curve, and let $L=\mathcal{O}_{\mathbf{P}^r}(1)|_C$. In this section we define the Gaussian map $\gamma_{C,L}$, and compute its corank in a number of cases, thus proving Theorem 1.3.

(5.1) The Gaussian map $\gamma_{C,L}$. Let $R_{C,L}$ be the kernel of the multiplication map

$$\mu_{C,L}:H^0(K_C)\otimes H^0(L)\to H^0(K_C+L).$$

The Gaussian map

$$\gamma_{C,L}:R_{C,L}\to H^0(2K_C+L)$$

is the map locally defined as $\sum_i s_i\otimes t_i\mapsto\sum_i(s_i\cdot dt_i-t_i\cdot ds_i)$.

We are interested in the corank of $\gamma_{C,L}$, i.e., the dimension of its cokernel in $H^0(2K_C+L)$. Most of the time, Castelnuovo Theorem 2.1 tells us that the multiplication map $\mu_{C,L}$ is surjective, which readily gives the dimension of $R_{C,L}$; then it suffices to compute the dimension of the kernel of $\gamma_{C,L}$ to find its corank. To do so, we shall use the canonical identification which we now explain.

(5.2) Restriction maps to the diagonal. We shall interpret the maps $\mu_{C,L}$ and $\gamma_{C,L}$ in terms of operations on the product $C\times C$. We let pr_1 and pr_2 be the two projections,

$$\begin{tikzcd} & C\times C \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ C & & C \end{tikzcd}$$

and $\Delta \subseteq C \times C$ be the diagonal. The multiplication map $\mu_{C,L}$ identifies with the restriction map r_1 to the diagonal Δ, as indicated in the diagram below.

$$\begin{tikzcd} H^0(\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L) \arrow[d, "r_1"'] & = & H^0(K_C) \otimes H^0(L) \arrow[d, "\mu_{C,L}"] \\ H^0\big((\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L)|_\Delta\big) & = & H^0(K_C + L) \end{tikzcd}$$

From this we deduce the following identification of the kernel of the multiplication map $\mu_{C,L}$,

$$R_{C,L} \cong H^0(C \times C, \mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - \Delta).$$

In turn, the Gaussian map $\gamma_{C,L}$ identifies with yet again a restriction map to the diagonal, namely the following map r_2,

$$r_2 : H^0(C \times C, \mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - \Delta) \to H^0\big(\Delta, (\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - \Delta)|_\Delta\big) \cong H^0(C, 2K_C + L).$$

For the last identification, note that $\Delta|_\Delta \cong N_{\Delta/C\times C} \cong -K_C$. The upshot is the following identification of the kernel of $\gamma_{C,L}$,

$$\ker(\gamma_{C,L}) \cong H^0(C \times C, \mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta). \tag{5.2.1}$$

Proposition 5.3. *Let C be a hyperelliptic curve of genus g, and L an effective line bundle. There is an identification*

$$\ker(\gamma_{C,L}) \cong H^0\big(C, (g-3)\mathfrak{g}\big) \otimes H^0\big(C, L - 2\mathfrak{g}\big), \tag{5.3.1}$$

where $\mathfrak{g}$ denotes the class of the g^1_2 on C.

Proof. We shall prove that the right-hand-sides of (5.3.1) and (5.2.1) are isomorphic, which suffices to prove the proposition. First assume that there exists an effective divisor

$$E \in |\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta|$$

in $C \times C$. Then for all $p \in C$, we have the following equality as divisors on the fibre $\mathrm{pr}_2^* p \cong C$,

$$E \cap \mathrm{pr}_2^* p = \textstyle\sum_{i=1}^{2g-4} p_i,$$

with the points $p_1, \dots, p_{2g-4}$ subject to the condition that $2p + \sum_{i=1}^{2g-4} p_i \in |K_C|$. In general $2p \notin \mathfrak{g}$, and then there must be two of the points p_i's, say p_1 and p_2, such that $p+p_1, p+p_2 \in \mathfrak{g}$. Then necessarily $p_1 = p_2$, and thus

$$E \cap \mathrm{pr}_2^* p = 2p_1 + \textstyle\sum_{i=3}^{2g-4} p_i \quad \text{with} \quad \begin{cases} p + p_1 \in \mathfrak{g}, \text{ and} \\ \sum_{i=3}^{2g-4} p_i \in |K_C - 2\mathfrak{g}| = |(g-3)\mathfrak{g}|. \end{cases}$$

This shows that E must contain the graph $I \subseteq C \times C$ of the hyperelliptic involution with multiplicity 2; in other words, $2I$ is a fixed part of the linear system $|\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta|$, and thus

$$H^0\big(C \times C, \mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta\big) = H^0\big(C \times C, \mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta - 2I\big).$$

The graph I is the unique divisor in $|\mathrm{pr}_1^* \mathfrak{g} + \mathrm{pr}_2^* \mathfrak{g} - \Delta|$, hence

$$\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta - 2I \sim \mathrm{pr}_1^*\big((g-3)\mathfrak{g}\big) + \mathrm{pr}_2^*\big(L - 2\mathfrak{g}\big), \tag{5.3.2}$$

and the conclusion follows if $\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta$ is effective. On the other hand, the linear equivalence (5.3.2) implies that $\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta$ is effective if $\mathrm{pr}_1^*\big((g-3)\mathfrak{g}\big) + \mathrm{pr}_2^*\big(L - 2\mathfrak{g}\big)$ is effective, hence the proposition holds unconditionally. $\square$

Proposition 5.4. *Let C be a non-hyperelliptic curve of genus 3, and L an effective line bundle on C. There is an identification*

$$\ker(\gamma_{C,L}) \cong H^0\big(C, L - 3K_C\big). \tag{5.4.1}$$

Proof. As in proof of Proposition 5.3, we will identify the right-hand-side of (5.4.1) with that of (5.2.1), and this will end the proof. Assume there exists an effective divisor

$$E \in |\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta|$$

on $C \times C$. Then for all $p \in C$, we have the following equality as divisors on the fibre $\mathrm{pr}_2^* p \cong C$,

$$E \cap \mathrm{pr}_2^*(p) = p_1 + p_2,$$

where $p_1, p_2 \in C$ are the only two points on C such that $2p + p_1 + p_2 \in K_C$: seeing C as a plane quartic, p_1, p_2 are the residual intersection points of C and its tangent line at p. Thus E must contain the tangential correspondence

$$T = \overline{\{(q,p) \in C \times C : q \neq p \text{ and } q \in \mathbf{T}_p C\}},$$

where $\mathbf{T}_p C$ denotes the tangent line to C at p in its model as a plane quartic. In other words, the linear system $|\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta|$ has T as a fixed part, and therefore

$$H^0\big(C \times C, \mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta\big) = H^0\big(C \times C, \mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta - T\big).$$

Let us now compute the class of T. For all $(q,p) \in C \times C$, we have the equalities as divisors on C,

$$T \cap \mathrm{pr}_1^*(q) = C \cap \mathrm{D}^q C - 2q \quad \text{and} \quad T \cap \mathrm{pr}_2^*(p) = C \cap \mathbf{T}_p C - 2p,$$

where $\mathrm{D}^q C$ is the first polar of q with respect to C (seen as a plane quartic), see, e.g., [18, Appendix A]. First polars with respect to a plane quartic are plane cubics, hence $\mathrm{D}^q C$ cuts out the divisor class $3K_C$ on C, since the latter is canonically embedded. We thus find

$$T \cdot \mathrm{pr}_1^*(q) \sim 3K_C - 2q \quad \text{and} \quad T \cdot \mathrm{pr}_2^*(p) \sim K_C - 2p,$$

hence

$$T \sim \mathrm{pr}_1^* K_C + p_2^*(3K_C) - 2\Delta.$$

Eventually, we find

$$\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta - T \sim \mathrm{pr}_2^*(L - 3K_C), \tag{5.4.2}$$

and this ends the proof if $\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta$ is effective. On the other hand it follows from (5.4.2) that $\mathrm{pr}_1^* K_C + \mathrm{pr}_2^* L - 2\Delta$ is effective if $L - 3K_C$ is effective, hence the result holds unconditionally. $\square$

Proposition 5.5. *Let C be a smooth projective curve of genus $g \geqslant 2$, and L an effective line bundle on C of degree $d > 0$.*

(a) *If C is hyperelliptic, then*

$$\mathrm{cork}(\gamma_{C,L}) = 2g + 2 - g \cdot h^1(L) + (g-2) \cdot h^1(L - 2\mathfrak{g}) - \mathrm{cork}(\mu_{C,L}),$$

where $\mathfrak{g}$ is the g^1_2 on C.

(b) *If C is non-hyperelliptic of genus 3, then*

$$\mathrm{cork}(\gamma_{C,L}) = h^0(4K_C - L) - 3 \cdot h^1(L) - \mathrm{cork}(\mu_{C,L}).$$

Note that if L is not effective of positive degree, the multiplication map $\mu_{C,L}$ has kernel $R_{C,L} = 0$, so that

$$\operatorname{cork}(\gamma_{C,L}) = h^0(2K_C + L).$$

Moreover we emphasize that the corank of the multiplication map $\mu_{C,L}$ may be computed in virtually any situation, using for instance Castelnuovo's Theorem 2.1 or the Base-Point-Free Pencil Trick; see also [13, §1], [24, Thm. (4.e.1)], and [35]. Part (b) of the above proposition had already appeared as [29, Prop. 2.9(a)].

Proof. One has

$$\begin{aligned}
\operatorname{cork}(\gamma_{C,L}) &= h^0(2K_C + L) - \dim(R_{C,L}) + \dim(\ker \gamma_{C,L}) \\
&= h^0(2K_C + L) - h^0(K_C)h^0(L) + h^0(K_C + L) - \operatorname{cork}(\mu_{C,L}) + \dim(\ker \gamma_{C,L}) \\
&= (3g - 3 + d) - g\bigl(1 - g + d + h^1(L)\bigr) + (g - 1 + d) - \operatorname{cork}(\mu_{C,L}) + \dim(\ker \gamma_{C,L}) \\
&= (g + 4)(g - 1) + d(2 - g) - gh^1(L) - \operatorname{cork}(\mu_{C,L}) + \dim(\ker \gamma_{C,L})
\end{aligned}$$

by Riemann–Roch and the fact that L is effective of positive degree. For hyperelliptic C, one has

$$h^0\bigl((g-3)\mathfrak{g}\bigr) = g - 2 \quad \text{and} \quad h^0(L - 2\mathfrak{g}) = d - g - 3 + h^1(L - 2\mathfrak{g}),$$

and thus Proposition 5.3 gives the result. For C a genus 3 curve, Proposition 5.4 gives the result, noting that

$$14 - d + h^0(C, L - 3K_C) = h^0(C, 4K_C - L)$$

by Riemann–Roch and Serre duality. □

(5.6) Proof of Theorem 1.3. Let us first consider the case when C is hyperelliptic. If $d \geqslant 2g + 3$, then L is very ample and non-special, hence $\mu_{C,L}$ is surjective by Castelnuovo's Theorem 2.1; moreover $h^1(L - 2\mathfrak{g}) = 0$ for degree reasons as well, so that the result follows from Proposition 5.5.

If $d \geqslant g + 4$ and L is general, then

$$L \sim p_1 + \cdots + p_g + 2\mathfrak{g} + D_0$$

for some general points $p_1, \ldots, p_g \in C$ and some effective divisor D_0. In particular we may assume that $p_1, \ldots, p_g$ impose independent conditions to the canonical series $|K_C|$, hence $h^1(L) = h^1(L - 2\mathfrak{g}) = 0$ by Serre duality, and moreover $h^1(L - q) = h^1(L - q - q') = 0$ for all $q, q' \in C$. It follows that L is very ample, hence $\mu_{C,L}$ is surjective by Castelnuovo's Theorem 2.1, and the result follows from Proposition 5.5.

We now consider the case when C is non-hyperelliptic of genus 3. If $d \geqslant 2g + 1 = 7$, then L is very ample and non-special, hence $\mu_{C,L}$ is surjective by Castelnuovo's Theorem 2.1, and the result follows from Proposition 5.5.

If $d = 2g = 6$, then L is base-point-free and non-special, and the map induced by $|L|$ may identify at most two points p and q, which happens if and only if $L = K_C + p + q$ (all this can be seen by standard considerations involving the Riemann–Roch Theorem). It follows that $\mu_{C,L}$ is surjective in this case as well, and then Proposition 5.5 gives the result.

If $g + 1 = 4 \leqslant d \leqslant 5$ and L is general, then $|L|$ is non-special, and the following may happen: if $d = 5$, $|L|$ is base-point-free and maps C to a plane quintic; if $d = 4$, $|L|$ is a base-point-free g^1_4. In all cases, it follows from Theorem 2.1 that $\mu_{C,L}$ is surjective, and the result follows from Proposition 5.5 as in the previous cases. □

6 – Ribbons and extensions

In this section we interpret the extensions of a smooth polarized curve (C, L) in terms of the integration of ribbons over (C, L), under the assumption that it satisfies Property N_2. This leads to a necessary condition for (C, L) to be extendable. We also define universal extensions, and give a criterion for their existence. Most results in this section are essentially an adaptation of some in [16], and we will thus be brief; we also propose various enhancements with respect to [16].

Definition 6.1. *Let (X, L) be a polarized variety such that L is very ample, and consider $X \subseteq \mathbf{P}^N$ the projective embedding defined by $|L|$. For all $k \in \mathbf{N}^*$, the polarized variety (X, L) is* k-extendable *if there exists $Y \subseteq \mathbf{P}^{N+k}$, not a cone, and an N-dimensional linear subspace $\Lambda \subseteq \mathbf{P}^{N+k}$ such that $X = Y \cap \Lambda$.*

In the above situation, we say that Y is a *non-trivial k-extension*, or simply a *non-trivial extension*, of (X, L). We say that (X, L) is *extendable* if it is k-extendable for some $k > 0$. The trivial extension of (X, L) denotes the cone with vertex a point over X, in its embedding defined by $|L|$.

Definition 6.2. *Let (X, L) be a polarized variety. A* ribbon *over (X, L), a.k.a. a ribbon over X with normal bundle L, is a scheme $\tilde{X}$ such that $\tilde{X}_{\mathrm{red}} = X$, and the ideal $\mathcal{I}_{X/\tilde{X}}$ defining X in $\tilde{X}$ verifies the two conditions $\mathcal{I}^2_{X/\tilde{X}} = 0$ and $\mathcal{I}_{X/\tilde{X}} = L^{-1}$.*

If $X \subseteq \mathbf{P}^N$ is smooth and $Y \subseteq \mathbf{P}^{N+1}$ is a 1-extension of $(X, \mathcal{O}_X(1))$, then the first infinitesimal neighbourhood of X in Y is a ribbon over $(X, \mathcal{O}_X(1))$ which we denote by $2X_Y$. A ribbon $\tilde{X}$ over (X, L) is *integrable* if there exists an extension Y such that $\tilde{X} = 2X_Y$; in this situation we say that the variety Y is an *integral* of the ribbon $\tilde{X}$.

A ribbon over (X, L) is uniquely determined by its extension class $e_{\tilde{X}} \in \mathrm{Ext}^1(\Omega^1_X, L^{-1})$, and two ribbons are isomorphic if and only if their extension classes are proportional. We will say a ribbon is *trivial* if its extension class is zero.

Let (X, L) be a smooth polarized manifold with L very ample, consider the corresponding embedding $X \subseteq \mathbf{P}^N$, and identify this $\mathbf{P}^N$ with a hyperplane $H \subseteq \mathbf{P}^{N+1}$. If $\tilde{X}$ is a ribbon over (X, L) contained in the first infinitesimal neighbourhood $2H_{\mathbf{P}^{N+1}}$, then its extension class lies in the kernel of the map

$$\eta : \mathrm{Ext}^1(\Omega^1_X, L^{-1}) \to \mathrm{Ext}^1(\Omega^1_{\mathbf{P}^N}\big|_X, L^{-1})$$

induced by the restriction map $\Omega^1_{\mathbf{P}^N}\big|_X \to \Omega^1_X$, as has been first observed in [44]. When X is a curve, the map η identifies with ${}^{\mathsf{T}}\gamma_{C,L}$, the transpose of the Gaussian map defined in (5.1).

Theorem 6.3. *Let (C, L) be a smooth polarized curve of genus $g \geqslant 2$ and degree $d \geqslant 2g + 3$. Then for all $v \in \ker({}^{\mathsf{T}}\gamma_{C,L})$, the ribbon $\tilde{C}_v$ with extension class v is the first infinitesimal neighbourhood of C in at most one surface, up to automorphism (see below). In particular, if (C, L) is extendable then $\gamma_{C,L}$ is not surjective.*

The precise meaning of the unicity statement above is the following. Consider $C \subseteq \mathbf{P}^{d-g}$ in its embedding defined by $|L|$, and identify this $\mathbf{P}^{d-g}$ with a hyperplane $H \subseteq \mathbf{P}^{d-g+1}$. Let $S, S' \subseteq \mathbf{P}^{d-g+1}$ be two surfaces, such that $S \cap H = S' \cap H = C$. If $2C_S \cong 2C_{S'}$, then there is exists a projectivity of $\mathbf{P}^{d-g+1}$ acting as the identity on H and mapping S to S'.

One gets the necessary condition for the integrability of (C, L) by applying the unicity statement to the zero vector $0 \in \ker({}^{\mathsf{T}}\gamma_{C,L})$. Indeed, the trivial ribbon $\tilde{C}_0$ is the first infinitesimal neighbourhood of C in the cone over C, so the unicity statement tells us that if S is a non-trivial extension of C (thus, S is not a cone), then the two ribbons $2C_S$ and $\tilde{C}_0$ are distinct, hence $2C_S$ comes from a non-zero vector in $\ker({}^{\mathsf{T}}\gamma_{C,L})$.

(6.4) We now outline the proof of Theorem 6.3, as this will be needed later on. It follows a construction given in [45]. By Theorem 2.2, the curve (C, L) satisfies Property N_2. Thus the homogeneous ideal of C in its embedding in $\mathbf{P}^{d-g}$ defined by $|L|$ has a minimal resolution as follows,

$$\cdots \longrightarrow \mathcal{O}_{\mathbf{P}^{d-g}}(-3)^{\oplus m_1} \xrightarrow{\mathbf{r}} \mathcal{O}_{\mathbf{P}^{d-g}}(-2)^{\oplus m} \xrightarrow{\mathbf{f}} \mathcal{I}_{C/\mathbf{P}^{d-g}} \longrightarrow 0. \tag{6.4.1}$$

We view $\mathbf{f}$ as a vector of quadratic equations defining scheme theoretically C, in the homogeneous coordinates $\mathbf{x} = (x_0 : \ldots : x_{d-g})$ on $\mathbf{P}^{d-g}$.

On the other hand, there is an exact sequence of vector spaces

$$0 \longrightarrow H^0(C, L)^\vee \longrightarrow H^0(C, N_{C/\mathbf{P}^{d-g}}(-1)) \longrightarrow \ker({}^\mathsf{T}\gamma_{C,L}) \longrightarrow 0, \tag{6.4.2}$$

cf. [16, Lem. 3.2] and the references given there. Let $v \in \ker({}^\mathsf{T}\gamma_{C,L})$, and choose a lift in $H^0(C, N_{C/\mathbf{P}^{d-g}}(-1))$. By (6.4.1) the latter space is a subspace of $H^0(C, \mathcal{O}_C(1))^{\oplus m}$, so we can represent the lift of v as a length m vector $\mathbf{f}_v$ of linear forms in the variable $\mathbf{x}$.

Then the ribbon $\tilde{C}_v$ with extension class v is the subscheme of $\mathbf{P}^{d-g+1}$ defined by the equations

$$\mathbf{f}(\mathbf{x}) + t\mathbf{f}_v(\mathbf{x}) = \mathbf{0}, \quad t^2 = 0 \tag{6.4.3}$$

in the homogeneous coordinates $(\mathbf{x} : t)$. In turn, any surface $S \subseteq \mathbf{P}^{d-g+1}$ containing $\tilde{C}_v$ is defined by the equations

$$\mathbf{f}(\mathbf{x}) + t\mathbf{f}_v(\mathbf{x}) + t^2\mathbf{h} = \mathbf{0}, \tag{6.4.4}$$

where $\mathbf{h}$ is a length m vector of constants, subject to conditions that we will not discuss here (see paragraph (4.9) in [16]). The upshot of these conditions however is that two vectors $\mathbf{h}$ and $\mathbf{h}'$ defining two surfaces S and S' containing the ribbon $\tilde{C}_v$ differ by an element of $H^0(C, N_{C/\mathbf{P}^{d-g}}(-2))$. Then the unicity statement follows from the vanishing of this space when (C, L) satisfies Property N_2, cf. [16, Lem. 3.6] and the references given there. $\square$

One should keep in mind the following conclusions from the above considerations. The projective space $\mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$ parametrizes isomorphism classes of non-trivial ribbons over (C, L) likely to be integrated to a non-trivial extension S of (C, L). Each such ribbon may be integrated to at most one extension, and each 1-extension conversely corresponds to a point in $\mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$.

In analogy with the terminology from deformation theory[1], we will say that the extension theory of (C, L) is *unobstructed* if every ribbon corresponding to a point of $\mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$ is integrable; otherwise we say it is *obstructed.* When the extension theory is unobstructed, we will see that we can construct a universal extension, in the following sense.

Definition 6.5. *Let (C, L) be a smooth polarized curve of genus $g \geqslant 2$ and degree $d \geqslant 2g + 3$. Let $r = \operatorname{cork}(\gamma_{C,L})$. An r-extension $Y \subseteq \mathbf{P}^{d-g+r}$ of (C, L) is* universal *if the following condition holds: for all $[v] \in \mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$, there exists a unique $(d-g+1)$-plane $\Lambda \subseteq \mathbf{P}^{d-g+r}$ containing C, such that the surface $Y \cap \Lambda$ is an integral of the ribbon over (C, L) defined by the extension class v.*

[1] in fact, if one looks at the construction in [45], one sees that this is more than a mere analogy.

(6.6) Note that, under the above assumptions, if $Y \subseteq \mathbf{P}^{d-g+k}$ is a k-extension of (C, L), then it is defined by equations

$$\mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x}) \cdot {}^{\mathsf{T}}\mathbf{t} + \mathbf{H}(\mathbf{t}) = 0,$$

in homogeneous coordinates $(\mathbf{x} : \mathbf{t})$, $\mathbf{x} = (x_0 : \ldots : x_{d-g})$ and $\mathbf{t} = (t_1 : \ldots : t_k)$, such that the span $\langle C \rangle$ is defined by $\mathbf{t} = 0$; see [36, Thm. 20.3] for instance. Here, $\mathbf{F}$ is an $m \times k$ matrix of linear forms in $\mathbf{x}$, and $\mathbf{H}$ is a length m vector, constant in $\mathbf{x}$ and quadratic in $\mathbf{t}$. One thus sees that the map

$$\Lambda \in \mathbf{P}^{d-g+k}/\langle C \rangle \longmapsto [2C_{Y\cap\Lambda}] \in \mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$$

is linear, given by the matrix $\mathbf{F}$. (Here $\mathbf{P}^{d-g+k}/\langle C \rangle$ denotes the $(k-1)$-dimensional projective space of $(d-g+1)$-planes Λ containing C, and the map associates a $(d-g+1)$-plane $\Lambda \subseteq \mathbf{P}^{d-g+r}$ containing C with the isomorphism class of the ribbon $2C_{Y\cap\Lambda}$).

Lemma 6.7. *Let (C, L) be a smooth polarized curve of genus $g \geqslant 2$ and degree $d \geqslant 2g + 3$. Let $Y \subseteq \mathbf{P}^N$ be an extension of (C, L), of dimension $1 + \operatorname{cork}(\gamma_{C,L})$. Assume that for general $[e] \in \mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$, there is a linear subspace $\Lambda \subseteq \mathbf{P}^N$ containing C and cutting out a surface on Y such that the ribbon $2C_{Y\cap\Lambda}$ has extension class e (in other words, $2C_{Y\cap\Lambda} \cong \tilde{C}_e$). Then Y is a universal extension of C.*

Proof. We consider the map

$$\Lambda \in \mathbf{P}^{d-g+\operatorname{cork}(\gamma_{C,L})}/\langle C \rangle \longmapsto [2C_{Y\cap\Lambda}] \in \mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$$

as in (6.6) above. The assumption made on Y means that this map is dominant. It is moreover linear, as has been observed in (6.6). Since its target and its source are projective spaces of the same dimension, namely $\operatorname{cork}(\gamma_{C,L}) - 1$, it is an isomorphism, which means that Y is a universal extension of (C, L). ☐

Theorem 6.8. *Let (C, L) be a smooth polarized curve of genus $g \geqslant 2$ and degree $d \geqslant 2g+3$. If the general ribbon in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$ is integrable, then all such ribbons are integrable, and there exists a universal extension of (C, L).*

Proof. Let us first prove, to fix ideas, that if all ribbons in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$ are integrable, then there exists a universal extension of (C, L). The proof is identical to that of [16, §5], so we will be brief. The idea is that we can package together all ribbons in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$ and their integrals in a projective bundle $\mathbf{P}(\mathcal{O}_{\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))}^{\oplus d-g} \oplus \mathcal{O}_{\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))}(1))$, and then the universal extension is the image of the family of all surface integrals by the map defined by the relative $\mathcal{O}(1)$ of this projective bundle. This works as follows.

One first chooses a section

$$v \in \ker({}^{\mathsf{T}}\gamma_{C,L}) \longmapsto \mathbf{f}_v \in H^0(C, N_{C/\mathbf{P}^{d-g}}(-1))$$

of (6.4.2). For all $v \in \ker({}^{\mathsf{T}}\gamma_{C,L})$, we let $\mathbf{h}_v$ be the unique vector of constants such that the integral of the ribbon $\tilde{C}_v$ is defined by the equations (6.4.4) with $\mathbf{h} = \mathbf{h}_v$. For all $\lambda \in \mathbf{C}$, one has

$$\mathbf{f}_{\lambda v} = \lambda \mathbf{f}_v, \tag{6.8.1}$$

so the ribbon $\tilde{C}_{\lambda v}$ (isomorphic to $\tilde{C}_v$) is defined by the equations

$$\mathbf{f}(\mathbf{x}) + t\mathbf{f}_{\lambda v}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \lambda t \mathbf{f}_v(\mathbf{x}) = \mathbf{0}, \quad t^2 = 0.$$

One may thus deduce the equations of the integral of $\tilde{C}_{\lambda v}$ from those of $\tilde{C}_v$, namely they are:

$$\mathbf{f}(\mathbf{x}) + \lambda t \mathbf{f}_v(\mathbf{x}) + \lambda^2 t^2 \mathbf{h}_v = \mathbf{0}.$$

By the unicity of the vector of constants $\mathbf{h}$ attached to λv, we conclude that

$$\mathbf{h}_{\lambda v} = \lambda^2 \mathbf{h}_v. \tag{6.8.2}$$

Next, let us construct the family $\mathcal{S}$ of all surface extensions of (C, L) in $\mathbf{P}(\mathcal{O}_{\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))}^{\oplus d-g} \oplus \mathcal{O}_{\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))}(1))$ by glueing affine pieces. We choose a basis $v_1, \ldots, v_r$ of $\ker({}^{\mathsf{T}}\gamma_{C,L})$, $r = \mathrm{cork}(\gamma_{C,L})$, and for $i = 1, \ldots, r$ we consider the subscheme $\mathcal{S}_i$ of $\mathbf{P}^{d-g+1} \times \mathbf{A}^{r-1}$ defined by the equations

$$\mathbf{f}(\mathbf{x}) + t\mathbf{f}(\mathbf{x})_{a_1 v_1 + \cdots + v_i + \cdots + a_r v_r} + t^2 \mathbf{h}_{a_1 v_1 + \cdots + v_i + \cdots + a_r v_r} = \mathbf{0},$$

in the homogeneous coordinates $(\mathbf{x} : t)$ on $\mathbf{P}^{d-g+1}$ and affine coordinates $(a_1, \ldots, \widehat{a_i}, \ldots, a_n)$ on $\mathbf{A}^{r-1}$ (with the convention that the term under the hat should be omitted); it is flat over $\mathbf{A}^{r-1}$. The homogeneity properties (6.8.1) and (6.8.2) ensure that any two pieces $\mathcal{S}_i$ and $\mathcal{S}_j$ glue along their open subsets defined by $(a_j \neq 0)$ and $(a_i \neq 0)$, via the isomorphism

$$([\mathbf{x} : t], a_1, \ldots, \widehat{a_i}, \ldots, a_n) \mapsto ([\mathbf{x} : a_i t], \frac{a_1}{a_j}, \ldots, \widehat{a_j}, \ldots, \frac{a_n}{a_j}).$$

The glueing of all $\mathcal{S}_1, \ldots, \mathcal{S}_r$ gives the family $\mathcal{S} \subseteq \mathbf{P}(\mathcal{O}_{\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))}^{\oplus d-g} \oplus \mathcal{O}_{\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))}(1))$ as we wanted. Eventually, the construction of the universal extension as the image of $\mathcal{S}$ by the relative $\mathcal{O}(1)$ of the projective bundle is exactly the same as in [16, Cor. 5.5].

It remains to prove that the integrability of the general ribbon implies that of all ribbons in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$. Let $[v_0] \in \mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$. If the general ribbon is integrable, we can find an arc $\mathbf{D} \subseteq \mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$ centered at $[v_0]$, such that for all $[v] \in \mathbf{D}^\circ = \mathbf{D} - [v_0]$, the corresponding ribbon $\tilde{C}_v$ is integrable. Then we have the two following families, defined by equations as above:

(i) the flat family $\tilde{\mathcal{C}} \subseteq \mathbf{P}^{d-g+1} \times \mathbf{D}$ of all ribbons $\tilde{C}_v$, $[v] \in \mathbf{D}$, and

(ii) the flat family $\mathcal{S}_{\mathbf{D}}^\circ \subseteq \mathbf{P}^{d-g+1} \times \mathbf{D}^\circ$ of the surface integrals of the ribbons $\tilde{C}_v$, $[v] \neq [v_0]$.

Taking the closure $\mathcal{S}_{\mathbf{D}}$ of $\mathcal{S}_{\mathbf{D}}^\circ$ in $\mathbf{P}^{d-g+1} \times \mathbf{D}$, we obtain a flat family of surfaces over $\mathbf{D}$. Since $\mathcal{S}_{\mathbf{D}}^\circ$ contains $\tilde{\mathcal{C}}\big|_{\mathbf{D}^\circ}$, $\mathcal{S}_{\mathbf{D}}$ will contain $\tilde{\mathcal{C}}$, hence the central fibre of $\mathcal{S}_{\mathbf{D}}$ is an integral of the ribbon $\tilde{C}_{v_0}$. Therefore all ribbons in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$ are integrable, and the theorem is proved. $\square$

(6.9) In the following sections we shall apply the above Theorem 6.8 to various specific situations, in which we know the dimension of $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$. Our strategy to verify that the general ribbon over (C, L) is integrable is to produce a family of extensions of (C, L) of the same dimension as $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$. Then, by the Unicity Theorem 6.3, there is an injective map from the parameter space of this family of extensions to $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$, which is dominant for dimension reasons.

Eventually, let us note that all the above considerations may be adapted to polarized manifolds (X, L) of arbitrary dimension. The only difference is that the exact sequence (6.4.2) should be slightly modified, see [16, Lem. 3.5].

7 – Extensions of polarized genus three curves

In this section, we study closely the extensions of polarized curves of genus 3 and degree $d \geqslant 2g + 3$, in order to determine whether their ribbons are obstructed or not. Our main output in this direction is Theorem 1.7.

(7.1) Classification of surfaces with sectional genus 3. The classification of rational surfaces with hyperplane sections non-hyperelliptic curves of genus 3 had been classically worked out by Castelnuovo [9]. He proved that all such surfaces are represented by a linear system of plane quartics. More recently Lanteri and Livorni [31] have classified all pairs (S, C) where S is a smooth surface, $C \subseteq S$ is a smooth genus 3 curve, and the linear system $|C|$ is globally generated and ample. For $d \geqslant 8$, Theorem 1.2 provides more generally the classification of surfaces, possibly singular, with one hyperplane section a linearly normal non-hyperelliptic curve of genus 3.

Corollary 7.2. *Let (C, L) be a non-hyperelliptic polarized curve of genus $g = 3$ and degree $d \geqslant 4g - 4 = 8$. Then all surface extensions of (C, L) are rational. If $d \geqslant 9$, they are all realized by a linear system of plane quartics; if $d = 8$, they are realized either by a linear system of plane quartics, or by a complete linear system of plane sextics with seven base points of multiplicity two as in Example 3.3.*

Proof. This is a direct application of Theorem 1.2. □

Lemma 7.3. *Let C be a non-hyperelliptic curve of genus 3, and L be a line bundle of degree $d \geqslant 0$ on it. Then $h^0(C, 4K_C - L)$ takes the following values:*
— *if $d > 16$, then $h^0(4K_C - L) = 0$;*
— *if $d = 16$, then $h^0(4K_C - L) = 1$ if $L = 4K_C$ and 0 otherwise;*
— *if $d = 15$, then $h^0(4K_C - L) = 1$ if $L = 4K_C - p$ for some $p \in C$, and 0 otherwise;*
— *if $d = 14$, then $h^0(4K_C - L) = 1$ if $L = 4K_C - p - q$ for some $p, q \in C$, and 0 otherwise;*
— *if $d = 13$, then L may always be written as $4K_C - p - q - r$ for some $p, q, r \in C$, and $h^0(4K_C - L) = 2$ if these three points are aligned on the canonical model of C, and 1 otherwise;*
— *if $d = 12$, then L may always be written as $4K_C - p - q - r - s$ for some $p, q, r, s \in C$, and $h^0(4K_C - L) = 3$ if these four points are aligned on the canonical model of C, and 2 otherwise;*
— *if $d < 12$, then $h^0(4K_C - L) = 14 - d$.*

Proof. If $d \geqslant 16$, then $\deg(4K_C - L) \leqslant 0$ and the result is clear. If $d = 15$, we can always write $L = 4K_C - p_0 + N$ for some arbitrarily chosen point $p_0 \in C$ and some degree 0 line bundle N. If $h^0(4K_C - L) > 0$, then $p_0 - N \sim p$ for some $p \in C$, hence $L = 4K_C - p$. In this case, $h^0(4K_C - L) = h^0(p) = 1$. If $d = 14$, it follows as in the previous case that $L = 4K_C - p - q$ if $h^0(4K_C - L) > 0$. In this case, $h^0(4K_C - L) = h^0(p + q) = 1$ since C is non-hyperelliptic. If $d \leqslant 13$, by Jacobi's inversion theorem (see [1, p. 19]) we can always write $L = 4K_C - \sum_{i=1}^{16-d} p_i$ for some points $p_1, \ldots, p_{16-d}$, and then the result follows by Riemann–Roch and Serre duality. □

(7.4) Proof of Theorem 1.7. The only if part of (1.7.1) is a direct consequence of Theorem 6.3, taking into account Theorem 1.3 and Lemma 7.3. On the other hand, Example 3.2 provides an extension for all (C, L) with $L = 4K_C - \sum_{i=1}^{16-d} p_i$, thus proving the if part of (1.7.1): explicitly, this goes as follows. Consider C in its canonical embedding as a smooth plane quartic, and let $\varepsilon : S \to \mathbf{P}^2$ be the blow-up of the plane at the points $p_1, \ldots, p_{16-d} \in C \subseteq \mathbf{P}^2$, with exceptional divisors $E_1, \ldots, E_{16-d}$ (when some p_i's coincide, this means that we blow-up infinitely near points). Let H denote the pull-back to S of the line class on $\mathbf{P}^2$. The linear system $|4H - \sum_{i=1}^{16-d} E_i|$ on S cuts out the complete linear series $|4K_C - \sum_{i=1}^{16-d} p_i| = |L|$ on the proper transform of C, which is very ample; it is thus base point free, and defines a birational morphism from S to an extension of (C, L).

Let us now prove (1.7.2). If $d \geqslant 14$, there is nothing to add to (1.7.1), since in these cases $\operatorname{cork}(\gamma_{C,L})$ is either 0 or 1 by Lemma 7.3, so we will suppose $d \leqslant 13$. Given $[Z] \in (\mathbf{P}^2)^{[16-d]}$

a length $16-d$, 0-dimensional subscheme $Z \subseteq \mathbf{P}^2$, we let S_Z be the blow-up of $\mathbf{P}^2$ along Z, with total exceptional divisor E_Z, and call H the pull-back of the line class on $\mathbf{P}^2$. The locus $\mathcal{S} \subseteq (\mathbf{P}^2)^{[16-d]}$ parametrizing those Z such that the linear system $|4H - E_Z|$ on S_Z contains a smooth curve is dense. Moreover, for all $[Z] \in \mathcal{S}$, since $d \geqslant 9$, this linear system has dimension $d-2$, is base point free, and defines a birational morphism. We consider the universal family

$$\mathcal{L} \to \mathcal{S}$$

of these linear systems, and the dense open subset $\mathcal{L}^\circ \subseteq \mathcal{L}$ consisting of those pairs (Z, C) such that C is a smooth member of $|4H - E_Z|$ on S_Z. After dividing out by the automorphism group of $\mathbf{P}^2$, we get the moduli space $\mathcal{SC}$ of such pairs, which has dimension

$$\dim\big((\mathbf{P}^2)^{[16-d]}\big) + (d-2) - 8 = 22 - d.$$

Next we consider the universal Jacobian $\mathcal{J}_3^d$ parametrizing degree d line bundle on genus 3 curves, which has dimension $4g - 3 = 9$, and its dense subset $\mathcal{J}^\circ$ corresponding to non-hyperelliptic curves. We shall examine the map

$$c : (Z, C) \in \mathcal{SC} \mapsto \big[C, \mathcal{O}_{S_Z}(4H - E_Z)|_C\big] \in \mathcal{J}^\circ,$$

the fiber of which over a point (C, L) consists of distinct isomorphism classes of extensions of (C, L). By our proof of the if part of (1.7.1), the image of c is the locus of those (C, L) such that L may be written as $4K_C - \sum_{i=1}^{16-d} p_i$, which is the whole $\mathcal{J}^\circ$ since we are assuming $d \leqslant 13$. Therefore all fibres of c have dimension at least

$$\dim(\mathcal{SC}) - \dim(\mathcal{J}^\circ) = 13 - d.$$

If $d < 12$, this proves that for all (C, L), the general ribbon in $\mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$ is integrable, by Lemma 7.3 and the argument given in (6.9). We conclude by Theorem 6.8 that all ribbons are integrable, and there exists a universal extension.

If $d = 12$ or 13, the same argument proves (1.7.2) for all (C, L) such that $L = 4K_C - \sum_{i=1}^{16-d} p_i$ with $p_1, \ldots, p_{16-d}$ not all on a line in $\mathbf{P}^2$.

We now treat the two remaining cases separately, in the same spirit as the others. First assume $d = 13$, and let S_1 be the blow-up of $\mathbf{P}^2$ along three pairwise distinct points lying on a line, with total exceptional divisor E_1. Consider the locus of smooth quartics in the linear system $|4H - E_1|$, which has dimension 11, and take its quotient by the automorphism group of S_1, which is the subgroup of the projectivities of $\mathbf{P}^2$ acting as the identity on a line, hence has dimension 3 (it is the group of homotheties and translations of the affine plane). We thus get the moduli space $\mathcal{SC}_1$ of pairs (S_1, C), of dimension 8. The image of $\mathcal{SC}_1$ by the map $c : \mathcal{L}^\circ \to \mathcal{J}^\circ$ is the locus $\mathcal{J}^1$ of pairs (C, L) such that $L = 4K_C - D$ with $|D|$ a g^1_3, which has dimension 7 (note that if $p_1, p_2, p_3 \in C \subseteq \mathbf{P}^2$ are aligned, then they move in a base-point-free g^1_3, and thus up to linear equivalence we may always assume that they are pairwise distinct). The upshot is that the fibers of the map

$$c|_{\mathcal{SC}_1} : \mathcal{SC}_1 \to \mathcal{J}^1$$

all have dimension at least 1, and we conclude as in the previous cases.

In the case $d = 12$, we fix three pairwise distinct points q_1, q_2, q_3 on a line in $\mathbf{P}^2$, and let $\mathcal{S}_2$ be the complement of $\{q_1, q_2, q_3\}$ in this line. For all $q \in \mathcal{S}_2$, we let S_q be the blow-up of $\mathbf{P}^2$ at q_1, q_2, q_3 and q, with total exceptional divisor E_q. The S_q's form a 1-dimensional family of pairwise non-isomorphic surfaces, all with automorphism group the subgroup of the projectivities of $\mathbf{P}^2$ acting as the identity on a line, of dimension 3. Quotienting the universal family of the linear systems $|4H - E_q|$, each of dimension 10, by this automorphism group, we get the moduli space $\mathcal{SC}_2$ of pairs (S_q, C), of dimension $10 + 1 - 3 = 8$. The image of $\mathcal{SC}_2$ by the map $c : \mathcal{L}^\circ \to \mathcal{J}^\circ$ is the locus $\mathcal{J}^2$ of pairs $(C, 3K_C)$, which has dimension 6. Therefore the fibers of the map

$$c|_{\mathcal{SC}_2} : \mathcal{SC}_2 \to \mathcal{J}^2$$

all have dimension at least 2, and we conclude as in the previous cases. □

Remark 7.5. In the case when $L = 4K_C - D$ with $|D|$ a g^1_3, there exist extensions of (C, L) supported on a surface different from S_1. Indeed, choosing a member of $|D|$ of the form $2p_1 + p_2$, we see that the blow-up S'_1 of $\mathbf{P}^2$ along three aligned points, two of which are infinitely near, provides an extension of (C, L). This surface is rigid as S_1 is, but its automorphism group is larger: it is the subgroup of those projectivities fixing the two points p_1 and p_2 (hence also leaving the line $\langle p_1, p_2\rangle$ stable), which has dimension 4. We thus get a moduli space $\mathcal{SC}'_1$ of dimension $11 - 4 = 7$ which surjects onto $\mathcal{J}^1$. Thus for all $(C, L) \in \mathcal{J}^1$, there is at least one extension supported on the surface S'_1.

For all C, there are finitely many g^1_3's having a member of the form $3p_1$, and the corresponding pairs (C, L) form a 6-dimensional locus in $\mathcal{J}^\circ$. These pairs have an extension supported on S''_1, the blow-up of $\mathbf{P}^2$ at three infinitely near points lying on a line. The surface S''_1 is rigid, and has an automorphism group of dimension 5.

Similar considerations may be made about the extensions of the polarized curves $(C, 3K_C)$.

Remark 7.6. One may want to prove Corollary 7.2 for $d \geqslant 9$ directly with the above considerations, without resorting to Theorem 1.2. Our proof of (1.7.2) shows that for all (C, L), the general extension of (C, L) is rational and realized by a linear system of plane quartics. However it is not clear to us why these two properties should be preserved when one specializes to an arbitrary extension of (C, L).

(7.7) Remarks on the extensions in degree $d < 2g + 3$. In this case Green's Theorem 2.2 no longer applies to guarantee that (C, L) verifies Property N_2, so that in particular one ribbon may a priori have several different integrals.

We will mostly concentrate on the case $d = 2g + 2 = 8$. If $L \neq 2K_C$ then all extensions of (C, L) are realized by a linear sytem of plane quartics by Corollary 7.2, and the analysis carried out in the proof of Theorem 1.7 applies mutatis mutandis. We find that if (C, L) is general, then the extensions of (C, L) form a family of the expected dimension

$$5 = \operatorname{cork}(\gamma_{C,L}) - 1 = h^0(4K_C - 2L) - 1.$$

We cannot say much more, however because of the possible failure of Property N_2.

The case $L = 2K_C$ is more interesting. In this case we still have, for general C, a 5-dimensional family of extensions of $(C, 2K_C)$ given by linear systems of plane quartics: these extensions are obtained by projecting $v_4(\mathbf{P}^2)$ from eight points that, in $\mathbf{P}^2$, lie on a conic, so that they sum to a bicanonical divisor of any quartic containing them. There is however another family of extensions, given by complete linear systems of plane sextics with seven base points of multiplicity two. One finds, using similar arguments as in the proof of Theorem 1.7 above, that for general C these form a 6-dimensional family. We thus find two independent families of extensions of $(C, 2K_C)$, one of the expected dimension and one superabundant.[2]

In degree $d < 8$ the situation is similar. In fact, linear systems of sextics with seven double base points of degree $d < 8$ necessarily have further base points, so that the funny situation for

[2]We note in addition that those surfaces obtained by a linear system of quartics with eight base points lying on a conic have a singularity of type $\frac{1}{4}(1, 1)$, i.e., it is locally the cone over a rational normal quartic curve. We emphasize also that sextics with seven double points are Cremona-minimal hence not Cremona-equivalent to smooth quartics, see [7] and [33].

$2K_C$ analyzed above now happens for more line bundles (e.g., in degree 7 it happens for those line bundles that may be written as $2K_C - p$ for some $p \in C$, which form a 1-dimensional family in the Jacobian of C). Note however that for $d < 8$ we are out of the range of application of Theorem 1.2 and Corollary 7.2 so that there may be even more families of extensions.

(7.8) Invariants of the universal extensions. We now list the degrees and dimensions of the universal extensions of genus 3 curves gotten above, limiting ourselves to the case when the dimension is at least 3:

— if $d = 13$, and $L = 4K_C - p - q - r$ for some $p, q, r \in C$ on a line in the canonical model of C, we find a threefold X of degree $d = 13$ in $\mathbf{P}^{12}$;

— if $d = 12$ and $L = 4K_C - p - q - r - s$ for some $p, q, r, s \in C$ not on a line in the canonical model of C, we find a threefold X of degree $d = 12$ in $\mathbf{P}^{11}$;

— if $d = 12$ and $L = 3K_C$, we find a fourfold X of degree $d = 12$ in $\mathbf{P}^{12}$;

— if $9 \leqslant d \leqslant 11$, we find a $(15 - d)$-dimensional variety of degree d in $\mathbf{P}^{11}$ (note that $4 \leqslant 15 - d \leqslant 6$).

Smooth projective varieties of degree $d = 9, 10, 11$ have been classified in [21], [22], and [6] respectively. Running through the correspondings lists, we notice that there is no smooth 6-fold of degree 9 in $\mathbf{P}^{11}$, hence the universal extension in degree 9 is certainly singular. By contrast, in degree 10 and 11, there are smooth 5-fold and 4-fold of sectional genus 3 in $\mathbf{P}^{11}$, namely $\mathbf{P}^1 \times \mathbf{Q}^4$ in the Segre embedding, and a scroll over $\mathbf{P}^2$, respectively. It is possible that these coincide with the universal extensions above, but we will not investigate this now.

8 – Extensions of polarized hyperelliptic curves

In this section, we study closely the extensions of polarized hyperelliptic curves of genus g and degree $d \geqslant 2g + 3$, in order to determine whether their extension theory is obstructed or not. Our main output in this direction is Theorem 1.8. Our general strategy is similar to that employed for genus 3 curves in Section 7, but the situation for hyperelliptic curves is slightly more complicated and requires more care.

(8.1) For all $\mu = g + 1, g, \ldots, 0$, we let $\mathcal{H}_\mu$ be the complete linear system

$$|2H + \mu F| = |2E + (2g + 2 - \mu)F| \quad \text{on } \mathbf{F}_{g+1-\mu}.$$

It defines a projective surface of degree $4g + 4$ in $\mathbf{P}^{3g+5}$, which is the maximum possible degree for a non-trivial extension of a linearly normal curve of genus $g \geqslant 2$, see Corollary 2.6. Since we focus on polarized curves of degree $d \geqslant 2g + 3$, in order for Property N_2 to hold, the maximal number of points from which we may project these surfaces is

$$b_{\max} = 2g + 1. \tag{8.1.1}$$

Proposition 8.2. *Let C be a hyperelliptic curve of genus g, and $\mathfrak{e}$ an effective divisor of degree μ on C, $0 \leqslant \mu \leqslant g + 1$. There exists an embedding of C as a member of the linear system $|2H + \mu F|$ on $\mathbf{F}_{g+1-\mu}$ such that $E|_C = \mathfrak{e}$ if and only if no two points of $\mathfrak{e}$ are conjugate with respect to the hyperelliptic involution on C.*

Proof of the only if part of Proposition 8.2. Suppose C is a smooth member of the linear system $|2H + \mu F|$ on $\mathbf{F}_{g+1-\mu}$, and let

$$E|_C = e_1 + \cdots + e_\mu.$$

Consider two points e_i and e_j of $E|_C$, distinct in the sense that $i \neq j$. Since $E \cdot F = 1$ and the g^1_2 on C is cut out by F, the only possibility for e_i and e_j to be conjugate is that $e_i = e_j$ and it is a ramification point of the g^1_2. However, the condition that $e_i = e_j$ means that C is tangent to E at this point, and the condition that it is a ramification point of the g^1_2 means that C is tangent to the fiber at this point, so these two conditions may not be realized simultaneously as C is smooth. The conclusion is thus that for all $i \neq j$, e_i and e_j are not conjugate. $\square$

In case $\mu \leqslant g$, this can also be proved by cohomological considerations, as the condition we want to prove is then equivalent to $h^0(C, E|_C) = 1$, which in turn is equivalent to $h^1(\mathbf{F}_{g+1-\mu}, E - C) = 0$, as can be seen by considering the restriction exact sequence. The latter vanishing holds because $E - C$ has vanishing h^0 and h^2 (the latter by Serre duality), and $\chi(E - F) = 0$ by Riemann–Roch.

Proof of the if part of Proposition 8.2. Let C be a hyperelliptic curve of genus g, and $\mathfrak{e}$ an effective divisor of degree μ on C, no two points of which are conjugate. We let $\mathfrak{f}$ be the class of the g^1_2, and $\mathfrak{h} = \mathfrak{e} + (g + 1 - \mu)\mathfrak{f}$. One has

$$K_C - \mathfrak{h} = (g-1)\mathfrak{f} - \mathfrak{e} - (g+1-\mu)\mathfrak{f} = (\mu - 2)\mathfrak{f} - \mathfrak{e},$$

so $\mathfrak{h}$ is non-special since our assumption on $\mathfrak{e}$ implies that it imposes independent conditions to any multiple of $\mathfrak{f}$. Therefore, $h^0(C, \mathfrak{h}) = g - \mu + 3$ by Riemann–Roch.

We first consider the case $\mu \leqslant g$. Let v be a basis of $H^0(C, \mathfrak{e})$, and s, t be a basis of $H^0(C, \mathfrak{f})$. Then $vs^{g+1-\mu}, \ldots, vt^{g+1-\mu}$ are linearly independent, and span a hyperplane in $H^0(C, \mathfrak{h})$. Eventually, we choose u such that $u, vs^{g+1-\mu}, \ldots, vt^{g+1-\mu}$ form a basis of $H^0(C, \mathfrak{h})$. Arguing in the same way as above, we find that $H^0(C, \mathfrak{h} + \mathfrak{f})$ has dimension $g - \mu + 5$, and $us, ut, vs^{g-\mu+2}, \ldots, vt^{g-\mu+2}$ is a basis of this space. Thus the linear system $|\mathfrak{h} + \mathfrak{f}|$ maps C in $\mathbf{P}^{g-\mu+4}$ in such a way that it sits in a rational normal scroll built on a line and a rational normal curve of degree $g - \mu + 2$ spanning complementary subspaces, that is, $\mathbf{F}_{g+1-\mu}$ in its embedding given by $|H + F|$, and $F|_C = \mathfrak{f}$ and $H|_C = \mathfrak{h}$. This ends the proof in this case, as $H = E + (g + 1 - \mu)F$. The argument is similar when $\mu = g + 1$, and we leave it to the reader. $\square$

(8.3) Proof of (1.8.1)**.** Let (C, L) be a polarized hyperelliptic curve of genus g and degree d. We first prove the result in case $d = 4g + 4$. We shall see that there are two ways of proceeding in order to write L as $2\mathfrak{h} + \mu\mathfrak{f}$, so as to be able to apply Proposition 8.2. We call these the even and odd way, respectively (note that one way is sufficient to prove (1.8.1)). We keep the notation $\mathfrak{f}$ for the class of the g^1_2 on C, and write $g = 2\gamma + \varepsilon$, $\varepsilon \in \{0, 1\}$ and $\gamma \in \mathbf{N}$.

In the even way, one first chooses a line bundle M^+ of (even) degree $2g + 2$ such that $L = 2M^+$. Then, there is a unique integer $k \in \{0, \ldots, \gamma + \varepsilon\}$ such that M^+ may be written as

$$M^+ = \mathfrak{e} + (\gamma + 1 + k)\mathfrak{f},$$

with the condition that $\mathfrak{e}$ is the sum of $g + \varepsilon - 2k$ points pairwise not conjugate with respect to the g^1_2 (k is thus the largest integer such that $M^+ - (\gamma + 1 + k)\mathfrak{f}$ is effective). We set

$$\mu = g + \varepsilon - 2k \text{ (even)} \quad \text{and} \quad \mathfrak{h} = \mathfrak{e} + (g + 1 - \mu)\mathfrak{f}.$$

By Proposition 8.2 there exists an embedding of C as a member of the linear system $|2H + \mu F|$ on $\mathbf{F}_{g+1-\mu}$ such that $E|_C = \mathfrak{e}$ and $F|_C = \mathfrak{f}$. The normal bundle of C in this embedding is

$$\begin{aligned} N_{C/\mathbf{F}_{g+1-\mu}} &= 2\mathfrak{h} + \mu\mathfrak{f} \\ &= 2\mathfrak{e} + (2g + 2 - \mu)\mathfrak{f} \\ &= 2\mathfrak{e} + 2(\gamma + 1 + k)\mathfrak{f} = 2M^+ = L, \end{aligned}$$

hence the embedding of $\mathbf{F}_{g+1-\mu}$ defined by the complete linear system $|2H+\mu F|$ is an extension of (C,L), as we wanted.

In the odd way, one chooses instead a line bundle M^- of (odd) degree $2g+1$ such that $L = 2M^- + \mathfrak{f}$. Then, there is a unique integer $k \in \{0,\dots,\gamma\}$ such that M^- may be written as

$$M^- = \mathfrak{e} + (\gamma + \varepsilon + k)\mathfrak{f},$$

with $\mathfrak{e}$ is the sum of $g+1-\varepsilon-2k$ points pairwise not conjugate with respect to the g^1_2. We set

$$\mu = g+1-\varepsilon-2k \text{ (odd)} \quad \text{and} \quad \mathfrak{h} = \mathfrak{e} + (g+1-\mu)\mathfrak{f}.$$

By Proposition 8.2 there exists an embedding of C as a member of the linear system $|2H+\mu F|$ on $\mathbf{F}_{g+1-\mu}$ such that $E|_C = \mathfrak{e}$ and $F|_C = \mathfrak{f}$. The normal bundle of C in this embedding is

$$\begin{aligned} N_{C/\mathbf{F}_{g+1-\mu}} &= 2\mathfrak{h} + \mu\mathfrak{f} \\ &= 2\mathfrak{e} + (2g+2-\mu)\mathfrak{f} \\ &= 2\mathfrak{e} + [2(\gamma+\varepsilon+k)+1]\mathfrak{f} = 2M^- + \mathfrak{f} = L, \end{aligned}$$

which proves that (C,L) is extendable in the same fashion as in the even way.

It remains to prove (1.8.1) when $d < 4g+4$. In this case, we choose $b = 4g+4-d$ distinct points $p_1,\dots,p_b$ on C, to the effect that

$$L^\sharp = L(p_1 + \cdots + p_b)$$

is a line bundle of degree $4g+4$ on C. Thus there exists a non-trivial extension $S^\sharp$ of $(C,L^\sharp)$ by the case $d = 4g+4$, and we get an extension of (C,L) by simple internal projection of $S^\sharp$ from the points $p_1,\dots,p_b$, as in Example 3.4. ☐

To prove the remaining parts of Theorem 1.8, we will apply Proposition 8.2 in the following way.

Corollary 8.4. *Let C be a hyperelliptic curve of genus g, $\mu \in \{g+1,g,\dots,0\}$, and b a non-negative integer. The locus in the Jacobian $J^{4g+4-b}(C)$ of degree $4g+4-b$ line bundles L on C such that L is the normal bundle of C in a simple internal projection from b points of the surface $(\mathbf{F}_{g+1-\mu}, 2H+\mu F)$ has dimension* $\min(g,\mu+b)$.

More precisely, the locus in the corollary parametrizes those L such that the following holds: there exists an embedding of C as a smooth member of the linear system $|2H+\mu F|$ on $\mathbf{F}_{g+1-\mu}$ passing through b points $p_1,\dots,p_b \in \mathbf{F}_{g+1-\mu}$, such that L is the normal bundle of the proper transform of C in the blow-up of $\mathbf{F}_{g+1-\mu}$ at $p_1,\dots,p_b$.

Proof. It follows from Proposition 8.2 that the locus $\mathcal{Z} \subseteq J^{2g+2-\mu}(C)$ of degree $2g+2-\mu$ line bundles $\mathfrak{h}$ such that there exists an embedding of C as a member of $|2H+\mu F|$ on $\mathbf{F}_{g+1-\mu}$ such that $H|_C = \mathfrak{h}$ has dimension $\min(g,\mu)$. In turn, the locus we are interested in is the image of $\mathcal{Z} \times C^{[b]}$, where $C^{[b]}$ denotes the b-th symmetric power of C, by the map

$$j_{\mu,b} : (\mathfrak{h}, D) \in J^{2g+2-\mu}(C) \times C^{[b]} \longmapsto 2\mathfrak{h} + \mu\mathfrak{f} - D \in J^{4g+4-b}(C), \tag{8.4.1}$$

with $\mathfrak{f}$ the class of the g^1_2, hence its dimension is $\min(g,\mu+b)$. ☐

(8.5) Proof of (1.8.2) **and** (1.8.3)**.** For all $\mu = g+1, g, \ldots, 0$, we consider as in the proof of Theorem 1.7 the universal family $\mathcal{L}_{\mu,b} \to \mathcal{S}_{\mu,b}$ of the linear systems $|2H + \mu F - E_D|$ on the blow-up of $\mathbf{F}_{g+1-\mu}$ along $D \in \mathcal{S}_{\mu,b} = (\mathbf{F}_{g+1-\mu})^{[b]}$, with total exceptional divisor E_D. The total space $\mathcal{L}_{\mu,b}$ has dimension $3g+5+b$. Then we divide out by the automorphism group of $\mathbf{F}_{g+1-\mu}$, which has dimension $5+\max(1, g+1-\mu)$, and end up with a moduli space $\mathcal{SC}_{\mu,b}$ of pairs (S, C) which has dimension

$$2g + b - 1 + \min(g, \mu).$$

Next we consider the map

$$c_{\mu,b} : (S, C) \in \mathcal{SC}_{\mu,b} \mapsto (C, N_{C/S}) \in \mathcal{J}_g^{4g+4-b}$$

to the universal Jacobian, which by Corollary 8.4 surjects onto an irreducible locus of dimension

$$2g - 1 + \min(g, \mu + b)$$

(recall that the hyperelliptic locus in $\mathcal{M}_g$ has dimension $2g-1$). Thus, the general fibers of $c_{\mu,b}$ have dimension

$$\delta_{\mu,b} = b + \min(g, \mu) - \min(g, \mu + b) = \begin{cases} 0 & \text{if } \mu + b \leqslant g \\ b + \mu - g & \text{if } g - b \leqslant \mu \leqslant g \\ b & \text{if } \mu = g+1, \end{cases}$$

and all fibres of $c_{\mu,b}$ have dimension at least this.

Now, by Corollary 4.6 the family of all possible extensions of (C, L) is parametrized by the unions of the fibers of the maps $c_{\mu,b}$, $\mu = g+1, g, \ldots, 0$ and $b = 4g+4-d$. To prove Theorem 1.8 it is thus sufficient, by the argument given in (6.9), to compare the dimension $\delta_{\mu,b}$ above with the dimension of $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L}))$, which by Theorem 1.3 happens to equal $b_{\max}$, cf. (8.1.1). After that the conclusion will follow as in the proof of Theorem 1.7.

One finds that $\delta_{\mu,b} \leqslant b$ always holds, and thus

$$\delta_{\mu,b} = \operatorname{cork}(\gamma_{C,L}) - 1 \quad \Longleftrightarrow \quad b = b_{\max} \text{ and } \mu = g+1, g,$$

which ends the proof of the theorem. □

(8.6) Remarks on the extensions in degree $d < 2g+3$. We can extend the above analysis to degrees lower than $2g+3$, even though in this range Green's Theorem 2.2 no longer applies to guarantee that (C, L) verifies Property N_2, and as a consequence much of our results about ribbons and extensions are no longer usable.

We find that for general (C, L) there is a family of extensions of dimension

$$4g + 4 - \deg(L) > 2g + 1,$$

whereas the corank of $\gamma_{C,L}$ still takes the same value $2g+2$ by Theorem 1.3, provided we don't consider degrees lower than $g+4$. Thus in this case the extensions of (C, L) form a superabundant family.

(8.7) Discussion of universal extensions. Let us compare the universal extensions of degree $2g+3$ and dimension $2g+3$ in $\mathbf{P}^{3g+5}$ gotten in Theorem 1.8 with the classification results in degree $d \leqslant 11$ available in the literature [28, 21, 22, 6]. The only relevant case occur in genus $g = 2, 3, 4$, and no smooth variety with the appropriate invariants exists in these cases. Therefore the corresponding universal extensions are certainly singular.

Remark 8.8. When L has degree $4g+4$, there are in general only finitely many extensions of (C,L), even though $\mathbf{P}(\ker({}^\mathsf{T}\gamma_{C,L}))$ has dimension $2g+1$ (indeed, $\delta_{\mu,0}=0$ in the above notation). From the arguments given in (8.3), it follows that one can always find at least two different extensions of (C,L), with underlying surfaces respectively the rational ruled surfaces $\mathbf{F}_{2k+1-\varepsilon}$ (from the even way), and $\mathbf{F}_{2k+\varepsilon}$ (from the odd way); in addition there are 2^{2g} possible choices for M^+ and M^- respectively. In this case, it is clear that there cannot exist a universal extension of (C,L).

Moreover, it is in general possible to find two M's with the same parity but leading to different values of μ. For instance, in the even case, if we have

$$M_1=\mathfrak{e}_1+(\gamma_1+1+k_1)\mathfrak{f} \quad \text{and} \quad M_2=\mathfrak{e}_2+(\gamma_2+1+k_2)\mathfrak{f},$$

the condition that $2M_1=2M_2$ amounts to

$$2\mathfrak{e}_1=2\mathfrak{e}_2+(k_2-k_1)\mathfrak{f},$$

which is fulfilled if the difference $\mathfrak{e}_1-\mathfrak{e}_2$ consists of ramification points of the g^1_2.

When the degree of L gets smaller than $4g+4$, i.e., when the number b of points from which one projects becomes positive, a given surface may be obtained by projecting two scrolls with different values of μ. Indeed, the sublinear system of $|2H+\mu F|$ on $\mathbf{F}_{g+1-\mu}$ of curves passing through a point p off $E\subseteq\mathbf{F}_{g+1-\mu}$ corresponds to the sublinear system of $|2H+(\mu+1)F|$ on $\mathbf{F}_{g-\mu}$ of curves passing through a point p' on $E\subseteq\mathbf{F}_{g-\mu}$ via the elementary transformation $\mathbf{F}_{g+1-\mu}\dashrightarrow\mathbf{F}_{g-\mu}$ based at p.

Thus one sees that the various loci $j_{\mu,b}(\mathcal{Z}\times C^{[b]})$ (in the notation of the proof of Corollary 8.4), are in general not disjoint as μ ranges from 0 to $g+1$.

9 – Extensions of pluricanonical curves

In this section we study the extensions of pluricanonical curves, i.e., polarized curves (C,mK_C) with $m>1$. In particular we will prove Theorems 1.4 and 1.9. The case of canonical curves is of a different nature and will not be discussed here; we refer for instance to [2] and [16].

9.1 – Corank of the Gaussian map

In this subsection we give the corank of the Gaussian map γ_{C,mK_C} for all non-hyperelliptic curves and $m>1$, which will prove Theorem 1.4. We rely on the following identification.

(9.1) Let (C,L) be a polarized curve, with L very ample, and N_C the normal bundle in the embedding given by $|L|$. By [17, Prop. 1.2] we have:

$$\forall m\geqslant 2:\quad H^0\big(N_C(-m)\big)\cong\ker({}^\mathsf{T}\phi_{K_C+(m-1)L,L}), \tag{9.1.1}$$

where for any two line bundles M,N on C, $\phi_{M,N}$ is a Gaussian map $R(M,N)\to H^0(K_C+M+N)$ defined in a similar way as in (5.1), with $R(M,N)$ the kernel of the multiplication map $H^0(M)\otimes H^0(N)\to H^0(M+N)$.

In particular, for $L=K_C$ the map $\phi_{K_C+(m-1)K_C,K_C}=\phi_{mK_C,K_C}$ is exactly the map γ_{C,mK_C} defined in (5.1) and considered in the present article. Thus we obtain, as a particular case of (9.1.1): if C is non-hyperelliptic, then

$$\forall m\geqslant 2:\quad H^0\big(N_{C/\mathbf{P}^{g-1}}(-m)\big)\cong\ker({}^\mathsf{T}\gamma_{C,mK_C}), \tag{9.1.2}$$

where the normal bundle $N_{C/\mathbf{P}^{g-1}}$ is that of the canonical embedding of C.

(9.2) Proof of Theorem 1.4. The identification (9.1.2) will enable us to compute γ_{C,mK_C} in all cases, using the values of $h^0(N_{C/\mathbf{P}^{g-1}}(-m))$ given in [30]. First of all, one has

$$h^0(N_{C/\mathbf{P}^{g-1}}(-m)) = 0 \quad \text{if} \quad m > 2 \text{ and } g \geqslant 5,$$

see [30, pp.58–59] and [29]. Moreover, one has

$$h^0(N_{C/\mathbf{P}^{g-1}}(-m)) = 0 \quad \text{if} \quad m \geqslant 2 \text{ and } \mathrm{Cliff}(C) > 2,$$

since canonical curves with Clifford index larger than 2 satisfy Property N_2 by [43, 38], which is well-known to imply the asserted vanishing, see [16, Lem. 3.6] and the references given there.

Thus, to prove the theorem it only remains to prove that $\mathrm{cork}(\gamma_{C,2K_C})$ takes the asserted values for curves of Clifford index 1 or 2. This is readily given in [30] in the following cases:
— plane quintics [30, Prop. 3.2] and sextics [30, Prop. 4.3];
— tetragonal curves of genus $g \geqslant 6$ that are not a plane quintic [30, Prop. 4.1] (this comprises the cases (b) and (c) for curves of Clifford index two in Theorem 1.4).

For trigonal curves, [30, Prop. 3.1] gives the following values:

(9.2.1)

g	$h^0(N_{C/\mathbf{P}^{g-1}}(-2))$
5	3
6	2
7	$1 + h^0(K_C - 4\mathfrak{g}) \leqslant 2$
8	$h^0(K_C - 4\mathfrak{g}) \leqslant 1$
9	$h^0(K_C - 5\mathfrak{g}) \leqslant 1$
10	$h^0(K_C - 6\mathfrak{g}) \leqslant 1$
$\geqslant 11$	0

where $\mathfrak{g}$ stands for the class of the g^1_3. We prove in (9.3) below that these values always equal $h^0(K_C - (g-4)\mathfrak{g})$. There remains the case of non-trigonal genus 5 curves, which is treated in (9.4) below. $\square$

(9.3) Classification of trigonal curves. Here we classify trigonal curves of genus g with $5 \leqslant g \leqslant 10$ along the lines of [34, §9][3], in order to prove that for such curves one has

$$\mathrm{cork}(\gamma_{C,2K_C}) = h^0(N_{C/\mathbf{P}^{g-1}}(-2)) = h^0(K_C - (g-4)\mathfrak{g}).$$

This may also be seen with [29, Prop. 2.9(e)], but the following self-contained proof is more in the spirit of our text. Let $f : C \to \mathbf{P}^1$ be a genus g triple cover of $\mathbf{P}^1$. It holds that

$$f_*\mathcal{O}_C = \mathcal{O}_{\mathbf{P}^1} \oplus V, \quad \text{with} \quad V = \mathcal{O}_{\mathbf{P}^1}(-a) \oplus \mathcal{O}_{\mathbf{P}^1}(-b)$$

such that

$$a + b = g + 2 \quad \text{and} \quad 0 < a \leqslant b \leqslant 2a.$$

Then C may be realized as a divisor in the rational ruled surface $\mathbf{F}_{b-a}$, with class

$$3E + (2b-a)F \sim 3H + (2a-b)F.$$

Moreover, one has

$$\begin{aligned} h^0\big((g-4)\mathfrak{g}\big) &= h^0\big(C, f^*\mathcal{O}_{\mathbf{P}^1}(g-4)\big) \\ &= h^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-4) \oplus \mathcal{O}_{\mathbf{P}^1}(g-4-a) \oplus \mathcal{O}_{\mathbf{P}^1}(g-4-b)\big) \\ &= h^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-4) \oplus \mathcal{O}_{\mathbf{P}^1}(b-6) \oplus \mathcal{O}_{\mathbf{P}^1}(a-6)\big), \end{aligned}$$

[3] there is a typo in [34], as the fourth formula on top of p.1153 should read $t = 2n - m$ instead of $t = 2m - n$.

from which one deduces

$$\begin{aligned} h^0\big(K_C-(g-4)\mathfrak{g}\big) &= h^1\big((g-4)\mathfrak{g}\big) \\ &= h^0\big((g-4)\mathfrak{g}\big)-3(g-4)+g-1 \\ &= h^0\big((g-4)\mathfrak{g}\big)-2g+11. \end{aligned}$$

This gives the following complete classification.

	a	b	$h^0\big((g-4)\mathfrak{g}^1_3\big)$	$h^0\big(K-(g-4)\mathfrak{g}^1_3\big)$	class
$g=5$	3	4	2	3	$3E+5F$ on $\mathbf{F}_1$
$g=6$	4	4	3	2	$3E+4F$ on $\mathbf{F}_0$
	3	5	3	2	$3E+7F$ on $\mathbf{F}_2$
$g=7$	4	5	4	1	$3E+6F$ on $\mathbf{F}_1$
	3	6	5	2	$3E+9F$ on $\mathbf{F}_3$
$g=8$	5	5	5	0	$3E+5F$ on $\mathbf{F}_0$
	4	6	6	1	$3E+8F$ on $\mathbf{F}_2$
$g=9$	5	6	7	0	$3E+7F$ on $\mathbf{F}_1$
	4	7	8	1	$3E+10F$ on $\mathbf{F}_3$
$g=10$	6	6	9	0	$3E+6F$ on $\mathbf{F}_0$
	5	7	9	0	$3E+9F$ on $\mathbf{F}_2$
	4	8	10	1	$3E+12F$ on $\mathbf{F}_4$

From this classification one readily deduces that the values given in (9.2.1) indeed equal $h^0(K_C-(g-4)\mathfrak{g})$. The only non-trivial case is when $g=7$; then one finds as above that

$$h^0(4\mathfrak{g}) = h^0(\mathbf{P}^1, \mathcal{O}_{P^1}(4)\oplus\mathcal{O}_{P^1}(4-a)\oplus\mathcal{O}_{P^1}(4-b)) = \begin{cases} 6 & \text{if } (a,b)=(4,5) \\ 7 & \text{if } (a,b)=(3,6), \end{cases}$$

hence by Riemann–Roch

$$h^0(K_C-4\mathfrak{g}) = \begin{cases} 0 & \text{if } (a,b)=(4,5) \\ 1 & \text{if } (a,b)=(3,6) \end{cases}$$

as required.

(9.4) Complete intersection canonical curves. We conclude the subsection by giving the values of $\operatorname{cork}(\gamma_{C,mK_C})$ for non-hyperelliptic curves of genus $g\leqslant 4$, and non-trigonal curves of genus 5.

These curves are complete intersections in their canonical embedding, and thus one finds, using (9.1.2) once again:

$$\operatorname{cork}(\gamma_{C,mK_C}) = h^0(N_{C/\mathbf{P}^{g-1}}(-m)) = \begin{cases} h^0(\mathcal{O}_C(4-m)) & \text{if } g=3; \\ h^0(\mathcal{O}_C(3-m)\oplus\mathcal{O}_C(2-m)) & \text{if } g=4; \\ h^0(\mathcal{O}_C(2-m)^{\oplus 3}) & \text{if } g=5. \end{cases} \tag{9.4.1}$$

9.2 – Surface extensions

In this subsection we study the surface extensions of pluricanonical curves. Comparing their number of moduli with the values for $\operatorname{cork}(\gamma_{C,mK_C})$ found in the previous subsection, we will prove Theorem 1.9, to the effect that the extension theory of pluricanonical curves satisfying Property N_2 is unobstructed.

We shall prove Theorem 1.9 by considering separately the various cases for which we have found in Subsection 9.1 above that $\mathrm{cork}(\gamma_{C,mK_C})$ is nonzero (if $\mathrm{cork}(\gamma_{C,mK_C})$ is zero, then there is only the trivial ribbon over (C, mK_C), and the statement is empty). The strategy is the same as for Theorems 1.7 and 1.8, namely we prove that the general ribbon in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,mK_C}))$ is integrable by a dimension count, after what the conclusion follows by Theorem 6.8. We will be brief and only outline the dimension count. Note that all extensions of pluricanonical curves appear in the classification of Theorem 1.2.

(9.5) Note that

$$2g+3 \leqslant 4g-4 \quad \iff \quad g \geqslant 4,$$

so by the Green Theorem 2.2 all pluricanonical curves of genus $g \geqslant 4$ satisfy Property N_2. Similarly, in genus 3, m-canonical curves satisfy Property N_2 if $m \geqslant 3$.

For bicanonical curves of genus 3, the sufficient condition for Property N_2 provided by Theorem 2.2 does not hold, and indeed a direct computation using Macaulay2 [23] shows that, if $C \subseteq \mathbf{P}^5$ is a non-hyperelliptic curve of genus 3 embedded by the complete linear series $|2K_C|$, its ideal $\mathcal{I}_{C/\mathbf{P}^5}$ has the minimal resolution

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbf{P}^2}(-6)^{\oplus 3} \to \mathcal{O}_{\mathbf{P}^2}(-5)^{\oplus 8} \oplus \mathcal{O}_{\mathbf{P}^2}(-4)^{\oplus 3} \to \\ \mathcal{O}_{\mathbf{P}^2}(-4)^{\oplus 6} \oplus \mathcal{O}_{\mathbf{P}^2}(-3)^{\oplus 8} \to \mathcal{O}_{\mathbf{P}^2}(-2)^{\oplus 7} \to \mathcal{I}_{C/\mathbf{P}^5} \to 0, \end{aligned}$$

so that $(C, 2K_C)$ does not satisfy Property N_2.

9.2.1 Complete intersection curves

(9.6) Genus 3. In this case the result is contained in Theorem 1.7. Let us briefly recall how it goes in this particular case, and add a few comments.

Let C be a smooth plane quartic. The 4-canonical model of C is $v_4(C)$, which is a hyperplane section of the 4-Veronese surface $v_4(\mathbf{P}^2)$. Since $\mathrm{cork}(\gamma_{C,4K_C}) = 1$, we conclude that the unique ribbon over $(C, 4K_C)$ is integrable.

We obtain extensions of the 3-canonical model of C by projecting the surface $v_4(\mathbf{P}^2)$ from four points on $v_4(C)$ that, in $\mathbf{P}^2$, are cut out by a line on C, so that they sum to a canonical divisor of C. There is a 2-dimensional family of such divisors, and correspondingly a 2-dimensional family of non-isomorphic extensions of $(C, 3K_C)$, in agreement with $\mathrm{cork}(\gamma_{C,3K_C}) = 3$. In order to avoid possible misunderstandings, note that the curve $v_3(C)$ is contained in the 3-Veronese surface $v_3(\mathbf{P}^2)$ but it is not a hyperplane section, so $v_3(\mathbf{P}^2)$ is not an extension of $(C, 3K_C)$.

Bicanonical curves of genus 3 are out of the range of Theorem 1.9 because they don't satisfy Property N_2. Their extensions have been analyzed in (7.7) above, where we have found for general C two families of extensions, one of the expected dimension 5 and one superabundant of dimension 6. In this case we cannot apply Theorem 6.3 to guarantee that each surface is the integral of a unique ribbon, and thus we have no proof of the fact that the general ribbon in $\mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,2K_C}))$ is integrable.

(9.7) Genus 4. Let $C \subseteq \mathbf{P}^3$ be a smooth complete intersection of a quadric X_2 and a cubic X_3. The surface $v_3(X_2) \subseteq \mathbf{P}^{15}$ is an extension of $v_3(C)$, in fact the only one since $\mathrm{cork}(\gamma_{C,3K_C}) = 1$.

Similarly, for each cubic X_3' containing $C \subseteq \mathbf{P}^3$, the surface $v_2(X_3') \subseteq \mathbf{P}^9$ has $v_2(C)$ as a hyperplane section, and it is thus an extension of $(C, 2K_C)$. This provides a 4-dimensional family of extensions of $(C, 2K_C)$, in agreement with $\mathrm{cork}(\gamma_{C,2K_C}) = 5$.

(9.8) Genus 5. Let $C \subseteq \mathbf{P}^4$ be a smooth complete intersection of three quadrics. For each surface $X_2 \cap X_2'$ complete intersection of two quadrics containing C, the surface $v_2(X_2 \cap X_2')$ has $v_2(C)$ as a hyperplane section. This provides a 2-dimensional family of extensions of $(C, 2K_C)$, in agreement with $\operatorname{cork}(\gamma_{C,2K_C}) = 3$.

9.2.2 Clifford index 1

(9.9) Trigonal curves. Let C be a smooth trigonal curve of genus $g \geqslant 5$, non-hyperelliptic. In its canonical model, it sits in a rational normal scroll $Y \subseteq \mathbf{P}^{g-1}$ of degree $g-2$, with class $C \sim 3\mathcal{H} - (g-4)F$, with $\mathcal{H}$ the hyperplane section class of $Y \subseteq \mathbf{P}^{g-1}$, and F the class of a ruling, cf. Example 3.5. Extensions of $(C, 2K_C)$ are to be found as simple projections of the image of Y by the map $\phi_{|C|}$. The center of the projection must be an effective, degree $10-g$, divisor D_{10-g} on C, such that:

$$\big(3\mathcal{H} - (g-4)F\big)\big|_C - D_{10-g} \sim 2K_C \quad \Longleftrightarrow \quad D_{10-g} \sim K_C - (g-4)\mathfrak{g},$$

where $\mathfrak{g}$ is the class of the g^1_3 on C (recall that $\mathcal{H}|_C \sim K_C$). Thus each $D \in |K_C - (g-4)\mathfrak{g}|$ gives an extension of $(C, 2K_C)$, in agreement with $\operatorname{cork}(\gamma_{C,2K_C}) = h^0(K_C - (g-4)\mathfrak{g})$.

(9.10) Plane quintics. Let $C \subseteq \mathbf{P}^2$ be a smooth plane quintic. Then $K_C = 2L|_C$, where L denotes the line class on $\mathbf{P}^2$, and the extensions of $(C, 2K_C)$ are to be found as simple projections of the Veronese surface $v_5(\mathbf{P}^2)$. The center of the projection must be an effective, degree 5, divisor D on C, such that:

$$5L|_C - D \sim 2K_C \quad \Longleftrightarrow \quad D \sim L|_C .$$

Thus each $D \in |\, L|_C \,|$ gives an extension of $(C, 2K_C)$, in agreement with $\operatorname{cork}(\gamma_{C,2K_C}) = h^0(\mathcal{O}_{\mathbf{P}^2}(1))$.

9.2.3 Clifford index 2

(9.11) Quadric sections of Del Pezzo surfaces. Let C be a genus g curve such that in its canonical model C is a quadric section of a Del Pezzo surface $S \subseteq \mathbf{P}^{g-1}$. Then $v_2(S)$ is an extension of $(C, 2K_C)$, which proves the theorem in this case since $\operatorname{cork}(\gamma_{C,2K_C}) = 1$.

(9.12) Bielliptic curves. Let $f : C \to E$ be a genus g double cover of the elliptic curve E. Then the 2-Veronese re-embedding of the cone in $\mathbf{P}^{g-1}$ over the elliptic normal curve $E \subseteq \mathbf{P}^{g-2}$ is an extension of $(C, 2K_C)$ as in Example 3.1, which proves the theorem in this case since $\operatorname{cork}(\gamma_{C,2K_C}) = 1$.

(9.13) Plane sextics. Let $C \subseteq \mathbf{P}^2$ be a smooth plane sextic. Then $K_C = 3L|_C$, where L denotes the line class on $\mathbf{P}^2$, and the Veronese surface $v_6(\mathbf{P}^2)$ is an extension of $(C, 2K_C)$, which proves the theorem in this case since $\operatorname{cork}(\gamma_{C,2K_C}) = 1$.

9.3 – Universal extensions

Eventually, we consider those pluricanonical curves (C, mK_C) for which $\operatorname{cork}(\gamma_{C,2K_C}) > 1$, and provide a construction of the universal extension, except in the trigonal case. The constructions are similar to those in [15] and [32, Appendix], see also [19], and inspired by examples of Totaro (private communication, see [14]). When $\operatorname{cork}(\gamma_{C,2K_C}) = 1$, the universal extension is a surface and there is nothing to add to the analysis carried out in the previous Subsection 9.2.

(9.14) Genus 3. We shall construct a 4-dimensional variety $X \subseteq \mathbf{P}^{12}$ of degree 12 having tricanonical curves of genus 3 as curve sections, and projections of $v_4(\mathbf{P}^2)$ from 4 points lying on a line in the plane as surface sections.

We start from the following basic fact, the proof of which we leave to the reader. Let $f, \ell \in \mathbf{C}[\mathbf{x}]$, $\mathbf{x} = (x_0, x_1, x_2)$, be two homogeneous polynomials of degrees 4 and 1 respectively. Then the hypersurface S of the weighted projective space $\mathbf{P}(1^3, 3)$ defined by the homogeneous equation

$$f(\mathbf{x}) + y\ell(\mathbf{x}) = 0,$$

of weighted degree 4 in the homogeneous coordinates $(\mathbf{x}, y)$, is isomorphic to the surface obtained by first blowing up $\mathbf{P}^2$ at the four points defined by the equations $f(\mathbf{x}) = \ell(\mathbf{x}) = 0$, and then contracting the proper transform of the line defined by $\ell(\mathbf{x}) = 0$.

Now, we claim that the weighted quartic hypersurface

$$X: \quad x_0y_0 + x_1y_1 + x_2y_2 = 0 \quad \text{in} \quad \mathbf{P}(1^3, 3^3)_{(\mathbf{x}:\mathbf{y})},$$

in its embedding $X \subseteq \mathbf{P}^{12}$ defined by weighted cubics, is the universal extension we are looking for. To prove our claim, we consider a canonical curve C of genus 3 defined by a quartic equation $f(\mathbf{x}) = 0$ in $\mathbf{P}^2$. Up to a change of coordinates, we may assume that f has no term in x_0^4, so that it is possible to write it as

$$f = x_1f_1 + x_2f_2.$$

First, we note that C is defined by the three cubic equations

$$y_0 = y_1 - f_1(\mathbf{x}) = y_2 - f_2(\mathbf{x}) = 0$$

in X, so that in its tricanonical embedding it is the section of $X \subseteq \mathbf{P}^{12}$ by three hyperplanes. Next, we consider a general line $L \subseteq \mathbf{P}^2$ defined by an equation

$$x_0 + a_1x_1 + a_2x_2 = 0$$

in $\mathbf{P}^2$. Then, the extension of $(C, 3K_C)$ given by the projection of $v_4(\mathbf{P}^2)$ from the four points in $C \cap L$ is the section of $X \subseteq \mathbf{P}^{12}$ by the two hyperplanes corresponding to the cubic equations

$$y_1 - f_1(\mathbf{x}) - a_1y_0 = y_2 - f_2(\mathbf{x}) - a_2y_0 = 0,$$

since the latter is isomorphic to the hypersurface

$$(x_1f_1 + x_2f_2) + y_0(x_0 + a_1x_1 + a_2x_2) = 0$$

in $\mathbf{P}(1^3, 3)$. It follows that the map

$$\Lambda \in \mathbf{P}^{12}/\langle C\rangle \longmapsto [2C_{X\cap\Lambda}] \in \mathbf{P}(\ker({}^{\mathsf{T}}\gamma_{C,L})),$$

in the notation of (6.6), is dominant. It thus follows from Lemma 6.7 that $X \subseteq \mathbf{P}^{12}$ is a universal extension of $(C, 3K_C)$.

One can perform the same construction for bicanonical curves of genus three, even though they don't verify Property N_2. Thus, the weighted quartic hypersurface

$$X': \quad x_0^2y_0 + x_0x_1y_1 + x_0x_2y_2 + x_1^2y_3 + x_1x_2y_3 + x_2^2y_5 = 0 \quad \text{in} \quad \mathbf{P}(1^3, 2^6)_{(\mathbf{x}:\mathbf{y})},$$

in its embedding in $\mathbf{P}^{11}$ defined by weighted quadrics, has as surface sections all projections of $v_4(\mathbf{P}^2)$ from eight points complete intersection of a quartic and a conic.

(9.15) Genus 4. We shall construct a 6-dimensional variety $X \subseteq \mathbf{P}^{13}$ of degree 12 having bicanonical curves of genus 4 and their extensions as sections by linear spaces. This is the weighted cubic

$$X: \quad x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0 \quad \text{in} \quad \mathbf{P}(1^4, 2^4)_{(\mathbf{x}:\mathbf{y})},$$

in its embedding $X \subseteq \mathbf{P}^{13}$ defined by weighted quadrics.

Indeed, let C be the canonical genus 4 curve defined by the equations $f, g \in \mathbf{C}[\mathbf{x}]$ in $\mathbf{P}^3$, of degrees 2 and 3 respectively. We may write the degree 3 equation as

$$g = x_0g_0 + x_1g_1 + x_2g_2 + x_3g_3,$$

which enables us to see C as being cut out in X by the five degree 2 equations

$$f(\mathbf{x}) = y_0 - g_0(\mathbf{x}) = y_1 - g_1(\mathbf{x}) = y_2 - g_2(\mathbf{x}) = y_3 - g_3(\mathbf{x}) = 0.$$

In turn, each cubic surface $X_3' \subseteq \mathbf{P}^3$ containing C has an equation of the form $g + (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3)f$, hence it is cut out in X by the four degree 2 equations

$$y_0 - g_0(\mathbf{x}) - a_0f(\mathbf{x}) = y_1 - g_1(\mathbf{x}) - a_1f(\mathbf{x}) = y_2 - g_2(\mathbf{x}) - a_2f(\mathbf{x}) = y_3 - g_3(\mathbf{x}) - a_3f(\mathbf{x}) = 0,$$

so that $v_2(X_3') \subseteq \mathbf{P}^9$ is the section of $X \subseteq \mathbf{P}^{13}$ by four hyperplanes containing C, as required.

(9.16) Genus 5. The universal extension of non-trigonal bicanonical curves of genus 5 is the Veronese 4-fold $v_2(\mathbf{P}^4) \subseteq \mathbf{P}^{14}$.

(9.17) Plane quintics. The universal extension for bicanonical models of plane quintics may be constructed analogously to what we have done in (9.14) above for plane quartics. Thus the universal extension is the weighted quintic hypersurface

$$X: \quad x_0y_0 + x_1y_1 + x_2y_2 = 0 \quad \text{in} \quad \mathbf{P}(1^3, 4^3)_{(\mathbf{x}:\mathbf{y})},$$

in its embedding $X \subseteq \mathbf{P}^{17}$ defined by weighted cubics, a projective variety of dimension 4 and degree 20.

(9.18) Bicanonical trigonal curves. We find a universal extension of dimension larger than two in the following cases:

— $g = 5$ and C is a member of $|3E + 5F|$ on $\mathbf{F}_1$, the universal extension of $(C, 2K_C)$ has dimension 4 and degree 16 in $\mathbf{P}^{14}$;

— $g = 6$ and C is a member of $|3E + 4F|$ on $\mathbf{F}_0$ or of $|3E + 7F|$ on $\mathbf{F}_2$, the universal extension of $(C, 2K_C)$ has dimension 3 and degree 20 in $\mathbf{P}^{16}$;

— $g = 7$ and C is a member of $|3E + 9F|$ on $\mathbf{F}_3$, the universal extension of $(C, 2K_C)$ has dimension 3 and degree 24 in $\mathbf{P}^{19}$.

We believe it should be possible to give an explicit contruction of these universal extensions along the same lines as in the other cases, but we don't dwell on this and leave it as an open project. It is plausible that the universal extension in genus 6 will be the same for both kinds of curves.

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der mathematischen Wissenschaften, vol. 267, Springer-Verlag, New York, 1985.

[2] E. Arbarello, A. Bruno and E. Sernesi, *On hyperplane sections of K3 surfaces*, Algebr. Geom. **4** (2017), no. 5, 562–596.

[3] A. Beauville and J.-Y. Mérindol, *Sections hyperplanes des surfaces $K3$*, Duke Math. J. **55** (1987), no. 4, 873–878.

[4] M. Beltrametti and A. J. Sommese, *Zero cycles and kth order embeddings of smooth projective surfaces*, in *Problems in the theory of surfaces and their classification (Cortona, 1988)*, Sympos. Math., XXXII, Academic Press, London, 1991, With an appendix by Lothar Göttsche, 33–48.

[5] A. Bertram, L. Ein and R. Lazarsfeld, *Surjectivity of Gaussian maps for line bundles of large degree on curves*, in *Algebraic geometry (Chicago, IL, 1989)*, Lecture Notes in Math., vol. 1479, Springer, Berlin, 1991, 15–25.

[6] G. M. Besana and A. Biancofiore, *Degree eleven projective manifolds of dimension greater than or equal to three*, Forum Math. **17** (2005), no. 5, 711–733.

[7] A. Calabri and C. Ciliberto, *Birational classification of curves on rational surfaces*, Nagoya Math. J. **199** (2010), 43–93.

[8] G. Castelnuovo, *Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche.*, Rend. Circ. Mat. Palermo **4** (1890), 73–88.

[9] ———, *Sulle superficie algebriche le cui sezioni sono curve di genere 3.*, Torino Atti **25** (1890), 695–715.

[10] ———, *Sui multipli di una serie lineare di gruppi di punti appartenenti ad una curva algebrica*, Rend. del Circolo Mat. di Palermo **7** (1893), 89–110.

[11] F. Catanese and C. Ciliberto, *Symmetric products of elliptic curves and surfaces of general type with $p_g = q = 1$*, J. Algebraic Geom. **2** (1993), no. 3, 389–411.

[12] L. Chiantini and C. Ciliberto, *Weakly defective varieties*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 151–178.

[13] C. Ciliberto, *Sul grado dei generatori dell'anello canonico di una superficie di tipo generale*, Rend. Sem. Mat. Univ. Politec. Torino **41** (1983), no. 3, 83–111.

[14] C. Ciliberto and T. Dedieu, *Double covers and extensions*, arXiv:2008.03109, to appear in Kyoto J. Math.

[15] ———, *$K3$ curves with index $k > 1$*, Boll. Unione Mat. Ital. **15** (2022), no. 1-2, 87–115.

[16] C. Ciliberto, T. Dedieu and E. Sernesi, *Wahl maps and extensions of canonical curves and $K3$ surfaces.*, J. Reine Angew. Math. **761** (2020), 219–245.

[17] C. Ciliberto and R. Miranda, *On the Gaussian map for canonical curves of low genus*, Duke Math. J. **61** (1990), no. 2, 417–443.

[18] T. Dedieu, *Some classical formulae for curves and surfaces*, `https://hal.archives-ouvertes.fr/hal-02914653`.

[19] B. Dewer, *Extensions of gorenstein weighted projective 3-spaces and characterization of the primitive curves of their surface sections*, preprint, arXiv:2303.01882.

[20] D. H. J. Epema, *Surfaces with canonical hyperplane sections*, CWI Tract, vol. 1, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1984.

[21] M. L. Fania and E. L. Livorni, *Degree nine manifolds of dimension greater than or equal to* 3, Math. Nachr. **169** (1994), 117–134.

[22] ———, *Degree ten manifolds of dimension n greater than or equal to* 3, Math. Nachr. **188** (1997), 79–108.

[23] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[24] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. **19** (1984), no. 1, 125–171.

[25] R. Hartshorne, *Curves with high self-intersection on algebraic surfaces*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 111–125.

[26] ———, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[27] T. Horowitz, *Varieties of low Δ-genus*, Duke Math. J. **50** (1983), no. 3, 667–683.

[28] P. Ionescu, *Embedded projective varieties of small invariants*, in *Algebraic geometry, Bucharest 1982 (Bucharest, 1982)*, Lecture Notes in Math., vol. 1056, Springer, Berlin, 1984, 142–186.

[29] A. L. Knutsen and A. F. Lopez, *Surjectivity of gaussian maps for curves on enriques surfaces*, Adv. Geom. **7** (2007), no. 2, 215–247.

[30] A. L. Knutsen, *Global sections of twisted normal bundles of $K3$ surfaces and their hyperplane sections*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **31** (2020), no. 1, 57–79.

[31] A. Lanteri and E. L. Livorni, *Complex surfaces polarized by an ample and spanned line bundle of genus three*, Geom. Dedicata **31** (1989), no. 3, 267–289.

[32] A. F. Lopez, *On the extendability of projective varieties: a survey*, with an appendix by T. Dedieu, to appear in *The art of doing algebraic geometry*, Birkhäuser Trends in Mathematics.

[33] M. Mella and E. Polastri, *Equivalent birational embeddings. II: Divisors*, Math. Z. **270** (2012), no. 3-4, 1141–1161.

[34] R. Miranda, *Triple covers in algebraic geometry*, Amer. J. Math. **107** (1985), no. 5, 1123–1158.

[35] G. Pareschi, *Gaussian maps and multiplication maps on certain projective varieties*, Compositio Math. **98** (1995), no. 3, 219–268.

[36] I. Peeva, *Graded syzygies*, Algebra and Applications, vol. 14, Springer-Verlag London, Ltd., London, 2011.

[37] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. (2) **127** (1988), no. 2, 309–316.

[38] F.-O. Schreyer, *A standard basis approach to syzygies of canonical curves*, J. Reine Angew. Math. **421** (1991), 83–123.

[39] C. Segre, *Ricerche sulle rigate ellittiche di qualunque ordine*, Atti Accad. Sci. Torino **21** (1885–86), 628–651.

[40] ———, *Recherches générales sur les courbes et les surfaces réglées algébriques. II.*, Math. Ann. **34** (1889), 1–25.

[41] F. Serrano, *The adjunction mappings and hyperelliptic divisors on a surface*, J. Reine Angew. Math. **381** (1987), 90–109.

[42] A. J. Sommese and A. Van de Ven, *On the adjunction mapping*, Math. Ann. **278** (1987), no. 1-4, 593–603.

[43] C. Voisin, *Courbes tétragonales et cohomologie de Koszul*, J. Reine Angew. Math. **387** (1988), 111–121.

[44] ———, *Sur l'application de Wahl des courbes satisfaisant la condition de Brill-Noether-Petri*, Acta Math. **168** (1992), no. 3-4, 249–272.

[45] J. Wahl, *On cohomology of the square of an ideal sheaf*, J. Algebraic Geom. **6** (1997), no. 3, 481–511.

Ciro Ciliberto. Dipartimento di Matematica. Università degli Studi di Roma Tor Vergata. Via della Ricerca Scientifica, 00133 Roma, Italy. `cilibert@mat.uniroma2.it`

Thomas Dedieu. Institut de Mathématiques de Toulouse ; UMR5219. Université de Toulouse ; CNRS. UPS IMT, F-31062 Toulouse Cedex 9, France. `thomas.dedieu@math.univ-toulouse.fr`

CC is a member of GNSAGA of INdAM. He acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.