Degenerations and enumerations of curves on surfaces

Incomplete *preliminary version*

Seminar held in Roma Tor Vergata 2015–2017

edited by Th. Dedieu

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Warning. Some texts from the collection have been omitted from this version, as they are still missing some parts; they should be available soon, and you can contact me if you wish to consult a draft version.

Tout ce qu'on savait, c'est que, lorsqu'il revint d'Italie, il était prêtre.

Victor Hugo, *Les misérables*

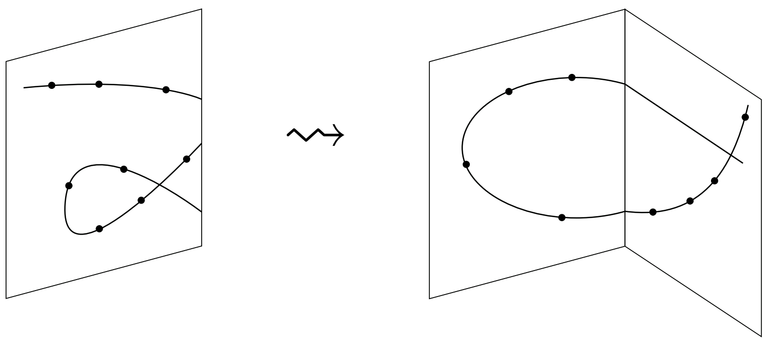
Contents

Short Presentation

These notes originate in a workshop held in Rome, University “Tor Vergata”, from October 2015 to June 2017, in the framework of the Marie Skłodowska Curie project “Families of Subvarieties in Complex Algebraic Varieties”.[1] We would meet on a weekly basis, on Tuesday afternoons. The talks were scheduled to last two hours, but it happened frequently that we discussed over the whole afternoon. Let me take the occasion to thank all colleagues and friends in Rome for the lively, enthusiastic, and dedicated atmosphere in the workshop, which hopefully reflects on these notes.

The main theme of these ntoes is the enumeration of singular curves on surfaces. The first instance of this problem is arguably the enumeration of plane curves: consider the family V_g^d of degree d and genus g curves in the projective plane; it has dimension dimension $D_g^d = 3d+g-1$; now, the question is: given D_g^d fixed general points in the projective plane, how many curves are there in the family V_g^d that pass through these D_g^d points? For example, take $d = 3$ and $g = 0$, which is the first non-trivial case. Smooth plane cubics are elliptic curves, hence rational plane cubics have one singular point, either an ordinary double point or an ordinary cusp. The question is to find how many of these rational cubics pass through 8 general points; the answer is 12, as we will prove in many different ways in the course of the notes.

We give a particular focus to the approach to this problem by degeneration. The idea is to let the surface degenerate to, say, the transverse union of two surfaces, and in the same time let some of the fixed points go on one side and some on the other side, as indicated in the figure below.



The initial enumerative problem is thus reduced to a collection of simpler enumerative problems (the degrees and genera of these auxiliary problems are smaller than those of the initial problem).

There are other guises of approaching enumerative problems by degeneration. In these notes, we also consider such approaches using tropical geometry, and Gromov–Witten invariants. We also study solutions coming from intersection theory, both in a very classical form involving calculus on polynomials, as well as in a nowadays more standard form involving Chern classes and the likes.

The problem of counting curves on the projective plane is now fairly well understood, and its solution is discussed at length in this volume. The current research is more concerned with counting curves on varieties with trivial canonical bundle; we give particular attention to the

[1] Project FOSICAV, which has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 652782.

problem of counting curves on $K3$ surfaces, which indeed was the goal of the workshop these notes arose from.

The chapters in this volume are of different natures: some offer a detailed account with complete proofs of existing results (e.g., Lecture VI), while others propose a more streamlined presentation (e.g., Lecture XI); some chapters make the most of by now classical articles by pushing their ideas as far as possible (typically, Lecture III), and the appendices contain original material (Lectures B and C). All along we have striven to provide a comprehensive treatment, and as a result we haven't hesitated to work out numerous detailed examples.

It is in principle possible to read each chapter independently, although some make use of others.

List of Talks

- Seminario panoramico
  (Ciro Ciliberto, 13 ottobre 2015)
- Deformazioni di curve su superficie
  (Edoardo Sernesi, 27 ottobre 2015)
- Dimostrazione della formula di Caporaso–Harris: sezioni iperpiane delle varietà di Severi, I
  (Concettina Galati, 3 novembre 2015)
- Dimostrazione della formula di Caporaso–Harris: sezioni iperpiane delle varietà di Severi, II
  (Concettina Galati, 17 novembre 2015)
- Dimostrazione della formula di Caporaso–Harris: geometria delle varietà di Severi
  (Thomas Dedieu, 24 novembre 2015)
- Deformazioni d'un m-tacnodo in $m-1$ nodi
  (Margherita Lelli-Chiesa, 1º dicembre 2015)
- Prodotti di spazi di deformazioni di tacnodi e applicazione allo studio delle sezioni iperpiane delle varietà di Severi
  (Francesco Bastianelli, 15 dicembre 2015)
- Degenerazioni di curve nodali in una degenerazione normal crossings di superficie e applicazioni
  (Concettina Galati, 19 gennaio 2016)
- Enumerazione di curve piane mediante diagrammi a gradini, I
  (Thomas Dedieu, 2 febbraio 2016)
- Enumerazione di curve piane mediante diagrammi a gradini, II
  (Thomas Dedieu, 9 febbraio 2016)
- Enumerazione di curve piane: esempi!
  (Thomas Dedieu, 16 febbraio 2016)
- Introduzione agli invarianti di Gromov–Witten, I
  (Filippo Viviani, 5 aprile 2016)
- Introduzione agli invarianti di Gromov–Witten, II
  (Filippo Viviani, 12 aprile 2016)
- La formula di degenerazione di Jun Li, I
  (Margarida Melo, 26 aprile 2016)
- La formula di degenerazione di Jun Li, II
  (Margarida Melo, 3 maggio 2016)
- Geometria enumerativa delle superficie K3, I: la formula di Yau–Zaslow/Beauville
  (Thomas Dedieu, 10 maggio 2016)

- Geometria enumerativa delle superficie K3, II: la formula di Göttsche–Yau–Zaslow/Bryan–Leung
  (Thomas Dedieu, 24 maggio 2016)
- Geometria enumerativa delle superficie K3, III: risultati di Maulik–Pandharipande, Maulik–Pandharipande–Thomas, Klemm–Maulik–Pandharipande–Scheidegger, ecc.
  (Thomas Dedieu, 31 maggio 2016)
- Geometria enumerativa delle superficie K3, IV: numeri di Noether–Lefschetz e grande linee della dimostrazione della congettura di Yau–Zaslow secondo Maulik–Pandharipande e Klemm–Maulik–Pandharipande–Scheidegger
  (Thomas Dedieu, 14 giugno 2016)
- Enumerazione di curve in superficie toriche mediante geometria tropicale secondo Mikhalkin e Brugallé, I
  (Thomas Dedieu, 22 novembre 2016)
- Enumerazione di curve in superficie toriche mediante geometria tropicale secondo Mikhalkin e Brugallé, II
  (Thomas Dedieu, 30 novembre 2016)
- Enumerazione di curve in superficie toriche mediante geometria tropicale secondo Mikhalkin e Brugallé, III
  (Thomas Dedieu, 13 dicembre 2016)
- La congettura di Göttsche e le sue dimostrazioni, I
  (Francesco Bastianelli, 2 maggio 2017)
- Alcune formule classiche per superficie in $\mathbf{P}^3$
  (Thomas Dedieu, 9 maggio 2017)
- La congettura di Göttsche e le sue dimostrazioni, II
  (Francesco Bastianelli, 16 maggio 2017)
- "Polinomi nodi" per curve in superficie, secondo Vainsencher, Kleiman–Piene, Qviller
  (Thomas Dedieu, 30 maggio 2017)
- La congettura di Göttsche generalizzata
  (Filippo Viviani, 6 giugno 2017)
- "Polinomi nodi" per curve in superficie, II; conlusione.
  (Thomas Dedieu, 27 giugno 2017)

Lecture I
Limits of nodal curves: generalities and examples

by Ciro Ciliberto and Thomas Dedieu

1 – The basic question

The objective of this lecture is to introduce to the subject of interest in this series of seminars. This exposition is mainly based on [4]. In this text we will work over the complex field.

Let $f : S \to \mathbf{D}$ be a flat, projective family of surfaces of degree d in $\mathbf{P}^r$ (with $r \geqslant 3$), with S a smooth threefold, and $\mathbf{D}$ the complex unit disc, usually called a *degeneration* of the *general fibre* $S_t := f^{-1}(t)$, for $t \in \mathbf{D}^* := \mathbf{D} - \{0\}$, which we assume is a smooth irreducible surface, to the *central fibre* S_0, which may be singular and even reducible. This is also called a *smooth deformation* of S_0. One has $S \subseteq \mathbf{D} \times \mathbf{P}^r$, so that S is endowed with the line bundle $\mathcal{L} = p^*(\mathcal{O}_{\mathbf{P}^r}(1))$, where $p : S \to \mathbf{P}^r$ is the projection to the second factor: this is called the *polarizing (hyperplane) bundle* on S.

(1.1) Question. *What are the limits of tangent, or in general of multitangent, hyperplanes to S_t, for $t \neq 0$, as t tends to 0?*

The same question makes sense, more generally, for varieties of any dimension. In the case of degeneration of (plane) curves we refer to [15, pp. 134–135] for a glimpse on this subject. In dimension higher than 2 the problem is quite complicated and few results are known (however, B. Segre's paper [23] is an interesting classical source of basic information on the subject).

One of the main interests in Question (1.1) arises form enumerative geometry, since, by degenerating a surface to a reducible one, whose components are much simpler than the original surface (e.g., a degeneration into a union of planes), we may hope that the configuration of limits of pluritangent hyperplanes is easier to understand and their number (if finite), or in general the degree of the variety parameterising them, may be computed. This is indeed the subject of the foundational work by L. Caporaso and J. Harris [1, 2], and independently by Z. Ran

[18, 19, 20], which were both aimed to the study, with degeneration techniques, of the so–called *Severi varieties*, i.e., the families of irreducible nodal plane curves of a given degree.

Section 2 contains classical enumerative formulae for surfaces. Sections 3–5 present general facts about Question (1.1), following the presentation given in [4]. Finally, Sections 6–8 present several examples in detail in order to illustrate the general theory.

2 – Classical results

Enumerative results on the number of (pluri)tangent planes to a *general* surface S of degree $d > 1$ in $\mathbf{P}^3$ are quite classical, and probably the first of them go back to G. Salmon, see [21, 22].

The set $\check{S} \subseteq \check{\mathbf{P}}^3$ of tangent planes to S is the *dual* of S, which is birational to S. One has

$$\deg(\check{S}) = d(d-1)^2.$$

As soon as $d > 2$, $\check{S}$ is singular. How do the singularities of $\check{S}$ look like if S is general enough? There is a *nodal curve* D_b, whose general points correspond to *bitangent planes* to S. Salmon computes

$$\deg(D_b) = \frac{1}{2}d(d-1)(d-2)(d^3-d^2+d-12).$$

There is also a *cuspidal curve* D_s, whose points correspond to *stationary tangent planes* to S, i.e., tangent planes to S in *parabolic points*, namely points $x \in S$ such that the tangent plane to S at x cuts S in a curve with a cusp at x. Salmon computes

$$\deg(D_s) = 4d(d-1)(d-2),$$

whereas the curve Γ_p of parabolic points on S is the complete intersection of S with its *Hessian surface*, hence

$$\deg(\Gamma_p) = 4d(d-2).$$

Finally, there are finitely many triple points of $\check{S}$, which are triple points of D_b too. Salmon computes their number

$$t = \frac{1}{6}d(d-2)(d^7-4d^6+7d^5-45d^4+114d^3-111d^2+548d-960). \tag{2.0.1}$$

For all these classical formulae, the reader is invited to consult [XII, C].

All these numbers have been computed in recent times by I. Vainsencher [41, 26] and by S. Kleiman and R. Piene [12, 13], via Chern class computations, see [XIII]. However several questions are still open, like:

(2.1) Question. *What are the singularities of D_b and of D_s? Are these curves irreducible? How do they intersect? Is Γ_p smooth? What is the monodromy of the covering $T \to U$, given by the triple points of $\check{S}$, when S moves in the open subset $U \subseteq |\mathcal{O}_{\mathbf{P}^3}(d)|$ of smooth surfaces?*

Partial answers to some of these question have been given for $d = 4$ in [4].

3 – Semistable degenerations

A degeneration $f : S \to \mathbf{D}$ as in §1 is said to be *semistable* if the central fibre S_0 is reduced and with *local normal crossing* singularities. We will often assume that all irreducible components of S_0 are smooth, so the singularities of S_0 are: *double curves*, along which only two irreducible components of S_0 intersect transversally and *triple points*, where only three irreducible components of S_0 intersect transversally.

As in §1, we will assume there is a *polarising line bundle* $\mathcal{L}$ on S (although, in principle, it could be neither ample nor relatively ample). Denote by L_t the restriction of $\mathcal{L}$ to S_t for all $t \in \mathbf{D}$. Then we will say that (S_0, L_0) is *a limit* of (S_t, L_t) (with $t \in \mathbf{D}^*$), when $t \to 0$ (or that (S_t, L_t) with $t \in \mathbf{D}^*$ is a deformation of (S_0, L_0)).

The limit (S_0, L_0) *is not unique* if S_0 is reducible. Indeed, if W is an effective divisor supported on the central fibre S_0, consider the line bundle $\mathcal{L}(-W)$, which is said to be obtained from $\mathcal{L}$ by *twisting* by W. For $t \in \mathbf{D}^*$, its restriction to S_t is the same as L_t, but in general this is not the case for S_0; any such line bundle $\mathcal{L}(-W)|_{S_0}$ is called a *limit line bundle* of $\mathcal{L}_t$ for $t \in \mathbf{D}^*$. If Q is an irreducible component of S_0, the restriction $\mathcal{L}_{W,Q} := \mathcal{L}(-W)_{|Q}$ is called the *aspect* of $\mathcal{L}(-W)$ on Q.

(3.1) Remark. Since $\mathrm{Pic}(\mathbf{D})$ is trivial, the divisor $S_0 \subseteq S$ is linearly equivalent to 0. So if W is a divisor supported on S_0, one has $\mathcal{L}(-W) \cong \mathcal{L}(mS_0 - W)$ for all integers m. In particular if $W + W' = S_0$ then $\mathcal{L}(-W) \cong \mathcal{L}(W')$.

Another family of surfaces $f' : S' \to \mathbf{D}$ as above is said to be a *model of* $f : S \to \mathbf{D}$ if there is a commutative diagram

$$\begin{tikzcd} S' \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \bar{S}' \arrow[l] \arrow[r, dashed, "p"] \arrow[d] & \bar{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "f"] \\ \mathbf{D} & \mathbf{D} \arrow[l, "t^{d'} \leftarrow\!\shortmid t"] \arrow[r, equal] & \mathbf{D} \arrow[r, "t \mapsto t^d"'] & \mathbf{D} \end{tikzcd}$$

where the two squares marked with a $\square$ are Cartesian, and p is a birational map, which is an isomorphism over $\mathbf{D}^*$. The family $f' : S' \to \mathbf{D}$, if semistable, is a *semistable model of* $f : S \to \mathbf{D}$ if in addition $d' = 1$ and p is a morphism. The *semistable reduction theorem* of [12] asserts that any family $f : S \to \mathbf{D}$ as in §1 has a semistable model.

(3.2) Example (Families of surfaces in $\mathbf{P}^3$)**.** Consider a *linear pencil* of degree d surfaces in $\mathbf{P}^3$, generated by a *general* surface S_∞ and a *special* one S_0. We will usually consider the case in which S_0 has local normal crossing singularities, and S_∞ intersects transversally the double curve of S_0 at smooth points of it. This pencil gives rise to a flat, proper family $\varphi : \mathcal{S} \to \mathbf{P}^1$, with $\mathcal{S}$ a hypersurface of type $(d, 1)$ in $\mathbf{P}^3 \times \mathbf{P}^1$, isomorphic to the blow–up of $\mathbf{P}^3$ along the *base locus scheme* $S_0 \cap S_\infty$ of the pencil, and it has S_0, S_∞ as fibres over $0, \infty \in \mathbf{P}^1$, respectively.

We will study the family obtained by restricting $\mathcal{S}$ to a disk $\mathbf{D} \subseteq \mathbf{P}^1$ centered at 0, that by abuse of notation we will still denote by $f : \mathcal{S} \to \mathbf{D}$, such that S_t is smooth for all $t \in \mathbf{D}^*$, and we will consider a semistable model of $f : S \to \mathbf{D}$. To do so, we resolve the singularities of S which occur only in the central fibre of f, at the points mapped by $\mathcal{S}_0 \to S_0 \subseteq \mathbf{P}^3$ to the intersection points of S_∞ with the double curves of S_0 (they are the singular points of the curve $S_0 \cap S_\infty$). These are *ordinary double points* of S, i.e., singularities analytically equivalent to the one at the origin of the hypersurface $xy = zt$ in $\mathbf{A}^4$. Such a singularity is resolved by a single blow–up, which produces an exceptional divisor $E \cong \mathbf{P}^1 \times \mathbf{P}^1$, and then it is possible to contract E in the direction of either one of its rulings without introducing any singularity: the result is called a *small resolution* of the ordinary double point.

Let $\tilde{f} : \tilde{S} \to \mathbf{D}$ be a semistable model thus obtained. One has $\tilde{S}_t \cong S_t$ for $t \in \mathbf{D}^*$. If S_0 has irreducible components $Q_1, \ldots, Q_r$, then $\tilde{S}_0$ consists of irreducible components $\tilde{Q}_1, \ldots, \tilde{Q}_r$ which are suitable blow–ups of $Q_1, \ldots, Q_r$, respectively. If q is the number of ordinary double points of the original total space S, we will denote by $E_1, \ldots, E_q$ the exceptional curves on $\tilde{Q}_1, \ldots, \tilde{Q}_r$ arising from the small resolution process.

The simplest example of this situation is when S_0 is the union of a general surface S' of degree h with a general surface S'' of degree $d-h$ (and $1 \leqslant h \leqslant \lfloor \frac{d}{2} \rfloor$), intersecting along a smooth curve R of degree $h(d-h)$. In this case $\tilde{S}_0$ consists of two components Q_1, Q_2, and we may assume that $Q_2 = S''$, whereas $Q_1 = \tilde{S}'$ is the blow–up of S' at the $q = dh(d-h)$ intersection points of R with S_∞. Then Q_1 and Q_2 intersect along a curve which can be identified with R and the exceptional curves $E_1, \ldots, E_q$ all lie on Q_1.

The case $h = 1$ in which $S' = P$ is a general plane, is particularly interesting since it shows the geometric significance of twisting the polarising line bundle $\mathcal{L} = p^*(\mathcal{O}_{\mathbf{P}^3}(1))$ on $\tilde{S}$. The polarising line bundle L_0 maps $\tilde{S}_0$ to the reducible surface $S'' \cup P$. If we twist by $Q_1 = \tilde{P}$ and consider $\mathcal{L}' = \mathcal{L}(-\tilde{P})$, then its aspects are:
- the trivial bundle on $Q_2 = S''$;
- $|\mathcal{O}_{\tilde{P}}(dH - \sum_{i=1}^{d(d-1)} E_i)|$ on $Q_1 = \tilde{P}$.

Hence L_0' contracts S'' to a point x. Moreover $\dim(|\mathcal{O}_{\tilde{P}}(dH - \sum_{i=1}^{d(d-1)} E_i|) = 3$ and this linear system maps $\tilde{P}$ to a *monoid* Σ, i.e., to a rational surface of degree d with a point of multiplicity $d-1$ at x (see Remark (7.3) for the case $d = 4$).

In conclusion, by twisting, we see that the degeneration of $(S_t, \mathcal{O}_{S_t}(1))$ to $(S'' \cup \tilde{P}, L_0)$ is also a semistable model of the degeneration of $(S_t, \mathcal{O}_{S_t}(1))$ to a general monoid $(\Sigma, \mathcal{O}_\Sigma(1))$.

(3.3) Example (Families of polarised K3 surfaces)**.** A special case of the previous example is the one of a general quartic surface in $\mathbf{P}^3$ degenerating to the general union of two quadrics. This is also a special case of degenerations of polarised K3 surfaces to a union of two scrolls (see [7]), which are type II degenerations according to the Kulikov-Persson-Pinkham classification of semistable degenerations of $K3$ surfaces (see [11, 17]).

Let $\mathcal{K}_p$ be the moduli space of primitively polarized $K3$ surfaces (S, L) of genus $p \geqslant 3$, i.e., the dualising sheaf ω_S of S is trivial, $h^1(S, \mathcal{O}_S) = 0$, and L is big, nef and indivisible in $\mathrm{Pic}(S)$, with $L^2 = 2p-2$. Write $p = 2l + \varepsilon$, with $\varepsilon = 0, 1$ and $l \in \mathbf{N}$. If $E' \subseteq \mathbf{P}^p$ is an elliptic normal curve of degree $p+1$, set $L_{E'} = \mathcal{O}_{E'}(1)$. Consider two *general* line bundles $L_1, L_2 \in \mathrm{Pic}^2(E')$ with $L_1 \neq L_2$. Denote by Σ_i' the rational normal scroll of degree $p-1$ in $\mathbf{P}^p$ described by the secant lines to E' spanned by the divisors in $|L_i|$, for $1 \leqslant i \leqslant 2$. One has

$$\Sigma_i' \cong \begin{cases} \mathbf{P}^1 \times \mathbf{P}^1 & \text{if } p = 2l+1 \text{ is odd,} \\ \mathbf{F}_1 & \text{if } p = 2l \text{ is even.} \end{cases}$$

The surfaces Σ_1' and Σ_2' are $\mathbf{P}^1$–bundles on $\mathbf{P}^1$. Denote by σ_i and f_i a minimal section and a fiber of the ruling of Σ_i', respectively, so that $\sigma_i^2 = \varepsilon - 1$, $f_i^2 = 0$, and

$$L_{\Sigma_i'} := \mathcal{O}_{\Sigma_i'}(1) \simeq \mathcal{O}_{\Sigma_i'}(\sigma_i + l f_i), \quad \text{for} \quad 1 \leqslant i \leqslant 2. \tag{3.3.1}$$

By [7, Thm. 1], Σ_1' and Σ_2' intersect transversally along E', which is anticanonical on Σ_i', i.e.

$$E' \sim -K_{\Sigma_i'} \sim 2\sigma_i + (3-\varepsilon) f_i \quad \text{for} \quad 1 \leqslant i \leqslant 2, \tag{3.3.2}$$

where $\sim$ is the linear equivalence. Hence $\Sigma' = \Sigma_1' \cup \Sigma_2'$ has normal crossings and its dualising sheaf $\omega_{\Sigma'}$ is trivial. Set $L_{\Sigma'} := \mathcal{O}_{\Sigma'}(1)$. The first cotangent sheaf $T^1_{\Sigma'}$ (cf. [4, § 1]) is the degree 16 line bundle on E'

$$T^1_{\Sigma'} \cong \mathbf{N}_{E'/\Sigma_1'} \otimes \mathbf{N}_{E'/\Sigma_2'} \cong L_{E'}^{\otimes 4} \otimes (L_1 \otimes L_2)^{\otimes(3-2l-\varepsilon)}, \tag{3.3.3}$$

where $\mathbf{N}_{E'/\Sigma_i'}$ is the normal sheaf of E' in Σ_i', for $i = 1, 2$, and the last isomorphism comes from (3.3.1) and (3.3.2).

The surface Σ' is a flat limit of smooth $K3$ surfaces in $\mathbf{P}^p$. Namely, if $\mathcal{H}_p$ is the component of the Hilbert scheme of surfaces in $\mathbf{P}^p$ containing $K3$ surfaces S such that $(S, \mathcal{O}_S(1)) \in \mathcal{K}_p$, then Σ' sits in $\mathcal{H}_p$ and, for general choices of E', L_1, L_2, the Hilbert scheme is smooth at Σ' (see [7]). The fact that $T^1_{\Sigma'}$ is non-trivial implies that Σ' is not a semi–stable limit of $K3$ surfaces: indeed, the total space $\mathfrak{S}'$ of every flat deformation of Σ' to $K3$ surfaces in $\mathcal{H}_p$ is singular along a divisor $T \in |T^1_{\Sigma'}|$ (cf. [4, Prop. 1.11 and § 2]). More precisely (see again [7] for details), if

(3.3.4)
$$\begin{tikzcd} \Sigma' \arrow[r, hook] \arrow[d] & \mathfrak{S}' \arrow[r, hook] \arrow[d, "\pi'"] \arrow[drr, "\mathrm{pr}_2"] & \mathbf{D} \times \mathbf{P}^p \arrow[dr] & \\ 0 \arrow[r, hook] & \mathbf{D} & & \mathbf{P}^p \end{tikzcd}$$

is a deformation of Σ' in $\mathbf{P}^p$ whose general member is a smooth $K3$ surface, then $\mathfrak{S}'$ has double points at the support of a divisor $T \in |T^1_{\Sigma'}|$ associated to the tangent direction to $\mathcal{H}_p$ at Σ' determined by the deformation (3.3.4), via the map

$$T_{\mathcal{H}_p,\Sigma'} \cong H^0(\Sigma', \mathbf{N}_{\Sigma'/\mathbf{P}^p}) \to H^0(T^1_{\Sigma'}) \tag{3.3.5}$$

induced by the surjective sheaf map

$$\mathbf{N}_{\Sigma'/\mathbf{P}^p} \to T^1_{\Sigma'}.$$

If T is reduced (this is the case if (3.3.4) is general enough), then the tangent cone to $\mathfrak{S}$ at each of the 16 points of T has rank 4. If $\Pi : \mathfrak{S} \to \mathfrak{S}'$ is a small resolution of singularities, one gets a semistable degeneration $\pi := \pi' \circ \Pi : \mathfrak{S} \longrightarrow \mathbf{D}$ of $K3$ surfaces, with central fiber $\Sigma := \Sigma_1 \cup \Sigma_2$, where $\Sigma_i = \Pi^{-1}(\Sigma'_i)$, for $i = 1, 2$, and still $\omega_\Sigma \cong \mathcal{O}_\Sigma$. The polarising line bundle on $\mathfrak{S}$ is $\mathcal{L} := \Pi^*(\mathrm{pr}_2^*(\mathcal{O}_{\mathbf{P}^p}(1)))$.

Set $E = \Sigma_1 \cap \Sigma_2$; then $E \cong E'$ and

$$T^1_\Sigma \simeq \mathcal{O}_E. \tag{3.3.6}$$

Equation (3.3.6) is a particular case of the following general fact (see [1, 4] for a more general formulation).

(3.4) Lemma (Triple Point Formula)**.** *Assume $f : S \to \mathbf{D}$ is semistable. Let Q, Q' be smooth irreducible components of S_0 intersecting along the* double curve *R. Then*

$$\deg(\mathbf{N}_{R/Q}) + \deg(\mathbf{N}_{R/Q'}) + \mathrm{Card}\left\{ \begin{array}{c} \textit{triple points of } S_0 \\ \textit{along } R \end{array} \right\} = 0,$$

where a triple point *is the intersection $R \cap Q''$ with a component Q'' of S_0 different from Q, Q'.*

4 – Limit linear systems

Consider a semistable degeneration $f : \mathcal{S} \to \mathbf{D}$ as in §3. Suppose there is a polarising, ample, fixed components free line bundle $\mathcal{L}$ on the total space $\mathcal{S}$, such that $h^0(S_t, L_t)$ is constant for $t \in \mathbf{D}$.

Consider the subscheme $\mathrm{Hilb}(\mathcal{L})$ of the *relative Hilbert scheme* of curves of $\mathcal{S}$ over $\mathbf{D}$, which is the Zariski closure of the set of all curves $C \in |L_t|$, for $t \in \mathbf{D}^*$. Assume that $\mathrm{Hilb}(\mathcal{L})$ is a component of the relative Hilbert scheme, a condition satisfied if $\mathrm{Pic}(S_t)$ has no torsion for $t \in \mathbf{D}^*$. One has a natural projection morphism $\varphi : \mathrm{Hilb}(\mathcal{L}) \to \mathbf{D}$, which is a projective bundle over $\mathbf{D}^*$, because $\mathrm{Hilb}(\mathcal{L})$ is isomorphic to $\mathfrak{P} := \mathbf{P}(f_*(\mathcal{L}))$ over $\mathbf{D}^*$. We call the fibre of φ over 0 the *limit linear system* of $|L_t|$ as $t \in \mathbf{D}^*$ tends to 0, and we will denote it by $\mathfrak{L}$.

(4.1) Remark. In general, *the limit linear system is not a linear system.* One would be tempted to say that $\mathfrak{L}$ coincides with $|L_0|$. This is the case if S_0 is irreducible, but it is in general no longer true when S_0 is reducible. Indeed in this case, there may be non–zero sections of L_0 whose zero–locus contains some irreducible component of S_0, and accordingly points of $|L_0|$ which do not correspond to points in the Hilbert scheme of curves. This is related to the non–uniqueness of the limit line bundle and to the twisting procedure mentioned in § 3 (see, e.g., Example (4.2) below). We will come back to this in a while.

The variety $\mathrm{Hilb}(\mathcal{L})$ is birational to $\mathfrak{P}$, and $\mathfrak{L}$ is a suitable degeneration of the projective space $|L_t|$, $t \in \Delta^*$.

As the following example shows, in passing from $\mathfrak{P}$ to $\mathrm{Hilb}(\mathcal{L})$, one has to perform a series of blow–ups along smooth centres contained in the central fibre, which correspond to spaces of non–trivial sections of some (twisted) line bundles which vanish on divisors contained in the central fibre. The exceptional divisors one gets in this way give rise to components of $\mathfrak{L}$, and may be identified with birational modifications of sublinear systems of twisted linear systems restricted to S_0.

(4.2) Example (See [9])**.** Consider a family of degree d surfaces $f : \mathcal{S} \to \mathbf{D}$ arising, as in Example (3.2), from a linear pencil generated by a general surface S_∞ and by $S_0 = S'' \cup P$, where P is a plane and S'' a general surface of degree $d-1$. One has a semistable model $\tilde{f} : \tilde{S} \to \mathbf{D}$ as described in Example (3.2), from which we keep the notation.

Let $\mathcal{L}$ be the polarising hyperplane bundle. The component $\mathrm{Hilb}(\mathcal{L})$ of the Hilbert scheme is gotten from the projective bundle $\mathbf{P}(\mathcal{L})$, by blowing up the point of the central fibre $|\mathcal{O}_{S_0}(1)|$ corresponding to the 1–dimensional space of non–zero sections vanishing on the plane P. The limit linear system $\mathfrak{L}$ is the union of $\mathfrak{L}_1$, the blown–up $|\mathcal{O}_{S_0}(1)|$, and of the exceptional divisor $\mathfrak{L}_2 \cong \mathbf{P}^3$, identified as the twisted linear system $|\mathcal{O}_{\tilde{S}_0}(1) \otimes \mathcal{O}_{\tilde{S}}(-\tilde{P})|$, whose aspects have been described in Example (3.2).

The components $\mathfrak{L}_1$ and $\mathfrak{L}_2$ of $\mathfrak{L}$ meet along the exceptional divisor $\mathfrak{E} \cong \mathbf{P}^2$ of the morphism $\mathfrak{L}_1 \to |\mathcal{O}_{S_0}(1)|$. The elements of $\mathfrak{E} \subseteq \mathfrak{L}_1$ identify as the points of $|\mathcal{O}_R(1)| \cong |\mathcal{O}_P(1)|$, whereas the plane $\mathfrak{E} \subseteq \mathfrak{L}_2$ is the set of elements $C \in |\mathcal{O}_{\tilde{P}}(d) \otimes \mathcal{O}_{\tilde{P}}(-\sum_{i=1}^{d(d-1)} E_i)|$ containing the proper transform $\hat{R} \cong R$ of R on $\tilde{P}$. The corresponding element of $|\mathcal{O}_R(1)|$ is cut out on $\hat{R}$ by the further component of C, which is the pull–back to $\tilde{P}$ of a line in P.

All this will be made explicit in the case $d = 4$ in §7.2.

5 – Severi varieties and their limits

Let $f : \mathcal{S} \to \mathbf{D}$ be a (not necessarily semistable) family as in §3, polarised by a line bundle $\mathcal{L}$ on $\mathcal{S}$. Fix a non–negative integer δ, and consider the locally closed subset $\mathring{V}_\delta(\mathcal{S}, \mathcal{L})$ of $\mathrm{Hilb}(\mathcal{L})$ formed by all curves $D \in |L_t|$, for $t \in \mathbf{D}^*$, such that D is irreducible, nodal, and has exactly δ nodes. Define $V_\delta(\mathcal{S}, \mathcal{L})$ (resp. $V_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L})$) as the Zariski closure of $\mathring{V}_\delta(\mathcal{S}, \mathcal{L})$ in $\mathrm{Hilb}(\mathcal{L})$ (resp. in $\mathfrak{P} = \mathbf{P}(f_*(\mathcal{L}))$). This is the *relative Severi variety* (resp. the *crude relative Severi variety*). Sometimes we may write $\mathring{V}_\delta$, V_δ, and V_δ^{cr}, rather than $\mathring{V}_\delta(\mathcal{S}, \mathcal{L})$, $V_\delta(\mathcal{S}, \mathcal{L})$ and $V_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L})$, respectively.

There is a natural map $f_\delta : V_\delta \to \mathbf{D}$. If $t \in \mathbf{D}^*$, the fibre $V_{\delta,t}$ of f_δ over t is the *Severi variety* $V_\delta(S_t, L_t)$ of δ–nodal curves in the linear system $|L_t|$ on S_t. The degree of $V_{\delta,t}$ as a subvariety of $|L_t|$ is independent on $t \in \mathbf{D}^*$, and will be denoted by $d_\delta(\mathcal{L})$ (or simply by d_δ). Let $\mathfrak{V}_\delta(\mathcal{S}, \mathcal{L})$ (or simply $\mathfrak{V}_\delta$) be the central fibre of $f_\delta : V_\delta \to \mathbf{D}$; it is the *limit Severi variety* of $V_\delta(S_t, \mathcal{L}_t)$ as $t \in \mathbf{D}^*$ tends to 0. This is a subscheme of the limit linear system $\mathfrak{L}$, which has been studied by various authors (in the present setting by Z. Ran [18, 19, 20], then by L. Caporaso, J. Harris [1, 2], more recently by C. Galati and C. Galati and A. Knutsen [9, 11] following Z. Ran; this theme is however quite classical as B. Segre's paper [23] shows, and it appears in disguise in work of other authors, like S. Kleiman [13]).

In a similar way, one defines the *crude limit Severi variety* $\mathfrak{V}^{\mathrm{cr}}_\delta(\mathcal{S},\mathcal{L})$ (or $\mathfrak{V}^{\mathrm{cr}}_\delta$), sitting in $|L_0|$.

(5.1) Remark. For $t \in \mathbf{D}^*$, the *expected dimension* of the Severi variety $V_\delta(S_t,\mathcal{L}_t)$ is $\dim(|L_t|)-\delta$ (see [24, Thm. 6.3]). We will always assume that the dimensions of (all components of) $V_\delta(S_t,L_t)$ equal the expected dimension for all $t \in \mathbf{D}^*$. This is a strong assumption, implying that $\mathring{V}_\delta$ is smooth; it will be satisfied in our applications.

Let us come now to the description of the limit Severi variety, under the assumption that the family $f : \mathcal{S} \to \mathbf{D}$ is semistable. We will freely use the notation introduced above and follow the approach in [9, 11]. We will suppose that the central fibre S_0 has smooth irreducible components $Q_1,\dots,Q_r$, with double curves $R_1,\dots,R_s$. We will assume, in addition, that there are q exceptional curves $E_1,\dots,E_q$ on S_0, arising from a small resolution of an original family with singular total space, as discussed in §3.

(5.2) Notation. Let $\underline{\mathbf{N}}$ be the set of sequences $\underline{\tau} = (\tau_m)_{m\geqslant 2}$ of non–negative integers with only finitely many non–vanishing terms. Define two maps $\nu,\ \mu : \underline{\mathbf{N}} \to \mathbf{N}$ as follows

$$\nu\,(\underline{\tau}) = \sum\nolimits_{m\geqslant 2} \tau_m \cdot (m-1), \quad \text{and} \quad \mu\,(\underline{\tau}) = \prod\nolimits_{m\geqslant 2} m^{\tau_m}.$$

Given a p-tuple $\underline{\boldsymbol{\tau}} = (\underline{\tau}_1,\dots,\underline{\tau}_p) \in \underline{\mathbf{N}}^p$, set

$$\nu(\underline{\boldsymbol{\tau}}) = \nu(\underline{\tau}_1)+\cdots+\nu(\underline{\tau}_p), \quad \text{and} \quad \mu(\underline{\boldsymbol{\tau}}) = \mu(\underline{\tau}_1)\cdots\mu(\underline{\tau}_p),$$

defining two maps $\nu,\ \mu : \underline{\mathbf{N}}^p \to \mathbf{N}$. Given $\boldsymbol{\delta} = (\delta_1,\dots,\delta_r) \in \mathbf{N}^r$, set

$$|\boldsymbol{\delta}| := \delta_1+\cdots+\delta_r.$$

Given a subset $I \subseteq \{1,\dots,q\}$, $|I|$ will denote its cardinality.

(5.3) Definition. *Consider a divisor W on $\mathcal{S}$, supported on the central fibre S_0, such that the twist $\mathcal{L}(-W)$ is* centrally effective, *i.e., all the aspects of $\mathcal{L}(-W)$ on the components $Q_1,\dots,Q_r$ of S_0 are effective. Fix*

$$\boldsymbol{\delta} \in \mathbf{N}^r, \quad \underline{\boldsymbol{\tau}} \in \underline{\mathbf{N}}^p \quad \textit{and} \quad I \subseteq \{1,\dots,r\}.$$

Let $\mathring{V}(W,\boldsymbol{\delta},I,\underline{\boldsymbol{\tau}})$ be the Zariski locally closed subset in $|\mathcal{L}(-W)\otimes\mathcal{O}_{S_0}|$ parametrizing curves D such that:
(i) *D contains no double curve R_l, with $l \in \{1,\dots,s\}$, and no triple point of S_0;*
(ii) *D contains E_i, with multiplicity 1, if and only if $i \in I$, and has a node on it;*
(iii) *off the singular locus of S_0 the curve $D-\sum_{i\in I}E_i$ has only nodes as singularities, and their number is δ_s on Q_s, for $s\in\{1,\dots,r\}$;*
(iv) *for every $l\in\{1,\dots,s\}$ and $m\geqslant 2$, there are exactly $\tau_{l,m}$ points on R_l, off the intersections with $\sum_{i\in I}E_i$ and off the triple points of S_0 along R_l, at which D has an m-*tacnode *(see below for the definition), with reduced tangent cone equal to the tangent line of R_l there.*

We let $V(W,\boldsymbol{\delta},I,\underline{\boldsymbol{\tau}})$ be the Zariski closure of $\mathring{V}(W,\boldsymbol{\delta},I,\underline{\boldsymbol{\tau}})$ in $|\mathcal{L}(-W)\otimes\mathcal{O}_{S_0}|$.

Recall that an *m-tacnode* is an A_{2m-1}-double point, i.e., a plane curve singularity locally analytically isomorphic to the curve of $\mathbf{C}^2$ defined by the equation $y^2 = x^{2m}$ at the origin. Condition (iv) in Definition (5.3) requires that D is a Cartier divisor on S_0, having $\tau_{l,m}$ m–th order tangency points with the curve R_l, at points of R_l which are neither points on $\sum_{i\in I}E_i$ nor triple points of S_0.

(5.4) Notation. Rather than using the notation $V(W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$, we may sometimes use a more expressive one like, e.g., $V(W, \delta_{Q_1} = 2, E_1, \tau_{R_1,2} = 1)$ for the variety parametrizing curves in $|\mathcal{L}(-W) \otimes \mathcal{O}_{S_0}|$, with two nodes on Q_1, one simple tacnode along R_1, and containing the exceptional curve E_1. Similarly $V(\delta_{Q_1} = 1)$ is the variety parametrizing curves in $|\mathcal{L} \otimes \mathcal{O}_{S_0}|$, with only one node on Q_1, etc.

(5.5) Proposition ([9, 11], see [VIII])**.** *Let $W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}}$ be as above, and set*

$$\delta = |\boldsymbol{\delta}| + |I| + \nu(\underline{\boldsymbol{\tau}}).$$

Let V be an irreducible component of $V(W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$. If
(i) the linear system $|\mathcal{L}(-W) \otimes \mathcal{O}_{S_0}|$ has the same dimension as $|\mathcal{L}_t|$ for $t \in \mathbf{D}^*$, and
(ii) V has (the expected) codimension δ in $|\mathcal{L}(-W) \otimes \mathcal{O}_{S_0}|$,
then V is an irreducible component of multiplicity $\mu(V) := \mu(\underline{\boldsymbol{\tau}})$ of the limit Severi variety $\mathfrak{V}_\delta(S, \mathcal{L})$.

(5.6) Remark. Same assumptions as in Proposition (5.5). If there is at most one tacnode (i.e. all $\tau_{l,m}$ but possibly one vanish, and this is equal to 1), the relative Severi variety V_δ is smooth at the general point of V (see [9, 11]), and thus V belongs to only one irreducible component of V_δ. There are other cases in which such a smoothness property holds (see [1] or, in this volume, [VI]).

If V_δ is smooth at the general point $D \in V$, the multiplicity of V in the limit Severi variety $\mathfrak{V}_\delta$ is the minimal integer m such that there are local analytic *m–multisections* of $V_\delta \to \mathbf{D}$, i.e. analytic smooth curves in V_δ, passing through D and intersecting the general fibre $V_{\delta,t}$, $t \in \mathbf{D}^*$, at m distinct points.

Proposition (5.5) does not provide a complete picture of the limit Severi variety. For instance, curves passing through a triple point of S_0 could (and in fact do; see [4], and §8.1 below) play a role in this limit. It would be desirable to know that one can always obtain a semistable model of the original family, where *every* irreducible component of the limit Severi variety is realized as a family of curves of the kind stated in Definition (5.3).

(5.7) Definition. *Let $f : \mathcal{S} \to \mathbf{D}$ be a semistable family as in §3, with a polarising line bundle $\mathcal{L}$, and δ a positive integer. The* regular part of the limit Severi variety *$\mathfrak{V}_\delta(\mathcal{S}, \mathcal{L})$ is the cycle in the limit linear system $\mathfrak{L} \subseteq \mathrm{Hilb}(\mathcal{L})$ defined as*

$$\mathfrak{V}_\delta^{\mathrm{reg}}(\mathcal{S}, \mathcal{L}) := \sum_W \sum_{|\boldsymbol{\delta}|+|I|+\nu(\underline{\boldsymbol{\tau}})=\delta} \mu(\underline{\boldsymbol{\tau}}) \cdot \left(\sum_{V \in \mathrm{Irr}^\delta(V(W,\boldsymbol{\delta},I,\underline{\boldsymbol{\tau}}))} V \right), \tag{5.7.1}$$

(sometimes simply denoted by $\mathfrak{V}_\delta^{\mathrm{reg}}$) where:
(i) W varies among all effective divisors on S supported on the central fibre S_0, such that $\mathcal{L}(-W)$ is centrally effective and $h^0(\mathcal{L}_0(-W)) = h^0(L_t)$ for $t \in \mathbf{D}^*$;
(ii) $\mathrm{Irr}^\delta(Z)$ denotes the set of all codimension δ irreducible components of a scheme Z.

Proposition (5.5) asserts that the cycle $Z(\mathfrak{V}_\delta) - \mathfrak{V}_\delta^{\mathrm{reg}}$ is effective where $Z(\mathfrak{V}_\delta)$ is the cycle associated to $\mathfrak{V}_\delta$. We call the irreducible components of the support of $\mathfrak{V}_\delta^{\mathrm{reg}}$ the *regular components* of the limit Severi variety.

Let now $\tilde{f} : \tilde{\mathcal{S}} \to \mathbf{D}$ be a semistable model of a (not necessarily semistable) degeneration $f : \mathcal{S} \to \mathbf{D}$, and $\tilde{\mathcal{L}}$ the pull–back on $\tilde{\mathcal{S}}$ of a polarising line bundle $\mathcal{L}$ on $\mathcal{S}$. There is a natural map $\mathrm{Hilb}(\tilde{\mathcal{L}}) \to \mathrm{Hilb}(\mathcal{L})$, which induces a morphism $\phi : \tilde{\mathfrak{L}} \to |L_0|$.

(5.8) Definition. *The semistable model $\tilde f : \tilde{\mathcal{S}} \to \mathbf{D}$ is a δ–*good model *of $f : \mathcal{S} \to \mathbf{D}$ (or simply a* good model*, if it is clear which δ we are considering), if the following equality of cycles holds*

$$\phi_*\big(\mathfrak{V}_\delta^{\mathrm{reg}}(\tilde{\mathcal{S}}, \tilde{\mathcal{L}})\big) = \mathfrak{V}_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L}).$$

Note that the cycle $\mathfrak{V}_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L}) - \phi_*\big(\mathfrak{V}_\delta^{\mathrm{reg}}(\tilde{\mathcal{S}}, \tilde{\mathcal{L}})\big)$ is effective. The family $f : \mathcal{S} \to \mathbf{D}$ is said to be *δ–well behaved* (or simply *well behaved* if δ is understood) if it has a δ-good model.

(5.9) Remark. Suppose that $f : \mathcal{S} \to \mathbf{D}$ is δ–well behaved, with δ–good model $\tilde f : \tilde{\mathcal{S}} \to \mathbf{D}$. It is possible that some components in $\mathfrak{V}_\delta^{\mathrm{reg}}(\tilde{\mathcal{S}}, \tilde{\mathcal{L}})$ are contracted by ϕ to varieties of smaller dimension, and therefore their push–forwards to $\mathfrak{V}_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L})$ are zero. Hence these components of $\mathfrak{V}_\delta(\tilde{\mathcal{S}}, \tilde{\mathcal{L}})$ are *not visible* in $\mathfrak{V}_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L})$. They are however usually visible in the crude limit Severi variety of some other model $f' : \mathcal{S}' \to \mathbf{D}$, obtained from $\tilde{\mathcal{S}}$ via an appropriate twist of $\mathcal{L}$. The central fibre S'_0 is then a flat limit of S_t, as $t \in \mathbf{D}^*$ tends to 0, different from S_0 (a situation met in Example (3.2)).

(5.10) Conjecture (See [4]). *Let $f : \mathcal{S} \to \mathbf{D}$ be a degeneration of surfaces, endowed with a line bundle $\mathcal{L}$ as above, and δ a positive integer. Then $f : \mathcal{S} \to \mathbf{D}$ is δ–well behaved.*

The local computations in [9, 11] provide a criterion for being well behaved:

(5.11) Proposition. *Assume there is a semistable model $\tilde f : \tilde{\mathcal{S}} \to \mathbf{D}$ of $f : \mathcal{S} \to \mathbf{D}$, with a limit linear system $\tilde{\mathfrak{L}}$ which does not contain, in codimension $\delta + 1$, curves of the following types:*
(i) curves containing double curves of $\tilde S_0$;
(ii) curves passing through a triple point of $\tilde S_0$;
(iii) non–reduced curves.
If in addition, for $W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}}$ as in Definition (5.3), every irreducible component of $V(W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$ has the expected codimension in $|L_0(-W)|$, then $\tilde f : \tilde{\mathcal{S}} \to \mathbf{D}$ is δ–well behaved.

(5.12) Remark. While there are good reasons to believe that Conjecture (5.10) has an affirmative answer, one cannot expect that, in the algebraic category, there is a good model of a given family, which *universally* works for all polarising line bundles $\mathcal{L}$. This in fact could require infinitely many birational modifications of the total space.

On the other hand, it is hopeless to ask for a semistable model on which all irreducible components of the limit Severi variety would parametrize nodal curves (i.e., such that there are no tacnodes as in Item (iv) of Definition (5.3)): we explain this in [B], together with C. Galati.

The three final sections are devoted to examples which will hopefully clarify the considerations and results in Section 5.

6 – Example: Limits of 1–nodal plane sections for surfaces in $\mathbf{P}^3$

Consider a degeneration of a general degree d surface S in $\mathbf{P}^3$ to a reducible surface $S' \cup S''$, as in Example (3.2), from which we keep the notation. Here $S_0 = S' \cup S''$, and $R = S' \cap S''$ is a smooth curve of degree $h(d-h)$. Then in the semistable model $\tilde f : \tilde{\mathcal{S}} \to \mathbf{D}$ constructed in Example (3.2), $\tilde S_0 = Q_1 \cup Q_2$, and Q_1 and Q_2 intersect transversally along a curve which may be identified with R.

(6.1) Assume $h \geqslant 2$. Then the only twist which is centrally effective is the trivial one and $\mathfrak{V}_1^{\mathrm{reg}}$ consists of $3+q$ components (remember that $q = hd(d-h)$), precisely:
(1) $V(\delta_{Q_i} = 1)$, with $i = 1, 2$, with multiplicity 1;
(2) $V(\tau_{R,2} = 1)$, with multiplicity 2;
(3) $V(E_i)$, for $i = 1, \ldots, q$, with multiplicity 1.

This tells us that in the above degeneration, the limit of the dual variety $\check{S}$, which sits in $\mathcal{L}_0 = \check{\mathbf{P}}^3$ and coincides with the crude limit of the Severi variety $\mathfrak{V}_1^{\mathrm{cr}}$, contains:
(i) the dual varieties $\check{S}'$ and $\check{S}''$ with multiplicity 1 (corresponding to (1));
(ii) the dual variety $\check{R}$ with multiplicity 2 (corresponding to (2));
(iii) the q planes $x_i^{\perp} \subseteq \check{\mathbf{P}}^3$, orthogonal to the points x_i, for $i = 1, \ldots, q$, with multiplicitiy 1 (corresponding to (3)).

One computes:
(a) $\deg(\check{S}') = h(h-1)^2$ and $\deg(\check{S}'') = (d-h)(d-h-1)^2$;
(b) $\deg(\check{R}) = h(d-h)(d-2)$.
So adding up, we see that the sum of the degrees of the components in (i)–(iii) above, counted with the appropriate multiplicities, is $d(d-1)^2$, hence the components in (1)–(3) exhaust the limit Severi variety $\mathfrak{V}_1$, i.e., the above degeneration is 1–well behaved.

(6.2) If $h = 1$, one needs to change the analysis, due to the presence of the non–trivial twist by Q_1 which is centrally effective. In this case $\mathfrak{V}_1^{\mathrm{reg}}$ consists of the following components:
(α) $V(\delta_{Q_2} = 1)$, with multiplicity 1 (note that $V(\delta_{Q_1} = 1)$ is zero);
(β) $V(Q_1, \delta_{Q_1} = 1)$, with multiplicity 1;
(γ) $V(\tau_{R,2} = 1)$, with multiplicity 2;
(δ) $V(E_i)$, for $i = 1, \ldots, q$, with multiplicity 1.
However, only the components in $(\alpha), (\gamma)$ and (δ) contribute to the crude limit of the Severi variety $\mathfrak{V}_1^{\mathrm{cr}}$, and exhaust it (same computation as above), because the component in (β) is not visible in the crude limit, since it is contracted to a point (see Remark (5.9)). In conclusion also this degeneration is 1–well behaved and the limit of $\check{S}$ is the union of $\check{S}'$ plus $d(d-1)$ planes plus the dual of the plane curve R counted with multiplicity 2.

7 – Example: Limits of 3–nodal plane sections for quartic surfaces in $\mathbf{P}^3$

The number of 3–nodal plane sections of a general quartic surface S in $\mathbf{P}^3$ (or of triple points of the dual $\check{S}$) given by Salmon's formula (2.0.1) is 3200.

7.1 – Degeneration to two quadrics

Let us recover this first by considering a degeneration to two quadrics: thus we will look at the case $h = d - h = 2$ in Example (3.2) (from which we keep the notation; this is the same as the degeneration in Example (3.3) for $p = 3$).

In this case R is an elliptic quartic curve, the semistable degeneration has central fibre $Q_1 \cup Q_2$, with $R = Q_1 \cap Q_2$, Q_1 is a quadric, and Q_2 a quadric blown up at the complete intersection points $x_1, \ldots, x_{16}$ of R with a general quartic surface S (see Figure 1).

For the computation, one has to consider the various cases for $V(W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$ as in Definition (5.3). No non–trivial twist is centrally effective, so $W = 0$. If $|\boldsymbol{\delta}| = 3$, then $V(W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$ is empty, because one cannot have two (or more) singular points on a plane section of a smooth quadric. Hence:

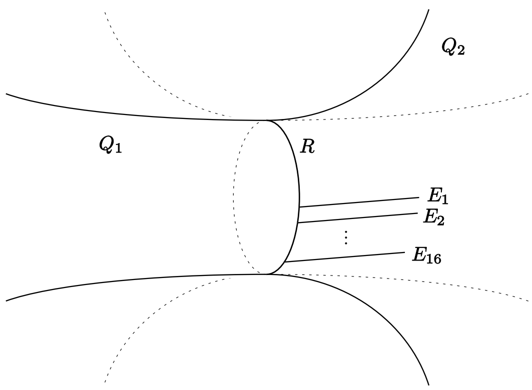


Figure 1: Degeneration into two quadrics

• $0 \leqslant |\boldsymbol{\delta}| \leqslant 2$, and if $|\boldsymbol{\delta}| = 2$ then $\delta_1 = \delta_2 = 1$;
• $\tau = (\tau_2, \tau_3, \tau_4)$ and $|I| + 2\tau_2 + 3\tau_3 + 4\tau_4 \leqslant 4$;
• $|\boldsymbol{\delta}| + |I| + \tau_2 + 2\tau_3 + 3\tau_4 = 3$.
Therefore the cases to be discussed are the following.

Case 1: $\delta_1 = \delta_2 = 1$, $|I| = 1$.

For each $i = 1, \ldots, 16$, a plane tangent to both S' and S'', and passing through x_i in $\mathbf{P}^3$ is entirely determined by the choice of a line passing through x_i in both Q_1 and Q_2. There are exactly $2 \times 2 = 4$ such choices (two rulings on each quadric). We get $16 \times 4 = 64$ curves.

Case 2: $\delta_1 = \delta_2 = 1$, $\tau_2 = 1$.

The intersection $\check{S}' \cap \check{S}'' \cap \check{R}$ consists of $2 \times 2 \times 8 = 32$ points. From this, we want to remove the points corresponding to planes tangent to one quadric at some point lying on R. Each of them has multiplicity 2 in the intersection $\check{S}' \cap \check{S}'' \cap \check{R}$ by the:

(7.1) Lemma (Lemma 3.5, [4]). *Let R be a smooth, irreducible curve contained in a smooth surface S in $\mathbf{P}^3$. Let $\check{R}_S$ be the irreducible curve in $\check{\mathbf{P}}^3$ parametrizing planes tangent to S along R. Then the dual varieties $\check{S}$ and $\check{R}$ both contain $\check{R}_S$, and do not intersect transversally at its general point.*

Proof. Clearly $\check{R}_S$ is contained in $\check{S} \cap \check{R}$. If either $\check{S}$ or $\check{R}$ are singular at the general point of $\check{R}_S$, there is nothing to prove. Assume that $\check{S}$ and $\check{R}$ are both smooth at the general point of $\check{R}_S$. One has to show that they are tangent there. Let $x \in R$ be general. Let H be the tangent plane to S at x. Then $H \in \check{R}_S$ is the general point. The biduality theorem (see , e.g., [10, Example 16.20]) says that the tangent planes to $\check{S}$ and $\check{R}$ at H both coincide with the set of planes in $\mathbf{P}^3$ containing x, hence the assertion. □

The curve parametrizing planes tangent to S' at some point of R has degree 4 (to see this, intersect R with a general polar plane of S', cf. [XII, Appendix **??**]). Hence this curve intersects $\check{S}''$ at 8 distinct points (without multiplicities). Eventually, we have to remove $2 \times 2 \times 8$ points from $\check{S}' \cap \check{S}'' \cap \check{R}$, and there does not remain anything.

Case 3: $\delta_i = 1$ *for an* $i = 1, 2$, $|I| = 2$.

For each $i = 1, 2$ and each set $\{r, s\} \subseteq I$, $r \neq s$, we count points of intersection of $\check{Q}_i$ with the line of $\check{\mathbf{P}}^3$ orthogonal to the line $\langle x_r, x_s \rangle$. There are $2 \times \binom{16}{2} \times 2 = 480$ such points.

Case 4: $\delta_i = 1$ *for an* $i = 1, 2$, $|I| = 1$, $\tau_2 = 1$.

For each $i = 1, 2$ and $r = 1, \dots, 16$, there are two lines on Q_i through the point x_r. They determine two pencils of planes tangent to Q_i and passing through x_r.

The planes of each of these pencils cut out a g^1_2 on R: the intersection of such a plane with R consists of 4 points, two of which are fixed (the two intersection points of the chosen line through x_r with R).

Since a g^1_2 on an elliptic curve has 4 double points, we get 2×4 planes satisfying our conditions for each i and r. These give a contribution of $2 \times 16 \times 2 \times 8 = 512$ to our computation.

Case 5: $\delta_i = 1$ for an $i = 1, 2$, $\tau_2 = 2$.

Planes satisfying these conditions are spanned by two lines belonging respectively to the two rulings of Q_i, both tangent to R. By genericity, two such lines do not meet on R.

Each ruling of Q_i cuts out a g^1_2 on R, and thus contains exactly 4 tangent lines to R (since a g^1_2 on R has 4 ramification points). We thus get 16 planes satisfying our conditions for each i. This contributes $2 \times 4 \times 16 = 128$ to our computation.

Case 6: $\delta_i = 1$ for an $i = 1, 2$, $\tau_3 = 1$.

This requires planes spanned by two lines on Q_i, one of which is tangent to R, and meets the other line on R at this tangency point. Such a plane is tangent to Q_i at a point lying on R, and therefore cuts a curve with worse singularity than required. Hence they do not contribute to our computation.

Case 7: $\tau_4 = 1$.

Since R is a smooth degree 4 elliptic curve in $\mathbf{P}^3$, the set of its hyperplane sections is

$$\{(a, b, c, d) \in \mathrm{Sym}^4(R) \mid a + b + c + d = 0 \in (R, +)\}.$$

Hence, the planes we are looking for are in 1-1 correspondence with 4-torsion points of R and so there are 16 of them, each to be counted with multiplicity 4, so we get a contribution of 64.

Case 8: $|I| = 1$, $\tau_3 = 1$.

Fix $r = 1, \dots, 16$, and consider the projection π from x_r. It maps R to a smooth cubic plane curve. Since this has 9 flexes, this gives us 9 planes through x_r satisfying the required condition. Each of them has to be counted with multiplicity 3, and this gives a contribution of $16 \times 3 \times 9 = 432$ to our computation.

Case 9: $|I| = 2$, $\tau_2 = 1$.

For given r, s distinct in $\{1, \dots, 16\}$, the pencil of planes through x_r and x_s cuts out (off of $x_r + x_s$) a g^1_2 on R. It has 4 double points, hence the pencil contains 4 tangent planes to R. By genericity, the tangency points are distinct from x_r and x_s. This gives a contribution of $\binom{16}{2} \times 2 \times 4 = 960$ to the computation.

Case 10: $|I| = 3$.

For each choice of three different indices r, s and l in $\{1, \dots, 16\}$, there is one plane through x_r, x_s and x_l. So we get a contribution of $\binom{16}{3} = 560$.

Adding up all the non–trivial contributions as in the following table, we get exactly 3200, which is the number predicted by Salmon's formula. This shows that this degeneration is 3–good.

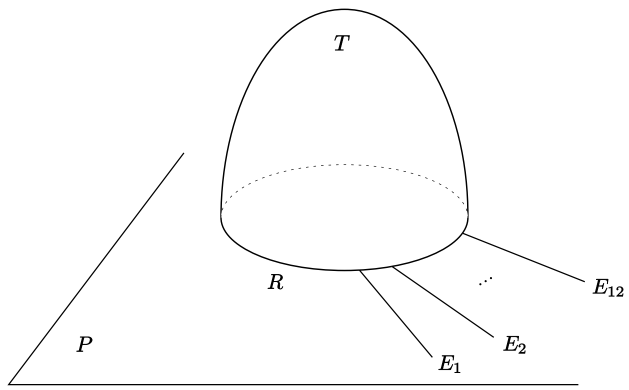


Figure 2: Degeneration into a cubic plus a plane

Cases	Contribution
Case 1	64
Case 3	480
Case 4	512
Case 5	128
Case 7	64
Case 8	432
Case 9	960
Case 10	560
Total	**3200**

7.2 – Degeneration to a cubic and a plane

We will now do the same enumeration as in Section 7.1, this time by considering a degeneration to a cubic and a plane. Thus, we want to count 3–nodal plane sections of a general quartic surface S in $\mathbf{P}^3$, and we will look at the degeneration in Example (3.2), with $d = 4$ and $h = 1$. This will be instructive, because it will require to make one of the first non–trivial cases of computation of the degree of a Severi variety of plane curves with nodes.

We simplify a bit the notation of Example (3.2). We denote by $T + P_0$ the reducible surface S_0 in Example (3.2) (so T is a general cubic, P_0 a general plane, with $P_0 \cap T = R$ a general plane cubic), and we denote by $T + P$ the central fibre of the semistable degeneration, where P is the plane P_0 blown–up at 12 points $x_1, \ldots, x_{12}$ forming a general divisor in $|\mathcal{O}_R(4)|$ (see Figure 2).

Now we explain in some detail in this case what we stated in Example (3.2) about the limit linear system $\mathfrak{L}$ of the plane sections of S_t.

(7.2) Proposition. *Let $\tilde{\mathcal{H}}_{\mathbf{P}^3}$ be the blow up of $|\mathcal{O}_{\mathbf{P}^3}(1)| = \check{\mathbf{P}}^3$ at the point corresponding to the plane $P_0 \subseteq \mathbf{P}^3$. Let $D_{P_0} \cong \mathbf{P}^2$ be the exceptional divisor of this blow up.*

Set $\mathcal{H}_P = |\mathcal{O}_P(4H - \sum_{i=1}^{12} E_i)| \cong \mathbf{P}^3$ and denote by $D_R \subseteq \mathcal{H}_P$ the plane parametrizing strict transforms of plane quartic curves that contain R (H denotes the pull back to P of a general line of P_0).

Then the limit of the limit linear system $\mathfrak{L}$ of $|\mathcal{O}_{S_t}(1)|$ as t tends to 0 is obtained by glueing $\tilde{\mathcal{H}}_{\mathbf{P}^3}$ and $\mathcal{H}_P$ along D_{P_0} and D_R.

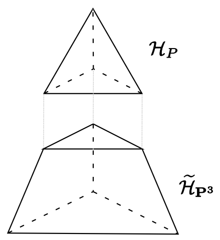


Figure 3: Limit hyperplane linear system for a degeneration into a cubic and plane

Proof. One has $\dim(|4H - \sum_{i=1}^{12} E_i|) = 3$ (see Example (3.2) and Remark (7.3) below).

There is only one non–trivial twist of the polarising hyperplane bundle $\mathcal{L}$, i.e., the twist by P, which is centrally effective. As we saw in Example (3.2), the twist $\mathcal{L}(-P)$ is trivial on T, whereas on P it restricts to $\mathcal{O}_P(H+R) = \mathcal{O}_P(4H - \sum_{i=1}^{12} E_i)$.

Up to a multiplicative constant, there is only one non–zero section of $\mathcal{L}_{|T+P}$ vanishing on P, hence it gives no element in $\mathfrak{L}$. To see the corresponding curves in $\mathfrak{L}$ we have to blow up $|\mathcal{L}_{|T+P}| = |\mathcal{O}_{\mathbf{P}^3}(1)|$ at P_0. One has

$$T_{\check{\mathbf{P}}^3,[P_0]} = H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))/H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}) \cong H^0(\mathbf{P}^3, \mathcal{O}_{P_0}(1)).$$

This says that $D_{P_0} \cong \mathbf{P}(T_{\check{\mathbf{P}}^3,[P_0]})$ identifies with the 2-dimensional linear system $|\mathcal{O}_{P_0}(1)|$. To give a geometric interpretation of this, let us look at the limits of the curves cut out on $T + P_0$ by a plane Π of $\mathbf{P}^3$ tending to P_0. The limit curve $\Pi \cap T$ tends to R, while the limit curve of $\Pi \cap P_0$ is a line moving freely in $|\mathcal{O}_{P_0}(1)|$.

On the other hand, the plane D_R is

$$|4H - R| \cong |\mathcal{O}_P(H)| \subseteq |4H - \textstyle\sum_{i=1}^{12} E_i|.$$

There is therefore a natural identification between D_{P_0} and D_R, with the glueing as in the statement. $\square$

(7.3) Remark. As mentioned in Example (3.2), the surface P is mapped to a quartic with a triple point in $\mathbf{P}^3$ by the linear system $|4H - \sum_{i=1}^{12} E_i|$. This mapping is an isomorphism outside R, and contracts R to the triple point.

Indeed, one has an exact sequence

$$0 \to \mathcal{O}_P(H) \to \mathcal{O}_P(4H - \sum_{i=1}^{12} E_i) \to \mathcal{O}_R \to 0.$$

Since $h^1(P, \mathcal{O}_P(H)) = h^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) = 0$, the sequence is exact on global sections, which implies $h^0(P, \mathcal{O}_P(4H - \sum_{i=1}^{12} E_i)) = 4$.

The linear system $|4H - \sum_{i=1}^{12} E_i|$ has no base points and the morphism determined by it is birational onto its image (we leave this to the reader to check). The image of P via this map has degree 4, since $(4H - \sum_{i=1}^{12} E_i)^2 = 4$. The map contracts $R \equiv 3H - \sum_{i=1}^{12} E_i$ because $(4H - \sum_{i=1}^{12} E_i) \cdot (3H - \sum_{i=1}^{12} E_i) = 0$, and its image is a triple point because $R^2 = (3H - \sum_{i=1}^{12} E_i)^2 = -3$.

7.2.1 Curves in the hyperplane bundle

According to the structure of the limit linear system $\mathfrak{L}$, one has to compute the limit of trinodal curves appearing in both $\tilde{\mathcal{H}}_{\mathbf{P}^3}$ and in $\mathcal{H}_P$. First we perform the computation for $\tilde{\mathcal{H}}_{\mathbf{P}^3}$.

Here one has to consider the various cases for $V(0, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$ as in Definition (5.3), with $I \subseteq \{1, \ldots, 12\}$. Since there cannot be any node on P in this linear system, one has $|\boldsymbol{\delta}| = \delta$, the number of nodes on T. Then:

- $0 \leqslant \delta \leqslant 3$ and $\tau = (\tau_2, \tau_3)$;
- $|I| + 2\tau_2 + 3\tau_3 \leqslant 3$;
- $\delta + |I| + \tau_2 + 2\tau_3 = 3$.

Hence the cases to be discussed are the following.

Case 1: $\delta = 3$.

Tritangent planes to the general cubic surface T correspond to the triangles contained in T. These are 45: the reader may apply Salmon's formula (2.0.1) or, better, directly compute this number as an exercise.

Case 2: $\delta = 2$, $|I| = 1$.

Binodal hyperplane sections of T must contain a line. Since T contains 27 lines, there are 27 pencils of such curves. In each of these pencils, and for every $r \in \{1, \ldots, 12\}$, there is exactly one curve passing in addition through the point x_r. This gives a contribution of $27 \times 12 = 324$ to the computation.

Case 3: $\delta = 2$, $\tau_2 = 1$.

Each of the 27 pencils of bitangent planes to T cuts out a g^1_2 on R, off the fixed intersection of the line with R.

A g^1_2 on an elliptic curve possesses 4 ramification points. Each of this contributes to the computation, with multiplicity 2, hence the total contribution is $27 \times 4 \times 2 = 216$.

Case 4: $\delta = 1$, $|I| = 2$.

For any pair of distinct indices r and s in $\{1, \ldots, 12\}$, the pencil of planes passing through x_r and x_s contains $12 = \deg(\check{T})$ tangent planes to T. The contribution is then $\binom{12}{2} \times 12 = 792$.

Case 5: $\delta = 1$, $|I| = 1$, $\tau_2 = 1$.

For each $r = 1, \ldots, 12$, consider the projection π_r from the point x_r. It gives a double cover $\tilde{T} \to \mathbf{P}^2$ branched along a plane quartic curve B, where $\tilde{T}$ is the blow up of T at x_r (the curve B is the projection from x_r of the curve $T \cap \mathrm{D}^{x_r}T$, which has degree 6 and multiplicity 2 at x_r, see [XII, Section **??**]; by $\mathrm{D}^{x_r}T$ we denote the first polar of T with respect to the point x_r). Being contained in the plane P_0, which passes through x_r, R is mapped 2-1 to a line $\ell_R \subseteq \mathbf{P}^2$, which is nowhere tangent to B.

Tangent planes to T containing x_r in $\mathbf{P}^3$ map via π_r to tangent lines to B. Tangent planes to R containing x_r map to lines passing through one of the 4 points of $\ell_R \cap B$.

Let p be a point in $\ell_R \cap B$. The pencil of lines through p in $\mathbf{P}^2$ contains $12 = \deg(\check{B})$ tangent lines to B. Among them there is the tangent line to B at q, with multiplicity 2 (this can be seen as in Lemma (7.1)), which corresponds to a plane tangent to T at a point lying on R in $\mathbf{P}^3$. Such a plane gives a section of $T + P$ with a singularity worse than a tacnode, so it has to be discarded from computation.

Finally, the contribution in this case is $12 \times 4 \times 2 \times 10 = 960$.

Case 6: $\delta = 1$, $\tau_3 = 1$.

The plane cubic R possesses 9 flexes. For each of them, there is the pencil of planes containing the tangent line to R at that flex. Such a pencil is a line in $\check{\mathbf{P}}^3$, meeting $\check{T}$ at 12 points. One needs to remove from these 12 points the one corresponding to the tangent plane to T at the flex of R. It is of multiplicity 3 (the proof of this fact is similar to the one of Lemma (7.1) and we leave it to the reader).

Planes satisfying the given conditions have to be counted with multiplicity 3. So we get a contribution of $3 \times 9 \times 9 = 243$.

The table below shows the result of the total summation.

Cases	Contribution
Case 1	45
Case 2	324
Case 3	216
Case 4	792
Case 5	960
Case 6	243
Total	**2580**

7.2.2 Curves in the twisted hyperplane bundle: plane quartics

Since Salmon's formula (2.0.1) gives a total of 3200 tritangent planes to a general quartic, we are missing a total of $3200 - 2580 = 620$ limit trinodal curves. These have to be seen in $\mathcal{H}_P = |4H - \sum_{i=1}^{12} E_i|$. This system however contains the 2–dimensional system D_R of trinodal curves, which is the plane along which $\mathcal{H}_P$ and $\widetilde{\mathcal{H}}_{\mathbf{P}^3}$ glue. So these curves do not contribute to the computation. Hence, we have to compute the number of trinodal curves in $|4H - \sum_{i=1}^{12} E_i|$ off D_R. All such curves will be irreducible, since, by the genericity assumptions, the only reducible curves in $|4H - \sum_{i=1}^{12} E_i|$ are the ones in D_R (the reader is invited to prove this).

The linear system $|4H - \sum_{i=1}^{12} E_i|$ is mapped to the plane P_0 to $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_D|$, where $D = \{x_1, \dots, x_{12}\}$ and $x_1 + \cdots + x_{12} \in |\mathcal{O}_R(4)|$ is a general divisor. The varieties of δ-nodal curves not containing R in this linear system are logarithmic Severi varieties of the pair $(\mathbf{P}^2, R)$: on logarithmic Severi varieties, see [III], and in particular [III, Section 5.3] about the particular kind we are considering right now.

Since $\dim(|\mathcal{O}_{\mathbf{P}^2}(4)|) = 14$, D imposes only 11 conditions to $|\mathcal{O}_{\mathbf{P}^2}(4)|$. Since passing through a point imposes at most one condition, there is a set Y of 11 points in D that imposes the 11 conditions (in fact, by genericity of $x_1 + \cdots + x_{12} \in |\mathcal{O}_R(4)|$, any subset of 11 points does). Hence

$$|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_D| = |\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Y|.$$

Finally, Y can be seen as the limit of $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Z|$, where Z is a general set of 11 points in the plane.

We will compute the number of trinodal curves in $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Z|$. Their limits, when Z tends to Y, will be trinodal curves in $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Y| = |\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_D|$, and this gives the required information on the number we want to compute.

Finally what we have to compute is the number of trinodal quartics passing through 11 general points in the plane, which is the degree of the codimension 3 family of trinodal curves in $|\mathcal{O}_{\mathbf{P}^2}(4)| \cong \mathbf{P}^{14}$, i.e., this is the degree of the appropriate Severi variety. This has been done, in general, by Caporaso and Harris in [1, 2]. Their result in this case reads as follows:

(7.4) Proposition. *If Z is a general set of 11 points in $\mathbf{P}^2$, then the 3–dimensional linear system $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Z|$ contains 675 trinodal curves:*
(i) 620 of them are irreducible;
(ii) $55 = \binom{11}{2}$ are reducible in a line joining 2 of the points of Z and the cubic through the remaining 9 points of Z.

When Z tends to Y as above, the 55 reducible quartics as in (ii) of Proposition (7.4) tend to curves in D_R, which ought to be discarded. The limit of the 620 curve in (i) is the missing contribution we need.

It would be however unsatisfactory at this point to rely on (the quite difficult) general Caporaso–Harris' result to finish the computation. And in fact we can perform it *our way*, by using a new degeneration.

The degeneration we want to use is the following. Consider the blow up $\mathcal{S}$ of $\mathbf{D} \times \mathbf{P}^2$ along a general line E of $\mathbf{P}^2$ in the *central fibre* over $0 \in \mathbf{D}$. Then there is a morphism $f : \mathcal{S} \to \mathbf{D}$ which is a semistable degeneration of $S_t = \mathbf{P}^2$ for $t \neq 0$ to $S_0 = P \cup F$, with $P \cong \mathbf{P}^2$ the proper transform of the original central fibre and F the exceptional divisor of the blow up (see Figure 4).

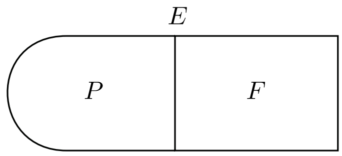


Figure 4: the degeneration of the plane

Note that $F \cong \mathbf{F}_1$, i.e., the plane blown up at a point p, and $P \cap F = E$, with $E^2 = -1$ on F (i.e., the exceptional divisor corresponding to p), according to the Triple Point Formula in Lemma (3.4).

There is an obvious polarizing bundle $\mathcal{L}$ on $\mathcal{S}$, which restrict to $\mathcal{O}_{\mathbf{P}^2}(1)$ on the general fibre and to L_0 on the central fibre, with the aspects

$$\mathcal{L}_P = \mathcal{O}_{\mathbf{P}^2}(1), \quad \mathcal{L}_F = \mathcal{O}_F(f),$$

where f is a fibre of the morphism $\pi : F \cong \mathbf{F}_1 \to \mathbf{P}^1$.

Now pick a set $A = \{\alpha_1, \ldots, \alpha_6\}$ of six general points on P and a set $B = \{\beta_1, \ldots, \beta_5\}$ of five general points on F. Then $Z_0 := A \cup B$ is the limit of a set Z_t of 11 general points on the general fibre S_t. Denote by $\mathcal{Z}$ the curve $\{Z_t\}_{t\in\mathbf{D}}$ in $\mathcal{S}$. Consider $\mathcal{L}^{\otimes 4} \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$, which restricts to $\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_{Z_t}$ on the general fibre S_t. In order to have information about trinodal curves in $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_{Z_t}|$, we have to look at their limits in the limit linear system $\mathfrak{L}$ of $\mathcal{L}^{\otimes 4} \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$.

(7.5) Lemma. *There is only one twist of $\mathcal{L}^{\otimes 4} \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$ which is centrally effective, namely $\mathcal{L}^{\otimes 4} \otimes \mathcal{O}_S(-F) \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$.*

Proof. Any twist is of the form $\mathcal{L}^{\otimes 4} \otimes \mathcal{O}_S(-aF) \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$ and on S_0 this resticts to:
(i) $\mathcal{O}_{\mathbf{P}^2}(4-a) \otimes \mathcal{I}_A$ on $P \cong \mathbf{P}^2$;
(ii) $\mathcal{O}_{\mathbf{F}_1}(4f + aE) \otimes \mathcal{I}_B$ on $F \cong \mathbf{F}_1$. As $\mathbf{F}_1$ is the blow up of $\mathbf{P}^2$ at p, then (with a slight abuse of notation) one has $|\mathcal{O}_{\mathbf{F}_1}(4f + aE) \otimes \mathcal{I}_B| \cong |\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_B \otimes \mathcal{I}_p^{\otimes 4-a}|$.

Since B is a general set of 5 points on F, for the effectivity on F one needs $a \geqslant 1$, because $\dim(|\mathcal{O}_{\mathbf{F}_1}(4f)|) = 4$. Similarly, for the effectivity on P we need $a \leqslant 1$, because $|\mathcal{O}_{\mathbf{P}^2}(h) \otimes \mathcal{I}_A|$ is empty for $h \leqslant 2$. This proves the assertion. $\square$

In conclusion, we have to look at limits of trinodal curves only in $|\mathcal{L}^{\otimes 4} \otimes \mathcal{O}_{\mathcal{S}}(-F) \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}} \otimes \mathcal{O}_{S_0}|$, which is:
(i) $\mathcal{L}_P := |\mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_A|$, the aspect on P, of dimension 3;
(ii) $\mathcal{L}_F := |\mathcal{O}_{\mathbf{F}_1}(4f + E) \otimes \mathcal{I}_B|$, the aspect on F, which is the same as $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_p^{\otimes 3} \otimes \mathcal{I}_B|$ in the identification of F with the blow-up of $\mathbf{P}^2$ at a point p, also of dimension 3;
(iii) the two curves on P and F in the two aspects have to *match* along E, i.e., they have to cut out on E the same divisor. This *matching condition* implies dimension 3 for $|\mathcal{L}^{\otimes 4} \otimes \mathcal{O}_{\mathcal{S}}(-F) \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}} \otimes \mathcal{O}_{S_0}|$: indeed, given a curve in $|\mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_A|$, which depends on 3 parameters, this fixes the degree 3 divisor it cuts on E and there is only one curve in $|\mathcal{O}_{\mathbf{F}_1}(4f - E) \otimes \mathcal{I}_B|$ matching it.

Next we introduce the self-explanatory notation $\boldsymbol{\delta} = (\delta_P, \delta_F)$ and $\tau = (\tau_2, \tau_3)$ (in this case there is no I to be considered) and

$$|\boldsymbol{\delta}| + \tau_2 + 2\tau_3 = 3.$$

It is useful to notice that a curve in $|\mathcal{O}_{\mathbf{F}_1}(4f+E)| \otimes \mathcal{I}_B$ with a node at a point $x \in F$, splits in the unique element f_x of $|f|$ through x, and in a curve of $|\mathcal{O}_{\mathbf{F}_1}(3f+E) \otimes \mathcal{I}_B|$ (which is the same as $|\mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_B \otimes \mathcal{I}_p^{\otimes 2}|$) passing through x.

Now the cases to be analyzed are the following. We will be sketchy, leaving the details to the reader. The corresponding analysis for the simpler enumeration of 2-nodal plane quartics is carried out in [VII, Section 4].

Case 1: $\delta_F = 3$.

A trinodal curve on F contains three curves in $|f|$ with residual in $|\mathcal{O}_{\mathbf{F}_1}(f+E)|$, which is the same as $|\mathcal{O}_{\mathbf{P}^2}(1)|$ (with the curves in $|f|$ mapping to lines through p). This implies that the three splitting curves in $|f|$ have to contain three points of B, and the residual in $|\mathcal{O}_{\mathbf{F}_1}(f+E)|$ contains the remaining 2. This fixes the curve on F and accordingly also the curve on P. This shows that the contribution in this case is $\binom{5}{2} = 10$.

Case 2: $\delta_F = 2, \delta_P = 1$.

Two curves in $|f|$ split from the curve on F and either only one or exactly two points of B lie on splitting elements of $|f|$. If two points β_1, β_2 of B lie on splitting elements of $|f|$, these curves cut E each in one point b_1, b_2. Then the matching curves on P lie in the pencil of cubics $|\mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_{A \cup \{b_1, b_2\}}|$ which, as well known, contains exactly 12 nodal curves (the reader may be asked to verify this as an exercise). Each such curve cuts a divisor on E (which contains $b_1 + b_2$), and the matching uniquely determines the remaining component of the curve on F, which lies in the pencil $|\mathcal{O}_{\mathbf{F}_1}(2f+E) \otimes \mathcal{I}_{B - \{\beta_1, \beta_2\}}| = |\mathcal{O}_{\mathbf{P}^2}(2) \otimes \mathcal{I}_{B - \{\beta_1, \beta_2\} \cup \{p\}}|$. This gives a contribution of $\binom{5}{2} \times 12 = 120$ to our computation.

The reader may check, with similar arguments, that if only one point of B lies on a splitting element of $|f|$ (and there are 5 possibilities for this), then one also has a well determined pencil of curves on P with 12 nodal curves, hence we have $5 \times 12 = 60$ more limit trinodal curves of this type, for a total of 180.

Case 3: $\delta_F = 2, |\tau| \neq 0$.

Since, as above, two curves in $|f|$ split from the curve on F, it is not possible to have a tangency to E, so this case gives no contribution to the computation.

Case 4: $\delta_F = 1, \delta_P = 2$.

Only one curve in $|f|$ splits from the curve on F, and there are two possible cases:
(a) this curve does not contain any point of B;
(b) this curve contains one point of B (this depends on 5 choices).

Also the curve on P is reducible in a conic and a line, and there are two possible cases:
(a') 5 of the points in A lie on the conic, and one lies on the line (this depends of 6 choices);
(b') 4 of the points in A lie on the conic, and two lie on the line (this depends of 15 choices).

One has that:
Case (a,a') contributes with 12 curves,
Case (a,b') contributes with 30 curves,
Case (b,a') contributes with 30 curves,
Case (b,b') contributes with 75 curves, for a total contribution of 147.
Let us check case (b,b'), and leave the others to the reader. Each of the 15 choices of 2 points in A determines a unique line, and then the residual conic through the 4 remaining points moves in a pencil. Each choice of one point in B determines a unique curve in $|f|$, and then the residual curve in $|3f+E|$ through the 4 remaining points moves in a 2-dimensional linear system. Finally,

the residual conic on P is determined by its matching with the fixed curve in $|f|$ on F, and then the residual curve in $|3f+E|$ on F is fixed with its matching with the conic on P at its remaining point of intersection with E and with the fixed line on P. Hence the contribution is of $5 \times 15 = 75$.

Case 5: $\delta_F = 1, \delta_P = 1, \tau_2 = 1$.

Again one has the two alternatives (a) and (b) as in Case 4. Case (a) contributes with 40 curves, case (b) with 160. Let us check case (b), and leave case (a) to the reader.

Once the splitting curve of $|f|$ has been fixed (which depends on 5 choices, i.e., the choice of the point in B), it intersects E at a point q. The matching condition on P gives the net of cubics through A and q. This net maps P to $\mathbf{P}^2$ as a double cover, branched along a quartic D which, by genericity, is smooth. The curve E is mapped to a conic C. We have to count the number of lines tangent to both, D and C, which is the intersection number 24 of $\check{C}$ (a conic) and $\check{D}$ (a curve of degree 12), minus the number of points where D and C are tangent to each other (which is 4, the number of branch points of the double cover $E \to C$), counted with multiplicity 2 (this is similar to Lemma (7.1)). Since each of these curves contributes with multiplicity 2 to the total, we have a total of $2 \times 5 \times 16 = 160$.

Case 6: $\delta_P = 1, \tau_3 = 1$.

Each of the curves in question counts with multiplicity 3. The linear system $\mathcal{L}_P$ maps P to a smooth cubic surface Φ in $\mathbf{P}^3$, and E is mapped to a rational normal cubic $\Gamma \subseteq \Phi$. Consider the duals:

- $\check{\Phi}$, which is a surface of degree 12;
- $\check{\Gamma}$, which is a scroll of degree 4, with a cuspidal curve Γ^* of degree 3, which is the rational normal cubic of $\check{\mathbf{P}}^3$ described by all osculating planes to Γ in $\mathbf{P}^3$ (see [XII]).

The surface $\check{\Gamma}$ is the projection of a smooth rational normal scroll Σ of degree 4 in $\mathbf{P}^5$, and we denote by H the hyperplane class in this embedding and $|\mathfrak{f}|$ the pencil of lines on Σ. One has $H^2 = 4, \mathfrak{f}^2 = 0, H \cdot \mathfrak{f} = 1$. [1]

The strict transform of Γ^* on Σ is clearly unisecant with the curves in $|\mathfrak{f}|$ and therefore $\Gamma^* \equiv H - \mathfrak{f}$.

The curve Γ' described by the tangent planes to Φ at the points of Γ sits in $\check{\Phi} \cap \check{\Gamma}$. It is a rational curve of degree 6 (to see this, intersect Γ with a general polar quadric of S'), meeting the curves of $|\mathfrak{f}|$ in 1 point. Hence on Σ one has $\Gamma' \equiv H + 2\mathfrak{f}$, and therefore $\Gamma' \cdot \Gamma^* = 5$.

We have to compute the intersection number of $\check{\Phi}$ and Γ^*, which is 36, and subtract the number of osculating planes to Γ tangent to Φ at a point of Γ, which is $\Gamma' \cdot \Gamma^* = 5$, counted with multiplicity 3 (to see this, imitate the proof of Lemma (7.1)). In conclusion we get a contribution of $3 \times 21 = 63$.

Case 7: $\delta_P = 2, \tau_2 = 1$.

Each of the curves in question counts with multiplicity 2. The curve on P splits, and we have the two alternatives (a') and (b') as in Case 4. However, case (a') gives no contribution to the computation, because in a pencil of lines there is no line tangent to E. Each of the situations in case (b') (which depend on 15 choices, i.e., the pairs of points of the splitting line ℓ on P) gives rise to 2 curves, corresponding to the conics of the residual pencil to ℓ tangent to E. In conclusion, one has a contribution of $2 \times 15 \times 2 = 60$.

Case 8: $\delta_P = 3$.

The curve on P splits in 3 lines, which have to contain the points in A. The reader will readily check that their number is 15.

The following table shows the result of the total summation of non–trivial contributions.

[1] One has that $\Sigma \cong \mathbf{F}_0$ embedded in $\mathbf{P}^5$ via the morphism determined by the linear system $|H|$ of curves of type $(2,1)$. Indeed it cannot be $\Sigma \cong \mathbf{F}_2$ because otherwise there would be a pencil of tangent planes to Γ along moving points, which is not possible.

Cases	Contribution
Case 1	10
Case 2	180
Case 4	147
Case 5	200
Case 6	63
Case 7	60
Case 8	15
Total	**675**

From this total we have to subtract the limit of reducible quartic curves passing through 11 general points of the plane. Their number is $55 = \binom{11}{2}$, since they are the ones appearing in (ii) of Proposition (7.4). We leave it to the reader to check that the limits of these 55 curves are the ones in Case 1 and Case 4 (b,a') and (a,b').

(7.6) Remark. In conclusion, also the degeneration to a cubic plus a plane is 1–good. However the reader will have noticed that, both in this case and in the previous one, we do not have an a priori proof of this (as in the intentions of Conjecture (5.10)), but the proof relies on Salmon's formula (2.0.1), rather than proving it.

8 – A non-good case

8.1 – Degeneration of cubics to three planes

Consider a degeneration as in Example (3.2), where $d = 3$, with S_0 consisting of the union of three distinct planes P_0, P_1, P_2. Let us denote by ℓ_i the intersection line of P_i and P_{i+1} (for $i = 0, 1, 2$; the indices will now be considered to vary in $\mathbf{Z}/3\mathbf{Z}$), and by p the intersection point of P_0, P_1, P_2. The central fibre of the corresponding semistable degeneration constructed in Example (3.2) is $\tilde{S}_0 = Q_0 \cup Q_1 \cup Q_2$ as in Figure 5, and we may assume that each of the surfaces Q_i is the blow up of P_i at three points along ℓ_i.

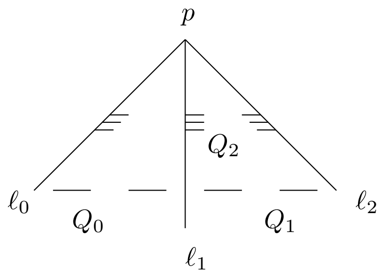


Figure 5: the degeneration of a cubic to the union of three planes

If we look at limits of nodal curves in the polarising hyperplane bundle, according to Proposition (5.5) we have to consider only planes passing by one of the aforementioned blown up points. This gives a contribution of 9 to the degree of the dual of a general cubic surface in $\mathbf{P}^3$, whereas we know that this degree is 12. So we are missing something of degree 3, which shows that this degeneration is not 1–good. In what follows we will construct a 1–good model, which will show that the degeneration under consideration is 1–well behaved.

(8.1) Remark. This degeneration is however 3–good. To see this, notice that the limit of the 27 lines on a general cubic can be seen here as the lines different from the ℓ_i's joining pairs of blown up points. Hence we see all triangles formed by these lines, which are 45, so we see also the limit of all 3–nodal planes sections of a general cubic surface.

For the same reason this degeneration is also 2–good.

The obvious guess is that the limit of the dual of the general cubic consists of the 9 aforementioned planes in $\check{\mathbf{P}}^3$ plus the plane $p^\perp$ counted with multiplicity 3, i.e., *there is something of degree 3 hidden at* p. This is indeed the case and to see it, one has to look at another model of the degeneration. This is explained in detail in [4] in the more complicated situation of a general quartic degenerating to a tetrahedron (i.e., the union of four linearly independent planes in $\mathbf{P}^3$). Let us now see how this 1–good model is gotten in the present case.

(8.2) Construction of a good model. Let $\bar{f} : \bar{S} \to \mathbf{D}$ be the family obtained from $\tilde{f} : \tilde{S} \to \mathbf{D}$ (notation as in Example (3.2)) by the base change $t \in \mathbf{D} \mapsto t^3 \in \mathbf{D}$. The central fibre $\bar{S}_0$ is isomorphic to $\tilde{S}_0$, so we will keep the above notation for it.

Analytically locally around p, the total space $\bar{S}$ is isomorphic to the hypersurface of $\mathbf{C}^4$ defined by the equation $xyz = t^3$ at the origin. Blow-up $\bar{S}$ at p. The blown–up total space locally sits in $\mathbf{C^4} \times \mathbf{P^3}$. Let $[\xi : \eta : \zeta : \vartheta]$ be the homogeneous coordinates in $\mathbf{P^3}$. Then the new total space is locally defined in $\mathbf{C^4} \times \mathbf{P^3}$ by the equations

$$\mathrm{rk}\begin{pmatrix} x & y & z & t \\ \xi & \eta & \zeta & \vartheta \end{pmatrix} \leqslant 1 \qquad \text{and} \qquad \xi\eta\zeta = \vartheta^3. \tag{8.2.1}$$

Denote the exceptional divisor by T; it is isomorphic to the cubic surface with equation $\xi\eta\zeta = \vartheta^3$ in the $\mathbf{P}^3$ with coordinates $[\xi : \eta : \zeta : \vartheta]$. This cubic contains three lines, along which it intersects the proper transforms $\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$ of Q_0, Q_1, Q_2 respectively; thus, each line on T identifies with the exceptional (-1)-curve on one of $\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$. The cubic T has three A_2 double points, located at the intersections of T with the proper transforms $\tilde{\ell}_0, \tilde{\ell}_1, \tilde{\ell}_2$ of ℓ_0, ℓ_1, ℓ_2 respectively. See Remark (8.3) below for more about this cubic. The new central fibre is shown in Figure 6.

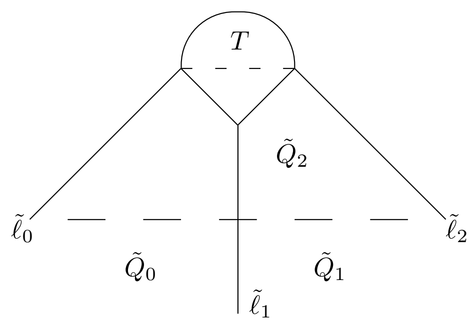


Figure 6: the central fibre after base change and blow–up of the vertex p

The model we have arrived at at this point is good enough to see all limits of 1-nodal curves, but it is not yet a good model because it is not semistable. Indeed, the total space is singular along the proper transforms $\tilde{\ell}_0, \tilde{\ell}_1, \tilde{\ell}_2$ of ℓ_0, ℓ_1, ℓ_2: analytically locally around the general point of one of these curves, it is isomorphic to a neighbourhood of the origin in the hypersurface defined by $xy = t^3$ in $\mathbf{C}^4$; thus, the total space is locally the product of an A_2 surface singularity with a smooth curve.

To resolve these singularities, we blow up along the three (disjoint) curves $\tilde{\ell}_0, \tilde{\ell}_1, \tilde{\ell}_2$. This has the effect of replacing each of these curves by a chain of two ruled surfaces; correspondingly, all three A_2 double points of T are resolved, each being replaced by a chain of two (-2)-curves. Thus, T is replaced by its minimal resolution, which we will denote by $\tilde{T}$. The three surfaces

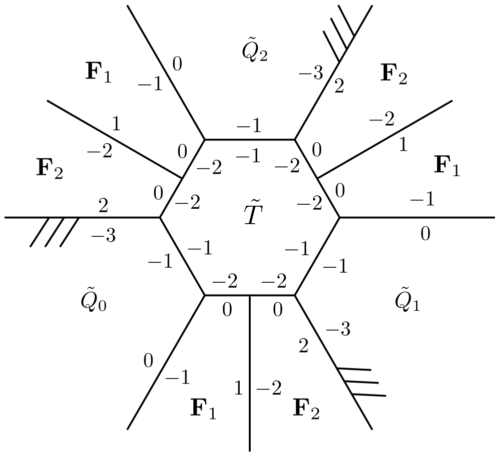


Figure 7: the central fibre of the good model (seen from above)

$\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$, on the other hand, remain unchanged, and we will denote their proper transforms by the same symbols.

The new central fibre is shown in Figure 7. We leave to the reader the verification of the self-intersections of the double curves of the central fibre indicated on the figure, as an exercise in the triple point formula. With our choice of distribution of the (-1)-curves in the small resolution at the beginning, the chains of rational ruled surfaces are made of an $\mathbf{F}_1$ and an $\mathbf{F}_2$ surface, as indicated on the figure.

(8.3) Remark. The cubic surface T is the image of the plane via the linear system $\mathcal{C}$ of cubic curves with two base points x_1, x_2, and infinitely near base points such that the cubics $C \in \mathcal{C}$ have two common flexes along lines L_1 and L_2. One has to blow up the plane at x_1 and x_2 three consecutive times to make the proper transform $\tilde{\mathcal{C}}$ of $\mathcal{C}$ base point free. This is shown in Figure 8, where we denote by $\tilde{C}$ the proper transform of the general curve $C \in \mathcal{C}$, by $\tilde{L}_i$ the proper transforms of the lines L_i, for $i = 1, 2$, and by E_{ij}, for $i = 1, 2$, $j = 1, 2, 3$, the exceptional curves of the blow–ups, and one has $E_{i3}^2 = -1$ for $i = 1, 2$ and $E_{ij}^2 = -2$, for $i, j = 1, 2$. [2]

The cubic T has three A_2–double points, at the vertices of the triangle $\vartheta = 0$, $\xi\eta\zeta = 0$. They are the images of $\tilde{L}_1 + \tilde{L}_2$ and of $E_{i1} + E_{i2}$, for $i = 1, 2$. The three lines pairwise joining the A_2–double points, lie on T and are easily seen to be the only lines on T. They correspond to the two exceptional divisors E_{13} and E_{23} and to the line L_3 joining x_1 and x_2.

The cubic T is *self–dual*, i.e., $\check{T}$ is projectively equivalent to T, hence it has degree 3. This (heuristically) explains the contribution by 3 to the degree of the dual hidden at p.

(8.4) Goodness of the new model. The degeneration we have just constructed is 1–good. Indeed now there is a new centrally effective twist, which corresponds to a degeneration of the cubics S_t, $t \in \mathbf{D} \setminus \{0\}$, to the cubic $T \subseteq \mathbf{P}^3$. It provides the missing contribution by 3, as we will now briefly explain.

Let $\mathcal{L}$ be the pull-back to the new degeneration of the polarising hyperplane bundle on the initial degeneration. On the central fibre, it restricts to the pull-back H of the line class on the surfaces $\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$, to the trivial class on $\tilde{T}$, and to the class of the ruling on the chains of

[2] In [4] we gave a different plane linear system of cubics representing the same surface T. It is the linear system of cubics that pass through three independent points y_1, y_2, y_3 in the plane and are tangent there to the lines $\langle y_i, y_{i+1} \rangle$, with $i =\in \mathbf{Z}/3\mathbf{Z}$. To pass from $\mathcal{C}$ to this linear system, just make a quadratic transformation based at x_1 and x_2 and at the point infinitely near to x_1 along L_1.

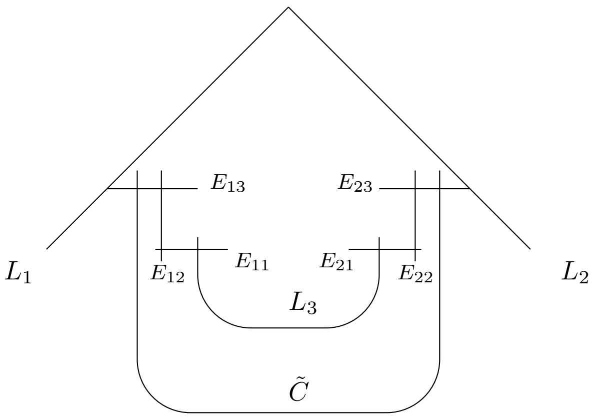


Figure 8: the linear system of curves $\tilde{C}$

ruled surfaces. The new relevant twist is $\mathcal{L}(-\tilde{T})$: it restricts to the class $H - E_i$ on $\tilde{Q}_i$ for all $i = 0, 1, 2$ (recall the notation from (8.2)), to the pull-back of the hyperplane class of $T \subseteq \mathbf{P}^3$ on $\tilde{T}$, and it is trivial on the chains of ruled surface; thus, it maps $\tilde{T}$ to T, contracts $\tilde{Q}_i$ to the line E_i on T for all $i = 0, 1, 2$, and contracts the three chains of ruled surfaces to the three A_2 double points of T.

Now, let us fix a general pencil of planes in the $\mathbf{P}^3$ spanned by S_0, and let us count the "tangent planes to S_0" in this pencil. Nine of them are the planes in the pencil passing through each of the special points on the lines ℓ_i, $0 \leqslant i \leqslant 2$. As for the remaining three, consider the plane in the pencil passing through the triple point; it cuts out three coplanar lines meeting at the point p, the triple point of S_0. Thus its proper transform determines three points on T, one on each line of T, and these three points are collinear. The planes in the $\mathbf{P}^3$ spanned by T which contain these three points form a pencil, and in this pencil there are three planes which are tangent to T in simple points. This provides us with the remaining three "tangent planes to S_0". For further details, see [4].

8.2 – Degeneration of quartics to four planes

We conclude this text by a brief discussion of degenerations, as in Example (3.2) and Section 8.1 above, of smooth quartics S_t to the union S_0 of four planes P_0, P_1, P_2, P_3 in linear general position, which we call a *tetrahedron*. This is one of the main topics in [4], and many examples in this volume are inspired by this situation. An interesting feature is that this is a degeneration of $K3$ surfaces.

There are four different kinds of alternative degenerations of the S_t's to be considered in order to reveal all limits of δ-nodal hyperplane sections, $1 \leqslant \delta \leqslant 3$. Each corresponds to a kind of twist in some suitable good model of the initial degeneration.

— The contributions of the *faces* of the tetrahedron (i.e., the planes P_i themselves) are visible in degenerations to monoid quartic surfaces, similarly to what we have seen for the degeneration to a cubic and a plane in Section 7.2.

— The contributions of the sections by a hyperplane containing a *vertex* (i.e., a triple point $P_i \cap P_j \cap P_k$) are visible in degenerations to the union of a cubic $T \subseteq \mathbf{P}^3$ as in the previous Section 8.1 and the plane spanned by the three lines of T; the plane corresponds to the face of the tetrahedron opposite to the vertex in consideration.

— The contributions of the sections by a hyperplane containing an *edge* (i.e., a double line $P_i \cap P_j$) are visible in degenerations to a rational quartic surface with a double line and two triple points on the latter; the double line is the image of the opposite edge, and the two triple points are the respective images of the two faces adjacent to the edge.

More can be found on the rational surfaces appearing in these degenerations in [X, Section **??**].

The construction of a good model follows the same lines as in Section 8.1; it is rather complicated in practice, but can be organized so as to be manageable, see [4]. Starting from the degeneration to $S_0 = P_0 + P_1 + P_2 + P_3$ as in Example (3.2), one performs the base change $t \mapsto t^6$ and then resolve the singularities. Morally, a degree 3 base change is needed to unravel the contributions involving the vertices, as in Section 8.1, and a degree 2 base change is needed for those involving the edges. The central fibre of this good model is represented in Figure 9; the reader may also see [4, p. 143] for another picture, with a different design. The dotted curve represents (in a tropical fashion, see [X, Section **??**]) the pull-back of a hyperplane section of the initial tetrahedron.

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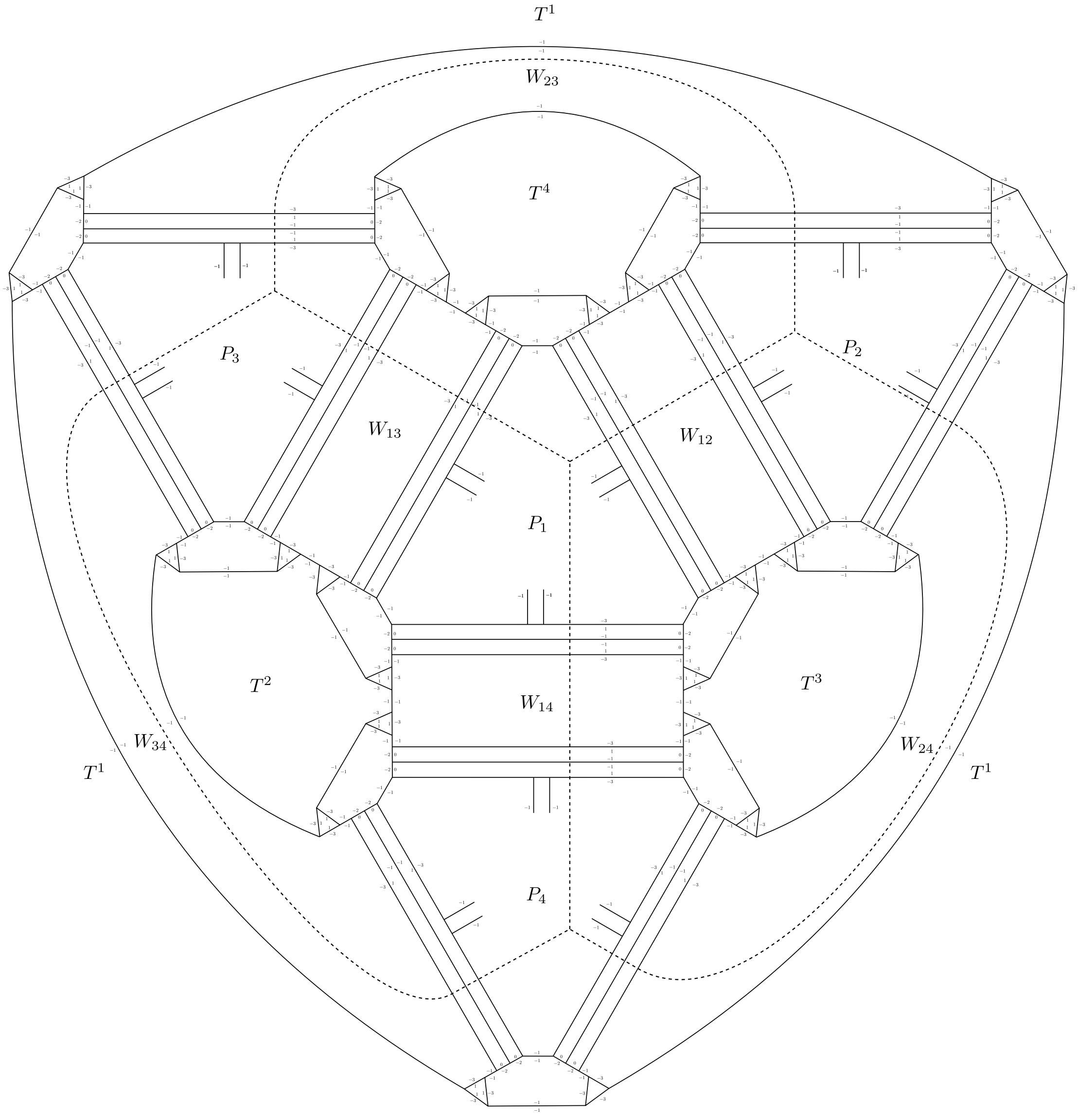

Figure 9: Central fibre of a good model of the degeneration of quartic surfaces to a tetrahedron

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Lecture II
Deformations of curves on surfaces

by Thomas Dedieu and Edoardo Sernesi

This text surveys some foundational material from the deformation theory of singular curves on surfaces, of ubiquitous use and necessity in this whole collection. The included topics are best described by the above table of contents.

1 – Algebroid plane curves

A *curve singularity* is a scheme of the form $\mathrm{Spec}(B)$, where $(B, \mathfrak{m})$ is the local ring at a singular point of a reduced algebraic curve. If the curve is contained in a nonsingular algebraic surface then we speak of a *planar curve singularity*. We will be mostly interested in this second case and in some of our considerations it will be convenient to replace such a B by its $\mathfrak{m}$-adic completion. In this section we will focus on this case and therefore we will consider local rings of the form

$$B = \mathbb{C}[[X, Y]]/(f)$$

for some power series f without constant term and without multiple factors. The scheme $C := \mathrm{Spec}(B)$ for such a ring is called an *algebroid plane curve*. If f is irreducible then C is called an *analytic branch*.

Assume that f is irreducible. Then the normalization $\widetilde{B}$ of B is a complete DVR and therefore it is isomorphic to a formal power series ring in one variable. Thus there exists $t \in K(B)$, the field of fractions of B, such that $\widetilde{B} = \mathbb{C}[[t]]$ and $K(B) = \mathbb{C}((t))$, the fraction field of $\mathbb{C}[[t]]$, which coincides with the field of formal Laurent power series in t.

We can write:

$$f(X, Y) = f_n(X, Y) + f_{n+1}(X, Y) + \cdots$$

for some $n \geq 1$, where f_i is a homogeneous polynomial of degree i. The integer n is the *multiplicity* of f (or of C) at $0 = [\mathfrak{m}]$. The polynomial $f_n(X, Y)$ is called the *initial form* of f.

We have two ideals associated to the algebroid plane curve C:

$$\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right) = (f_X, f_Y)$$

is called the *Milnor ideal* or *gradient ideal* of f. The corresponding algebra:

$$M_f := \mathbb{C}[[X,Y]]/\,(f_X, f_Y)$$

is the *Milnor algebra* of f.

The ideal

$$(f, f_X, f_Y)$$

is called the *Tjurina ideal* of f and

$$T_f := \mathbb{C}[[X,Y]]/\,(f, f_X, f_Y)$$

is the *Tjurina algebra* of f. It will be convenient to describe the Tjurina algebra as follows:

$$T_f = B/J$$

where $J = (f, f_X, f_Y)/(f) \subseteq B$ is the so-called *jacobian ideal.* Viewed as a B-module T_f is also called the *first cotangent module* of B, and denoted by T^1_B. Since C is either nonsingular or has an isolated singularity at $[\mathfrak{m}]$, the Tjurina ideal is either (1) or supported at $[\mathfrak{m}]$, and therefore T_f is a finite dimensional $\mathbb{C}$-vector space. Then

$$\tau(f) := \dim(T_f) < \infty$$

is called the *Tjurina number* of f (or of B). On the other hand it is also true that M_f is finite dimensional because of the following:

(1.1) Lemma (Risler [12])**.** *If $f \in (X,Y)^2 \subseteq \mathbb{C}[[X,Y]]$ has no multiple factors then f_X, f_Y form a regular sequence.*

Proof. It suffices to prove that f_X and f_Y have no common factors because they are contained in (X,Y). Assume that they have a non-constant irreducible factor p and consider the algebra $A = \mathbb{C}[[X,Y]]/(p)$. Let $\overline{f} \in A$ be the image of f. Then $\overline{f}$ is annihilated by every $\mathbb{C}$-derivation $D : K(A) \to K(A)$. Let $t \in K(A)$ be such that $K(A) = \mathbb{C}((t))$ and consider the derivation $D = \frac{d}{dt}$. Then $\ker(D) = \mathbb{C}$. Therefore $\overline{f} \in \mathbb{C}$ and therefore $\overline{f} = 0$, which means that $f = pq$ for some $q \in (X,Y)$. Therefore, since $f_X = p_X q + p q_X$ is divisible by p but p_X is not, it follows that q is divisible by p, and therefore f is divisible by p^2, a contradiction. □

For a proof of Lemma (1.1) in the case of convergent power series see [8], Lemma 2.3 p. 113. The dimension

$$\mu(f) = \dim(M_f)$$

is called the *Milnor number* of f (or of B). By construction we have:

$$\tau(f) \leq \mu(f)$$

and equality holds if and only if $f \in (f_X, f_Y)$. Notice also that

$$\mu(f) = i(f_X, f_Y)$$

where $i(-,-)$ denotes intersection multiplicity.

(1.2) Definition. *A polynomial* $P(X,Y) = \sum_\alpha a_\alpha X^{\alpha_1} Y^{\alpha_2}$ *is called* quasi-homogeneous *(qh) of weights* $w = (w_1, w_2) \in \mathbb{N}^2$ *and degree* d *if*

$$\alpha_1 w_1 + \alpha_2 w_2 = d$$

for all α *such that* $a_\alpha \neq 0$. *An algebroid plane curve* $\mathrm{Spec}\,(\mathbb{C}[[X,Y]]/(f)))$ *is quasi homogeneous if there is a q.h. polynomial* P *such that* $\mathbb{C}[[X,Y]]/(f) \cong \mathbb{C}[[X,Y]]/(P)$.

If $P(X,Y)$ is qh then we have the generalized *Euler relation* in $\mathbb{C}[X,Y]$:

$$dP = w_1 X P_X + w_2 Y P_Y$$

and therefore $P \in (P_X, P_Y)$. Therefore an algebroid qh plane curve satisfies $\mu(f) = \tau(f)$. The converse is true, by the following:

(1.3) Theorem (Saito [13])**.** *The algebroid plane curve* $\mathrm{Spec}\,(\mathbb{C}[[X,Y]]/(f))$ *is qh if and only if* $\mu(f) = \tau(f)$.

(1.4) Example ([8, p. 111])**.** The polynomial $P(X,Y) = X^5 + Y^5 + X^2Y^2$ is not qh. One can compute that a $\mathbb{C}$-basis of T_f is defined by the monomials $1, X, \dots, X^4, XY, Y, \dots, Y^4$ and a $\mathbb{C}$-basis of M_f has the additional monomial Y^5. Thus $\tau(P) = 10 < \mu(P) = 11$.

(1.5) Example. The polynomial $f = X^a \pm Y^b$ is qh of weights (b,a) and degree ab. Therefore:

$$\tau(f) = \mu(f) = (a-1)(b-1)$$

(1.6) Example. The *simple singularities* (also called *ADE singularities*) are qh. They have the following equations:

- A_k: $x^{k+1} + y^2 = 0, \quad k \geq 1.$
- D_k: $x(y^2 + x^{k-2}) = 0, \quad k \geq 4.$
- E_6: $x^3 + y^4 = 0.$
- E_7: $x(x^2 + y^3) = 0.$
- E_8: $x^3 + y^5 = 0.$

We have:

$$\mu(A_k) = \mu(D_k) = \mu(E_k) = k.$$

2 – The conductor

Let B be a ring and let $\overline{B} \subseteq \mathrm{Frac}(B)$ be a subring of the total ring of fractions containing B. The homomorphism

$$\mathrm{Hom}_B(\overline{B}, B) \to B, \quad \varphi \mapsto \varphi(1)$$

induces an isomorphism

$$\mathrm{Hom}_B(\overline{B}, B) \cong \mathrm{Ann}_B(\overline{B}/B).$$

Then $\mathrm{Ann}_B(\overline{B}/B)$ is an ideal both in B and in $\overline{B}$. It is called the *conductor* of $\overline{B}$ in B. We define the *δ-invariant* of $B \subseteq \overline{B}$ as

$$\delta(\overline{B}/B) := \dim_{\mathbb{C}}(\overline{B}/B).$$

If $\overline{B}$ is the integral closure of B in $\mathrm{Frac}(B)$ then $\mathrm{Ann}_B(\overline{B}/B)$ is simply called the *conductor of* B and denoted by $\mathcal{A}(B)$; $\delta(\overline{B}/B)$ is denoted by $\delta(B)$ and called *δ-invariant of* B.

In case $C = \mathrm{Spec}(B)$ is either a curve singularity or an algebroid plane curve, $\delta(B)$ is denoted by $\delta(C)$ or $\delta(f)$ in case $B = \mathbb{C}[[X,Y]]/(f)$. It is called the *$\delta$-invariant* of B (of C).

Notice that if $C = \mathrm{Spec}(B)$ is any curve singularity then $\delta(B) = \delta(\widehat{B})$, by the flatness of $\widehat{B}$ over B.

(2.1) Definition. *Let $\varphi : \widetilde{C} \longrightarrow C$ be a birational morphism of reduced curves. The* conductor of φ *is the sheaf of $\mathcal{O}_C$-ideals:*

$$\mathcal{A}(\varphi) := \mathcal{A}nn(\varphi_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C)$$

or, equivalently:

$$\mathcal{A}(\varphi) = \mathcal{H}om(\varphi_* \mathcal{O}_{\widetilde{C}}, \mathcal{O}_C).$$

For each $x \in C$, the finite nonnegative integer

$$\delta(\varphi, x) := \delta\big((\varphi_* \mathcal{O}_{\widetilde{C}})_x / \mathcal{O}_{C,x}\big) = \dim_{\mathbb{C}} \Big((\varphi_* \mathcal{O}_{\widetilde{C}})_x / \mathcal{O}_{C,x}\Big)$$

*is called the (local) δ-*invariant of φ at x, *and*

$$\delta(\varphi) := h^0(\varphi_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C) = \sum_{x \in C} \delta(\varphi, x)$$

*is the (global) δ-*invariant of φ. *In case φ is the normalization of C (i.e., $\widetilde{C}$ is nonsingular) we write $\mathcal{A}(\varphi) = \mathcal{A}(C)$ and $\delta(\varphi) = \delta(C)$ and call them the* conductor of C *and the* δ-invariant of C *respectively.*

The exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_* \mathcal{O}_{\widetilde{C}} \longrightarrow \nu_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C \longrightarrow 0$$

gives

$$\chi(\mathcal{O}_{\widetilde{C}}) = \chi(\nu_* \mathcal{O}_{\widetilde{C}}) = \chi(\mathcal{O}_C) + \delta(\varphi),$$

which can be rephrased as:

$$\delta(\varphi) = p_a(C) - p_a(\widetilde{C}) \tag{2.1.1}$$

where as usual $p_a(-)$ denotes *arithmetic genus* of $-$. In case φ is the normalization of C, the above identity takes the form:

$$\delta(C) = p_a(C) - p_g(C) \tag{2.1.2}$$

which shows that the δ-invariant of C measures the difference between the arithmetic genus and the geometric genus of the curve.

For the next lemma, observe that $\mathcal{A}(\varphi)$ is naturally a sheaf of $\varphi_* \mathcal{O}_{\widetilde{C}}$-modules. Therefore, since φ is affine, we have $\widetilde{C} = \mathrm{Spec}(\varphi_* \mathcal{O}_{\widetilde{C}})$ and $\mathcal{A}(\varphi)$ corresponds to a sheaf of $\mathcal{O}_{\widetilde{C}}$-modules which we denote by $\widetilde{\mathcal{A}}(\varphi)$.

(2.2) Lemma. *Let $\varphi : \widetilde{C} \to C$ be a birational morphism of reduced projective curves. Then there is a canonical isomorphism:*

$$\varphi_* \omega_{\widetilde{C}} \cong \mathcal{H}om_{\mathcal{O}_C}(\varphi_* \mathcal{O}_{\widetilde{C}}, \omega_C) \tag{2.2.1}$$

If moreover C is Gorenstein, then

$$\varphi_* \omega_{\widetilde{C}} \cong \mathcal{A}(\varphi) \otimes_{\mathcal{O}_C} \omega_C \tag{2.2.2}$$

and

$$\omega_{\widetilde{C}} = \widetilde{\mathcal{A}}(\varphi) \otimes_{\mathcal{O}_{\widetilde{C}}} \varphi^* \omega_C. \tag{2.2.3}$$

Proof. The proof of (2.2.1) follows from standard facts and is valid for finite morphisms of projective schemes. Recall the two following facts from [9, III, Ex. 6.10, p. 239]. For each quasi-coherent sheaf $\mathcal{G}$ on C, the module $\mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \mathcal{G})$ has a natural structure of quasi-coherent sheaf on $\widetilde{C}$, and as such it is denoted by $\varphi^!\mathcal{G}$. For each quasi-coherent sheaf $\mathcal{F}$ on $\widetilde{C}$, there is a natural isomorphism:

$$\varphi_*\mathcal{H}om_{\mathcal{O}_{\widetilde{C}}}(\mathcal{F}, \varphi^!\mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{F}, \mathcal{G}).$$

Taking $\mathcal{G} = \omega_C$ and $\mathcal{F} = \mathcal{O}_{\widetilde{C}}$ gives

$$\varphi_*(\varphi^!\omega_C) \cong Hom_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \omega_C).$$

Therefore it suffices to prove that $\varphi^!\omega_C = \omega_{\widetilde{C}}$ (which is [9, III, Ex. 7.2(a)]. For all coherent $\mathcal{F}$ on $\widetilde{C}$ we have:

$$H^1(\widetilde{C}, \mathcal{F})^\vee = H^1(C, \varphi_*\mathcal{F})^\vee = \mathrm{Hom}_{\mathcal{O}_C}(\varphi_*\mathcal{F}, \omega_C) \cong \mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\mathcal{F}, \varphi^!\omega_C),$$

and this isomorphism is functorial in $\mathcal{F}$. This implies that $\varphi^!\omega_C$ is a dualizing sheaf for $\widetilde{C}$ in the sense of [9, Definition p. 241] (see [18, Exercise 29.1.A]) , and thus (2.2.1) is proved.

If ω_C is invertible, then we have:

$$\varphi_*\omega_{\widetilde{C}} = \mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \omega_C) \cong \mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \mathcal{O}_C) \otimes \omega_C = \mathcal{A}(\varphi) \otimes \omega_C,$$

which is (2.2.2). Then, by the Projection Formula we have

$$\varphi_*\big(\widetilde{\mathcal{A}}(\varphi) \otimes \varphi^*\omega_C\big) = \varphi_*\widetilde{\mathcal{A}}(\varphi) \otimes \omega_C = \mathcal{A}(\varphi) \otimes \omega_C,$$

where the equality $\varphi_*\widetilde{\mathcal{A}}(\varphi) = \mathcal{A}(\varphi)$ follows from [9, II, Ex. 5.17e]. Thus

$$\varphi_*\omega_{\widetilde{C}} = \varphi_*\big(\widetilde{\mathcal{A}}(\varphi) \otimes \varphi^*\omega_C\big),$$

hence $\omega_{\widetilde{C}} = \widetilde{\mathcal{A}}(\varphi) \otimes \varphi^*\omega_C$, again by [9, II, Ex. 5.17e]. □

We refer the reader to [18, §29.3–4] for versions of the formula $\varphi^!\omega_C = \omega_{\widetilde{C}}$ valid in a more general framework.

(2.3) Corollary. *(a) Let $C = \mathrm{Spec}(B)$ be a Gorenstein reduced curve singularity. Then:*

$$\delta(B) = \dim_{\mathbb{C}}(B/\mathcal{A}(B)).$$

(b) Let C be a Gorenstein reduced projective curve and $\varphi : \widetilde{C} \to C$ a birational morphism. Then

$$\delta(\varphi) = h^0(\mathcal{O}_C/\mathcal{A}(\varphi)).$$

Proof. It suffices to prove (b). Let $\mathcal{A} = \mathcal{A}(\varphi)$. By definition we have an exact sequence:

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \mathcal{O}_C) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C/\mathcal{A} \longrightarrow 0\ .$$

Tensoring by ω_C, using the fact that C is Gorenstein and recalling (2.2.1), we obtain:

$$0 \longrightarrow \varphi_*\omega_{\widetilde{C}} \longrightarrow \omega_C \longrightarrow [\mathcal{O}_C/\mathcal{A}] \otimes \omega_C \longrightarrow 0\ .$$

From this exact sequence we deduce:

$$\begin{aligned} h^0(\mathcal{O}_C/\mathcal{A}) &= \chi([\mathcal{O}_C/\mathcal{A}]\otimes\omega_C) \\ &= \chi(\omega_C) - \chi(\varphi_*\omega_{\widetilde{C}}) \\ &= \chi(\omega_C) - \chi(\omega_{\widetilde{C}}) \\ &= p_a(C) - p_a(\widetilde{C}) \\ &= \delta(\varphi) \end{aligned}$$

and comparing with (2.1.1) we conclude. □

(2.4) Corollary. *Let $\varphi : \widetilde{C} \to C$ be a birational morphism of projective reduced curves and assume that $C \subseteq Y$, a projective nonsingular surface. Then*

$$\mathcal{A}(\varphi)\otimes\mathcal{O}_C(C) = \varphi_*\omega_{\widetilde{C}}\otimes\omega_Y^{-1}$$

and

$$\widetilde{\mathcal{A}}(\varphi)\otimes\varphi^*\mathcal{O}_C(C) = \omega_{\widetilde{C}}\otimes\varphi^*\omega_Y^{-1}.$$

Proof. The assumptions imply that C is Gorenstein. Then the corollary is an easy consequence of Lemma (2.2). □

(2.5) Lemma. *Let C be a reduced curve contained in a nonsingular algebraic surface Y. Let $p \in C$ be a point of multiplicity m and $b : Z \to Y$ the blow-up of Y at p. Then, letting $\overline{C} \subseteq Z$ be the proper transform of C and $E \subseteq Z$ the exceptional curve we have:*

$$b_*\omega_{\overline{C}} = \omega_C \otimes b_*\mathcal{O}_Z\big(-(m-1)E\big).$$

Proof. Left to the reader. □

The Milnor number and the δ-invariant are related as follows:

(2.6) Proposition (Milnor formula)**.** *Let $f = f_1\cdots f_r \in \mathbb{C}[[X,Y]]$, where $f_1,\dots,f_r$ are pairwise distinct analytic branches. Then:*

$$\delta(f) = \frac{1}{2}\big[\mu(f)+r-1\big].$$

Proof. See [8, Prop. 3.35, p. 208]. □

(2.7) Remark. Let $\varphi : \widetilde{C} \longrightarrow C$ be a birational morphism of reduced curves, and denote by $\Delta \subseteq C$ and $\widetilde{\Delta} \subseteq \widetilde{C}$ the two subschemes defined by $\mathcal{A}(\varphi)$ and $\widetilde{\mathcal{A}}(\varphi)$ respectively. If both C and $\widetilde{C}$ are Gorenstein, then it follows from (2.2.3) that $\widetilde{\Delta}$ is Cartier. However it is not true in general that Δ is a Cartier divisor (for instance, at a node $xy = 0$ the conductor is the ideal generated by x and y).

(2.8) Remark. In general the two sheaves of $\mathcal{O}_{\widetilde{C}}$-modules $\widetilde{\mathcal{A}}(\varphi)$ and $\varphi^*\mathcal{A}(\varphi)$ are different, even though, in the words of [11, (1.6), p. 350], "many writers who should know better write $\varphi^*\mathcal{A}(\varphi)$ for the ideal $\mathcal{A}(\varphi)\cdot\mathcal{O}_{\widetilde{C}}$" (the latter being $\widetilde{\mathcal{A}}(\varphi)$ in our situation).

For the remainder of this remark, we write $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ for $\mathcal{A}(\varphi)$ and $\widetilde{\mathcal{A}}(\varphi)$ respectively. By definition, $\varphi^*\mathcal{A} = \varphi^{-1}\mathcal{A}\otimes_{\varphi^{-1}\mathcal{O}_C}\mathcal{O}_{\widetilde{C}}$, and then the multiplication map induces an exact sequence of $\mathcal{O}_{\widetilde{C}}$-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \varphi^*\mathcal{A} \longrightarrow \widetilde{\mathcal{A}} \longrightarrow 0.$$

We shall see in the two following examples that in general $\mathcal{K}$ is non-trivial and $\varphi^*\mathcal{A}$ has a non-trivial torsion part, which is a manifestation of the fact that φ is not flat. The conclusion is that the object we want to consider is indeed $\widetilde{\mathcal{A}}$ and not $\varphi^*\mathcal{A}$.

Our first example is the normalization of the node $xy = 0$. The adjoint ideal is (x, y). Locally on the branch $y = 0$ in the normalization, the multiplication map $\varphi^{-1}\mathcal{A} \otimes_{\varphi^{-1}\mathcal{O}_C} \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{C}}$ corresponds to the multiplication map

$$(x, y) \otimes_{\mathbb{C}[x,y]/(xy)} \mathbb{C}[x] \longrightarrow \mathbb{C}[x],$$

which has non-trivial kernel, generated by $y \otimes 1$ and thus killed by multiplication by x. The upshot is that the torsion part of $\varphi^*\mathcal{A}$ is $\mathcal{O}_{\widetilde{\Delta}}$, where $\widetilde{\Delta}$ consists of the two points in the preimage of the node.

Our second example is the normalization of the cusp $y^2 - x^3 = 0$, which we see as $t \mapsto (t^2, t^3)$. The adjoint ideal is again (x, y), equivalently (t^2, t^3). The multiplication map $\varphi^{-1}\mathcal{A} \otimes_{\varphi^{-1}\mathcal{O}_C} \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{C}}$ corresponds to the multiplication map

$$(t^2, t^3) \otimes_{\mathbb{C}[t^2,t^3]} \mathbb{C}[t] \longrightarrow \mathbb{C}[t],$$

which has non-trivial kernel generated by $t^2 \otimes t - t^3 \otimes 1$ and thus killed by multiplication by t^2. Again the torsion part of $\varphi^*\mathcal{A}$ is $\mathcal{O}_{\widetilde{\Delta}}$, where this time $\widetilde{\Delta}$ is defined by the ideal (t^2) of $\mathbb{C}[t]$.

3 – Equivalence of singularities

We say that two algebroid plane curves:

$$C = \mathrm{Spec}(\mathbb{C}[[X, Y]]/(f), \quad D = \mathrm{Spec}(\mathbb{C}[[X, Y]]/(g)$$

are *isomorphic* or *analytically equivalent* if there is an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[[X, Y]]/(f) \cong \mathbb{C}[[X, Y]]/(g)$$

Because of the following result it is not restrictive to assume that f is a polynomial when studying isomorphism classes of algebroid plane curves.

(3.1) Theorem (Samuel [14])**.** *Given a reduced non-constant $f \in \mathbb{C}[[X, Y]]$ there is a polynomial g such that $\mathbb{C}[[X, Y]]/(f) \cong \mathbb{C}[[X, Y]]/(g)$.*

A weaker equivalence relation has been introduced by Zariski as follows. Consider and analytic branch $C = \mathrm{Spec}(\mathbb{C}[[X, Y]]/(f)$ and let $n = e(C)$ be its multiplicity (at 0). The blowing up of $\mathrm{Spec}(\mathbb{C}[[X, Y]])$ at 0 is defined by the substitution

$$X = X', \quad Y = X'Y'$$

Then

$$f(X, Y) = f(X', X'Y') = X'^n f'(X', Y')$$

and f' is the *proper transform* of f. One easily shows (see [20], p. 3) that f' is irreducible, i.e., defines an analytic branch C'. Let $e(C')$ its multiplicity. Then $e(C') \le e(C)$. The theorem of resolution of singularities guarantees that after a finite number of iterations of this operation we obtain a nonsingular branch $\overline{C}$, i.e., such that $e(\overline{C}) = 1$. Let

$$C, C', \dots, C^{(i)}, \dots$$

be the successive proper transforms of C and $e_i = e(C^{(i)})$. Let $N \geq 1$ be such that $e_N = 1$ but $e_{N-1} \neq 1$ and let

$$e^*(C) = \{e_0, e_1, \dots, e_{N-1}\}$$

The sequence $e^*(C)$ is uniquely determined by C.

(3.2) Definition. *Two analytic branches C, D are said to be* equivalent *or* equisingular*, if*

$$e^*(C) = e^*(D)$$

It is possible to show that two simple singularities are equisingular if and only if they are analytically equivalent. The simplest examples for which the two notions differ are ordinary n-fold points, with $n \geq 4$. For example, any two ordinary 4-fold points are equisingular, but they are isomorphic if and only if their respective 4-tuples of principal tangent lines have equivalent cross ratios. For more details about equisingularity we refer to [20].

4 – Generalities on deformations

Deformation theory studies local deformations of a given algebro-geometric object X (a projective scheme, a singularity, etc.). Under good hypotheses it produces a family containing all sufficiently small deformations of X, up to isomorphism, and having certain functorial properties. Here we are interested in deformations of planar curve singularities, a case where the theory simplifies consistently. Before adventuring into the details we need a general preamble about what we mean by (local) deformations.

Consider the category $\mathfrak{Loc}$ of local $\mathbb{C}$-algebras with residue field $\mathbb{C}$. A *deformation* of X parametrized by $\mathrm{Spec}(R)$, where $R \in \mathrm{Ob}(\mathfrak{Loc})$ is a pullback diagram:

$$\zeta : \quad \begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

where π is flat. Given another deformation of X parametrized by $\mathrm{Spec}(R)$:

$$\eta : \quad \begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{Y} \arrow[d, "\rho"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

we say that ζ and η are *isomorphic* if there is an isomorphism of R-schemes

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & \mathcal{Y} \arrow[dl, "\rho"] \\ & \mathrm{Spec}(R) & \end{tikzcd}$$

compatible with the identifications of X with the fibres over $\mathrm{Spec}(\mathbb{C})$.

Deformation theory associates to X a covariant functor $M_X : \mathfrak{Loc} \to \mathfrak{Sets}$ taking values in the category of sets. It is defined by:

$$M_X(R) = \{\text{isom. classes of def.s of } X \text{ parametrized by } \mathrm{Spec}(R)\}$$

for all $R \in \mathrm{Ob}(\mathfrak{Loc})$, and $M_X(f : R \to S)$ is defined by pullback. This functor can be extended to the category of pointed $\mathbb{C}$-schemes in an obvious way. Since deformation theory is unable to give a satisfactory information on either of these functors what can be done is to restrict M_X to more amenable subcategories of $\mathfrak{Loc}$. Two full subcategories are relevant in deformation theory. The first one is $\widehat{\mathfrak{Loc}}$, the category of complete local $\mathbb{C}$-algebras with residue field $\mathbb{C}$. The second one is $\mathfrak{Art}$, whose objects are the local artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$.

Covariant functors $F : \mathfrak{Art} \to \mathfrak{Sets}$ are called *functors of Artin rings.* The restriction of M_X to $\mathfrak{Art}$ is such a functor. If $A \in \mathrm{Ob}(\mathfrak{Art})$ then the elements of $M_X(A)$ are called *infinitesimal deformations* of X parametrized by A.

A *formal deformation* of X is a pair $(R, \widehat{\xi})$ where $(R, \mathfrak{m})$ is a complete local $\mathbb{C}$-algebra, i.e., $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$, and $\widehat{\xi} = \{\xi_n \in M_X(R/\mathfrak{m}^{n+1})\}_{n\geq 1}$ is a sequence of infinitesimal deformations which are compatible, i.e., such that $\xi_n \mapsto \xi_{n-1}$ under the map

$$M_X(R/\mathfrak{m}^{n+1}) \to M_X(R/\mathfrak{m}^n)$$

for each $n \geq 1$. Associating to every $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$ the set of formal deformations $(R, \widehat{\xi})$ we define a new functor

$$\widehat{M}_X : \widehat{\mathfrak{Loc}} \to \mathfrak{Sets}.$$

A formal deformation $(R, \widehat{\xi})$ is called *effective* if there is a deformation $\xi \in M_X(R)$ such that $\xi \mapsto \xi_n$ under the map:

$$M_X(\pi_n) : M_X(R) \to M_X(R/\mathfrak{m}^{n+1})$$

induced by the projection $\pi_n : R \to R/\mathfrak{m}^{n+1}$, for each $n \geq 1$.

Important point: Not every formal deformation is effective, i.e., the functor $\widehat{M}_X$ is not isomorphic to the restriction of M_X to $\widehat{\mathfrak{Loc}}$. Namely in general $\widehat{M}_X(R) \neq M_X(R)$.

A formal deformation $(R, \widehat{\xi})$ is called *algebraizable* if there is a deformation ζ of X parametrized by a pointed algebraic scheme (S, s_0) and an isomorphism $u : R \cong \widehat{\mathcal{O}_{S,s_0}}$ and such that the sequence of deformations

$$\widehat{\zeta} = \{\zeta_n \in M_X(\mathcal{O}_{S,s_0}/\mathfrak{m}^{n+1})\}$$

obtained by pulling back ζ is mapped to $\widehat{\xi}$ under u. Every algebraizable formal deformation is effective because ζ can be pulled back to $\widehat{\mathcal{O}_{S,s_0}}$, but the converse is not true.

To every $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$ one can associate the functor of Artin rings:

$$h_R : \mathfrak{Art} \to \mathfrak{Sets}, \quad A \mapsto \mathrm{Hom}(R, A).$$

Functors of this form are called *prorepresentable.* Consider a formal deformation $\widehat{\xi} \in \widehat{M}_X(R)$, for some $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$. Let $A \in \mathrm{Ob}(\mathfrak{Art})$ and $f \in h_R(A) = \mathrm{Hom}(R, A)$. Since A is artinian there is an $n \gg 0$ such that f factors as

$$\begin{tikzcd} R \arrow[r] \arrow[dr, "f"'] & R/\mathfrak{m}^{n+1} \arrow[d, "f_n"] \\ & A \end{tikzcd}$$

One can then associate to f the element $M_X(f_n)(\xi_n) \in M_X(A)$. Since this is independent of n we have defined a morphism of functors of Artin rings that we denote with the same symbol:

$$\widehat{\xi} : h_R \to M_X.$$

(4.1) Definition. *If a formal deformation $\widehat{\xi} \in \widehat{M}_X(R)$ is such that $\widehat{\xi}$ is an isomorphism of functors then we say that the pair $(R, \widehat{\xi})$* prorepresents *M_X. In this case M_X is prorepresentable.*

In practice if a pair $(R, \widehat{\xi})$ prorepresents M_X it means that for every $A \in \mathrm{Ob}(\mathfrak{Art})$ every $\xi \in M_X(A)$ is induced as above by a unique $f : R \to A$.

The prorepresentability of M_X is a strong property which is often not satisfied even by very simple objects X. For example, if X is the singularity $XY = 0$ then M_X is not prorepresentable (see [15, Example 2.6.8, p. 95]). The condition can be weakened in two ways as follows.

(4.2) Definition. *A formal deformation $\widehat{\xi} \in \widehat{M}_X(R)$ is called* versal *if for every $A \in \mathrm{Ob}(\mathfrak{Art})$ the induced map*

$$\widehat{\xi} : h_R(A) \to M_X(A)$$

is surjective.

A formal deformation $\widehat{\xi} \in \widehat{M}_X(R)$ is called miniversal *(or* semiuniversal*) if it is versal and in addition*

$$\widehat{\xi}(\mathbb{C}[\varepsilon]) : h_R(\mathbb{C}[\varepsilon]) \to M_X(\mathbb{C}[\varepsilon])$$

is bijective.

The map $\widehat{\xi}(\mathbb{C}[\varepsilon])$ is called the *differential* of $\widehat{\xi}$. Versality and miniversality are satisfied in most natural geometrical situations. Finally we give the following:

(4.3) Definition. *An algebraic deformation ζ of X parametrized by a pointed algebraic scheme (S, s_0) is called* formally versal *(resp.* formally semiuniversal*) if the associated algebraizable formal deformation $(\widehat{\mathcal{O}}_{S,s_0}, \widehat{\zeta})$ is versal (resp. semiuniversal).*

For more details on the contents of this section we refer to [15] and [10].

5 – The semiuniversal deformation of an isolated singularity

The case we will consider is $X = \mathrm{Spec}(B)$ where $(B, \mathfrak{m})$ is a noetherian local $\mathbb{C}$-algebra with residue field $B/\mathfrak{m} = \mathbb{C}$. Assume that B is either essentially of finite type (eft), i.e., a localization of a $\mathbb{C}$-algebra of finite type, or complete, and that $\mathrm{Spec}(B)$ has an isolated singularity at $[\mathfrak{m}]$. Under these conditions deformation theory ensures the existence of a formal semiuniversal family of deformations of B, or of $\mathrm{Spec}(B)$ (see [15]). In the cases of interest for us such a deformation can be easily constructed explicitly. Note firstly the following:

(5.1) Proposition. *If B and B' are local $\mathbb{C}$-algebras as above such that $\widehat{B} = \widehat{B}'$, then their deformation functors are isomorphic.*

Proof. [10, ex. 18.6, p. 127]. ☐

This means that it is not restrictive to study deformations of a complete local $\mathbb{C}$-algebra. Since we are interested in planar curve singularities, i.e., in the local ring B of a reduced curve C contained in a nonsingular surface Y at a point $p \in C$, we can replace B by its completion and therefore we may assume that $= \mathbb{C}[[X, Y]]/(f)$, i.e., that C is an algebroid plane curve.

Consider indeterminates $t_1, \dots, t_N$ where $N = \tau(f)$ and let

$$p_1(X, Y), \dots, p_N(X, Y) \in \mathbb{C}[[X, Y]]$$

be polynomials inducing a basis of the Tjurina algebra T_f. Consider the following family:

(5.1.1)
$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathrm{Spec}(\mathcal{B}) =: \mathcal{C} \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{C}[[t_1, \dots, t_N]]) =: M(B) \end{tikzcd}$$

where

$$\mathcal{B} := \mathbb{C}[[t_1, \dots, t_N, X, Y]]/(f(Y,Y) + \sum_1^N t_i p_i(X,Y)).$$

(5.2) Proposition. *The family* (5.1.1) *is a formally semiuniversal deformation of* C.

Replacing $\mathbb{C}[[t_1, \dots, t_N]]$ *by* $\mathbb{C}[t_1, \dots, t_N]$ *we obtain an algebraic formally semiuniversal family of deformations of* C*:*

(5.2.1)
$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathrm{Spec}\left(\mathbb{C}[t_1, \dots, t_N][[X,Y]])/(f(Y,Y) + \sum_1^N t_i p_i(X,Y))\right) \arrow[d, "\pi_a"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{C}[t_1, \dots, t_N]) \end{tikzcd}$$

Therefore, letting

$$S_n := \mathrm{Spec}(\mathbb{C}[[t_1, \dots, t_N]]/(t_1, \dots, t_N)^{n+1}), \quad n \geq 1,$$

the restrictions of (5.1.1) *(or, what is the same, of* (5.2.1)*) to* S_n *define a sequence of (infinitesimal) deformations of* C*:*

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C}_n \arrow[d, "\pi_n"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & S_n \end{tikzcd}$$

which is an algebraizable formal semiuniversal deformation $\widehat{\pi} = \{\pi_n\}$ *of* C.

Proof. See [10], §14, for a detailed explicit proof. □

The support of $\mathcal{C}$ is different from the support of π_n, which is C, and a priori there is no information on how to recover $\mathcal{C}$ starting from $\widehat{\pi}$. Nevertheless from general properties of formal schemes it follows that π is uniquely determined by $\widehat{\pi} := \{\pi_n\}$ (see e.g. [15], Theorem 2.5.11, p. 81). This fact implies that Proposition (5.2) can be strengthened as follows.

(5.3) Corollary. *a) For every deformation*

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

of C *where* $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$*, there is a morphism* $\varphi : \mathrm{Spec}(R) \to M(B)$ *(not unique, but whose differential is unique) inducing an isomorphism*

$$\mathcal{D} \cong \mathrm{Spec}(R) \times_{M(B)} \mathcal{C}$$

compatible with the isomorphisms of the closed fibres with C.

b) Given a family of deformations of C:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "s_0"] & S \end{tikzcd}$$

parametrized by an algebraic *scheme* S *(or by the spectrum of an eft local* $\mathbb{C}$*-algebra) there is a morphism (not unique, but whose differential is unique)*

$$\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0}) \to \mathrm{Spec}(\mathbb{C}[[t_1,\dots,t_N]]) = M(B)$$

which induces an isomorphism of deformations over $\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0})$:

$$\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0}) \times_S \mathcal{X} \cong \mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0}) \times_{M(B)} \mathcal{C}.$$

For another discussion of this property see [10], Prop. 15.2, p. 108.

(5.4) Remarks. (1) If C is nonsingular then $T^1_B = 0$ and the semiuniversal deformation is:

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d] & C \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, equal] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

(see §1 for the definition of T^1_B).

(2) If $C : f = 0$ is singular then it is immediate to verify that $f + \varepsilon = 0$ is nonsingular for small values of ε. This implies that C has nonsingular deformations, i.e., that it is *smoothable.* The same proof shows that any germ of hypersurface in $\mathbb{C}^n$ with an isolated singularity at the origin is smoothable. An obvious modification of the argument shows that a germ of complete intersection isolated singularity is smoothable.

(3) Versality is an open property, but not semiuniversality. In other words, the semiuniversal family (5.1.1) is versal in a neighborhood of 0. Obviously it is not semiuniversal, e.g. where the fibres are smooth, because there the fibres are rigid and their semiuniversal deformation just consists of the fibre itself (see Remark (1) above). The standard reference for openness of versality is [7].

(5.5) Example. (i) The semiuniversal deformation of $A_1 : X^2 - Y^2 = 0$ is $X^2 - Y^2 + t = 0$. The only singular fibre is for $t = 0$.

(ii) Consider $A_2 : X^2 - Y^3 = 0$. Then the semiuniversal deformation is

$$F(t,u,X,Y) = X^2 - Y^3 + tY + u = 0$$

The singular fibres are determined by the conditions $F = F_X = F_Y = 0$ which are equivalent to $t = 3Y^2$, $u = -2Y^3$. We obtain the locus $\Delta : 27u^2 - 4t^3 = 0$ inside $M(B)$. The fibres are the cusp at $(0,0)$ and one node over the other points of Δ. The fibres over $M(B) \setminus \Delta$ are nonsingular.

6 – Equigenericity and equisingularity

It is possible to define two interesting loci inside $M(B)$, the equigeneric locus and the equisingular locus.

The *equigeneric locus* $EG \subseteq M(B)$ is supported on the points of $M(B)$ whose fibre is a singularity imposing the same number of conditions to adjoints as C does, i.e., on the points having the same δ-invariant as C. The locus $EG \subseteq M(B)$ is only defined set-theoretically, it does not have a functorial definition. As a locally closed subset of $M(B)$ it is analytically irreducible near $0 \in M(B)$ and its tangent cone is a vector subspace of T_f. Identifying $T_f = B/J$ the tangent cone to EG at 0 is identified with A/J. In particular A contains J and EG has codimension $\delta(C)$ in $M(B)$. In Example (5.5)(ii) we have $EG = \Delta$.

The *equisingular locus* $ES \subseteq M(B)$ is roughly the locus where the topological type of the singularity is the same as that of C. In particular $ES \subseteq EG$. In example (5.5)(ii) we have $ES = \{0\}$.

The locus ES has a scheme structure because it has a (complicated) functorial definition that we do not need to recall here. ES is nonsingular and its tangent space at $0 \in M(B)$ is the vector space $I/J \subseteq B/J = T_f$ where $I \subseteq A \subseteq B$ is an ideal called the *equisingular ideal.*

(6.1) Lemma. *(i) $I = A$ if and only if B is an A_1-singularity (a node).*

(ii) All the simple singularities satisfy $I = J$, i.e., they have no nontrivial equisingular deformations. Equivalently $ES = \{0\}$.

(iii) The simplest example such that $\dim(ES) > 0$, i.e., $J \neq I$, is an ordinary 4-ple point.

For detailed information on equisingularity and many examples we refer to [8]. The following results due to Teissier are important in this context.

(6.2) Theorem (Teissier [17]). *Let $\mathcal{C} \to S$ be an equigeneric family of reduced curves, S reduced. Then there is a Zariski dense open set $U \subseteq S$ such that $\mathcal{C} \times_S U \to U$ is equisingular.*

(6.3) Theorem (Teissier [17]). *Let $p : \mathcal{C} \to S$ be a flat family of reduced curves with $\mathcal{C}, S$ reduced of finite type. If S is normal then the following are equivalent:*

(a) $p : \mathcal{C} \to S$ is equigeneric.

(b) The composition

$$\mathcal{C}' \xrightarrow{\nu} \mathcal{C} \xrightarrow{p} S$$

where ν is the normalization, is smooth and for all $s \in S$ the induced morphism on the fibres $\nu_s : \mathcal{C}'_s \to \mathcal{C}_s$ is the normalization.

Note: if S is not assumed to be normal the theorem is false: see [5], Ex. (2.6).

7 – Families of curves on surfaces

We now fix a projective nonsingular algebraic surface Y, a class $\xi \in \mathrm{NS}(Y)$ and the scheme of curves of class ξ in Y, denoted Curves_Y^ξ. We have a universal family:

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d] & \mathrm{Curves}_Y^\xi \times Y \\ \mathrm{Curves}_Y^\xi & \end{tikzcd}$$

whose fibres have arithmetic genus $p_a(\xi)$ depending only of ξ. If $[C] \in \mathrm{Curves}_Y^\xi$ then

$$p_a(\xi) = \frac{(C + K_Y) \cdot C}{2} + 1.$$

For a given $0 \leq g \leq p_a(\xi)$ one can consider the (possibly empty) locally closed locus $V_g^\xi \subseteq \mathrm{Curves}_Y^\xi$ parametrizing reduced curves having geometric genus g. Inside V_g^ξ one has the locus $V^{\xi,\delta}$ parametrizing reduced curves having $\delta = p_a(\xi) - g$ nodes and no other singularities. We want to compare these two loci, exploring the conditions under which we have $\overline{V^{\xi,\delta}} = \overline{V_g^\xi}$.

Let $C \subseteq Y$ be a reduced curve parametrized by a point of V_g^ξ. Dualizing the conormal sequence we obtain an exact sequence of sheaves on C:

$$0 \longrightarrow T_C \longrightarrow T_{Y|C} \longrightarrow \mathcal{O}_C(C) \longrightarrow T_C^1 \longrightarrow 0$$

where T_C^1 is the first cotangent sheaf of C, which has been described locally in §1. Note that in §1 we considered the case of algebroid plane curves. The module T_B^1 has an analogous definition for eft local rings and, being a torsion module in our case (isolated singularities), it is invariant under completion.

Locally at the point $v \in V_g^\xi$ parametrizing C the restricted universal family

$$p : \mathcal{C}_g \to V_g^\xi$$

induces a family of deformations of the singular points of C. The semiuniversal families induce a morphism from $\mathrm{Spec}(\widehat{\mathcal{O}}_{V_g^\xi,v})$ to the product of the bases of the semiuniversal families of deformations of the singularities of C. The differential of this morphism turns out to be precisely the map:

$$H^0(C, \mathcal{O}_C(C)) \to H^0(T_C^1)$$

appearing in the above exact sequence. Let's denote, as before, by $\mathcal{J} \subseteq \mathcal{I} \subseteq \mathcal{A} \subseteq \mathcal{O}_C$ respectively the jacobian, equisingular, and conductor ideal sheaves. Then this observation implies the following:

(7.1) Proposition. *If we identify $T_v(\mathrm{Curves}_Y^\xi) = H^0(C, \mathcal{O}_C(C))$ then:*

(i) $H^0(C, \mathcal{J} \otimes \mathcal{O}_C(C))$ is the tangent space to the subscheme of formally locally trivial deformations of C in Y.

(ii) $H^0(C, \mathcal{I} \otimes \mathcal{O}_C(C))$ is the tangent space to the subscheme of equisingular deformations of C in Y.

(iii) $H^0(C, \mathcal{A} \otimes \mathcal{O}_C(C))$ contains the reduced tangent cone to the subscheme of equigeneric deformations of C in Y.

8 – Deformations of morphisms

Given $C \subseteq Y$ as in the previous section we may consider a partial normalization morphism:

$$\varphi : \widetilde{C} \to C \subseteq Y$$

where we assume that $\widetilde{C}$ has at worst local complete intersection singularities. There is a well defined deformation theory for such a morphism, which produces a semiuniversal deformation of φ:

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "\Phi"] \arrow[d] & M(\varphi) \times Y \\ M(\varphi) & \end{tikzcd}$$

where $M(\varphi) = \mathrm{Spec}(R)$ for some complete local $\mathbb{C}$-algebra. Semiuniversality here means a property analogous to the one considered in §5 for deformations of local rings; in particular φ is the fibre of Φ over the the closed point $0 := [\mathfrak{m}_R] \in M(\varphi)$. The properties of $M(\varphi)$ are determined by the deformation functor Def_φ of φ. For the infinitesimal study of $M(\varphi)$ one considers the complex:

$$\Omega^\bullet_\varphi := [\varphi^*\Omega^1_Y \to \Omega^1_{\widetilde{C}}].$$

The hyperext spaces $\mathrm{Ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}})$ and $\mathrm{Ext}^2_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}})$ are respectively the tangent space and the obstruction space of Def_φ. Then deformation theory implies the following:

(8.1) Proposition. *There is a canonical identification*

$$T_0M(\varphi) = \mathrm{Ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}})$$

and inequalities:

$$\mathrm{ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}) - \mathrm{ext}^2_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}) \leq \dim(M(\varphi)) \leq \mathrm{ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}).$$

For details we refer to [6, Sec. 3.2]. Our assumptions imply that φ is unramified at the general point of every component of $\widetilde{C}$. Therefore a simple computation (see [3, Lemma 11]) implies that there is a sheaf N_φ which fits into an exact sequence:

(8.1.1) $$0 \to \mathcal{H}om(\Omega^1_{\widetilde{C}}, \mathcal{O}_{\widetilde{C}}) \to \mathcal{H}om(f^*\Omega^1_Y, \mathcal{O}_{\widetilde{C}}) \to N_\varphi \to \mathcal{E}xt^1(\Omega^1_{\widetilde{C}}, \mathcal{O}_{\widetilde{C}}) \to 0$$

and such that

$$H^0(C, N_\varphi) = \mathrm{Ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}), \quad H^1(C, N_\varphi) = \mathrm{Ext}^2_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}).$$

The sheaf N_φ is called *normal sheaf* of φ.

We will be interested in two special cases:

(a) $\widetilde{C}$ is nonsingular, i.e., φ is the normalization map.

(b) φ is unramified.

In case (a) the sheaf $\Omega^1_{\widetilde{C}}$ is locally free and therefore the sequence (8.1.1) becomes:

(8.1.2) $$0 \longrightarrow T_{\widetilde{C}} \longrightarrow \varphi^*T_Y \longrightarrow N_\varphi \longrightarrow 0.$$

The sheaf N_φ has torsion in general. More precisely we have an exact sequence:

(8.1.3) $$0 \longrightarrow \mathcal{H}_\varphi \longrightarrow N_\varphi \longrightarrow \overline{N}_\varphi \longrightarrow 0$$

where $\mathcal{H}_\varphi = \mathcal{O}_Z$ for some effective divisor Z supported on the locus where the differential $d\varphi$ degenerates. Therefore $\varphi(Z) \subseteq C$ is the set of points where C has a non-linear branch. The exact sequences (8.1.2) and (8.1.3) imply that:

$$c_1(N_\varphi) = \omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}$$

and

$$\overline{N}_\varphi \cong \omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}(-Z).$$

We have the following useful:

(8.2) Lemma (Arbarello-Cornalba [4]). *If $\theta \in H^0(\mathcal{H}_\varphi)$ then the corresponding infinitesimal (first order) deformation of φ leaves the image fixed.*

In case (b) we have an exact sequence

$$0 \to \mathcal{K} \longrightarrow \varphi^*\Omega^1_Y \longrightarrow \Omega^1_{\widetilde{C}} \to 0 \tag{8.2.1}$$

where $\mathcal{K}$ is invertible because $\Omega^1_{\widetilde{C}}$ has homological dimension 1 since $\widetilde{C}$ has l.c.i. singularities (which implies that $\Omega^1_{\widetilde{C}}$ is resolved by the conormal exact sequence of a local embedding). Then in the sequence (8.1.1) we have an identification $N_\varphi = \mathcal{K}^\vee$.

(8.3) Lemma. *In case (b) we have*

$$N_\varphi \cong \omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}.$$

Proof. Since $\widetilde{C}$ is a local complete intersection, we have

$$\omega_{\widetilde{C}} = \det(\Omega^1_{\widetilde{C}})$$

by [9, Thm. III.7.11]. On the other hand the exact sequence (8.2.1) yields

$$\det(\Omega^1_{\widetilde{C}}) = \det(\varphi^*\Omega^1_Y) \otimes \mathcal{K}^{-1} = \varphi^*\omega_Y \otimes N_\varphi.$$

The result follows since $\varphi^*\omega_Y$ is invertible. □

(8.4) Corollary. *In case (b) we have, in the notation of Section 2 for the conductor:*

$$N_\varphi \cong \varphi^* N_{C/Y} \otimes \widetilde{\mathcal{A}}(\varphi)$$

and

$$\varphi_* N_\varphi \cong N_{C/Y} \otimes \mathcal{A}(\varphi).$$

Proof. It is a direct consequence of the previous lemma and Corollary (2.4). □

9 – A criterion for the density of nodal curves

We keep the same notations of the previous section.

(9.1) Theorem. *Let $V \subseteq V^\xi_g$ be an irreducible component and let $[C] \in V$ be a general point with normalization $\varphi : \widetilde{C} \to Y$. Consider the following conditions:*

(a) $\omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}$ *is globally generated.*

(b) $\dim(V) \geq h^0(\widetilde{C}, \omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1})$.

(c) $h^0(\omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}(-\mathfrak{a})) = h^0(\omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}) - 3$ *for every effective divisor $\mathfrak{a}$ of degree 3 on $\widetilde{C}$.*

If (a) and (b) are satisfied then C is immersed, i.e., all its local branches are linear. If also (c) is satisfied then C is nodal.

For the proof we refer to [11]. Let's see an application.

(9.2) Corollary (Arbarello-Cornalba [3], Zariski [19]). *The general element of any component of the Severi variety $V_{d,g}$ of integral plane curve of degree d and genus g is a nodal curve.*

Proof (outline). Let $[C] \in V_{d,g}$ be a general element of a component. We have

$$\deg(\omega_{\widetilde{C}} \otimes \varphi^* \omega_{\mathbb{P}^2}^{-1}) = 2g - 2 + 3d$$

Since $d \geq 3$ (otherwise there is nothing to prove) conditions (a) and (c) of the Theorem are clearly satisfied. One can construct a family of curves in $V_{d,g}$ as the image of the semiuniversal deformation of φ. Using (8.1.3) one can easily deduce that this family satisfies (b) as well. $\square$

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Lecture III
Geometry of logarithmic Severi varieties at a general point

by Thomas Dedieu

This lecture is devoted to the study of logarithmic Severi varieties of a pair (S, R), where S is a surface and R is a curve on S: these are the families of curves on S with prescribed homology class and geometric genus, and prescribed contact pattern with R, meaning that contact orders with R are prescribed both at fixed assigned points and at unassigned, a priori mobile, points. Logarithmic Severi varieties are often referred to as relative Severi varieties.

Logarithmic Severi varieties naturally appear as irreducible components of limit Severi varieties in degenerations of smooth surfaces, see [I]. The emblematic example is that of logarithmic Severi varieties of the pair $(\mathbf{P}^2, \text{line})$ appearing in degenerations of Severi varieties of plane curves of a given degree and genus, see [VI] and [VII] in this collection. On the other hand, in degenerations of $K3$ surfaces we will rather encounter logarithmic Severi varieties of pairs (S, R) such that $K_S + R = 0$, see Section 5.3.

The central result of this text is Theorem (1.7). On the one hand it gives the expected dimension of logarithmic Severi varieties, together with a condition ensuring that this is indeed the actual dimension. On the other hand it gives various regularity properties of the general members of logarithmic Severi varieties, under suitable assumptions on the positivity of $-K_S$ with respect to the prescribed homology class and contact conditions with R.

Another important result is Proposition (4.3), which characterizes logarithmic Severi varieties as essentially the families of curves of maximal dimension with respect to the homology class, the genus, and the number of moving contact points with R, under a positivity assumption on the logarithmic canonical divisor $-(K_S + R)$. It will be used in subsequent lectures to prove (in specific situations) that the irreducible components of a degeneration of logarithmic Severi varieties are again logarithmic Severi varieties (or rather, irreducible components thereof).

The main source for this lecture is Caporaso and Harris' [1, §2]; slightly earlier important works are [20] and [4]. I have adapted the presentation given in [1], which is restricted to the pair consisting of the projective plane and a line, in order to make it fit to the more general situation studied here, and needed further in this volume. My treatment is very much inspired by [11], which I worked out together with Edoardo Sernesi, see also [II] earlier in this volume. I have included many examples throughout, with the two purposes of clarifying the subtleties of the various definitions and statements, and of putting these results in perspective.

The organization of the text is as follows. Section 1 is devoted to the definition of logarithmic Severi varieties and the statement of the main theorem (1.7), and Section 2 contains the material in deformation theory necessary for the proof of the main theorem, which is given in Section 3. Section 4 is devoted to the aforementioned Proposition (4.3) characterizing logarithmic Severi varieties by the maximality of their dimensions, and its proof. Finally, Section 5 is devoted to expanded examples, featuring many examples of superabundant logarithmic Severi varieties, and highlighting some specificities of the case when the logarithmic canonical divisor $K_S + R$ is trivial; we also point out some interesting applications of logarithmic Severi varieties (e.g., to the tropical vertex group, and to $\mathbf{A}^1$-curves on open surfaces obtained as the complement of an anticanonical curve).

1 – Definitions and main results

(1.1) Let S be a nonsingular, projective, connected, algebraic surface over the field $\mathbf{C}$ of complex numbers, and let $R \subseteq S$ be a reduced curve.

By a curve on S, we always mean a closed subscheme of S of pure dimension 1. The *geometric genus* of a reduced curve C is the (arithmetic) genus of its normalization $\bar{C}$, namely

$$1 - \chi(\mathcal{O}_{\bar{C}}) = \sum_{i=1}^{n} g_i - n + 1$$

where $g_1, \ldots, g_n$ are the respective genera of the connected components $\bar{C}_1, \ldots, \bar{C}_n$ of $\bar{C}$. Note that with this definition, denoting by g the geometric genus of C, the canonical bundle of the normalization $\bar{C}$ has degree $2g-2$, and one has the usual Riemann–Roch Formula on $\bar{C}$, i.e.,

$$\chi(L) = 1 - g + \deg(L)$$

for all line bundles on $\bar{C}$.

For every $\xi \in \mathrm{NS}(S)$ (thus, ξ is a homology class on S that can be represented by a divisor) and integer g, we consider the space $M_g^{\xi,\mathrm{bir}}(S)$ parametrizing morphisms

$$\phi : C \to S$$

from a smooth genus g curve C (projective, but possibly disconnected) that are birational on their image, and such that $\phi_*[C] = \xi$. The space $M_g^{\xi,\mathrm{bir}}(S)$ is defined more precisely and studied in Subsection 2.1.

We also consider the locally closed subscheme $V_g^{\xi}(S)$ of $\mathrm{Curve}(S)$, consisting of those points $[C]$ such that C is a reduced curve having geometric genus g and homology class ξ; by $\mathrm{Curve}(S)$, we denote the Hilbert scheme of curves on S.

(1.2) We denote by $\underline{\mathbf{N}}$ the set of sequences $\alpha = [\alpha_1, \alpha_2, \ldots]$ of non-negative integers with all but finitely many α_i non-zero. In practice, we may omit the infinite sequence of zeros at the end. We may also drop the brackets if only α_1 is non-zero.

For all $\alpha \in \underline{\mathbf{N}}$, we let

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots ;$$
$$I\alpha = \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n + \cdots .$$

For all $\alpha, \alpha' \in \underline{\mathbf{N}}$, we say that $\alpha \geqslant \alpha'$ if $\alpha_i \geqslant \alpha'_i$ for all $i \geqslant 1$.

By a set Ω of cardinality $\alpha \in \underline{\mathbf{N}}$, we mean a sequence of sets $\Omega = (\Omega_1, \Omega_2, \ldots)$ such that each Ω_i has cardinality α_i.

(1.3) Definition. *Let $g \in \mathbf{Z}$, $\xi \in \mathrm{NS}(S)$, and $\alpha, \beta \in \underline{\mathbf{N}}$ be such that*

$$I\alpha + I\beta = \xi \cdot R,$$

and consider a set $\Omega = \big(\{p_{i,j}\}_{1 \leqslant j \leqslant \alpha_i}\big)_{i \geqslant 1}$ of α points on R.[1]

We define $M^{\xi}_g(\alpha,\beta)(\Omega)$ as the locally closed subset of $M^{\xi,bir}_g(S)$ consisting of those $[\phi : C \to S]$ such that the intersection $\phi(C) \cap R$ is proper and contained in the smooth locus of R, and there exist α points $q_{i,j} \in C$, $1 \leqslant j \leqslant \alpha_i$, and β points $r_{i,j} \in C$, $1 \leqslant j \leqslant \beta_i$, such that

$$\forall\, 1 \leqslant j \leqslant \alpha_i : \quad \phi(q_{i,j}) = p_{i,j} \qquad \textit{and} \tag{1.3.1}$$

$$\phi^* R = \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} i\, r_{i,j}. \tag{1.3.2}$$

Note that this definition is functorial. In other words, $M^{\xi}_g(\alpha,\beta)(\Omega)$ represents a certain functor, see (2.10).

(1.4) Remark. We emphasize that we do not assume the points in Ω to be general, and neither do we in the Main Theorem (1.7) below. It will be important for some applications to have this flexibility.

(1.5) Definition. *Continuing with the situation of Definition (1.3), we let $V^{\xi}_g(\alpha,\beta)(\Omega)$ be the locally closed subscheme of $V^{\xi}_g(S)$ consisting of those points $[C]$ such that the normalization $\nu : \bar{C} \to C \subseteq S$ belongs to $M^{\xi}_g(\alpha,\beta)(\Omega)$. We call it a* logarithmic[2] Severi variety *of the pair (S,R).*

(1.6) Notation, examples, and comments. In practice we will try to find a balance between rigorous and decipherable notation. For instance we will frequently drop the Ω, and replace ξ with an adequate shorthand.

Let us consider the emblematic case when $S = \mathbf{P}^2$ and R is a line. Let $\xi = d[H]$ with $[H]$ the hyperplane class; we shall simply denote ξ by d. For example, for all g, we may write $V^d_g\big([0,1], d-2\big)$ for the family of plane d-ics of genus g, tangent to the line R at some prescribed general point $p \in R$ (in this case, $\Omega = \{p\}$).

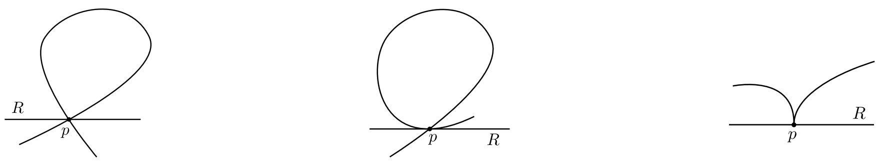
Let $\mathrm{p}(d)$ be the arithmetic genus of plane curves of degree d. The logarithmic Severi variety $V^d_{\mathrm{p}(d)}\big([0,1], d-2\big)$ parametrizes smooth d-ics tangent to R at p; it has codimension 2 in the

[1] If R is reducible one should consider more precise data, specifying the distribution of the contact points with R on its various components: let $R_1, \ldots, R_n$ be the irreducible components of R; one should consider $\alpha_1, \ldots, \alpha_n \in \underline{\mathbf{N}}$ and $\beta_1, \ldots, \beta_n \in \underline{\mathbf{N}}$ such that $I\alpha_i + I\beta_i = \xi \cdot R_i$ for all $i = 1, \ldots, n$, and $\Omega_1, \ldots, \Omega_n$ sets of points on $R_1, \ldots, R_n$ respectively, with cardinalities $\alpha_1, \ldots, \alpha_n$. For the purposes of this chapter we can safely ignore this, and we will do so for obvious reasons of simplicity. (There will be, however, some examples with reducible R, in Section 5 most notably, and then of course we will consider data with the necessary precision).

[2] in [1] it is a called a *generalized* Severi variety; the terminology *relative* Severi variety is nowadays quite popular; I prefer the term *logarithmic* which refers to the fact that we consider pairs instead of “absolute” surfaces, whereas *relative* often refers to the consideration of a map (or a family) instead of a surface.

linear system $|dH|$. The logarithmic Severi variety $V^d_{\mathrm{p}(d)-1}\big([0,1], d-2\big)$ parametrizes plane d-ics of cogenus 1, tangent to R at p; it has codimension 3 in the linear system $|dH|$ (hence $V^d_{\mathrm{p}(d)-1}\big([0,1], d-2\big)$ is a divisor in $V^d_{\mathrm{p}(d)}\big([0,1], d-2\big)$), and its general member is a curve with one node at a general point of $\mathbf{P}^2$, as we will see.

We emphasize that Definition (1.3) requires that curves in $V^d_{\mathrm{p}(d)-1}\big([0,1], d-2\big)$ have one local branch with the prescribed order two of contact with R at p; we illustrate this below.



not in $V^d_{p(d)-1}\big([0,1]\big)$ in $V^d_{p(d)-1}\big([0,1]\big)$ in $V^d_{p(d)-1}\big([0,1]\big)$

Another subtlety of Definition (1.3) worth noting is the following. Consider the logarithmic Severi variety $V = V^d_{\mathrm{p}(d)}\big(1, [d-3, 1]\big)$: loosely speaking, it parametrizes curves passing through an assigned point on R, call it p, and tangent to R. However, looking closely at the definition, one sees that a curve that is simply tangent to R at p and otherwise intersects it transversely does not belong V; a curve having an order 3 contact at p and otherwise transverse does belong to V, on the other hand.

The following is the main result of this chapter. It is a vast generalization of the result proved independently by Arbarello and Cornalba [3] and Zariski [20] for plane curves of fixed degree and genus.

(1.7) Theorem. *Let $g \in \mathbf{Z}$, $\xi \in \mathrm{NS}(S)$, $\alpha, \beta \in \underline{\mathbf{N}}$, and $\Omega = \{p_{i,j}\}_{1\leqslant j\leqslant \alpha_i} \subseteq R$ be as in Definition (1.3), and consider an irreducible component V of $V^\xi_g(\alpha,\beta)(\Omega)$. Let $[C]$ be a general member of V, $\phi : \bar{C} \to C \subseteq S$ its normalization, $q_{i,j}$ $(1 \leqslant j \leqslant \alpha_i)$, $r_{i,j}$ $(1 \leqslant j \leqslant \beta_i)$ points on $\bar{C}$ such that* (1.3.1) *and* (1.3.2) *hold. Set*

$$D = \sum_{1\leqslant j\leqslant \alpha_i} i\, q_{i,j} + \sum_{1\leqslant j\leqslant \beta_i} (i-1)\, r_{i,j}.$$

(1.7.00) One has

$$\dim V \geqslant -(K_S + R)\cdot \xi + g - 1 + |\beta|.$$

(1.7.0) If $-K_S\cdot C_i - \deg \phi_* D|_{C_i} \geqslant 1$ for every irreducible component C_i of C, then the inequality in (1.7.00) above is an equality.

(1.7.1) If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 2$ for every irreducible component C_i of C, then:

(a$^{\flat\flat}$) *the normalization map ϕ is an immersion, except possibly at the points $r_{i,j}$;*

(b) *the points $q_{i,j}$ and $r_{i,j}$ of $\bar{C}$ are pairwise distinct;*

(c$^\flat$) *none of the points $s_{i,j} := \phi(r_{i,j})$ belongs to Ω;*

(d) *for every curve $G \subseteq S$ and finite set $\Gamma \subseteq S$ such that $(G \cup \Gamma) \cap \Omega = \varnothing$, if $[C]$ is general with respect to G and Γ, then C intersects G transversely and does not intersect Γ.*

(1.7.2) If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 3$ for every irreducible component C_i of C, then:

(a$^\flat$) *the normalization map ϕ is an immersion;*

(c) *the points $p_{i,j}$ and $s_{i,j} = \phi(r_{i,j})$ on C are pairwise distinct;*

(e) *the curve C is smooth at its intersection points with R.*

(1.7.3) If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 4$ for every irreducible component C_i of C, then
(a) *the curve C is nodal.*

The tangent space of $V^\xi_g(\alpha,\beta)(\Omega)$ at the general point $[C]$ of V is described in Remark (3.9) under the assumptions of (1.7.2), and in Paragraph (3.6) under the weaker assumptions of (1.7.0).

We emphasize once again that it is not required in the above statement that the set Ω be general. The only requirement is that the points in Ω be smooth points of R, which in turn is imposed by the requirement, in Definitions (1.3) and (1.5), that the intersection with R be proper and contained in the smooth locus of R.

We will see examples in Section 5 of the effect that imposing the passing through a set Ω of non-general points may have. We will also see examples of components of Severi varieties parametrizing reducible curves, for which the assumption of (1.7.0) fails for one irreducible component C_i, and having dimension strictly larger than expected (i.e., the inequality in (1.7.00) is strict).

2 – Background from deformation theory

Throughout this section, we consider a nonsingular, connected, projective surface S, defined over $\mathbf{C}$.

2.1 – Deformations of maps with fixed target

(2.1) Let $\phi : C \to S$ be a non-constant morphism from a smooth projective curve C. A *deformation of ϕ with fixed target* over a pointed base $(B,0)$ is the data of a deformation $\mathcal{C} \xrightarrow{\pi} B$ of C over $(B,0)$ together with a morphism $\Phi : \mathcal{C} \to S \times B$ of B-schemes, such that the restriction of Φ over 0 equals ϕ.

This defines the *deformation functor* $\mathrm{Def}_{\phi/S}$ of ϕ with fixed target S. It is prorepresented by a complete local $\mathbf{C}$-algebra R_ϕ, see [15, Thm. 3.4.8].

(2.2) The deformations of ϕ with fixed target S are controlled by the *normal sheaf of ϕ*, i.e., the sheaf N_ϕ of $\mathcal{O}_C$-modules defined by the exact sequence on C

(2.2.1) $$0 \to T_C \xrightarrow{d\phi} \phi^* T_S \to N_\phi \to 0 \ :$$

the spaces $H^0(C, N_\phi)$ and $H^1(C, N_\phi)$ are respectively the Zariski tangent space and an obstruction space for the deformations of ϕ with fixed target S. In particular, we have

(2.2.2) $$\chi(N_\phi) \leqslant \dim R_\phi \leqslant h^0(N_\phi).$$

(2.3) The rank 1 sheaf N_ϕ may have torsion. We denote by $\mathcal{H}_\phi$ its torsion part and by $\bar{N}_\phi$ its maximal torsion-free quotient; they fit in an exact sequence

(2.3.1) $$0 \to \mathcal{H}_\phi \to N_\phi \to \bar{N}_\phi \to 0.$$

The torsion sheaf $\mathcal{H}_\phi$ is supported on the divisor Z of zeroes of the differential $d\phi$, and it is zero if and only if $Z = 0$, i.e., if and only if the differential $d\phi$ is everywhere non-vanishing; in this case, we say that ϕ is an immersion. Moreover, there is an exact sequence of locally free sheaves on C

(2.3.2) $$0 \to T_C(Z) \to \phi^* T_S \to \bar{N}_\phi \to 0,$$

which readily implies the identification of line bundles on C

$$\bar{N}_\phi \cong \omega_C \otimes \phi^* \omega_S^{-1}(-Z). \tag{2.3.3}$$

(2.4) The scheme of morphisms. We will pretend in this text that the global deformation functor of maps from smooth genus g curves to the surface S is represented by a scheme, which we will denote by $M_g(S)$. This comports the existence of a family $\mathcal{U} \to M_g(S)$ of smooth genus g curves defined over $M_g(S)$, and of a universal morphism $\Phi : \mathcal{U} \to S \times M_g(S)$.

It is not strictly true that the global deformation functor of genus g maps to S is representable in the category of schemes, but it will be harmless for our purposes to pretend it is: one possible way to make things rigorously work out is to use Hartshorne's modular families [12, Def. p.171], as in [11, Subsec. 2.1]. A modular family of curves of genus g is a surrogate of a universal family of curves of genus g. Let us consider such a modular family $\mathcal{C}_g/\mathcal{M}_g$, with the abuse of notation of denoting its base $\mathcal{M}_g$ although it is not the moduli space of genus g curves. Then we may work with the scheme $\mathrm{Hom}_{\mathcal{M}_g}(\mathcal{C}_g, S \times \mathcal{M}_g)$ as a surrogate for $M_g(S)$, with Hom defined as in [14, Thm. I.1.10]. We will not enter in these technicalities here, and refer the reader to [11, Subsec. 2.1] for the complete details.

In this text, we will be mostly interested in the following subspace of $M_g(S)$. For all $\xi \in \mathrm{NS}(S)$, we consider $M_g^{\xi,\mathrm{bir}}(S)$ the subspace of $M_g(S)$ parametrizing morphisms ϕ that are birational on their image, and such that $\phi_*[C] = \xi$.

2.2 – Comparison of the spaces of maps and curves

Let $\xi \in \mathrm{NS}(S)$, and g be a non-negative integer.

(2.5) From maps to curves. Consider the universal morphism $\Phi : \mathcal{U}_M \to S \times M_g^{\xi,\mathrm{bir}}$ defined over $M_g^{\xi,\mathrm{bir}}$. Let $\bar{M}$ be the semi-normalization of the reduced scheme underlying $M_g^{\xi,\mathrm{bir}}$, $\bar{\mathcal{U}}_M := \mathcal{U}_M \times_{M_g^{\xi,\mathrm{bir}}} \bar{M}$ the base change of the universal family $\mathcal{U}_M$, and $\bar{\Phi} : \bar{\mathcal{U}}_M \to S \times \bar{M}$ the induced morphism of $\bar{M}$-schemes. I claim that the scheme-theoretic image $\bar{\Phi}(\bar{\mathcal{U}}_M)$ is flat over $\bar{M}$.

Indeed, the morphism $\varpi := \mathrm{pr}_2 : \bar{\Phi}(\bar{\mathcal{U}}_M) \to \bar{M}$ is a well-defined family of codimension 1 algebraic cycles of S in the sense of [14, I.3.11]. Since $\bar{M}$ is normal, the claim follows from [14, I.3.23.2].

It follows that there is a morphism from $\bar{M}$ to the Hilbert scheme of curves on S. It factorizes through $V_g^\xi(S)$, and actually through the normalization $\bar{V}_g^\xi \to V_g^\xi$ by the universal property of the normalization. Since by definition the semi-normalization morphism $\bar{M} \to M_g^{\xi,\mathrm{bir}}$ is $1:1$, two points $[\phi : C \to S], [\phi' : C' \to S] \in \bar{M}$ are mapped to the same point $[\Gamma] \in V_g^\xi$ if and only if there exists an isomorphism $\iota : C \cong C'$ such that $\phi = \phi' \circ \iota$.

(2.6) From curves to maps. On the other hand, consider the universal family $\mathcal{U}_V \to V_g^\xi$ of curves gotten from the universal family over the Hilbert scheme of curves on S. Let $\bar{V}$ be the normalization of V_g^ξ, and $\bar{\mathcal{U}}_V$ the normalization of $\mathcal{U}_V \times_{V_g^\xi} \bar{V}$. Teissier's Résolution Simultanée Theorem [17] asserts that $\bar{\mathcal{U}}_V \to \bar{V}$ is a family of smooth genus g curves; it comes with a morphism of $\bar{V}$-schemes

$$\bar{\mathcal{U}}_V \to \mathcal{U}_V \times_{V_g^\xi} \bar{V} \subseteq S \times \bar{V}.$$

It follows that there is a morphism from $\bar{V}$ to the space $M_g^{\xi,\mathrm{bir}}$. It is generically injective, because the universal family of curves over V_g^ξ is nowhere isotrivial.

From the considerations in (2.5) and (2.6) above, one deduces the following.

(2.7) Proposition. *Let $[\phi : C \to S] \in M_g^{\xi,bir}(S)$ be a general point (i.e., a general point of any irreducible component of $M_g^{\xi,bir}(S)$). Let $\Gamma := \phi(C)$, $\xi \in \mathrm{NS}(S)$ the homology class of Γ, and g its geometric genus. Then $[\Gamma]$ belongs to a unique irreducible component of V_g^ξ and*

$$\dim_{[\Gamma]} V_g^\xi = \dim R_\phi.$$

(Recall that R_{ϕ_0} is the complete local $\mathbf{C}$-algebra that prorepresents $\mathrm{Def}_{\phi/S}$).

The next result provides a sharper upper bound on the dimension of the Severi varieties than that given by the inequality $\dim R_\phi \leqslant h^0(N_\phi)$ in (2.2.2).

Let $\phi : C \to S$ be a morphism from a smooth projective curve C, birational onto its image Γ. Let $\xi \in \mathrm{NS}(S)$ be the homology class of Γ, and g its geometric genus. We consider $\Phi : \mathcal{C} \to S \times B$ a deformation of ϕ over a pointed normal connected scheme $(B, 0)$. Then $\Phi(\mathcal{C}) \subseteq S \times B$ is a deformation of Γ over $(B, 0)$, see (2.5). There are thus two classifying morphisms κ and γ from $(B, 0)$ to $M_g^{\xi,\mathrm{bir}}(S)$ (or R_ϕ) and $\mathrm{Curve}(S)$ respectively, with differentials

$$d\kappa : T_{B,0} \to H^0(C, N_\phi) \quad \text{and} \quad d\gamma : T_{B,0} \to H^0(\Gamma, N_{\Gamma/X}).$$

(2.8) Lemma. *The inverse image by $d\kappa$ of the torsion $H^0(C, \mathcal{H}_\phi) \subseteq H^0(C, N_\phi)$ is contained in the kernel of $d\gamma$.*

Proof. Given a non-zero section $\sigma \in H^0(C, N_\phi)$, the first order deformation of ϕ defined by σ can be described in the following way: consider an affine open cover $\{U_i\}_{i\in I}$ of C, and for each $i \in I$ consider a lifting $\theta_i \in C(U_i, \phi^* T_X)$ of the restriction $\sigma_{|U_i}$. Each θ_i defines a morphism

$$\tilde{\phi}_i : U_i \times \mathrm{Spec}(\mathbf{C}[\varepsilon]) \to S$$

extending $\phi_{|U_i} : U_i \to X$. The morphisms $\tilde{\phi}_i$ are then made compatible after gluing the trivial deformations $U_i \times \mathrm{Spec}(\mathbf{C}[\varepsilon])$ into the first order deformation of C defined by the coboundary $\partial(\sigma) \in H^1(C, T_C)$ of the exact sequence (2.2.1). In case $\sigma \in \mathrm{H}^0(C, \mathcal{H}_\phi)$, everyone of the maps $\tilde{\phi}_i$ is the trivial deformation of $\sigma_{|U_i}$ over an open subset. This implies that the corresponding first order deformation of ϕ leaves the image fixed, hence the vanishing of $dq_0(\sigma)$. □

(2.9) Corollary. *Let $g \in \mathbf{Z}$, $\xi \in \mathrm{NS}(S)$. Let $[C]$ be a general point of V_g^ξ, and $\phi : \bar{C} \to C \subseteq S$ its normalization. Then*

$$\dim V_g^\xi \leqslant h^0(\bar{C}, \bar{N}_\phi).$$

Proof. By generality we may assume that $[C]$ is a smooth point of V_g^ξ. Then $\dim V_g^\xi = \dim T_{[C]} V_g^\xi$, and by (2.6) there is a map

$$d\kappa_{[\phi]} : T_{[C]} V_g^\xi \to H^0(\bar{C}, N_\phi).$$

It is injective because to every tangent vector $\theta \in T_{[C]}V$ corresponds a non-trivial deformation of C. On the other hand, it follows from Lemma (2.8) that $\mathrm{Im}\, d\kappa_{[\phi]} \subseteq H^0(\bar{C}, \bar{N}_\phi)$. □

Lemma (2.8) (see also [II, Lemma (8.2)]) is a crucial observation (and indeed one of the cornerstones of the proof of Theorem (1.7)) that was made by Arbarello and Cornalba [4, p. 26], who deemed it a *fenomeno assai curioso.* They write: « nel caso in cui ϕ sia una birazionalità tra C e Γ, la presenza di “cuspidi” su Γ, comporta l’esistenza, dal punto di vista infinitesimo, di più di un modello liscio della curva Γ, se così ci possiamo esprimere. »[3]

[3] a very curious phenomenon; in the case when ϕ is birational between C and Γ, the presence of “cusps” on Γ comports the existence, at the infinitesimal level, of more than one smooth model of the curve Γ, if we may say so.

Next, paraphrasing them, in order to use this phenomenon constructively they establish [4, Cor. 6.11]: in the above notation, if B is the complex unit disk and if the family of curves is *equisingular*, then $d\kappa(\partial/\partial t)$ belongs to $H^0(C,\mathcal{H}_\phi)$ if and only if it is zero. Later, Caporaso and Harris (together with J. de Jong, they write) state and prove [1, Lem. 2.3]. They add the remark that this is linked to the notion of equisingularity, even though they make absolutely no use of that notion, neither in their statement nor in its proof.

The treatment I give above is that of Sernesi and myself in [11]. Although essentially equivalent to that of [1], it slightly differs in its formulation. This formulation, I hope, sheds some light on what is actually going on, and in particular shows that it is not necessary to invoke equisingularity in order to prove Corollary (2.9).

The standard application of Corollary (2.9) is to the proof that curves in a given family are immersed, i.e., they have no cusps: we will use it in Paragraph (3.7) below to prove assertion $(\mathrm{a}^{\flat\flat})$ of Theorem (1.7).

2.3 – Tangency conditions with respect to a fixed curve

We consider R a fixed reduced curve on S. In this subsection we study deformations of curves on S satisfying some tangency conditions with R; it follows from our Definition (1.3) that it suffices to treat the case when R is smooth.

(2.10) Let m be a non-negative integer. Let $\phi : C \to S$ be a non-constant morphism from a smooth projective curve C. A deformation of ϕ with fixed target *preserving a tangency of order m with R* over a pointed base $(B,0)$ is a deformation $\Phi : \mathcal{C} \to S \times B$ of ϕ with fixed target as in (2.1), such that there exists a section Q of $\mathcal{C} \to B$ such that the pulled-back divisor Φ^*R contains Q with multiplicity m (i.e., $\Phi^*R - mQ \geqslant 0$).

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & S \times B \arrow[dl, "\mathrm{pr}_2"] \\ B \arrow[u, bend left, "Q"] & \end{tikzcd}$$

The tangency is said to be respectively at a *fixed point* $p \in R$ if $\Phi(Q) = \{p\} \times B$, and at a *variable point* if $\mathrm{pr}_1 \circ \Phi(Q)$ is a curve.

We say that a family of maps $\Phi : \mathcal{C} \to S \times B$ preserves a tangency of order m with R if for all $b \in B$ it is locally around b a deformation of maps preserving a tangency of order m.

The following result displays the additional conditions the class of a deformation of maps has to meet for this deformation to preserve a tangency with R. It is [1, Lem. 2.6]. Let B be a reduced scheme, and $\Phi : \mathcal{C} \to S \times B$ be a family of maps preserving a tangency with R of order exactly m. Let $b \in B$ be a general point, and $\phi : C \to S$ be the corresponding map. This comes with a classifying map κ, with differential $d\kappa : T_0B \to H^0(C, N_\phi)$; we call $\bar{d}\kappa$ its composition with the projection $H^0(C,N_\phi) \to H^0(C,\bar{N}_\phi)$.

Let $q := Q \cap C$, and $l-1$ be the order of vanishing of the differential $d\phi$ at q (i.e., l is the multiplicity of the point $p := \phi(q)$ in the local branch of $\phi(C)$ corresponding to q). Note that necessarily $l \leqslant m$.

(2.11) Lemma. *Let $\sigma \in \mathrm{Im}(\bar{d}\kappa) \subseteq H^0(C,\bar{N}_\phi)$ be a non-zero section, and denote by $v_q(\sigma)$ its order of vanishing at $q = Q \cap C$.*
(a) *One has $v_q(\sigma) \in \{m-l\} \cup \llbracket m, +\infty\llbracket$.*[4]
(b) *If the tangency is at a fixed point of R, then actually $v_q(\sigma) \in \llbracket m, +\infty\llbracket$.*

[4]I use the notation $\llbracket a,b\rrbracket = [a,b] \cap \mathbf{Z}$ for $a,b \in \mathbf{R} \cup \{\pm\infty\}$; albeit being standard probably only in France, it is convenient.

Proof. This is a local computation. Let (x, y) be (analytic) local coordinates on S at $p = \phi(q)$, such that R is defined by the equation $y = 0$. Then the vector fields $\partial/\partial x$ and $\partial/\partial y$ generate T_S near p, and their pull-backs generate $\phi^* T_S$ near q; by abuse of notation we shall denote them by $\partial/\partial x$ and $\partial/\partial y$ as well.

We may assume that B is a curve. Let ε be a local coordinate on B centered at b, and t be a local equation of the section Q near q. Thus (t, ε) are local coordinates on $\mathcal{C}$ at q. We may assume that t gives a local coordinate on C at q, in such a way that ϕ is given locally by

$$\phi(t) = \begin{cases} \big(t^l + a_{l+1}t^{l+1} + \cdots, t^m\big), & \text{if } l < m \\ \big(t^n + a_{n+1}t^{n+1} + \cdots, t^m\big) \text{ for some } n \geqslant m, & \text{if } l = m. \end{cases}$$

From now on we assume $l < m$ and leave the other, similar, case to the reader. Then the differential of ϕ at q is

$$\begin{aligned} \tfrac{\partial}{\partial t} \longmapsto & (lt^{l-1} + (l+1)a_{l+1}t^l + \cdots) \cdot \tfrac{\partial}{\partial x} + mt^{m-1} \cdot \tfrac{\partial}{\partial y} \\ &= t^{l-1}\big((l + (l+1)a_{l+1}t + \cdots) \cdot \tfrac{\partial}{\partial x} + mt^{m-l} \cdot \tfrac{\partial}{\partial y}\big). \end{aligned}$$

We see that around q on C, the torsion part $\mathcal{H}_\phi$ of N_ϕ is a skycraper sheaf of length $l - 1$ concentrated at q, generated by the image in N_ϕ of the local section

$$\tau : t \mapsto \big(l + (l+1)a_{l+1}t + \cdots\big) \cdot \tfrac{\partial}{\partial x} + mt^{m-l} \cdot \tfrac{\partial}{\partial y}$$

of $\phi^* T_S$; thus the torsion-free quotient $\bar{N}_\phi$ is generated by the class of $\partial/\partial y$ modulo τ. Observe that modulo τ, $\partial/\partial x$ equals $t^{m-l} \cdot \partial/\partial y$ times an invertible, hence the image of $\partial/\partial x$ in $\bar{N}_\phi$ vanishes to the order exactly $m - l$ at q.

In turn the family Φ may be written locally as

$$\Phi(t, \varepsilon) = \big((t^l + a_{l+1}t^{l+1} + \cdots) + \varepsilon(u_0 + u_1 t + \cdots) + O(\varepsilon^2), t^m, \varepsilon\big),$$

since $\Phi^* R$ contains Q with multiplicity m. By definition, the corresponding section $\bar{d}\kappa(\partial/\partial\varepsilon)$ of $\bar{N}_\phi$ is

$$\bar{d}\kappa(\tfrac{\partial}{\partial \varepsilon}) = (u_0 + u_1 t + \cdots) \cdot \tfrac{\partial}{\partial x} \bmod \tau.$$

Since $\partial/\partial x$ itself vanishes modulo τ to the order $m - l$ as we have seen, one has $v_q\big(\bar{d}\kappa(\partial/\partial\varepsilon)\big) \geqslant m - l$ in any event.

Moreover, by generality of $b \in B$ we may assume that all $\Phi(\cdot, \varepsilon)$ have their differential vanishing to the order l at $Q \cap \mathcal{C}_\varepsilon$, which translates into the fact that $u_1 = \cdots = u_{l-1} = 0$. Then $\bar{d}\kappa(\partial/\partial\varepsilon)$ vanishes either to the order $m - l$, if $u_0 \neq 0$, or to some order larger than m, if $u_0 = 0$. When the tangency is maintained at a fixed point we are necessarily in the latter case. The lemma is proved. ☐

Let $\mathcal{C} \subseteq S \times B$ be a family of a reduced curves over a reduced base B. It is said to preserve a tangency of order m with R if the corresponding family of normalization maps over the normalization $\bar{B}$ of B (see (2.6)) does.

(2.12) Corollary. *Let $V \subseteq \mathrm{Curve}(S)$ be a family of curves of genus g having a tangency of order m with the divisor R. Let $[C]$ be a general member of V, $\phi : \bar{C} \to C \subseteq S$ be the normalization of C, $q \in \bar{C}$ the tangency point, and $l-1$ the order of vanishing of the differential $d\phi$ at q. Then*

$$\dim V \leqslant h^0\big(\bar{C}, \bar{N}_\phi(-(m-l)_+ \cdot q)\big),$$

where $(m-l)_+$ denotes $\max(m-l, 0)$. If the tangency is at a fixed prescribed point on R, then actually

$$\dim V \leqslant h^0\big(\bar{C}, \bar{N}_\phi(-m \cdot q)\big).$$

Proof. As in the proof of Corollary (2.9), we have $\dim V = \dim T_{[C]}V$ by generality of $[C]$, and there is an injective map $T_{[C]}V \to H^0(\bar{C}, \bar{N}_\phi)$. By Lemma (2.11) the image of this map is contained in $H^0\big(\bar{C}, \bar{N}_\phi(-a \cdot q)\big)$ with $a = (m-l)_+$ if the tangency is mobile, and $a = m$ if the tangency is fixed. This ends the proof. ☐

3 – Proof of the Main Theorem

In this section we prove Theorem (1.7). The proof itself is in Subsection 3.2, after we give some lemmas in Subsection 3.1.

3.1 – Applications of the Riemann–Roch formula

Lemma (3.1) below is standard, but we will also use the more clever Lemma (3.2), which amounts to [1, Observation 2.5].

(3.1) Lemma. *Let X be a smooth (possibly disconnected) projective curve, and L a line bundle on X. Let $k \in \mathbf{N}^*$. If $\deg\big(L \otimes \omega_X^{-1}\big|_{X_i}\big) > k$ for all irreducible component X_i of X, then the linear system $|L|$ separates any k points on X.*

Proof. Let Z be a subscheme of X of length k, and let Z' be another subscheme of X such that $Z' \subsetneq Z$. The assumption on the degree of L ensures that both $L(-Z)$ and $L(-Z')$ are non-special, hence $h^0\big(L(-Z)\big) < h^0\big(L(-Z')\big)$ by the Riemann–Roch formula. ☐

(3.2) Lemma. *Let X be a smooth (possibly disconnected) projective curve of genus $g = 1 - \chi(\mathcal{O}_X)$, and L, M two line bundles on X such that*

$$\forall\, X_i \text{ irreducible component of } X: \quad \deg(M|_{X_i}) \leqslant \deg(L|_{X_i}).$$

(a). *If* $\deg L \otimes \omega_X^{-1}\big|_{X_i} > 0$ *for every component X_i of X, then*

$$h^0(X, M) \leqslant h^0(X, L) = \deg(L) - g + 1. \tag{3.2.1}$$

(b). *If* $\deg L \otimes \omega_X^{-1}\big|_{X_i} > 1$ *for every component X_i of X, then equality holds in* (3.2.1) *if and only if* $\deg M = \deg L$.

Proof. Assumption (a) ensures that L is non-special, hence the right-hand-side equality in (3.2.1), by the Riemann–Roch formula. If M is non-special as well, then $h^0(L) - h^0(M) = \deg(L) - \deg(M)$ again by the Riemann–Roch formula, which gives the result in both cases (a) and (b).

We thus assume from now on that M is special. Then,

$$h^0(X, M) \leqslant h^0(X, \omega_X) = g + n - 1.$$

Under Assumption (a),

$$\deg(L) = \sum_{i=1}^{n} \deg(L|_{X_i}) \geqslant \sum_{i=1}^{n} (2g_i - 1) = 2g - 2 + n,$$

where n is the number of components of X and g_i is the genus of X_i for all $i = 1, \ldots, n$; recall that $g = \sum g_i - n + 1$. By the Riemann–Roch formula, one has

$$h^0(X, L) = \deg(L) - g + 1 \geqslant g - 1 + n$$

hence $h^0(X,L) \geqslant h^0(X,M)$.

Under Assumption (b) one has in the same fashion $\deg(L) \geqslant 2g-2+2n$, hence

$$h^0(X,L) \geqslant g+2n-1 > h^0(X,M).$$

This ends the proof, as the specialty of M implies $\deg(M) < \deg(L)$ under the assumption of (a), and a fortiori under the assumption of (b). $\square$

3.2 – Proof of the main theorem

This section is dedicated to the proof of Theorem (1.7). We consider an irreducible component V of $V_g^\xi(\alpha,\beta)(\Omega)$ and a general point $[C]$ of V, as in the theorem. We use freely the notation introduced in the statement.

(3.3) *We start by proving that V has expected dimension*

$$\operatorname{expdim} V_g^\xi(\alpha,\beta)(\Omega) = -(K_S+R)\cdot\xi + g - 1 + |\beta|. \tag{3.3.1}$$

By (2.2.2) the expected dimension of V_g^ξ is $\chi(N_\phi)$, which by the Riemann–Roch formula and the exact sequence (2.2.1) equals

$$\operatorname{expdim} V_g^\xi = \deg\omega_{\bar C} - \deg\phi^*\omega_S + 1 - g = -K_S\cdot C + g - 1. \tag{3.3.2}$$

Now requiring that a curve C have tangency of order m with R at a specified point p is m linear conditions on the coefficients of the equation of C, and if we let the point p vary along R the expected codimension of the corresponding locus of curves C is one less, i.e. $m-1$. We thus end up with

$$\begin{aligned}\operatorname{expdim}\bigl(V_g^\xi(\alpha,\beta)(\Omega)\bigr) &= \operatorname{expdim}\bigl(V_g^\xi\bigr) - \sum_i \sum_{1\leqslant j\leqslant\alpha_i} i - \sum_i \sum_{1\leqslant j\leqslant\beta_i} (i-1)\\ &= \operatorname{expdim}\bigl(V_g^\xi\bigr) - I\alpha - (I\beta - |\beta|),\end{aligned}$$

which together with (3.3.2) gives (3.3.1) after one remarks that $I\alpha + I\beta = R\cdot\xi$.

Note that this proves that, in any event,

$$\dim(V) \geqslant -(K_S+R)\cdot\xi + g - 1 + |\beta|, \tag{3.3.3}$$

which is assertion (1.7.00). $\square$

(3.4) *We now turn to the proof that the dimension of V equals its expected dimension under assumption (1.7.0).*

Note that the points $q_{i,j}$ are necessarily pairwise distinct because they have distinct images $p_{i,j}\in R$. Let us first assume in addition that the points $q_{i,j}$ and $r_{i,j}$ are all together pairwise distinct; the case when this does not hold will be dealt with in (3.5).

We set

$$D := \sum_{1\leqslant j\leqslant\alpha_i} i\, q_{i,j} + \sum_{1\leqslant j\leqslant\beta_i} (i-1)\, r_{i,j}, \tag{3.4.1}$$

the divisor on $\bar C$ of "infinitesimal tangency conditions with R" (compare (3.3)), and

$$D_0 := \sum_{1\leqslant j\leqslant\beta_i} (l_{i,j}-1)\, r_{i,j}, \tag{3.4.2}$$

where $l_{i,j} := v_{r_{i,j}}(d\phi)$ is the order of vanishing of the differential $d\phi$ at the point $r_{i,j}$, i.e., D_0 is the "ramification divisor of ϕ in the points $r_{i,j}$". We then decompose the difference of these two divisors as

$$D - D_0 = D_1 - D_1' \tag{3.4.3}$$

where D_1 and D_1' are non-negative divisors on $\bar{C}$ with disjoint supports; note that D_1' may be nonzero only at the points $r_{i,j}$, and that it is so if and only if $l_{i,j} > i$.

It follows from Corollary (2.12) that

$$\dim(V) \leqslant h^0\big(\bar{C}, \bar{N}_\phi(-D_1)\big). \tag{3.4.4}$$

Let Z_0 be the non-negative divisor on $\bar{C}$ such that the ramification divisor of ϕ is $D_0 + Z_0$. Then by (2.3.3) we have $\bar{N}_\phi \cong \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D_0 - Z_0)$, and therefore (3.4.4) above reads

$$\begin{aligned}
\dim(V) &\leqslant h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D_0 - Z_0 - D_1)\big) && (3.4.5)\\
&= h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D - D_1' - Z_0)\big) && (3.4.6)\\
&\leqslant h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big). && (3.4.7)
\end{aligned}$$

Now by assumption (1.7.0) the line bundle $\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)$ is non-special, hence

$$\begin{aligned}
h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big) &= h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S(R)^{-1}(\phi^*R - D)\big) && (3.4.8)\\
&= 2g - 2 - (K_S + R)\cdot\xi + |\beta| + 1 - g && (3.4.9)\\
&= \operatorname{expdim} V_g^\xi(\alpha,\beta)(\Omega). && (3.4.10)
\end{aligned}$$

We thus have $\dim V \leqslant \operatorname{expdim} V_g^\xi(\alpha,\beta)(\Omega)$ which, together with (3.3.3) implies that V has the expected dimension, if indeed the points $q_{i,j}$ and $r_{i,j}$ are all together pairwise distinct.

(3.5) Now if it is not true that the points $q_{i,j}$ and $r_{i,j}$ are all together pairwise distinct, then V is actually a component of some Severi variety $V_g^\xi(\alpha',\beta')(\Omega')$ with $|\beta'| < |\beta|$ for which the corresponding points $q'_{i,j}$ and $r'_{i,j}$ are indeed pairwise disjoint (as sets $\Omega = \Omega'$, i.e., $\bigcup_i \Omega_i = \bigcup_i \Omega'_i$, and $\Omega_i \subseteq \bigcup_{k\geqslant i} \Omega'_k$).

Then, setting correspondingly D' as in (3.4.1), one gets $\dim V \leqslant h^0(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D'))$ exactly as in (3.4). Now $\deg D' > \deg D$ because $|\beta'| < |\beta|$, and it therefore follows from Lemma (3.2), part (a) that

$$h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D')\big) \leqslant h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big), \tag{3.5.1}$$

so that it still holds that $\dim V \leqslant \operatorname{expdim} V_g^\xi(\alpha,\beta)(\Omega)$, hence V has the expected dimension. □

(3.6) *Note that the above proof gives the additional fact that, under the assumption of (1.7.0), the tangent space of $V_g^\xi(\alpha,\beta)(\Omega)$ at the general point $[C]$ of V identifies with*

$$H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big) \cong H^0\big(\bar{C}, \bar{N}_\phi(-D_1)\big) \cong H^0\big(\bar{C}, \bar{N}_\phi(D_0 - D)\big) \subseteq H^0\big(\bar{C}, \bar{N}_\phi\big).$$

(3.7) *We now prove that under Assumption (1.7.1) the assertions* (a$^{\flat\flat}$), (b), (c$^\flat$), *and* (d) *hold.*

Suppose by contradiction that (b) doesn't hold. Then we argue as in (3.5). In this case, part (b) of Lemma (3.2) applies thanks to Assumption (1.7.1), and we get that the inequality (3.5.1) is strict, which is in contradiction with (3.3.3).

The same argument shows that none of the points $\phi(r_{i,j})$ can be fixed on R. This implies in particular that ($\mathrm{c}^\flat$) holds.

The proof of (d) is similar: if C were tangent to G, then it would belong to an irreducible component W of some Severi variety of the pair $(S, R+G)$. Assumption (1.7.1) implies that W is liable for part (1.7.0) of Theorem (1.7), hence $\dim(W) < \dim(V)$, in contradiction with the fact that $[C]$ is a general member of V. The same argument shows that C avoids Γ (pick some random curve on S containing Γ).

Finally, we note that equality holds in (3.4.7) if and only if $D_1' = Z_0 = 0$ since the line bundle $\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)$ is globally generated by assumption (1.7.1). Now it is indeed the case that equality holds in (3.4.7) since we have proved that $\dim(V) = \operatorname{expdim}\big(V_g^\xi(\alpha,\beta)\big)$. We conclude that $D_1' = Z_0 = 0$, which means that (i) ϕ is an immersion outside of the points $r_{i,j}$ (this is assertion ($\mathrm{a}^{\flat\flat}$)) and (ii) $l_{i,j} \leqslant i$ for $1 \leqslant j \leqslant \beta_i$. □

Remark. It is not enough to assume that $\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)$ is non-special and globally generated because of the argument we made to assume that the points $q_{i,j}$ and $r_{i,j}$ are pairwise distinct. We actually need to know something about every possible $\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D')$, where $D' = \sum iq'_{i,j} + \sum (i-1)r'_{i,j}$ in the notation used for this argument.

(3.8) *We now prove that, under the assumption (1.7.2), ϕ is an immersion also at the points $r_{i,j}$, i.e. that $l_{i,j} = 1$ for $1 \leqslant j \leqslant \beta_i$, thus completing the proof of assertion* ($\mathrm{a}^\flat$).

Let $i \geqslant 1$ and $1 \leqslant j \leqslant \beta_i$. It follows from the assumption that the linear series $\big|\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big|$ separates any two points, so there exists a section $\sigma \in H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big)$ with vanishing order $v_{r_{i,j}}(\sigma) = 1$ at the point $r_{i,j}$. Seen as a section $\tilde{\sigma} \in H^0(\bar{C}, \bar{N}_\phi)$, it vanishes at $r_{i,j}$ with order $v_{r_{i,j}}(\tilde{\sigma}) = 1 + (i - l_{i,j})$ (see (2.3.3)). By Lemma (2.11) this implies

$$1 + i - l_{i,j} \in \{i - l_{i,j}\} \cup [\![i, +\infty [\![$$

and therefore $l_{i,j} = 1$ as required. □

(3.9) Remark. Under the conditions of (1.7.2) the map ϕ is an immersion, hence $N_\phi = \bar{N}_\phi \cong \omega_C \otimes \phi^*\omega_S^{-1}$, see Paragraph (2.2). Therefore, the tangent space of $V_g^\xi(\alpha,\beta)(\Omega)$ at the general point $[C]$ of V identifies with

$$H^0\big(\bar{C}, N_\phi(-D)\big) \cong H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big)$$

(the definition of D is in (3.4.1) above).

(3.10) *Let us prove that Assumption (1.7.2) implies Assertion* (c)*, i.e., the points $p_{i,j}$ and $s_{i,j} = \phi(r_{i,j})$ are pairwise distinct.*

By (3.7), we already know that ($\mathrm{c}^\flat$) holds, i.e., none of the $s_{i,j} = \phi(r_{i,j})$ belongs to $\Omega = \{p_{i,j}\}$. We thus only need to show that no two of the points $s_{i,j}$ coincide.

Suppose there exist (i,j) and (i',j') distinct such that $\phi(r_{i,j}) = \phi(r_{i',j'})$. Assumption (1.7.2) implies that the series $|\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)| = |T_{[C]}V|$ separates any two points. Therefore there exists a section $\sigma \in T_{[C]}V \cong H^0(\bar{N}_\phi(-D))$ such that

$$v_{r_{i,j}}(\sigma) = 1 \quad \text{and} \quad v_{r_{i',j'}}(\sigma) = 0.$$

This implies the existence of a deformation of C in which the points $\phi(r_{i,j})$ and $\phi(r_{i',j'})$ no longer coincide, a contradiction to the generality of $[C]$ in V. □

(3.11) *Let us prove that Assumption (1.7.2) implies Assertion* (e)*, i.e., C is smooth at its intersection points with R.*

At this point we know that ($a^\flat$) and (c) hold under Assumption (1.7.2), i.e., the curve C is immersed and the points $p_{i,j}$ and $s_{i,j}$ are pairwise distinct. Because the intersection $C \cap R$ is set-theoretically the union of all the points $p_{i,j}$ and $s_{i,j}$, this implies that C is smooth at its intersection points with R. □

(3.12) *We eventually prove that under the Assumption (1.7.3) the curve C is nodal, which is Assertion* (a) *of Theorem (1.7).*

Since we already know that the curve C is immersed, it is enough to show that for all point $p \in C$, C has neither three or more local branches, nor two or more tangent local branches.

To exclude the former possibility, suppose by contradiction that there exist $a, b, c \in \bar{C}$ pairwise distinct such that $\phi(a) = \phi(b) = \phi(c)$. The assumption (1.7.3) implies that the linear series $\left|\omega_{\bar{C}} \otimes \phi^* \omega_S^{-1}(-D)\right|$ separates any three points, so there exists a section $\sigma \in H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^* \omega_S^{-1}(-D)\big)$ such that

$$\sigma(a) = \sigma(b) = 0 \quad \text{and} \quad \sigma(c) \neq 0;$$

it corresponds to a first-order deformation of ϕ leaving both $\phi(a)$ and $\phi(b)$ fixed while moving $\phi(c)$. By generality of $[C]$, there is correspondingly an actual deformation of the curve C for which the three local branches under consideration are no longer intersecting in a common point, a contradiction to the generality of $[C]$ in V (see, e.g., [11, Prop. 1.4]).

We exclude the second possibility in a similar fashion. Suppose by contradiction that there exist $a, b \in \bar{C}$ distinct such that $\phi(a) = \phi(b)$ and $\operatorname{Im} d\phi_a = \operatorname{Im} d\phi_b$. Again since $\left|\omega_{\bar{C}} \otimes \phi^* \omega_S^{-1}(-D)\right|$ separates any three points, there exists $\sigma \in H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^* \omega_S^{-1}(-D)\big)$ such that $\sigma(a) = 0$ and $\sigma(b) \notin \operatorname{Im} d\phi_a$. It corresponds to a first-order deformation of ϕ leaving $\phi(a)$ fixed while moving $\phi(b)$ in a direction transverse to the common tangent to the two local branches of C under consideration. This contradicts the generality of $[C]$ as before. □

This ends the proof of Theorem (1.7).

4 – A dimensional characterization of logarithmic Severi varieties

We consider a pair (S, R) consisting of a smooth surface S and a reduced boundary divisor R, as described in 1. In this Section we give an upper bound for the dimension of families V of (not necessarily reduced) curves in S with prescribed homology class and genus, together with a dimensional characterization of logarithmic Severi varieties: it turns out that the irreducible components of logarithmic Severi varieties are essentially those families for which the upper bound is reached. The content of this section is basically [1, Cor. 2.7]. The main result, Proposition (4.3), is a generalization of Zariski's dimensional characterization of (classical) Severi varieties [20].

(4.1) Let V be an irreducible locally closed subset of the Hilbert scheme of curves on S, and suppose that it parametrizes genus g curves in the following sense: for a general member X of V, there exists a smooth curve C of genus g, and a morphism $\phi : C \to X$, not constant on any component of C, and such that the push-forward in the sense of cycles $\phi_*[C]$ equals the fundamental cycle of X.

(4.2) Let $\xi \in \mathrm{NS}(S)$ be the homology class of members of V. Letting the general member X of V have irreducible components $X_1, \ldots, X_n$, with respective classes $\xi_1, \ldots, \xi_n$, we assume furthermore:

$$\forall i = 1, \ldots, n : \quad -(K_S + R) \cdot \xi_i > 0,$$

so that we may apply (1.7.0) without any restriction on the contact conditions with R.

(4.3) Proposition. *Consider a finite subset $\Omega \subseteq R$. Under the above assumptions in (4.1) and (4.2), for all general member X of V, one has*

$$\dim(V) \leqslant -(K_S + R) \cdot \xi + g - 1 + \mathrm{Card}\big((X \cap R) \setminus \Omega\big), \tag{4.3.1}$$

where the last number is defined set-theoretically (i.e., we do not count with multiplicities).

Moreover, if equality holds in (4.3.1)*, then there exist α, β such that V is a dense subset of a component of a log-Severi variety $V_g^{\xi}(\alpha, \beta)(\Omega)$ if and only if*

$$\mathrm{Card}\big(\phi^{-1}(R)\big) = \mathrm{Card}(X \cap R), \tag{4.3.2}$$

with $\phi : C \to X$ a genus g morphism as above.

Note that α and β are of course uniquely determined by the contacts of X and L. Before proving this statement we make a few observations, in the hope that they will clarify some of its subtleties.

(4.4) Remark. Proposition (4.3) is really a result about families of embedded curves in S, not families of maps.

Indeed, if a general member X is not reduced, then genus g maps $\phi : C \to X$ as above involve multiple covers, and there is in general a positive dimensional family of maps giving the same X, so that the family of maps corresponding to V has in general dimension larger than V.

When equality holds in (4.3.1), the map ϕ is necessarily a birational isomorphism on each component of C. If moreover $g \geqslant 1$, then the general member of V is actually reduced. See (4.9) for precisions about this.

(4.5) Remark. A straightforward corollary of Proposition (4.3) which may be useful for the applications is that the inequality (4.3.1) still holds if we only assume C to be reduced and replace g by the *arithmetic* genus of C; this alternative inequality is always strict when C is not smooth.

(4.6) Remark. Assumption (4.3.2) ensures that the normalization of X is unibranch over the points in $X \cap R$.

(4.6.1) Example. Set $(S, R) = (\mathbf{P}^2, L)$ whith L a line, and $\xi = 3[H]$ with $[H]$ the hyperplane class. The family V of plane cubics with one node on L and otherwise smooth has dimension 7, which equals

$$\underbrace{-(K_S + R) \cdot \xi}_{=2[H]\cdot 3[H]=6} + \underbrace{g}_{=0} -1 + \underbrace{\mathrm{Card}\big((X \cap R) \setminus \Omega\big)}_{=2} \quad \text{with } \Omega = \varnothing.$$

It is a divisor in the log-Severi variety $V_0^{3[H]}(0, 3)(\varnothing)$, but it is not a component of the family $V_0^{3[H]}(0, [1, 1])(\varnothing)$ of rational plane cubics with one variable tangency along L, which has dimension 7 as well.

(4.7) Remark. In the equality case for (4.3.1), if one replaces (4.3.2) with the weaker condition that

$$\text{Card}\big(\phi^{-1}(R \setminus \Omega)\big) = \text{Card}\big((X \cap L) \setminus \Omega\big), \tag{4.7.1}$$

the families that we get that are not log-Severi varieties may be considered as "log-Severi varieties with Ω containing repeated points", see Example (4.7.2) below. (These are not log-Severi varieties according to Definition (1.3)).

Beware that in [1, Cor. 2.7], the weaker (4.7.1) is used instead of (4.3.2), which is slightly inaccurate.

(4.7.2) Example. Set $(S, R) = (\mathbf{P}^2, L)$ as above, and $\xi = 4[H]$, and fix $p \in L$. The family of plane quartics with one triple point at p has dimension $\binom{6}{2} - 1 - 6 = 8$, which equals

$$\underbrace{-(K_S + R) \cdot \xi}_{=2[H]\cdot 4[H]=8} + \underbrace{g}_{=0} - 1 + \underbrace{\text{Card}\big((X \cap L) \setminus \Omega\big)}_{=1} \quad \text{with } \Omega = \{p\}.$$

One may wish to consider it as $V_0^{3[H]}(3,1)(p,p,p)$.

Proof of Proposition (4.3)

We divide the proof into several steps, which correspond to paragraphs (4.8)–(4.10). We first treat the case when X is irreducible; the general case is taken care of by induction in (4.10).

(4.8) We first prove the proposition under the assumption that X is reduced and irreducible; in this case ϕ is the normalization of X and $C \cong \bar{X}$.

If $e := \text{Card}\big((X \cap R) \setminus \Omega\big) = 0$, then the statement is a slight variant of part (1.7.0) of the main Theorem (1.7): we get the required inequality (4.3.1) exactly as in paragraph (3.4), with $D = \phi^* R$ and $D_0 = 0$ in (3.4.1) and (3.4.2) respectively. The only difference with the setting of (3.4) is that here two distinct points of the support of $\phi^* R \subseteq \bar{X}$ may have the same image by ϕ in X; this makes absolutely no difference in the argument.

Now if (4.3.2) holds, then $\bar{X}$ is unibranch over the points of $X \cap R$. This implies that V is contained in a certain log-Severi variety $V_g^\xi(\alpha, 0)(\Omega)$. If moreover equality holds in (4.3.1), then V is dense in an irreducible component of this same log-Severi variety by (1.7.0).

In the case when $e > 0$, we consider the map

$$\rho : V \to \text{Sym}^e R$$

sending a curve to its reduced intersection scheme with $R \setminus \Omega$; this may not be well-defined everywhere since e may drop along certain closed subschemes of V, but it is in a neighbourhood of $[X]$.

Then we can apply the $e = 0$ case of the Proposition to the fibres of ρ; for a general $\Sigma \in \text{Sym}^e R$, setting $\Omega' = \Omega \cup \text{Supp}\, \Sigma$ one gets that the fibre $\rho^{-1}(\Sigma)$ has dimension at most $-(K_S + R) \cdot \xi + g - 1$. Inequality (4.3.1) follows, and the equality case of the Proposition as well, again applying the $e = 0$ case to the fibres of ρ.

Remark. Note that the above reasoning also proves that, if V is dense in a suitable irreducible component of a log-Severi variety, then the map $\rho : V \to \text{Sym}^e R$ is dominant.

(4.9) Let us now consider the case when X is non-reduced, but still irreducible; we shall show that inequality (4.3.1) holds, and that it is almost always strict; when equality holds in (4.3.1), condition (4.3.2) does not hold. This will also prove the second part of the proposition, since V is not a dense subset of a component of a log-Severi variety when X is non-reduced.

We have to consider $\phi : C \to X$ where X is non-reduced but irreducible, and C may be reducible. We let $m > 1$ be the degree of ϕ, i.e. the sum of the degrees of the various $\phi_i : C_i \to X$. The key is to write the appropriate version of the Riemann–Hurwitz Formula: we have

$$2g - 2 = \deg(\omega_C) \geqslant m \deg(\omega_{X_{\mathrm{red}}}) = m(2q - 2),$$

with q the arithmetic genus of X_{red}. Let h be the geometric genus of X_{red}: we have $h \leqslant q$, so that eventually $g - 1 \geqslant m(h - 1)$.

Then we apply Proposition (4.3) in the reduced case, which we have already proven above, to the reduced curve X_{red} underlying X: it has class $\frac{1}{m}\xi$, so we get

$$\begin{aligned}\dim(V) &\leqslant -(K_S + R) \cdot \frac{1}{m}\xi + h - 1 + \mathrm{Card}\big((X \cap R) \setminus \Omega\big)\\ &\leqslant -(K_S + R) \cdot \frac{1}{m}\xi + \frac{g-1}{m} + \mathrm{Card}\big((X \cap R) \setminus \Omega\big);\end{aligned}$$

recall that $X \cap R$ is considered as a set, so that

$$\mathrm{Card}\big((X \cap R) \setminus \Omega\big) = \mathrm{Card}\big((X_{\mathrm{red}} \cap R) \setminus \Omega\big).$$

Now the required inequality (4.3.1) follows from the basic inequality:

$$(\star) \qquad -(K_S + R) \cdot \frac{1}{m}\xi + \frac{g-1}{m} \leqslant -(K_S + R) \cdot \xi + g - 1.$$

If $g \geqslant 1$, the latter inequality $(\star)$ is straightforward, and always strict, as $-(K_S + R) \cdot \xi > 0$ and $m > 1$. If $g \leqslant 0$ however, one has $-(K_S + R) \cdot \xi \geqslant m$ on the one hand, because $\frac{1}{m}\xi$ is an integral class, and $g \geqslant -m + 1$, with equality holding if and only if C is the disjoint union of m smooth rational curves; thus inequality $(\star)$ holds also in this case, and it is an equality if and only if $-(K_S + R) \cdot \frac{1}{m}\xi = 1$ and C is the disjoint union of m smooth rational curves. When the latter condition is realized, condition (4.3.2) does not hold.

(4.9.1) Example. Let S be the projective plane blown-up at eight general points, with exceptional divisors $E_1, \ldots, E_8$, and take $R = 0$. We consider the class

$$\xi = m(3H - \textstyle\sum_{i=1}^{8} E_i)$$

where H as usual denotes the pull-back to S of the line class on $\mathbf{P}^2$. Let X_0 be an irreducible rational curve in the linear system $|3H - \sum_{i=1}^{8} E_i|$, and consider the family V consisting solely of $X = mX_0$ as a 0-dimensional family of curves of genus $-m + 1$, by means of the morphism $\phi : C \to X$ consisting of m disjoint copies of the normalisation of X_0. Equality holds for V in (4.3.1).

(4.10) It remains to consider the case when X is reducible. Proceeding by induction on the number of irreducible components, we may assume that $X = X_1 \cup X_2$ where X_1 and X_2 move in two families V_1 and V_2 of curves of genera g_1 and g_2 respectively, such that $V \subseteq V_1 \times V_2$ and the Proposition holds for V_1 and V_2. Adding the two corresponding inequalities gives

(4.10.1)

$$\begin{aligned}&\dim(V) \leqslant \dim(V_1) + \dim(V_2)\\ &\qquad = -(K_S + R) \cdot \xi + (g_1 + g_2 - 1) - 1 + \mathrm{Card}\big((X_1 \cap R) \setminus \Omega\big) + \mathrm{Card}\big((X_2 \cap R) \setminus \Omega\big).\end{aligned}$$

If $(X_1 \cap X_2 \cap R) \setminus \Omega$ is empty then this readily gives the required inequality, and the second part of the proposition in the equality case also follows.

So let us consider the case when $(X_1 \cap X_2 \cap R) \setminus \Omega$ is non-empty, and let us assume for simplicity that this set consists of only one point p, the general case being strictly similar. In this case condition (4.3.2) does not hold because p has at least two preimages.

If p is a fixed point of the intersection with R for either one of the two families, say V_1, then it is also a fixed point for the other family: indeed, otherwise, by the generality of X we could perturb X_2 a little so that it does not pass through p, and then p is no longer a point in $X_1 \cap X_2 \cap R$. In this case, applying the proposition to V_1 and V_2 with $\Omega' := \Omega \cup \{p\}$, one gets

$$\dim(V) \leqslant -(K_S + R) \cdot \xi + (g_1 + g_2 - 1) - 1 + \underbrace{\mathrm{Card}\big((X_1 \cap R) \setminus \Omega'\big) + \mathrm{Card}\big((X_2 \cap R) \setminus \Omega'\big)}_{=\mathrm{Card}\big((X \cap R)\setminus\Omega\big) - 1},$$

and the result follows; note that inequality (4.3.1) is strict in this case.

Otherwise p is variable for both V_1 and V_2; in this case V necessarily has codimension at least 1 in $V_1 \times V_2$ (this may be proved for instance as in (4.8) by applying the Proposition to the fibres of the projection $V \to V_1$), and therefore (4.10.1) gives the required inequality (4.3.1). Equality may hold in (4.3.1), but V cannot be dense in a component of a log-Severi variety, since it already has positive codimension in $V_1 \times V_2$.

(4.10.2) Example. Take $S = \mathbf{P}^2$ and R a line, let $V_1 = V_2 = V_0^{[H]}(0,1)(\varnothing)$, and consider V the sub-family of $V_1 \times V_2$ parametrizing the sums of two lines, the intersection point of which lies on R. Then V has dimension 3, and parametrizes curves of genus -1, so that equality holds in (4.3.1) because $\mathrm{Card}(X \cap R) = 1$ for a general membre X of V. It has codimension 1 in $V_1 \times V_2$, which is an irreducible component of $V_{-1}^{[2H]}(0,2)(\varnothing)$. Concretely, on V we are artificially decreasing $\mathrm{Card}(X \cap R)$ by imposing that two contact points with R coincide, and the latter prescription is not of Severi type.

The proof of Proposition (4.3) is now over. □

5 – Examples

5.1 – An example with contact points in special position

In this short subsection I give an example illustrating (i) the effect of having the points in Ω in special position (which is also a theme in Subsection 5.3 below), and (ii) the importance of considering the positivity of $-K_S \cdot C - \deg \phi_* D$ (in the notation of Theorem (1.7)) on all irreducible components of C separately.

(5.1) Example. Let S be the projective plane, R be a smooth cubic, and Ω be cut out on R by another cubic R'. We consider the Severi variety $V_0^4(9,3)(\Omega)$ of rational quartics passing through the 9 intersection points of R and R' (and thus with 3 unassigned additional contact points with R); it has expected dimension

$$-\underbrace{(K_S + R)}_{=0} \cdot \xi + \underbrace{g}_{=0} - 1 + \underbrace{|\beta|}_{=3} = 2.$$

It has a 3-dimensional irreducible component V_{ni} parametrizing reducible quartics $C_1 + C_2$ where C_1 is a cubic in the pencil generated by R and R', and C_2 is a line. The component V_{ni}

thus has dimension larger than expected, and indeed (1.7.0) does not apply in this case because, in the notation of Theorem (1.7),

$$-K_S \cdot C_1 - \deg \phi_* D|_{C_1} = 0.$$

The Severi variety $V_0^4(9,3)(\Omega)$ has another component V_{irr}, which parametrizes irreducible quartics C. This time,

$$-K_S \cdot C - \deg \phi_* D = 3,$$

so (1.7.0) applies and all irreducible components of V_{irr} have dimension 2 as expected. It is however not obvious that there indeed exist irreducible rational quartics passing through Ω (so that V_{irr} is non-empty): we give an argument in the next paragraph.

(5.2) Le me now show that V_{irr} introduced above is non-empty, following a suggestion of Edoardo Sernesi. Consider the family W of reducible quartics $C_1 + C_2$, where C_1 and C_2 are two conics, passing respectively through the first five points of Ω, and through the remaining four points. Since no four points of Ω are aligned, C_1 is fixed and C_2 moves in a pencil, hence W has dimension 1 as expected. Besides,

$$-K_S \cdot C_1 - \deg \phi_* D|_{C_1} = 1 \quad \text{and} \quad -K_S \cdot C_2 - \deg \phi_* D|_{C_2} = 2,$$

so (1.7.0) indeed applies to W.

The fact that the dimension of W equals the expected dimension implies that the nodes of a general member of W may be smoothed independently, i.e., a general member of W may be deformed in such a way that an arbitrary subset of its nodes are preserved while the other are smoothed. Indeed the assumption implies that W is smooth at a general point $[C]$, and the conditions defining its tangent space as a linear subspace of $H^0(C, \mathcal{O}_C(C))$ are independent, hence these conditions may be relaxed independently. This is a standard argument going back to Severi himself, of which a modern account may be found in [16].

The upshot is that one may smooth one and only one node of a general member $C_1 + C_2$ of W, while preserving the condition of passing through the points in Ω: the result is a necessarily irreducible quartic curve with three nodes, passing through Ω, as required.

5.2 – Superabundant log Severi varieties coming from double covers

In this subsection I give examples of superabundant logarithmic Severi varieties, i.e., such that the dimension exceeds the expected dimension: the inequality in (1.7.00) is strict, and (1.7.0) does not apply. These examples live on the projective plane, and come from linear systems on double covers. This is taken from [8]. The examples are given in (5.5) below, after a few recaps on double covers.

(5.3) We shall use the following elementary facts about double covers. Let d be a positive integer, and B be a degree $2d$ curve in $\mathbf{P}^2$. We consider the double cover $\pi : S \to \mathbf{P}^2$ branched over B. Let H be the line class on $\mathbf{P}^2$, and L be its pull-back to S. For all $k \in \mathbf{N}$ we have

$$H^0(S, kL) = \pi^* H^0(\mathbf{P}^2, kH) \oplus \pi^* H^0(\mathbf{P}^2, kH - \tfrac{1}{2}B),$$

which is the isotypical decomposition of $H^0(S, kL)$ as a representation of $\mathbf{Z}/2\mathbf{Z}$. The first summand corresponds to divisors that are double covers of degree k curves in $\mathbf{P}^2$, and the second to divisors that decompose as B (seen as the ramification divisor in S) plus the double cover of a degree $k - d$ curve in $\mathbf{P}^2$.

(5.4) Proposition. *For $k \geqslant d$, the general member C of $|kL|$ is not a double cover of some hypersurface in $\mathbf{P}^2$, the restriction $\pi|_C$ is birational on its image, a degree $2k$ hypersurface $C^\flat$ in $\mathbf{P}^2$ everywhere tangent to B, with a node at every point of a complete intersection Z of type $(k, k-d)$.*

Proof. The divisor C belongs to a unique pencil $\langle A', B+D' \rangle$, with A' and D' the double covers of curves A and D in $\mathbf{P}^2$ of respective degrees k and $k-d$. Thus $C^\flat := \pi(C)$ belongs to the pencil $\langle 2A, B+2D \rangle$, from which it follows that $C^\flat$ is double along $Z := A \cap D$, and touches B doubly along $A \cap B$, which accounts for the whole intersection scheme of $C^\flat$ and B. The base locus of this pencil is the scheme defined by the ideal sheaf $\mathcal{I}_Z^2(\mathcal{I}_A^2 + \mathcal{I}_B)$.

The pull-back $\pi^* C^\flat \in |2kL|$ splits as $C + i(C)$, with i the involution on V associated to π ; it has a double singularity along $Z' := \pi^{-1}(Z)$ and $\pi^{-1}(B \cap A)$, with at each point one local sheet belonging to C and another to $i(C)$. The union $Z' \cup \pi^{-1}(B \cap A)$ is the base locus of the pencil $\langle A', B+D' \rangle$. □

(5.5) Example. We consider the image $V_{B,k}$ in $|2kH|$ of the linear system $|kL|$ on S. It has dimension

$$h^0(S, kL) - 1 = h^0(\mathbf{P}^2, kH) + h^0(\mathbf{P}^2, (k-d)H) - 1,$$

and parametrizes curves of geometric genus

$$g_{k,d} = \tfrac{1}{2}(2k-1)(2k-2) - k(k-d),$$

everywhere tangent to B; the number of contact points is thus $2kd$.

The family of curves $V_{B,k}$ is therefore contained in the log-Severi variety $V^{2k}_{g_{k,d}}(0, [0, 2kd])$ of the pair $(\mathbf{P}^2, B)$, for which the expected dimension is

$$-(K_{\mathbf{P}^2} + B) \cdot 2kH + g_{k,d} - 1 + 2kd = k(k+3-d).$$

By (1.7.0) a component of the Severi variety has the expected dimension if it has an irreducible member and

$$-K_{\mathbf{P}^2} \cdot 2kH - 2kd \geqslant 1 \quad \Longleftrightarrow \quad 2k(3-d) \geqslant 1;$$

we note that the latter inequality holds if and only if $d \leqslant 2$.

It turns out that the dimension of our family $V_{B,k}$ exceeds the expected dimension of the log-Severi variety. Indeed a direct computation shows that

$$\begin{aligned} \dim(V_{B,k}) - \operatorname{expdim}\big(V^{2kH}_{g_{k,d}}(0, [0, 2kd])\big) &= \tfrac{(d-1)(d-2)}{2} \\ &= p_g(S) \end{aligned}$$

(cf. [5, V.22 p.237] for the last equality).

5.3 – Logarithmic $K3$ surfaces

In this subsection I discuss logarithmic Severi varieties of pairs (S, R) with $K_S + R = 0$, in relation with Severi varieties of $K3$ surfaces. This is a central theme of this volume, and as such it will be revisited in various other chapters.

(5.6) $K3$ surfaces. Let us first consider the case of "absolute" $K3$ surfaces: let S be a $K3$ surface, and $R = 0$. In this case Theorem (1.7) is not quite accurate, a (well-known) prominent issue being that the condition in order for (1.7.0) to apply is not verified, and indeed the expected dimension given in (1.7.0) is not the actual dimension.

Suppose S is equipped with a polarisation L of genus p (i.e., $L^2 = 2p - 2$). The expected dimension of V_g^L as defined in (3.3) is $g - 1$, whereas its actual dimension is g, if $0 \leqslant g \leqslant p$ and S is very general, say. Technically, the deformations of $[C] \in V_g^L$ are governed by the invertible sheaf $L|_C \cong \omega_C$, hence the obstruction space is $H^1(C, \omega_C)$ which is 1-dimensional; it turns out however that the equigeneric deformations of C are unobstructed, even though the obstruction space is non-trivial. A conceptual explanation for this is that there exist non-algebraic $K3$ surfaces, which do not carry any curve at all; the expected dimension for Severi varieties on $K3$ surfaces thus becomes accurate if considered in family over all $K3$ surfaces, including the non-algebraic ones. I refer to [11, §4.2] for a detailed account; in the present volume, this phenomenon will be considered again in [XI, subsection 3.1], in the context of Gromov–Witten theory.

We shall now describe some analogous phenomena for $K3$ pairs, by looking at typical examples.

(5.7) Surfaces with canonical curve sections. Let S be $\mathbf{P}^2$, and R be a smooth cubic; note that in this case one has $K_S + R = 0$. Let C be a smooth curve of degree $d \geqslant 4$ on S, and set $\Omega = C \cap R$ (for simplicity we may assume that C and R intersect transversely). The blow-up S' of $\mathbf{P}^2$ at Ω is a smooth surface having a unique anticanonical divisor, namely the proper transform R' of R. The linear system $|C'|$ of the proper transform C' of C gives a birational model of S' in $\mathbf{P}^p$, in which the proper transform of R is contracted, and whose hyperplane sections are the canonical models of the degree d plane curves passing through Ω (we have let p equal $\mathrm{p}(d)$, the genus of smooth plane d-ics).

In many aspects the surface S' behaves like a $K3$ surface. Beware, however, that it may not always be deformed to a $K3$ surface: this is the case if and only if $d \leqslant 6$; if $d > 6$ the elliptic singularity on S' gotten from the contraction of the proper transform of R is not smoothable, see [2].

Surfaces of this kind naturally occur in degenerations of $K3$ surfaces. For instance, surfaces S' as above with $d = 4$ appear in degenerations of smooth quartics in $\mathbf{P}^3$ to the union of a smooth cubic and a plane, see [I, Section 7.2]; there we have seen that the logarithmic Severi variety $V_g^4(12, 0)(\Omega)$ of the pair $(\mathbf{P}^2, R)$ appears in the limit of the Severi varieties of genus g hyperplane sections of smooth quartics.

There are of course many variants of this situation. For instance, Du Val surfaces[5] have been brought in the foreground recently in [1]. In this volume we shall often consider the case when S is a toric surface, and R is a cycle of smooth rational curves, the sum of all toric divisors of S. An emblematic instance of this situation is that of the pair consisting of $\mathbf{P}^2$ and a triangle, endowed with the linear system of plane quartics passing through twelve points cut out as above on the triangle by a quartic; it occurs in the degeneration of quartic $K3$ surfaces in $\mathbf{P}^3$ to a tetrahedron, see [I, 8.2] in which it corresponds to the contribution of a face (see ibid.). Other examples are given in [X, Section **??**].

[5] a genus p Du Val surface is the projective plane blown-up at nine points on a smooth cubic R, endowed with the linear system of the proper transforms of plane curves of degree $3g$ with a g-tuple ordinary point at the first eight blown-up points, and a $(g-1)$-tuple ordinary point at the remaining blown-up point; these curves have geometric genus g, and the linear system has dimension g (note that it has a tenth base point lying on R).

(5.8) Logarithmic $K3$ surfaces. Let $S' \subseteq \mathbf{P}^p$ be as in (5.7). For all $g = 0, \dots, p$, we may view the Severi variety of genus g hyperplane sections of $S' \subseteq \mathbf{P}^p$ as the logarithmic Severi variety $V_p^d(3d, 0)(\Omega)$ of the pair $(\mathbf{P}^2, R)$. Its expected dimension, as defined in (3.3), is $g - 1$, whereas its actual dimension is g; note that once again (1.7.0) does not apply, because the required inequality does not hold. In this case the discrepancy between the actual and expected dimensions comes from the fact that the points in Ω are not general points on R. This is illustrated in Examples (5.8.1) and (5.8.2) below.

It is pleasant to think of the pair $(\mathbf{P}^2, R)$ endowed with a set Ω as a *logarithmic $K3$ surface*, *algebraic* when Ω is cut out on R by a plane curve as above, and *non-algebraic* otherwise, when the points in Ω are in general position. One may prefer to consider that the logarithmic $K3$ surface is the equivalent data of the pair (S', R'), where S' is the blow-up of $\mathbf{P}^2$ at Ω, and R' is the proper transform of R. For a more conceptual definition, as well as a classification, I refer to [13].

In the logarithmic context, the manoeuver of taking non-algebraic $K3$'s into account to adjust the expected dimension takes the following guise: one chooses an arbitrary point $a \in \Omega$, and considers $V_p^d(3d-1, 1)(\Omega - a)$ instead of $V_p^d(3d, 0)(\Omega)$. Statement (1.7.0) applies to the former, which thus has both expected and actual dimension equal to p, and it turns out that the mobile contact point with R remains in fact immobile, so that all members of $V_p^d(3d-1, 1)(\Omega - a)$ automatically pass through the point a as well.

(5.8.1) Example. Let us illustrate this in the concrete case $d = 4$ (then, $p = 3$). The linear system of plane quartics has dimension 14. If we take a set Ω of 12 general points on R, then the Severi variety $V_3^4(12, 0)(\Omega)$ is empty: Ω imposes 12 independent conditions on quartics, the linear system of quartics through Ω is 2-dimensional, and all its members are made of R plus a line. On the other hand if we take Ω the complete intersection of R with a smooth quartic C, then one sees using the restriction exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(1) \to \mathcal{O}_{\mathbf{P}^2}(4) \to \mathcal{O}_R(4) \to 0$$

that the linear system of quartics through Ω is 3-dimensional, generated by C and the net of reducible quartics containing R. If we take a set Ω of 11 general points on R, it imposes 11 independent conditions on quartics, and the linear system of quartics through Ω is 3-dimensional with a 12-th base point on R.

(5.8.2) Example. We may consider the linear system $|2C'|$ in a similar fashion. Seen on $\mathbf{P}^2$, it is the system of plane $(2d)$-ics with a node at each of the $3d$ points of $\Omega = C \cap R$.

Again we shall work this out in the case $d = 4$. One has $(2C')^2 = 4 \cdot (C')^2 = 16 = 2 \cdot 9 - 2$, so the adjunction formula on S', which is essentially the same as on a $K3$ surface, tells us that curves in $|2C'|$ have genus 9. Moreover the Riemann–Roch Formula, which works as on a $K3$ as well, tells us that $|2C'|$ has dimension 9.

On the other hand plane octics have arithmetic genus 21, so an octic with 12 nodes has geometric genus 9, confirming the above computations carried out on S'. The linear system of plane octics has dimension 44. Since a node at a prescribed point is 3 conditions, the expected dimension of a linear system of octics with 12 nodes at prescribed points is $44 - 36 = 8$. When the nodes are imposed at a general set of points Ω on R, this is indeed the correct dimension, and the linear system consists only of curves made of R plus a quintic curve passing (in general simply) through Ω. When Ω is cut out on R by a quartic curve C_1, the linear system has an extra generator (namely, the curve $2C_1$), and thus has one extra dimension.

This may also be verified using a resolution of the ideal sheaf $\mathcal{I}_\Omega^2$, where $\mathcal{I}_\Omega$ is the ideal sheaf of $\Omega \subseteq \mathbf{P}^2$. Let r and f be homogeneous equations of the curves R and C respectively. While for $\mathcal{I}_\Omega$ there is the Koszul resolution, for its square we have the exact sequence

$$0 \to \mathcal{O}(-10) \oplus \mathcal{O}(-11) \xrightarrow{\begin{pmatrix} -f & 0 \\ r & -f \\ 0 & r \end{pmatrix}} \mathcal{O}(-6) \oplus \mathcal{O}(-7) \oplus \mathcal{O}(-8) \xrightarrow{(r^2, rf, f^2)} \mathcal{I}_\Omega^2 \to 0,$$

which gives $h^0(\mathcal{O}_{\mathbf{P}^2}(8) \otimes \mathcal{I}_\Omega^2) = 10$ as required.

This carries over to all systems $|kC'|$, to the effect that the condition of having a k-uple point at all $3d$ points of $\Omega = C \cap R$ imposes one less condition on plane d-ics than if the $3d$ points of Ω were in general position. We leave this to the reader.

(5.9) Severi varieties of curves with full contact. Logarithmic Severi varieties of logarithmic $K3$ surfaces (S, R) parametrizing curves having one point of contact with high order with R have attracted the attention for several reasons. A first reason is the possibility, if (S, R) comes from a degeneration of $K3$ surfaces, to deform such a logarithmic Severi variety (roughly speaking) to a Severi variety on the nearby $K3$ surfaces, the contact point of high order m deforming to $m - 1$ nodes: this is the central theme of the present volume, in particular see [VIII]. Besides, this possibility is not specific to $K3$ surfaces.

Another reason is the search for $\mathbf{A}^1$ curves (i.e., curves isomorphic to the affine line, in other words affine rational curves) on logarithmic $K3$ surfaces, which is the analogue of the search for rational curves on $K3$ surfaces. Let me use an example to clarify this problem. Consider the projective plane together with a smooth cubic curve R (and an empty set of points Ω). The problem is to find $\mathbf{A}^1$ curves lying in the open surface $\mathbf{P}^2 \setminus R$. To this effect it is relevant to consider the logarithmic Severi varieties $V_0^d(0, [0, \dots, 0, 3d])$ of the pair $(\mathbf{P}^2, R)$. They have expected dimension 0, and of course for any such curve the contact point with R must be one of the $(3d)^2$ order $3d$ torsion points of R. The problem of the existence of $\mathbf{A}^1$ curves thus corresponds to that of the non-emptiness of these logarithmic Severi varieties. The characterization of the log-$K3$ surfaces for which there exists infinitely many $\mathbf{A}^1$ curves has been carried out in [7]. The interested reader may also consult [9] for other non-emptiness results, in particular [9, Prop. 3.12].

Logarithmic Severi varieties with one point of full contact have also been related in [11] with the tropical vertex group. Elements of this group are formal families of symplectomorphisms of the 2-dimensional algebraic torus $(\mathbf{C}^*)^2$. The relevant Severi varieties are the following: let S be a toric surface, and let R be the sum of all toric divisors $R_1, \dots, R_n, R_{\text{out}}$ (R_{out} is simply one of the toric divisors, to which we want to give a special role); one considers logarithmic Severi varieties of the pair (S, R) of the form

$$V_0^\xi\big((\alpha_1, \dots, \alpha_n, 0), (0, \dots, 0, [0, \dots, 0, m])\big)(\Omega_1, \dots, \Omega_n, \varnothing),$$

which parametrize rational curves having fixed intersection schemes with $R_1, \dots, R_n$, and one contact point of maximal order $m = \xi \cdot R_{\text{out}}$ with R_{out}. These logarithmic Severi varieties have expected and actual dimension 0, and the number of points they are made of may be interpreted as Gromov–Witten invariants. The main result of [11] is that these numbers are structure constants of the tropical vertex group; they are in fact determined by ordered product factorizations of commutators in the tropical vertex group, namely they appear as the coefficients of the logarithms of the factors.

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Lecture IV
Deformations of an m-tacnode into $m-1$ nodes

by Margherita Lelli-Chiesa

This text is dedicated to a somewhat technical-looking statement (Theorem (1.2)) which is the cornerstone of many results in this volume: it is the key to understand the phenomenon of, loosely speaking, $(m-1)$-nodal curves degenerating to m-tacnodal curves within a degeneration of smooth surfaces to the transverse union of two surfaces. About this phenomenon see, in particular, [I, VI, B].

1 – Statement of results

Let C be a planar curve. A point $P \in C$ is called an m-tacnode if the local equation of C at P has the form $y(y+x^m) = 0$, where (x, y) are local coordinates centered at P. The Jacobian ideal of C at P is $J = (2y + x^m, yx^{m-1})$ and

$$\mathcal{O}_{\mathbf{A}^2}/J = \langle y, xy, \ldots, x^{m-2}y, 1, x, \ldots, x^{m-1}\rangle.$$

We denote by $\pi : S \to \Delta$ the versal deformation space of $P \in C$, where $\Delta \simeq \mathbf{A}^{2m-1}_{\alpha_0,\ldots,\alpha_{m-2},\beta_0,\ldots,\beta_{m-1}}$, and $S \subseteq \Delta \times \mathbf{A}^2_{x,y}$ is defined by the equation

$$y^2 + yx^m + \alpha_{m-2}yx^{m-2} + \ldots + \alpha_1 yx + \alpha_0 y + \beta_{m-1}x^{m-1} + \ldots + \beta_1 x + \beta_0 = 0, \tag{1.0.1}$$

see [II, Section 5]. We set

$$\Delta_m := \{(\underline{\alpha}, \underline{\beta}) \in \Delta \,|\, \pi^{-1}(\underline{\alpha}, \underline{\beta}) \text{ has } m \text{ nodes}\}, \tag{1.0.2}$$

$$\Delta_{m-1} := \{(\underline{\alpha}, \underline{\beta}) \in \Delta \,|\, \pi^{-1}(\underline{\alpha}, \underline{\beta}) \text{ has } m-1 \text{ nodes}\}. \tag{1.0.3}$$

(1.1) Proposition. (1.1.1) *The locus Δ_m is smooth of dimension $m-1$, and it is defined in Δ by the equations $\beta_0 = \beta_1 = \ldots = \beta_{m-1} = 0$.*
(1.1.2) *The locus Δ_{m-1} has dimension m, and has m local sheets at a general point of Δ_m.*

Proof. The equation of $\pi^{-1}(\underline{\alpha}, \underline{\beta})$ has degree two in y and its discriminant is given by

$$\delta_{\underline{\alpha},\underline{\beta}}(x) = (x^m + \alpha_{m-2}x^{m-2} + \ldots + \alpha_1 x + \alpha_0)^2 - 4(\beta_{m-1}x^{m-1} + \ldots + \beta_1 x + \beta_0).$$

If V denotes the space of monic polynomials of degree $2m$ in x with no degree $2m-1$ term, the regular map $\delta : \Delta \to V$ mapping a point $(\underline{\alpha}, \underline{\beta})$ to its discriminant $\delta_{\underline{\alpha},\underline{\beta}}(x)$ is an isomorphism;

indeed, $\underline{\alpha}$ and $\underline{\beta}$ can be easily expressed in terms of the coefficients of $\delta_{\underline{\alpha},\underline{\beta}}(x)$. We remark that, for fixed $(\underline{\alpha},\underline{\beta})$, the locus $\pi^{-1}(\underline{\alpha},\underline{\beta})$ is a double cover of the x-axis branched along the vanishing locus of $\delta_{\underline{\alpha},\underline{\beta}}(x)$, which is a divisor of degree $2m$.

A point $(\underline{\alpha},\underline{\beta})$ lies in Δ_m whenever $\delta_{\underline{\alpha},\underline{\beta}}(x)$ has m double roots, that is, if and only if $\delta_{\underline{\alpha},\underline{\beta}}(x)$ is a square, equivalently if and only if $\beta_0 = \beta_1 = \ldots = \beta_{m-1} = 0$. This proves (1.1.1).

Analogously, $(\underline{\alpha},\underline{\beta})$ lies in Δ_{m-1} exactly when $\delta_{\underline{\alpha},\underline{\beta}}(x)$ has $m-1$ double roots, that is, we can rewrite the discriminant as

$$\delta_{\underline{\alpha},\underline{\beta}}(x) = \left(\prod_{i=1}^{m-1}(x-a_i)^2\right)\cdot(x^2+bx+c), \tag{1.1.3}$$

where in fact $b = 2\sum_{i=1}^{m-1} a_i$ because the degree $2m-1$ term of $\delta_{\underline{\alpha},\underline{\beta}}(x)$ vanishes. Hence, Δ_{m-1} has dimension m, and the m local sheets at a general point of Δ_m as in (1.1.2) reflect the possibilities for choosing $m-1$ double roots out of m. □

Our main goal is to prove the following theorem.

(1.2) Theorem ([2, Lemma 2.8]). *Let $\Lambda \subseteq \Delta$ be a smooth m-dimensional variety such that:*
(i) Λ contains Δ_m;
(ii) the tangent space $T_{\underline{0}}\Lambda$ of Λ at $\underline{0}$ is not contained in the hyperplane H defined by the equation $\beta_0 = 0$.
Then, we have $\Lambda \cap \Delta_{m-1} = \Delta_m \cup \Psi$, where Ψ is a smooth curve intersecting Δ_m at $\underline{0}$ with multiplicity m.

Note that Δ_{m-1} and Λ both have codimension $m-1$ in Δ, which has dimension $2m-1$, so that the expected dimension for their intersection is 1. The theorem thus states that if Λ contains Δ_m, under a suitable transversality assumption, namely (ii), the intersection $\Lambda \cap \Delta_{m-1}$ residual to the obvious superabundant component Δ_m has the expected dimension.

The reader may also consult [A] for complements and detailed examples.

2 – Proof of the main theorem in a model case

We first prove Theorem (1.2) in the special case where $\Lambda = \Lambda_0 = \{\beta_1 = \ldots = \beta_{m-1} = 0\}$.

Equation (1.0.1) of S restricted to Λ is:

$$y^2 + yx^m + \alpha_{m-2}yx^{m-2} + \ldots + \alpha_1 yx + \alpha_0 y + \beta_0 = 0, \tag{2.0.1}$$

and the discriminant of a point $(\underline{\alpha},\underline{\beta}) \in \Lambda$ has the form:

$$\delta_{\underline{\alpha},\underline{\beta}}(x) = (x^m + \alpha_{m-2}x^{m-2} + \ldots + \alpha_1 x + \alpha_0)^2 - 4\beta_0. \tag{2.0.2}$$

If $\beta_0 = 0$, then $(\underline{\alpha},\underline{\beta})$ lies in Δ_m. Therefore, $\Lambda \cap \Delta_{m-1}$ contains Δ_m with multiplicity m, since by Proposition (1.1) Δ_{m-1} has multiplicity m along Δ_m.

From now on, we assume $\beta_0 \neq 0$, set $\nu(x) := x^m + \alpha_{m-2}x^{m-2} + \ldots + \alpha_1 x + \alpha_0$ and rewrite (2.0.2) as:

$$\delta_{\underline{\alpha},\underline{\beta}}(x) = (\nu(x) - 2\sqrt{\beta_0})(\nu(x) + 2\sqrt{\beta_0}). \tag{2.0.3}$$

Note that the polynomials $\nu(x) - 2\sqrt{b}$ and $\nu(x) + 2\sqrt{b}$ have no common roots since $\beta_0 \neq 0$. We impose that $\delta_{\underline{\alpha},\underline{\beta}}(x)$ has $m-1$ double roots, taking advantage of the following lemma.

(2.1) Lemma. *Let* $\gamma := 2\sqrt{\beta_0}$ *with* $\beta_0 \neq 0$. *Then, for every integer* $m \geqslant 2$ *there exists a polynomial*

$$\nu_\gamma(x) := x^m + c_{m-2}(\gamma)x^{m-2} + \ldots + c_1(\gamma)x + c_0(\gamma)$$

in x *with complex coefficients* $c_i(\gamma)$ *such that* $c_{m-2}(\gamma) \neq 0$ *and the following are satisfied:*

(a) if $m = 2l+1$ *is odd, then both* $\nu_\gamma(x) + \gamma$ *and* $\nu_\gamma(x) - \gamma$ *have* l *double roots;*

(b) if $m = 2l$ *is even, then* $\nu_\gamma(x) + \gamma$ *(respectively,* $\nu_\gamma(x) - \gamma$*) has* l *(resp.,* $l-1$*) double roots.*

Furthermore, $\nu_\gamma(x)$ *is unique up to replacing it with* $\nu_\gamma(\xi x)$ *for some* m*-th root* ξ *of unity, and it is odd (respectively, even) if* m *is.*

Taking the lemma for granted, we set $m := 2l + \varepsilon$ with $\varepsilon \in \{0,1\}$ depending on the parity of m and consider the polynomial

$$\nu_1(x) = x^m + c_{m-2}(1)x^{m-2} + c_{m-4}(1)x^{m-4} + \ldots + c_\varepsilon(1)x^\varepsilon$$

obtained for $\gamma = 1$. For any other $\gamma \in \mathbf{C}^*$, by setting $\gamma := u^m$, we get $\nu_\gamma(x) = u^m \nu_1\left(\frac{x}{u}\right)$. The coefficients of $\nu_\gamma(x)$ are thus obtained from those of $\nu_1(x)$ in the following way:

$$c_{2h+\varepsilon}(\gamma) = c_{2h+\varepsilon}(u) = u^{2l-2h}c_{2h+\varepsilon}(1), \text{ for } 0 \leqslant h \leqslant l-1.$$

By the change of variable $t := u^2$, the equations

$$c_{2h+\varepsilon}(t) = t^{l-h}c_{2h+\varepsilon}(1), \text{ for } 0 \leqslant h \leqslant l-1$$

parametrize a curve Ψ_0 in $\Lambda \cap \Delta_{m-1}$. This curve is smooth because $c_{m-2}(1) \neq 0$, and its contact order with Δ_m at $\underline{0}$ is m because $\beta_0 = \frac{t^m}{4}$. Therefore, it only remains to prove the lemma.

Proof of Lemma (2.1). We first prove the existence of a degree m covering $f : \mathbf{P}^1 \to \mathbf{P}^1$ mapping ∞ to ∞ and with $\{\infty, \gamma, -\gamma\}$ as branch locus; we also require f to be totally ramified at ∞ and the number of ramification points in the fiber $f^{-1}(-\gamma)$ (respectively, in $f^{-1}(\gamma)$) to coincide with the number of double roots we are requiring for the polynomial $\nu_\gamma(x) + \gamma$ (respectively, $\nu_\gamma(x) - \gamma$).

Assume that $m = 2l+1$ is odd. By Riemann's Existence Theorem, the existence of f is equivalent to the existence of two permutations τ, σ in the symmetric group S_m, each one the product of l disjoint transpositions, such that $\tau \cdot \sigma$ is cyclic of order m in S_m. The permutations $\tau := (1\,2)(3\,4)\cdots(2l-1\,2l)$ and $\sigma := (2\,3)(4\,5)\cdots(2l\,2l+1)$ satisfy these requirements and they are the only elements of S_m doing that, up to conjugation. As a consequence, f exists and is unique up to automorphisms of the domain fixing ∞. The even case is analogous.

Since $f(\infty) = \infty$, the covering f defines a polynomial $\nu_\gamma(x)$ satisfying (a) or (b) respectively. By acting with an automorphism of $\mathbf{P}^1$ fixing ∞ (i.e., composing $\nu_\gamma(x)$ with a polynomial of degree 1), we can assume $\nu_\gamma(x)$ to be monic and with no degree $m-1$ term. The polynomial $\nu_\gamma(x)$ is then unique up to replacing it with $\nu_\gamma(\xi x)$ for some m-th root ξ of unity. By Proposition (A.1) in the Appendix, we can therefore assume that $\nu_1(x)$ coincides with the Chebyshev polynomial of the first kind $T_m(x)$; in particular, it is even or odd depending on the parity of m, and the coefficient $c_{m-2}(1)$ is nonzero, as one can easily check using (A.0.1) in the Appendix. The same properties hold for $\nu_\gamma(x)$ for any $\gamma = u^m \in \mathbf{C}^*$ because $\nu_\gamma(x) = u^m \nu_1\left(\frac{x}{u}\right)$.

□

3 – Proof of the main theorem in general

We now prove Theorem (1.2) in full generality.

We consider the blow-up $\tau : \widetilde{\Delta} \to \Delta$ of Δ along Δ_m and denote by $Z := \tau^{-1}(\Delta_m)$ the exceptional divisor and by $\Phi := \tau^{-1}(\underline{0})$ the inverse image of $\underline{0}$. Then $\Phi \simeq \mathbf{P}^{m-1}_{\beta_0,\ldots,\beta_{m-1}}$ can be identified with the space of polynomials of degree at most $m-1$ in x modulo scalars. The open subset $\Phi \supset \Phi_0 := \{\beta_0 \neq 0\}$ parametrizes polynomials that do not vanish in $x=0$ and the point $Q = [1 : 0 : \ldots : 0] \in \Phi_0$ corresponds to constants. The proof of Theorem (1.2) in general is based on the description of the strict transform $\widetilde{\Delta_{m-1}}$ of Δ_{m-1} in $\widetilde{\Delta}$ and proceeds by steps.

Step 1. *The point Q lies in $\widetilde{\Delta_{m-1}}$.*

Let $\widetilde{\Lambda_0}$ be the strict transform of the linear space $\Lambda_0 = \{\beta_1 = \ldots = \beta_{m-1} = 0\}$ under τ. We have already shown that $\Lambda_0 \cap \Delta_{m-1} = \Delta_m \cup \Psi_0$ as in Theorem (1.2). Since $\widetilde{\Lambda_0} \cap \Phi = \{Q\}$, then Q lies in the strict transform $\widetilde{\Psi_0}$ of the curve Ψ_0 and hence also in $\widetilde{\Delta_{m-1}}$.

Step 2. *The locus Φ is an irreducible component of $\widetilde{\Delta_{m-1}} \cap Z$ and is the only component containing Q.*

We identify Δ_m with the space of monic polynomials of degree m in x with no degree $m-1$ term. Let $\alpha(t) \subseteq \Delta_m$ be any arc such that $\alpha(0) = \underline{0}$. As t goes to 0, all the roots of the polynomial corresponding to $\alpha(t)$ tend to $x=0$ and hence the limit of $\tau^{-1}(\alpha(t)) \cap \widetilde{\Delta_{m-1}}$ is not contained in Φ_0. As a consequence, any component of $\widetilde{\Delta_{m-1}} \cap Z$ containing Q is contained in Φ, and thus coincides with it by dimensional reason.

Step 3. *The strict transform $\widetilde{\Delta_{m-1}}$ is smooth at Q.*

The previous step implies that the only irreducible component of $\widetilde{\Lambda_0} \cap \widetilde{\Delta_{m-1}}$ containing Q is the curve $\widetilde{\Psi_0}$. In particular, the intersection is proper in a neighborhood of Q (indeed, $\dim \widetilde{\Lambda_0} = \dim \widetilde{\Delta_{m-1}} = m$, and $\dim \widetilde{\Delta} = 2m-1$); smoothness of $\widetilde{\Psi_0}$ thus yields smoothness of $\widetilde{\Delta_{m-1}}$ at Q.

Step 4. *The strict transform $\widetilde{\Delta_{m-1}}$ is smooth at every point of Φ_0.*

For every $c \in \mathbf{C}^*$, let σ_c be the automorphism of Δ such that $\sigma_c(\alpha_i) = c^{m-i}\alpha_i$ for $0 \leqslant i \leqslant m-2$ and $\sigma_c(\beta_j) = c^{2m-j}\beta_j$ for $0 \leqslant j \leqslant m-1$. This defines an action of the multiplicative group $\mathbf{C}^*$ on Δ, under which Δ_m is clearly invariant. Since $\delta_{\sigma_c(\underline{\alpha}),\sigma_c(\underline{\beta})}(x) = c^{2m}\delta_{\underline{\alpha},\underline{\beta}}\left(\frac{x}{c}\right)$, then Δ_{m-1} is invariant, too. As a consequence, the action of $\mathbf{C}^*$ on Δ lifts to an action on $\widetilde{\Delta}$ preserving $\widetilde{\Delta_{m-1}}$. Furthermore, $\underline{0}$ is a fixed point lying in the closure of any orbit and hence the lifted action also preserves Φ. In particular, given $c \in \mathbf{C}^*$, we obtain an automorphism $\widetilde{\sigma_c}$ of Φ mapping a point P of homogeneous coordinates $[\beta_0 : \beta_1 : \ldots : \beta_{m-1}]$ to

$$\widetilde{\sigma_c}(P) = [c^{2m}\beta_0 : c^{2m-1}\beta_1 : \ldots : c^{m+1}\beta_{m-1}] = [\beta_0 : c^{-1}\beta_1 : \ldots : c^{-(m-1)}\beta_{m-1}].$$

Since the limit for c^{-1} going to 0 of $\widetilde{\sigma_c}(P)$ is Q as soon as $\beta_0 \neq 0$, this shows that Q lies in the closure of any orbit of $\widetilde{\Delta_{m-1}}$ intersecting Φ_0. The statement thus follows from Step 3.

Step 5. *The intersection multiplicity of $\widetilde{\Delta_{m-1}}$ and Z along Φ is m.*

Since the intersection multiplicity of Ψ_0 and Δ_m at $\underline{0}$ is m, then the same holds for that of $\widetilde{\Psi_0}$ and Z at Q. On the other hand, the intersection $\widetilde{\Psi_0} \cap Z$ is contained in $\widetilde{\Delta_{m-1}} \cap Z \cap \widetilde{\Lambda_0}$, and the intersection multiplicity at Q of Φ (which is the only component of $\widetilde{\Delta_{m-1}} \cap Z$ containing Q by Step 2) with $\widetilde{\Lambda_0}$ is 1, because $\widetilde{\Lambda_0} \cap Z$ is a section of $\tau|_Z : Z \to \Delta_m$. The statement follows.

Step 6. *Conclusion.*

Let $\Delta_m \subseteq \Lambda \subseteq \Delta$ be as in the hypotheses. Since $\widetilde{\Lambda} \cap Z$ is a section of $\tau|_Z : Z \to \Delta_m$, then the intersection $\widetilde{\Lambda} \cap \Phi$ is a point R with multiplicity one. The assumption $T_{\underline{0}}\Lambda \not\subseteq H = \{\beta_0 = 0\}$ ensures that R lies in Φ_0 and hence $\widetilde{\Delta_{m-1}}$ is smooth at R by Step 4. The m-dimensional tangent space $T_R\widetilde{\Delta_{m-1}}$ contains $T_R\Phi$, that has dimension $m-1$. Since $T_R\widetilde{\Lambda} \cap T_R\Phi = \{0\}$ and $\dim T_R\widetilde{\Lambda} = m$, then $T_R\widetilde{\Delta_{m-1}}$ and $T_R\widetilde{\Lambda}$ generate the $(2m-1)$-dimensional tangent space $T_R\widetilde{\Delta}$. Equivalently, $\widetilde{\Delta_{m-1}}$ and $\widetilde{\Lambda}$ intersect transversally along a smooth curve $\widetilde{\Psi}$ in a neighborhood of R. Moreover, the curve $\widetilde{\Psi}$ is not tangent at Φ in R (again because $T_R\widetilde{\Lambda} \cap T_R\Phi = \{0\}$) and its intersection multiplicity with Z at Q is m by Step 5. Therefore, $\Psi := \tau(\widetilde{\Psi})$ is a smooth curve having intersection multiplicity m with Δ_m at $\underline{0}$.

A – Appendix: Chebyshev polynomials

There exist four families of Chebyshev polynomials, T_m, U_m, V_m, W_m — called of the first, second, third and fourth kind — each satisfying

(A.0.1) $$P_0(x) = 1, \quad \text{and} \quad \forall m \geqslant 1 : \quad P_{m+1}(x) = 2xP_m(x) - P_{m-1}(x).$$

They may be defined by the following choice of initial conditions:

(A.0.2) $$T_1(x) = x,$$
(A.0.3) $$U_1(x) = 2x,$$
(A.0.4) $$V_1(x) = 2x - 1,$$
(A.0.5) $$W_1(x) = 2x + 1.$$

They are the unique polynomials satisfying

(A.0.6) $$T_m(\cos\theta) = \cos(m\theta),$$
(A.0.7) $$U_m(\cos\theta) = \frac{\sin((m+1)\theta)}{\sin\theta},$$
(A.0.8) $$V_m(\cos\theta) = \frac{\cos((m+1/2)\theta)}{\cos(\theta/2)},$$
(A.0.9) $$W_m(\cos\theta) = \frac{\sin((m+1/2)\theta)}{\sin(\theta/2)}$$

for all $\theta \in \mathbf{R}$. The Chebyshev polynomials of the first kind can be explicitly written down as

$$T_m(x) = \frac{1}{2}\Big[\big(x + \sqrt{x^2-1}\big)^m + \big(x - \sqrt{x^2-1}\big)^m\Big],$$

and are even (respectively, odd) whenever m is even (respectively, odd). Polynomials of the third and fourth kind satisfy $W_m(x) = (-1)^m V_m(x)$.

These are the first few values of Chebyshev polynomials:

$$\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_2(x) &= 2x^2 - 1, \\
T_3(x) &= 4x^3 - 3x, \\
T_4(x) &= 8x^4 - 8x^2 + 1, \dots
\end{aligned}$$

$$\begin{aligned}
U_0(x) &= 1, \\
U_1(x) &= 2x, \\
U_2(x) &= 4x^2 - 1, \\
U_3(x) &= 8x^3 - 4x, \\
U_4(x) &= 16x^4 - 12x^2 + 1, \dots
\end{aligned}$$

$$\begin{aligned}
V_0(x) &= 1, \\
V_1(x) &= 2x - 1, \\
V_2(x) &= 4x^2 - 2x - 1, \\
V_3(x) &= 8x^3 - 4x^2 - 4x + 1, \\
V_4(x) &= 16x^4 - 8x^3 - 12x^2 + 4x + 1, \dots
\end{aligned}$$

$$\begin{aligned}
W_0(x) &= 1, \\
W_1(x) &= 2x + 1, \\
W_2(x) &= 4x^2 + 2x - 1, \\
W_3(x) &= 8x^3 + 4x^2 - 4x - 1, \\
W_4(x) &= 16x^4 + 8x^3 - 12x^2 - 4x + 1 \dots
\end{aligned}$$

The next result summarizes some properties of Chebyshev polynomials.

(A.1) Proposition. *If $m = 2l$ is even, then*

(a1) $T_m(x) + 1 = 2(T_l(x))^2$,

(a2) $T_m(x) - 1 = 2(x^2 - 1)(U_l(x))^2$;

in particular, the polynomial $T_m(x) + 1$ is a perfect square, while $T_m(x) - 1$ is the product of a perfect square by a degree two polynomial.

If instead $m = 2l + 1$ is odd, then:

(b1) $T_m(x) + 1 = (x + 1)V_l(x)^2$,

(b2) $T_m(x) - 1 = (x - 1)W_l(x)^2$;

in particular, both $T_m(x) + 1$ and $T_m(x) - 1$ are the product of a perfect square by a linear polynomial.

Proof. Set $x = \cos\theta$. If $m = 2l$ is even, then $T_m(x) + 1 = \cos(2l\theta) + 1 = 2\cos^2(l\theta) = 2(T_l(x))^2$. Similarly,

$$T_m(x) - 1 = \cos(2l\theta) - 1 = -2\sin^2(l\theta) = -2\sin^2\theta \cdot \frac{\sin^2(l\theta)}{\sin^2\theta} = 2(x^2 - 1)(U_l(x))^2.$$

If instead $m = 2l + 1$ is odd, one has

$$T_m(x) + 1 = \cos(m\theta) + 1 = 2\cos^2(m\theta/2) = 2\cos^2(\theta/2) \cdot \frac{\cos^2(m\theta/2)}{\cos^2(\theta/2)} = (x+1)(V_l(x))^2,$$

and

$$T_m(x) - 1 = \cos(m\theta) - 1 = -2\sin^2(m\theta/2) = -2\sin^2(\theta/2) \cdot \frac{\sin^2(m\theta/2)}{\sin^2(\theta/2)} = (x-1)(W_l(x))^2.$$

□

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Lecture V
Products of deformation spaces of tacnodes

by Francesco Bastianelli

1 – Introduction

This lecture retraces [1, Section 4.2], and concerns Caporaso and Harris' description of products of deformation spaces of higher-order tacnodes. Of course, this description relies on the analysis included in [IV] about deformations of a single tacnode. The main result of this lecture shall be involved in the proof of [VI, **??**], which is [1, Theorem 1.3], in order to describe the local geometry of hyperplane sections of generalized Severi varieties $V^{d,\delta}(\alpha,\beta)$, around a point $[X_0]$ parameterizing a reducible curve $X_0 = X \cup L$ endowed with some tacnodes of orders m_j, each deforming to $m_j - 1$ nodes on the curves parameterized nearby.

We consider a sequence $\underline{m} := (m_1, m_2, \ldots, m_n)$ of integers $m_j \geqslant 2$, and we set

$$\lambda := \operatorname{lcm}\{m_j\}, \qquad \mu := \prod_j m_j, \qquad \kappa := \frac{\mu}{\lambda},$$

where lcm denotes the least common multiple.
For any $1 \leqslant j \leqslant n$, let (C_j, p_j) be a tacnode of order m_j, and let $\pi_j\colon \mathcal{S}_j \longrightarrow \Delta_j$ be the corresponding versal deformation family described in [IV]. Therefore, $\Delta_j \cong \mathbb{A}^{2m_j-1}$ is an affine space with coordinates $\left(\underline{a}_j, \underline{b}_j\right) = \left(a_{j,m_j-2}, \ldots, a_{j,0}, b_{j,m_j-1}, \ldots, b_{j,0}\right)$, the subscheme $\mathcal{S}_j \subseteq \mathbb{A}^2 \times \Delta_j$ has equation

$$y^2 + \left(x^{m_j} + a_{j,m_j-2}x^{m_j-2} \cdots + a_{j,0}\right) y + b_{j,m_j-1}x^{m_j-1} + \cdots + b_{j,0} = 0,$$

and $\pi_j\colon \mathcal{S}_j \longrightarrow \Delta_j$ is the second projection. Moreover, we introduce the subloci Δ_{j,m_j} and Δ_{j,m_j-1} of Δ_j as

$$\Delta_{j,m_j} := \overline{\left\{\left(\underline{a}_j, \underline{b}_j\right) \in \Delta_j \,\middle|\, \pi_j^{-1}\left(\underline{a}_j, \underline{b}_j\right) \text{ is a curve having } m_j \text{ nodes}\right\}}$$

and

$$\Delta_{j,m_j-1} := \overline{\left\{\left(\underline{a}_j, \underline{b}_j\right) \in \Delta_j \,\middle|\, \pi_j^{-1}\left(\underline{a}_j, \underline{b}_j\right) \text{ is a curve having } m_j - 1 \text{ nodes}\right\}}.$$

We recall that $\Delta_{j,m_j} \cong \mathbb{A}^{m_j-1}$ is the linear (m_j-1)-dimensional subspace given by the equations $b_{j,m_j-1} = \cdots = b_{j,0} = 0$, and it is the locus over which the fibers $\pi_j^{-1}\left(\underline{a}_j, \underline{b}_j\right)$ are reducible plane curves, whereas Δ_{j,m_j-1} is an m_j-dimensional subvariety containing Δ_{m_j}.

Then we define

$$\Delta := \Delta_1 \times \cdots \times \Delta_n,$$

$$\Delta_{\underline{m}} := \Delta_{1,m_1} \times \cdots \times \Delta_{n,m_n},$$

and

$$\Delta_{\underline{m}-1} := \Delta_{1,m_1-1} \times \cdots \times \Delta_{n,m_n-1},$$

so that $\Delta \cong \mathbb{A}^{\Sigma(2m_j-1)}$ is an affine space endowed with coordinates $(\underline{a}_1, \underline{b}_1, \ldots, \underline{a}_n, \underline{b}_n)$, the linear subspace $\Delta_{\underline{m}} \cong \mathbb{A}^{\Sigma(m_j-1)}$ is given by equations $\{b_{j,i} = 0,\ 1 \leqslant j \leqslant n,\ 0 \leqslant i \leqslant m_j - 1\}$, and $\Delta_{\underline{m}-1}$ is a subvariety of dimension $\sum m_j$ containing $\Delta_{\underline{m}}$. Finally, we set

$$H := \bigcup_j \{b_{j,0} = 0\},$$

which is a union of hyperplanes of Δ.

We now state the main result of this lecture (cf. [1, Lemma 4.3]). Along the same lines as [IV, (1.2)] (cf. [1, Lemma 4.1]), it describes the local intersection at the origin $\underline{0} \in \Delta$ between $\Delta_{\underline{m}-1}$ and varieties $W \subseteq \Delta$ containing $\Delta_{\underline{m}}$ as a subvariety of codimension 1 and satisfying a suitable transversality assumption.

(1.1) Lemma. *Let $W \subseteq \Delta$ be a smooth subvariety of dimension $\sum(m_j - 1) + 1$ containing $\Delta_{\underline{m}}$, and such that the tangent space $T_{\underline{0}}W$ at the origin $\underline{0} \in \Delta$ is not contained in H. Then, in an étale neighborhood of the origin,*

$$W \cap \Delta_{\underline{m}-1} = \Delta_{\underline{m}} \cup \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_\kappa,$$

where $\Gamma_1, \ldots, \Gamma_\kappa \subseteq W$ are distinct reduced unibranched curves such that each Γ_α has intersection multiplicity $\left(\Gamma_\alpha \cdot \Delta_{\underline{m}}\right)_{\underline{0}} = \lambda$ with $\Delta_{\underline{m}}$ at the origin, and the origin is a point of multiplicity $\mathrm{mult}_{\underline{0}}(\Gamma_\alpha) = \frac{\lambda}{\max_j\{m_j\}}$ of Γ_α.

Note that $\Delta_{\underline{m}-1}$ has codimension $\sum(m_j - 1)$ in Δ, so the expected dimension of the intersection $W \cap \Delta_{\underline{m}-1}$ is 1.

(1.2) Remark. (i) The intersection multiplicity at $\underline{0} \in \Delta$ between $\Delta_{\underline{m}}$ and the reducible curve $\Gamma := \Gamma_1 \cup \cdots \cup \Gamma_\kappa$ is $\left(\Gamma \cdot \Delta_{\underline{m}}\right)_{\underline{0}} = \kappa\lambda = \mu$, in analogy with the case of a single tacnode.

(ii) If $\lambda = m_j$ for some j, then all the curves Γ_α are smooth.

Before proving Lemma (1.1), we state another result which shall turn out to be equivalent to it. We denote by

$$\widetilde{\Delta} := \mathrm{Bl}_{\Delta_{\underline{m}}}\Delta \xrightarrow{\pi} \Delta$$

the blow-up of Δ along $\Delta_{\underline{m}}$. Let $E \subseteq \widetilde{\Delta}$ be the exceptional divisor, and let $F := \pi^{-1}(\underline{0}) \subseteq E$ be the fibre over the origin $\underline{0} \in \Delta$. Finally, let $\widetilde{\Delta}_{\underline{m}-1}$ and $\widetilde{H}$ denote the proper transforms of $\Delta_{\underline{m}-1}$ and H, respectively. Then the following holds (cf. [1, Lemma 4.4]).

(1.3) Lemma. *The intersection $\widetilde{\Delta}_{\underline{m}-1} \cap E$ contains F as a component of multiplicity μ. Moreover, in an étale neighborhood of any point $p \in F$ not contained in $\widetilde{H}$, the variety $\widetilde{\Delta}_{\underline{m}-1}$ consists of κ reduced branches, each having multiplicity $\frac{\lambda}{\max_j\{m_j\}}$ along F, intersection number λ with E along F, and tangent cone at p supported on a linear space contained in E.*

2 – Proofs

In analogy with the argument used to deduce [1, Lemma 4.1], the proof of Lemma (1.1) proceeds in three steps. The first one consists of proving the assertion when W is assumed to be a linear space. Then we use it to achieve Lemma (1.3). Finally, we conclude by deducing Lemma (1.1) for an arbitrary subvariety W satisfying the hypothesis of the statement.

(2.1) Proof of Lemma (1.1) with linear $W \subseteq \Delta$. We assume that $W \cong \mathbb{A}^{\Sigma(m_j-1)+1}$ is a linear subspace of Δ such that $\Delta_{\underline{m}} \subseteq W$ and $W = T_{\underline{0}}W \not\subseteq H$. Thus $\Delta_{\underline{m}} \subseteq H$ is a hyperplane in W, and we may choose a point $v \in \Delta \smallsetminus H$ such that $W = \langle \Delta_{\underline{m}}, v\rangle$, i.e., W is the linear span of $\Delta_{\underline{m}}$ and v.

For any $1 \leqslant j \leqslant n$, let $\rho_j \colon \Delta \longrightarrow \Delta_j$ be the j^{th} projection, and let $t \colon W \longrightarrow \mathbb{C}$ be a non-zero linear function vanishing along the hyperplane $\Delta_{\underline{m}}$. We note that $\rho_j\left(\Delta_{\underline{m}-1}\right) = \Delta_{m_j-1}$ and $\rho_j\left(\Delta_{\underline{m}}\right) = \Delta_{m_j}$. Moreover, setting $W_j := \rho_j(W)$ and $v_j := \rho_j(v)$, we have that $v_j \in W_j \smallsetminus \Delta_{m_j}$ as $v \notin H$. Thus $W_j = \langle \Delta_{m_j}, v_j\rangle \subseteq \Delta_j$ is an m_j-dimensional plane not contained in the hyperplane $\{b_{j,0} = 0\}$. In particular W_j satisfies the hypothesis of [IV, (1.2)], so the intersection between Δ_{m_j-1} and W_j in Δ_j consists of the union of Δ_{m_j} and a smooth curve Γ_j having intersection multiplicity $\left(\Gamma_j \cdot \Delta_{m_j}\right)_{\underline{0}} = m_j$ with Δ_{m_j} at the origin $\underline{0} \in \Delta_j$. Therefore, in some étale neighborhood of the origin $\underline{0} \in W_j$, we may choose local coordinates $\left(x_{j,0}, \dots, x_{j,m_j-2}, t_j\right)$ such that $\rho_j^*\left(t_j\right) = t$,[1] Δ_{m_j} is the hyperplane with equation $t_j = 0$, and the curve Γ_j is defined by the equations

$$x_{j,1} = \cdots = x_{j,m_j-2} = 0 \quad \text{and} \quad t_j = \left(x_{j,0}\right)^{m_j}.$$

Then the functions $t = \rho_j^*\left(t_j\right)$ and $y_{j,i} := \rho_j^*\left(x_{j,i}\right)$, with $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant m_j - 2$, define a system of local coordinates on some étale neighborhood of W at the origin. It follows from their very definition that the intersection $W \cap \Delta_{\underline{m}-1}$ consists of the plane $\Delta_{\underline{m}} = \{t = 0\}$ and of the subvariety Γ defined by the equations

$$t = \left(y_{j,0}\right)^{m_j} \quad \text{and} \quad y_{j,i} = 0, \qquad \text{for } 1 \leqslant j \leqslant n \text{ and } 1 \leqslant i \leqslant m_j - 2. \tag{2.1.1}$$

We point out that Γ is a curve, as for any fixed value of $t \in \mathbb{C}$, there exist finitely many $\left((\sum m_j - 1) + 1\right)$-tuples $(y_{1,0}, \dots, y_{n,m_n-2}, t)$ satisfying (2.1.1), which depend only on the choice of an m_j^{th} root of t for each coordinate $y_{j,0}$.

In order to parameterize any branch of Γ, let us consider a sequence $\underline{\zeta} := (\zeta_1, \zeta_2, \dots, \zeta_n)$ such that each ζ_j is a m_j^{th} root of unity. Then we define a local parameterization $\varphi_{\underline{\zeta}} \colon \mathbb{C} \longrightarrow W$ as

$$z \longmapsto \begin{cases} t &= z^{\lambda} \\ y_{j,0} &= \dfrac{z^{\frac{\lambda}{m_j}}}{\zeta_j} \quad \text{for } 1 \leqslant j \leqslant n \\ y_{j,i} &= 0 \qquad \text{for } 1 \leqslant j \leqslant n \text{ and } 1 \leqslant i \leqslant m_j - 2, \end{cases} \tag{2.1.2}$$

and we denote by $\Gamma_{\underline{\zeta}}$ its image. Notice that the map $\varphi_{\underline{\zeta}}$ is injective.[2] Furthermore, every branch of Γ can be parameterized in this way: for any point $\underline{y} \in \Gamma$ with coordinates $(y_{1,0}, \dots, y_{n,m_n-2}, t)$, it is enough to choose a λ^{th} root z of t and set $\zeta_j = z^{\frac{\lambda}{m_j}}/y_{j,0}$, which is an m_j^{th} root of unity by (2.1.1). We also point out that the multiplicity of $\Gamma_{\underline{\zeta}}$ at the origin is the least power of z appearing in the parameterization, i.e., $\operatorname{mult}_{\underline{0}}(\Gamma_{\underline{\zeta}}) = \frac{\lambda}{\max_j\{m_j\}}$. On the other hand, the intersection multiplicity at the origin between $\Gamma_{\underline{\zeta}}$ and the hyperplane $\Delta_{\underline{m}} = \{t = 0\}$ is $\left(\Gamma_{\underline{\zeta}} \cdot \Delta_{\underline{m}}\right)_{\underline{0}} = \lambda$, as $t = z^{\lambda}$ on $\Gamma_{\underline{\zeta}}$. Therefore the behavior of the branches of Γ satisfies the statement, and it remains to enumerate them.

To this aim, we note that two distinct sequences $\underline{\zeta} := (\zeta_1, \zeta_2, \dots, \zeta_n)$ and $\underline{\eta} := (\eta_1, \eta_2, \dots, \eta_n)$ parameterize the same branch if and only if $\zeta_j = \varepsilon^{\frac{\lambda}{m_j}}\eta_j$ for all j, where ε is some λ^{th} root of

[1] It suffices to choose $t_j \colon W_j \longrightarrow \mathbb{C}$ to be a linear function such that $t_j(\Delta_{m_j}) = 0$ and $t_j(v_j) = t(v)$.

[2] Suppose that $\varphi_{\underline{\zeta}}(z) = \varphi_{\underline{\zeta}}(w)$ for some $z, w \in \mathbb{C}$. Then $(z/w)^{\frac{\lambda}{m_j}} = 1$ for any j, hence z/w is a ν^{th} root of unity, where ν divides any $\frac{\lambda}{m_j}$. Thus $\nu = 1$ as λ is the least common multiple of the integers m_j, hence $z = w$.

unity.[3]

Moreover, $\zeta_j = \varepsilon^{\frac{\lambda}{m_j}}\zeta_j$ for any j if and only if $\varepsilon = 1$ [4]. We conclude that the number of branches of Γ equals the number of sequences $\underline{\zeta} := (\zeta_1, \zeta_2, \ldots, \zeta_n)$ divided by the number of λ^{th} roots of unity, that is $\kappa = \frac{\mu}{\lambda}$. Thus we achieved Lemma (1.1) when W is assumed to be a linear subspace of Δ. $\square$

We now turn to prove Lemma (1.3).

(2.2) Proof of Lemma (1.3). Let G be the Grassmannian of vector subspaces of $\Delta \cong \mathbb{A}^{\Sigma(2m_j-1)}$ of dimension $\sum(m_j-1)+1$, and let $B \cong \mathbb{P}^{\Sigma m_j - 1}$ be the subvariety of G parameterizing those subspaces containing $\Delta_{\underline{m}} \cong \mathbb{A}^{\Sigma(m_j-1)}$. Let $W_{\underline{b}}$ denote the subspace parameterized by $\underline{b} \in B$, and consider the incidence variety $\Lambda := \left\{ (\underline{y}, \underline{b}) \in \Delta \times B \middle| \underline{y} \in W_{\underline{b}} \right\} \subseteq \Delta \times B$. Then Λ is the restriction to B of the universal family over G, and it fits in the following diagram

(2.2.1)
$$\begin{tikzcd} \Lambda \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \Delta \arrow[r, dashed, "\psi"] & B, \end{tikzcd}$$

where pr_1 and pr_2 are the natural projections, while ψ is the map sending any $\underline{y} \in \Delta \smallsetminus \Delta_{\underline{m}}$ to the point parameterizing the linear span $\langle \underline{y}, \Delta_{\underline{m}} \rangle \subseteq \Delta$ [5]. We point out that $\mathrm{pr}_1 \colon \Lambda \longrightarrow \Delta$ is an isomorphism outside $(\mathrm{pr}_1)^{-1}\left(\Delta_{\underline{m}}\right)$, whereas the fibre $(\mathrm{pr}_1)^{-1}(\underline{y})$ over any point $\underline{y} \in \Delta_{\underline{m}}$ is a section of pr_2, that is a projective space of dimension $\sum m_j - 1$. Therefore $\mathrm{pr}_1 \colon \Lambda \longrightarrow \Delta$ coincides with the blow-up $\widetilde{\Delta} := \mathrm{Bl}_{\Delta_{\underline{m}}}\Delta \xrightarrow{\pi} \Delta$ of Δ along $\Delta_{\underline{m}}$ (see, e.g., [2, p.604]). [6].

Consider the proper transforms $\widetilde{\Delta}_{\underline{m}-1}, \widetilde{H} \subseteq \widetilde{\Delta}$ of $\Delta_{\underline{m}-1}$ and H. Let $\widetilde{\Delta}^0_{\underline{m}-1} \subseteq \widetilde{\Delta}_{\underline{m}-1}$ and $\Delta^0_{\underline{m}-1} \subseteq \Delta_{\underline{m}-1}$ denote the open subsets of points not lying on $\widetilde{H}$ and H, respectively, and let $B_0 \subseteq B$ be the open subvariety parameterizing subspaces not contained in H. Then the diagram (2.2.1) restricts to

(2.2.2)
$$\begin{tikzcd} \widetilde{\Delta}^0_{\underline{m}-1} \arrow[d, "\pi_0"'] \arrow[dr, "\tau_0"] & \\ \Delta^0_{\underline{m}-1} \arrow[r, "\psi_0"] & B_0. \end{tikzcd}$$

We note that the closure $\overline{(\psi_0)^{-1}(\underline{b})} \subseteq \Delta_{\underline{m}-1}$ of the fibre over a point $\underline{b} \in B_0$ is the residual intersection of $\Delta_{\underline{m}-1}$ and the subspace $W_{\underline{b}}$ when we remove $\Delta_{\underline{m}}$. By Lemma (1.1) applied to $W_{\underline{b}} \cong \mathbb{A}^{\Sigma(m_j-1)+1}$, in some étale neighborhood of the origin $\underline{0} \in \Delta$, the latter intersection consists of κ distinct reduced unibranched curves $\Gamma_1, \ldots, \Gamma_\kappa$, each satisfying $\left(\Gamma_\alpha \cdot \Delta_{\underline{m}}\right)_{\underline{0}} = \lambda$

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[3]Indeed, $\varphi_{\underline{\zeta}}(z) = \varphi_{\underline{\eta}}(w)$ for some $z, w \in \mathbb{C}$ if and only if $(z/w)^\lambda = 1$ and $\eta_j z^{\frac{\lambda}{m_j}} = \zeta_j w^{\frac{\lambda}{m_j}}$, that is $z = \varepsilon w$ for some λ^{th} root ε of unity and $\zeta_j = \varepsilon^{\frac{\lambda}{m_j}}\eta_j$.

[4]Notice that $\varepsilon^{\frac{\lambda}{m_j}} = 1$ for any j if and only if ε is a ν^{th} root of unity, where ν divides any $\frac{\lambda}{m_j}$. Hence $\nu = 1$.

[5]We note that B is naturally identified with the projectivization $\mathbb{P}\left(\Delta/\Delta_{\underline{m}}\right) \cong \mathbb{P}^{\Sigma m_j - 1}$ of the quotient $\Delta/\Delta_{\underline{m}}$. Under this identification, we may view ψ as the map sending a point $\underline{y} = \left(\underline{a}_1, \underline{b}_1, \ldots \underline{a}_n, \underline{b}_n\right) \in \Delta \smallsetminus \Delta_{\underline{m}}$ to the point $\underline{b} \in \mathbb{P}\left(\Delta/\Delta_{\underline{m}}\right)$ having homogeneous coordinates $\underline{b} = \left[\underline{b}_1, \ldots, \underline{b}_n\right]$.

[6]In other terms, the dominant rational map $\psi \colon \Delta \dashrightarrow \mathbb{P}\left(\Delta/\Delta_{\underline{m}}\right)$ can be resolved to a morphism by blowing up Δ along the indeterminacy locus $\Delta_{\underline{m}}$.

and $\mathrm{mult}_{\underline{0}}(\Gamma_\alpha) = \frac{\lambda}{\max_j\{m_j\}}$. Notice that the proper transform $\widetilde{W}_b = (\mathrm{pr}_2)^{-1}(\underline{b})$ of W_b meets the fibre $F := \pi^{-1}(\underline{0})$ at a single point $p \in \widetilde{\Delta} \smallsetminus \widetilde{H}$ (recall that $\pi : \widetilde{\Delta} \to \Delta$ is the blow-up is the blow-up along $\Delta_{\underline{m}}$, which identifies with $\mathrm{pr}_1 : \Lambda \to \Delta$). It follows that in some étale neighborhood of p, the closure of the fibre $(\tau_0)^{-1}(\underline{b}) \subseteq \widetilde{\Delta}_{\underline{m}-1}$ consists of the proper transforms $\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_\kappa$ of the curves $\Gamma_\alpha \subseteq \Delta_{\underline{m}-1}$. Moreover, any $\widetilde{\Gamma}_\alpha$ is a reduced unibranched curve such that

$$(2.2.3) \qquad \left(\widetilde{\Gamma}_\alpha \cdot E\right)_p = \left(\Gamma_\alpha \cdot \Delta_{\underline{m}}\right)_{\underline{0}} = \lambda$$

and

$$(2.2.4) \qquad \left(\widetilde{\Gamma}_\alpha \cdot F\right)_p = \mathrm{mult}_{\underline{0}}\left(\Gamma_\alpha\right) = \frac{\lambda}{\max_j\{m_j\}},$$

by the Projection Formula and [3, Lemma 1.40 p.28].

Finally, as we vary $\underline{b} \in B_0$, each $\widetilde{\Gamma}_\alpha$ describes a reduced branch $\widetilde{\Delta}^\alpha$ of $\widetilde{\Delta}_{\underline{m}-1}$ having multiplicity $\frac{\lambda}{\max_j\{m_j\}}$ along F, intersection number λ with E along F, and since the curves $\widetilde{\Gamma}_\alpha$ are unibranched, the tangent cone to $\widetilde{\Delta}^\alpha$ at p is supported on a linear space contained in E. In particular, the intersection $\widetilde{\Delta}_{\underline{m}-1} \cup E$ contains the whole fibre F, with multiplicity given by the sum of the intersection multiplicities along F between E and each branch of $\widetilde{\Delta}_{\underline{m}-1}$, that is $\kappa\lambda = \mu$. □

Finally, we conclude this lecture by proving Lemma (1.1) for an arbitrary subvariety $W \subseteq \Delta$ satisfying the hypothesis of the statement.

(2.3) Proof of Lemma (1.1). Let $W \subseteq \Delta$ be a smooth $(\sum(m_j-1)+1)$-dimensional variety containing $\Delta_{\underline{m}}$ and such that $T_{\underline{0}}W \not\subseteq H$. By the smoothness of W, its proper transform $\widetilde{W} \subseteq \widetilde{\Delta}$ intersects the exceptional divisor E transversally, and cuts out on the latter a section over $\widetilde{\Delta}_{\underline{m}}$. Moreover, since $T_{\underline{0}}W \not\subseteq H$, we have that $\widetilde{W}$ meets $F := \pi^{-1}(\underline{0})$ at a single point $p \in \widetilde{\Delta}$ which lies off $\widetilde{H}$, hence $T_p\widetilde{W} \cap T_pF = \{0\}$. By Lemma (1.3), in some étale neighborhood of p, the variety $\widetilde{\Delta}_{\underline{m}-1}$ consists of κ reduced sheets $\widetilde{\Delta}^1, \dots, \widetilde{\Delta}^n$, and the tangent cone to a branch $\widetilde{\Delta}^\alpha$ at p is supported on a linear space $L \subseteq E$. In particular, $T_pF \subseteq L \subseteq E$. We notice that $\dim L = \dim \widetilde{\Delta}^\alpha = \sum m_j$, and $\dim T_pF = \dim F = \sum m_j - 1 = \dim \widetilde{\Delta}^\alpha - 1$. Thus $\dim\left(T_p\widetilde{W} \cap L\right) \leqslant 1$, so that the linear span of $T_p\widetilde{W}$ and L satisfies

$$\begin{aligned} \dim\langle T_p\widetilde{W}, L\rangle \quad &\geqslant \quad \dim T_p\widetilde{W} + \dim L - 1 = \sum(m_j-1)+1+\sum m_j - 1 \\ &= \quad \sum(2m_j-1) = \dim \widetilde{\Delta}, \end{aligned}$$

that is $\langle T_p\widetilde{W}, L\rangle = T_p\widetilde{\Delta}$. Therefore, the intersection around p between $\widetilde{W}$ and $\widetilde{\Delta}^\alpha$ is transverse, and it consists of a unibranched reduced curve $\widetilde{\Gamma}_\alpha$ such that $\left(\widetilde{\Gamma}_\alpha \cdot F\right)_p$ equals the multiplicity $\frac{\lambda}{\max_j\{m_j\}}$ of $\widetilde{\Delta}^\alpha$ along F, and $\left(\widetilde{\Gamma}_\alpha \cdot E\right)_p$ is the intersection number λ between $\widetilde{\Delta}^\alpha$ and E along F.

For any $1 \leqslant \alpha \leqslant \kappa$, we consider the reduced curve $\Gamma_\alpha := \pi\left(\widetilde{\Gamma}_\alpha\right)$, and in some étale neighborhood of the origin, we have

$$\Delta_{\underline{m}-1} \cap W = \Delta_{\underline{m}} \cup \Gamma_1 \cup \dots \cup \Gamma_\kappa.$$

Since each $\widetilde{\Gamma}_\alpha$ meets E only at the point p, the curve Γ_α is unibranched as well. Furthermore, by arguing as in (2.2.3) and (2.2.4), we deduce that each Γ_α satisfies $\mathrm{mult}_{\underline{0}}(\Gamma_\alpha) = \left(\widetilde{\Gamma}_\alpha \cdot F\right)_p = \frac{\lambda}{\max_j\{m_j\}}$ and $\left(\Gamma_\alpha \cdot \Delta_{\underline{m}}\right)_{\underline{0}} = \left(\widetilde{\Gamma}_\alpha \cdot E\right)_p = \lambda$. Thus Lemma (1.1) follows. □

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Appendix A
Geometry of the deformation space of an m-tacnode into $m-1$ nodes

by Thomas Dedieu

1 – Introduction

This appendix complements the previous lecture [IV], of which we shall keep the notation: let $\pi : S \to \Delta$ be the versal deformation space of the m-tacnode defined by the equation $y(y + x^m) = 0$ in the affine plane; we have $\Delta \cong \mathbf{A}^{2m-1}_{\alpha_0,\dots,\alpha_{m-2},\beta_0,\dots,\beta_{m-1}}$, and S is defined in $\mathbf{A}^2_{x,y} \times \Delta$ by the equation

$$y^2 + y(x^m + \alpha_{m-2}x^{m-2} + \cdots + \alpha_0) + \beta_{m-1}x^{m-1} + \cdots + \beta_0.$$

Inside Δ we consider the two subvarieties Δ_{m-1} and Δ_m, the closures of the two loci of those $t \in \Delta$ such that the curves $\pi^{-1}(t)$ have $m-1$ nodes and m nodes, respectively.

The goal of [IV] was to establish some properties of Δ_{m-1} around the origin in Δ, of fundamental importance in order to carry out the program of [1], see [VI]. Here we shall provide complementary results, and work out detailed examples for $m = 2$ and 3. The first motivation is that this will hopefully give more insight into the main result of [IV], as well as help to follow the arguments given there. On the other hand, the variety Δ_{m-1} for $m = 2$ will show up under another guise in this volume, see [XII], namely as the local model for the dual variety $X^\vee \subseteq \check{\mathbf{P}}^3$ of a surface $X \subseteq \mathbf{P}^3$, around a point corresponding to a tacnodal hyperplane section; this makes Δ_{m-1} one of the main characters of this volume. We are therefore interested in more properties of Δ_{m-1} than those strictly necessary for the Caporaso–Harris Program.

Finally, I would like to point out before I start that the singularity of Δ_{m-1} at the origin is known for $m = 2$ as a swallowtail singularity, *queue d'aronde* in french, one of seven elementary situations in René Thom's *théorie des catastrophes*.

2 – General results

Let $\tau : \tilde{\Delta} \to \Delta$ be the blow-up along Δ_m; recall that $\Delta_m \subseteq \Delta$ is the affine plane with equations

$$\beta_0 = \cdots = \beta_{m-1} = 0,$$

and it is contained in Δ_{m-1}. Let $Z \subseteq \tilde{\Delta}$ be the exceptional divisor, a $\mathbf{P}^{m-1}$-bundle over Δ_m, and let Φ be the fibre of Z over the origin. Let $\tilde{\Delta}_{m-1}$ be the proper transform of Δ_{m-1}. Let H be the hyperplane $\beta_0 = 0$ in Δ, and let $\tilde{H}$ be its proper transform in $\tilde{\Delta}$.

The main result of [IV] may be reformulated as follows, cf. [1, Lemma 4.2], where the equivalence of the two statements is proved (all the ingredients for this proof are present in [IV]).

(2.1) Proposition. *Let Z_{m-1} be the intersection $\tilde{\Delta}_{m-1} \cap Z$ (this is the exceptional locus of $\tau : \tilde{\Delta}_{m-1} \to \Delta_{m-1}$). Then Z_{m-1} contains Φ as a component of multiplicity m. Moreover, $\tilde{\Delta}_{m-1}$ is smooth at all points of Φ not in $\tilde{H}$.*

We shall now prove the result below, following [2, §2.4.4]. It is interesting in itself, but also as a pretext to give a local description of Δ_{m-1} at an arbitrary point of Δ_m.

(2.2) Proposition. *The fibres of Z_{m-1} over Δ_m are unions of linear spaces.*

Before we prove this, we introduce a natural stratification of Δ_m. Each stratum is the locus in which the m-tacnode deforms into k non-infinitely near tacnodes of orders $m_1, \ldots, m_k$ respectively, with $m_1 + \cdots + m_k = m$.

The space Δ_m identifies with the space of monic degree m polynomials in x with no x^{m-1}-term, and we shall consider in this space the loci of polynomials having roots with given multiplicities. For all partition $m = m_1 + \cdots + m_k$, we let

$$\Delta\{m_1, \ldots, m_k\} \subseteq \Delta_m \subseteq \Delta$$

be the locus corresponding to monic degree m polynomials with no x^{m-1}-term of the form $(x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}$, with $\lambda_1, \ldots, \lambda_k$ pairwise distinct. Note that $\Delta\{m_1, \ldots, m_k\}$ has dimension $k-1$ (because we impose that there is no x^{m-1}-term, equivalently the λ_i's sum up to zero), so its codimension in Δ_m is $m - k = \sum_{i=1}^{k}(m_i - 1)$.

For $\alpha \in \Delta\{m_1, \ldots, m_k\}$, the curve $S_\alpha = \pi^{-1}(\alpha) \subseteq \mathbf{A}^2_{x,y}$ is a reducible curve with two branches, defined by the two equations $y = 0$ and $y = (x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}$ respectively; these two branches intersect at the k points $r_1 = (\lambda_1, 0), \ldots, r_k = (\lambda_k, 0)$, with multiplicities $m_1, \ldots, m_k$ respectively.

For all $i = 1, \ldots, k$ we consider the versal deformation space $\Delta(S_\alpha, r_i)$ of the singular point $r_i \in S_\alpha$; note that this is an m_i-tacnode. There is then a natural map

$$\sigma : U \to \prod_{i=1}^{k} \Delta(S_\alpha, r_i),$$

where U is an open neighbourhood U of α in Δ, and by the openness of versality it has surjective differential at α; the fibre of σ over the origin parametrizes the equisingular deformations of S_α, for which only the location of the points r_i varies along the x-axis.

For all $i = 1, \ldots, k$ we consider the subvarieties $\Delta_{m_i-1}(r_i)$ and $\Delta_{m_i}(r_i)$ of $\Delta(S_\alpha, r_i)$, defined as the closures of the loci where r_i deforms to $m_i - 1$ nodes and m_i nodes respectively. Then, in the neighbourhood U of α in Δ, we have

$$\Delta_m = \sigma^{-1}\big(\Delta_{m_1}(r_1) \times \cdots \Delta_{m_k}(r_k)\big)$$

and

$$\Delta_{m-1} = \bigcup_{l=1}^{k} \sigma^{-1}\big(\Delta_{m_1}(r_1) \times \cdots \times \Delta_{m_l-1}(r_l) \times \cdots \Delta_{m_k}(r_k)\big).$$

From this we see that Δ_{m-1} has k local sheets in a neighbourhood of α, each containing Δ_m. The l-th sheet looks like the product of $\Delta_{m_l-1} \subseteq \Delta_{m_l}$ with some smooth factor, and along this sheet the fibres of $\pi : S \to \Delta$ have m_i nodes tending to r_i for $i \neq l$, and $m_l - 1$ nodes tending to r_l.

We are now ready to prove the statement.

Proof of Proposition (2.2). In the above setup, applying Proposition (2.1) to the spaces $\Delta(S_\alpha, r_i)$, we will be able to describe the fibre of Z_{m-1} over $\alpha \in \Delta_m$. Note that Φ, the fibre of Z over the origin, identifies with the projectivization of the space of degree $m-1$ polynomials in x (with coefficients $\beta_0, \dots, \beta_{m-1}$), equivalently with the projectivization of the space of polynomials modulo those vanishing to order m at the origin (i.e., $\mathbf{C}[x]/(x^m)$).

Then, for all $l = 1, \dots, k$, by Proposition (2.1) applied to $\Delta_{m_l-1}(r_l)$, the proper transform of the l-th sheet of Δ_{m-1} intersects the fibre Φ_α of Z over α in the linear subspace of Φ_α corresponding to polynomials in x vanishing to order m_h at r_h for all $h \neq l$; in other words, if we identify Φ_α with the projectivization of

$$\mathbf{C}[x]/\big((x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}\big) \;\cong\; \prod_{h=1}^{k} \mathbf{C}[x]/\big((x-\lambda_h)^{m_h}\big),$$

then the proper transform of the l-th sheet of Δ_{m-1} intersects Φ_α in its linear subspace which is the projectivization of $\mathbf{C}[x]/\big((x-\lambda_l)^{m_l}\big)$ (with multiplicity m_l).

In turn, the intersection of $\tilde{\Delta}_{m-1}$ with the fibre Φ_α of Z over α is the union of these linear spaces for l ranging from 1 to k. $\square$

(2.3) A complement. Consider a small arc $t \in \mathbf{D} \mapsto \alpha(t) \in \Delta_m$ tending to the origin as t tends to 0, where $\mathbf{D}$ stands for the unit disc in $\mathbf{C}$. If this arc is small enough, then all points $\alpha(t)$ with $t \neq 0$ lie in the same stratum $\Delta\{m_1, \dots, m_k\}$. Moreover, as t goes to $0 \in \mathbf{D}$ the singular points $r_i(t)$ of the curve $S_{\alpha(t)} \subseteq \mathbf{A}^2_{x,y}$ tend to the origin. It follows that the limiting position in Φ of the intersection with $\Phi_{\alpha(t)}$ of the l-th sheet of Δ_{m-1} along $\Delta\{m_1, \dots, m_k\}$ is the space of polynomials vanishing to order $m - m_l$ at 0 (in the identification of Φ with polynomials in x modulo those vanishing to order m at the origin).

This implies, as in [IV, 2], that Φ is an irreducible component of Z_{m-1}, the general point of which corresponds to the tangent direction of an arc in Δ_{m-1} which intersects Δ_m only in the origin.

3 – Complete description in the case $m = 2$

When $m = 2$ we are considering the versal deformation space of the ordinary tacnode, defined in $\mathbf{A}^2_{x,y}$ by the equation $y(y + x^2) = 0$; this is $S \to \Delta \cong \mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$, with S defined in $\mathbf{A}^2_{x,y} \times \Delta$ by the equation

$$y^2 + y(x^2 + \alpha_0) + \beta_1 x + \beta_0 = 0.$$

We are interested in the surface $\Delta_1 \subseteq \Delta$ of deformations of cogenus at least one of the tacnode, especially along the curve $\Delta_2 \subseteq \Delta$ of deformations of cogenus two, which is the line $\beta_1 = \beta_0 = 0$, along wich the tacnode deforms to two nodes. The picture looks as follows, as we shall see.

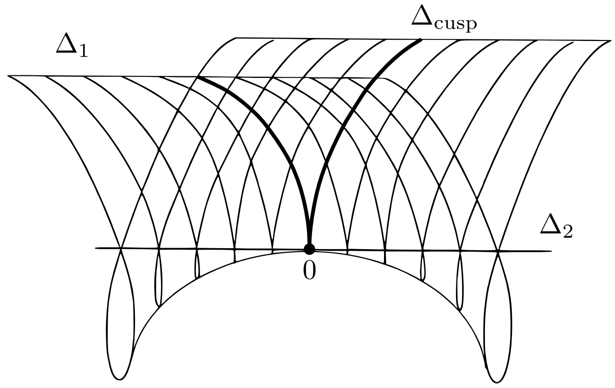

Figure 1: Geometry of Δ_1 along Δ_2 (beware the singularity along Δ_{cusp}) (reproduction from [2, p.181])

The surface Δ_1 has two local sheets along the line Δ_2; they intersect transversely along the complement of the origin in Δ_1, but they have the same tangent plane at the origin. Also, it is important to keep in mind that the surface Δ_1 is singular along the curve Δ_{cusp}, which is the locus along which the tacnode deforms to an ordinary cusp: the surface Δ_1 has a cuspidal singularity at the generic point of Δ_{cusp}, although this not pictured on the above figure.

The proper transform of Δ_1 under the blow-up of Δ along the line Δ_2 looks as follows.

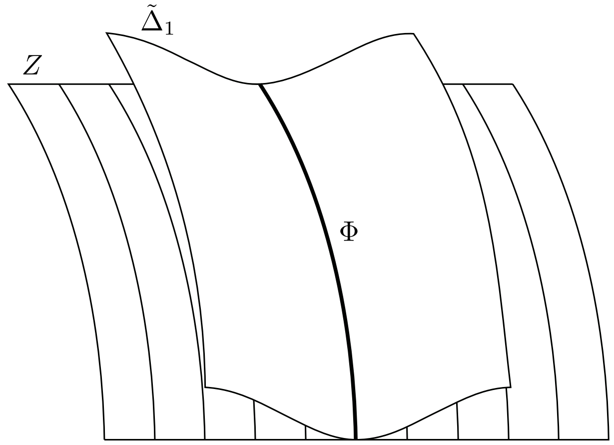

Figure 2: Intersection of $\tilde{\Delta}_1$ with the exceptional divisor (reproduction from [1, p.379])

(3.1) Equation of Δ_1**.** The surface Δ_1 is defined in $\mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$ by the condition that the polynomial

$$\delta_{\alpha_0,\beta_0,\beta_1}(x) = (x^2+\alpha_0)^2 - 4(\beta_1 x + \beta_0) = x^4 + 2\alpha_0 x^2 - 4\beta_1 x + \alpha_0^2 - 4\beta_0$$

has at least one non-simple root (see [IV, Section 1]), equivalently its discriminant vanishes. Using the formula for the discriminant of monic degree 4 polynomials with no x^3-term, namely

$$\operatorname{Disc}(x^4+ax^2+bx+c) = -4a^3b^2 + 16a^4c - 27b^4 + 144ab^2c - 128a^2c^2 + 256c^3,$$

we find the equation for Δ_1 in $\mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$,

(3.1.1) $$256 \cdot (16\alpha_0^3\beta_1^2 + 16\alpha_0^2\beta_0^2 - 72\alpha_0\beta_0\beta_1^2 - 27\beta_1^4 - 64\beta_0^3) = 0.$$

Note that it is quasi-homogeneous of degree 12 for the weights $2, 4, 3$ of $\alpha_0, \beta_0, \beta_1$ respectively.

(3.2) Equations of the curve Δ_{cusp}. We consider the locus Δ_{cusp} of those $t \in \Delta$ such that the curve S_t has (at least) a cuspidal singularity. The locus Δ_{cusp} is defined by the condition that the polynomial $\delta_{\alpha_0,\beta_0,\beta_1}$ has at least one triple root, in other words that there exists $u \in \mathbf{A}^1$ such that

$$\begin{aligned}\delta_{\alpha_0,\beta_0,\beta_1}(x) &= (x-u)^3 \cdot (x+3u) \\ &= x^4 - 6u^2x^2 + 8u^3x - 3u^4\end{aligned}$$

(recall that the roots of $\delta_{\alpha_0,\beta_0,\beta_1}$ sum up to zero as it has no x^3-term). Therefore, Δ_{cusp} is the projection on the second factor of the complete intersection in $\mathbf{A}^1_u \times \mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$ defined by the equations

$$\begin{cases} 2\alpha_0 = -6u^2 \\ -4\beta_1 = 8u^3 \\ \alpha_0^2 - 4\beta_0 = -3u^4. \end{cases}$$

Eliminating u from these equations (note that this can be done algorithmically), one finds the following equations defining Δ_{cusp} in $\mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$,

$$\begin{cases} \alpha_0^2 - 3\beta_0 = 0 \\ 4\alpha_0^3 + 27\beta_1^2 = 0. \end{cases}$$

(3.3) Singularities of the surface Δ_1. To compute the equations of the singularities of Δ_1, we consider the Jacobian ideal $(\partial_{\alpha_0}F, \partial_{\beta_0}F, \partial_{\beta_1}F)$, where F is the equation of Δ_1 given in (3.1.1). It readily defines the singular subscheme of Δ_1, since F is quasi-homogeneous hence liable to the generalized Euler formula, cf. [II, Section 1]. Using a software for symbolic computation with polynomials, e.g., Macaulay2 [3] and its command `primaryDecomposition`, one finds the ideal defining the singular subscheme of Δ_1 is the intersection of the two following primary ideals,

$$(\beta_0, \beta_1) \quad \text{and} \quad (9\alpha_0\beta_0^2 - 4\alpha_0^2\beta_1 + 24\beta_1^2,\ 8\alpha_0^3 - 27\beta_0^2 - 36\alpha_0\beta_1,\ 27\beta_0^4 + 16\alpha_0^2\beta_1^2 - 64\beta_1^3)$$

The former is the defining ideal of Δ_2, while the latter is not prime, with radical the defining ideal of the curve Δ_{cusp}.

Also, one sees directly from the equation of Δ_1, (3.1.1), that it has a triple point at the origin, with tangent cone the triple plane at the origin $\beta_0^3 = 0$.

(3.4) Blow-up along the line Δ_2. The blow-up of $\mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$ along the line Δ_2 is the hypersurface in $\mathbf{P}^1_{(u_0:u_1)} \times \mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$ defined by the equation $u_0\beta_1 - u_1\beta_0 = 0$. Pulling back the equation of Δ_1, (3.1.1), we find

$$\beta_1^2 \cdot \left(16\alpha_0^2u_0^2 - 64\beta_1u_0^3 + 16\alpha_0^3 - 72\alpha_0\beta_1u_0 - 27\beta_1^2\right) = 0$$

in the affine chart $u_1 \neq 0$, and

$$\beta_0^2 \cdot \left(16\alpha_0^2 - 64\beta_0 + 16\alpha_0^3u_1^2 - 72\alpha_0\beta_0u_1^2 - 27\beta_0^2u_1^4\right) = 0$$

in the affine chart $u_0 \neq 0$; the factors between parentheses are thus the defining equations of the proper transform $\tilde{\Delta}_1$ in $\mathbf{A}^1_{u_0} \times \mathbf{A}^3_{\alpha_0,\beta_1}$ and $\mathbf{A}^1_{u_1} \times \mathbf{A}^3_{\alpha_0,\beta_0}$ respectively. From these equations we see that $\tilde{\Delta}_1$ is smooth in the latter affine chart, and has a double point at the origin of the former, with tangent cone there given by $\beta_1^2 = 0$.

To obtain the intersection with the exceptional divisor Z, we plug $\beta_1 = 0$ and $\beta_0 = 0$ respectively in these two equations; we find

$$16\alpha_0^2(u_0^2 + \alpha_0) = 0 \quad \text{and} \quad 16\alpha_0^2(1 + \alpha_0 u_1^2) = 0$$

for the two equations of $Z_1 = \tilde{\Delta}_1 \cap Z$ in the two charts $\mathbf{A}^1_{u_0} \times \mathbf{A}^3_{\alpha_0,\beta_1}$ and $\mathbf{A}^1_{u_1} \times \mathbf{A}^3_{\alpha_0,\beta_0}$ respectively. The exceptional divisor Z_1 thus has two irreducible component, the fibre Φ_0, and the hypersurface $u_0^2 + \alpha_0 u_1^2) = 0$; the latter maps $2:1$ onto the line Δ_2, and corresponds to the normal directions of the two sheets of Δ_1 along Δ_2.

4 – The case $m = 3$ and beyond

From this point on it is no longer advisable to do the computations by hand, and I have used Macaulay2 [3]. In fact, even with a computer it is not realistic to go much further. Still we can make some interesting observations, that are summed up in Subsection 4.2.

4.1 – The case $m = 3$

When $m = 3$, we are considering the versal deformation space of the 3-tacnode, defined in $\mathbf{A}^2_{x,y}$ by the equation $y(y + x^3) = 0$; this is $S \to \Delta \cong \mathbf{A}^5_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}$, with S defined in $\mathbf{A}^2_{x,y} \times \Delta$ by the equation

$$y^2 + y(x^3 + \alpha_1 x + \alpha_0) + \beta_2 x^2 + \beta_1 x + \beta_0 = 0.$$

We are interested in the solid $\Delta_2 \subseteq \Delta$ of deformations of cogenus at least two of the 3-tacnode, especially along the locus $\Delta_3 \subseteq \Delta$ of deformations of cogenus three, which is the plane $\beta_2 = \beta_1 = \beta_0 = 0$ along wich the 3-tacnode deforms to three (possibly infinitely near) nodes.

The singularities of Δ_2 are more complicated than those of Δ_1 in the case $m = 2$, and we will content ourselves with the description of the exceptional divisor of $\tilde{\Delta}_2$.

(4.1) Equations of Δ_2. We shall proceed by elimination, as in (3.2). The solid Δ_2 is defined by the condition that the polynomial

$$\begin{aligned} (4.1.1) \quad \delta_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}(x) &= (x^3 + \alpha_1 x + \alpha_0)^2 - 4(\beta_2 x^2 + \beta_1 x + \beta_0) \\ &= x^6 + 2\alpha_1 x^4 + 2\alpha_0 x^3 + (\alpha_1^2 - 4\beta_2)x^2 + (2\alpha_0\alpha_1 - 4\beta_1)x + (\alpha_0^2 - 4\beta_0) \end{aligned}$$

has at least two non-simple roots, equivalently that there exist $(u, v, w) \in \mathbf{A}^3$ such that

$$(4.1.2) \qquad \delta_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}(x) = (x-u)^2(x-v)^2(x-w)(x + 2u + 2v + w).$$

It is thus the projection to the second factor of the complete intersection in $\mathbf{A}^3_{u,v,w} \times \mathbf{A}^5_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}$ defined by the four non-trivial equations obtained by identifying the x-terms in (4.1.1) and (4.1.2). Eliminating u, v, w from these equations, for instance by using the function `eliminate` from Macaulay2 [3], one finds the ideal of Δ_2 in Δ is the following:

$$\begin{aligned} \big(&48\alpha_0\alpha_1^2\beta_2^2 + 3\alpha_1^2\beta_0\beta_1 + 45\alpha_0\alpha_1\beta_1^2 - 72\alpha_0\alpha_1\beta_0\beta_2 + 27\alpha_0^2\beta_1\beta_2 \\ &\qquad - 128\alpha_1\beta_1\beta_2^2 + 64\alpha_0\beta_2^3 + 27\alpha_0\beta_0^2 - 125\beta_1^3 + 180\beta_0\beta_1\beta_2, \\ &9\alpha_1^3\beta_0\beta_1 - 9\alpha_0\alpha_1^2\beta_1^2 + 63\alpha_0\alpha_1^2\beta_0\beta_2 - 54\alpha_0^2\alpha_1\beta_1\beta_2 - 81\alpha_0^3\beta_2^2 + 16\alpha_1^2\beta_1\beta_2^2 - 128\alpha_0\alpha_1\beta_2^3 - 54\alpha_0\alpha_1\beta_0^2 \\ &\qquad - 81\alpha_0^2\beta_0\beta_1 + 25\alpha_1\beta_1^3 - 252\alpha_1\beta_0\beta_1\beta_2 + 135\alpha_0\beta_1^2\beta_2 + 324\alpha_0\beta_0\beta_2^2 + 192\beta_1\beta_2^3 + 405\beta_0^2\beta_1, \\ &16\alpha_1^4\beta_2^2 + 16\alpha_1^3\beta_1^2 - 16\alpha_1^3\beta_0\beta_2 - 216\alpha_0^2\alpha_1\beta_2^2 - 128\alpha_1^2\beta_2^3 - 81\alpha_0\alpha_1\beta_0\beta_1 - 135\alpha_0^2\beta_1^2 + 135\alpha_0^2\beta_0\beta_2 \\ &\qquad - 120\alpha_1\beta_1^2\beta_2 + 576\alpha_1\beta_0\beta_2^2 + 288\alpha_0\beta_1\beta_2^2 + 256\beta_2^4 + 675\beta_0\beta_1^2 - 432\beta_0^2\beta_2, \end{aligned}$$

$$
\begin{aligned}
&16\alpha_1^4\beta_0\beta_2 - 16\alpha_0\alpha_1^3\beta_1\beta_2 - 16\alpha_1^3\beta_0^2 - 81\alpha_0\alpha_1^2\beta_0\beta_1 + 81\alpha_0^2\alpha_1\beta_1^2 - 216\alpha_0^2\alpha_1\beta_0\beta_2 + 135\alpha_0^3\beta_1\beta_2 \\
&\quad + 16\alpha_1^2\beta_1^2\beta_2 - 128\alpha_1^2\beta_0\beta_2^2 + 192\alpha_0\alpha_1\beta_1\beta_2^2 + 135\alpha_0^2\beta_0^2 + 240\alpha_1\beta_0\beta_1^2 - 225\alpha_0\beta_1^3 + 576\alpha_1\beta_0^2\beta_2 \\
&\qquad - 252\alpha_0\beta_0\beta_1\beta_2 - 320\beta_1^2\beta_2^2 + 256\beta_0\beta_2^3 - 432\beta_0^3, \\
&81\alpha_0\alpha_1^3\beta_0\beta_2 - 81\alpha_0^2\alpha_1^2\beta_1\beta_2 - 1215\alpha_0^3\alpha_1\beta_2^2 + 240\alpha_1^3\beta_1\beta_2^2 - 81\alpha_0\alpha_1^2\beta_0^2 - 486\alpha_0^2\alpha_1\beta_0\beta_1 - 729\alpha_0^3\beta_1^2 \\
&\quad + 240\alpha_1^2\beta_1^3 + 729\alpha_0^3\beta_0\beta_2 - 420\alpha_1^2\beta_0\beta_1\beta_2 + 261\alpha_0\alpha_1\beta_1^2\beta_2 + 1980\alpha_0\alpha_1\beta_0\beta_2^2 + 2052\alpha_0^2\beta_1\beta_2^2 \\
&\quad - 2240\alpha_1\beta_1\beta_2^3 + 2560\alpha_0\beta_2^4 + 243\alpha_1\beta_0^2\beta_1 + 3645\alpha_0\beta_0\beta_1^2 - 1836\alpha_0\beta_0^2\beta_2 - 2300\beta_1^3\beta_2 + 3312\beta_0\beta_1\beta_2^2\big).
\end{aligned}
$$

It has five generators, quasi-homogeneous of degrees $15, 17, 16, 18, 19$ respectively, for the weights $3, 2$ for α_0, α_1 and $6, 5, 4$ for $\beta_0, \beta_1, \beta_2$.

(4.2) Tangent cone at the origin. Using the ideal of Δ_2 in Δ, one finds (e.g., with the function `tangentCone` of Macaulay2) its tangent cone at the origin is defined by the primary ideal

$$
\big(25\beta_0\beta_1^2 - 16\beta_0^2\beta_2, \beta_0^2\beta_1, \beta_0^3, 27\alpha_0\beta_0^2 - 125\beta_1^3 + 180\beta_0\beta_1\beta_2\big).
$$

It defines a degree 6 projective scheme, supported over the 3-plane $\beta_1 = \beta_2 = 0$.

(4.3) Blow-up along the plane Δ_3. We consider the blow-up of Δ along Δ_3 as the subvariety of $\mathbf{P}^2_{(u_0:u_1:u_2)} \times \mathbf{A}^5_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}$ defined by the equations $u_j\beta_i - u_i\beta_j = 0$ for all distinct i and j in $\{0, 1, 2\}$.

Let us consider the affine chart $u_2 = 1$ to fix ideas; it may be seen in $\mathbf{A}^2_{u_0,u_1} \times \mathbf{A}^3_{\alpha_0,\alpha_1,\beta_2}$ by eliminating β_0 and β_1, since $\beta_0 = u_0\beta_2$ and $\beta_1 = u_1\beta_2$ in this chart. The pull-back of the ideal of Δ_2 in Δ is the intersection of the two primary ideals (β_2^2) and

$$
\begin{aligned}
&\big(3\alpha_1^2u_0u_1 + 45\alpha_0\alpha_1u_1^2 - 125\beta_2u_1^3 + 48\alpha_0\alpha_1^2 - 72\alpha_0\alpha_1u_0 + 27\alpha_0u_0^2 + 27\alpha_0^2u_1 - 128\alpha_1\beta_2u_1 + 180\beta_2u_0u_1 + 64\alpha_0\beta_2, \\
&16\alpha_1^3u_1^2 + 16\alpha_1^4 - 16\alpha_1^3u_0 - 81\alpha_0\alpha_1u_0u_1 - 135\alpha_0^2u_1^2 - 120\alpha_1\beta_2u_1^2 + 675\beta_2u_0u_1^2 - 216\alpha_0^2\alpha_1 - 128\alpha_1^2\beta_2 \\
&\qquad + 135\alpha_0^2u_0 + 576\alpha_1\beta_2u_0 - 432\beta_2u_0^2 + 288\alpha_0\beta_2u_1 + 256\beta_2^2, \\
&144\alpha_0\alpha_1^2u_1^2 - 400\alpha_1\beta_2u_1^3 + 144\alpha_0\alpha_1^3 - 279\alpha_0\alpha_1^2u_0 + 135\alpha_0\alpha_1u_0^2 + 135\alpha_0^2\alpha_1u_1 - 400\alpha_1^2\beta_2u_1 + 81\alpha_0^2u_0u_1 \\
&\qquad + 792\alpha_1\beta_2u_0u_1 - 405\beta_2u_0^2u_1 - 135\alpha_0\beta_2u_1^2 + 81\alpha_0^3 + 320\alpha_0\alpha_1\beta_2 - 324\alpha_0\beta_2u_0 - 192\beta_2^2u_1, \\
&\alpha_1^4u_0 - \alpha_1^3u_0^2 - \alpha_0\alpha_1^3u_1 + 81\alpha_0^2\alpha_1u_1^2 + \alpha_1^2\beta_2u_1^2 + 15\alpha_1\beta_2u_0u_1^2 - 225\alpha_0\beta_2u_1^3 + 81\alpha_0^2\alpha_1^2 - 135\alpha_0^2\alpha_1u_0 \\
&\qquad - 8\alpha_1^2\beta_2u_0 + 54\alpha_0^2u_0^2 + 36\alpha_1\beta_2u_0^2 - 27\beta_2u_0^3 + 54\alpha_0^3u_1 - 204\alpha_0\alpha_1\beta_2u_1 \\
&\qquad\qquad + 288\alpha_0\beta_2u_0u_1 - 20\beta_2^2u_1^2 + 108\alpha_0^2\beta_2 + 16\beta_2^2u_0\big).
\end{aligned}
$$

The latter has four generators, and defines the proper transform $\tilde{\Delta}_2$ of Δ_2 in this affine chart. To obtain the intersection Z_2 of $\tilde{\Delta}_2$ with the exceptional divisor, we plug in the equation $\beta_2 = 0$. We find the ideal of Z_2 in $\mathbf{A}^2_{u_0,u_1} \times \mathbf{A}^3_{\alpha_0,\alpha_1,\beta_2}$ is the intersection of the three primary ideals

$$
\begin{gathered}
\big(\alpha_0\alpha_1, \alpha_0^2, \alpha_1^2u_1 + 9\alpha_0u_0, \alpha_1^3\big), \\
\big(u_1^2 + \alpha_1 - u_0, u_0u_1 + \alpha_0, \alpha_1u_0 - u_0^2 - \alpha_0u_1\big), \quad \text{and} \\
\big(\alpha_1u_0u_1^2 - \alpha_0u_1^3 + \alpha_1^2u_0 - 2\alpha_1u_0^2 + u_0^3 - \alpha_0\alpha_1u_1 + 3\alpha_0u_0u_1 + \alpha_0^2, \\
16\alpha_1^2u_1^2 + 16\alpha_1^3 - 31\alpha_1^2u_0 + 15\alpha_1u_0^2 + 15\alpha_0\alpha_1u_1 + 9\alpha_0u_0u_1 + 9\alpha_0^2, \\
\alpha_1^2u_0u_1 + 15\alpha_0\alpha_1u_1^2 + 16\alpha_0\alpha_1^2 - 24\alpha_0\alpha_1u_0 + 9\alpha_0u_0^2 + 9\alpha_0^2u_1, \\
\alpha_1^3u_0 - \alpha_1^2u_0^2 - \alpha_0\alpha_1^2u_1 - 6\alpha_0\alpha_1u_0u_1 - 9\alpha_0^2u_1^2 - 15\alpha_0^2\alpha_1 + 9\alpha_0^2u_0, \\
16\alpha_1^4 - 15\alpha_1^2u_0^2 - 96\alpha_0\alpha_1^2u_1 - 90\alpha_0\alpha_1u_0u_1 - 135\alpha_0^2u_1^2 - 360\alpha_0^2\alpha_1 + 216\alpha_0^2u_0\big).
\end{gathered}
$$

The first one has radical (α_0, α_1), and defines the fibre Φ_0 of Z over the origin with multiplicity 3. The second one is radical, and defines a locus that dominates Δ_3 (this can be seen by eliminating the variables u_0 and u_1, which gives the zero ideal); moreover, one finds there are three points in this locus over the generic point of Δ_3. As for the third one, if we eliminate u_0 and u_1, we find the ideal

$$\big((4\alpha_1^3 + 27\alpha_0^2)^2\big),$$

which is generated by the square of the discriminant of the polynomial $x^3 + \alpha_1 x + \alpha_0$; over the generic point of the locus defined by this discriminant in Δ_3, we find the the third ideal defines a line with multiplicity 2.

Computations in the two other affine charts are similar, although they take a little more time to the computer. In conclusion we find that the exceptional divisor Z_2 of $\tilde{\Delta}_2 \to \Delta_2$ has three irreducible components: (i) the fibre Φ_0 of Z over the origin; (ii) a surface dominating Δ_3, and corresponding to the normal directions of the three sheets of Δ_2 along Δ_3; (iii) a surface fibered in lines over the cuspidal curve $\Delta\{2,1\} \subseteq \Delta_3$ (in the notation of Section 2).

4.2 – General conclusions

(4.4) Some computations in the case $m = 4$. For $m = 4$ it is still possible to compute the defining ideal of the fourfold Δ_3 in $\mathbf{A}^7_{\alpha_0,\ldots,\alpha_2,\beta_0,\ldots,\beta_3}$, although it took about half an hour on my personal computer.

This ideal has 20 generators, all quasihomogenous if we assign the weights $4, 3, 2$ to $\alpha_2, \alpha_1, \alpha_0$ and $8, \ldots, 5$ to $\beta_3, \ldots, \beta_0$ respectively. For what it's worth, the degrees of the generators are $18, 19, 20, 20, 21, 21, 21, 22, 22, 22, 23, 23, 24, 24, 28, 28, 29, 30, 30, 31$.

The tangent cone to Δ_2 at the origin is supported on the 4-plane $\beta_0 = \beta_1 = \beta_2 = 0$, and has degree 10.

I have not been able to compute a primary decomposition of the intersection of $\tilde{\Delta}_3$ with the exceptional divisor Z of the blow-up of Δ along Δ_4.

(4.5) General conjectures. From the above computations one can conjecture that the defining ideal of Δ_{m-1} in Δ will always be quasihomogeneous for the weights $m, \ldots, 2$ for α_0, α_{m-2} and $2m, \ldots, m+1$ for $\beta_0, \ldots, \beta_{m-1}$, respectively.

The origin will be a point of multiplicity $\sum_{k=1}^{m} k$, with tangent cone there supported on the m-plane $\beta_0 = \cdots = \beta_{m-2} = 0$.

The exceptional divisor of $\tilde{\Delta}_{m-1} \to \Delta_{m-1}$ will have $m-1$ irreducible components, respectively dominating the following loci in Δ_m : $\Delta\{1, \ldots, 1\} = \Delta_m$, $\Delta\{2, 1, \ldots, 1\}$, $\ldots$, $\Delta\{m-1, 1\}$.

The latter two facts are probably not very hard to prove using the setup of Section 2. It is plausible that the first conjecture may be approached using reduced discriminant theory (see [C]).

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Lecture VI
Degenerations and deformations of generalized Severi varieties

by Thomas Dedieu and Concettina Galati

This text is dedicated to the recursive formula obtained by Caporaso and Harris [1] as a solution to the problem of "counting plane curves of any genus", see Corollary (1.5) and Theorem (5.4). Fix a degree $d \in \mathbf{N}^*$, and denote by $p_a(d) = \frac{1}{2}(d-1)(d-2)$ the genus of smooth plane curves of degree d. For all $g = 0, \ldots, p_a(d)$, (reduced) plane curves of degree d and geometric genus g form a family of pure dimension $\nu_g^d = 3d + g - 1$. The problem is to find the number of plane curves of degree d and geometric genus g passing through ν_g^d general points in the plane. It turns out that this number equals the number of δ-nodal plane curves of degree d which pass through ν_g^d general points in the plane, with $\delta = p_a(d) - g$ (a curve is δ-nodal if its singularities are exactly δ nodes, i.e., δ ordinary double points).

Caporaso and Harris obtain their inductive formula by specializing one after another the aforementioned ν_g^d imposed crossing points to general points on a fixed line L. This may be conveniently formulated in terms of a degeneration of the projective plane $\mathbf{P}^2$ to the transverse union along a line of $\mathbf{P}^2$ and a minimal rational ruled surface $\mathbf{F}_1$, using the formalism presented in [I]. We will do it on an example in [VII], but here we stick to the presentation given in [1]. A similar result for curves on minimal rational ruled surfaces has been obtained by Vakil [6], following a similar degeneration procedure, and around the same period of the time. We shall nevertheless concentrate on the case of plane curves.

The above described degeneration procedure makes it necessary to generalize the problem. Thus, one is led to count curves of given degree and genus with, in addition, a prescribed intersection pattern with the line L, both at assigned and unassigned points. We shall call *generalized Severi varieties* the families formed by these curves: they are a particular case of the logarithmic Severi varieties studied in [III].

We give precise definitions and statements in Section 1 below. There, we shall also provide more details about the general strategy of the proof, which will explain the title of this text.

1 – The counting problem and its solution

Let us fix once and for all a line $L \subseteq \mathbf{P}^2$. We shall use freely the notation and definitions introduced in [III, Section 1], in the special case when the pair (S, R) is $(\mathbf{P}^2, L)$; every class $\xi \in \mathrm{NS}(\mathbf{P}^2)$ is d times the linear equivalence class of a line for some integer d, and therefore it shall be denoted simply by d.

Let $d \in \mathbf{N}$ and $g \in \mathbf{Z}$, let $\alpha = (\alpha_i)_{i\geqslant 1}$ and $\beta = (\beta_i)_{i\geqslant 1}$ be two sequences in $\underline{\mathbf{N}}$ such that $I\alpha + I\beta = d$, and let $\Omega = \big(\{p_{i,j}\}_{1\leqslant j\leqslant \alpha_i}\big)_{i\geqslant 1}$ be a set of α points on the line L. The *logarithmic Severi variety* $V^d_g(\alpha,\beta)(\Omega)$ is the locally closed subset of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ (the projective space of dimension $\frac{1}{2}d(d+3)$ parametrizing all plane curves of degree d) parametrizing those reduced plane curves of degree d and geometric genus g that intersect L at the points of Ω with the multiplicities prescribed by α and at further unassigned points with the multiplicities prescribed by β; for the rigorous definition, see [III, Definition (1.5)]. The points of Ω shall be referred to as the *fixed*, or *assigned*, contact points, and the other intersection points with L as *mobile*, or *unassigned*, contact points (as it turns out, in the present situation, the unassigned contact points indeed move as the curves moves in the Severi variety, see Proposition (2.2) below). The relevant notation for these points and their relatives is explained in Paragraph (2.1). Forgetting about the location of assigned intersection points at Ω, we may also loosely say that the curves in $V^d_g(\alpha,\beta)(\Omega)$ have intersection pattern with L prescribed by α and β.

As it turns out (see Proposition (2.2) below), the general element of every irreducible component of $V^d_g(\alpha,\beta)(\Omega)$ has δ nodes and no further singularities, with

$$\delta = p_a(d) - g = \frac{1}{2}(d-1)(d-2) - g.$$

The upshot is that it is indifferent to require either that the geometric genus be g or that the singularities be exactly δ nodes.

(1.1) Definition. *For all natural integers d and δ, and all $\alpha, \beta \in \underline{\mathbf{N}}$ and Ω as above, the* generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ *is the Zariski closure in the projective space $|\mathcal{O}_{\mathbf{P}^2}(d)|$ of the logarithmic Severi variety $V^d_{p_a(d)-\delta}(\alpha,\beta)(\Omega)$ of the pair $(\mathbf{P}^2, L)$.*

In the particular case when $\alpha = 0$, and thus $\Omega = \varnothing$, and $\beta = (d, 0, \ldots)$ (i.e., when no conditions are imposed on the intersection with L), the Severi varieties $V^d_g(\alpha,\beta)(\Omega)$ and $V^{d,\delta}(\alpha,\beta)(\Omega)$ will simply be denoted by V^d_g and $V^{d,\delta}$, respectively, and called *plain* Severi varieties.

Although this should not create any problem, beware that in our notation V^d_g and $V^{d,\delta}$ (with any additional decorations) are respectively locally closed and closed in $|\mathcal{O}_{\mathbf{P}^2}(d)|$.

For any point $p \in \mathbf{P}^2$, the degree d plane curves passing through the point p form a hyperplane $p^\perp$ in the projective space $|\mathcal{O}_{\mathbf{P}^2}(d)|$. Therefore, the problem of counting curves of degree d and genus g (and no further conditions) mentioned in the introduction amounts to the problem of computing the degree of the plain Severi variety $V^{d,\delta} = \overline{V}^d_g$, with $\delta = p_a(d) - g$. We shall denote this number by $N^{d,\delta}$. More generally, for all $\alpha, \beta \in \underline{\mathbf{N}}$, we shall denote by $N^{d,\delta}(\alpha,\beta)$ the common degree of all $V^d_g(\alpha,\beta)(\Omega)$, where Ω is a general, cardinality α, set of points of L.

The recursion procedure of Caporaso and Harris consists in computing the number $N^{d,\delta}$ by cutting $V^{d,\delta}$ by hyperplanes $p^\perp$ with $p \in L$. Their main result is Theorem (1.4) below. We need some additional notation to state it.

(1.2) Notation. For all $k \in \mathbf{N}^*$, we let $e_k = (0, \ldots, 0, 1) \in \underline{\mathbf{N}}$: it is the sequence with only one non-zero entry, in k-th position and equal to 1 (recall the convention made in [III] that we may omit the infinite sequence of zeros at the end of elements of $\underline{\mathbf{N}}$). For all $\beta, \beta' \in \underline{\mathbf{N}}$, we let

$$I^\beta = \prod_{k\geqslant 1} k^{\beta_k} \qquad \text{and} \qquad \tbinom{\beta'}{\beta} = \prod_{k\geqslant 1} \tbinom{\beta'_k}{\beta_k}.$$

If $\Omega = (\Omega_1, \Omega_2, \ldots)$ is a set of cardinality $\alpha \in \underline{\mathbf{N}}$, a subset of Ω of cardinality $\alpha' \leqslant \alpha$ is simply a set $\Omega' = (\Omega'_1, \Omega'_2, \ldots)$ of cardinality $\alpha' \in \underline{\mathbf{N}}$ such that $\Omega'_k \subseteq \Omega_k$ for all $k \geqslant 1$.

The following notation is not needed to state the main theorem below, but we shall use it for the proof and in the more precise Theorem (4.1). For all $\beta \in \underline{\mathbf{N}}$, we let

$$\mathrm{lcm}(\beta) = \mathrm{lcm}\{i : \beta_i \neq 0\} \qquad \text{and} \qquad \max(\beta) = \max\{i : \beta_i \neq 0\}.$$

(1.3) Convention. In the statement below, generalized Severi varieties $V^{d-1,\delta'}(\alpha', \beta')(\Omega') \subseteq |\mathcal{O}_{\mathbf{P}^2}(d-1)|$ are considered as the projective subvarieties of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ parametrizing all curves of the form $C + L$ with $C \in V^{d-1,\delta'}(\alpha', \beta')(\Omega')$. We will make this identification freely all along the text, where needed.

(1.4) Theorem (Caporaso–Harris). *Let d and δ be natural integers, and let $\alpha, \beta \in \underline{\mathbf{N}}$ be sequences such that $I\alpha + I\beta = d$, and Ω be a set of α general points on the line L. Let p be a general point of L. Then one has the following equality of cycles:*

$$\begin{aligned}(1.4.1)\quad V^{d,\delta}(\alpha,\beta)(\Omega) \cap p^\perp = &\sum_{k\geqslant 1:\, \beta_k>0} k \cdot V^{d,\delta}(\alpha+e_k, \beta-e_k)(\Omega \cup \{p\}) \\ &+ \sum_{\substack{\Omega'\subseteq\Omega,\, \mathrm{Card}(\Omega')=\alpha' \\ \beta'\geqslant\beta:\, I\alpha'+I\beta'=d-1 \\ \delta'=\delta+|\beta'-\beta|-d+1}} I^{\beta'-\beta}\binom{\beta'}{\beta} \cdot V^{d-1,\delta'}(\alpha',\beta')(\Omega')\end{aligned}$$

where, in the first sum, $\Omega \cup \{p\}$ means that we add the point p to the k-th term of $\Omega = (\Omega_1, \Omega_2, \ldots)$.

Note that, in the above theorem, if we start with a plain Severi variety for $V^{d,\delta}(\alpha,\beta)(\Omega)$ (i.e., if $\alpha = 0$ and thus $\Omega = \varnothing$, and $\beta = (d)$), then the intersection with $p^\perp$ decomposes as a sum of genuinely generalized (i.e., non-plain) Severi varieties. This is why one needs to generalize the problem of counting curves of fixed degree and genus for the inductive solution proposed by Caporaso and Harris.

It follows from Proposition (2.2) below that if p is outside of Ω, then no irreducible component of $V^{d,\delta}(\alpha,\beta)(\Omega)$ is contained in the hyperplane $p^\perp$. Therefore, all irreducible varieties appearing on the right-hand-side of (1.4.1) above have codimension one in $V^{d,\delta}(\alpha,\beta)(\Omega)$, and indeed we shall verify this by hand later in the text. The number $N^{d,\delta}(\alpha,\beta)$ we are looking for is the degree of $V^{d,\delta}(\alpha,\beta)(\Omega)$ as well as that of the cycle on the right-hand-side of (1.4.1). Thus, Theorem (1.4) has the following direct corollary.

(1.5) Corollary. *Let d and δ be natural integers, and let $\alpha, \beta \in \underline{\mathbf{N}}$ be sequences such that $I\alpha + I\beta = d$. Then one has the following equality:*

$$N^{d,\delta}(\alpha,\beta) = \sum_{k\geqslant 1:\, \beta_k>0} k \cdot N^{d,\delta}(\alpha+e_k, \beta-e_k) + \sum_{\substack{\alpha'\leqslant\alpha \\ \beta'\geqslant\beta:\, I\alpha'+I\beta'=d-1 \\ \delta'=\delta+|\beta'-\beta|-d+1}} I^{\beta'-\beta}\binom{\beta'}{\beta}\binom{\alpha'}{\alpha} \cdot N^{d-1,\delta'}(\alpha',\beta').$$

Since the numbers $N^{1,\delta}(\alpha,\beta)$ are readily computed, the above recursive formula enables one to compute all numbers $N^{d,\delta}(\alpha,\beta)$, and thus to completely solve our counting problem. Note that the recursion indeed terminates, as those terms appearing in the first sum are all of the form $N^{d,\delta}(\alpha'', \beta'')$ with $\beta'' < \beta$, so that unassigned intersection with L are exhausted after finitely many applications of the formula, and then only terms of the form $N^{d-1,\delta'}(\alpha', \beta')$ appear. It is of course helpful to watch the inductive process work on a concrete example. For this we refer to [VII], or [1, p. 349]; see also Example (5.1).

Although, arguably, the relevant enumerative problem is to count irreducible plane curves of any genus, there is no irreducibility requirement in our definition of Severi varieties, and therefore the numbers $N^{d,\delta}(\alpha, \beta)$ count curves regardless of their being irreducible or not. It is yet possible to derive a recursion formula counting irreducible curves using the inductive process of Caporaso and Harris (and so do they in the original article). We postpone this until Section 5, as the formula is yet more complicated than the above one, and having the geometry of the inductive process in mind helps in understanding it.

The proof of Theorem (1.4) is in two parts. The first one consists in using stable reduction in order to limit the possible irreducible components of the intersection of $V^{d,\delta}(\alpha, \beta)(\Omega)$ and $p^\perp$. This is the degeneration part; it is carried out in Section 3 and the result is Theorem (3.1). The dimensional characterization of Severi varieties (Proposition (2.3)) is fundamental in identifying the components of the intersection as generalized Severi varieties.

The second part consists in showing that all the generalized Severi varieties found in the first part indeed occur as irreducible components of the intersection of $V^{d,\delta}(\alpha, \beta)(\Omega)$ and $p^\perp$, and to compute their multiplicites in this intersection. This is the deformation part; it is carried out in Section 4 and the result is Theorem (4.1). A fundamental result for this part is the study of the deformation space of an m-tacnode into $m-1$ nodes, also due to Caporaso and Harris, and described in [IV]. The latter result is certainly the central result of this whole volume of notes.

Theorems (3.1) and (4.1) together directly imply Theorem (1.4).

2 – Recap on Severi varieties

(2.1) We shall use consistent notation to denote assigned and unassigned contact points with the line L. Consider a generalized Severi variety $V^{d,\delta}(\alpha, \beta)(\Omega)$ as in Definition (1.1). The assigned contact points with L are the elements of Ω, which we write as $\Omega = \big(\{p_{i,j}\}_{1\leqslant j\leqslant \alpha_i}\big)_{i\geqslant 1}$. Let $[C]$ be a general member of any irreducible component of $V^{d,\delta}(\alpha, \beta)(\Omega)$. Denote by $\bar{C} \to C$ the normalization of C, and by $\phi : \bar{C} \to \mathbf{P}^2$ its composition with the inclusion $C \subseteq \mathbf{P}^2$. By definition of the Severi varieties, there exist α points $q_{i,j} \in \bar{C}$, for all $i \geqslant 1$ and $1 \leqslant j \leqslant \alpha_i$, and β points $r_{i,j} \in \bar{C}$, for all $i \geqslant 1$ and $1 \leqslant j \leqslant \beta_i$, such that

$$\forall\, i \geqslant 1,\ \forall\, j = 1, \dots, \alpha_i : \quad \phi(q_{i,j}) = p_{i,j}$$

and

$$\phi^* L = \sum_{i\geqslant 1} \sum_{1\leqslant j\leqslant \alpha_i} i\, q_{i,j} + \sum_{i\geqslant 1} \sum_{1\leqslant j\leqslant \beta_i} i\, r_{i,j}.$$

Moreover, we let $s_{i,j} = \phi(r_{i,j})$ for all $i \geqslant 1$ and $1 \leqslant j \leqslant \beta_i$.

Thus, the assigned and unassigned contact points of C with L are, respectively, the $p_{i,j}$'s and the $s_{i,j}$'s, and their respective counterparts on the normalization $\bar{C}$ are the $q_{i,j}$'s and the $r_{i,j}$'s.[1]

The two following results are key ingredients in the proof of both Theorems (3.1) and (4.1). They are direct applications of Theorem (1.7) and Proposition (4.3) in [III], respectively.

[1] The reader may wish to keep in mind that $\phi(q_{i,j}) = p_{i,j}$ whereas $\phi(r_{i,j}) = s_{i,j}$; thus, beware, ϕ goes backwards with the alphabetical order for the assigned contact points, but forward for the unassigned ones.

(2.2) Proposition. *Let the notation be as in (2.1) above. The generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ has pure dimension $2d+g-1+|\beta|$, with $g = p_a(d)-\delta$. Morever:*
— *the curve $C \subseteq \mathbf{P}^2$ is δ-nodal, and smooth at its intersection points with the line L;*
— *the contact points of C with L, i.e., all the points $p_{i,j}$ and $s_{i,j}$, are altogether pairwise distinct;*
— *the counterparts on the normalization $\bar{C}$ of the contact points of C with L, i.e., all the points $q_{i,j}$ and $r_{i,j}$, are altogether pairwise distinct;*
— *for any curve G and finite set Γ in $\mathbf{P}^2$, if $(G \cup \Gamma)\cap\Omega = \varnothing$ and $[C]$ is general with respect to G and Γ, then C intersects G transversely and does not intersect Γ.*

(2.3) Proposition. *Let $V \subseteq |\mathcal{O}_{\mathbf{P}^2}(d)|$ be an irreducible variety, parametrizing genus g curves in the following sense: for a general member $[X]$ of V, there exists a smooth curve $\tilde{X}$ of genus g, and a morphism $f : \tilde{X} \to X$, not constant on any component of $\tilde{X}$, and such that the push-forward in the sense of cycles $f_*\tilde{X}$ equals the fundamental cycle of X.*

Let $\Omega \subseteq L$ be a finite set of points. For any general member $[X]$ of V, one has

$$\dim(V) \leqslant 2d+g-1+\mathrm{Card}\big((X\cap L)\setminus\Omega\big),$$

where the last number is defined set-theoretically (i.e., multiplicities do not count).

Moreover, if equality holds, then there exist $\alpha,\beta \in \underline{\mathbf{N}}$ such that V is a dense subset of a component of the generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$, $\delta = p_a(d)-g$, if and only if

$$\mathrm{Card}\big(f^{-1}(X\cap L)\big) = \mathrm{Card}(X\cap L),$$

with $f : \tilde{X} \to X$ a genus g morphism as above.

Lastly, let us mention that, although we pretend not to know it in this text, all generalized Severi varieties $V^{d,\delta}(\alpha,\beta)(\Omega)$ have their irreducible components in bijection with the possible splittings of degree d, δ-nodal curves, into several irreducible components. Thus, for instance, there is at most one irreducible component of $V^{d,\delta}(\alpha,\beta)(\Omega)$ parametrizing irreducible curves. This is proved in [12] for plain Severi varieties, and in [7] for generalized Severi varieties.

3 – Degenerations of generalized Severi varieties

This section is dedicated to the analysis of how curves in a generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ degenerate when one imposes the passing through a general point of the line L. The main result of this section is Theorem (3.1) below, but we will also provide in Section 3.4 a geometric decription of the degeneration at play, of fundamental importance for the proof of the theorem in the other direction, i.e., Theorem (4.1) on deformations of generalized Severi varieties, which is the object of the next section.

In the following statement, and as in Convention (1.3), for all $\delta',\alpha',\beta',\Omega'$, the generalized Severi variety $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$ is identified with a variety parametrizing degree d curves containing the line L, namely

$$\big\{[C\cup L]:\ [C]\in V^{d-1,\delta'}(\alpha',\beta')(\Omega')\big\} \ \subseteq\ |\mathcal{O}_{\mathbf{P}^2}(d)|.$$

(3.1) Theorem ([1, Theorem 1.2])**.** *Let $p \in L$ be a general point, and let V be an irreducible component of the intersection $V^{d,\delta}(\alpha,\beta)(\Omega)\cap p^\perp$. Then V is an irreducible component of one of the following varieties:*
a) for all k such that $\beta_k > 0$, the variety

$$V^{d,\delta}(\alpha+e_k,\beta-e_k)(\Omega\cup\{p_{k,\alpha_k+1}\})$$

with $p_{k,\alpha_k+1} = p$;

b) for all $\alpha' \leqslant \alpha$, $\beta' \geqslant \beta$, and $\delta' \leqslant \delta$ such that (i) $I\alpha' + I\beta' = d-1$, and (ii) $\delta - \delta' + |\beta' - \beta| = d-1$, the variety

$$V^{d-1,\delta'}(\alpha', \beta')(\Omega'),$$

where Ω' is an arbitrary cardinality α' subset of Ω.

(3.2) Remark. The condition $\delta - \delta' + |\beta' - \beta| = d - 1$ in case b), is equivalent to the condition $\dim(V^{d-1,\delta'}(\alpha', \beta')(\Omega')) = \dim(V^{d,\delta}(\alpha, \beta)(\Omega)) - 1$. Indeed,

$$\begin{aligned}\dim(V^{d,\delta}(\alpha, \beta)) - \dim(V^{d-1,\delta'}(\alpha', \beta')) &= (2d + g - 1 + |\beta|) - (2(d-1) + g' - 1 + |\beta'|) \\ &= g - g' + |\beta - \beta'| - 2,\end{aligned}$$

where we let g and g' be the genera of the members of $V^{d,\delta}(\alpha, \beta)$ and $V^{d-1,\delta'}(\alpha', \beta')$ respectively, and thus

$$g - g' = \binom{d-1}{2} - \delta - \binom{d-2}{2} + \delta' = d - 2 - \delta + \delta'$$

by the Pascal Formula.

Later on we will be able to explain geometrically where the $\delta - \delta'$ nodes degenerate, see Subsection 3.4.

(3.3) Remark. Note moreover that, still in case b), the conditions $\alpha' \leqslant \alpha$, $\beta' \geqslant \beta$, and $I\alpha' + I\beta' = d - 1$ altogether imply that $\alpha' < \alpha$, i.e., there exists an i_0 such that $\alpha'_{i_0} < \alpha_{i_0}$.

It is however possible that $\beta' = \beta$, but in that case all the irreducible components of $V^{d,\delta}(\alpha, \beta)$ having V in their intersection with $p^\perp$ parametrize reducible curves, see Remark (4.21).

Proof of Theorem (3.1). Let $[X_0]$ be a general point of V. There are two cases to be considered. The first case is when the curve X_0 does not contain L. Then, in order for $[X_0]$ to sit in $p^\perp$, we must have one of the mobile contact points (which we may loosely refer to as points of type β) be at the prescribed point p (note that by generality, p is off Ω). Thus one mobile contact point must be turned into an assigned contact point (i.e., a point of type β is turned into one of type α), hence V must be contained in a variety

$$V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p_{k,\alpha_k+1} = p\}).$$

Since the latter varieties all have pure dimension $\dim(V^{d,\delta}(\alpha, \beta)(\Omega)) - 1 = \dim(V)$, this implies that V is an irreducible component of a generalized Severi variety as in case a) of the theorem.

The remaining case is when X_0 has L as an irreducible component; then we shall loosely say that $X_0 = L \cup C$. In this case the content of Theorem (3.1) is Proposition (3.15) below. $\square$

The proof of Proposition (3.15) occupies most of this section. The general idea is to apply (a suitable version of) semistable reduction to a family of general members of $V^{d,\delta}(\alpha, \beta)(\Omega)$ degenerating to X_0, in order to derive some necessary conditions on X_0 which will limit the number of dimensions in which it can move. Comparing this with the fact that X_0 has to move in dimension $\dim(V^{d,\delta}(\alpha, \beta)(\Omega)) - 1$, and taking advantage of the dimensional characterization of generalized Severi varieties (Corollary (2.3)), we will then be able to conclude.

(3.4) Setup. Let V be an irreducible component of $V^{d,\delta}(\alpha, \beta) \cap p^\perp$, the general member of which has L as an irreducible component. Let X_0 be a general member of V, and let C be the sum of the components of X_0 supported on curves different from L; we shall loosely write $X_0 = L \cup C$.

(3.5) Semistable reduction. Let Γ be a general analytic curve in $V^{d,\delta}(\alpha,\beta)$ passing through the point $[X_0]$, and let $\mathcal{X} \to \Gamma$ be the corresponding family of curves. By the generality of Γ, its general member $[X_\gamma]$ is general in an irreducible component of $V^{d,\delta}(\alpha,\beta)$, hence it corresponds to a reduced curve X_γ with δ nodes as its only singularities.

Let $\nu : \Gamma^\nu \to \Gamma$ be the normalization of Γ, and let $b_0 \in \Gamma^\nu$ be a point with $\nu(b_0) = [X_0]$. We let $\mathcal{X}^\nu$ be the normalization of the total space $\mathcal{X} \times_\Gamma \Gamma^\nu$, obtained from $\mathcal{X}$ by the base change $\Gamma^\nu \to \Gamma$. The general fibre of $\mathcal{X}^\nu \to \Gamma^\nu$ is smooth of genus $g = p_a(d) - \delta$, by the Teissier Simultaneous Resolution Theorem, see [III].

Applying a suitable version of stable reduction to the family $\mathcal{X}^\nu \to \Gamma^\nu$, we arrive at a family of curves $f : \mathcal{Y} \to B$, with special fibre $Y_0 = f^{-1}(b_0)$, satisfying the following conditions:

- $\mathcal{Y} \to B$ is a nodal reduction of $\mathcal{X}^\nu \to \Gamma^\nu$, i.e., all fibres of f are reduced, nodal curves, and B is a branched cover of Γ^ν;
- the total space $\mathcal{Y}$ is smooth;
- there are $|\alpha|$ sections $Q_{i,j}$ $(1 \leqslant j \leqslant \alpha_i)$ and $|\beta|$ sections $R_{i,j}$ $(1 \leqslant j \leqslant \beta_i)$ of $\mathcal{Y} \to B$, altogether pairwise disjoint, such that $\pi(Q_{i,j}) = p_{i,j}$ (the points of Ω), and for all $b \neq b_0$ one has
$$(\pi|_{Y_b})^* L = \sum i\, Q_{i,j}|_{Y_b} + \sum i\, R_{i,j}|_{Y_b}\,;$$
- the map sending the general fibre Y_b of $\mathcal{Y}$ to the corresponding plane curve X_γ extends to a regular map $\pi : \mathcal{Y} \to \mathbb{P}^2$ (here $\gamma = \nu(\eta(b))$, see diagram (3.5.1) below);
- the family $\mathcal{Y} \to B$ is minimal with respect to the above properties.

Note that, since $\mathcal{Y}$ is smooth and the $Q_{i,j}$'s and $R_{i,j}$'s are sections, they do not pass through any singular point of the central fibre Y_0.

The family $\mathcal{Y} \to B$ is obtained from $\mathcal{X}^\nu \to \Gamma^\nu$ by applying a finite number of base changes and blow-ups. A number of additional such operations may be required with respect to the usual nodal reduction in order to first organize the points $q_{i,j}$ and $p_{i,j}$ on the fibres into sections (a priori they may be permuted by the monodromy), and then separate these sections. The situation is summarized in the diagram below.

(3.5.1)
$$\begin{tikzcd}
Q_{i.j}, R_{i,j} \arrow[r, hook] & \mathcal{Y} \arrow[d, "f"] \arrow[dr] \arrow[drrrr, bend left, "\pi"] & & & & \\
& B \arrow[ul, bend left] \arrow[dr, "\eta"] & \mathcal{X}^\nu \arrow[r] \arrow[d] & \mathcal{X} \arrow[r, hook] \arrow[d] & \Gamma \times \mathbb{P}^2 \arrow[r] & \mathbb{P}^2 \\
& & \Gamma^\nu \arrow[r] & \Gamma & &
\end{tikzcd}$$

3.1 – General picture of the degeneration

(3.6) Components of the central fibre of $\mathcal{Y}$. The irreducible components of the central fibre Y_0 of $\mathcal{Y}$ come in three types. Namely, we let:

- $\tilde{L}$ be the union of the components of Y_0 on which π is nonconstant and maps to L;
- $\tilde{C}$ be the union of the components of Y_0 on which π is nonconstant and maps to C;
- Z be the union of the components of Y_0 on which π is constant.

(3.7) Degenerations of assigned and mobile contact points. We shall now relabel the sections $(Q_{i,j})_{1 \leqslant j \leqslant \alpha_i}$ and $(R_{i,j})_{1 \leqslant j \leqslant \beta_i}$ according to the type of the irreducible component of the central fibre they pass through. We advise the reader to read this paragraph with Figure 1 at hand.

For all i, we let α_i^C and α_i^L be the number of the sections $Q_{i,j}$, $1 \leqslant j \leqslant \alpha_i$, passing through $\tilde{C}$ and $\tilde{L}$, respectively. We package all these numbers as sequences $\alpha^C = (\alpha_i^C)_{i\geqslant 1} \in \underline{\mathbf{N}}$ and $\alpha^L = (\alpha_i^L)_{i\geqslant 1} \in \underline{\mathbf{N}}$, and label the corresponding sections as $(Q_{i,j}^C)_{1\leqslant j\leqslant \alpha_i^C}$ and $(Q_{i,j}^L)_{1\leqslant j\leqslant \alpha_i^L}$, their respective intersection points with the central fibre as $(q_{i,j}^C)_{1\leqslant j\leqslant \alpha_i^C}$ and $(q_{i,j}^L)_{1\leqslant j\leqslant \alpha_i^L}$, and the images by π in L of the latter as $(p_{i,j}^C)_{1\leqslant j\leqslant \alpha_i^C}$ and $(p_{i,j}^L)_{1\leqslant j\leqslant \alpha_i^L}$. (Note that, by definition, also the $q_{i,j}^C \in \tilde{C}$ are mapped to L by π).

Similarly, for all i we let β_i^C be the number of the sections $R_{i,j}$, $1 \leqslant j \leqslant \beta_i$, passing through $\tilde{C}$. We also let $\beta^C = (\beta_i^C)_{i\geqslant 1} \in \underline{\mathbf{N}}$, label the corresponding sections $(R_{i,j}^C)_{1\leqslant j\leqslant \beta_i^C}$, and name $(r_{i,j}^C)_{1\leqslant j\leqslant \beta_i^C}$ their respective points on the central fibre Y_0, and $(s_{i,j}^C)_{1\leqslant j\leqslant \beta_i^C}$ their images on L by π.

The aware reader will have noticed that we have not considered all possibilities, as there are neither α^Z, nor β^Z or β^L. The reason for this is that these turn out not to exist, as we will see in Claims (3.9) and (3.10) below.

(3.8) New mobile contact points. We now identify some new points on $\tilde{C}$ which, as we shall see, induce new mobile contact points with L as C moves in the family V. These are the intersection points of $\tilde{C}$ with the vertical part of π^*L, i.e., the part of the divisor $\pi^*L \subseteq \mathcal{Y}$ with support contained in the central fibre Y_0, which is

$$(\pi^*L)^{\mathrm{vert}} = \pi^*L - \sum iQ_{i,j} - \sum iR_{i,j}.$$

Thus, for all i, we let $\beta_i^{C\cap L}$ be the number of points in $\tilde{C} \cap (\pi^*L)^{\mathrm{vert}}$ appearing with multiplicity i in the divisor $(\pi^*L)^{\mathrm{vert}}$, and label these points as $(r_{i,j}^{C\cap L})_{1\leqslant j\leqslant \beta_i^{C\cap L}}$. We also let $(s_{i,j}^{C\cap L})_{1\leqslant j\leqslant \beta_i^{C\cap L}}$ be the images of these points by π, and $\beta^{C\cap L} = (\beta_i^{C\cap L})_{i\geqslant 1} \in \underline{\mathbf{N}}$.

The reason for the notation is that it will turn out that the support of $(\pi^*L)^{\mathrm{vert}}$ is the union of $\tilde{L}$ and those connected components of Z which intersect $\tilde{L}$ (this follows from Claim (3.9) below and the argument given in (3.19)). Therefore, on the surface $\mathcal{Y}^\flat$ obtained from $\mathcal{Y}$ by contracting Z, we can see the points $r_{i,j}^{C\cap L}$ as the intersection points of (the images in $\mathcal{Y}^\flat$ of) $\tilde{C}$ and $\tilde{L}$.

The situation is summarized in Figure 1 below. In fact, we want this figure to picture the situation as it occurs in reality, and therefore we shall make a number of claims before we draw the picture. The proof of these claims occupies a substantial part of the remainder of this section.

The final interpretation in terms of degenerations of Severi varieties of the data introduced above is given in Proposition (3.15) below.

(3.9) Claim. The sections $(Q_{i,j})_{1\leqslant j\leqslant \alpha_i}$ and $(R_{i,j})_{1\leqslant j\leqslant \beta_i}$ are disjoint from Z.

(3.10) Claim. All sections $(R_{i,j})_{1\leqslant j\leqslant \beta_i}$ pass through $\tilde{C}$.

Note that Claims (3.9) and (3.10) imply that

$$\{Q_{i,j}\}_{1\leqslant i\leqslant \alpha_i} = \{Q_{i,j}^C\}_{1\leqslant i\leqslant \alpha_i'} \cup \{Q_{i,j}^L\}_{1\leqslant i\leqslant \alpha_i''} \quad \text{and} \quad \{R_{i,j}\}_{1\leqslant i\leqslant \alpha_i} = \{R_{i,j}^C\}_{1\leqslant i\leqslant \alpha_i'}.$$

(3.11) Claim. The curve $\tilde{L}$ consists of a unique irreducible component, on which π is an isomorphism.

This claim implies that the line L appears with multiplicity 1 in X_0, so $X_0 = C \cup L$ with C a degree $d-1$ curve.

(3.12) Claim. The curve Z consists of a disjoint union of chains of rational curves joining $\tilde{L}$ to $\tilde{C}$.

This Claim implies that the surface $\mathcal{Y}^\flat$ obtained from the smooth surface $\mathcal{Y}$ by contracting Z has only rational double points as singularities.

(3.13) Claim. The map π is a birational isomorphism on each irreducible component of $\tilde{C}$.

This Claim implies that the curve C is reduced.

(3.14) Claim. The curve $\tilde{C}$ is smooth (though not connected, in general).

We may now draw a picture of the situation.

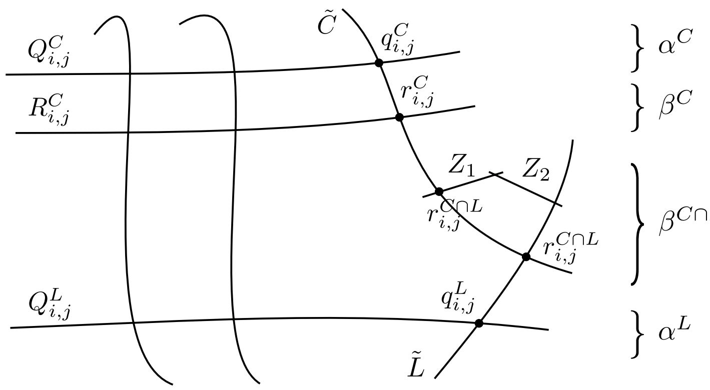

Figure 1: General picture of the degeneration

We shall prove the following result, which encapsulates Theorem (3.1) in the case of an irreducible component having general member of the form $X_0 = C \cup L$.

(3.15) Proposition. *In the situation of (3.1), the curve C is a general member of the generalized Severi variety $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$, where:*
— $\alpha' = \alpha^C$*, and* $\Omega' = (p^C_{i,j})_{1\leqslant j\leqslant \alpha^C_i}$*;*
— $\beta' = \beta^C + \beta^{C\cap L}$*;*
— $\delta' = \delta - d + 1 + |\beta' - \beta|$*.*

In other words, we will prove that the assigned contact points for the curves X_γ (recall the notation from (3.5)) that end up on $\tilde{C}$ transfer to assigned contact points for C, whereas those that end up on $\tilde{L}$ vanish, being accounted for by the component L of $X_0 = C \cup L$; moreover, all mobile contact points of X_γ end up on $\tilde{C}$ and thus transfer to mobile contact points on C; the curve C has additional mobile contact points with L, which come out of the intersection points of $\tilde{C}$ and $\tilde{L}$ in $\mathcal{Y}^\flat$.

We will provide an explicit geometric description of the degeneration of the curves X_γ in a neigbourhood of all these points in Subsection 3.4 below. Then, we will be able to track the limits of the δ nodes of the curves X_γ, and thus provide a geometric explanation for the formula relating δ' and δ.

3.2 – Simplified proof of Proposition (3.15)

We follow the presentation of [1], and first give a proof of Proposition (3.15) under some simplifying assumptions, in order to describe as clearly as possible what is actually going on in reality. We shall prove later on that these simplifying assumptions do indeed hold, in Subsection 3.3.

We have made six claims in Subsection 3.1 above, with numbers (3.9)–(3.14). Our simplifying assumptions consist in taking the three Claims (3.9), (3.11), and (3.12) for granted, for the moment.

(3.16) Proof of Proposition (3.15) assuming Claims (3.9), (3.11), and (3.12). Claim (3.12) ensures that the surface $\mathcal{Y}^\flat$ obtained from $\mathcal{Y}$ by contracting Z has at worst rational double points as singularities, and is still a nodal reduction of the family $\mathcal{X} \to \Gamma$. From now on we work on this new family, and denote any curve in $\mathcal{Y}^\flat$ image of a curve in $\mathcal{Y}$ by the same symbol (with the exception of the central fibre of $\mathcal{Y}^\flat$, which we will denote by $Y_0^\flat$).

Since Y_0 and $Y_0^\flat$ are semistable limits of the curves X_γ, which are smooth of genus g, it holds that
$$p_a(Y_0) = p_a(Y_0^\flat) = g.$$
Claim (3.11) ensures that $\tilde{L} \simeq L$ is rational. On the other hand, $Y_0^\flat = \tilde{L} \cup \tilde{C}$ is nodal, and the total number of points in $\tilde{C} \cap \tilde{L}$ is $|\beta^{C\cap L}|$. Thus one finds

$$g = p_a(Y_0^\flat) = p_a(\tilde{L} \cup \tilde{C}) = p_a(\tilde{C}) + |\beta^{C\cap L}| - 1. \tag{3.16.1}$$

We shall now apply Corollary (2.3), which gives a dimension bound on families of genus g curves and a dimensional characterization of generalized Severi varieties. Let V be the irreducible component of $V^{d,\delta}(\alpha,\beta) \cap p^\perp$ of which $X_0 = C \cup L$ is a general member. The dimension bound tells us that

$$\dim(V) \leqslant 2(d-1) + g(\tilde{C}) - 1 + |\beta^C + \beta^{C\cap L}|, \tag{3.16.2}$$

since the number of unassigned contact points of C with L is at most $|\beta^C + \beta^{C\cap L}|$, and C has degree at most $d-1$ (indeed, the unassigned contact points of C with L are the points in the support of $\pi_*\tilde{C} \cdot L$ off Ω, one has $\pi_*\tilde{C} \cdot L = \pi_*(C \cdot \pi^* L)$ by the projection formula, and the intersection points of $\tilde{C}$ with the sections Q_{ij} are mapped to Ω; besides, Claim (3.9) ensures that the number $|\beta^C + \beta^{C\cap L}|$ does not count any point in $\tilde{C} \cap \tilde{L} \subseteq Y_0^\flat$ several times).

But $p^\perp$ is a hyperplane which does not contain any irreducible component of $V^{d,\delta}(\alpha,\beta)(\Omega)$, by Proposition (2.2), and since $p \notin \Omega$. Hence

$$\dim(V) = \dim\big(V^{d,\delta}(\alpha,\beta)\big) - 1 = 2d + g - 2 + |\beta|. \tag{3.16.3}$$

In the upshot, we have the following sequence of equalities and inequalities:

$$\begin{aligned} 2d + g + |\beta| - 2 &\leqslant 2(d-1) + g(\tilde{C}) - 1 + |\beta^C + \beta^{C\cap L}| && \text{by (3.16.2) and (3.16.3)} \\ &\leqslant 2(d-1) + p_a(\tilde{C}) - 1 + |\beta^C + \beta^{C\cap L}| && \text{since } g(\tilde{C}) \leqslant p_a(\tilde{C}) \\ &= 2(d-1) + g - |\beta^{C\cap L}| + |\beta^C + \beta^{C\cap L}| && \text{by (3.16.1)} \\ &= 2(d-1) + g + |\beta^C| \\ &\leqslant 2(d-1) + g + |\beta|. \end{aligned} \tag{3.16.4}$$

We conclude that equality holds throughout. In particular, (3.16.2) is an equality and therefore, by Corollary (2.3), V is an irreducible component of the Severi variety $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$ where α', Ω', and β' are as stated in Proposition (3.15), and δ' is determined by the geometric genus of C (note in particular that the orders of contacts of C with L are given by the multiplicities in $\pi^* L \cdot \tilde{C}$, and therefore they are indeed α' and β'). Yet a couple of words are in order as to why the condition $\mathrm{Card}(C \cap L) = \mathrm{Card}(\pi^{-1}(C \cap L))$ of Corollary (2.3) is indeed verified: on the one hand, it follows from Claim (3.9) that the sections $Q_{i,j}$ are still mutually disjoint in $\mathcal{Y}^\flat$, so that there is a unique point in $\tilde{C}$ over each point of Ω'; on the other hand, the equality in (3.16.2) implies (by Corollary (2.3) again) that C indeed intersects L at $|\beta^C + \beta^{C\cap L}|$ pairwise distinct points, so that eventually all points in $\pi^{-1}(C \cap L)$ have distinct images in L, as required.

To finish, let us explicitly derive the formula for δ': it follows from the series of equalities in (3.16.4) that the geometric genus of C is

$$g(C) = g(\tilde{C}) = p_a(\tilde{C}) = g - |\beta^{C\cap L}| + 1,$$

so that

$$\begin{aligned}\delta' = \binom{d-2}{2} - g(C) &= \binom{d-2}{2} - g + |\beta^{C\cap L}| - 1 \\ &= \binom{d-2}{2} - \binom{d-1}{2} + \delta + |\beta^{C\cap L}| - 1 = \delta - (d-1) + |\beta^{C\cap L}|,\end{aligned}$$

which is the value asserted in Proposition (3.15), as $\beta^{C\cap L} = \beta' - \beta$, since $\beta^C = \beta$ by the series of equalities in (3.16.4). □

(3.17) Proof of Claims (3.10), (3.13), and (3.14) assuming Claims (3.9), (3.11), and (3.12). As we have already observed, it follows from the series of equalities in (3.16.4) that $\beta = \beta^C$, which proves Claim (3.10).

On the other hand, the fact that equality holds in (3.16.2) implies by Corollary (2.3) that C is reduced. Thus, Claim (3.13) holds. Eventually, we have already observed that the series of equalities in (3.16.4) implies the equality $g(\tilde{C}) = p_a(\tilde{C})$, which proves that $\tilde{C}$ is smooth, i.e., Claim (3.14) holds. □

3.3 – Complete proof of Proposition (3.15)

(3.18) Labelling of the remaining sections. In order to prove Claims (3.9), (3.11), and (3.12), we need to consider the situation as complicated as it could a priori be, and therefore to introduce some more notation with respect to that already given in Subsection 3.1.

Thus, for all i we define (in the surface $\mathcal{Y}$) $\beta_i^{C,Z}$ as the number of the sections $(R_{i,j})_{1\leqslant j\leqslant \beta_i}$ passing through a connected component of Z that intersects $\tilde{C}$, and package these numbers in the sequence $\beta^{C,Z} = (\beta_i^{C,Z})$.

On the other hand we redefine $|\beta^{C\cap L}|$ as the sum of the number of points of $\tilde{C}$ meeting $\tilde{L}$ plus the number of connected components of Z meeting both $\tilde{C}$ and $\tilde{L}$. Thus it is the number of intersection points between $\tilde{C}$ and $\tilde{L}$ after we contract Z. Note that $|\beta^{C\cap L}|$ no longer corresponds to an actual sequence $\beta^{C\cap L} = (\beta_i^{C\cap L})$; the point is, without Claim (3.12), there could be a connected component of Z meeting $\tilde{C}$ at several distinct points with different multiplicities in $(\pi^*L)^{\mathrm{vert}}$.

(3.19) Bounding the number of mobile contact points of C and L. To this end, we note that every connected component of Z mapping to a point of L by π necessarily intersects either $\tilde{L}$, or a section among $Q_{i,j}$ and $R_{i,j}$. Indeed, otherwise there would exist a connected component of $\pi^{-1}(L)$ entirely contained in Z, and thus after contracting Z we would obtain a surface with an isolated point in the preimage of L, which is impossible.

Therefore the number of mobile contact points of C and L (in $\mathbf{P}^2$) is bounded from above as follows:

$$\operatorname{Card}\bigl(C \cap (L - \Omega)\bigr) \leqslant |\beta^C + \beta^{C,Z}| + |\beta^{C\cap L}|. \tag{3.19.1}$$

(3.20) Bounding the genus of C. As in (3.16), Y_0 is a semistable limit of the curves X_γ, and therefore $p_a(Y_0) = g$. Since the number of intersection points between $\tilde{C}$ and $\tilde{L}$ after we contract Z is $|\beta^{C\cap L}|$, we find that

$$g = p_a(Y_0) \geqslant p_a(\tilde{L}) + p_a(\tilde{C}) + |\beta^{C\cap L}| - 1. \tag{3.20.1}$$

Note that this inequality is strict if some connected component of Z has positive arithmetic genus, or intersects $\tilde{C}$ or $\tilde{L}$ at more than one point.

(3.21) Proof of Claim (3.11). Let e be the multiplicity of L in $X_0 = C \cup L$. Then C has degree $d - e$. On the other hand, $\tilde{L}$ is mapped $e : 1$ to L by π, so by Riemann–Hurwitz we have the inequality $p_a(\tilde{L}) \geqslant -e + 1$. Together with (3.20.1), this gives the bound:

$$p_a(\tilde{C}) \leqslant g - |\beta^{C\cap L}| + e. \tag{3.21.1}$$

Now, we argue exactly as in (3.16): by Corollary (2.3), we find that the irreducible component V of $V^{d,\delta}(\alpha,\beta) \cap p^\perp$ that C moves in has dimension at most

$$\begin{aligned} 2(d-e) + p_a(\tilde{C}) - 1 + |\beta^C + \beta^{C,Z}| + |\beta^{C\cap L}| &\leqslant 2(d-e) + g + |\beta^C + \beta^{C,Z}| + e - 1 \\ &= 2d + g + |\beta^C + \beta^{C,Z}| - e - 1. \end{aligned} \tag{3.21.2}$$

But since
$$\dim(V) = \dim\bigl(V^{d,\delta}(\alpha,\beta)\bigr) - 1 = 2d + g + |\beta| - 2,$$
and $|\beta| \leqslant |\beta^C + \beta^{C,Z}|$, one necessarily has $e = 1$. □

(3.22) Proof of Claim (3.12). By the fact that (3.21.2) is in fact an equality, we see that also (3.16.1) must be an equality. Thus, as already observed in (3.20), every connected component of Z has arithmetic genus 0, i.e., it is a tree of rational curves, and intersects each of $\tilde{L}$ and $\tilde{C}$ at most once.

In order to prove that such a connected component is a chain of rational curves connecting $\tilde{L}$ and $\tilde{C}$, it is therefore sufficient to prove that any "end component" of a connected component of Z meets either $\tilde{C}$ or $\tilde{L}$. By an end component, we mean an irreducible component Z_1 that meets at most one other irreducible component of Z (one if the corresponding connected component of Z has several irreducible components, zero if Z_1 is itself a connected component of Z).

The key argument is that if an end component Z_1 of Z intersects neither $\tilde{C}$ nor $\tilde{L}$, then by the minimality of $\mathcal{Y}$ there must be at least two of the sections $(Q_{i,j})_{1\leqslant j\leqslant\alpha_i}$ and $(R_{i,j})_{1\leqslant j\leqslant\beta_i}$ meeting Z_1, for otherwise we could blow down Z_1 in $\mathcal{Y}$ and still satisfy all the conditions imposed on $\mathcal{Y}$. (Note that Z_1, being a rational curve intersecting $Y_0 - Z_1$ in exactly one point since Y_0 is connected, is a (-1)-curve).

These two sections cannot be both of type α, for otherwise they would be contracted by π to the same point of L, which is excluded by the definition of generalized Severi varieties. To rule out the other possibilities, we use the fact that since (3.21.2) is an equality, then also (3.19.1) must be an equality. But if two sections either both of type β, or of types α and β respectively, meet the same connected component of Z, then the two corresponding points of C in $\mathbf{P}^2$ are the same, hence (3.19.1) is a strict inequality. This ends the proof of Claim (3.12). □

(3.23) Proof of Claim (3.9). Now we know that Claim (3.12) holds, i.e., the connected components of Z are chains of rational curves connecting $\tilde{C}$ and $\tilde{L}$. Therefore we can indeed consider the points $(r^{C\cap L}_{i,j})_{1\leqslant j\leqslant\beta^{C\cap L}_i}$ introduced in (3.8), and their respective images by π, the points $(s^{C\cap L}_{i,j})_{1\leqslant j\leqslant\beta^{C\cap L}_i}$ (see also (3.18)).

Our main argument here is that (3.19.1) is an equality, as we have already observed in (3.22) above. If a section of type α intersects a connected component of Z, then the image in $\mathbf{P}^2$ of the point $r_{i,j}^{C\cap L}$ corresponding to this component must lie in Ω, which implies that (3.19.1) is a strict inequality, a contradiction.

Similarly, if a section of type β intersects a connected component of Z, then the point $r_{i,j}^{C,Z}$ corresponding to this section and the point $r_{i,j}^{C\cap L}$ corresponding to the component of Z have the same image in $\mathbf{P}^2$, hence (3.19.1) is a strict inequality, a contradiction. □

We have now proved all three Claims (3.9), (3.11), and (3.12). In Subsection 3.2 we had proved Proposition (3.15) assuming these three claims. Thus, we now have a complete proof of Proposition (3.15), and the proof of Theorem (3.1) is over.

3.4 – Local picture of the degeneration of curves

We consider the situation set up in (3.1) and (3.5), when a general member X_γ of $V^{d,\delta}(\alpha,\beta)$ degenerates to a curve $X_0 = L\cup C$. We shall now give a complete description of this degeneration, in its concrete incarnation as a degeneration of plane curves. This will be useful in the next section, when we prove the theorem going the other direction than Theorem (3.1) above.

We carry on with the notation introduced above for Proposition (3.15). Something non-trivial happens only at the points where C and L intersect, which come in the following four types: (3.24)–(3.27).

(3.24) Vanishing assigned contact points. These are the points of $\Omega \setminus \Omega'$, i.e., the points $p_{i,j}^L$. Since C is general in the component V of $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$, by Proposition (2.2) it does not pass through any point $p_{i,j}^L$. Thus the curve X_0 is smooth at those points, i.e., it has only one local branch there, which is an open subset of the line L itself.

Therefore the degeneration happens like this: locally at a point $p_{i,j}^L$, the curves X_γ, $\gamma \neq 0$, have one smooth local branch tangent to the order i with L, which degenerates in X_0 to an open neighbourhood of $p_{i,j}^L$ in L.

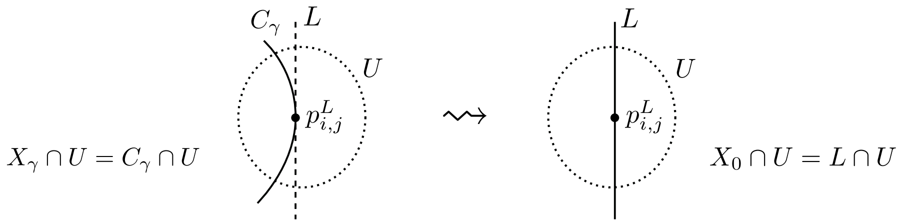

Figure 2: Degeneration in a neighbourhood U of a vanishing assigned contact point

(3.25) Non-vanishing assigned contact points. These are the points of Ω', i.e., the points $p_{i,j}^C$. Such a point is by definition the image of the point $q_{i,j}^C \in \tilde{C}$, and C is smooth at $p_{i,j}^C$ by Proposition (2.2). On the other hand $p_{i,j}^C$ also has a preimage in $\tilde{L}$, and exactly one as $\tilde{L} \to L$ is an isomorphism: let's call it $(q_{i,j}^C)'$. It follows that X_0 has two smooth local branches at $p_{i,j}^C$, tangent to i-th order, so that $p_{i,j}^C$ is an i-tacnode of X_0.

As we have just seen, the map $Y_0 \to X_0$ factors through the normalization of $p_{i,j}^C$ in X_0 (recall that Y_0 is the central fibre of the semistable reduction introduced in (3.5)), and thus the family $\mathcal{Y}$ is smooth in a neighbourhood of the two preimages of $p_{i,j}^C$ in Y_0. This implies that the curves X_γ, as they approach X_0, have two local branches analytically locally around $p_{i,j}^C \in \mathbf{P}^2$.

The difference in arithmetic genus between Y_b and X_γ over a neighbourhood of $p^C_{i,j}$ in $\mathbf{P}^2$ is constant as b moves (with $\gamma = \nu(\eta(b)$, see diagram (3.5.1)), because both $\mathcal{Y}$ and $\mathcal{X}$ are flat families. Moreover, $\mathcal{X}$ is equigeneric over $\Gamma \setminus \{0\}$, and $\mathcal{Y}$ is a nodal reduction of $\mathcal{X}$, so for $b \neq 0$, the difference in arithmetic genus between Y_b and X_γ is just the sum of the δ-invariants of the singularities of X_γ; since the curves X_γ are nodal for $\gamma \neq 0$, this is therefore the number of nodes of X_γ tending to the point $p^C_{i,j}$ as γ tends to 0. On the other hand, over a neighbourhood of $p^C_{i,j}$, $Y_0 \to X_0$ is the normalization of an i-tacnode, so the difference in arithmetic genus between Y_0 and X_0 over a neighbourhood of $p^C_{i,j}$ equals i.

The upshot is that locally at $p^C_{i,j}$, the curves X_γ, $\gamma \neq 0$, have two local branches intersecting transversely in i points tending to the i-tacnode of X_0 at $p^C_{i,j}$; one of the two branches of X_γ is tangent to order i with L at the fixed point $p^C_{i,j} \in \Omega' \subseteq \Omega$ (this is the branch corresponding to the branch of Y_b intersecting the section $Q^C_{i,j}$, namely that tending to the branch of Y_0 contained in $\tilde{C}$).

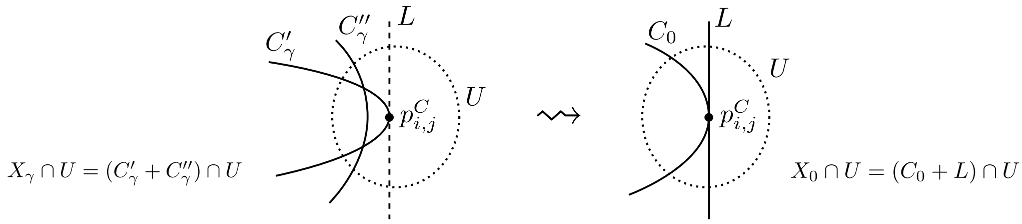

Figure 3: Degeneration in a neighbourhood U of a non-vanishing assigned contact point

(3.26) Limits of unassigned contact points. These are the points $s^C_{i,j}$. At those points the situation is almost exactly the same as that at the points $p^C_{i,j}$ described above, the key point being that the points $s^C_{i,j} \in L$ have two points in their preimage in Y_0, namely $r^C_{i,j} \in \tilde{C}$ and an additional point on $\tilde{L}$, which we may call $(r^C_{i,j})'$ so that, as in (3.25), the map $Y_0 \to X_0$ factors through the normalization of the i-tacnode of X_0 at $s^C_{i,j}$. The only difference is that here the branch of X_γ which is tangent to order i with L touches L at a mobile point, different from $s^C_{i,j}$ by Proposition (2.2) (this is the branch corresponding to the branch of Y_b intersecting the section $R^C_{i,j}$, which tends to a local branch of $\tilde{C}$).

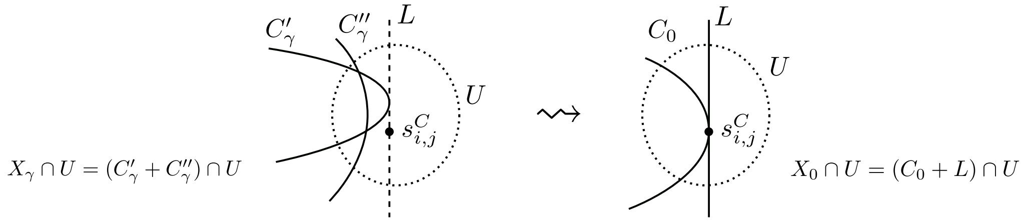

Figure 4: Degeneration in a neighbourhood U of a limit of unassigned contact points

(3.27) New unassigned contact point. These are the points $s^{C\cap L}_{i,j}$. Such a point is the image of a point $r^{C\cap L}_{i,j}$ which, in the surface $\mathcal{Y}^\flat$ obtained from $\mathcal{Y}$ by contracting Z, lies at the intersection of $\tilde{C}$ and $\tilde{L}$; it is thus an ordinary node of the central fibre $Y^\flat_0$ (in fact, seen in $\mathcal{Y}$ it is a node as well, although there it may lie on $\tilde{C} \cap Z$ rather than on $\tilde{C} \cap \tilde{L}$), and the point $s^{C\cap L}_{i,j}$ therefore has only one point in its preimage in Y_0. Since the nearby fibres Y_b are smooth (because $\mathcal{X}$ is equigeneric over $\Gamma \setminus \{0\}$, and $\mathcal{Y}$ is a nodal reduction of $\mathcal{X}$), they have only one branch in an analytic neighbourhood of the preimage of $s_{i,j}^{C\cap L}$ in $\mathcal{Y}$, and therefore the curves X_γ, $\gamma \neq 0$, have only one local branch in a neighbourhood of $s_{i,j}^{C\cap L}$ in $\mathbf{P}^2$.

The points $s_{i,j}^{C\cap L}$ are again i-tacnodes of the curve X_0, this time because both L and C are smooth at $s_{i,j}^{C\cap L}$ by Proposition (2.2), and intersect there with multiplicity i because by definition $s_{i,j}^{C\cap L}$ lies on a component of multiplicity i of $(\pi^*L)^{\mathrm{vert}}$. In particular the difference in arithmetic genus between Y_0 and X_0 over $s_{i,j}^{C\cap L}$ is $i-1$. Since this difference for the families $\mathcal{Y}$ and $\mathcal{X}$ is constant over a neighbourhood of $s_{i,j}^{C\cap L}$, and the curves X_γ are nodal for $\gamma \neq 0$, we see that the latter will have $i-1$ nodes tending to $s_{i,j}^{C\cap L}$.

To sum up, analytically locally around $s_{i,j}^{C\cap L}$, the curves X_γ, $\gamma \neq 0$, have one irreducible, $(i-1)$-nodal local branch, transverse to L and not passing by $s_{i,j}^{C\cap L}$, degenerating to the i-tacnode at $s_{i,j}^{C\cap L}$ formed by C and L.

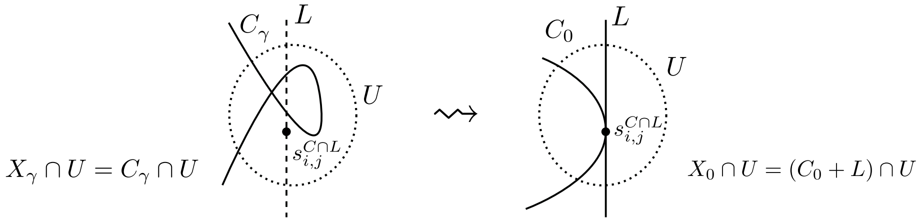

Figure 5: Degeneration in a neighbourhood U of a new unassigned contact point

We emphasize that this phenomenon is arguably the central topic of this whole volume; its local model has been studied in detail in the previous chapters [IV] and [A].

Whereas the above description of the degeneration will be needed in the next section, the remainder of this subsection is not necessary for the proof of the main theorem. I believe it is nevertheless helpful.

First of all, having the above descriptions at hand, we may now give the promessed geometric interpretation for the relation between δ and δ' in Theorem (3.1) and Proposition (3.15).

(3.28) Book-keeping of the nodes in the degeneration. As a result of (3.24)–(3.27) above, we see that as X_γ degenerates to $X_0 = C \cup L$ there are i nodes of X_γ tending to each point of types α_i^C and β_i^C (respectively the preserved tangencies at assigned points, and the limits of tangencies at unassigned points, which are all preserved), and $i-1$ nodes of X_γ tending to each point of type $\beta_i^{C\cap L}$ (the new tangency conditions, all at unassigned points); the degeneration is otherwise equisingular. Therefore, we have:

$$\delta - \delta' = I\alpha^C + I\beta^C + \big(I\beta^{C\cap L} - |\beta^{C\cap L}|\big).$$

On the other hand we know that, in the notation of Proposition (3.15), $\alpha' = \alpha^C$ and $\beta' = \beta^C + \beta^{C\cap L}$, so that

$$I\alpha^C + I\beta^C + I\beta^{C\cap L} = I\alpha' + I\beta' = C \cdot L = d-1;$$

since moreover $\beta = \beta^C$, we also have $|\beta^{C\cap L}| = |\beta' - \beta|$. Finally, we find

$$\delta - \delta' = (d-1) - |\beta' - \beta|$$

as required.

Lastly, let us analyze the multiplicities of the irreducible components of Y_0 in the divisor π^*L. This will be helpful in understanding the multiplicities of the components of the type $V^{d-1,\delta'}(\alpha', \beta')$ in the intersection of $V^{d,\delta}(\alpha, \beta)$ with the hyperplane $p^\perp$, cf. Theorem (4.1).

(3.29) Multiplicities of the divisor π^*L in $\mathcal{Y}$. Recall that the central fibre Y_0 of $\mathcal{Y}$ is reduced and consists of (i) $\tilde{L}$, an irreducible curve mapped isomorphically to L, (ii) $\tilde{C}$, the normalizations of the various irreducible components of C, and (iii) Z, the union of disjoint chains of rational curves joining $\tilde{L}$ to $\tilde{C}$.

Let m be the multiplicity of $\tilde{L}$ in the divisor π^*L on $\mathcal{Y}$. Let us remind that the integer m records the ramification of $\pi : \mathcal{Y} \to \mathbf{P}^2$ along $\tilde{L}$ in the direction normal to $\tilde{L}$, hence $m > 1$ does not contradict $\pi|_{\tilde{L}}$ being an isomorphism onto L; instead, if $\pi|_{\tilde{L}} : \tilde{L} \to L$ had degree k, this would translate into $\pi_*\tilde{L} = kL$.

The main output of this paragraph will be the divisibility of m by $\mathrm{lcm}(\beta^{C\cap L}) = \mathrm{lcm}(\beta' - \beta)$. To see this, consider a point $r_{i,j}^{C\cap L} \in \tilde{C}$. By definition, see Paragraph (3.8), $r_{i,j}^{C\cap L}$ sits both on $\tilde{C}$ and on a multiplicity i component of $(\pi^*L)^{\mathrm{vert}}$, and we have seen that $(\pi^*L)^{\mathrm{vert}}$ is supported on $\tilde{L} + Z$. Thus, if $i < m$, then $r_{i,j}^{C\cap L}$ sits on a smooth irreducible rational curve $Z_1 \subseteq Z$ which has multiplicity i in π^*L. Let Z_2 be the next curve in the chain of rational curves joining $\tilde{C}$ to $\tilde{L}$ and containing Z_1 (here, 'next' is intended relative to the direction pointing to $\tilde{L}$; if Z_1 directly joins $\tilde{C}$ to $\tilde{L}$, then we take $Z_2 = \tilde{L}$), and let i_2 be its multiplicity in π^*L. Since π contracts Z_1, we have $\pi^*L \cdot Z_1 = 0$, hence

$$(iZ_1 + i_2 Z_2) \cdot Z_1 = 0.$$

Since Z_1 is a (-2)-curve[2] and $Z_1 \cdot Z_2 = 1$, we find that $i_2 = 2i$. We continue by induction to compute the successive multiplicities: the connected component of Z containing Z_1 is a chain of (-2)-curves; let us denote them by $Z_1, \dots, Z_l$, in that order, and let $Z_{l+1} = \tilde{L}$. Let $i_1, \dots, i_{l+1}$ be their multiplicities in π^*L. Once we know, for some $k \leqslant l$, that $i_s = si$ for all $s = 1, \dots, k$, we have

$$\pi^*L \cdot Z_k = (i_{k-1}Z_1 + i_k Z_k + i_{k+1}Z_{k+1}) \cdot Z_k = 0,$$

hence $i_{k+1} = 2i_k - i_{k-1} = (k+1)i$, as required. Finally, the multiplicity m of $\tilde{L} = Z_{l+1}$ in π^*L equals $(l+1)i$; thus, m is divisible by i, and the length of the chain of (-2)-curves is $l = m/i - 1$.

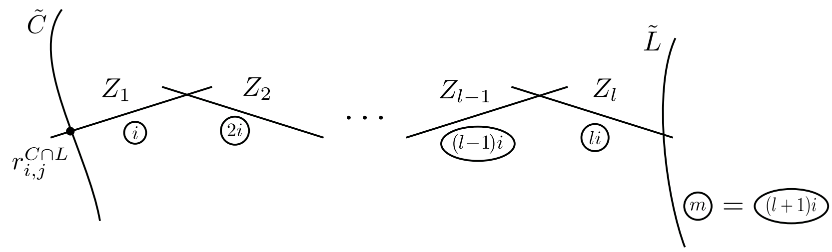

Figure 6: Multiplicities of the components of $(\pi^*L)^{\mathrm{vert}}$

The same argument shows that there are no points $r_{i,j}^{C\cap L}$ with $i > m$, and all points $r_{m,j}^{C\cap L}$ appear directly on $\tilde{L}$.

In conclusion, for all i such that $\beta_i^{C\cap L} = (\beta' - \beta)_i > 0$, the multiplicity m is divisible by i; equivalently, m is divisible by $\mathrm{lcm}(\beta' - \beta)$.

4 – Deformations of generalized Severi varieties

In this section we prove the result going the other direction than Theorem (3.1). It asserts that all the generalized Severi varieties which, according to Theorem (3.1), may appear in the hyperplane section $V^{d,\delta}(\alpha, \beta) \cap p^{\perp}$, do indeed appear.

[2] this is well-known and may be seen as follows: all fibres of $\mathcal{Y}$ are linearly equivalent, and have intersection number 0 with Z_1; therefore, $Y_0 \cdot Z_1 = (\tilde{C} + Z_1 + Z_2) \cdot Z_1 = 0$, hence $Z_1^2 = -(\tilde{C} + \tilde{L}) \cdot Z_1 = -2$.

(4.1) Theorem ([1, Theorem 1.3]). *Let p be a general point of L.*

a) Let V' be an irreducible component of $V^{d,\delta}(\alpha+e_k,\beta-e_k)(\Omega\cup\{p\})$ as in part a) of Theorem (3.1). Then V' is a component of the intersection $V^{d,\delta}(\alpha,\beta)(\Omega)\cap p^\perp$, and at a general point of V', the variety $V^{d,\delta}(\alpha,\beta)$ is smooth and has intersection multiplicity k with $p^\perp$ along V'.

b) Let V' be an irreducible component of $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$ as in part b) of Theorem (3.1). Then V' is a component of the intersection $V^{d,\delta}(\alpha,\beta)(\Omega)\cap p^\perp$ and, at a general point of V', the variety $V^{d,\delta}(\alpha,\beta)$ has $\binom{\beta'}{\beta}I^{\beta'-\beta}/\operatorname{lcm}(\beta'-\beta)$ local sheets, each of which has intersection multiplicity $\operatorname{lcm}(\beta'-\beta)$ with $p^\perp$ along V'. Moreover, each of these sheets has the generic point of V' as a point of multiplicity $\operatorname{lcm}(\beta'-\beta)/\max(\beta'-\beta)$,

In both cases a) and b) the general strategy will be to show that the general member of V' may be deformed to general members of $V^{d,\delta}(\alpha,\beta)(\Omega)$. We treat the two cases separately, in Subsections 4.1 and 4.2 respectively.

The multiplicity in case a) displays a phenomenon reminiscent of the fundamental principle of projective duality. The deformation argument for case b) is much more demanding, and eventually relies on the local material on deformations of tacnodes gathered in Chapters [IV] and [V], which is arguably the keystone of the present volume.

4.1 – Fixing an unassigned contact point

This section is devoted to the proof of part a) of Theorem (4.1), which is the case when V' parametrizes curves that do not contain the line L, and for which the condition that they lie on the hyperplane $p^\perp$ is accounted for by the fact that p is one of the assigned contact points.

It is fairly clear that $V^{d,\delta}(\alpha+e_k,\beta-e_k)(\Omega\cup\{p\})$ is contained in the hyperplane section of $V^{d,\delta}(\alpha,\beta)(\Omega)$ by $p^\perp$, so the point here is to establish the assertion on the smoothness and intersection multiplicity.

The multiplicity is as one would expect it to be from the point of view of projective duality. Indeed, the latter tells us the following. Consider a variety $X\subseteq\mathbf{P}^N$, and its dual $X^\vee\subseteq\check{\mathbf{P}}^N$ in the dual projective space, which parametrizes hyperplanes in $\mathbf{P}^N$ tangent to X. Then for a general point $p\in X$, the hyperplane $p^\perp\subseteq\check{\mathbf{P}}^N$ (consisting of those hyperplanes of $\mathbf{P}^N$ that contain the point p) is tangent to $X^\vee$ at the points $T^\perp\in\check{\mathbf{P}}^N$ corresponding to hyperplanes that are tangent to X at p. More generally, one may consider "osc-dual" varieties $X^{\vee_\beta}\subseteq\check{\mathbf{P}}^N$ of X, parametrizing hyperplanes of $\mathbf{P}^N$ osculating X to the order β, and then $p^\perp$ osculates $X^{\vee_\beta}$ to the order β at points $O^\perp$ corresponding to osculating hyperplanes to X at p. Some of these are discussed in Chapter [XII].

In the situation under consideration, this has to be applied to the case when X is the degree $d-I\alpha$ rational normal curve image of L by the linear system cut out by degree d plane curves with the contact conditions at assigned points determined by α and Ω. In fact this is exactly what we do in practice, following the approach of [1, §4.3], and establishing along the way the appropriate biduality principle for osculating hyperplanes to rational normal curves, which is the following.

(4.2) Proposition ([1, Lemma 4.7]). *Consider the line $L\subseteq\mathbf{P}^2$, and a general point p of L. We assume the data of d,α,β, and $\Omega=(p_{i,j})_{1\leqslant j\leqslant\alpha_i}$ is given as in Theorem (4.1); in particular, $I\alpha+I\beta=d$. We define the following three loci in $|\mathcal{O}_L(d)|$:*

- $p^\perp=\{D: p\in D\}$;
- $\Phi_{\alpha,\beta}(\Omega)=\big\{D=\sum_{1\leqslant j\leqslant\alpha_i} i\cdot p_{i,j}+\sum_{1\leqslant j\leqslant\beta_i} i\cdot s_{i,j}$ *for some $s_{i,j}$'s in L*$\big\}$;
- $\Psi_{\alpha,\beta}(\Omega)=\big\{D=k\cdot p+\sum_{1\leqslant j\leqslant\alpha_i} i\cdot p_{i,j}+\sum_{1\leqslant j\leqslant(\beta-e_k)_i} i\cdot s_{i,j}$ *for some $s_{i,j}$'s in L*$\big\}$.

In a neighbourhood of a general point $[D_0] \in \Psi_{\alpha,\beta}(\Omega)$*, the variety* $\Phi_{\alpha,\beta}(\Omega)$ *is smooth and has intersection multiplicity* k *with the hyperplane* $p^\perp$ *along* $\Psi_{\alpha,\beta}(\Omega)$*.*

Proof. We will proceed by an explicit local computation. Let x be an affine coordinate on L centered at the point p. Let the points $p_{i,j}$ have respective coordinates $\lambda_{i,j}$, and let the points $s^0_{i,j}$ such that

$$D_0 = k \cdot p + \sum_{1 \leqslant j \leqslant \alpha_i} i \cdot p_{i,j} + \sum_{1 \leqslant j \leqslant (\beta - e_k)_i} i \cdot s^0_{i,j}$$

have respective coordinates $\mu_{i,j}$. By generality of D_0, we may assume that these coordinates are altogether mutually distinct.

Then we can parametrize a neighbourhood of $[D_0]$ in $\Phi_{\alpha,\beta}(\Omega)$ by

$$(\varepsilon, \varepsilon_{i,j}) \mapsto [f(x)] = \Big[(x-\varepsilon)^k \cdot \prod_{1 \leqslant j \leqslant \alpha_i} (x - \lambda_{i,j})^i \cdot \prod_{1 \leqslant j \leqslant (\beta - e_k)_i} (x - \mu_{i,j} - \varepsilon_{i,j})^i\Big] \in \mathbf{C}_d[x],$$

from which we see first of all that $\Phi_{\alpha,\beta}(\Omega)$ is smooth at $[D_0]$. On the other hand, writing

$$f(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0 \in \mathbf{C}_d[x],$$

we get a system of affine coordinates on some chart of $|\mathcal{O}_L(d)|$, in which the hyperplane $p^\perp$ is defined by the linear equation $b_0 = 0$. Then, in the local coordinates $(\varepsilon, \varepsilon_{i,j})$ on $\Phi_{\alpha,\beta}(\Omega)$ around $[D_0]$, the intersection with $p^\perp$ is defined by the equation

$$\prod \lambda^i_{i,j} \cdot \varepsilon^k \cdot \prod (\varepsilon_{i,j} + \mu_{i,j})^i = 0 \iff \varepsilon^k \cdot \prod (\varepsilon_{i,j} + \mu_{i,j})^i = 0,$$

hence, locally around $[D_0]$, it is k times the divisor defined by the equation $\varepsilon = 0$, and the latter divisor is exactly $\Psi_{\alpha,\beta}(\Omega)$. □

In order to relate the local geometry of generalized Severi varieties to the model situation of Proposition (4.2), we consider the restriction map $\pi : |\mathcal{O}_{\mathbf{P}^2}(d)| \dashrightarrow |\mathcal{O}_L(d)|$. Note that this is merely the projection from the codimension $d+1$ linear subspace $L^\perp$ of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ parametrizing curves wich contain L. Finally, observe that the hyperplane $p^\perp$ of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ is the preimage by π of the hyperplane $p^\perp$ of $|\mathcal{O}_L(d)|$.

Let W be the plain Severi variety $V^{d,\delta}$, and denote by π_W the restriction of π to W. By definition, using the notation of Proposition (4.2), we have

$$V^{d,\delta}(\alpha,\beta)(\Omega) = \pi_W^{-1}\big(\Phi_{\alpha,\beta}(\Omega)\big) \qquad \text{and} \qquad V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p\}) = \pi_W^{-1}\big(\Psi_{\alpha,\beta}(\Omega)\big).$$

Therefore, part a) of Theorem (4.1) follows directly from the following proposition together with Proposition (4.2) above.

(4.3) Proposition. *Let* X_0 *be a general member of (any irreducible component of)* $V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p\})$*. The differential of the map* $\pi_W : W \dashrightarrow |\mathcal{O}_L(d)|$ *is surjective at the point* $[X_0]$*.*

Proof. First note that, being general, X_0 does not contain L and therefore π is well defined at $[X_0]$. Since the map π is a linear projection, it suffices to show that the projective tangent space $\mathbf{T}_{[X_0]}W$ to $W = V^{d,\delta}$ at the point $[X_0]$ intersects the center of the projection transversely, i.e., that the intersection $\mathbf{T}_{[X_0]}W \cap L^\perp$ has codimension $d+1$ in $\mathbf{T}_{[X_0]}W$.

Now, the projective tangent space $\mathbf{T}_{[X_0]}W$ identifies with the linear series of degree d curves passing through the nodes of X_0, and thus its intersection with $L^\perp$ identifies with the linear series of degree $d-1$ curves passing through the nodes of X_0. By Lemma (4.4), these two linear series have dimensions $\frac{d(d+3)}{2} - \delta$ and $\frac{(d-1)(d+2)}{2} - \delta$ respectively, so the codimension of $\mathbf{T}_{[X_0]}W \cap L^\perp$ in $\mathbf{T}_{[X_0]}W$ equals that of $|\mathcal{O}_{\mathbf{P}^2}(d-1)|$ in $|\mathcal{O}_{\mathbf{P}^2}(d)|$, which is $d+1$, as we wanted. □

The following lemma is the classical fact that, on a regular surface, the adjoint series cut out the complete canonical series. We include the proof for completeness.

(4.4) Lemma. *Let $C \subseteq \mathbf{P}^2$ be a δ-nodal curve of degree d. For all $e \geqslant d-3$, the nodes of C impose independent conditions on curves of degree e.*

Proof. It is sufficient to prove that the nodes of C impose independent conditions on curves of degree $d-3$. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbf{P}^2}$ be the ideal sheaf of the nodes of C as a subscheme of $\mathbf{P}^2$, and let $\mathcal{A} \subseteq \mathcal{O}_C$ be the conductor ideal sheaf of C (with respect to its normalization $\bar{C}$). The ideal $\mathcal{A}$ is the restriction of $\mathcal{I}$ to C (see, e.g., [II, Lemma (2.5)]), and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbf{P}^2}(d-3) \otimes \mathcal{I} \longrightarrow \mathcal{O}_C(d-3) \otimes \mathcal{A} \longrightarrow 0.$$

It follows that there is an isomorphism $H^0(\mathcal{O}_{\mathbf{P}^2}(d-3) \otimes \mathcal{I}) \cong H^0(\mathcal{O}_C(d-3) \otimes \mathcal{A})$. Now, by [II, Corollary (2.4)], $H^0(\mathcal{O}_C(d-3) \otimes \mathcal{A})$ is isomorphic to $H^0(\bar{C}, \omega_{\bar{C}})$, where $\bar{C}$ still denotes the normalization of C, hence it has dimension g, the geometric genus of C, so that finally

$$\begin{aligned} h^0\big(\mathcal{O}_{\mathbf{P}^2}(d-3) \otimes \mathcal{I}\big) &= g = p_a(d) - \delta \\ &= h^0\big(\mathcal{O}_{\mathbf{P}^2}(d-3)\big) - \delta, \end{aligned}$$

as we wanted to prove. □

4.2 – Merging a line

This section is devoted to the proof of part b) of Theorem (4.1), which is the case when V' parametrizes curves containing the line L. The proof will be divided in two parts (Sections 4.2.1 and 4.2.2) which we shall describe shortly, but we first need to introduce some notation. We advise the reader that he should have Sections 3.1 and 3.4 clear in mind in order to understand what is going on.

(4.5) The situation. We consider a curve $X_0 = C \cup L$, where $[C]$ is a general member of the component V' of $V^{d-1,\delta'}(\alpha', \beta')(\Omega')$ we have started with.

We have $\alpha' \leqslant \alpha$, and correspondingly Ω' is a subset of Ω; by this we mean that $\Omega' = (p'_{i,j})_{1 \leqslant j \leqslant \alpha'_i}$ such that for all i, $(p'_{i,j})_{1 \leqslant j \leqslant \alpha'_i}$ is a subsequence of $(p_{i,j})_{1 \leqslant j \leqslant \alpha_i}$.

On the other hand we have $\beta' \geqslant \beta$, and we choose a subset $\Lambda = (s_{i,j})_{1 \leqslant j \leqslant \beta_i}$ of cardinality β of the sequence $(s'_{i,j})_{1 \leqslant j \leqslant \beta'_i}$ of unassigned contact points of C with L.

(4.6) General strategy. The first part of the proof will consist in deforming the curve X_0 to degree d curves locally equigenerically around the points of Ω' and Λ, while maintaining the contact conditions encoded in α, β, and Ω. At a point $p'_{i,j}$ or $s_{i,j}$, the curve $C \cup L$ has an order i tacnode; the requirement that the deformation be locally equigeneric around such a point amounts in pratice to the requirement that the i-th order tacnode deforms into i nodes.

In the second part of the proof we will take care of the $\beta' - \beta$ remaining mobile contact points, i.e., those points $s'_{i,j}$ that are not in Λ. At these points, the curve $C \cup L$ has order i tacnodes as well, but these ones will be deformed into $i-1$ nodes only. This will ensure in particular that the line L be merged with C along the deformation, so that for instance if C is irreducible, then the curves deformed from $C \cup L$ will be irreducible as well.

The reader should consult Section 3.4 to see that these are indeed the appropriate deformations. The reason for this two steps deformation process is that we want to isolate the deformations of i-tacnodes to $i-1$ nodes from the rest, in order to apply the results from the two previous lectures [IV] and [V].

4.2.1 The relaxed Severi variety

Here we perform the first step of the proof, as described in (4.6) above. The deformation procedure described there will amount to defining a relaxed Severi variety, seeing $X_0 = C \cup L$ as one of its members, and showing that our relaxed Severi variety is smooth at $[X_0]$.

We use the notation introduced in (4.5) above, and in particular we fix the choice of a subsequence Λ. Moreover we label the points in $\Omega \setminus \Omega'$ as $(p_{i,j}^L)_{1 \leqslant j \leqslant \alpha_i^L}$; then, letting $\alpha^L = (\alpha_i^L)_{i \geqslant 1}$, we have $\alpha = \alpha' + \alpha^L$. Note that around a point $p_{i,j}^L$, the curve X_0 has only one local branch, which is an open subset of the line L.

On the other hand, we label the mobile contact points of C with L that have not been included in Λ as $(s_{i,j}^{C\cap L})_{1 \leqslant j \leqslant \beta_i^{C\cap L}}$; then, letting $\beta^{C\cap L} = (\beta_i^{C\cap L})_{i \geqslant 1}$, we have $\beta' = \beta + \beta^{C\cap L}$.

(4.7) The relaxed Severi variety. In an analytic neighbourhood of the point $[X_0]$ in $|\mathcal{O}_{\mathbf{P}^2}(d)|$, we define the *relaxed Severi variety* W_Λ to be the closure of the locus of reduced, degree d curves X_t satisfying the following six conditions:

(i) X_t preserves the δ' nodes of C, i.e., for every node of X_0 off L, X_t will have a node nearby;

(ii) at each point $p_{i,j}^L \in \Omega \setminus \Omega'$, X_t has contact of order i with L;

(iii) in a neighbourhood of each point $p'_{i,j} \in \Omega'$, X_t has singularities with total cogenus i, i.e., if we let $\bar{X}_t$ be the normalization of exactly these singularities, then $p_a(\bar{X}_t) = p_a(X_t) - i$;

(iv) in a neighbourhood of each point $s_{i,j} \in \Lambda$, X_t has singularities with total cogenus i;

(v) in a neighbourhood of each point $p'_{i,j} \in \Omega'$, it follows from condition (iii) that X_t has two local branches; we require that the one branch that is a deformation of an open subset of C has contact of order i with L at $p'_{i,j}$;

(vi) in a neighbourhood of each point $s_{i,j} \in \Lambda$, the curve X_t has two local branches; we require that the one branch that is a deformation of an open subset of C has contact of order i with L at a point near $s_{i,j}$.

Just to be sure, we recall that the curve $X_0 = C \cup L$ has an i-th order tacnode at each point $p'_{i,j} \in \Omega'$ or $s_{i,j} \in \Lambda$, with two local branches that are open subsets of C and L respectively. Conditions (iii) and (iv) imply that in a neighbourhood of such a point, the curves X_t have two local branches as well, which are deformations of open subsets of C and L respectively.

With this definition for the relaxed Severi variety, it is clear that $[X_0] \in W_\Lambda$. We could have defined W_Λ alternatively, by requiring in (iii) and (iv) that in a neighbourhood of each point $p'_{i,j}$ or $s_{i,j}$, the curve X_t has i nodes. The reason why W_Λ will be smooth at $[X_0]$ is because the only special feature of X_0 with respect to a general member of W_Λ is that it has several groups of i nodes coming together as i-th order tacnodes (for various i's), and this is perfectly harmless from the point of view of the deformation theory of nodal curves: somehow, this is the classical view that an i-th order tacnode is just i infinitely near ordinary nodes. This will be made precise when we compute the Zariski tangent space to W_Λ at the point $[X_0]$.

In order to show the smoothness of W_Λ at $[X_0]$ we will compare the dimension of W_Λ with that of its tangent space at $[X_0]$. We proceed to compute the dimension of W_Λ in the next paragraph.

(4.8) The relaxed Severi variety as a local sheet of a Severi variety. The basic observation in this paragraph is that the six conditions defining the relaxed Severi variety W_Λ in fact essentially define a generalized Severi variety in the plain sense. Namely, we are requiring that the curves X_t, (i) have a certain genus (or, equivalently, cogenus), defined by the δ' nodes deformations of those of C and the various groups of i nodes deformations of the points $p'_{i,j} \in \Omega'$ and $s_{i,j} \in \Lambda$, and (ii) satisfy certain contact conditions with L. Specifically, the cogenus, i.e., the difference in arithmetic genus between the curves X_t and their normalizations, is required to be:

$$\begin{aligned}\delta'' &:= \delta' + I\alpha' + I\beta = \delta' + I\alpha' + I\beta' - I(\beta' - \beta) \\ &= \delta' + (d-1) - I(\beta' - \beta) = \delta - \big(I(\beta' - \beta) - |\beta' - \beta|\big)\end{aligned} \tag{4.8.1}$$

(for the last equality, recall the relation between δ and δ', given for instance in Theorem (3.1)). On the other hand the contact conditions with L are exactly those prescribed by α, β and Ω. Altogether, these are the conditions defining the (generalized) Severi variety $V^{d,\delta''}(\alpha,\beta)(\Omega)$.

The additional requirements in the definition of W_Λ have the effect of selecting an open subset of some local sheets of $V^{d,\delta''}(\alpha,\beta)(\Omega)$ at $[X_0]$. We are first of all requiring that the i nodes corresponding to a point $p'_{i,j}$ or $s_{i,j}$ be in a neighbourhood of this same point; in other words we prescribe among the tacnodes of X_0 which deform to i nodes, and which are ignored. This already selects local sheets of $V^{d,\delta''}(\alpha,\beta)(\Omega)$ at $[X_0]$, determined by Λ: indeed, if we consider W_Λ as a space of maps from curves of cogenus δ'' to $\mathbf{P}^2$, then the member of W_Λ corresponding to X_0 is the partial normalization of $X_0 = C \cup L$ at the points of Ω' and Λ; if one considers another partial normalization of X_0 at the points of Ω' and some other Λ', one gets a curve member of the space of maps corresponding to $V^{d,\delta''}(\alpha,\beta)(\Omega)$, which however does not belong to W_Λ.

We are moreover requiring that, locally around a point $p'_{i,j}$ or $s_{i,j}$, the order i contact with L be supported on the one local branch that is a deformation of the local branch of C. This again, and in the same way as above, has the effect of operating a selection among the local sheets of $V^{d,\delta''}(\alpha,\beta)(\Omega)$ at $[X_0]$.

The upshot is that the relaxed Severi variety W_Λ is an open subset in some local sheets of the generalized Severi variety $V^{d,\delta''}(\alpha,\beta)(\Omega)$. Since the latter is equidimensional, we have

$$\dim(W_\Lambda) = \dim\big(V^{d,\delta''}(\alpha,\beta)\big) = 2d + g'' + |\beta| - 1,$$

with $g'' = p_a(d) - \delta'' = \binom{d-1}{2} - \delta''$.

(4.9) The ideal defining the tangent space to the relaxed Severi variety. We shall prove, in Proposition (4.10) below, that the tangent space at $[X_0]$ of the relaxed Severi variety W_Λ identifies with the subspace $H^0(X_0, \mathcal{I}(d))$ of $H^0(X_0, \mathcal{O}(d))$ defined by an ideal sheal $\mathcal{I}$ of $\mathcal{O}_{X_0}$ which we shall now describe, based on the description of the tangent space of generalized Severi varieties given in [III].

If X is a δ-nodal curve member of the ordinary Severi variety $V^{d,\delta}$, then the tangent space $T_{[X]}V^{d,\delta}$ is $H^0(X, \mathcal{A}(d))$, with $\mathcal{A}$ the ideal sheaf of the points supporting the nodes of X. When i of the δ nodes become infinitely near so as to form an i-th order tacnode, the ideal sheaf $\mathcal{A}$ has to be modified in the most straightforward way, i.e., it is merely the ideal sheaf of the possibly infinitely near points supporting the nodes of X. In other words, it is cut out locally at an order i tacnode by curves osculating to the order i the two local branches of X at the tacnode.

On the other hand we have seen in a previous Lecture, see [III, Lemma (2.11)], that an order i contact condition with L at an assigned (resp. unassigned) point translates into the vanishing to the order i (resp. $i-1$) of the sections in $H^0(X, N_X)$ corresponding to vectors tangent to such deformations.

These considerations motivate the definition of the ideal sheaf $\mathcal{I}$.

(4.9.1) Definition. We consider the curve $X_0 = C \cup L$ introduced in (4.5) above. For all i, j with $1 \leqslant j \leqslant \alpha'_i$, we let ${}^{\alpha}\mathcal{I}_{i,j}$ be the subsheaf of $\mathcal{O}_{X_0}$ of those regular functions f, such that the restrictions $f|_C$ and $f|_L$ vanish at $p'_{i,j}$ to the orders i and $2i$ respectively.

Similarly, for all i, j with $1 \leqslant j \leqslant \beta_i$, we let ${}^{\beta}\mathcal{I}_{i,j}$ be the sheaf of regular functions f on X_0, such that $f|_C$ and $f|_L$ vanish at $s_{i,j}$ to the orders i and $2i - 1$ respectively.

For all points $x \in X_0$, we denote by $\mathfrak{m}_x$ the ideal sheaf defining x in X_0. Lastly, we call $u_1, \ldots, u_{\delta'}$ the δ' nodes of C. Finally, we set:

$$\mathcal{I} = \prod_{1 \leqslant i \leqslant \delta'} \mathfrak{m}_{u_i} \cdot \prod_{1 \leqslant j \leqslant \alpha_i^L} \mathfrak{m}^i_{p^L_{i,j}} \cdot \prod_{1 \leqslant j \leqslant \alpha'_i} {}^{\alpha}\mathcal{I}_{i,j} \cdot \prod_{1 \leqslant j \leqslant \beta_i} {}^{\beta}\mathcal{I}_{i,j}.$$

(4.10) Proposition. *The relaxed Severi variety W_Λ is smooth at $[X_0]$, with tangent space*

$$T_{[X_0]} W_\Lambda = H^0(X_0, \mathcal{I}(d)) \ \subseteq \ T_{[X_0]}\big(|\mathcal{O}_{\mathbf{P}^2}(d)|\big) = H^0(X_0, \mathcal{O}_{X_0}(d)),$$

where $\mathcal{I}$ is the sheaf of ideals defined in (4.9.1) above.

Proof. We will compare the deformations of X_0 along W_Λ with those of the map given by a certain partial normalization of X_0. Let $\mu : \tilde{X}_0 \to X_0$ be the normalization of $X_0 = C \cup L$ at the nodes u_i of C and at the tacnodes $p'_{i,j} \in \Omega'$ and $s_{i,j} \in \Lambda$ (at which C and L touch), but not at the tacnodes $s'_{i,j}$, because the latter are off Λ. We will consider the deformations of the map $\phi : \tilde{X}_0 \to \mathbf{P}^2$ obtained as the composition of μ with the inclusion $X_0 \subseteq \mathbf{P}^2$.

The map ϕ is unramified, hence its normal sheaf N_ϕ is locally free, equal to $\omega_{\tilde{X}_0} \otimes \phi^*\omega_{\mathbf{P}^2}^{-1}$, see [II, Lemma (8.3)]. By Lemma (4.11) below, $H^1(\tilde{X}_0, N_\phi)$ vanishes, hence the deformation space of ϕ is smooth of dimension

$$h^0(\tilde{X}_0, N_\phi) = h^0(\tilde{X}_0, \omega_{\tilde{X}_0} \otimes \phi^*\omega_{\mathbf{P}^2}^{-1}).$$

The smoothness of this space ensures in particular the existence of a map F from a neighbourhood of $[\phi]$ to $|\mathcal{O}_{\mathbf{P}^2}(d)|$, mapping a deformation of ϕ to its image in $\mathbf{P}^2$, as in [III, (2.6)].

The image of F is what we will call the *superrelaxed Severi variety* $\tilde{W}_\Lambda$, which we define as W_Λ but only with the three requirements (i), (iii), and (iv) out of the six defining W_Λ; simply put, we just drop the contact conditions with L. Arguing as in (4.8) above, we see that $\tilde{W}_\Lambda$ is a local sheet of the plain Severi variety $V^{d,\delta''}$ at $[X_0]$, which implies that it has dimension

$$\dim(\tilde{W}_\Lambda) = \dim(V^{d,\delta''}) = 3d + g'' - 1.$$

That F surjects onto a neighbourhood of $[X_0]$ in $\tilde{W}_\Lambda$ is a consequence of Teissier's Résolution Simultanée Theorem, as in [III, (2.6)].

In order to analyze the tangent space to $\tilde{W}_\Lambda$ at $[X_0]$, we let $\mathcal{A} = (\mathcal{O}_{X_0} : \mu_*\mathcal{O}_{\tilde{X}_0})$ be the conductor of $\tilde{X}_0$ in X_0, see [II, §2]. It is a sheaf of ideals of $\mathcal{O}_{X_0}$, trivial on the open subset onto which μ is an isomorphism, equal to $\mathfrak{m}_{u_i}$ locally at the nodes u_i of C, while locally at the points $p'_{i,j} \in \Omega'$ and $s_{i,j} \in \Lambda$ it is the sheaf of regular functions vanishing to the order i on both C and L; this can be seen for instance with [II, Lemma (2.5)].

By [II, Corollary (8.4)], one has $\mu_* N_\phi = \mathcal{A} \otimes N_{X_0/\mathbf{P}^2}$. Therefore, there is a canonical isomorphism

$$H^0(\tilde{X}_0, N_\phi) \cong H^0(X_0, \mathcal{A} \otimes N_{X_0/\mathbf{P}^2}).$$

This isomorphism identifies with the differential of F at $[\phi]$, and thus we see that F is an isomorphism onto a neighbourhood of $[X_0]$ in $\tilde{W}_\Lambda$; in particular, the superrelaxed Severi variety $\tilde{W}_\Lambda$ is smooth at $[X_0]$, with tangent space $H^0(X_0, \mathcal{A} \otimes N_{X_0/\mathbf{P}^2})$.

Now, it follows from [III, (2.11)] that the Zariski tangent space at $[\phi]$ to the image by F^{-1} of the relaxed Severi variety $W_\Lambda \subseteq \tilde{W}_\Lambda$ is contained in

$$H^0\bigg(\tilde{X}_0, N_\phi\Big(-\sum_{1 \leqslant j \leqslant \alpha_i^L} i \cdot q^L_{i,j} - \sum_{1 \leqslant j \leqslant \alpha'_i} i \cdot q'_{i,j} - \sum_{1 \leqslant j \leqslant \beta_i} (i-1) \cdot r_{i,j}\Big)\bigg),$$

where (i) $q^L_{i,j}$ is the unique point of $\tilde{X}_0$ over $p^L_{i,j}$, and (ii) $q'_{i,j}$ and $r_{i,j}$ are the points on the proper transform of C in $\tilde{X}_0$ lying over $p'_{i,j}$ and $s_{i,j}$ respectively. Moreover, it follows from the description of $\mathcal{A}$ given above that

$$\mathcal{A} \otimes \phi_* \mathcal{O}_{\tilde{X}_0}\Big(-\sum_{1\leqslant j\leqslant \alpha_i^L} i\cdot q^L_{i,j} - \sum_{1\leqslant j\leqslant \alpha'_i} i\cdot q'_{i,j} - \sum_{1\leqslant j\leqslant \beta_i} (i-1)\cdot r_{i,j}\Big) = \mathcal{I},$$

hence

$$\phi_* N_\phi\Big(-\sum_{1\leqslant j\leqslant \alpha_i^L} i\cdot q^L_{i,j} - \sum_{1\leqslant j\leqslant \alpha'_i} i\cdot q'_{i,j} - \sum_{1\leqslant j\leqslant \beta_i} (i-1)\cdot r_{i,j}\Big) = \mathcal{I}\otimes N_{X_0/\mathbf{P}^2}.$$

Therefore, the Zariski tangent space to the relaxed Severi variety W_Λ at $[X_0]$ is contained in $H^0(X_0, \mathcal{I}\otimes N_{X_0/\mathbf{P}^2})$. Then, the proposition follows from the equality

$$h^0(X_0, \mathcal{I}\otimes N_{X_0/\mathbf{P}^2}) = \dim(W_\Lambda),$$

which is given by (4.12.2) below. □

The rest of the present Section 4.2.1 is devoted to some elementary considerations leading to the non-speciality of the invertible sheaves on $\tilde{X}_0$ involved in the above proof, and the computation of the dimensions of their spaces of global sections with the Riemann–Roch Formula.

(4.11) Lemma. *Let p be any point of the partial normalization $\tilde{X}_0$, and*

$$D = \sum_{1\leqslant j\leqslant \alpha_i^L} i\cdot q^L_{i,j} + \sum_{1\leqslant j\leqslant \alpha'_i} i\cdot q'_{i,j} + \sum_{1\leqslant j\leqslant \beta_i} (i-1)\cdot r_{i,j},$$

with the notation as in the proof of Proposition (4.10) above. The two line bundles $N_\phi(-D)$ and $N_\phi(-D-p)$ on $\tilde{X}_0$ are non-special.

Proof. Let $\tilde{C}$ and $\tilde{L}$ be the proper transforms in $\tilde{X}_0$ of C and L respectively. Since $N_\phi = \omega_{\tilde{X}_0}\otimes \phi^*\omega^{-1}_{\mathbf{P}^2}$ as we have seen above, one has

$$\omega_{\tilde{X}_0}\otimes \big(N_\phi(-D)\big)^{-1} = \phi^*\mathcal{O}_{\mathbf{P}^2}(-3)\otimes \mathcal{O}_{\tilde{X}_0}(D).$$

The degree of the restriction of this line bundle to $\tilde{C}$ has degree

$$\begin{aligned} -3(d-1) + I\alpha' + \big(I\beta - |\beta|\big) &\leqslant -3(d-1) + I\alpha' + I\beta' \\ &= -2(d-1) < 0, \end{aligned}$$

hence every global section of $\omega_{\tilde{X}_0}\otimes N_\phi(-D)^{-1}$ vanishes identically on $\tilde{C}$. One proves similarly that the same holds for $\omega_{\tilde{X}_0}\otimes N_\phi(-D-p)^{-1}$, noting that it restricts on $\tilde{C}$ to a line bundle of degree $-2(d-1)+\varepsilon$, where ε equals 0 or 1 depending on whether p sits on $\tilde{C}$ or not.

Thus, every global section of $\omega_{\tilde{X}_0}\otimes N_\phi(-D)^{-1}$ restricts on $\tilde{L}$ to a global section of the line bundle

$$\phi^*\mathcal{O}_{\mathbf{P}^2}(-3)\otimes \mathcal{O}_{\tilde{L}}(D|_{\tilde{L}})\otimes \mathcal{O}_{\tilde{L}}(-\tilde{C}|_{\tilde{L}}),$$

which has degree

$$\begin{aligned} -3 + I\alpha^L - I\beta^{C\cap L} &= -3 + I(\alpha-\alpha') - I(\beta'-\beta) \\ &= -3 + (I\alpha + I\beta) - (I\alpha' + I\beta') \\ &= -3 + d - (d-1) = -2. \end{aligned}$$

Therefore every global section of $\omega_{\tilde{X}_0} \otimes N_\phi(-D)^{-1}$ vanishes identically on $\tilde{L}$ as well, hence

$$h^0(\tilde{X}_0, \omega_{\tilde{X}_0} \otimes N_\phi(-D)^{-1}) = h^1(\tilde{X}_0, N_\phi(-D)) = 0.$$

One concludes in the same way that every global section of $\omega_{\tilde{X}_0} \otimes N_\phi(-D-p)^{-1}$ vanishes identically, hence $N_\phi(-D-p)$ is non-special. □

(4.12) Lemma. *The following equalities hold:*

$$h^0\big(\tilde{X}_0, N_\phi\big) = 3d + g'' - 1; \tag{4.12.1}$$

$$h^0\big(\tilde{X}_0, N_\phi(-D)\big) = 2d + g'' - 1 + |\beta|. \tag{4.12.2}$$

Moreover, the line bundle $N_\phi(-D)$ has no base point.

Proof. Both equalities follow from the application of the Riemann–Roch Formula on the curve $\tilde{X}_0$, which has arithmetic genus g'', see (4.8). The line bundle N_ϕ equals $\omega_{\tilde{X}_0} \otimes \phi^*\omega_{\mathbf{P}^2}^{-1}$ which is obviously non-special, hence

$$\begin{aligned} h^0(N_\phi) &= 1 - g'' + \deg\big(\omega_{\tilde{X}_0} \otimes \phi^*\omega_{\mathbf{P}^2}^{-1}\big) \\ &= 1 - g'' + (2g'' - 2) + 3d \\ &= 3d + g'' - 1. \end{aligned}$$

The line bundle $N_\phi(-D)$ is non-special by Lemma (4.11) above, hence

$$\begin{aligned} h^0\big(N_\phi(-D)\big) &= 1 - g'' + \deg(N_\phi) - \deg(D) \\ &= 3d + g'' - 1 - \big(I\alpha^L + I\alpha' + I\beta - |\beta|\big) \\ &= 2d + g'' - 1 + |\beta|, \end{aligned}$$

because

$$I\alpha^L + I\alpha' + I\beta = I\alpha + I\beta = d.$$

Eventually, since for all $p \in \tilde{X}_0$ the line bundle $N_\phi(-D-p)$ is non-special as well, we have

$$h^0\big(\tilde{X}_0, N_\phi(-D-p)\big) = h^0\big(\tilde{X}_0, N_\phi(-D)\big) - 1,$$

hence not all global sections of $N_\phi(-D)$ vanish at p. □

4.2.2 Deformation of order i tacnodes to $i-1$ nodes

Here we perform the second step of the proof of part b) of Theorem (4.1), as described in (4.6) above. We assume that $\beta' > \beta$, for otherwise there is nothing left to do in the second step (see Remark (4.21) below for what happens in this case).

We consider the map

$$F_\Lambda : W_\Lambda \to \Delta,$$

from the relaxed Severi variety W_Λ, introduced and studied in the previous Section 4.2.1, to the product Δ of the versal deformation spaces of the tacnodes $s_{i,j}^{C\cap L}$ of $X_0 = C \cup L$, i.e., those tacnodes of X_0 not included in Λ.

We refer to [IV] and [V] for background and the necessary material on deformation spaces of tacnodes and their product. The result we shall need here is [V, (1.1)], which we now recall in a form adapted to our context.

(4.13) The tacnodes $s_{i,j}^{C\cap L}$ are indexed by $\beta'-\beta$, and thus

$$\Delta = \prod_{i\in\mathbf{N}^*} \prod_{1\leqslant j\leqslant \beta'_i-\beta_i} \Delta_{i,j},$$

where each $\Delta_{i,j}$ is the versal deformation space of the i-th order tacnode $s_{i,j}^{C\cap L}$. For all i and j, we let $\Delta_{i,j;i}$ and $\Delta_{i,j;i-1}$ be the subvarieties of $\Delta_{i,j}$ of those deformations of $s_{i,j}^{C\cap L}$ into i and $i-1$ nodes respectively. We let

$$\mathbf{m} = (m_l)_{1\leqslant l\leqslant |\beta'-\beta|} = \big(\underbrace{1,\dots,1}_{\beta'_1-\beta_1}, \quad \dots, \quad \underbrace{n,\dots,n}_{\beta'_n-\beta_n}\big)$$

with $n=\max(\beta'-\beta)$, and consider

$$\Delta_{\mathbf{m}} = \prod_{i\in\mathbf{N}^*} \prod_{1\leqslant j\leqslant \beta'_i-\beta_i} \Delta_{i,j;i}, \quad \text{and} \quad \Delta_{\mathbf{m-1}} = \prod_{i\in\mathbf{N}^*} \prod_{1\leqslant j\leqslant \beta'_i-\beta_i} \Delta_{i,j;i-1}.$$

Each $\Delta_{i,j}$ is a $(2i-1)$-dimensional affine space, with coordinates $a_0,\dots,a_{i-2},b_0,\dots,b_{i-1}$, such that the corresponding versal deformation is the hypersurface

$$y^2 + y(x^i + a_{i-2}x^{i-2} + \cdots + a_0) + b_{i-1}x^{i-1} + \cdots + b_0 \tag{4.13.1}$$

in $\mathbf{A}^2_{x,y}\times\mathbf{A}^{2i-1}_{\mathbf{a},\mathbf{b}}$. In these terms, $\Delta_{i,j;i}$ is the dimension $i-1$ affine subspace defined by the equations $b_0=\cdots=b_{i-1}=0$. We consider in addition the hyperplane $\mathcal{H}_{i,j}\subseteq\Delta_{i,j}$ defined by the equation $b_0=0$ in this system of coordinates. Observe that $\mathcal{H}_{i,j}$ parametrizes those deformations of the tacnode defined by $y(y+x^m)=0$ that still pass through the origin in $\mathbf{A}^2_{x,y}$; in other words, these are the deformations maintaining one assigned contact point of order 1 with the line $y=0$. The general member of $\mathcal{H}_{i,j}$ is an irreducible curve. Eventually we let

$$\mathcal{H} = \bigcup_{i,j}\mathcal{H}_{i,j} \ \subseteq\ \Delta.$$

Now, the result is the following. Let $W\subseteq\Delta$ be a smooth subvariety of dimension $\dim(\Delta_{\mathbf{m-1}})+1$, containing $\Delta_{\mathbf{m-1}}$, and such that the tangent space at the origin T_0W is not contained in $\mathcal{H}$. Then, in an étale neighbourhood of the origin $0\in\Delta$,

$$W\cap\Delta_{\mathbf{m-1}} = \Delta_{\mathbf{m}}\cup\Gamma_1\cup\cdots\cup\Gamma_\kappa,$$

where

$$\kappa = \frac{\prod\limits_{1\leqslant l\leqslant|\beta'-\beta|} m_l}{\operatorname*{lcm}\limits_{1\leqslant l\leqslant|\beta'-\beta|} m_l} = \frac{I^{\beta'-\beta}}{\operatorname{lcm}(\beta'-\beta)},$$

and $\Gamma_1,\dots,\Gamma_\kappa$ are distinct reduced unibranched curves, each of which has the origin as a point of multiplicity

$$\frac{\operatorname*{lcm}\limits_{1\leqslant l\leqslant|\beta'-\beta|} m_l}{\max\limits_{1\leqslant l\leqslant|\beta'-\beta|} m_l} = \frac{\operatorname{lcm}(\beta'-\beta)}{\max(\beta'-\beta)},$$

and has intersection multiplicity $\operatorname{lcm}_{1\leqslant l\leqslant|\beta'-\beta|} m_l = \operatorname{lcm}(\beta'-\beta)$ with $\Delta_{\mathbf{m}}$ at the origin.

We now proceed to prove a series of claims in order to see that the hypotheses of the above statement are verified in our situation, taking $W=F_\Lambda(W_\Lambda)$.

(4.14) Claim. The inverse image $W_{\Lambda,\mathbf{m}} = F_\Lambda^{-1}(\Delta_\mathbf{m})$ is the locus of those members of W_Λ that contain L. It is also the intersection of W_Λ with the hyperplane $p^\perp \subseteq |\mathcal{O}_{\mathbf{P}^2}(d)|$, for general $p \in L$.

Proof. Let us first prove the first assertion. The inverse image $F_\Lambda^{-1}(\Delta_\mathbf{m})$ is the locus in W_Λ along which all the tacnodes $s_{i,j}^{C\cap L}$ of X_0 deform equigenerically. Taking in addition into account the requirements (i), (iii), and (iv) in the definition of the relaxed Severi variety W_Λ, we see that the curves in $F_\Lambda^{-1}(\Delta_\mathbf{m})$ are equigeneric deformations of X_0. Therefore they all come from deformations of the map $\bar{\phi} : \bar{X}_0 \to \mathbf{P}^2$ obtained from the total normalization $\nu : \bar{X}_0 \to X_0$, by the same argument that has been used in the proof of Proposition (4.10), which is essentially Teissier's Résolution Simultanée. Now since the normalization $\bar{X}_0$ is disconnected, the domain of any deformation of $\bar{\phi}$ is disconnected as well by the Principle of Connectedness (see, e.g., [13, Exercise III.11.4]). The upshot is that all curves in $F_\Lambda^{-1}(\Delta_\mathbf{m})$ consist of a deformation of C plus a deformation of L.

On the other hand we are assuming here that $\beta' > \beta$, for otherwise the second step of the proof of part b) of Theorem (4.1) is pointless. Since

$$I\alpha + I\beta = d \quad \text{and} \quad I\alpha' + I\beta' = d - 1,$$

it follows that $I(\alpha - \alpha') \geqslant 2$, i.e., $I\alpha^L \geqslant 2$. Now by requirement (ii) in the definition of the relaxed Severi variety, for each member of $F^{-1}(\Delta_\mathbf{m})$, the component that is a deformation of L must satisfy the contact conditions with L encoded in α^L. Therefore it intersects L in at least two points (possibly infinitely near), hence it is L itself, and our first assertion about $F^{-1}(\Delta_\mathbf{m})$ is proved.

As for the second assertion, the first assertion gives us of course the inclusion $F_\Lambda^{-1}(\Delta_\mathbf{m}) \subseteq W_\Lambda \cap p^\perp$ for all $p \in L$. To prove the other inclusion, we note that for all $p \in L$ neither in Ω nor in Λ, any curve in $W_\Lambda \cap p^\perp$ in a neighbourhood of $[X_0]$ has a total of

$$I\alpha + I\beta + 1 = d + 1$$

imposed points of intersection with L whereas it has degree d, so that it must contain L. □

(4.15) Claim. The image of F_Λ contains the locus $\Delta_\mathbf{m}$.

Proof. The projection of $F_\Lambda(W_\Lambda)$ to the factor $\Delta_{i,j}$ of Δ contains $\Delta_{i,j;i}$ if and only if we can find a deformation X_t of X_0 along W_Λ in which a neighbourhood of the tacnode $s_{i,j}^{C\cap L} \in X_0$ deforms to a reducible open curve $X_t^{\circ\prime} + X_t^{\circ\prime\prime}$, such that $X_t^{\circ\prime}$ and $X_t^{\circ\prime\prime}$ intersect transversely in i points that can be chosen arbitrarily on $X_t^{\circ\prime}$.

To see that this holds simultaneously for all tacnodes $s_{i,j}^{C\cap L} \in X_0$, we note that $F_\Lambda^{-1}(\Delta_\mathbf{m})$ identifies as in (4.8) with a local sheet at $[X_0]$ of the Severi variety

$$V^{d-1,\delta'}\big(\alpha', \beta + I(\beta' - \beta)e_1\big)(\Omega') :$$

we have seen that $F_\Lambda^{-1}(\Delta_\mathbf{m})$ consists of those curve in W_Λ having L as an irreducible component, and these curves correspond to equigeneric deformations of C maintaining the contact conditions with L in the points $p'_{i,j} \in \Omega'$ and $s_{i,j} \in \Lambda$, but dropping those in the points encoded in the points $s_{i,j} \notin \Lambda$; the latter dropping so to speak amounts to turn $\beta' - \beta$ into $I(\beta' - \beta)e_1$.

Then the condition we need to verify essentially follows from Proposition (2.2) applied to $V^{d-1,\delta'}\big(\alpha', \beta + I(\beta' - \beta)e_1\big)(\Omega')$. More explicitly, the latter Severi variety contains locally around $[X_0]$ the Severi varieties

$$V^{d-1,\delta'}\big(\alpha' + I(\beta' - \beta)e_1, \beta\big)(\Omega' \cup \Xi_t),$$

where Ξ_t is an arbitrary choice of i points on L around each of the points $s_{i,j} \notin \Lambda$, which is exactly the condition we needed to verify. □

(4.16) Claim. The inverse image $W_{\Lambda,\mathbf{m}} = F_\Lambda^{-1}(\Delta_{\mathbf{m}})$ has codimension one in W_Λ.

Proof. This follows from the identification given above of $F_\Lambda^{-1}(\Delta_{\mathbf{m}})$ with a local sheet at $[X_0]$ of the Severi variety $V^{d-1,\delta'}\big(\alpha', \beta + I(\beta' - \beta)e_1\big)$. Explicitly, this identifications tells us the dimension of $F_\Lambda^{-1}(\Delta_{\mathbf{m}})$ as in (4.8), then

$$\begin{aligned}\dim(W_\Lambda) - \dim\big(F_\Lambda^{-1}(\Delta_{\mathbf{m}})\big) \\ &= \Big[2d + \tbinom{d-1}{2} - \delta'' - 1 + |\beta|\Big] - \Big[2(d-1) + \tbinom{d-2}{2} - \delta' - 1 + |\beta| + I(\beta' - \beta)\Big] \\ &= 2 + (d-2) - (\delta'' - \delta') - I(\beta' - \beta),\end{aligned}$$

and this equals one by (4.8.1). ☐

(4.17) Remark. It is instructive, although not strictly necessary for our purposes, to identify the fibre of F_Λ over a general point of $\Delta_{\mathbf{m}}$. We shall obtain such an identification from the considerations in the proof of Claim (4.15) above, after we observe that a point of $\Delta_{i,j;i} \subseteq \Delta_{i,j}$ corresponds to the choice of i points on one branch of the tacnode at $s_{i,j}$, but only up to reparametrization. This amounts to the fact that, in (4.13.1), the polynomial

$$x^i + a_{i-2}x^{i-2} + \cdots + a_0$$

defining the unions of these i points has no x^{i-1} term; any degree i polynomial in x can be put in this form by a change of variable, which amounts to translating the roots so that they sum up to zero.

Then the fibre of F_Λ over a general point of $\Delta_{\mathbf{m}}$ identifies with an open subset of the Severi variety

$$V^{d-1,\delta'}\Big(\alpha' + \big[I(\beta' - \beta) - |\beta' - \beta|\big]e_1,\ \beta + |\beta' - \beta|e_1\Big)(\Omega' \cup \Xi_t^\circ), \tag{4.17.1}$$

with Ξ_t° an arbitrary choice of $i-1$ points on L around each of the points $s_{i,j} \notin \Lambda$. The fibre of F_Λ over the origin of Δ on the other hand identifies with an open subset of

$$V^{d-1,\delta'}(\alpha', \beta')(\Omega'), \tag{4.17.2}$$

which was indeed our starting point. It may be useful for the conclusion in (4.20) below to recall that it has dimension one less than $V^{d,\delta}(\alpha, \beta)$.

The fibre over the general point of $\Delta_{\mathbf{m}}$ and the fibre over the origin have the same dimension: indeed the Severi varieties (4.17.1) and (4.17.2) have the same dimension

$$2(d-1) + g' - 1 + |\beta| + |\beta' - \beta| \ = \ 2(d-1) + g' - 1 + |\beta'|,$$

with $g' = p_a(d-1) - \delta' = \binom{d-2}{2} - \delta'$.

We leave it to the reader to verify that the latter common dimension of the fibre over the general point of $\Delta_{\mathbf{m}}$ and the fibre over the origin also equals

$$\dim(W_\Lambda) - \dim(\Delta_{\mathbf{m}}) - 1;$$

this is yet another straightforward computation after one notes that

$$\dim(\Delta_{\mathbf{m}}) = \sum\nolimits_{1 \leqslant l \leqslant |\beta' - \beta|} (m_l - 1) = I(\beta' - \beta) - |\beta' - \beta|.$$

(4.18) Claim. Let $dF_\Lambda : T_{[X_0]}W_\Lambda \to T_0\Delta$ be the differential map of F_Λ at the point $[X_0]$. One has

$$\dim\big(dF_\Lambda^{-1}(T_0\Delta_{\mathbf{m}})\big) = \dim\big(F_\Lambda^{-1}(\Delta_{\mathbf{m}})\big) = \dim(W_\Lambda) - 1.$$

Proof. The second equality is Claim (4.16) above. For the first one, the key observation is that, under the isomorphism
$$T_{[X_0]}W_\Lambda \cong H^0\big(X_0, \mathcal{I}(d)\big),$$
the inverse image of $T_0\Delta_{\mathbf{m}} \subseteq T_0\Delta$ by the differential dF_Λ at $[X_0]$ is the subspace of sections vanishing identically along $L \subseteq X_0$.

The restriction $\mathcal{I}(d)|_L$ has degree
$$d - I\alpha - I\beta = 0,$$
and is therefore trivial. Now, the restriction map on global sections,
$$H^0\big(X_0, \mathcal{I}(d)\big) \to H^0\big(L, \mathcal{I}(d)|_L\big) \cong \mathbf{C},$$
is non-zero by Lemma (4.12), hence
$$\dim\big(dF_\Lambda^{-1}(T_0\Delta_{\mathbf{m}})\big) = h^0\big(X_0, \mathcal{I}(d)\big) - 1 = \dim(W_\Lambda) - 1,$$
where the last equality comes from the smoothness of W_Λ at $[X_0]$. □

(4.19) Claim. The image of the differential map $dF_\Lambda : T_{[X_0]}W_\Lambda \to T_0\Delta$ is not contained in $\mathcal{H}$.

Proof. From the description of $\mathcal{H}$ in (4.13), the inverse image $(dF_\Lambda)^{-1}(\mathcal{H})$ is the union of the subspaces in $H^0\big(X_0, \mathcal{I}(d)\big)$ of sections vanishing at one point $s_{i,j}^{C\cap L}$. By Lemma (4.12) these are all proper subspaces (in fact, hyperplanes) of $H^0\big(X_0, \mathcal{I}(d)\big)$, and the claim is proved. □

(4.20) Conclusion. We may now finish the proof of part b) of Theorem (4.1). It follows from the description given in Section 3.4 that, locally around the point $[X_0]$, the sum of the local sheets of the Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ corresponding to the choice of Λ is given as the closure in the relaxed Severi variety W_Λ of the inverse image
$$F_\Lambda^{-1}(\Delta_{\mathbf{m-1}} \setminus \Delta_{\mathbf{m}}).$$

Now, it follows from Claim (4.18) that the differential map
$$dF_\Lambda : T_{[X_0]}W_\Lambda \to T_0\Delta$$
has rank $\dim(\Delta_{\mathbf{m}}) + 1$, equal to the dimension of $W = F_\Lambda(W_\Lambda)$. Thus W is smooth at $[X_0]$, of dimension $\dim(\Delta_{\mathbf{m}}) + 1$, and the map F_Λ is a submersion over W, locally around the origin in Δ.

The subvariety $W \subseteq \Delta$ is smooth and, by Claim (4.19), its tangent space is not contained in $\mathcal{H}$. We may thus apply the main result of [V], to the effect that
$$W \cap \Delta_{\mathbf{m-1}} = \Delta_{\mathbf{m}} \cup \Gamma_1 \cup \cdots \cup \Gamma_\kappa,$$
with $\Gamma_1, \ldots, \Gamma_\kappa$ curves as in (4.13) above. The upshot is thus that the local sheets of $V^{d,\delta}(\alpha,\beta)(\Omega)$ corresponding to the choice of Λ are in the number $\kappa = I^{\beta'-\beta}/\mathrm{lcm}(\beta'-\beta)$, each a fibration over a curve Γ_l, $l = 1, \ldots, \kappa$, with the fibre $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$ over the origin. The local sheet corresponding to the curve Γ_l has multiplicity $\mathrm{lcm}(\beta'-\beta)/\max(\beta'-\beta)$ at the generic point of the fibre over the origin, i.e., the generic point of $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$, and intersection multiplicity $\mathrm{lcm}(\beta'-\beta)$ with the inverse image $F_\Lambda^{-1}(\Delta_{\mathbf{m}})$, i.e., with the hyperplane $p^\perp \subseteq |\mathcal{O}_{\mathbf{P}^2}(d)|$ for general $p \in L$ by Claim (4.14), along the fibre over the origin.

Finally, note that $\binom{\beta'}{\beta}$ is the number of possible choices for the subsequence Λ of the sequence of unassigned contact points of C with L. □

The proof of Theorem (4.1) is now over. We end this section with a remark about what happens when $\beta' = \beta$.

As we have already observed, in this case the first step performed in Subsubsection 4.2.1 above is sufficient to prove Theorem (4.1). Indeed, if $\beta = \beta'$, then the relaxed Severi variety W_Λ itself is the unique local sheet of the Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ at $[X_0]$, we already know that it is smooth at $[X_0]$, and there is nothing else to prove.

(4.21) Remark. If $\beta' = \beta$, then all the irreducible components of $V^{d,\delta}(\alpha,\beta)$ having an irreducible component of $V^{d-1,\delta'}(\alpha',\beta')$ in their intersection with the hyperplane $p^\perp \subseteq |\mathcal{O}_{\mathbf{P}^2}(d)|$ parametrize reducible curves.

Indeed, the members of such a component W in a neighourhood of $[X_0] = [C \cup L]$ with $[C] \in V^{d-1,\delta'}(\alpha',\beta')$ correspond to deformations of the total normalization $\bar{X}_0 \to X_0 \subseteq \mathbf{P}^2$, and therefore they are reducible curves because $\bar{X}_0$ is disconnected. This is the same argument we have used in the proof of Claim (4.14) above.

Conversely, if $\beta' > \beta$, and if V is an irreducible component of $V^{d-1,\delta'}(\alpha',\beta')$ that parametrizes irreducible degree $d-1$ curves, then all the irreducible components of $V^{d,\delta}(\alpha,\beta)$ having V in their intersection with $p^\perp$ parametrize irreducible curves.

Indeed, if $\beta' > \beta$, then there is at least one order i tacnode of $X_0 = C \cup L$ that is deformed to $i-1$ nodes only, so that locally around such a point the two local branches of X_0, corresponding respectively to an irreducible component C_1 of C and to L, deform to only one, irreducible, local branch, and thus C_1 and L merge into the same irreducible component of the curves X_t neighbouring X_0.

5 – The formula for irreducible curves

In this section we give and explain the version of Caporaso and Harris' recursion formula enabling the counting of irreducible curves. The formula is given in Theorem (5.4) below. The key observation to pass from the standard formula to the formula for irreducible curves is Remark (4.21) at the end of the previous section. Let us start with an example.

(5.1) Example. We consider the enumeration of quartics with three nodes and two assigned simple contact points with the line L, i.e., curves parametrized by $V^{4,3}(2,2)(\Omega)$. By the recursion formula, we have

$$N^{4,3}(2,2) = N^{4,3}(3,1) + 3N^{3,1}(0,3) + 2N^{3,0}(1,2). \tag{5.1.1}$$

The irreducible components of $V^{4,3}(2,2)(\Omega)$ parametrize either irreducible quartics, or quartics that are the sum of a cubic and a line. We let $N^{4,3}(2,2)_{\mathrm{irr}}$ and $N^{4,3}(2,2)_{3+1}$ be the respective contributions to $N^{4,3}(2,2)$ of irreducible and decomposable quartics. Thus,

$$N^{4,3}(2,2) = N^{4,3}(2,2)_{\mathrm{irr}} + N^{4,3}(2,2)_{3+1}.$$

Formula (5.1.1) splits accordingly into

$$\begin{aligned} N^{4,3}_{\mathrm{irr}}(2,2) &= N^{4,3}_{\mathrm{irr}}(3,1) + 3N^{3,1}_{\mathrm{irr}}(0,3) + 2N^{3,0}_{\mathrm{irr}}(1,2) \qquad \text{and} \\ N^{4,3}_{3+1}(2,2) &= N^{4,3}_{3+1}(3,1) + 3N^{3,1}_{3+1}(0,3) + 2N^{3,0}_{3+1}(1,2). \end{aligned} \tag{5.1.2}$$

Note that all components of $V^{3,1}(0,3)$ and $V^{3,0}(1,2)$ parametrize irreducible cubics, hence the distribution of $N^{3,1}(0,3)$ into $N^{3,1}_{\mathrm{irr}}(0,3)$ and $N^{3,1}_{3+1}(0,3)$ (and similarly that of $N^{3,0}(1,2)$) depends on whether the merging of these irreducible cubics with the line L produces an irreducible or a decomposable quartic. The upshot of Remark (4.21) is that

$$N^{3,0}(1,2) = N^{3,0}(1,2)_{3+1} \qquad \text{and} \qquad N^{3,1}(0,3) = N^{3,1}(0,3)_{\mathrm{irr}},$$

because $\beta' = \beta$ for the former and $\beta' > \beta$ for the latter, in the notation of Remark (4.21). Therefore, (5.1.2) reduces to

$$N^{4,3}_{\mathrm{irr}}(2,2) = N^{4,3}_{\mathrm{irr}}(3,1) + 3N^{3,1}(0,3) \qquad \text{and}$$
$$N^{4,3}_{3+1}(2,2) = N^{4,3}_{3+1}(3,1) + 2N^{3,0}(1,2).$$

Now let us show these formulas in action. First note that

$$N^{3,1}(0,3) = 12 \qquad \text{and} \qquad N^{3,0}(1,2) = 1$$

(the former is the plain number of rational cubics, while the latter is merely the number of cubics passing through one assigned point on L, and passing through eight general crossing points in both cases).

Let us compute the contributions $N^{4,3}(2,2)_{3+1}$ and $N^{4,3}(3,1)_{3+1}$. Since $V^{4,3}(2,2)$ has dimension 9, $N^{4,3}(2,2)_{3+1}$ is the number of quartics decomposed as a cubic plus a line passing through two assigned points p_1, p_2 on L, and through nine general points $a_1, \ldots, a_9$ in the plane. There are only the two following possibilities (recall that in the definition of Severi varieties it is requested that its members should not contain L itself):

(i) the line passes through one of p_1 and p_2 and through one of the a_i's, and the cubic passes through the remaining nine points among p_1, p_2, and $a_1, \ldots, a_9$;

(ii) the line passes through two of the a_i's, and the cubic passes through p_1, p_2 and the seven points remaining among $a_1, \ldots, a_9$.

We thus find

$$N^{4,3}(2,2)_{3+1} = 2 \times 9 + \binom{9}{2} = 54.$$

Similarly, $N^{4,3}(3,1)_{3+1}$ counts decomposed quartics passing through three assigned points p_1, p_2, p_3 on L and eight general points $a_1, \ldots, a_8$ on the plane. We have the same possibilities as in the previous case, and thus

$$N^{4,3}(3,1)_{3+1} = 3 \times 8 + \binom{8}{2} = 52.$$

We can now observe that, indeed,

$$N^{4,3}(2,2)_{3+1} = 54 = N^{4,3}(3,1)_{3+1} + 2N^{3,0}(1,2) = 52 + 2 \cdot 1.$$

Besides, one has $N^{4,3}(2,2)_{\mathrm{irr}} = N^{4,3}(0,4)_{\mathrm{irr}}$ because putting two crossing points on L doesn't put the crossing points in special position, and therefore $N^{4,3}(2,2)_{\mathrm{irr}} = 620$, see [I, Section 7.2.2], or [1, p. 349]. As a sanity check, note that

$$N^{4,3}(2,2) = 674 = N^{4,3}(2,2)_{\mathrm{irr}} + N^{4,3}(2,2)_{3+1} = 620 + 54$$

(see [1, p. 349] for $N^{4,3}(2,2) = 674$). On the other hand, $N^{4,3}(3,1) = 636$ by [1, p. 349], hence $N^{4,3}(3,1)_{\mathrm{irr}} = 636 - 52 = 584$. We can now observe that, indeed,

$$N^{4,3}_{\mathrm{irr}}(2,2) = 620 = N^{4,3}_{\mathrm{irr}}(3,1) + 3N^{3,1}(0,3) = 584 + 3 \cdot 12.$$

The main point in passing from the standard formula to the formula for irreducible curves is to reconsider Remark (4.21) carefully in order to make it cover all possible cases. The upshot is the following.

(5.2) Let the notation be as in Section 4.2 above. In particular, we assume that $V^{d-1,\delta'}(\alpha',\beta')$ appears in the intersection of $V^{d,\delta}(\alpha,\beta)$ with the hyperplane $p^{\perp}$, and let V' be an irreducible component of $V^{d-1,\delta'}(\alpha',\beta')$. Let C be a general member of V', and assume that it decomposes as a union of irreducible curves $C_1,\dots,C_k$.

Then, a local sheet of $V^{d,\delta}(\alpha,\beta)$ at the point $[C\cup L]$ parametrizes irreducible curves if and only if the corresponding choice of Λ as in Paragraph (4.5) is such that, for all $l=1,\dots,k$, there is at least one unassigned contact point of C with L on the component C_l that is off Λ.

Indeed, Λ is the subset of cardinality β of the set of unassigned contact points of C with L around which two local branches are maintained in the deformation of $C\cup L$. Moreover, two local branches are maintained at all assigned contact points of C with L. Therefore, the contact points of C with L at which the two local branches deform to a unique irreducible branch are exactly the points $s_{i,j}^{C\cap L}$ in the notation introduced at the beginning of Section 4.2.1, which deform to $i-1$ nodes only; those form the complement of Λ in the set of unassigned contact points of C with L.

The other ingredient of the proof of the recursive formula for irreducible curves is the following basic enumerative theoretic computation.

(5.3) Let V be the sum of irreducible components of $V^{d,\delta}(\alpha,\beta)$ that parametrizes all the curves that decompose as unions of irreducible curves $C_1,\dots,C_k$, such that for all $l=1,\dots,k$, the curve C_l is a general member of an irreducible component of $V^{d_l,\delta_l}(\alpha^l,\beta^l)$. For all l, let $V_l=V_{\mathrm{irr}}^{d_l,\delta_l}(\alpha^l,\beta^l)$ be the sum of irreducible components of $V^{d_l,\delta_l}(\alpha^l,\beta^l)$ that parametrizes all irreducible members of $V^{d_l,\delta_l}(\alpha^l,\beta^l)$. Then, V is the image of the product $\prod_{l=1}^k V_l$ by the Segre map

$$\Sigma:\prod_{l=1}^k |\mathcal{O}_{\mathbf{P}^2}(d_l)| \longrightarrow |\mathcal{O}_{\mathbf{P}^2}(d)|.$$

The degree of V is the intersection number $V\cdot H^{\dim(V)}$, where H denotes the hyperplane class in the Chow ring of $|\mathcal{O}_{\mathbf{P}^2}(d)|$. It may be computed by pulling-back by Σ. One has $\Sigma^*H=H_1+\cdots+H_k$, where H_l is the hyperplane class of $|\mathcal{O}_{\mathbf{P}^2}(d_l)|$ for all l, and then

$$\begin{aligned}\Big(\prod_{l=1}^k V_l\Big)\cdot(\Sigma^*H)^{\dim(V)} &= \Big(\prod_{l=1}^k V_l\Big)\cdot(H_1+\cdots+H_k)^{\dim(V)}\\ &= \Big(\prod_{l=1}^k V_l\Big)\cdot\sum_{\nu_1+\cdots+\nu_k=\dim(V)}\binom{\dim V}{\nu_1,\dots,\nu_k}H_1^{\nu_1}\cdots H_k^{\nu_k}\\ &= \binom{\dim V}{\dim V_1,\dots,\dim V_k}\prod_{l=1}^k V_l\cdot H_l^{\dim(V_l)},\end{aligned}$$

by computing in the Chow ring of $\prod_{l=1}^k|\mathcal{O}_{\mathbf{P}^2}(d_l)|$, as explained in [3, Section 2.1.4].

The above number equals $V\cdot H^{\dim(V)}$ if the restriction of Σ to $\prod_{l=1}^k V_l$ is birational, which happens if and only if the V_l's are pairwise distinct; otherwise the restriction of Σ has finite degree, equal to the order of the group of permutations of the factors that leaves the product unchanged. The upshot is that, denoting by σ the order of this group (we shall in fact use a refined version of this in the recursion formula),

$$\deg(V)=\frac{1}{\sigma}\binom{\dim V}{\dim V_1,\dots,\dim V_k}\prod_{l=1}^k\deg(V_l).$$

For the application below, recall that the dimensions of generalized Severi varieties is given in Proposition (2.2). Now, fasten your seatbelts, the formula is the following.

(5.4) Theorem. *For all d, δ, α, β as in Corollary (1.5), let $N_{\mathrm{irr}}^{d,\delta}(\alpha,\beta)$ be the number of irreducible members of the generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ passing through $2d+g-1+|\beta|$ general points of the plane ($g = p_a(d)-\delta$), for any general Ω. Then,*

$$N_{\mathrm{irr}}^{d,\delta}(\alpha,\beta) = \sum_{k\geqslant 1:\, \beta_k>0} k \cdot N_{\mathrm{irr}}^{d,\delta}(\alpha+e_k,\beta-e_k)$$
$$+ \sum \Bigg[\frac{1}{\sigma}\binom{2d+g-1+|\beta|}{2d_1+g_1-1+|\beta^1|,\,\ldots,\,2d_k+g_k-1+|\beta^k|} \cdot \binom{\alpha}{\alpha^1,\ldots,\alpha^k,\alpha-\alpha'}$$
$$\cdot \prod_{l=1}^{k}\binom{\beta^l}{\beta^l-\gamma^l} \cdot \prod_{l=1}^{k} I^{\gamma^l} \cdot \prod_{l=1}^{k} N_{\mathrm{irr}}^{d_l,\delta_l}(\alpha^l,\beta^l)\Bigg],$$

where the second sum is taken over all integers $k > 0$, and over all collections of integers $d_1,\ldots,d_k$, and $\delta_1,\ldots,\delta_k$, and all collections of sequences $\alpha^1,\ldots,\alpha^k$, $\beta^1,\ldots,\beta^k$, and $\gamma^1,\ldots,\gamma^k$, subject to the relations

$$\begin{aligned} \alpha' = \alpha^1 + \cdots + \alpha^k &\leqslant \alpha \\ \beta' = \beta^1+\cdots+\beta^k &= \beta + \gamma^1 + \cdots + \gamma^k \\ \forall l = 1,\ldots,k: \qquad \gamma^l &\neq 0 \\ d_1+\cdots+d_k &= d-1 \\ \delta_1+\cdots+\delta_k &= \delta + |\beta'-\beta| - d + 1 - \sum_{h<l} d_h d_l, \end{aligned}$$

and σ is defined as follows: define an equivalence relation on $\{1,\ldots,k\}$ by declaring $h \sim l$ if

$$d_h = d_l, \quad \delta_h = \delta_l, \quad \alpha^h = \alpha^l, \quad \beta^h = \beta^l, \quad \text{and} \quad \gamma^h = \gamma^l;$$

then σ is the product of the factorials of the cardinalities of the equivalence classes.

In the recursion formula, we have used multinomials for elements of $\underline{\mathbf{N}}$; we define them as

$$\binom{\alpha}{\alpha^1,\ldots,\alpha^k,\alpha^0} = \prod_{i\geqslant 1}\binom{\alpha_i}{\alpha_i^1,\ldots,\alpha_i^k,\alpha_i^0}.$$

The interpretation of the first sum in the formula should be fairly clear; let us decipher the second sum. Each summand corresponds to the contribution of

$$\prod_{l=1}^{k} V_{\mathrm{irr}}^{d_l,\delta_l}(\alpha^l,\beta^l) \;\subseteq\; V^{d_1+\cdots+d_k,\,\delta_1+\cdots+\delta_k+\sum_{h<l} d_h d_l}\big(\textstyle\sum_l \alpha^l, \sum_l \beta^l\big)$$

to $V_{\mathrm{irr}}^{d,\delta}(\alpha,\beta) \cap p^{\perp}$ determined by the sequences $\gamma^1,\ldots,\gamma^k \in \underline{\mathbf{N}}$ in the following way: for all $l = 1,\ldots,k$, $\gamma^l \in \underline{\mathbf{N}}$ is the number of unassigned contact points of the general member C_l of $V_{\mathrm{irr}}^{d_l,\delta_l}(\alpha^l,\beta^l)$ at which an open neighbourhood of $C_l \cup L$ deforms to only one irreducible branch; the deformation of $C_1 \cup \cdots \cup C_k \cup L$ is irreducible if and only if there is at least one such point for all l (see Paragraph (5.2) above), which corresponds to the requirement that $\gamma^l \neq 0$ for all l. Note that the number of nodes of $C_1 \cup \cdots \cup C_k$ is indeed $\delta_1 + \cdots + \delta_k + \sum_{h<l} d_h d_l$.

This contribution in fact means the sum of the contributions of all $\prod_{l=1}^{k} V_{\mathrm{irr}}^{d_l,\delta_l}(\alpha^l,\beta^l)(\Omega^l)$, for all sets $\Omega^1,\ldots,\Omega^k$ of cardinalities $\alpha^1,\ldots,\alpha^k \in \underline{\mathbf{N}}$ such that $\Omega = \bigcup_{l=1}^{k} \Omega^l$; the multinomial $\binom{\alpha}{\alpha^1,\ldots,\alpha^k,\alpha-\alpha'}$ corresponds to the number of possible $\Omega^1,\ldots,\Omega^k$. On the other hand, the factor

$\frac{1}{\sigma}$ times the first multinomial is the degree of $\prod_{l=1}^{k} V_{\mathrm{irr}}^{d_l,\delta_l}(\alpha^l, \beta^l, \gamma^l)$ (see Paragraph (5.3)); note that we have added the decoration γ^l (with the meaning explained above), and the "stabilizer term" σ has been changed accordingly in the theorem with respect to Paragraph (5.3).

We leave to the reader the excellent exercise of filling out the details of the proof of Theorem (5.4).

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Lecture VII
The Caporaso–Harris recursion within a degeneration of the plane

by Thomas Dedieu

In this text we show on examples how the Caporaso–Harris recursion procedure [1], explained in details in [VI], can be formulated in terms of a degeneration of the projective plane to the union of a projective plane and a rational ruled surface $\mathbf{F}_1$ intersecting transversely along a line.

We formulate the degeneration procedure in a general framework in Section 1, and work it out explicitly on two examples in Sections 2 and 3, namely for the enumerations of 2-nodal plane quartics, and of 1-nodal cubics tangent to a line, respectively. In Section 4 we follow a different degeneration procedure, by way of comparison, in order to enumerate 2-nodal plane quartics.

We assume the reader is familiar with the two chapters [I] and [VI], and will freely use terminology and notation from them.

1 – The degeneration procedure

(1.1) Consider the projective plane $\mathbf{P}^2$ and a line R in it. Let d, δ be integers and $\alpha, \beta \in \underline{\mathbf{N}}$ be sequences such that $d > 0$, $\delta \geqslant 0$, and $I\alpha + I\beta = d$. Let Ω be a (sequence of) set(s) of α points on the line R. Let Z be the union of $2d + p_a(d) - \delta - 1 + |\beta|$ general points in the plane, where $p_a(d) = \frac{1}{2}(d-1)(d-2)$. We want to count the (finitely many) plane curves of degree d with δ nodes, intersecting R at the points of Ω with the multiplicities prescribed by α and at further unassigned points with the multiplicities prescribed by β, and passing through all points of Z. The result is the number $N^{d,\delta}(\alpha, \beta)$ in the notation of [VI] and [1].

(1.2) Let $\mathcal{S}$ be the blow-up of $\mathbf{P}^2 \times \mathbf{A}^1$ along the line $R \times \{0\}$, and let $\pi : \mathcal{S} \to \mathbf{A}^1$ be the composition of the blow-up map with the second projection. For $t \neq 0$, the fibre S_t of π over t is a projective plane, whereas the central fibre S_0 is the union of a projective plane and a rational ruled surface $\mathbf{F}_1$, which we shall call $\mathbf{P}$ and $\mathbf{F}$ respectively, intersecting transversely along a curve E which is a line in $\mathbf{P}$, and the section of the ruling with self-intersection -1 in $\mathbf{F}$; we shall write $S_0 = \mathbf{P} \cup_E \mathbf{F}$.

Let $\mathcal{L}$ be the line bundle on $\mathcal{S}$ obtained by pulling-back $\mathcal{O}_{\mathbf{P}^2}(1)$ successively by the first projection $\mathbf{P}^2 \times \mathbf{A}^1 \to \mathbf{P}^2$ and by the blow-up map. The line bundle $\mathcal{L}$ restricts to $\mathcal{O}_{\mathbf{P}}(H)$ and $\mathcal{O}_{\mathbf{F}}(F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively, where H is the line class and F is the class of the ruling.

Let $\mathcal{R}$ be the proper transform of $R \times \mathbf{A}^1$ in $\mathcal{S}$. For $t \neq 0$, the fibre R_t of $\mathcal{R}$ over $t \in \mathbf{A}^1$ is the line R in $\mathbf{P}^2$, whereas the central fibre R_0 is a section of the ruling of $\mathbf{F}$ that has self-intersection $+1$. Let $\mathcal{W}$ be the proper transform of $\Omega \times \mathbf{A}^1$, and denote by Ω_0 its central fibre: this is merely Ω seen on the section $R_0 \cong \mathbf{P}^1$ of $\mathbf{F}$.

In order to perform the Caporaso–Harris degeneration procedure, we let the set of points Z degenerate as follows. Let p be one point of Z, and consider a general section σ_p of $\mathbf{P}^2 \times \mathbf{A}^1 \to \mathbf{A}^1$ which intersects $R \times \{0\}$, and whose fibre over $1 \in \mathbf{A}^1$ coincides with p. Then we consider the proper transform $\mathcal{Z}$ in $\mathcal{S}$ of $(Z-p) \times \mathbf{A}^1 + \sigma_p$; its fibre over $1 \in \mathbf{A}^1$ is $Z \subseteq \mathbf{P}^2$, whereas its central fibre is the sum of $Z - p \subseteq \mathbf{P}$ and a general point p_0 of $\mathbf{F}$.

(1.3) We consider the family of surfaces $\pi : \mathcal{S} \to \mathbf{A}^1$, equipped with the sheaf $\mathcal{L}^{\otimes d}$. It provides a degeneration to $S_0 = \mathbf{P} \cup_E \mathbf{F}$ of the linear system $|dH|$ on $\mathbf{P}^2$. Moreover, $\mathcal{R}$ provides a degeneration of the line $R \subseteq \mathbf{P}^2$, and $\mathcal{W}$ and $\mathcal{Z}$ provide degenerations of Ω and Z respectively. We want to solve the enumerative problem stated in (1.1) by considering its limit within this degeneration.

Let $\mathcal{I}_{\mathcal{Z}/\mathcal{S}}$ be the ideal sheaf defining $\mathcal{Z}$ in $\mathcal{S}$, and let $\mathcal{I}_{\mathcal{W}}$ be the ideal sheaf encoding the contact conditions with $\mathcal{R}$ at $\mathcal{W}$ corresponding to $\alpha \in \underline{\mathbf{N}}$. It follows from the results in [VI], which are taken from [1], that the pair $(\mathcal{S}/\mathbf{A}^1, \mathcal{L}^{\otimes d} \otimes \mathcal{I}_{\mathcal{Z}/\mathcal{S}} \otimes \mathcal{I}_{\mathcal{W}})$ is δ-well behaved, in the terminology of [I, Section 5]. Thus, our enumerative problem stated in (1.1) can be solved by computing the regular part of the limit Severi variety

$$\mathfrak{V}_\delta(\mathcal{S}/\mathbf{A}^1, \mathcal{L}^{\otimes d} \otimes \mathcal{I}_{\mathcal{Z}/\mathcal{S}} \otimes \mathcal{I}_{\mathcal{W}});$$

in other words, the number we are looking for equals the number of curves in the linear systems on $\mathbf{P} \cup \mathbf{F}$ corresponding to all possible twists of $\mathcal{L}^{\otimes d}$, passing through $Z - p$ on $\mathbf{P}$ and through p_0 on $\mathbf{F}$, with contact conditions with R_0 determined by $\alpha \in \underline{\mathbf{N}}$ at the points Ω_0 and by $\beta \in \underline{\mathbf{N}}$ at unassigned points, and with $\delta_{\mathbf{P}}$ nodes on $\mathbf{P}$, $\delta_{\mathbf{F}}$ nodes on $\mathbf{F}$, and τ tacnodes along $E = \mathbf{P} \cap \mathbf{F}$ ($\tau \in \underline{\mathbf{N}}$), such that

$$\delta = \delta_{\mathbf{P}} + \delta_{\mathbf{F}} + \nu(\tau) \tag{1.3.1}$$

with $\nu(\tau) = \sum_{m \geqslant 2} \tau_m \,(m-1)$; it is understood that these curves are counted with the multiplicity $\mu(\tau) = \prod_{m\geqslant 2} m^{\tau_m}$.[1]

The framework of [I] does not allow to formally take in consideration the contact conditions at unprescribed points; however this appendix is intended to be illustrative, so I will remain informal. Moreover, the framework of [I] would require to have an invertible sheaf instead of $\mathcal{L}^{\otimes d} \otimes \mathcal{I}_{\mathcal{Z}/\mathcal{S}} \otimes \mathcal{I}_{\mathcal{W}}$; this can be easily remedied by considering a suitable blow-up of $\mathcal{S}$, but I prefer not to do it here.

2 – Two-nodal quartics

In this section we enumerate 2-nodal plane quartics by repeatedly applying the procedure described in Section 1.

[1] Beware the difference in notation between $\alpha = (\alpha_m)_{m\geqslant 1}$ and $\beta = (\beta_m)_{m\geqslant 1}$ on the one hand, and $\tau = (\tau_m)_{m\geqslant 2}$ on the other hand: while the former encode all contact points with R_0, including the transverse ones, the latter only encodes m-tacnodes with $m > 1$. One can extend τ to $\tilde{\tau}$ to put it in α/β style, by setting $\tau_1 = \deg(\mathcal{L}^{\otimes d}(-W)\big|_E) - \sum_{m\geqslant 2} m\,\tau_m$ and $\tilde{\tau} = (\tau_1, \tau)$, where $\mathcal{L}^{\otimes d}(-W)$ is the twist of $\mathcal{L}^{\otimes d}$ under consideration; then, $\nu(\tau) = I\tilde{\tau} - |\tilde{\tau}|$, and $\mu(\tau) = I^{\tilde{\tau}}$.

(2.1) Our initial problem is to count 2-nodal plane quartics passing through 12 general points. In the notation of (1.1), we consider $d = 4$, $\delta = 2$, $\beta = 4$, and $\alpha = 0$. The result is the number $N^{4,2} = N^{4,2}(0, 4)$.

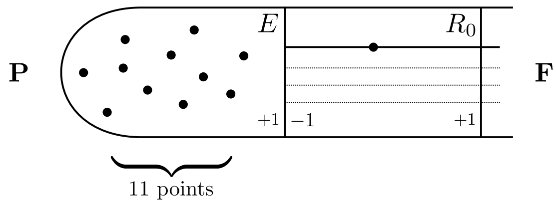
(2.2) We will let our 12 points degenerate one by one to general points on $\mathbf{F}$, and we will need to consider the restriction to $\mathbf{P} \cup \mathbf{F}$ of all possible twists of $\mathcal{L}^{\otimes 4}$. The line bundle $\mathcal{L}^{\otimes 4}$ itself restricts to $\mathcal{O}_{\mathbf{P}}(4H)$ and $\mathcal{O}_{\mathbf{F}}(4F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively. Thus, members of the corresponding linear system on S_0 consist of a plane quartic on $\mathbf{P}$ and four fibres of the ruling of $\mathbf{F}$, which have to match along their intersections with $E = \mathbf{P} \cap \mathbf{F}$, which consist of four points.

The only other relevant twist will be $\mathcal{L}^{\otimes 4}(-\mathbf{F})$, which restricts to $\mathcal{O}_{\mathbf{P}}(3H)$ and $\mathcal{O}_{\mathbf{F}}(E + 4F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively. The linear systems $|3H|_{\mathbf{P}}$ and $|E + 4F|_{\mathbf{F}}$ have dimensions 9 and 8 respectively. Members of the linear system on S_0 corresponding to $\mathcal{L}^{\otimes 4}(-\mathbf{F})$ consist of a member of $|3H|_{\mathbf{P}}$ and a member of $|E+4F|_{\mathbf{F}}$ which match along their intersections with E, which consist of three points; the dimension of the linear system on S_0 is therefore $9+8-3 = 14$ (both $|3H|_{\mathbf{P}}$ and $|E + 4F|_{\mathbf{F}}$ restrict to the complete linear system $|3H|_E$ on $E \simeq \mathbf{P}^1$), equal to that of the linear system of all plane quartics.

To rule out the other twists, we will use (i) that $\mathcal{L}^{\otimes 4}(-a\mathbf{F})$ with $a > 1$ restricts to $\mathcal{O}_{\mathbf{P}^2}(4-a)$ on $\mathbf{P}^2$, hence the corresponding curves on $\mathbf{P}$ have degree at most 2, and (ii) that $\mathcal{L}^{\otimes 4}(a\mathbf{F})$ with $a > 0$ restricts to $\mathcal{O}_{\mathbf{F}_1}(-aE + 4F)$ on $\mathbf{F}$, which is not effective, so that it is already clear that the latter twists will never give any contribution to our enumerations.

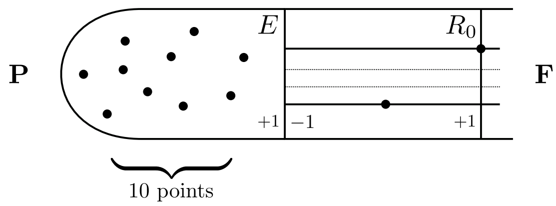
Now, let us start the enumeration. The relations we get at each step are all spelled out in Paragraph (2.8).

(2.3) $N^{4,2}(0, 4)$**.** Let us apply the procedure described in Section 1 to our initial enumerative problem: we end up with 11 general points on $\mathbf{P}$, and 1 general point on $\mathbf{F}$, as indicated on the figure below.



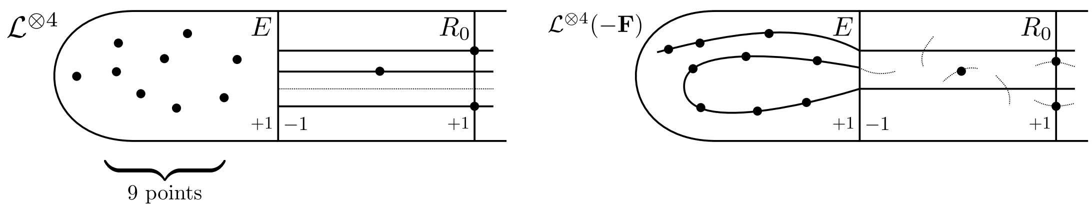
Only the trivial twist is relevant, because there is no curve of degree $d' < 4$ on $\mathbf{P}$ passing through the 11 base points. Thus we are only considering curves consisting of a plane quartic $C_{\mathbf{P}}$ on $\mathbf{P}$ and of four rulings of $\mathbf{F}$. One of these rulings must pass through the base point on $\mathbf{F}$, and has a fixed intersection point with E, which imposes a fixed passing point to $C_{\mathbf{P}}$ on E. On the other hand, the only way to fulfil Relation (1.3.1) with $\delta = 2$ with the curves under consideration is to have $\delta_{\mathbf{P}} = 2$ (and $\delta_{\mathbf{F}} = 0$ and $\tau = 0$). The upshot is that $N^{4,2}(0, 4) = N^{4,2}(1, 3)$.

(2.4) $N^{4,2}(1, 3)$**.** We now apply the procedure of Section 1 to the enumerative problem corresponding to the number $N^{4,2}(1, 3)$. We end up with 10 general points of $\mathbf{P}$, 1 general point of R_0, and 1 general point of $\mathbf{F}$, as pictured below.



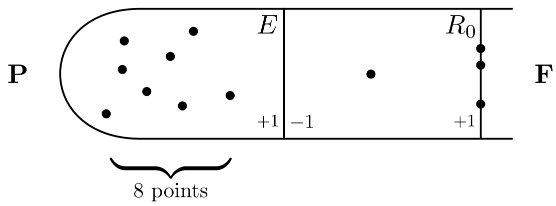
Again, only the trivial twist is relevant, and one must have $\delta_{\mathbf{P}} = 2$. This time, two of the rulings on $\mathbf{F}$ must pass through the two base points on $\mathbf{F}$, and they each impose one fixed passing point on E. The upshot is that $N^{4,2}(1,3) = N^{4,2}(2,2)$.

(2.5) $N^{4,2}(2,2)$**.** We repeat the procedure to compute this number. We end up with 9 general points of $\mathbf{P}$, 2 general point of R_0, and 1 general point of $\mathbf{F}$, as pictured below.



This time the two twists $\mathcal{L}^{\otimes 4}$ and $\mathcal{L}^{\otimes 4}(-\mathbf{F})$ will contribute. The trivial twist gives a contribution of $N^{4,2}(3,1)$ as in the previous cases. The contribution of the other twist comes from curves made of a cubic $C_{\mathbf{P}}$ on $\mathbf{P}$ and a curve $C_{\mathbf{F}}$ on $\mathbf{F}$ linearly equivalent to $E + 4F$. The cubic $C_{\mathbf{P}}$ is uniquely determined by the 9 base points on $\mathbf{P}$ (if one will, $N^{3,0}(0,3) = 1$), it is smooth and fixes 3 pairwise distinct passing points on E for $C_{\mathbf{F}}$. Therefore, the curve $C_{\mathbf{F}}$ must be 2-nodal. Since $C_{\mathbf{F}} \cdot F = 1$, this implies that it must be of the form $F_1 + F_2 + C'$ with F_1 and F_2 two rulings, and $C' \sim E + 2F$. The two rulings must take up two of the fixed points on E, which gives $\binom{3}{2}$ possibilities, and then C' is uniquely determined by the third point on E and the base points on $\mathbf{F}$, as the linear system $|E + 2F|$ has dimension 4.

(2.6) $N^{4,2}(3,1)$**.** We repeat the procedure, and end up with 8 general points of $\mathbf{P}$, 3 general point of R_0, and 1 general point of $\mathbf{F}$, as pictured below.



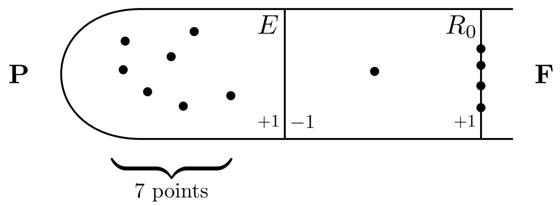
The trivial twist $\mathcal{L}^{\otimes 4}$ contributes by $N^{4,2}(4,0)$ as in the previous cases, and this time the twist $\mathcal{L}^{\otimes 4}(-\mathbf{F})$ contributes in several different ways: on the $\mathbf{P}$ side we have cubics through the 8 base points, which thus move in a pencil, and cut out a g^1_3 on E, while on the $\mathbf{F}$ side we have curves in $|E + 4F|$ through the $3 + 1$ base points, which impose independent linear conditions, so that we end up with a 4-dimensional linear system. Therefore we have the following three possibilities to fulfil (1.3.1) with $\delta = 2$.

(a) $\delta_{\mathbf{P}} = \delta_{\mathbf{F}} = 1$. The number of 1-nodal cubics through the 8 points on $\mathbf{P}$ is $N^{3,1}(0,3)$, and each of these fixes 3 points on E. In turn, the curve on $\mathbf{F}$ has to split as $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E + 3F$ in order to be 1-nodal; moreover, in order to match with the curve on $\mathbf{P}$, the ruling F_1 must pass through one of the 3 fixed points on E. Since $|E + 3F|$ has dimension 6, the curve C_1 is then uniquely determined by the two other fixed points on E and the $3 + 1$ base points on $\mathbf{F}$. The number of curves of this kind is thus $\binom{3}{1} N^{3,1}(0,3)$.

(b) $\tau_2 = 1$ (i.e., one 2-tacnode) and $\delta_{\mathbf{F}} = 1$. The number of cubics through the 8 points on $\mathbf{P}$ that are tangent to E is $N^{3,0}(0,[1,1])$ (this number is the degree of the dual surface of a twisted cubic, which is 4). The curve on $\mathbf{F}$ has to split as $F_1 + C'$ as in the previous case, and this time the ruling F_1 can only pass through the simple point of the divisor fixed on E by the curve on $\mathbf{P}$. Then, the curve C_1 is uniquely determined by its passing through the double point of the divisor on E and the $3+1$ base points on $\mathbf{F}$. The number of curves of this kind is thus $N^{3,0}(0,[1,1])$, to be counted with multiplicity 2 because of the tacnodal contribution in (1.3.1).

(c) $\delta_{\mathbf{F}} = 2$. In this case the curve on $\mathbf{F}$ has to split as $F_1 + F_2 + C'$ with $F_1, F_2 \sim F$ and $C' \sim E + 2F$. One of F_1 and F_2, say F_1, must pass through one of the three base points on R_0, and C' must pass through the other two: this gives $\binom{3}{1}$ choices. Then F_1 fixes a point on E, which determines a unique cubic through the 8 base points on $\mathbf{P}$ (which we can formulate as $N^{3,0}(1,2) = 1$), which fixes two other points on E. The ruling F_2 must pass through one of them, which gives $\binom{2}{1}$ possibilities, after what the curve C' is uniquely determined by the third point on E, the two remaining points on R_0, and the general point on $\mathbf{F}$. The number of curves of this kind is thus $\binom{3}{1}\binom{2}{1}N^{3,0}(1,2)$.

(2.7) $N^{4,2}(4,0)$**.** We repeat the procedure, and end up with 7 general points of $\mathbf{P}$, 4 general point of R_0, and 1 general point of $\mathbf{F}$, as pictured below.



The trivial twist $\mathcal{L}^{\otimes 4}$ no longer contributes, because there is no curve in $\mathbf{F}$ linearly equivalent to $4F$ and passing through the $4+1$ base points on F. This is thus the point in the recursion where "contributions of degree 4 curves have been entirely exhausted.". For the contribution of the twist $\mathcal{L}^{\otimes 4}(-\mathbf{F})$, we have the following possibilities.

(a) $\delta_{\mathbf{P}} = 2$. The number of 2-nodal cubics through the 7 points on $\mathbf{P}$ is $N^{3,2}(0,3)$ (which equals $\binom{7}{2}$ because all such cubics are decomposed as a conic plus a line). For each such cubic, there is a unique matching curve in $|E+4F|$ on $\mathbf{F}$ passing through the $4+1$ base points.

(b) $\delta_{\mathbf{P}} = \delta_{\mathbf{F}} = 1$. The curve on $\mathbf{F}$ must split as $F_1 + C'$, with $F_1 \sim F$ and $C' \sim E + 3F$. The ruling F_1 must pass through one of the base points on R_0 (which gives 4 possibilities), and therefore fixes a point on E. Then, there is a finite number $N^{3,1}(1,2)$ of 1-nodal cubics on $\mathbf{P}$ passing through this point and the 7 base points on $\mathbf{P}$ (the number $N^{3,1}(1,2)$ is easily seen to equal $N^{3,1}(0,3)$, which is 12). In turn, each of these fixes two new points on E, and the curve C' is uniquely determined by these two points and the remaining $3+1$ base points on $\mathbf{F}$. Curves of this kind are thus in the number $\binom{4}{1}N^{3,1}(1,2)$.

(c) $\delta_{\mathbf{F}} = 2$. The curve on $\mathbf{F}$ must split as $F_1 + F_2 + C'$, with $F_1, F_2 \sim F$ and $C' \sim E + 2F$, and the two rulings F_1 and F_2 each must pass through one of the 4 base points on R_0, which gives $\binom{4}{2}$ possibilities. These two rulings then fix a unique cubic on $\mathbf{P}$ through the 7 points (which we can phrase as $N^{3,0}(2,1) = 1$), which in turn fixes a third point on E. The curve C' is then uniquely determined by the latter point and the remaining $2+1$ base points on $\mathbf{F}$. We thus find $\binom{4}{2}N^{3,0}(2,1)$ curves.

(d) $\delta_{\mathbf{P}} = 1$ and $\tau_2 = 1$ (i.e., one 2-tacnode). There is a finite number of possible curves on $\mathbf{P}$, in the number $N^{3,1}(0,[1,1])$, and each of these determines a unique matching curve in $|E+4F|$ on $\mathbf{F}$. These curves count with multiplicity 2.

(e) $\delta_{\mathbf{F}} = 1$ and $\tau_2 = 1$. The curve on $\mathbf{F}$ must split as $F_1 + C'$, with $F_1 \sim F$ and $C' \sim E+3F$. The ruling F_1 must pass through one of the base points on R_0 (which gives 4 possibilities), and therefore fixes a point on E. Then, there is a pencil of cubics on $\mathbf{P}$ passing through this point and the 7 base points on $\mathbf{P}$; it cuts out (residually) a g^1_2 on E, which has two members made of a double point, meaning that there are two cubics in the pencil that are tangent to E: in other words, $N^{3,0}(1,[0,1]) = 2$. Finally, for each such cubic there is a unique matching curve C'

which also pass through the $3+1$ remaining base points on $\mathbf{F}$. Thus we find $\binom{4}{1}N^{3,0}(1,[0,1])$ curves; they count with multiplicity 2.

(f) $\tau_3 = 1$. There is a finite number of possible curves on $\mathbf{P}$, which is $N^{3,0}(0,[0,0,1])$. This number equals the number of divisors of the form $3q$ in the g^2_3 cut out on E by the 2-dimensional linear system of cubics through the 7 base points on $\mathbf{P}$, which is 3 (this is the number of flexes of a 1-nodal cubic, see [XII, Section **??**]). For each such cubic there is a unique matching curve in $|E+4F|$ on $\mathbf{F}$ which also passes through the $4+1$ base points. We thus find $N^{3,0}(0,[0,0,1])$ curves, and they count with multiplicity 3.

(2.8) Summary and conclusion. We have applied the degeneration procedure of Section 1 five times, and after that we are reduced to enumerations of curves of degrees $d' < 4$. The relations we have obtained are the following:

$$\begin{aligned}
N^{4,2}(0,4) &= N^{4,2}(1,3)\\
&= N^{4,2}(2,2)\\
&= N^{4,2}(3,1) + \tbinom{3}{2}\underbrace{N^{3,0}(0,3)}_{=1}\\
N^{4,2}(3,1) &= N^{4,2}(4,0) + \tbinom{3}{1}\underbrace{N^{3,1}(0,3)}_{=12} + 2\cdot\underbrace{N^{3,0}(0,[1,1])}_{=4} + \tbinom{3}{1}\tbinom{2}{1}\underbrace{N^{3,0}(1,2)}_{=1}\\
N^{4,2}(4,0) &= \underbrace{N^{3,2}(0,3)}_{=21} + \tbinom{4}{1}\underbrace{N^{3,1}(1,2)}_{=12} + \tbinom{4}{2}\underbrace{N^{3,0}(2,1)}_{=1} + 2\cdot\underbrace{N^{3,1}(0,[1,1])}_{=36}\\
&\quad + 2\cdot\tbinom{4}{1}\underbrace{N^{3,0}(1,[0,1])}_{=2} + 3\cdot\underbrace{N^{3,0}(0,[0,0,1])}_{=3}.
\end{aligned}$$

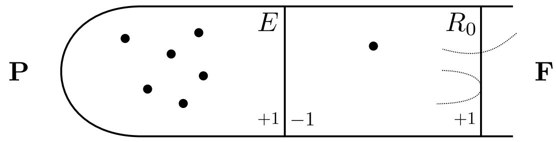
With the (possible) exception of $N^{3,1}(0,[1,1])$, which is computed in Section 3 below, the numbers of cubic curves appearing in the relations above can all be more or less elementarily computed; hints for these computations have been given in the course of the recursive application of the degeneration procedure. In the upshot, we get

$$\begin{aligned}
N^{4,2}(4,0) &= 21+48+6+72+16+9 = 172\\
N^{4,2}(3,1) &= 172+36+8+6 = 222\\
N^{4,2}(0,4) = N^{4,2}(1,3) = N^{4,2}(2,2) &= 222+3 = 225.
\end{aligned}$$

3 – An ancillary enumeration of cubics

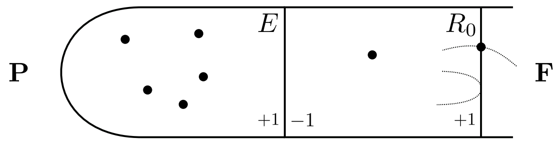
In this section we compute the number of 1-nodal plane cubics tangent to a line; the result is the number $N^{3,1}(0,[1,1])$. This is necessary for the enumeration of 2-nodal quartics in the previous section, and will serve as an additional illustration of the recursive degeneration procedure. We will need to consider the line bundles $\mathcal{L}^{\otimes 3}$, which restricts to $\mathcal{O}_{\mathbf{P}}(3H)$ and $\mathcal{O}_{\mathbf{F}}(3F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively, and $\mathcal{L}^{\otimes 3}(-\mathbf{F})$, which restricts to $\mathcal{O}_{\mathbf{P}}(2H)$ and $\mathcal{O}_{\mathbf{F}}(E+3F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively. The generalized Severi variety $V^{3,1}(0,[1,1])$ has dimension 7, so we start with 7 base points.

(3.1) $N^{3,1}(0,[1,1])$**.** We apply the degeneration procedure of Section 1, which gives 6 general base points on $\mathbf{P}$, and one general base point on $\mathbf{F}$.



Only the trivial twist contributes, because there is no conic on **P** through the 6 base points, and we must have $\delta_{\mathbf{P}} = 1$ in order to fulfil (1.3.1) with $\delta = 1$. To have the correct intersection pattern with R_0, the curve on **F** must be of the form $F_1 + 2F_2$ with $F_1, F_2 \sim F$. Then, either F_1 passes through the base point on **F**, after what there are $N^{3,1}(1,[0,1])$ matching 1-nodal cubics through the 6 base points on **P**, or F_2 passes through the base point on **F**, and then there are $N^{3,1}([0,1],1)$ matching 1-nodal cubics through the 6 base points on **P**; it follows from the analysis in [1], reported on in [VI], that the latter curves count with multiplicity 2.

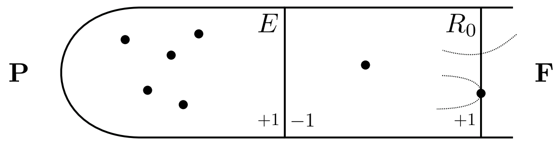
(3.2) $N^{3,1}(1,[0,1])$**.** Applying the degeneration procedure, we get 5 general base points on **P**, one general base point on R_0, and one general point on **F**.



A priori, the two twists $\mathcal{L}^{\otimes 3}$ and $\mathcal{L}^{\otimes 3}(-\mathbf{F})$ contribute in this case. In the contribution of the trivial twist $\mathcal{L}^{\otimes 3}$, the curve on **F** must once again be of the form $F_1 + 2F_2$ with $F_1, F_2 \sim F$ in order to meet the contact condition with R_0. Then, F_1 must past through the base point on R_0, and F_2 through the general base point on **F**. There are then $N^{3,1}([1,1],0)$ matching curves on **P**, and the curves we find in this way count with multiplicity 2 because of the assignement of the tangency point with R_0.

In the contribution of the twist $\mathcal{L}^{\otimes 3}(-\mathbf{F})$, we find a unique conic through the five base points on **P**. It is smooth and fixes two distinct points on E. The curve on **F** must be decomposed as $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E + 2F$ in order to be 1-nodal. The ruling F_1 must pass through one of the two points on E, and therefore does not pass through the base point on R_0. It follows that it is impossible to choose C' so that the contact conditions with R_0 are fulfilled (it is possible to choose C' tangent to R_0 at the base point, but this does not give an admissible curve, see the comment on the definition of logarithmic Severi varieties in [III, (1.6)]; this curve will contribute in (3.3) however).

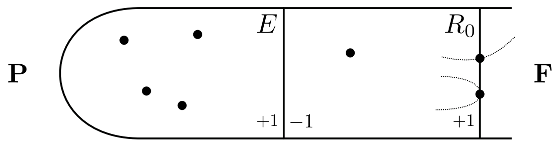
(3.3) $N^{3,1}([0,1],1)$**.** We apply the degeneration procedure, and end up with 5 general base points on **P**, one general base point on **F**, and a fixed tangency point with R_0.



We will have contributions from the two twists $\mathcal{L}^{\otimes 3}$ and $\mathcal{L}^{\otimes 3}(-\mathbf{F})$. For the trivial twist $\mathcal{L}^{\otimes 3}$, the curve on **F** must be of the form $F_1 + 2F_2$ with F_1 and F_2 two rulings, with F_2 passing through the fixed point on R_0 and F_1 through the general base point on **F**. This curve fixes a tangency point and a simple passing point on E, and the possible cubic curves on **P** are in the number $N^{3,1}([1,1],0)$.

In the situation defined by the twist $\mathcal{L}^{\otimes 3}(-\mathbf{F})$ the curve on **P** is a conic, and there is a unique one through the five base points (i.e., $N^{2,0}(0,2) = 1$), which fixes two points on E. In order to be 1-nodal, the curve on **F** must be decomposed as $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E + 2F$. The curve F_1 must pass through one of two points on E, which gives two possibilities, and then C' must pass through the second point on E, through the base point which is general on **F**, and be tangent to R_0 at the prescribed point: these are 4 independent linear conditions, hence the curve C' is uniquely determined. The contribution of $\mathcal{L}^{\otimes 3}(-\mathbf{F})$ is thus by $\binom{2}{1} N^{2,0}(0,2)$.

(3.4) $N^{3,1}([1,1],0)$**.** We apply the degeneration procedure and get 4 general base points on **P**, one general base point on **F**, and a fixed tangency plus a fixed simple intersection point with R_0.



This time only the twist $\mathcal{L}^{\otimes 3}(-\mathbf{F})$ may contribute, because it is impossible for a curve in $|3F|$ to fulfil all the conditions on **F**. This is therefore the point in the recursion procedure where there only remain numbers $N^{d',\delta'}(\alpha',\beta')$ with $d' < 3$. Thus, we have a conic on **P**, which moves in a pencil determined by the four base points. We have the following possibilities to fulfil (1.3.1) with $\delta = 1$.

(a) $\delta_{\mathbf{P}} = 1$. There are $N^{2,1}(0,2)$ 1-nodal conics through the 4 base points on **P**, and each fixes two points on E ($N^{2,1}(0,2) = 3$ because all such conics are decomposed in two lines). Then the curve on **F** must satisfy 6 independent linear conditions, and is thus uniquely determined.

(b) $\delta_{\mathbf{F}} = 1$. Then the curve on **F** must be of the form $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E+2F$. The ruling F_1 must pass through the fixed simple intersection point with R_0. This fixes a point on E, which determines a unique conic on **P** through the 4 base points ($N^{2,0}(1,1) = 1$), which in turn fixes a second point on E. Then the curve C' must pass through this point, through the general base point on **F**, and be tangent with R_0 at the prescribed point: it is thus uniquely determined.

(c) $\tau_2 = 1$. The curve on **P** must be a conic tangent to E: these are in the number $N^{2,0}(0,[0,1])$ (which equals 2, the number of double points in the g^1_2 cut out on E). Then the curve on **F** has an imposed tangency with E at a prescribed point, hence it must in total satisfy 6 independent linear conditions, and thus it is uniquely determined. These curve counts with the multiplicity 2.

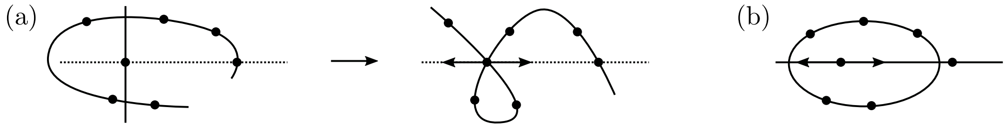
(3.5) Summary and conclusion. In the four above applications of the degeneration procedure, we have obtained the following relations:

$$
\begin{aligned}
&N^{3,1}(0,[1,1]) = N^{3,1}(1,[0,1]) + 2 \cdot N^{3,1}([0,1],1)\\
&N^{3,1}(1,[0,1]) = 2 \cdot N^{3,1}([1,1],0)\\
&N^{3,1}([0,1],1) = N^{3,1}([1,1],0) + \tbinom{2}{1} \underbrace{N^{2,0}(0,2)}_{=1}\\
&N^{3,1}([1,1],0) = \underbrace{N^{2,1}(0,2)}_{=3} + \underbrace{N^{2,0}(1,1)}_{=1} + 2 \cdot \underbrace{N^{2,0}(0,[0,1])}_{=2}
\end{aligned}
\tag{3.5.1}
$$

Putting these identities all together, we obtain $N^{3,1}([1,1],0) = 8$, $N^{3,1}([0,1],1) = 10$, $N^{3,1}(1,[0,1]) = 16$ and, finally, $N^{3,1}(0,[1,1]) = 36$.

(3.6) Remark. The numbers $N^{3,1}([1,1],0)$ and $N^{3,1}([0,1],1)$ may also be computed by appropriately enumerating singular cubics in a pencil, following [XI, Section 2.1], and doing so sheds some light on our degeneration procedure.

For the first number, we consider the pencil of plane cubics defined by 8 base points as follows: 3 of base points on a line R, 2 of which are infinitely near, and 5 general points of $\mathbf{P}^2$. There are 12 singular members (counted with multiplicities) in this pencil, as in any pencil of cubics, and the curves counted by $N^{3,1}([1,1],0)$ are all among them. Some singular members of the pencil must be excluded however, namely (a) the unique member of the pencil with a double point at the point where there are two infinitely near base points, and (b) the member of the pencil made of the line R itself plus the unique conic through the 5 general base points.



Both these curves have two nodes after we blow-up the base points of the pencil, so they contribute each by 2 to the number 12 of singular members in the pencil, and we thus find $N^{3,1}([1,1],0) = 12 - 2 - 2 = 8$.

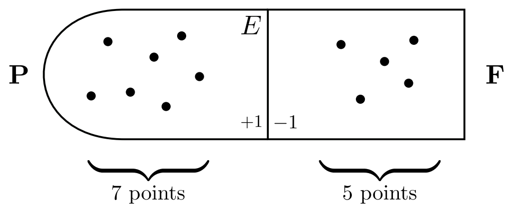
For the second number, we put the 8 base points as follows: 2 infinitely near base points on the line R, and 6 general points of $\mathbf{P}^2$. This time there is no member of the pencil made of the line R and a conic, and only the member as in case (a) above must be excluded. We thus find $N^{3,1}([1,1],0) = 12 - 2 = 10$.

The difference between these two enumerations explains Relation (3.5.1) above.

4 – Alternative degeneration procedure

In this section I present, by way of comparison, an alternative way of enumerating 2-nodal plane quartics by degeneration, which somehow packages all five degenerations in Paragraphs (2.3)–(2.7) in just one degeneration. This is the same degeneration procedure as in [I, Section 7.2.2], where 3-nodal quartics are enumerated, so I will be brief.

We consider always the same degeneration of the projective plane to the transverse union $\mathbf{P} \cup_E \mathbf{F}$, and this time we let 5 of the 12 base points on $\mathbf{P}^2$ degenerate to general points of $\mathbf{F}$.



This prevents the trivial twist $\mathcal{L}^{\otimes 4}$ to give any contribution, and therefore the total number $N^{4,2}$ will be accounted for by the contributions from the twist $\mathcal{L}^{\otimes 4}(-\mathbf{F})$. In this twist, we have a net of plane cubics on $\mathbf{P}$ defined by the 7 general base points on $\mathbf{P}$, which cuts out a g^2_3 on E; on the other hand, the 5 general base points on $\mathbf{F}$ define a 3-dimensional linear subsystem of $|E+4F|$, which cuts out a g^3_3 on E. We have the following possibilities to fulfil (1.3.1) with $\delta = 2$.

(a) $\delta_{\mathbf{P}} = 2$. There are $N^{3,2}(0,3) = \binom{7}{2}$ two-nodal cubics through the 7 base points on $\mathbf{P}$, and for each of those there is a unique matching member of $|E+4F|$ through the 5 base points on $\mathbf{F}$.

(b) $\delta_{\mathbf{P}} = \delta_{\mathbf{F}} = 1$. The 1-nodal curve on $\mathbf{F}$ necessarily decompose as $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E + 3F$. There are the two following combinatorial possibilities.

(i) The ruling F_1 passes through 1 base point and C' through the other 4 ($\binom{5}{1} = 5$ possibilities). There are then $N^{3,1}(1,2) = 12$ 1-nodal cubics on $\mathbf{P}$ matching with F_1 along E, and for each of those there is a unique matching curve C'.
(ii) The ruling F_1 moves freely in the pencil $|F|$, while C' passes through the 5 base points and thus moves in a pencil. The divisors cut out on E by curves of this kind form a quadric surface in the g^3_3 cut out by the complete system $|E+4F|$. It follows that the number of curves made of a curve of this kind and a nodal cubic on $\mathbf{P}$ through the 7 base points is $2 \times N^{3,1}(0,3) = 24$.

(c) $\delta_{\mathbf{F}} = 2$. The 2-nodal curve on $\mathbf{F}$ must decompose as $F_1 + F_2 + C'$ with $F_1, F_2 \sim F$ and $C' \sim E + 2F$, and there are the two following combinatorial possibilities.
(i) The two rulings F_1 and F_2 pass through 1 base point each, and C' moves in the pencil defined by the 3 remaining base points ($\binom{5}{2} = 10$ possibilities). There is a cubic through the 7 base points and matching with F_1 and F_2 ($N^{3,0}(2,1) = 1$), and it cuts out a third point on E. Then, there is a unique curve C' passing through this third point and the remaining 3 base points on $\mathbf{F}$.
(ii) The curves F_1 and C' pass through 1 and 4 base points respectively ($\binom{5}{1}$ possibilities), thus both are fixed, and F_2 moves freely in $|F|$. As in case (i), there is a unique cubic on $\mathbf{P}$ matching with $F_1 + C'$, and in turn a unique ruling F_2 through the third intersection point of this cubic with E.

(d) $\delta_{\mathbf{P}} = 1$ and $\tau_2 = 1$. There are $N^{3,1}(0,[1,1]) = 36$ one-nodal cubics tangent to E, and for each of those there is a unique matching curve on $\mathbf{F}$. These curves count with multiplicity 2.

(e) $\delta_{\mathbf{F}} = 1$ and $\tau_2 = 1$ (with multiplicity 2). The curve on $\mathbf{F}$ decomposes as $F_1 + C'$ with $F_1 \sim F$, and there are the two following possibilities.
(i) The ruling F_1 passes through 1 base point ($\binom{5}{1}$ possibilities). There are then $N^{3,0}(1,[0,1]) = 2$ cubics on $\mathbf{P}$ tangent to E and matching with F_1. Each of those determines a unique curve C'.
(ii) The ruling F_1 moves freely in $|F|$. There are finitely many curves C' in $|E+3F|$ passing through the 5 base points and tangent to E (in the number 2, which is the number of double points in a g^1_2 on E). For each of those there is a unique matching cubic on $\mathbf{P}$ ($N^{3,0}([0,1],1) = 1$), which cuts out a third point on E and thus fixes F_1.

(f) $\tau_3 = 1$ (with multiplicity 3). There are $N^{3,0}(0,[0,0,1]) = 3$ cubics on $\mathbf{P}$ triply tangent with E at some point, and each of those determines a unique matching curve on $\mathbf{F}$.

These eventually add up as :

$$
\begin{array}{lrl}
\quad N^{3,2}(0,3) & = & 21 \\
+ \binom{5}{1} N^{3,1}(1,2) + \binom{5}{0} \times 2 \times N^{3,1}(0,3) & & + 5 \cdot 12 + 2 \cdot 12 \\
+ \binom{5}{2} N^{3,0}(2,1) + \binom{5}{1} N^{3,0}(2,1) & & + 10 + 5 \\
+ 2 \cdot N^{3,1}(0,[1,1]) & & + 2 \cdot 36 \\
+ 2 \cdot \binom{5}{1} N^{3,0}(1,[0,1]) + 2 \cdot \binom{5}{5} \times 2 \times N^{3,0}([0,1],1) & & + 2 \cdot 5 \cdot 2 + 2 \cdot 2 \cdot 1 \\
+ 3 \cdot N^{3,0}(0,[0,0,1]) & & + 3 \cdot 3 \\
\hline
N^{4,2}(0,4) & = & 225.
\end{array}
$$

References

Chapters in this Volume

[I] C. Ciliberto and Th. Dedieu, *Limits of nodal curves: generalities and examples.*

[III] Th. Dedieu, *Geometry of logarithmic Severi varieties at a general point.*

[VI] Th. Dedieu and C. Galati, *Degenerations and deformations of generalized Severi varieties.*

[XI] Th. Dedieu, *Enumerative geometry of $K3$ surfaces.*

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces.*

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Lecture VIII
Proving the existence of curves on smooth surfaces by smoothing tacnodes and other singularities on reducible surfaces

by Concettina Galati

Abstract. Let $\mathcal{X} \to \mathbf{D}$ be any flat projective family of complex surfaces parametrized by the disc $\mathbf{D}$, with smooth total space $\mathcal{X}$, smooth general fiber $\mathcal{X}_t$, and reducible special fiber $\mathcal{X}_0 = \cup X_i$ with normal crossing singularities. Let $C = \cup_i C_i \subseteq \mathcal{X}_0$ be a reducible Cartier divisor with only nodes on the smooth locus of $\mathcal{X}_0$ and tacnodes and nodes on the singular locus E of $\mathcal{X}_0$. We will show how to study deformations of C to nodal curves $C_t \subseteq \mathcal{X}_t$ by using Caporaso and Harris' local analysis of the versal deformation space of an m-tacnode in [1] and [2]. Results of this paper are known. We especially refer to [7] and [10]. But they are usually proved in literature with some special assumptions on the surfaces $\mathcal{X}_t$. Here we work in full generality. This paper is in particular a generalization of [10, Section 3].

1 – Introduction

Let X be a smooth complex projective surface and let D be a smooth Cartier divisor of genus $p_a(D)$. Describing the singular elements in the linear system $|D|$ defined by D is not in general an easy problem, even if we restrict our interest to the existence of nodal divisors. There are various deformation arguments which may be helpful case by case. For example, if $X = \mathbf{P}^2$, we know that, for any $d \geqslant 3$, there exist degree d irreducible nodal curves of any genus $0 \leqslant g \leqslant \frac{(d-1)(d-2)}{2}$, cf. [19] and [12]. These curves are deformations of a rational nodal plane curve of degree d, obtained as a general projection of a rational normal curve $C_d \subseteq \mathbf{P}^d$ of degree d. Notice that $\frac{(d-1)(d-2)}{2}$ is the number of nodes of a rational nodal plane curve and so it is the maximal number of nodes of an irreducible plane curve of degree d. If X is any smooth complex projective surface and D is any smooth Cartier divisor, it is not in general obvious how to produce irreducible or even reducible curves $|D|$ with "the maximum allowed number of nodes". One very classical strategy is to degenerate the surface $X \simeq \mathcal{X}_t$ to a reducible normal crossing surface $\mathcal{X}_0$ and to prove the existence of divisors on $X = \mathcal{X}_t$ with certain singularities as deformations of suitable Cartier divisors on $\mathcal{X}_0$. This method was used in particular by Chen in [7] to prove the existence of irreducible rational nodal curves on a very general K3 surface, in the primitive linear system and in all its multiples. The key idea of Chen's theorem is to construct, on a suitable reducible K3 surface $\mathcal{X}_0 = A \cup B$, a curve with tacnodes at points of the singular locus of $\mathcal{X}_0$ and to study its deformations on a very general K3 surface $\mathcal{X}_t$, by using the analysis of the locus of $(m-1)$-nodal curves in the versal deformation space of the m-tacnode, carried out in [1] and [2], see [IV]. The local analysis in [1] and [2] of versal deformations of tacnodes has later been extended in [10], where the authors provide a method to construct curves with A_k singularities on smooth surfaces, for every $k \geqslant 1$. In particular [10] contains a new proof of the aforementioned Chen's theorem. Results in [10] have later been applied in [4] and [5].

In [10] the authors work in a family of regular surfaces. An output of the present paper is that this regularity assumption is unnecessary. Here we provide a generalization of [10, Section 3] using a simpler notation. We hope it will be clear to the reader how all results of this paper can be generalized to the study of deformations of reduced curves in a smooth total space with complete intersection singularities, see for example [11].

This paper is divided in four sections. In Section 2 we describe our degeneration argument and we state our deformation problem (Problem (2.2)). We moreover collect several local results from which, in Section 3, we will deduce our main result, Theorem (3.1). The latter proves in particular [I, Proposition (5.5)]; besides, [I] contains many applications of Theorem (3.1).

The degeneration argument which we are going to explain in the next section is useful to the study of many problems. For example, in [6], it has been applied to the study of the number of moduli of Severi varieties of nodal curves on K3 surfaces. Here we will restrict to the existence problem of nodal curves on smooth surfaces.

(1.1) Terminology and notation. We will work over $\mathbf{C}$. A curve will be a reduced separated scheme of finite type and dimension 1.

(1.2) Acknowledgments. I would like to thank Thomas Dedieu for reading the preliminar version of this paper.

2 – Our deformation problem

(2.1) Let $\mathcal{X} \to \mathbf{D}$ be a flat projective family of complex surfaces parametrized by the disc $\mathbf{D}$, with smooth total space $\mathcal{X}$, smooth general fiber $\mathcal{X}_t$, and reducible special fiber $\mathcal{X}_0$ with normal crossing singularities, i.e., locally at every point in the analytic topology, $\mathcal{X}_0$ is isomorphic to the closed subset of an open polydisk of $\mathbf{C}^3$ defined by $\{(x_1, x_2, x_3) | \Pi_{i=1}^k x_i = 0\}$, where k may be $1, 2$, or 3, cf. [8]. Thus every irreducible component X_i of $\mathcal{X}_0 = \cup_{i=1}^r X_i$ is smooth, and the singular locus of $\mathcal{X}_0$ consists of the intersection locus of its irreducible components, where $\mathcal{X}_0$ has singularities of multiplicity 2 or 3. We denote by $E = \cup_{1 \leqslant i < j \leqslant r} E_{ij}$ the singular locus of $\mathcal{X}_0$, where $E_{ij} = X_i \cap X_j$ (possibly empty), and by $E^o \subseteq E \subseteq \mathcal{X}_0$ the set of points of multiplicity exactly 2.

Let $p \in E^o$ be a double point of $\mathcal{X}_0$. Working locally at p, we may assume that $\mathcal{X}$ is isomorphic to a closed subset of an open polydisk of $\mathbf{C}^3 \times \mathbf{D}$, with coordinates (x, y, z, t), in such a way the local analytic equation of $\mathcal{X}$ at p is given by

$$xy = t,$$

and $xy = t = 0$ are the analytic equations of $\mathcal{X}_0$ at p.

Now we consider a reduced Cartier divisor

$$C = \cup_i C_i \subseteq \mathcal{X}_0 = \cup_{i=1}^r X_i,$$

where every $C_i = C \cap X_i \subseteq X_i$ is a reduced δ_i-nodal curve, that is smooth at every intersection point with E and does not contain any triple point of $\mathcal{X}_0$. The local equations of C at every intersection point $p \in E_{ij} \cap C = C_i \cap C_j$ are

$$\begin{cases} y + x - z^m = 0 \\ xy = t \\ t = 0, \end{cases} \tag{2.1.1}$$

for some $m \geqslant 1$ (depending on p). Thus the singularity of C at $p \in C \cap E_{ij}$ is analytically isomorphic to the planar curve singularity of local equation

$$f(y,z) = (y - z^m)y = 0, \tag{2.1.2}$$

which is called a *tacnode of order* m or an m*-tacnode*; observe that a 1-tacnode is a node.

Thus the curve

$$C = \cup_i C_i \subseteq \mathcal{X}_0 = \cup_{i=1}^r X_i,$$

has only planar singularities that are nodes and tacnodes.

(2.2) Problem. *We want to find sufficient conditions for $C \subseteq \mathcal{X}_0$ to be deformed to a nodal curve $C_t \subseteq \mathcal{X}_t$. More precisely we want to understand which is the maximal number of nodes of a curve $C_t \subseteq \mathcal{X}_t$ that is deformation of $C \subseteq \mathcal{X}_0$.*

(2.3) Remark. Every tacnode of C at a moving point $p \in E_{ij} = X_i \cap X_j \subseteq E$ imposes at most $m-1$ independent conditions to the linear system $|\mathcal{O}_{\mathcal{X}_0}(C)|$, being $m-1$ the expected number of conditions imposed to the linear system $|\mathcal{O}_{X_i}(C)|$ by a tangency of order m with $E_{ij} \subseteq E$ at an unprescribed point. Thus the naive expectation is that one may deform C to a curve $C_t \subseteq \mathcal{X}_t$ in such a way every m-tacnode of C on E deforms to $m-1$ nodes of C_t and all nodes of C on $\mathcal{X}_0 \setminus E$ are preserved. In this case we will say that *an m-tacnode of C is smoothed to $m-1$ nodes of C_t* because $m-1$ nodes impose one less condition to the arithmetic genus than an m-tacnode.

(2.4) Remark. Let us recall that an m-tacnode is an A_{2m-1} planar curve singularity. An A_k planar curve singularity has analytic equation $y^2 - x^{k+1}$. Every plane curve singularity of multiplicity 2 is an A_k-singularity, for some k and two plane curve singularities of multiplicity two are topologically equivalent if and only if they are analytically equivalent. We refer to [13] and [II] for the notion of equisingularity from the topological and the analytic point of view for a family of reduced curves on a smooth surface. In particular, for an A_k-singularity the equisingular ideal coincides with the Jacobian ideal, given by $I = J = (y, x^k)$ [13, Proposition 5.6]. This implies that an A_k-singularity at a moving point of a smooth surface imposes as most $k = \dim(\mathbf{C}[x,y]/J)$ independent conditions to a linear system. Now, an A_k-singularity locally deforms to an $A_{k'}$-singularity for every $k' \leqslant k$. Thus, in the same vein as in Problem (2.2), one could ask about deformations $C_t \subseteq \mathcal{X}_t$ of $C \subseteq \mathcal{X}_0$ with A_k-singularities. As in Remark (2.3), one expects that the curve singularity (2.1.1) deforms on $\mathcal{X}_t$ to d_i singularity of type A_i, for every multi-index $(d_1, ..., d_r)$ such that $\sum_i id_i = m-1$.

Deformations of C in $\mathcal{X}$ are parametrized by suitable subschemes of the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ of $\mathcal{X} \to \mathbf{D}$ [18]. We denote by $\mathcal{H} \subseteq \mathcal{H}^{\mathcal{X}|\mathbf{D}}$ the union of irreducible components containing the point $[C]$ corresponding to the curve C, and by $\mathcal{H}_t$ the fibre of $\mathcal{H}$ over $t \in \mathbf{D}$. We say that *a Cartier divisor $C_t \subseteq \mathcal{X}_t$ is a deformation of $C \subseteq \mathcal{X}_0$ if the point $[C_t] \in \mathcal{H}_t$ belongs to a curve $\gamma \subseteq \mathcal{H}$, which is a multi-section of order $r \geqslant 1$ of $\mathcal{H} \to \mathbf{D}$, totally ramified at the point $[C]$, i.e., intersecting $\mathcal{H}_0$ at $r[C]$.*

2.1 – Some local geometry: the versal deformation space of the singularities of C

Every deformations of C in $\mathcal{X}$ induces a deformation of each singularity of C. Isolated singularities, i.e., germs of isolated singularities in the analytic topology, have a moduli space with the property of versality, cf. [13, Section 3], [18, Chapter 2], [12, Chapter 2] and [II].

For a plane curve singularity (D, q), a versal deformation family is given by

$$\text{(2.4.1)} \qquad \begin{tikzcd} \mathcal{C}_p = \{g(x,y) + t_1 g_1(x,y) + ... + t_m g_m(x,y) = 0\} \arrow[r, hook] \arrow[dr] & \mathbf{A}^2 \times \Delta_{D,q} \arrow[d] \\ & \Delta_{D,q} = \mathrm{Spec}(\mathbf{C}[t_1, ..., t_m]) \end{tikzcd}$$

where, denoting by $J_p = (g, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z})$ the Jacobian ideal of the local equation $g(y, z)$ of D at p, the polynomials $g_1(x, y), ..., g_m(x, y)$ are such that their images in $\mathbf{C}(y, z)/J_q$ form a basis of $\mathbf{C}(y, z)/J_q$, as a $\mathbf{C}$-vector space. The parameter space

$$\Delta_{D,q} = \mathrm{Spec}(\mathbf{C}[t_1, ..., t_m]),$$

is a (mini)-versal deformation space of (D, q). Using the terminology of [13], we will refer to (2.4.1) as *the* étale versal deformation family of the plane curve singularity (D, q), and to $\Delta_{D,q}$ as *the* étale (mini)-versal deformation space of (D, q).

We now provide an explicit description of the étale versal deformation family of every singularity of our curve $C \subseteq \mathcal{X}$. At every node $p \in \mathcal{X}_0 \setminus E$, the curve C is analytically equivalent to the plane curve $f(y, z) = yz = 0$, thus

$$\mathbf{C}[y, z]/J_p \simeq \mathbf{C}[y, z]/(y, z) \simeq \mathbf{C},$$

and a versal family for (C, p) is $\mathcal{C}_p \to \Delta_{(C,p)} = \mathrm{Spec}(\mathbf{C}[\mathrm{t}])$, with

$$\mathcal{C}_p = \big\{xy + t = 0\big\} \subseteq \mathbf{A}^2 \times \mathrm{Spec}(\mathbf{C}[\mathrm{t}]).$$

The only singular fibre is that over $t = 0$, implying that a deformation of a nodal curve is either a nodal curve or a smooth curve.

At every tacnodal singularity $p \in E_{ij}$, a local equation f of C at p is given by (2.1.2), so that $J_f = (2y - z^m, mz^{m-1}y)$ and

$$\mathbf{C}[y, z]/J_f \simeq \mathbf{C}^{2m-1}.$$

In particular, at every tacnodal singularity $p \in E_{ij}$ of C, choosing

$$\{1,\, z,\, z^2,\, \ldots,\, z^{m-1},\, y,\, yz,\, yz^2,\, \ldots,\, yz^{m-2}\}$$

as a basis for $\mathbf{C}[y, z]/J_f$, a versal deformation family of (C, p) is given by $\mathcal{C}_p \to \Delta_{(C,p)}$, where

$$\Delta_{(C,p)} = \mathrm{Spec}(\mathbf{C}[\alpha_0, ..., \alpha_{m-2}, \beta_0, ..., \beta_{m-1}])$$

and $\mathcal{C}_p \subseteq \mathbf{A}^2 \times \Delta_{(C,p)}$ has equation

$$\text{(2.4.2)} \qquad \mathcal{C}_p : F(y, z; \underline{\alpha}, \underline{\beta}) = y^2 + \Big(\sum_{i=0}^{m-2} \alpha_i z^i + z^m\Big) y + \sum_{i=0}^{m-1} \beta_i z^i = 0,$$

where we are using the same notation as in [1] and [13], see [IV, V].

We now want to introduce the versal property of the versal family $\mathcal{C}_p \to \Delta_{(C,p)}$. Keeping in mind [13, Appendix B], we will do this in the analytic topology instead of the étale topology as in [13, Section 3], to simplify notation.

Let us denote by $\mathcal{D} \to \mathcal{H}$ the universal family parametrized by the Hilbert scheme $\mathcal{H}$. By versality, for every singular point p of C, there exist analytic neighborhoods U_p of $[C]$ in $\mathcal{H}$, U'_p of p in $\mathcal{D}$, and V_p of $\underline{0}$ in $\Delta_{(C,p)}$, and a map $\phi_p : U_p \to V_p$, such that the family $\mathcal{D}|_{U_p} \cap U'_p$ is isomorphic to the pull-back of $\mathcal{C}_p|_{V_p}$ by the map ϕ_p, making the following diagram

(2.4.3)
$$\begin{tikzcd} \mathcal{C}_p \arrow[d] & \mathcal{C}_p|_{V_p} \arrow[l, hook'] \arrow[d] & U_p \times_{V_p} \mathcal{C}_p|_{V_p} \arrow[l] \arrow[r, "\simeq"] \arrow[drr] & & \mathcal{D}|_{U_p} \cap U'_p \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[d] \\ T^1_{C,p} & V_p \arrow[l, hook'] & & & U_p \arrow[lll, "\phi_p"'] \arrow[r, hook] & \mathcal{H} \end{tikzcd}$$

commutative. Thus, answering Problem (2.2) is the same as describing the image of the versal map

(2.4.4)
$$\phi = \prod_{p \in \mathrm{Sing}(C)} \phi_p : \bigcap_{p \in \mathrm{Sing}(C)} U_p \to \prod_{p \in \mathrm{Sing}(C)} \Delta_{(C,p)},$$

where $\mathrm{Sing}(C)$ is the singular locus of C.

2.2 – Equisingular deformations of C

In order to understand the image of the versal map ϕ of (2.4.4), we describe its differential $d\phi_{[C]}$ at the point $[C] \in \bigcap_p U_p \subseteq \mathcal{H}$, which splits in the direct sum $d\phi_{[C]} = \bigoplus_{p \in \mathrm{Sing}(C)} d\phi_{p,[C]}$, where $d\phi_{p,[C]}$ is the differential of ϕ_p at the point $[C]$.

The differential map $d\phi_{[C]}$ can be identified with the map

(2.4.5)
$$H^0(\beta) = d\phi_{[C]} = \bigoplus_{p \in \mathrm{Sing}(C)} d\phi_{p,[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}}) \to H^0(C, T^1_C) = \bigoplus_{p \in \mathrm{Sing}(C)} T^1_{C,p}$$

induced by the exact sequence

(2.4.6)
$$0 \longrightarrow \Theta_C \longrightarrow \Theta_{\mathcal{X}}|_C \xrightarrow{\alpha} \mathcal{N}_{C|\mathcal{X}} \xrightarrow{\beta} T^1_C \longrightarrow 0,$$

where Θ_C and $\Theta_{\mathcal{X}}$ are the tangent sheaves of C and $\mathcal{X}$, respectively, $\mathcal{N}_{C|\mathcal{X}}$ is the normal bundle of C in $\mathcal{X}$, and T^1_C is the first cotangent sheaf of C, cf. [18, Prop. 1.1.9]. We are in particular using standard identifications of the tangent space $T_{[C]}\mathcal{H}$ at $[C]$ with $H^0(C, \mathcal{N}_{C|\mathcal{X}})$ and of $T_0\Delta_{(C,p)}$ with $T^1_{C,p}$.

The kernel of the sheaf map β in (2.4.6) is usually denoted by $\mathcal{N}'_{C|\mathcal{X}}$, and is called the *equisingular normal sheaf of C in $\mathcal{X}$*. This terminology is due to the fact that the origin $\underline{0} \in \Delta_{(C,p)}$ is the only point in the mini-versal deformation space corresponding to a singularity analytically equivalent to (C, p). Thus, by the short exact sequence

(2.4.7)
$$0 \longrightarrow \mathcal{N}'_{C|\mathcal{X}} \longrightarrow \mathcal{N}_{C|\mathcal{X}} \xrightarrow{\beta} T^1_C \longrightarrow 0,$$

we have that $H^0(C, \mathcal{N}'_{C|\mathcal{X}})$ can be identified with the kernel of the differential $d\phi_{[C]}$ in (2.4.5), and it parametrizes the tangent space to the analytically equisingular deformation locus of

C in $\mathcal{X}$. One says that $H^0(C, \mathcal{N}'_{C|\mathcal{X}})$ parametrizes first order infinitesimal deformations of C preserving all singularities of C from the analytic point of view. Similarly, the kernel

$$\ker(d\phi_{p,[C]})$$

of the differential $d\phi_{p,[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}}) \to T^1_{C,p}$ in (2.4.5) parametrizes first order infinitesimal deformations of C in $\mathcal{X}$ which are analytically equisingular at p, and there is an obvious inclusion $H^0(C, \mathcal{N}'_{C|\mathcal{X}}) \subseteq \ker(d\phi_{p,[C]})$.

(2.5) Definition. *We denote by*

$$\mathcal{ES}(C) = \phi^{-1}(\underline{0}, ..., \underline{0})$$

the (scheme theoretic) inverse image of the origin with respect to the versal map ϕ in (2.4.4). *We will call it the* equisingular deformation locus of C.

(2.6) Remark. As already observed, the inverse image $\phi^{-1}(\underline{0}, ..., \underline{0})$ of $(\underline{0}, ..., \underline{0})$ by the versal map (2.4.4) is, in general, the locus of analytically equisingular deformations of C in $\mathcal{X}$. Since we assume C with only nodes and tacnodes, analytically equisingular deformations of C in $\mathcal{X}$ coincide with topologically equisingular deformations of C in $\mathcal{X}$, cf. Remark (2.4). For this reason we call $\mathcal{ES}(C) = \phi^{-1}(\underline{0}, ..., \underline{0})$ the equisingular deformation locus of C, with no further specification. Similarly, we refer to its tangent space $H^0(C, \mathcal{N}'_{C|\mathcal{X}}) \simeq T_{[C]}(\mathcal{ES}(C))$ at $[C]$ just as the space of equisingular first order infinitesimal deformations of C in $\mathcal{X}$.

(2.7) Remark. The versal map ϕ in (2.4.4), as well as the description of its differential $d\phi_{[C]}$ in (2.4.5), and the exact sequences (2.4.6) and (2.4.7), are defined for any reduced Cartier divisor C on $\mathcal{X}_0$, whose singularities (C, p) are all reduced complete intersection singularities in $\mathbf{C}^3$. Moreover, the description of the étale versal deformation family of (C, p) provided in (2.4.1) under the assumption that (C, p) is a planar curve singularity, extends to complete intersection curve singularities in $\mathbf{C}^n$ (see [12], or [11, Section 3] for an example of explicit computation when (C, p) is a complete intersection curve singularity in $\mathbf{C}^3$).

(2.8) Lemma. *Let $\mathcal{X} \to \mathbf{D}$ be a flat projective family of complex surfaces as in (3.1).*

(a) For any *reduced Cartier divisor $C \subseteq \mathcal{X}_0$ with non-empty intersection with E, one has*

$$H^0(C, \mathcal{N}'_{C|\mathcal{X}}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}_0}) \subseteq H^0(C, \mathcal{N}_{C|\mathcal{X}_0}). \tag{2.8.1}$$

More generally, for every intersection point $p \in C \cap E$, the kernel of the differential $d\phi_{p,[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}}) \to T^1_{C,p}$ in (2.4.5) *is such that*

$$H^0(C, \mathcal{N}'_{C|\mathcal{X}}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}_0}) \subseteq \ker(d\phi_{p,[C]}) \subseteq H^0(C, \mathcal{N}_{C|\mathcal{X}_0}). \tag{2.8.2}$$

(b) If $C \subseteq \mathcal{X}_0$ is a reduced Cartier divisor as above with only $\delta = \sum_i \delta_i$ nodes on $\mathcal{X}_0 \setminus E$ and tacnodes on E, then $H^0(C, \mathcal{N}'_{C|\mathcal{X}}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}_0})$ is contained the linear subspace of sections of $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ vanishing at every node p of C on $\mathcal{X}_0 \setminus E$ and vanishing at every m-tacnode $p \in E$ of C with multiplicity $m - 1$.

Proof. We first prove part (a) of the lemma and, in particular, equality (2.8.1) for any reduced Cartier divisor C on $\mathcal{X}_0$ with non-empty intersection with the singular locus E of $\mathcal{X}_0$. Fix any intersection point $p \in E \cap C$. Keeping in mind Remark (2.7), we want to write equisingular infinitesimal first order deformations of C in $\mathcal{X}$ locally at the point p. We consider the localized exact sequence

$$0 \longrightarrow \mathcal{N}'_{C|\mathcal{X},p} \longrightarrow \mathcal{N}_{C|\mathcal{X},p} \longrightarrow T^1_{C,p} \longrightarrow 0. \tag{2.8.3}$$

Fix local analytic coordinates x, y, z, t at p in such a way that C has equations

$$t = 0, \quad f_1(x,y,z) = 0, \quad \text{and} \quad f_2(x,y,z) = xy = 0$$

at p. Then we may identify:

- the local ring $\mathcal{O}_{C,\,p} = \mathcal{O}_{\mathcal{X},p}/\mathcal{I}_{C|\mathcal{X},p}$ of C at p with the localization at the origin of $\mathbf{C}[x,y,z]/(f_1,f_2)$,
- the $\mathcal{O}_{C,p}$-module $\mathcal{N}_{C|\mathcal{X},\,p}$ with the free $\mathcal{O}_{\mathcal{X},\,p}$-module $\mathfrak{hom}_{\mathcal{O}_{\mathcal{X},\,p}}(\mathcal{I}_{C|\mathcal{X},\,p}, \mathcal{O}_{C,p})$ generated by the morphisms f_1^* and f_2^*, defined by

$$f_i^*\big(s_1(x,y,z)f_1(x,y,z) + s_2(x,y,z)f_2(x,y,z)\big) = s_i(x,y,z), \quad \text{for } i = 1,\, 2,$$

and, finally,

- the $\mathcal{O}_{C,p}$-module

$$\begin{aligned} (\Theta_{\mathcal{X}}|_C)_p &\simeq \Theta_{\mathcal{X},p}/(I_{C,p} \otimes \Theta_{\mathcal{X},p}) \\ &\simeq \big\langle \partial/\partial x,\, \partial/\partial y,\, \partial/\partial z,\, \partial/\partial t \big\rangle_{\mathcal{O}_{C,\,p}} \Big/ \big\langle \partial/\partial t - x\partial/\partial y - y\partial/\partial x \big\rangle \end{aligned}$$

  with the free $\mathcal{O}_{\mathcal{X},\,p}$-module generated by the derivatives $\partial/\partial x, \partial/\partial y, \partial/\partial z$.

With these identifications, the localization $\alpha_p : (\Theta_{\mathcal{X}}|_C)_p \to \mathcal{N}_{C|\mathcal{X},\,p}$ of the sheaf map α from (2.4.6) is defined by

$$\begin{aligned} \alpha_p(\partial/\partial x) &= \Big(s = s_1f_1 + s_2f_2 \longmapsto \partial s/\partial x =_{\mathcal{O}_{C,p}} s_1\partial f_1/\partial x + s_2\partial f_2/\partial x\Big) \\ &= \partial f_1/\partial x f_1^* + y f_2^*, \\ \alpha_p(\partial/\partial y) &= \partial f_1/\partial y f_1^* + x f_2^*, \text{ and} \\ \alpha_p(\partial/\partial z) &= \partial f_1/\partial z f_1^*. \end{aligned}$$

By definition of $\mathcal{N}'_{C|\mathcal{X}}$, a germ $s \in \mathcal{N}_{C|\mathcal{X},\,p}$ is equisingular at p, i.e., $s \in \mathcal{N}'_{C|\mathcal{X},\,p}$, if and only if there exists a germ

$$u = u_x(x,y,z)\partial/\partial x + u_y(x,y,z)\partial/\partial y + u_z(x,y,z)\partial/\partial z \in \left(\Theta_{\mathcal{X}}|_C\right)_p$$

such that $s = \alpha_p(u)$. Hence, locally at p, an equisingular first order infinitesimal deformation of C in $\mathcal{X}$ has equation

$$\text{(2.8.4)} \qquad \begin{cases} f_1(x,y,z) + \varepsilon(u_x\partial f_1/\partial x + u_y\partial f_1/\partial z + u_z\partial f_1/\partial z) = 0 \\ xy + \varepsilon(yu_x + xu_y) = 0. \end{cases}$$

We claim that the second equation of (2.8.4) is the local equation at p of an equisingular deformation of $\mathcal{X}_0$ in $\mathcal{X}$. Indeed, by the exact sequence

$$0 \longrightarrow \Theta_{\mathcal{X}_0} \longrightarrow \Theta_{\mathcal{X}}|_{\mathcal{X}_0} \longrightarrow \mathcal{N}_{\mathcal{X}_0|\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}_0} \longrightarrow T^1_{\mathcal{X}_0} \cong \mathcal{O}_E \longrightarrow 0,$$

one finds that $\mathcal{N}'_{\mathcal{X}_0|\mathcal{X}} \simeq \mathcal{I}_{E|\mathcal{X}_0}$.

Now if $s = \alpha_p(u)$ is the localization of a global section of $\mathcal{N}'_{C|\mathcal{X}}$, by the equality $H^0(\mathcal{X}_0, \mathcal{N}'_{\mathcal{X}_0|\mathcal{X}}) = H^0(\mathcal{X}_0, \mathcal{I}_{E|\mathcal{X}_0}) = 0$, we find that the analytic function

$$yu_x(x,y,z) + xu_y(x,y,z)$$

must be identically zero in the second equation of (2.8.4). This proves inclusion (2.8.1) and (2.8.2).

We now prove part (b) of the lemma. Assume that C is a Cartier divisor with only $\delta = \sum_i \delta_i$ nodes on $\mathcal{X}_0 \setminus E$ and tacnodes on E, as in (3.1), and consider $p \in E \cap C$. In this case we may assume

$$f_1(x, y, z) = x + y + z^m$$

so that the equations in (2.8.4) become

$$\text{(2.8.5)} \qquad \begin{cases} x + y + z^m + \varepsilon(u_x + u_y + mz^m u_z) = 0 \\ xy + \varepsilon(yu_x + xu_y) = 0. \end{cases}$$

If these equations are the equations of the localization of a global section in $H^0(C, \mathcal{N}_{C|\mathcal{X}})$, then, as above, we find that $yu_x + xu_y$ is identically zero. We deduce in particular that

- $u_x(0,0,0) = u_y(0,0,0) = 0$, and
- for every $n \geqslant 1$, no z^n-terms appear in $u_x(x,y,z)$ and $u_y(x,y,z)$, no y^n-terms and $y^n z^m$-terms appear in $u_x(x,y,z)$ and, finally, no x^n-terms and $x^n z^m$-terms appear in $u_y(x,y,z)$.

In particular, if X_i, with local equations $x = t = 0$, and X_j, with local equations $y = t = 0$, are the two irreducible components of $\mathcal{X}_0$ containing p, then the local equations at p on X_i of equisingular infinitesimal deformations of C are given by

$$\text{(2.8.6)} \qquad \begin{cases} y + z^m + \varepsilon(mz^{m-1}u_z(x,y,z) + yq(y,z)) = 0 \\ x = 0, \end{cases}$$

where $q(y,z)$ is an analytic function in the variables y and z, and similarly on X_j. This proves that all sections in $H^0(C, \mathcal{N}'_{C|\mathcal{X}})$ vanish with multiplicity $m-1$ at every m-tacnode of C on E.

The fact that all sections of $H^0(C, \mathcal{N}'_{C|\mathcal{X}_0})$ vanish at every node p of C on $\mathcal{X}_0 \setminus E$ is a very classical result, which can be proved here with the same approach as above, by using that at a node on $\mathcal{X}_0 \setminus E$ the local equations of C are $x - t = t = 0 = yz$ (see also [II]).

Thus the lemma is proved. □

2.3 – Image of the versal map

We now consider:

- $\mathcal{X} \to \mathbf{D}$ be a flat projective family of complex surfaces with smooth total space $\mathcal{X}$, smooth general fibre $\mathcal{X}_t$ and central fibre $\mathcal{X}_0$ with normal crossing singularities,
- $C \subsetneq \mathcal{X}_0 = \cup X_i$ a reduced Cartier divisor with only tacnodes at the intersection points with the singular locus $E = \cup_{ij} E_{ij}$ of $\mathcal{X}_0$ and nodes on $\mathcal{X}_0 \setminus E$,

as in (3.1).

(2.9) Lemma. *In the above setting, let $p \in C \cap E_{ij} \subseteq \mathcal{X}_0 \subseteq \mathcal{X}$ be an m-tacnode, with $m \geqslant 1$. Then, using the parametrization* (2.4.2) *for the mini-versal family of (C,p), we have that the image of the differential map*

$$d\phi_{p,[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}}) \to T^1_{C,p} = T_{\underline{0}}\Delta_{(C,p)}$$

is always contained in the linear subspace $H_p : \beta_1 = \cdots = \beta_{m-1} = 0$, while the image of the differential map

$$d\phi_{p,[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}_0}) \to T^1_{C,p} = T_{\underline{0}}\Delta_{(C,p)}$$

is always contained in the tangent space $\Gamma_p = T_{\underline{0}}EG_{(C,p)} : \beta_0 = \cdots = \beta_{m-1} = 0$ *at* $\underline{0}$ *to the equigeneric locus* $EG_{(C,p)}$ *in* $\Delta_{(C,p)}$ *(cf. [13] and [II]). In particular, the subspace*

$$\ker(d\phi_{p,[C]}) \subseteq H^0(C, \mathcal{N}'_{C|\mathcal{X}_0}) \tag{2.9.1}$$

has dimension

$$\dim(\ker(d\phi_{p,[C]})) \geqslant h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) - m + 1 \geqslant h^0(C, \mathcal{N}_{C|\mathcal{X}}) - m.$$

The differential map $d\phi_{p,[C]}$ *is surjective on* H_p *if and only if the subspace* $\ker(d\phi_{p,[C]})$ *of infinitesimal first order deformations of* C *in* $\mathcal{X}_0$ *that are equisingular at* p *has codimension exactly* $m-1$ *in* $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$.

Proof. We first observe that H_p contains in codimension 1 the tangent space

$$\Gamma_p = T_{\underline{0}}EG_{(C,p)} : \beta_0 = \cdots = \beta_{m-2} = 0$$

at $\underline{0}$ to the equigeneric locus $EG_{(C,p)} \subseteq \Delta_{(C,p)}$.

The image of $U_p \cap \mathcal{H}_0$ with respect to ϕ_p (recall the notation from Diagram (2.4.3)) is contained in the equigeneric locus $EG_{(C,p)} \subseteq \Delta_{(C,p)}$. Indeed, no matter how we deform C in $\mathcal{X}_0$, the smooth tangency of the irreducible components C_i and C_j with E_{ij} at p deforms to smooth tangencies with E_{ij} at $r \leqslant m$ points p_l, with $1 \leqslant l \leqslant r$, of multiplicity m_l with $\sum_l m_l = m$. This corresponds locally to a deformation of the m-tacnode of C at p into r tacnodes of orders $m_1, \dots, m_r$, each decreasing the geometric genus by m_l with respect to the arithmetic genus. Thus, $d\phi_{p,[C]}(H^0(C, \mathcal{N}_{C|\mathcal{X}_0})) \subseteq \Gamma_p = T_{\underline{0}}EG_{(C,p)}$.

Now, by the exact sequence

$$0 \to \mathcal{N}_{C|\mathcal{X}_0} \to \mathcal{N}_{C|\mathcal{X}} \to \mathcal{N}_{\mathcal{X}_0|\mathcal{X}} = \mathcal{O}_{\mathcal{X}_0} \to 0,$$

we have that $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ has codimension less or equal to 1 in $H^0(C, \mathcal{N}_{C|\mathcal{X}})$. If $h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) = h^0(C, \mathcal{N}_{C|\mathcal{X}})$, the lemma is true because $T_{\underline{0}}EG_{(C,p)} \subseteq H_p$. If $h^0(C, \mathcal{N}_{C|\mathcal{X}}) = h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) + 1$, it is enough to find a section

$$\sigma \in H^0(C, \mathcal{N}_{C|\mathcal{X}}) \setminus H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$$

such that $d\phi_{p,[C]}(\sigma) \in H_p \setminus T_{\underline{0}}EG_{(C,p)}$. In this case, one has

$$H^0(C, \mathcal{N}_{C|\mathcal{X}}) \simeq H^0(C, \mathcal{N}_{C|\mathcal{X}_0}) \oplus H^0(C, \mathcal{O}_{\mathcal{X}_0}) \simeq H^0(C, \mathcal{N}_{C|\mathcal{X}_0}) \oplus \mathbf{C}.$$

The section $\sigma \in H^0(C, \mathcal{N}_{C|\mathcal{X}}) \setminus H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ corresponding to the pair $(\underline{0}, 1)$ via the above isomorphism has local equations

$$\begin{cases} x + y + z^m = 0 \\ xy = \varepsilon. \end{cases} \tag{2.9.2}$$

It follows that the image of σ via $d\phi_{p,[C]}$ is the point corresponding to the curve $y(y + z^m) = \beta_0$. This proves the lemma. □

(2.10) Corollary. *In the same setting as above and with the same notation and hypotheses as in Lemma (2.9) and Diagram* (2.4.3)*, assume that:*

a) $h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) = h^0(C, \mathcal{N}_{C|\mathcal{X}}) - 1$;

b) the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ *is smooth at* $[C]$ *of dimension* $h^0(C, \mathcal{N}_{C|\mathcal{X}})$;

c) $\dim(\ker(d\phi_{p,[C]})) = h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) - m + 1$.

Then the image of the versal map

$$\phi_p : \mathcal{H} \cap U_p \to \Delta_{(C,p)}$$

is a smooth variety of dimension m in $\Delta_{(C,p)}$, with tangent space at $\underline{0}$ given by the linear subspace $H_p : \beta_1 = ... = \beta_{m-1} = 0$, while the image of the versal map

$$\phi_p : \mathcal{H}_0 \cap U_p \to \Delta_{(C,p)}$$

coincides with the equigeneric deformation locus $EG_{(C,p)} \subseteq \Delta_{(C,p)}$, which is smooth of dimension m at $\underline{0}$, with tangent space given by $\Gamma_p = T_{\underline{0}}EG_{(C,p)} : \beta_0 = ... = \beta_{m-1} = 0$, and whose general element corresponding to an m-nodal curve.

Moreover, $\phi_p(\mathcal{H} \cap U_p)$ intersects the locus of $(m-1)$-nodal curves in $\Delta_{(C,p)}$ along $\Gamma_p \cup \gamma$, where γ is a curve that is smooth at $\underline{0}$ and intersects Γ_p with multiplicity m. In particular the curve C may be deformed in $\mathcal{H}$ in such a way the tacnode at p is deformed into $m-1$ nodes.

Proof. First observe that, by hypothesis b), there is only one irreducible component $\mathcal{H}$ of $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ containing $[C]$, which is smooth at $[C]$ of dimension $h^0(C, \mathcal{N}_{C|\mathcal{X}})$. Moreover, by hypothesis a), the central fibre of $\mathcal{H}_0$ of $\mathcal{H}$ is smooth at $[C]$ of dimension $h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) = h^0(C, \mathcal{N}_{C|\mathcal{X}}) - 1$.

In particular, in Diagram (2.4.3), we have that the analytic varieties $\mathcal{H} \cap U_p$ and $\mathcal{H}_0 \cap U_p$ are irreducible. It follows that the image of $\phi_p : \mathcal{H} \cap U_p \to \Delta_{(C,p)}$ is an irreducible variety of dimension $\leqslant m$, since $\phi_p(\mathcal{H}_0 \cap U_p)$ is contained in the equigeneric locus, which has dimension $m-1$ and $\phi_p(\mathcal{H} \cap U_p)$ has dimension $\leqslant \dim(\phi_p(\mathcal{H}_0 \cap U_p)) + 1$. In particular, ϕ_p has general fibre of dimension at least $\dim_{[C]}(\mathcal{H}) - m$. Under the hypotheses of this corollary, the differential $d\phi_{p,[C]}$ has rank m. Hence, under hypotheses a), b) and c), the fibre $\phi_p^{-1}(\underline{0})$ has dimension exactly $\dim_{[C]}(\mathcal{H}) - m = h^0(C, \mathcal{N}_{C|\mathcal{X}}) - m$, and $\phi_p(\mathcal{H} \cap U_p)$ is irreducible of dimension m and smooth at $\underline{0}$. The fact that $\phi_p(\mathcal{H} \cap U_p)$ intersects the locus of $(m-1)$-nodal curves in $\Delta_{(C,p)}$ along $\Gamma_p \cup \gamma$, where γ is a curve that is smooth at $\underline{0}$ and intersects Γ_p with multiplicity m follows from [1, Lemma 4.1], cf. [IV, Theorem (1.2)]. □

(2.11) Remark. Under the hypotheses of Lemma (2.10), by [10, Proposition 3.7 and Corollary 3.12], one has more generally that there exist deformations $C_t \subseteq \mathcal{X}_t$ of $C \subseteq \mathcal{X}_0$ such that the m-tacnode of C at p deforms to d_i singularity of type A_i, for every multi-index $(d_1, ..., d_r)$ such that $\sum_i id_i = m-1$.

The following lemma provides a standard way to verify when the hypothesis c) in Corollary (2.10) is satisfied.

Let $W^{|C|}_{p,m-1} \subseteq |\mathcal{O}_{\mathcal{X}_0}(C)|$ be the linear system of Cartier divisors in $|\mathcal{O}_{\mathcal{X}_0}(C)|$ intersecting E at p with multiplicity $m-1$. We have that

$$\dim(W^{|C|}_{p,m-1}) \geqslant \dim(|\mathcal{O}_{\mathcal{X}_0}(C)|) - m + 1.$$

(2.12) Lemma. *In the same setting as above, and with the same notation and hypotheses as in Lemma (2.9), assume that:*

$$\dim(W^{|C|}_{p,m-1}) = \dim(|\mathcal{O}_{\mathcal{X}_0}(C)|) - m + 1. \tag{2.12.1}$$

Then we have

$$\dim(\ker(d\phi_{p,[C]})) = h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) - m + 1.$$

Proof. By the hypothesis that $\dim(W^{|C|}_{p,m-1}) = \dim(|\mathcal{O}_{\mathcal{X}_0}(C)|) - m + 1$, we have

$$\dim(W^{|C|}_{p,r}) = \dim(|\mathcal{O}_{\mathcal{X}_0}(C)|) - r + 1,$$

for every $r \leqslant m-1$. Via the natural inclusion

$$|\mathcal{O}_{X_0}(C)| = H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(C))/H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}) \subseteq H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$$

induced by the standard exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}_0} \to \mathcal{O}_{\mathcal{X}_0}(C) \to \mathcal{O}_C(C) \to 0,$$

we find a chain a codimension 1 linear subsystems

$$W^{|C|}_{p,m-1} \subsetneq W^{|C|}_{p,m-2} \subsetneq \ldots \subsetneq W^{|C|}_{p,1} \subsetneq |\mathcal{O}_{X_0}(C)| \subseteq H^0(C, \mathcal{N}_{C|\mathcal{X}_0}).$$

By Lemma (2.8), we know that $W^{|C|}_{p,r} \cap \ker(d\phi_{p,[C]}) \subseteq W^{|C|}_{p,m-1}$, for every $r \leqslant m-2$. Since $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ is mapped by $d\phi_{p,[C]}$ to the $(m-1)$-dimensional linear subspace $T_{\underline{0}}EG \subseteq T_{\underline{0}}\Delta_{(C,p)}$, we find that $d\phi_{p,[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}_0}) \to T_{\underline{0}}EG$ is surjective by standard linear algebra, and thus $\dim(\ker(d\phi_{p,[C]})) = h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) - m + 1$. $\square$

3 – Global deformations of C

Let $\mathcal{X} \to \mathbf{D}$ and $C = \cup_i C_i \subseteq \mathcal{X}_0 = \cup_i X_i$ be respectively the flat projective family of surfaces and the Cartier divisor C on $\mathcal{X}_0$ as in (3.1). We recall that $\mathcal{X}$ has smooth general fibre $\mathcal{X}_t$ and reducible central fibre $\mathcal{X}_0 = \cup_i X_i$ with normal crossing singularities. We denoted by X_i the irreducible components of $\mathcal{X}_0$, and by $E = \cup_{i,j} E_{ij}$ the singular locus of $\mathcal{X}_0$, where $E_{ij} = X_i \cap X_j$. Moreover, we assume that $C \cap X_i = C_i$ is a reduced δ_i-nodal curve intersecting E at smooth points, in such a way that the only singularities of C are nodes on $\mathcal{X}_0 \setminus E$ and tacnodes on E. For every tacnode p of C, we denote by m_p its order and by

$$\lambda = \operatorname{lcm}\{m_p |\, p \text{ tacnode of } C\} \tag{3.0.1}$$

the least common multiple of the m_p. We furthermore set

$$\mu = \prod_{p \text{ tacnode of } C} m_p \quad \text{and} \quad k = \frac{\mu}{\lambda}. \tag{3.0.2}$$

Finally, we assume that C has

$$\delta = \sum_i \delta_i \quad \text{nodes on } \mathcal{X}_0 \setminus E,$$

and we denote by

$$\tau_m \quad \text{the number of } \; m-\text{tacnodes of } C \text{ on } E,$$

for every $m \geqslant 1$.

Here we will provide sufficient conditions for $C \subseteq \mathcal{X}_0$ to be deformed to a nodal curve $C_t \subseteq \mathcal{X}_t$.

We recall that we denoted by $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ the relative Hilbert scheme of the family $\mathcal{X} \to \mathbf{D}$, by $\mathcal{H}$ the union of the irreducible components of $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ containing $[C]$, and by $\mathcal{H}_t$ the fibre of $\mathcal{H}$ over $t \in \mathbf{D}$. Let now $\mathbf{D}^o = \mathbf{D} \setminus \underline{0}$, $\mathcal{X}^o = \mathcal{X} \setminus \mathcal{X}_0$, and $\mathcal{H}^o = \mathcal{H} \setminus \mathcal{H}_0$. For all $\sigma \in \mathbf{N}$, we denote by $\mathcal{U}_\sigma^{\mathcal{X}^o|\mathbf{D}^o} \subseteq \mathcal{H}^o$ the relative Severi variety of σ-nodal curves, the fibre of which over $t \in \mathbf{D}^o$ is the locally closed (possibly empty) subscheme $\mathcal{U}_\sigma^{\mathcal{X}_t} \subseteq \mathcal{H}_t$ (endowed with its reduced structure) parametrizing reduced σ-nodal Cartier divisors on $\mathcal{X}_t$. We moreover denote by $\mathcal{V}_\sigma^{\mathcal{X}|\mathbf{D}} \subseteq \mathcal{H}$ the Zariski closure in $\mathcal{H}$ of $\mathcal{U}_\sigma^{\mathcal{X}^o|\mathbf{D}^o}$. Observe that, for every $t \neq 0$, the fiber $\mathcal{V}_\sigma^{\mathcal{X}_t}$ of $\mathcal{V}_\sigma^{\mathcal{X}|\mathbf{D}}$ over t is the Zariski closure of $\mathcal{U}_\sigma^{\mathcal{X}_t}$, while $\mathcal{V}_\sigma^{\mathcal{X}|\mathbf{D}} \cap \mathcal{H}_0$ will have several irreducible components, whose general point corresponds to a curve on $\mathcal{X}_0$ with singularities possibly different than nodes.

(3.1) Theorem. *In the above setting, let*

$$W_{[C],es}^{|C|} \subseteq |\mathcal{O}_{\mathcal{X}_0}(C)|$$

be the linear system of Cartier divisors in $|\mathcal{O}_{\mathcal{X}_0}(C)|$ passing through every node of C on $\mathcal{X}_0 \setminus E$, and intersecting E with multiplicity $m-1$ at every m-tacnode of C. Assume that:

1. *$h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) = h^0(C, \mathcal{N}_{C|\mathcal{X}}) - 1$;*
2. *the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ is smooth at $[C]$ of dimension $h^0(C, \mathcal{N}_{C|\mathcal{X}})$. In particular, there is only one irreducible component $\mathcal{H} \subseteq \mathcal{H}^{\mathcal{X}|\mathbf{D}}$ containing $[C]$, which is smooth at $[C]$;*
3. *$\dim(W_{[C],es}^{|C|}) = \dim\big(|\mathcal{O}_{\mathcal{X}_0}(C)|\big) - \delta - \sum_m \tau_m(m-1)$.*

Then $C \subseteq \mathcal{X}_0$ may be deformed to $(\delta + \sum_m \tau_m(m-1))$-nodal curves $C_t \subseteq \mathcal{X}_t$, in such a way every node of C on $\mathcal{X}_0 \setminus E$ is preserved, and every m-tacnode of C on E is smoothed to $m-1$ nodes. All nodal curves C_t obtained in this way are such that

$$h^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}_t}) = h^0(C_t, \mathcal{N}_{C_t|\mathcal{X}_t}) - \delta - \sum_m \tau_m(m-1), \tag{3.1.1}$$

thus, the corresponding point $[C_t] \in \mathcal{H}_t$ is a smooth point of the Severi variety $\mathcal{V}^{\mathcal{X}_t}_{\delta+\sum_m \tau_m(m-1)}$. In particular the nodes of C_t may be smoothed independently , i.e. the versal map (2.4.4) for C_t is surjective. This implies that

$$[C] \in \mathcal{V}_\sigma^{\mathcal{X}|\mathbf{D}}, \text{ for every } \sigma \leqslant \delta + \sum_m \tau_m(m-1).$$

Moreover, the equisingular deformation locus $\mathcal{ES}(C) \subseteq \mathcal{X}_0$ of $[C]$ in $\mathcal{X}_0$ (cf. Definition (2.5)) is an analytic open set of the smooth locus of an irreducible component V_0 of the central fibre

$$\mathcal{V}_0 = \mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)} \cap \mathcal{H}_0 = \mu V_0 + \cdots$$

of $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)} \subseteq \mathcal{H}$, of scheme theoretic multiplicity

$$\mu = \prod_{p \text{ tacnode of } C} m_p.$$

More precisely, at an analytic neighborhood of $[C]$, the universal Severi variety $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)}$ will be the union of k branches, each intersecting $\mathcal{H}_0$ at $[C]$ with multiplicity λ, where k and λ are defined by (3.0.2). These k local analytic branches of $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)}$ will all be smooth at $[C]$ if $\lambda = m_p$, for some tacnode p of C; otherwise they will all be singular at $[C]$, with multiplicity $\frac{\lambda}{\max\{m_p | p \text{ tacnode of } C\}}$.

(3.2) Remark. Hypothesis 3 in Theorem (3.1) can be weakened by asking

$$h^0(C,\mathcal{N}'_{C|\mathcal{X}}) = h^0(C,\mathcal{N}'_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) - \delta - \sum_m \tau_m(m-1). \tag{3.2.1}$$

As we will observe in the proof of the theorem, under hypotheses 1 and 2, we have that hypothesis 3 is a sufficient condition for the equality (3.2.1). The reason we preferred this statement of Theorem (3.1) is that hypothesis 3 in the theorem is usually easier to be verified than (3.2.1).

We finally observe that, if

$$H^0(C,\mathcal{N}_{C|\mathcal{X}_0}) = H^0(C,\mathcal{O}_C(C)), \tag{3.2.2}$$

which happens in particular if the family $\mathcal{X}\to\mathbf{D}$ is a family of regular surfaces (i.e., $h^1(\mathcal{X}_t,\mathcal{O}_{\mathcal{X}_t}) = 0$ for all t), then, under hypotheses 1, 2, and 3 of Theorem (3.1), we have that $H^0(C,\mathcal{N}'_{C|\mathcal{X}}) = W^{|C|}_{[C],es}$. Indeed, if (3.2.2) holds, then $H^0(C,\mathcal{N}'_{C|\mathcal{X}}) \subseteq W^{|C|}_{[C],es}$ by Lemma (2.8), and this inclusion is in fact an equality for dimensional reason.

(3.3) Remark. In Theorem (3.1), the nodes of C on E cannot be preserved by deforming C to a curve in $\mathcal{X}_t$, with $t\neq 0$. This is a very general result that holds for any reduced Cartier divisor on X_0 with a node on E under our hypotheses on $\mathcal{X}\to\mathbf{D}$, i.e., just assuming $\mathcal{X}\to\mathbf{D}$ a flat projective family of surfaces with smooth total space, smooth general fibre and central fibre $\mathcal{X}_0$ with normal crossing singularities (or more generally, with normal crossing singularity at a neighborhood of the node). A proof of this can be found in [9, Section 2]. This also follows from the hypotheses 1, 2 of Theorem (3.1), by Lemma (2.8). In order to see this, we fix a node p of C on E, and we consider the diagram (2.4.3). What we want to prove is that

$$\phi_p^{-1}(\underline{0}) = U_p \cap \mathcal{H}_0. \tag{3.3.1}$$

By Lemma (2.8), we know that

$$T_{\underline{0}}(\phi_p^{-1}(\underline{0})) = \ker(d\phi_{p,[C]}) \subseteq H^0(C,\mathcal{N}_{C|\mathcal{X}_0}),$$

and, by the hypothesis 1 of Theorem (3.1), we get that $\dim(T_{\underline{0}}(\phi_p^{-1}(\underline{0}))) = h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}}) - 1$. Now observe that $U_p\cap\mathcal{H}_0 \subseteq \phi_p^{-1}(\underline{0})$ because every deformation of C in $\mathcal{X}_0$ preserves the node of C at p. Finally, by the hypotheses 1 and 2 in Theorem (3.1), we have that $\mathcal{H}_0$ is smooth at $[C]$ of dimension $h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}}) - 1$. This proves (3.3.1), which in particular says that Theorem (3.1) completely answers Problem (2.2) for a curve C verifying the hypothesis (3.2.1).

Proof of Theorem (3.1). Let $\mathrm{Sing}(C)$ be the singular locus of C. For every singular point $p\in\mathrm{Sing}(C)$, we consider the mini-versal deformation space $\Delta_{(C,p)}$ described in Section 2.1, the versal map (2.4.4)

$$\phi = \prod_{p\in\mathrm{Sing}(C)} \phi_p : \bigcap_{p\in\mathrm{Sing}(C)} U_p \;\to\; \prod_{p\in\mathrm{Sing}(C)} \Delta_{(C,p)},$$

and its differential

$$d\phi_{[C]} = \bigoplus_{p\in\mathrm{Sing}(C)} d\phi_{p,[C]} : H^0(C,\mathcal{N}_{C|\mathcal{X}}) \;\to\; H^0(C,T^1_C) = \bigoplus_{p\in\mathrm{Sing}(C)} T^1_{C,p}$$

at $[C]\in\mathcal{H}$, described in Sections 2.2 and 2.3.

Arguing exactly as in the proof of Lemma (2.12), we see that

$$W^{|C|}_{[C],es} \subseteq |\mathcal{O}_{\mathcal{X}_0}(C)| \subseteq H^0(C, \mathcal{N}_{C|\mathcal{X}_0}),$$

and the hypothesis 3 of the theorem implies that the image of the differential $d\phi_{[C]}$ has dimension $\delta + \sum_m \tau_m(m-1)$, hence

$$\dim\big(\ker(d\phi_{[C]})\big) = h^0(C, \mathcal{N}'_{C|\mathcal{X}}) = h^0(C, \mathcal{N}'_{C|\mathcal{X}_0}) = h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) - \delta - \sum_m \tau_m(m-1). \tag{3.3.2}$$

This in turn implies that

$$\dim\big(\ker(d\phi_{p,[C]})\big) = h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) - m + 1, \quad \text{for every } m\text{-tacnode } p \in E \cap C \text{ of } C$$

and

$$\dim\big(\ker(d\phi_{p,[C]})\big) = h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) - 1, \quad \text{for every node of } C \text{ in } \mathcal{X}_0 \setminus E.$$

In particular, Corollary (2.10) applies at every m-tacnode p of C on E.

Now, by Remark (3.3), we know that every 1-tacnode (i.e., node) of C on E is necessarily smoothed as we deform $C \subseteq \mathcal{X}_0$ outside the central fibre of $\mathcal{X} \to \mathbf{D}$. Thus, denoting by

$$\mathrm{Sing}^\circ(C) = \mathrm{Sing}(C) \setminus \{\text{nodes on } E\},$$

we may restrict our attention to the versal map

$$\varphi = \prod_{p \in \mathrm{Sing}^\circ(C)} \phi_p : \bigcap_{p \in \mathrm{Sing}^\circ(C)} U_p \ \to \ \Big(\prod_{m \geqslant 2} \prod_{\substack{p \in E \cap C \\ m\text{-tacnode}}} \Delta_{(C,p)}\Big) \times \prod_{\substack{p \in C \setminus E \\ \text{node}}} \Delta_{(C,p)} \tag{3.3.3}$$

and its differential at $[C]$,

$$d\varphi_{[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}}) \ \to \ \Big(\bigoplus_{m \geqslant 2} \bigoplus_{\substack{p \in E \cap C \\ m\text{-tacnode}}} T^1_{C,p}\Big) \oplus \bigoplus_{\substack{p \in C \setminus E \\ \text{node}}} T^1_{C,p},$$

obtained by composing $d\phi_{[C]}$ with the natural projection map. By Lemma (2.9) and Corollary (2.10), and using the notation therein, we know that

$$d\varphi_{[C]}\big(H^0(C, \mathcal{N}_{C|\mathcal{X}})\big) \subseteq \Big(\bigoplus_{m \geqslant 2} \bigoplus_{\substack{p \in E \cap C \\ m\text{-tacnode}}} H_p\Big) \oplus \bigoplus_{\substack{p \in C \setminus E \\ \text{node}}} T^1_{C,p}$$

and

$$d\varphi_{[C]}(H^0(C, \mathcal{N}_{C|\mathcal{X}_0})) \subseteq \Big(\bigoplus_{m \geqslant 2} \bigoplus_{\substack{p \in E \cap C \\ m\text{-tacnode}}} \Gamma_p\Big) \oplus \bigoplus_{\substack{p \in C \setminus E \\ \text{node}}} T^1_{C,p}.$$

More precisely, observing that

$$\ker(d\varphi_{[C]}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}_0}) \subseteq H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$$

because every node on E is trivially preserved if we deform C in $\mathcal{X}_0$, by (3.3.2) we get that $d\varphi_{[C]}(H^0(C, \mathcal{N}_{C|\mathcal{X}_0}))$ has dimension

$$\sum_m \tau_m(m-1) + \delta = \dim\left[\Big(\bigoplus_{m \geqslant 2} \bigoplus_{\substack{p \in E \cap C \\ m\text{-tacnode}}} \Gamma_p\Big) \oplus \bigoplus_{\substack{p \in C \setminus E \\ \text{node}}} T^1_{C,p}\right].$$

Thus we obtain that

$$d\varphi_{[C]}\big(H^0(C,\mathcal{N}_{C|\mathcal{X}_0})\big) = \Big(\bigoplus_{m\geqslant 2}\bigoplus_{\substack{p\in E\cap C\\ m\text{-tacnode}}}\Gamma_p\Big)\oplus\bigoplus_{\substack{p\in C\setminus E\\ \text{node}}}T^1_{C,p}, \tag{3.3.4}$$

and

$$d\varphi_{[C]}(H^0(C,\mathcal{N}_{C|\mathcal{X}})) = \Omega\oplus\bigoplus_{\substack{p\in C\setminus E\\ \text{node}}}T^1_{C,p}, \tag{3.3.5}$$

for some linear subspace $\Omega\subseteq\bigoplus_{m\geqslant 2}\bigoplus_p H_p$ containing $\bigoplus_{m\geqslant 2}\bigoplus_p\Gamma_p$ in codimension 1. Arguing as in the proof of Corollary (2.10), we get that:

- the versal map φ defined in (3.3.3) maps $\mathcal{H}_0$ surjectively onto the product
$$\Big(\prod_{m\geqslant 2}\prod_{\substack{p\in E\cap C\\ m\text{-tacnode}}}\Gamma_p\Big)\times\underline{0}\times ...\times\underline{0},$$
- the image of φ is an analytic variety, smooth of dimension $\sum_m\tau_m(m-1)+\delta+1$ at $\underline{0}$, which is a product
$$\mathcal{W}\times\prod_{p\in C\setminus E\ \text{node}}\Delta_{(C,p)},$$
where $\mathcal{W}\subseteq\prod_{m\geqslant 2}\big(\prod_{p\in E\cap C\ \ m\text{-tacnode}}\Delta_{(C,p)}\big)$ may be identified with the image of the versal map
$$\psi=\prod_{p\in\mathrm{Sing}^\circ(C)\cap E}\phi_p:\bigcap_{p\in\mathrm{Sing}^\circ(C)\cap E}U_p\ \to\prod_{m\geqslant 2}\prod_{\substack{p\in E\cap C\\ m\text{-tacnode}}}\Delta_{(C,p)}. \tag{3.3.6}$$

In particular, $\mathcal{W}$ is a smooth analytic variety of dimension $\sum_m\tau_m(m-1)+1$, with tangent space $T_0\mathcal{W}=\Omega$ at $\underline{0}$, where Ω is as in (3.3.5). Keeping in mind the affine equations of the subspaces $\Gamma_p\subseteq H_p\subseteq\Delta_{(C,p)}$ in Lemma (2.9), the theorem follows by [1, Lemma 4.4] (see [V, Lemma (1.3)]), as we shall now explain.

To help the reader comparing our notation with that in [1], we observe that, for every m-tacnode (C,p), our subspace $\Gamma_p\subseteq\Delta_{(C,p)}$ is denoted in [1, Lemma 4.4] by Δ_{i,m_i}. Moreover, the product of the subspaces Γ_p is denoted in [1, Lemma 4.4] by Δ_m. Finally, the curves denoted by $\Gamma_1,...,\Gamma_k$ in the statement of [1, Lemma 4.4], will be denoted here by $\gamma_1,...,\gamma_k$.

By [1, Lemma 4.4], we have that the intersection of $\mathcal{W}$ with the locus of $\sum_m\tau_m(m-1)$-nodal curves in $\prod_{m\geqslant 2}\prod_{p\in E\cap C\ \ m\text{-tacnode}}\Delta_{(C,p)}$ is given by

$$\Big(\prod\nolimits_{m\geqslant 2}\prod\nolimits_{p\in E\cap C\ \ m\text{-tacnode}}\Gamma_p\Big)\cup\gamma_1\cup\cdots\cup\gamma_k,$$

where $\gamma_1,\dots,\gamma_k\subseteq\mathcal{W}$ are distinct, reduced, unibranched curves having intersection multiplicity exactly λ with $\prod_{m\geqslant 2}\prod_{p\in E\cap C\ \ m\text{-tacnode}}\Gamma_p$ at the origin, with λ and k as defined in (3.0.1) and (3.0.2). The curves $\gamma_1,...,\gamma_k$ are all smooth at $[C]$ if $\lambda=m_p$, for some tacnode p of C; otherwise they are all singular at $[C]$, with multiplicity $\lambda/\max\{m_p|\,p\text{ tacnode of }C\}$.

The proof of the theorem is now complete if we observe that the versal map φ defined by (3.3.3) has differential of maximal rank at $[C]$, thus

$$\varphi^{-1}(\gamma_1\cup...\cup\gamma_k\times\underline{0}\times...\times\underline{0})$$

is locally a fibration over $\gamma_1 \cup ... \cup \gamma_k \times \underline{0} \times ... \times \underline{0}$ with smooth fibre of dimension $h^0(C, \mathcal{N}_{C|\mathcal{X}}) - \delta - \sum_m \tau_m(m-1)$, and it actually consists of k analytic branches of $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)}$ at $[C]$. The fact that, if $[C_t]$ is general in one of these analytic branches of $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)}$, then (3.1.1) holds, follows by semicontinuity. $\square$

(3.4) Remark. Theorem (3.1) proves in particular [I, Proposition (5.5)]; besides, [I] contains many applications of Theorem (3.1). In order to help the reader compare Theorem (3.1) with [I, Proposition (5.5)], we observe that [I, Proposition (5.5)] is stated under the assumption that $\mathrm{Hilb}(\mathcal{L})$ (notation from loc. cit.) is an irreducible component of the relative Hilbert scheme of curves in the family of surfaces $S \to \mathbf{D}$, see [I, Section 4].

(3.5) Remark. The failure of hypothesis 3 in Theorem (3.1) is not always an obstruction to study deformations of C outside $\mathcal{X}_0$. It may happen for example that the singularities of C do not impose the expected number of conditions because several nodes of $\mathcal{X}_0 \setminus E$ are "disconnecting nodes", i.e., normalizing C at these nodes produces a reducible curve, and for some geometric reason it is not possible to deform C on $\mathcal{X}_t$ to a reducible curve. In this case the disconnecting nodes cannot be preserved by deforming C outside the central fibre, and the singularities of C do not impose the expected number of conditions. This happens for example for curves constructed in [10, Section 4].

When hypothesis 3 in (3.1) Theorem fails, one has to restrict the attention on the subset of singularities imposing the right number of conditions (namely, 1 condition for a node on $\mathcal{X}_0 \setminus E$, and $m-1$ conditions for an m-tacnode on E) to the linear system $|\mathcal{O}_{\mathcal{X}_0}(C)|$, and study the versal map only for this subset of singularities.

(3.6) Remark. Under the hypotheses of Theorem (3.1), by [10, Proposition 3.7 and Corollary 3.12], one has more generally that there exist deformations $C_t \subseteq \mathcal{X}_t$ of $C \subseteq \mathcal{X}_0$ so that every m-tacnode p of C is deformed to $d_{i,p}$ singularity of type A_i, for every multi-index $(d_{1,p}, ..., d_{r,p})$ such that $\sum_i id_{i,p} = m-1$, and every node of $C \subseteq \mathcal{X}_0$ is preserved. For these curves C_t the equality (3.1.1) still holds, thus the corresponding generalized universal Severi variety is smooth at $[C_t]$, but describing its local geometry at $[C_0]$ is much more complicated.

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Dipartimento di Matematica e Informatica, Università degli Studi della Calabria, Via Pietro Bucci, cubo 31B, 87036 Arcavacata di Rende (CS), Italy
galati@mat.unical.it

Appendix B
Flag semi-stable reduction of tacnodal curves at the limit of nodal curves

by Ciro Ciliberto, Thomas Dedieu, and Concettina Galati

0 – Introduction

The objective of this note is to provide down-to-earth explanations of the following phenomenon, which is a central theme of the present collection of articles. It is studied in detail in [IV], following Caporaso and Harris [2, 2], and in [VIII] in the same context that we are considering here; it also appears in various guises in many other places.

Let $\mathcal{S} \to \mathbf{A}^1$ be a family of surfaces such that the general member S_t, $t \neq 0$, is a smooth projective surface, and the central member S_0 is the union of two smooth projective surfaces intersecting transversely along a smooth curve B. Let $\mathcal{L}$ be a line bundle on $\mathcal{S}$, and consider the family of Severi varieties

$$\mathring{\mathcal{V}} = \coprod\nolimits_{t\neq 0} V^1(S_t, L_t) \longrightarrow \mathring{\mathbf{A}}^1 = \mathbf{A}^1 - \{0\},$$

where $V^1(S_t, L_t)$ is the family of 1-nodal curves in the linear system $|L_t|$ on the surface S_t, with $L_t = \mathcal{L}|_{S_t}$. Let V_0 be the family of curves in the linear system $|L_0|$ on S_0 having a *tacnode* along B, i.e., V_0 is the family of curves $C_0 \in |L_0|$, such that there exists a point $p \in B \cap C_0$, at which C_0 locally consists of two smooth branches $C_0' \subseteq S_0'$ and $C_0'' \subseteq S_0''$, both of which are tangent to B at p. Then, at least in principle, V_0 appears in the central fibre of the closure $\overline{\mathcal{V}}$ of $\mathring{\mathcal{V}}$, with multiplicity 2.

We carry out a detailed local study of the nodes degenerating to a tacnode in the above situation, with stable reduction as a central concept. We investigate in particular the following questions, with various points of view: (i) in which way is such a tacnodal curve of the appropriate genus? (ii) why does it count with multiplicity 2? (iii) how should one modify the family of surfaces in order to see a tacnodal limit curve in S_0 as a nodal curve in an alternative semi-stable limit of the S_t's?

The multiplicity 2 is fairly straightforward to understand, cf. (2.5). Seeing the tacnodal curve as a curve of the appropriate genus may be done in various ways, according to the chosen point of view, and is not always straightforward. We do provide a solution to problem (iii), but the conclusion is that although it is possible to see one tacnodal limit curve as a nodal curve in some semi-stable limit of the S_t's, it is impossible to do it simultaneously for all curves of the family V_0. Therefore, tacnodal curves in the degeneration cannot be avoided, and have to be considered as definitely acceptable limits of nodal curves in their own right: in [I, Section 5], a degeneration of surfaces is said to be well-behaved if the limits of Severi varieties can all be accounted for, up to some reduction, by families of nodal and tacnodal curves: the upshot of our discussion here is that it is hopeless to ask that the limits of Severi varieties be accounted for only by families of nodal curves.

The text is organized as follows. Section 1 is meant as an overview of this note, and hopefully clarifies in what sense we study the questions (i-ii-iii) mentioned above: in this section, we outline various ways of viewing a tacnodal curve as an appropriate limit of 1-nodal curves, in order to illustrate all the local computations performed in the subsequent sections. In Section 2, we give the equations of the local situation we want to study, namely 1-nodal curves in smooth surfaces degenerating to a tacnodal curve in the transverse union of two smooth surfaces. In Section 3, we perform the stable reduction of a family of 1-nodal curves degenerating to a tacnodal curve, forgetting about the ambient degeneration of surfaces. The stable reduction process in this section is rather simple (we first normalize the total space, which normalizes the general fibres, then do some modifications, and finally return to 1-nodal general fibres), but it is unfortunately not suitable to answer question (iii) above. Therefore, in Section 4, we compute the same stable reduction without normalizing the total space. Finally, we are able to answer question (iii) in Section 5, providing what we call a *flag semi-stable reduction* of the nodes degenerating to a tacnode inside surface sheets degenerating to two smooth transverse sheets. The name refers to the fact that we perform semi-stable reduction simultaneously for the family of curves and for the ambient family of surfaces.

1 – Synthetic descriptions of nodal curves degenerating to tacnode

In this section we outline various ways to see a tacnodal curve as the (equigeneric) limit of nodal curves, in the context described in the introduction above. We do not enter in too many details for the moment; a fully rigorous treatment will be given in the next sections.

(1.1) An example of ambient deformation space. We will follow throughout this section the guiding example of smooth quartics degenerating to two quadrics, and their hyperplane sections.

Let $\mathcal{S} \to \mathbf{A}^1$ be a pencil of quartic hypersurfaces in $\mathbf{P}^3$, generated by a general quartic S_∞, and the sum S_0 of two smooth quadrics S_0' and S_0'' intersecting transversely along a quartic elliptic normal curve B. The total space $\mathcal{S}$ has ordinary double points at the finitely many points of $S_\infty \cap B$, which one usually prefers to resolve in one way or another; in our context we can safely ignore this, as our study is local, away from the incriminated points. This example is described in more detail in [I, 7.1].

(1.2) Degeneration of smooth curves. A general hyperplane section C_t of S_t, $t \neq 0$, is a smooth plane quartic, and thus has genus 3. On the other hand, a general hyperplane section C_0 of S_0 is the union of two non-degenerate conics meeting transversely at 4 points: it has arithmetic genus 3, and it is a stable curve of genus 3.

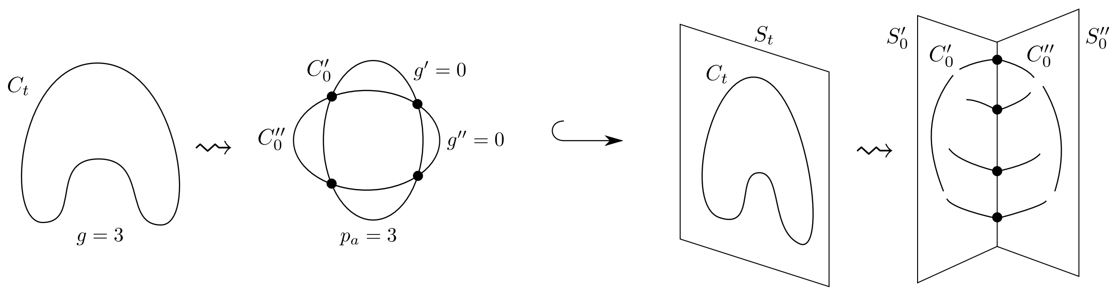

Figure 1: Degeneration of smooth curve sections

On Figure 1 above, the curves C_0' and C_0'' are pictured in $S_0 = S_0' \cup S_0''$ following a tropically inspired design, which is well suited for our drawings, see [X].

(1.3) Degeneration of 1-nodal curves. Continuing with our example of a pencil of quartic surfaces, we now consider for the rest of this section a family of 1-nodal hyperplane sections C_t of S_t, $t \neq 0$, degenerating to a tacnodal hyperplane section $C_0' \cup C_0''$ of S_0, i.e., C_0' and C_0'' are non-degenerate conics on S_0' and S_0'' respectively, tangent to B at some point q, and meeting transversely at two other points $p_1, p_2 \in B$ (recall that B is the curve $S_0' \cap S_0''$).

(1.3.1) An abuse of presentation. In fact, as we shall see in Section 2, such a family must be 2-valued, i.e., the curve C_t must consist of two 1-nodal hyperplane sections for $t \neq 0$, and the central fibre C_0 must be the double of $C_0' \cup C_0''$ on S_0. This is an important point but we shall ignore it for the moment, while we are outlining the situation, as it is not the aspect we want to focus on in this note. Thus, for the sake of keeping this outline simple, we shall pretend for the rest of this section that the curves C_t are irreducible 1-nodal curves, and C_0 is a reduced tacnodal curve, without any further mention. In particular, on Figure 2 and the other figures in this section, we only picture one 1-nodal curve tending to $C_0' \cup C_0''$, whereas in fact there are two. From Section 2 on, our treatment will be fully rigorous without any abuse of presentation.

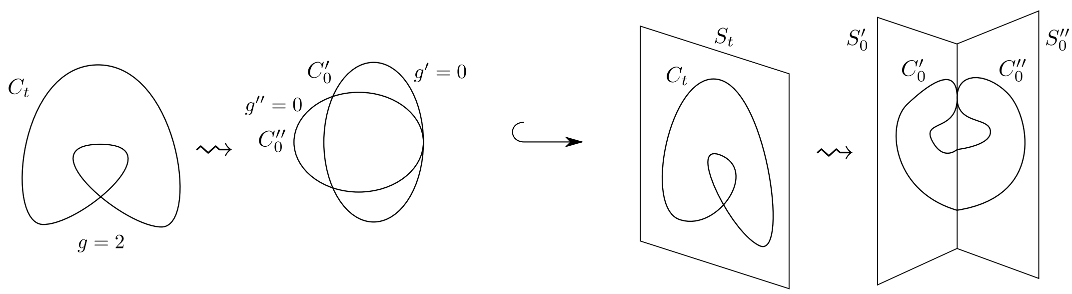

Figure 2: Degeneration of 1-nodal curves to a tacnodal curve

The curves C_t, $t \neq 0$, have geometric genus 2. In the next paragraphs we give various ways of appropriately seeing C_0 as a genus 2 limit of the C_t's.

(1.4) Degeneration of the normalizations. We may incarnate the 1-nodal curves C_t, $t \neq 0$, as the stable maps $\bar{C}_t \to S_t$ given by the normalizations $\bar{C}_t \to C_t$; these are maps from smooth, genus 2, sources. From this perspective, the relevant incarnation of the curve C_0 is as the image of a stable map as pictured on Figure 3 below, with source a stable, genus 2, curve consisting of two smooth rational components meeting transversely at three points (this source is a partial normalization of the tacnode of C_0).

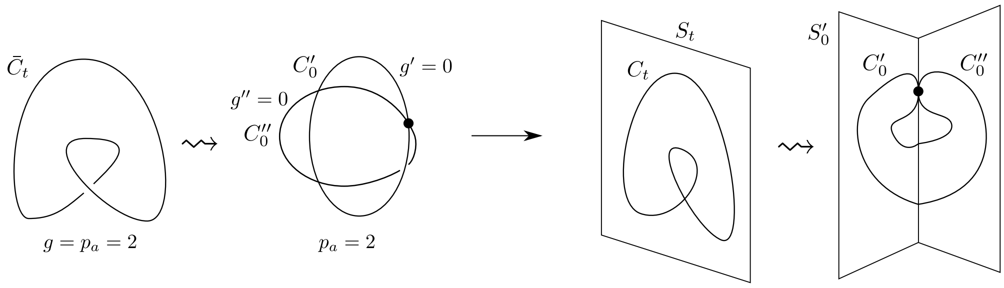

Figure 3: Degeneration of the normalizations of 1-nodal curve sections

To see this, one considers the family $\mathcal{C}$ of the curves C_t, $t \in \mathbf{A}^1$, and normalize it. It follows from Teissier's simultaneous resolution theorem [6] that the fiber of the normalization $\bar{\mathcal{C}}$ over $t \neq 0$ is the normalization $\bar{C}_t$. The fibre over 0 is computed in (3.2) below; it is singular over the tacnode, because there is a jump in δ-invariant when the node of C_t degenerates to the tacnode of C_0. This is compensated for by the jump in the number of irreducible components, which allows for the genus to be constant.

The latter point of view is arguably the more transparent. It is yet also desirable to understand the situation by seeing the curves C_t as singular curves in the surfaces S_t.

(1.5) Degeneration of the stable maps. The 1-nodal curves C_t are stable curves of arithmetic genus 3. As such, their limit C_0 can be modified into a stable curve of genus 3. In other words, the closed immersions $C_t \hookrightarrow S_t$ are stable maps of genus 3, and have a limit as such. These limits are pictured on Figure 4 below.

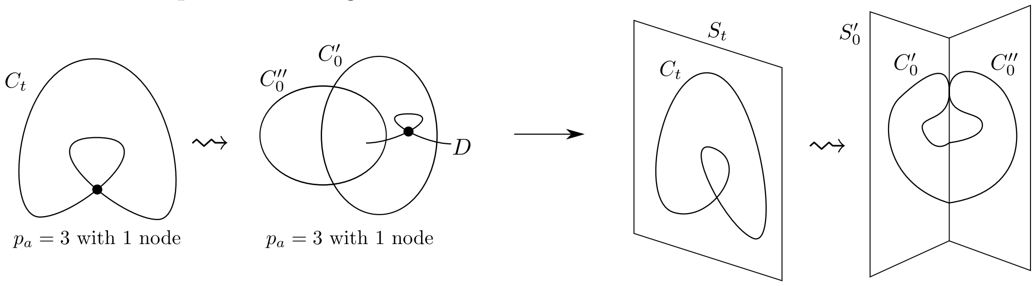

Figure 4: Degeneration as stable maps of the immersions of 1-nodal curve sections

The component D on Figure 4 is a 1-nodal rational curve ($p_a = 1$ and $p_g = 0$); it is contracted to the tacnode of C_0 by the stable map with target S_0 which is the limit of the maps $C_t \hookrightarrow S_t$. The computations are carried out in Section 3.

One nice feature of this point of view is that we see our limit of curves of arithmetic genus 3 with 1 node as a curve of arithmetic genus 3 with 1 node as well; the node in question is that of the component D (of course, the source of the limit stable map has other nodes; what we are saying is that the node of D is the limit of the node of C_t as t tends to 0). The price one pays is that this stable limit of maps has a contracted component.

(1.5.1) Observation. The above model may be used to recover that described in (1.4). To do so, one simultaneously normalizes the nodes of all the curves C_t: on the central fibre, this normalizes the node of D, which then becomes a destabilizing smooth rational component; the stable model of the central fibre is obtained by contracting this component, which gives the central fibre of the family of curves described in (1.4).

(1.6) Degeneration of the stable maps, II. It frequently happens in practice that one has to modify the family of surfaces $\mathcal{S} \to \mathbf{A}^1$ by adding one or more ruled surfaces over B between S_0' and S_0. Typically, one performs a base change $t \in \mathbf{A}^1 \mapsto t^m \in \mathbf{A}^1$, and then resolves the singularities of the obtained family. In particular, the description below is very useful to understand [IX].

Let us assume here that there is one ruled surface Σ_B over B between our two quadrics S_0' and S_0'', and that our family of surfaces has reduced central fibre. In this model of $\mathcal{S}$, our tacnodal curve C_0 is modified as indicated on Figure 5, by the adding of (i) reduced fibres of Σ_B to connect the points on C_0' and C_0'' that previously were transverse intersection points of C_0' and C_0'', and (ii) one double fibre to connect the two points that previously were the tacnode.

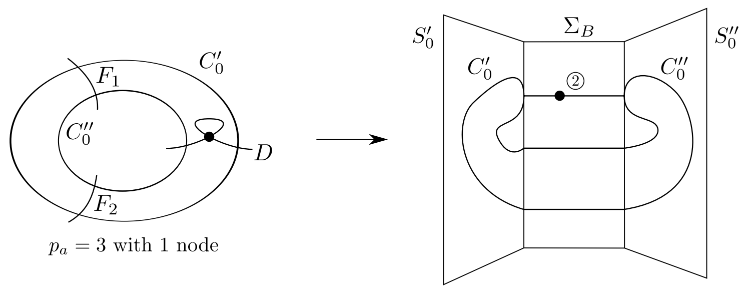

Figure 5: Degeneration as stable maps of the immersions of 1-nodal curve sections, II

The stable map limit of the immersions $C_t \hookrightarrow S_t$ in this context is easily deduced from the limit in paragraph (1.5). The two intersection points between C_0' and C_0'' are replaced by smooth rational curves F_1 and F_2, mapped isomorphically to the corresponding fibres of Σ_B. The nodal curve D on the other hand is mapped $2:1$ to the double fibre "at the tacnode", with ramification divisor the sum of the two points $D \cap C_0'$ and $D \cap C_0''$; the node of D is mapped to the limit of the nodes of the curves C_t as t approaches 0, which lies on the double fibre of Σ_D; the pre-image of this point in D consists only of the node of D.

Alternatively, this limit may be computed using the reduction which we discuss next, and which is constructed in Section 5.

(1.7) Flag semi-stable reduction. Ultimately, one may desire to see the stable limit of the 1-nodal curves C_t, depicted in Figure 4, not mapped to S_0 with some contracted components, but rather immersed in some semi-stable limit of the surfaces S_t. It is of course asserted by the general stable reduction theorems that this is possible, provided one allows the replacement of the stable limit of the C_t's by another semi-stable model. We carry this out explicitly in Section 5, and the result is depicted in Figure 13 there. We dub this a *flag semi-stable reduction* of the pairs (C_t, S_t), $t \neq 0$.

2 – The local situation

In this section, we set up an explicit incarnation of the situation that we study throughout this text, of a family of 1-nodal curves degenerating to a tacnodal curve inside a degeneration of smooth surfaces to the transverse union of two smooth surfaces. This is the same setup as in [IV].

(2.1) Notation. The notation $\mathbf{A}^4_{xyzt}$ denotes the affine 4-space equipped with affine coordinates (x, y, z, t); we will also use obvious variations, e.g., $\mathbf{A}^1_t$ denotes the affine line with affine coordinate t. Similarly, $\mathbf{P}^1_{x:y}$ denotes the projective line with homogeneous coordinates $(x : y)$, and obvious variations will be used.

The arrow $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$ without further indication denotes the affine projection map $(x, y, z, t) \in \mathbf{A}^4 \to t \in \mathbf{A}^1$.

(2.2) We consider the degeneration of surfaces $\mathcal{S} \to \mathbf{A}^1_t$, where the total space $\mathcal{S}$ is the hypersurface of $\mathbf{A}^4_{xyzt}$ defined by the equation

$$\mathcal{S} : xy = t,$$

and the map $\mathcal{S} \to \mathbf{A}^1_t$ is the restriction of the affine projection $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$.

For all $t \neq 0$, the fibre S_t of $\mathcal{S}$ over t is smooth and irreducible, whereas the central fibre S_0 is the union of two planes S_0' and S_0'', defined by the equations $t = y = 0$ and $t = x = 0$ respectively, which intersect transversely along the line defined by the equations $x = y = t = 0$.

(2.3) We want to study degenerations of nodal curves to a tacnodal curve inside this degeneration of surfaces, i.e., families of curves $\mathcal{C} \to \mathbf{A}^1$ fitting in a commutative diagram

$$\begin{tikzcd}[row sep=small, column sep=tiny] \mathcal{C} \arrow[dr] & \subseteq & \mathcal{S} \arrow[d] \\ & & \mathbf{A}^1 \end{tikzcd}$$

such that the fibres C_t with $t \neq 0$ are 1-nodal curves, and the central fibre C_0 is a curve with a tacnode along the double curve of S_0, by which we mean that C_0 consists of two smooth branches C_0' and C_0'' contained in S_0' and S_0'' respectively, which intersect in only one point where they are both tangent to the curve $S_0' \cap S_0''$.

(2.4) As a matter of fact, families of curves as in (2.3) cannot exist, because if they would, then the singularities of the curves would form a section of $\mathcal{S} \to \mathbf{A}^1$ meeting the central fibre along its double curve, which is impossible since the total space $\mathcal{S}$ is smooth.

As we shall now see, there exist instead families of curves with central fibre the double of the curve C_0 as in (2.3) above, and fibre over $t \neq 0$ the sum of two 1-nodal curves. We let, indeed, ${}^\sharp\mathcal{C} \to \mathbf{A}^1_t$ be defined in $\mathcal{S}$ by the equation

$$^\sharp\mathcal{C} : \left(x + y - z^2\right)^2 = 4t.$$

Its central fibre is indeed the double of a tacnodal curve C_0 as described in (2.3): it is defined by $t = 0$, and thus it is the double of the curve with the following equations in $\mathbf{A}^3_{xyz}$ (the latter being the central fibre of $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$):

$$\begin{cases} xy = 0 \\ x + y - z^2 = 0 \end{cases} \iff \begin{cases} y(y - z)^2 = 0 \\ -x = y - z^2. \end{cases}$$

On the other hand, for all $t_0 \neq 0$, the fibre ${}^\sharp C_{t_0}$ consists of two irreducible 1-nodal curves $C^{\pm}_{t_0}$, with the following equations in $\mathbf{A}^3_{xyz}$, the fibre of $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$ over t_0:

$$\text{(2.4.1)} \qquad \begin{cases} -xy + t_0 = 0 \\ x + y - z^2 \mp 2\sqrt{t_0} = 0 \end{cases} \iff \begin{cases} (y \mp \sqrt{t_0})^2 - yz^2 = 0 \\ -x = y - z^2 \mp 2\sqrt{t_0}. \end{cases}$$

(2.5) The family $^\sharp\mathcal{C}$ inside $\mathcal{S}$ over $\mathbf{A}^1$ is the universal local model for 1-nodal curves in smooth surfaces degenerating to a tacnodal curve along the transverse intersection of two smooth surface sheets.

The necessity of having two 1-nodal curves in each fibre S_t, $t \neq 0$, tending to the tacnodal curve C_0 explains why a tacnodal curve in the limit amounts for (at least) two 1-nodal curves.

(2.6) Next, we shall perform a degree 2 base change in order to separate $^\sharp\mathcal{C}$ in two distinct families $\mathcal{C}^+$ and $\mathcal{C}^-$ with the same central fibre C_0.

Thus, let $\tilde{\mathcal{S}} \to \mathbf{A}^1_t$ be the family of surfaces with total space defined by

$$\tilde{\mathcal{S}} : xy = t^2$$

in $\mathbf{A}^4_{xyzt}$, and with the map $\tilde{\mathcal{S}} \to \mathbf{A}^1_t$ given by the affine projection $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$. Note that $\tilde{\mathcal{S}}$ is singular along the double curve of its central fibre S_0. There are two families of 1-nodal curves tending to a curve with a tacnode along the double curve of S_0, namely $\mathcal{C}^\pm$ defined by

$$\mathcal{C}^\pm : x + y - z^2 = \pm 2t$$

inside $\tilde{\mathcal{S}}$. We will usually only look at one of these two families, namely $\mathcal{C}^-$, which we will simply denote by $\mathcal{C}$, in spite of the conflict of notation with (2.3).

3 – Abstract reduction of nodes degenerating to a tacnode, via normalization of the nodes

In this section, we compute the stable reduction of the family $\mathcal{C} \to \mathbf{A}^1$ of 1-nodal curves degenerating to a tacnodal curve described in the previous section. We start by normalizing the total space $\mathcal{C}$, which has the effect of normalizing the fibres C_t with $t \neq 0$. Then, we make some modifications in order for the pre-images of the nodes to form two disjoint sections. The last operation is to glue these two sections, so that we come back to a family with general member a 1-nodal curve.

(3.1) Setup. We consider the family $\mathcal{C}$ defined as $\mathcal{C}^-$ in (2.6) (recall also Notation (2.1)). It may be identified with its affine projection in $\mathbf{A}^3_{yzt}$, which is defined by the single equation

$$\mathcal{C}: \quad y(y - z^2 + 2t) + t^2 = 0 \iff (y+t)^2 - yz^2 = 0, \tag{3.1.1}$$

obtained by eliminating x from the two equations $xy = t^2$ and $x + y - z^2 + 2t = 0$ defining $\mathcal{C}^-$.

The total space $\mathcal{C}$ is a surface with a double curve along

$$R: \; y + t = z = 0,$$

and a pinch point at the origin (which is the intersection point of R with the central fibre), and is otherwise smooth. For all $t \neq 0$, C_t is a 1-nodal curve with its node at $C_t \cap R$, and C_0 is a 1-tacnodal curve with its tacnode at the pinch point $C_0 \cap R$. We let C_0', C_0'' be the two irreducible components of C_0, defined respectively by $y = t = 0$ and $y - z^2 = t = 0$.

(3.2) Normalization of the total space. The first move in the present approach is to normalize the total space $\mathcal{C}$. We do this by blowing-up the double curve R in $\mathcal{C}$. Let $\varepsilon_1 : \mathcal{C}_1 \to \mathcal{C}$ be the corresponding morphism.

The surface $\mathcal{C}_1$ lives in the subscheme of the product $\mathbf{A}^3_{yzt} \times \mathbf{P}^1_{t_1:z_1}$ defined by the equation

$$z_1(y+t) = t_1 z, \tag{3.2.1}$$

in which it is defined by the equation

$$t_1^2 - y z_1^2 = 0. \tag{3.2.2}$$

First note that the surface $\mathcal{C}_1$ is smooth. The exceptional locus of ε_1 (i.e., the total transform via ε_1 of R) is the curve R_1 defined by

$$y + t = z = 0$$

in $\mathcal{C}_1$; it is a bisection of $\mathcal{C}_1 \to \mathbf{A}^1_t$ (which is the composed map $\mathcal{C}_1 \to \mathcal{C} \to \mathbf{A}^1_t$), mapped $2:1$ onto R by ε_1 with ramification over the tacnode. The central fibre of $\mathcal{C}_1$ is defined by $t = 0$, which gives $z_1 y = t_1 z$ by (3.2.1), hence (3.2.2) becomes

$$t_1(t_1 - z z_1) = 0;$$

thus the central fibre of $\mathcal{C}_1$ consists of two branches intersecting transversely, which we shall still denote by C_0' and C_0''. The situation is summed up in Figure 6.

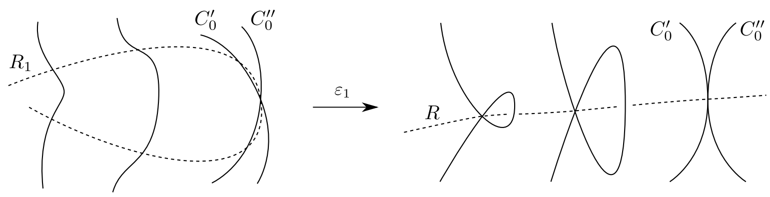

Figure 6: Normalization of the total space, $\varepsilon_1 : \mathcal{C}_1 \to \mathcal{C}$

(3.3) Base change. The next step is to perform the base change $t \in \mathbf{A}^1 \mapsto t^2 \in \mathbf{A}^1$ in order to separate the pre-images of the nodes of the C_t's in two sections. Let $\tilde{\mathcal{C}}_1 \to \mathcal{C}_1$ be the corresponding double cover. In practice, this amounts to replacing t by t^2 everywhere in the equations defining $\mathcal{C}_1$. The notable fact is that this yields an ordinary double point on the surface $\tilde{\mathcal{C}}_1$ at the point where C_0' and C_0'' intersect. We call R_1' and R_1'' the two irreducible components of the pre-image of R_1.

(3.4) Resolution. The next thing we do is to resolve the ordinary double point of $\tilde{\mathcal{C}}_1$. We call $\bar{\mathcal{C}}_1 \to \tilde{\mathcal{C}}_1$ the corresponding map. It is a birational morphism, with exceptional locus a (-2)-curve which we call E. The resulting family of curves is depicted on the right-hand-side of Figure 7.

(3.5) Reconstruction of the nodes. Finally, since what we are looking for is the stable reduction of the 1-nodal curves C_t, $t \neq 0$, we reconstruct the nodes that have been normalized in the first step by identifying the two disjoint sections R_1' and R_1'' of the family $\bar{\mathcal{C}}_1 \to \mathbf{A}^1$. We call $\mathcal{C}_2 \to \mathbf{A}^1$ the obtained family of curves, see the left-hand-side of Figure 7. Its central fibre is reduced, with a 1-nodal rational curve D (image of E) connecting C_0' and C_0''; the node of D is the limit position of the nodes of the curves C_t, $t \neq 0$.

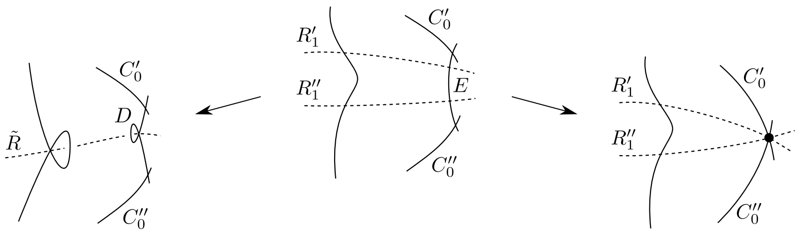

Figure 7: After the base change, $\mathcal{C}_2 \leftarrow \bar{\mathcal{C}}_1 \rightarrow \tilde{\mathcal{C}}_1$

(3.6) Why we're not satisfied with this. We want to perform the stable reduction of the nodal curves C_t (or possibly a semi-stable variant of it), while simultaneously performing the semi-stable reduction of the degeneration of surfaces $\tilde{\mathcal{S}} \to \mathbf{A}^1$, with notation as in Section 2. The main problem with the above described stable reduction process is that the normalization of the family of curves and later the reconstruction of the nodes necessitate unwanted modifications of the surfaces S_t, $t \neq 0$.

Indeed, the normalization of $\mathcal{C}$ inside $\tilde{\mathcal{S}}$ requires to blow-up every surface S_t, $t \neq 0$ at the point where the curve C_t has a node, which introduces a (-1)-curve. Finally, once the remaining steps of the reduction of the C_t's are performed, we have to recontract all these (-1)-curves, and this messes everything up, in particular this destabilizes the central fibre. We leave it to the reader to check for himself what indeed happens.

Our solution is to follow another path to the stable reduction of $\mathcal{C}$, which doesn't involve the normalization of the nodal C_t's. This is what we do in the next section.

4 – Abstract reduction of nodes degenerating to a tacnode, maintaining the nodes

4.1 – The construction

In this section we present our construction of the stable reduction of the family of 1-nodal curves C_t without normalizing the nodes. The setup is the same as that formulated in (3.1), and we don't repeat it here. We use Notation (2.1), as always.

(4.1) Blow-up of the tacnode. Let $\varepsilon_1 : \mathcal{C}_1 \to \mathcal{C}$ be the blow-up of $\mathcal{C}$ at the origin of $\mathbf{A}^3_{yzt}$ (here we use the same notation as in paragraph (3.2) for a different blow-up, but we believe this will not cause any confusion). We will see that the reduced exceptional locus of ε_1 is a smooth $\mathbf{P}^1$, which we shall call E_1, and the central fibre of $\mathcal{C}_1$ consists of the proper transform of C_0, which is the transverse union of two smooth curves, namely the proper transforms of C_0' and C_0'', which we continue to denote by the same symbols, plus E_1 with multiplicity 2; the total space $\mathcal{C}_1$ has a double curve along the proper transform of R, still with a pinch point at its intersection with the central fibre; besides this double curve, $\mathcal{C}_1$ has an ordinary double point at $C_0' \cap C_0''$, which lies on E_1, and no other singularities. See Figure 8.

We realize $\mathcal{C}_1$ as the proper transform of $\mathcal{C}$ in the subvariety of $\mathbf{A}^3_{yzt} \times \mathbf{P}^2_{y_1:z_1:t_1}$ defined by the equations

$$\mathrm{rk}\begin{pmatrix} y & z & t \\ y_1 & z_1 & t_1 \end{pmatrix} < 2.$$

In the affine chart $t_1 = 1$, $\mathcal{C}_1$ has the equation

$$\mathcal{C}_1 \text{ in } [t_1 = 1]: \quad (y_1 + 1)^2 - y_1 z_1^2 t = 0. \tag{4.1.1}$$

We thus see that $\mathcal{C}_1$ has a double curve along the proper transform R_1 of R, which has equations $y_1 + 1 = z_1 = 0$, with a pinch point at the point $(-1 : 0 : 1)$ on the exceptional divisor (the latter is the fibre of $\mathcal{C}_1$ over the origin in the projection $\mathbf{A}^3_{yzt} \times \mathbf{P}^2_{y_1:z_1:t_1} \to \mathbf{A}^3_{yzt}$). In this chart, the proper transform of C_0 is at infinity, and the central fibre $t = 0$ is the exceptional divisor, namely the double line $(y_1 + t_1)^2$ in $\mathbf{P}^2$.

In the affine chart $z_1 = 1$, we get the equation

$$\mathcal{C}_1 \text{ in } [z_1 = 1] : \quad (y_1 + t_1)^2 - y_1 z = 0. \tag{4.1.2}$$

We thus see that $\mathcal{C}_1$ has an ordinary double point at the point $(0 : 1 : 0)$ on the exceptional divisor. The central fibre is defined by the equation $t = 0$, which gives $t_1 z = 0$: $t_1 = 0$ is the proper transform of C_0, and has equation $y_1(y_1 - z) = 0$, while $z = 0$ defines the exceptional divisor: it is the divisor defined by $(y_1 + t_1)^2 = 0$, i.e., E_1 with multiplicity 2.

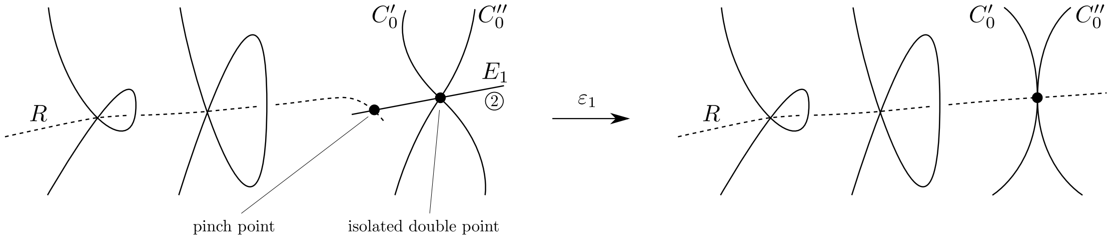

Figure 8: Blow-up of the tacnode, $\varepsilon_1 : \mathcal{C}_1 \to \mathcal{C}$

(4.2) Blow-up of the ordinary double point. Let $\varepsilon_2 : \mathcal{C}_2 \to \mathcal{C}_1$ be the blow-up at the point $C_0' \cap C_0'' \cap E_1$. The three curves C_0', C_0'', E_1 in $\mathcal{C}_1$ intersect transversely at this point, hence they are separated by this blow-up. The reduced exceptional locus is a smooth $\mathbf{P}^1$, which we shall call E_2; it appears with multiplicity 2 in the central fibre. See the right-hand-side of Figure 9 for a picture of the situation.

Note that the central fibre of $\mathcal{C}_1$ is $C_0' + C_0'' + 2E_1$, hence it has multiplicity 4 at the centre of the blow-up ε_2. However, as we shall now see, E_2 indeed appears only with multiplicity 2 in the central fibre of $\mathcal{C}_1$, because the centre of ε_2 is a singular point (of multiplicity 2) of $\mathcal{C}_1$.

This is fairly elementary, so we'll only sketch the computations. It is enough to consider the part of $\mathcal{C}_1$ in the affine chart $z_1 = 1$, which has equation (4.1.2). We add the new set of homogeneous variables $(y_2 : z_2 : t_2)$ with the new equations

$$\operatorname{rk} \begin{pmatrix} y_1 & z & y_1 + t_1 \\ y_2 & z_2 & t_2 \end{pmatrix} < 2.$$

Then, $\mathcal{C}_2$ has equation $t_2^2 - y_2 z_2 = 0$. Its central fibre has equation $t = 0$, which, in the affine chart $y_2 = 1$, since $t = t_1 z$, $t_1 = y_1(t_2 - 1)$, and $z = y_1 z_2$, reads

$$(t_2 - 1) z_2 y_1^2 = 0.$$

Now, $t_2 - 1 = 0$ defines one of the two branches C_0' and C_0'' (the other is invisible in this affine chart), $z_2 = 0$ is the equation of the proper transform of E_1, and $y_1^2 = 0$ that of the exceptional curve E_2 with multiplicity two.

(4.3) Base change and normalization. Let $\tilde{\mathcal{C}}_2 \to \mathcal{C}_2$ be the double covering of $\mathcal{C}_2$ ramified over the central fibre, and $\bar{\mathcal{C}}_2 \to \tilde{\mathcal{C}}_2$ the partial normalization obtained by leaving only the singularity along the proper transform of R. As we shall see, the family $\bar{\mathcal{C}}_2 \to \mathbf{A}^1$ is a semi-stable reduction of $\mathcal{C} \to \mathbf{A}^1$.

Algebraically, $\tilde{\mathcal{C}}_2$ is defined by adding a square root of t, i.e., a new variable s with the new equation $s^2 = t$, to the equations of $\mathcal{C}_2$. The normalization process may be chopped in several local computation units, which we gather in Section 4.2 below. The upshot is as follows. Denote by abuse of notation $E_1, E_2 \subseteq \tilde{\mathcal{C}}_2$ the reduced pre-images of $E_1, E_2 \subseteq \mathcal{C}_2$. The normalization may be obtained by successively blowing-up the Weil divisors E_1 and E_2.

First, the blow-up of E_1 produces two smooth rational curves E_1' and E_1'', disposed as pictured in Figure 9, and both mapped isomorphically onto E_1 (to see this, use (4.7) at the pinch point, and (4.6.1) at $E_1 \cap E_2$). Then, the blow-up of E_2 produces a double covering $\bar{E}_2 \to E_2$, with $\bar{E}_2$ a smooth rational curve disposed as pictured in Figure 9; this double covering branches at the two points $C_0' \cap E_2$ and $C_0'' \cap \bar{E}_2$, and the associated involution exchanges the two points $E_1' \cap \bar{E}_2$ and $E_1'' \cap \bar{E}_2$ (to see this use (4.5) at the two points $C_0' \cap E_2$ and $C_0'' \cap \bar{E}_2$, and (4.6.2) at the point $(E_1' + E_1'') \cap E_2$).

(4.3.1) A useful variant. Although arguably less natural, blowing-up first E_2 and then E_1 produces the same result. The blow-up of E_2 produces a double cover $\tilde{E}_2 \to E_2$, ramified at the two points $C_0' \cap \tilde{E}_2$ and $C_0'' \cap \tilde{E}_2$, and with a node over the point $E_1 \cap E_2$, as depicted in Figure 9. Then, the blow-up along E_1 produces two curves E_1' and E_1'' over E_1, and resolves the node of $\tilde{E}_2$.

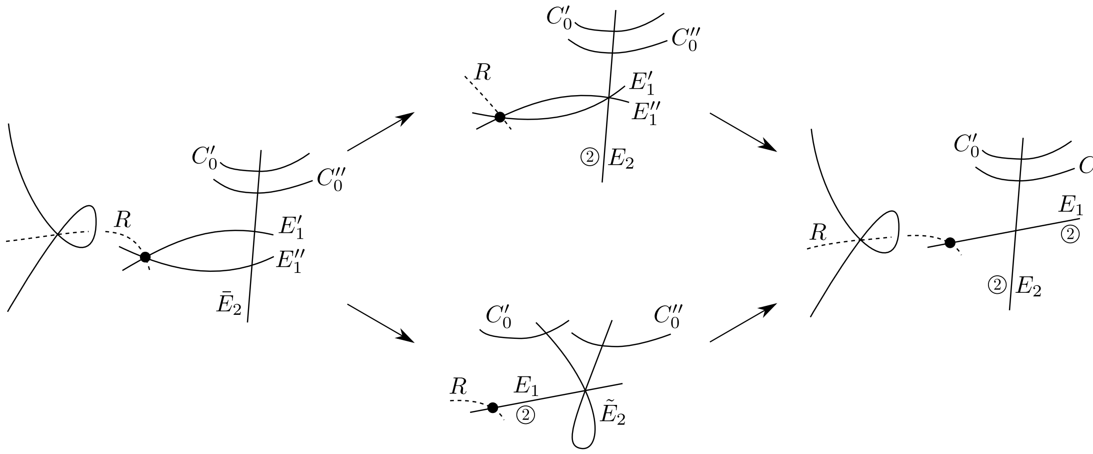

Figure 9: Removing the multiplicities, $\bar{\mathcal{C}}_2 \to \tilde{\mathcal{C}}_2$

(4.4) Conclusion. It follows from the previous considerations that the family $\bar{\mathcal{C}}_2 \to \mathbf{A}^1$ is indeed semi-stable. One may check that the final family has an ordinary double point all along the proper transform of R, i.e., the pinch point has disappeared. To build a stable model out of this semi-stable model, one has to contract the two rational curves E_1' and E_1'', which turns the curve $\bar{E}_2$ into a 1-nodal rational curve, with its node at the limit of the nodes of the curves C_t, $t \neq 0$.

4.2 – Ready for use toroidal computations

In the following paragraphs we gather some local computation, useful to normalize families of curves gotten by base change from families with non-reduced but normal crossing central fibre; the normalization will be expressed as a succession of blow-ups. We only include the computations needed for (4.3) above, but the reader will undoubtly figure out how to generalize them to similar situations. [5, p. 125] is a useful reference on this subject.

(4.5) Local situation $x^2y = t^2$. We consider the surface $\mathcal{C}$ defined by this equation in $\mathbf{A}^3_{xyt}$, together with the fibration over $\mathbf{A}^1_t$ induced by the projection $\mathbf{A}^3_{xyt} \to \mathbf{A}^1_t$. This is the family gotten by a degree two base change totally ramified at the origin from a central fibre with one double component $2A$ and one reduced component B meeting transversely, defined by the equations $x^2 = 0$ and $y = 0$ respectively.

The normalization of $\mathcal{C}$ amounts to the blow-up $\bar{\mathcal{C}} \to \mathcal{C}$ of the Weil divisor A, defined by the two equations $x = t = 0$ (note that $\mathcal{C}$ has an ordinary double point at the generic point of A, and a pinch point at the origin). To write down this blow-up, we add the set of homogeneous coordinates $(x_1 : t_1)$ with the new equation $x_1t = t_1x$.

The surface $\bar{\mathcal{C}}$ is defined by the homogeneous equation

$$x_1^2y = t_1^2. \tag{4.5.1}$$

One thus sees that $\bar{\mathcal{C}}$ is smooth. We claim that its central fibre has the form $\bar{A} + \bar{B}$, with two maps $\bar{A} \to A$ and $\bar{B} \to B$ induced by $\bar{\mathcal{C}} \to \mathcal{C}$, respectively a double covering of A ramified at the point $\bar{A} \cap \bar{B}$, and an isomorphism.

Indeed, the central fibre is defined by $t = 0$, which gives $t_1x = 0$. The equations $x = t = 0$ define the exceptional divisor in $\bar{\mathcal{C}}$; using (4.5.1), one sees that the exceptional divisor is a double cover of the line $x = t = 0$ in $\mathbf{A}^3_{xyt}$, which is the divisor A. The equation $t_1 = 0$, on the other hand, implies $y = 0$ by (4.5.1), and one sees that it defines a divisor projecting isomorphically to B.

(4.6) Local situation $x^2y^2 = t^2$. We consider the surface $\mathcal{C}$ defined by this equation in $\mathbf{A}^3_{xyt}$, together with the fibration $\mathcal{C} \to \mathbf{A}^1_t$ defined by affine projection. The central fibre of $\mathcal{C}$ consists of the two Weil divisors A ($y = t = 0$) and B ($x = t = 0$), each with multiplicity 2.

(4.6.1) Blow-up of the Weil divisor A. We add the homogenous coordinates $(y_1 : t_1)$, with the equation $y_1t = t_1y$; the blow-up $\mathcal{C}_1$ of $\mathcal{C}$ is defined by the equation $x^2y_1^2 = t_1^2$. Let us restrict to the chart $y_1 = 1$, to fix ideas; then $\mathcal{C}_1$ is defined by the equation $x^2 = t_1^2$. The central fibre is defined by $t = 0$, which implies $t_1y = 0$, hence it consists of the proper transform B_1 of B, still with multiplicity 2 (defined by $t = t_1 = 0$, hence $x^2 = 0$), plus the exceptional divisor, defined by $t = y = 0$ and $(x - t_1)(x + t_1) = 0$, which is thus a sum $A' + A''$ of two divisors mapping isomorphically to A.

(4.6.2) Successive blow-up of the second Weil divisor B. Now, we consider the blow-up $\mathcal{C}_2$ of $\mathcal{C}_1$ along the Weil divisor B_1, which is the proper transform of B in $\mathcal{C}_1$. Thus we blow-up along the subscheme defined by $t_1 = x = 0$, and take the proper transform $\mathcal{C}_2$ of $\mathcal{C}_1$: we add the homogenous coordinates $(x_2 : t_2)$, with the equation $x_2t_1 = t_2x$; the blow-up $\mathcal{C}_2$ of $\mathcal{C}_1$ is defined by the equation $x_2^2 = t_2^2$ which gives $(x_2 : t_2) = (1 : \pm 1)$. The central fibre is defined by $t = 0$, which implies $t_1y = 0$. If $t_1 = 0$, we get $x = 0$ and we obtain two reduced curves since $(x_2 : t_2) = (1 : \pm 1)$. These are the curves $B' + B''$ because $x = 0$. If $y = 0$, we get the equations $x = \pm t_1$, which give the two proper transforms of A' and A'' respectively.

Of course the situation in the present paragraph is more or less trivial, as the surface $\mathcal{C}$ has two smooth irreducible components, defined by $xy = t$ and $xy = -t$ respectively, hence its normalization is the disjoint union of these two components. However, we find it useful to describe the normalization as an explicit composition of blow-ups.

(4.7) Local situation $x^2 = y^2t^2$. We consider the surface $\mathcal{C}$ defined by this equation in $\mathbf{A}^3_{xyt}$, together with the fibration over $\mathbf{A}^1_t$ defined by the projection $\mathbf{A}^3_{xyt} \to \mathbf{A}^1_t$. The family $\mathcal{C} \to \mathbf{A}^1$ is gotten by a degree 2 base change from the family defined by $x^2 = y^2t$, which has a double curve dominating $\mathbf{A}^1$, and a pinch point on the central fibre; the central fibre is a smooth curve with multiplicity 2.

We blow-up the Weil divisor $x = t = 0$: this introduces homogeneous coordinates $(x_1 : t_1) \in \mathbf{P}^1$, with the equation $x_1 t = t_1 x$, and the blown-up surface is defined by the homogeneous equation $x_1^2 = y^2 t_1^2$. The exceptional divisor is given by the latter equation together with $x = t = 0$, hence it is the union of two lines meeting at the point $(0,0,0,(0:1))$. The central fibre is defined by $t = 0$, hence also $t_1 x = 0$, and one sees that it equals the exceptional divisor.

The total space is singular along the curve $y = x_1 = 0$, where it uniformly has an ordinary double point. In particular, the pinch point has been resolved by the blow-up.

5 – The flag reduction

We now use the construction of Section 4 in order to perform the semi-stable reduction of both families $\mathcal{C} \to \mathbf{A}^1$ and $\tilde{\mathcal{S}} \to \mathbf{A}^1$ at the same time (in the notation of (2.6)).

(5.1) Setup. We briefly recapitulate the situation. We consider the threefold $\tilde{\mathcal{S}}$ defined by the equation

$$\tilde{\mathcal{S}}: \quad xy = t^2 \tag{5.1.1}$$

in $\mathbf{A}^4_{xyzt}$, and the map $\tilde{\mathcal{S}} \to \mathbf{A}^1$ induced by the projection $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$. Thus $\tilde{\mathcal{S}} \to \mathbf{A}^1$ is a family of surfaces. We call S_0' and S_0'' the two components of its central fibre, defined by $y = t = 0$ and $x = t = 0$ respectively. The family of curves $\mathcal{C}$ studied in Sections 3 and 4 is a Cartier divisor of $\tilde{\mathcal{S}}$, defined in $\mathbf{A}^4_{xyzt}$ by the equations

$$\mathcal{C}: \quad \begin{cases} y + x - z^2 + 2t = 0 \\ \hfill xy = t^2. \end{cases} \tag{5.1.2}$$

(5.2) Blow-up of the tacnode. Let $\varepsilon_1 : \mathcal{S}_1 \to \tilde{\mathcal{S}}$ be the blow-up of $\tilde{\mathcal{S}}$ at the point $\mathbf{z} = (0,0,0,0) \in \mathbf{A}^4$, and let $\mathcal{C}_1$ be the proper transform of $\mathcal{C}$ in $\mathcal{S}_1$; the restriction $\varepsilon_1|_{\mathcal{C}_1}$ is the same map $\mathcal{C}_1 \to \mathcal{C}$ as in (4.1). We call Σ_1 the exceptional divisor of ε_1.

The blow-up amounts to adding the new homogeneous variables $(x_1 : y_1 : z_1 : t_1) \in \mathbf{P}^3$, with the equations

$$\operatorname{rk} \begin{pmatrix} x & y & z & t \\ x_1 & y_1 & z_1 & t_1 \end{pmatrix} < 2.$$

The total space $\mathcal{S}_1$ is defined by these equations plus the homogeneous equation

$$x_1 y_1 = t_1^2 \tag{5.2.1}$$

in $\mathbf{A}^4_{xyzt} \times \mathbf{P}^3_{x_1:y_1:z_1:t_1}$. The exceptional divisor Σ_1 is the corank 1 quadric defined by the equation (5.2.1) in $\mathbf{P}^3_{x_1:y_1:z_1:t_1}$ (and it sits over the origin of $\mathbf{A}^4_{xyzt}$). The central fibre of $\mathcal{S}_1$ is defined by the equation $t = 0$, which implies $t_1 x = t_1 y = t_1 z = 0$, hence it is the reduced sum of divisors $S_0' + S_0'' + \Sigma_1$, where we denote by abuse of notation $S_0', S_0'' \subseteq \mathcal{S}_1$ the respective proper transforms of $S_0', S_0'' \subseteq \tilde{\mathcal{S}}$ (the former are the respective blow-ups of the latter at their smooth point $\mathbf{z}$).

The surface $\mathcal{C}_1$ on the other hand is defined inside $\mathcal{S}_1$ by the same equations as in (4.1). We leave it to the reader to check that the central fibres of $\mathcal{S}_1$ and $\mathcal{C}_1$ are arranged as indicated on Figure 10 (compare also with Figure 8, right-hand-side).

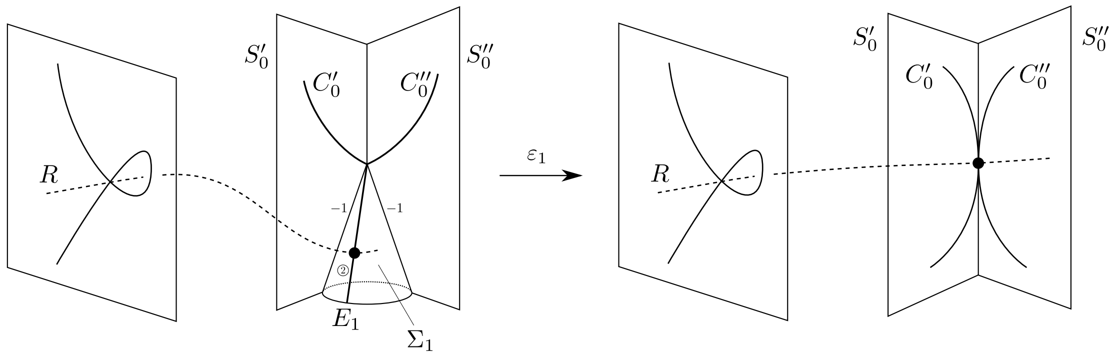

Figure 10: Blow-up of the tacnode, $\varepsilon_1 : \mathcal{S}_1 \to \mathcal{S}$

(5.3) Blow-up of the vertex of Σ_1. Let $\varepsilon_2 : \mathcal{S}_2 \to \mathcal{S}_1$ be the blow-up at the point $\mathbf{z}_1 = (0, 0, 0, 0, (0 : 0 : 1 : 0)) \in \Sigma_1$, $\mathcal{C}_2$ be the proper transform of $\mathcal{C}_1$, and Σ_2 be the exceptional divisor. As usual, we call Σ_1 its own proper transform in $\mathcal{S}_2$.

To fix ideas, we consider the affine chart $z_1 = 1$ with respect to the homogeneous coordinates introduced in (5.2) for the previous blow-up; there, $\mathcal{S}_1$ is defined by the affine equation $x_1 y_1 = t_1^2$ (compare (5.2.1)) in $\mathbf{A}^4$ with coordinates (x_1, y_1, z, t_1). We add the new homogeneous variables $(x_2 : y_2 : z_2 : t_2) \in \mathbf{P}^3$, with the new equations

$$\mathrm{rk} \begin{pmatrix} x_1 & y_1 & z & t_1 \\ x_2 & y_2 & z_2 & t_2 \end{pmatrix} < 2.$$

Then $\mathcal{S}_2$ is defined by the homogeneous equation

$$x_2 y_2 = t_2^2. \tag{5.3.1}$$

Exactly as in (5.2), the exceptional divisor Σ_2 is a corank 1 quadric surface. The central fibre of $\mathcal{S}_2$ is defined by $t = 0$, or equivalently (in the affine chart $z_1 = 1$) by $t_1 z = 0$. Let us work in the chart $x_2 = 1$. Then we have $z = x_1 z_2, t_1 = x_1 t_2$ and therefore $t_1 z = x_1^2 z_2 t_2$. Hence the central fibre of $\mathcal{S}_2$ is the sum of the three divisors respectively defined in $\mathcal{S}_2$ by the three equations $x_1^2 = 0, z_2 = 0, t_2 = 0$. The first equation gives a double component; the second a simple component; the third gives $x_2 y_2 = 0$, hence two reduced components. The double component is necessarily the exceptional divisor over the vertex of the cone. The upshot is that the central fibre is the non-reduced sum $S'_0 + S''_0 + \Sigma_1 + 2\Sigma_2$, with our usual abuse of notation: $S'_0, S''_0 \subseteq \mathcal{S}_2$ are the blow-ups of $S'_0, S''_0 \subseteq \mathcal{S}_1$ at their smooth point $\mathbf{z}_1$, and $\Sigma_1 \subseteq \mathcal{S}_2$ is the blow-up of $\Sigma_1 \subseteq \mathcal{S}_1$ at its vertex (in particular, it is isomorphic to an $\mathbf{F}_2$ minimal rational ruled surface).

Again, the surface $\mathcal{C}_2$ is defined inside $\mathcal{S}_2$ by the same equations as in (4.2), and we leave it to the reader to check that the central fibres of $\mathcal{S}_2$ and $\mathcal{C}_2$ are arranged as indicated on Figure 11, right-hand-side (compare also with Figure 9, right-hand-side). Note in particular that the curve E_2 is contained in Σ_2 and tangent to the curve $\Sigma_2 \cap \Sigma_1$.

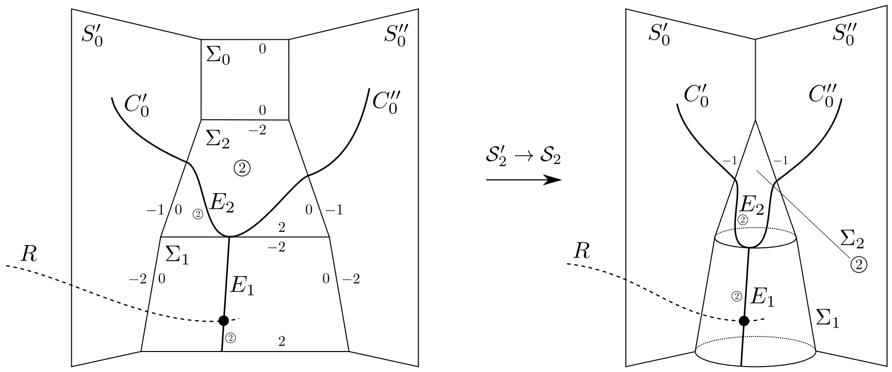

Figure 11: Blow-up of the vertex of Σ_1, $\varepsilon_2 : \mathcal{S}_2 \to \mathcal{S}_1$, and its resolution $\mathcal{S}'_2 \to \mathcal{S}_2$

(5.4) Resolution of singularities of $\mathcal{S}_2$. The threefold $\mathcal{S}_2$ is singular along the curve $S'_0 \cap S''_0$, at the generic point of which it has an A_1 singularity. Let $\mathcal{S}'_2 \to \mathcal{S}_2$ be the blow-up along this curve, with exceptional divisor Σ_0 a $\mathbf{P}^1$-bundle over $S'_0 \cap S''_0$. Then, $\mathcal{S}'_2$ is non-singular, with central fibre

$$S'_0 + S''_0 + \Sigma_1 + 2\Sigma_2 + \Sigma_0; \tag{5.4.1}$$

all summands are isomorphic to their counterparts in $\mathcal{S}_2$ except $\Sigma_2 \subseteq \mathcal{S}'_2$, which is the minimal resolution of the quadric cone $\Sigma_2 \subseteq \mathcal{S}_2$. On Figure 11, left-hand-side, we picture the central fibre of $\mathcal{S}'_2$ with the self-intersections of the intersection curves of any two of its irreducible components. One may check that they concord with the triple-point-formula, which we recall as Lemma (5.5) below.

One may prefer to skip step (5.4); this induces only minor changes.

(5.5) Lemma (Triple Point Formula, see [1, 4])**.** *Let $f : \mathcal{X} \to \mathbf{A}^1$ be a family of surfaces with smooth total space, such that the central fibre X_0 has simple normal crossing support but may be non-reduced. Let Q, Q' be irreducible components of X_0, with respective multiplicities m and m' in X_0, intersecting along the* double curve *B. Then*

$$m'B_Q^2 + mB_{Q'}^2 + \sum_{Q'' \neq Q, Q'} \mathrm{mult}_{X_0}(Q'') \cdot \mathrm{Card}(B \cap Q'') = 0,$$

where Q'' ranges through all irreducible components of X_0 besides Q' and Q''.

In the above statement, $\mathrm{mult}_{X_0}(Q'')$ denotes the multiplicity of Q'' in the central fibre X_0, and $B_Q^2 = \deg(N_{B/Q})$ (resp. $B_{Q'}^2 = \deg(N_{B/Q'})$) is the self-intersection of B as a curve in the surface Q (resp. Q').

(5.6) Base change and resolution. Let $\tilde{\mathcal{S}}_2 \to \mathcal{S}'_2$ be the double covering ramified over the central fibre. It is defined by adding a square root of t, i.e., we introduce a new variable s with the new equation $s^2 = t$, and consider the family of surfaces $\tilde{\mathcal{S}}_2 \to \mathbf{A}^1_s$ defined by the projection map. The proper transform of $\mathcal{C}_2$ in $\tilde{\mathcal{S}}_2$ is isomorphic to the family $\tilde{\mathcal{C}}_2$ of (4.3).

The central fibre of $\tilde{\mathcal{S}}_2$ is, of course, the same as that of $\mathcal{S}'_2$. We let $\tilde{\mathcal{S}}'_2 \to \tilde{\mathcal{S}}_2$ be the blow-up of the Weil divisor Σ_2. This has the effect of replacing Σ_2 by its double cover $\tilde{\Sigma}_2$ branched over the anticanonical divisor $(S'_0 + S''_0 + \Sigma_1 + \Sigma_0) \cap \Sigma_2$, and doesn't change anything else.

Let $\tilde{\mathcal{C}}'_2$ be the proper transform of $\tilde{\mathcal{C}}_2$ in $\tilde{\mathcal{S}}'_2$. It is obtained by replacing the curve E_2 with its double cover $\tilde{E}_2$ branched along the divisor $(C'_0 + C''_0 + 2E_1) \cap E_2$: this is a rational curve with an ordinary double point at the intersection with E_1 (see (4.3.1)). The central fibre is depicted on Figure 12, right-hand-side.

One checks with local computations in the style of those in subsection 4.2[1] that $\tilde{\mathcal{S}}_2'$ is singular only along the four curves at the intersection of one of S_0', S_0'' with one of Σ_0, Σ_1, where it has lines of A_1 double points. Correspondingly, the surface $\tilde{\Sigma}_2$ has four ordinary double points over the double points of the anticanonical divisor of Σ_2.

It follows that the blow-up of $\tilde{\mathcal{S}}_2'$ along the four aforementioned curves is non-singular. We let $\bar{\mathcal{S}}_2 \to \tilde{\mathcal{S}}_2'$ be the corresponding blow-up morphism. The central fibre of $\bar{\mathcal{S}}_2 \to \mathbf{A}^1_s$ is pictured on the left-hand-side of Figure 12 below (see also Figure 9 central-bottom). The self-intersections of the double curves of the central fibre are indicated as well.

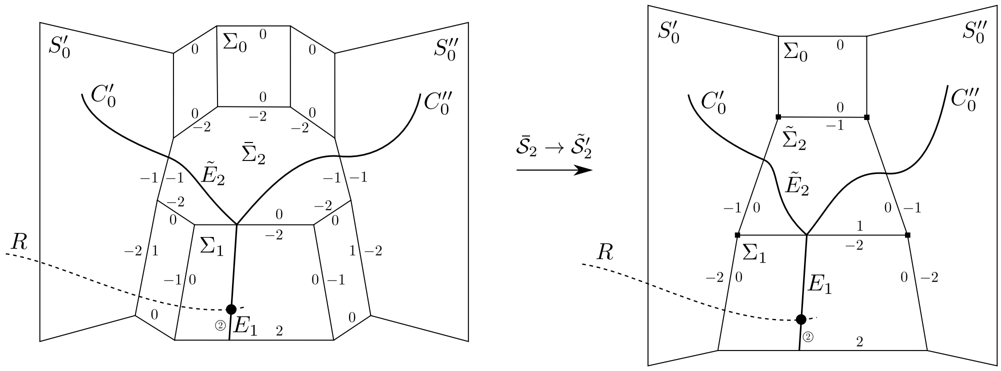

Figure 12: Base change and resolution

(5.7) Conclusion. Eventually, let $\mathcal{S}_3 \to \bar{\mathcal{S}}_2$ be the blow-up along the curve E_1, and Σ_1^1 its exceptional divisor. The curve E_1 is contained in $\tilde{\mathcal{C}}_2'$, hence the proper transform of the latter family of curves in $\mathcal{S}_3$ is isomorphic to $\bar{\mathcal{C}}_2$ in (4.3), see (4.3.1). The result is indicated on Figure 13 below (see also Figure 9, left-hand-side).

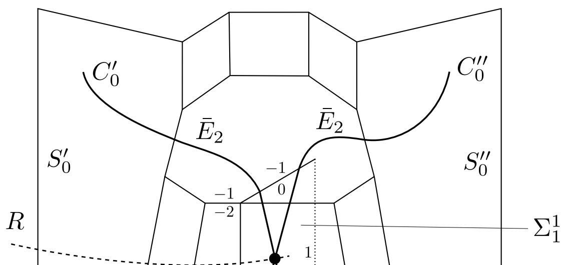

Figure 13: Flag semi-stable reduction, $\mathcal{S}_3$

This is the flag semi-stable reduction we were looking for: the family of surfaces $\mathcal{S}_3$ is semi-stable, and so is the family of curves $\bar{\mathcal{C}}_2$ it contains; thus, we have realized a tacnodal limit curve of 1-nodal curves as a curve with one distinguished node, in a semi-stable limit of the ambient surfaces.

[1] for instance, at the generic point of $S_0' \cap \Sigma_2$, $\tilde{\mathcal{S}}_2$ has an equation equivalent to $a^2 b = u^2$, and $\tilde{\mathcal{S}}_2' \to \tilde{\mathcal{S}}_2$ is locally the blow-up of $a = u = 0$.

One may in turn modify this flag semi-stable reduction in order to obtain whatever variant we fancy. For instance, it is possible to understand in this way the model described in (1.6).

References

Chapters in this Volume

[I] C. Ciliberto, *Limits of nodal curves: generalities and examples.*

[IV] M. Lelli-Chiesa, *Deformations of an m-tacnode into $m-1$ nodes.*

[VIII] C. Galati, *Proving the existence of curves on smooth surfaces by smoothing tacnodes and other singularities on reducible surfaces.*

[IX] Th. Dedieu, *Enumeration of curves by floor diagrams, via degenerations of the projective plane.*

[X] Th. Dedieu, *Enumeration of curves in toric surfaces, tropical geometry, and floor diagrams.*

External References

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Lecture IX
[coming soon]Enumeration of curves by floor diagrams, via degenerations of the projective plane

by Thomas Dedieu

Lecture X
[coming soon]Enumeration of curves in toric surfaces, tropical geometry, and floor diagrams

by Thomas Dedieu

Lecture XI
Enumerative geometry of $K3$ surfaces

by Thomas Dedieu

1 – Introduction

The scope of these notes is to explain various enumerative results about $K3$ surfaces without assuming familiarity with Gromov–Witten theory; in fact, they represent an attempt on my part to understand what these results mean in classical terms.

The enumerative results in question are due to Beauville, Bryan and Leung, Pandharipande, Maulik, Thomas, and others, and confirm conjectures made by Yau–Zaslow, Göttsche, and Katz–Klemm–Vafa. They are listed in (1.1) below.

They fall in three categories: (i) some don't really need Gromov–Witten theory at all either to be formulated or to be proved; (ii) others may be formulated without Gromov–Witten theory but their proofs we know so far heavily rely on techniques from this theory; (iii) the remaining ones require an understanding of Gromov–Witten theory to be fully apreciated. It was therefore unavoidable to assume that the reader nevertheless has a minimal idea of what Gromov–Witten invariants are; it should be more than enough to know the relevant facts listed in (1.2) below.

(1.1) Contents description.. In Section 2, I state a formula giving the number of rational curves in a primitive linear system on a $K3$ surface, and give its proof by Beauville using the universal compactified Jacobian, following the strategy suggested by Yau–Zaslow; this falls in category (i). I also give two geometric interpretations, due to Fantechi–Göttsche–van Straten of the multiplicity with which a given rational curve is counted, namely the topological Euler number of its compactified Jacobian.

This is generalized in Section 3 to a formula giving the number of genus g curves in a primitive linear system passing through g general points, which had been conjectured by Göttsche. I give an outline of its proof by degeneration to an elliptic $K3$ surface due to Bryan–Leung, as detailed as the scope of these notes and the ability of the author permit; it requires the formulation of the result in terms of twisted Gromov–Witten invariants specifically designed for algebraic $K3$ surfaces (see subsection 3.1), and relies among other things on a multiple cover formula for nodal rational curves.

Essentially all remaining results fall in category (iii). The goal of Section 4 is to explain the extension of the Yau–Zaslow formula to non-primitive linear systems, which has been proven by Klemm–Maulik–Pandharipande–Scheidegger (this proof is streamlined in subsection 6.3). This features the Aspinwall–Morrison multiple cover formula, and its application to define corrected Gromov–Witten invariants known as BPS states numbers. I also discuss other degenerate contributions, striving to sort out the relation between the number given by the formula and the actual number of integral rational curves.

Section 5 is devoted to various generalizations. Special care is accorded to the close connection between Gromov–Witten integrals on $K3$ surfaces and curve counts on threefolds. For instance I discuss the Katz–Klemm–Vafa formula, proved by Pandharipande–Thomas, which has to be seen as computing, in any genus, the excess contribution of a $K3$ surface to the Gromov–Witten invariants of any fibered threefold in which it appears as a fibre.

Section 6 introduces Noether–Lefschetz numbers for families of lattice-polarized $K3$ surfaces, and states a result due to Maulik–Pandharipande which shows, on a threefold fibered in lattice-polarized $K3$ surfaces, how these Noether–Lefschetz numbers give an explicit relation between Gromov–Witten invariants of the threefold and of the $K3$ fibres. Eventually, I discuss the application of this formula to the proof of the Yau–Zaslow formula for non-primitive linear systems. It involves a mirror symmetry theorem that enables the computation of Gromov–Witten invariants of anticanonical sections of toric 4-manifolds, as well as modularity results for Noether–Lefschetz numbers following from the work of Borcherds and Kudla–Millson; the latter enable the computation of all Noether–Lefschetz numbers of the family of lattice-polarized $K3$ surfaces considered in the proof.

(1.2) Gromov–Witten theory.. Let X be a projective manifold, say. The starting idea of Gromov–Witten theory is to view genus g curves on X as *stable maps*, i.e., morphisms $f : C \to X$ where C is a connected nodal curve of arithmetic genus g such that there are only finitely many automorphisms ϕ of C satisfying the identity $f \circ \phi = f$. The latter condition is called the *stability condition*, and amounts to the requirement that each irreducible component of arithmetic genus 0 (resp. 1) of C which is contracted by f carries at least 3 (resp. 1) special points, i.e., either intersection points with other irreducible components of C or, if relevant, marked points. An integral embedded curve $C \subseteq X$ is then encoded as the map $f : \bar{C} \to X$ obtained by composing the normalization of C with its embedding in X.

The point in choosing this point of view is to compactify the space of curves on X, which is a prerequisite to the definition of well-formed invariants counting curves on X. There are of course other possible ways to do so; they all come with some specific drawbacks, but this is inevitable. See the enlightening survey [29] for more on this question.

This being said, Gromov–Witten invariants are integrals (or intersection products if one prefers)

$$\int_{[\overline{M}_{g,k}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cup \ldots \cup \mathrm{ev}_k^*(\gamma_k), \tag{1.2.1}$$

where β is a homology class in $\mathrm{H}_2(X, \mathbf{Z})$, $\overline{M}_{g,k}(X,\beta)$ is the moduli space of genus g stable maps $f : C \to X$ such that $[f_*(C)] = \beta$ with k marked points $x_1, \ldots, x_k \in C$, $\mathrm{ev}_i : \overline{M}_{g,k}(X,\beta) \to X$ is the evaluation at the i-th marked point sending $(f : C \to X, x_1, \ldots, x_k)$ to $f(x_i) \in X$ for $i = 1, \ldots, k$, and $\gamma_1, \ldots, \gamma_k$ are cohomology classes in $\mathrm{H}^*(X, \mathbf{Z})$; the *virtual fundamental class* $[\overline{M}_{g,k}(X,\beta)]^{\mathrm{vir}}$ is a rational homology class in $\mathrm{H}_{2\,\mathrm{vdim}}\big(\overline{M}_{g,k}(X,\beta), \mathbf{Q}\big)$ where vdim is the virtual (or expected if one prefers) dimension of $\overline{M}_{g,k}(X,\beta)$

$$\mathrm{vdim}\, \overline{M}_{g,k}(X,\beta) = (\dim X - 3)(1-g) - K_X \cdot \beta + k^1, \tag{1.2.2}$$

and the integral (1.2.1) is defined to be 0 if the degree of the integrand does not match the dimension of the virtual class. The virtual class is the usual fundamental class when the moduli space $\overline{M}_{g,k}(X,\beta)$ has the expected dimension; otherwise it is given by an excess formula (it is the top Chern class of the obstruction bundle when $\overline{M}_{g,k}(X,\beta)$ is non-singular). Typically the cohomology classes $\gamma_1, \ldots, \gamma_k$ are the Poincaré duals to algebraic cycles $\Gamma_1, \ldots, \Gamma_k$ on X; in this case, the condition that the degrees of $\gamma_1, \ldots, \gamma_k$ sum up to $2\,\mathrm{vdim}\, \overline{M}_{g,k}(X,\beta)$ is equivalent to the equality

$$\sum\nolimits_{i=1}^{k} \big(\mathrm{codim}_X(\Gamma_i) - 1\big) = \mathrm{vdim}\, \overline{M}_{g,0}(X,\beta),$$

which means that the incidence conditions imposed by $\Gamma_1, \ldots, \Gamma_k$ to genus g curves in the class β are expected to define a finite number of curves. Therefore, under suitable transversality assumptions, and provided the moduli space $\overline{M}_{g,k}(X,\beta)$ (or equivalently $\overline{M}_{g,0}(X,\beta)$) has the expected dimension, the Gromov–Witten invariant (1.2.1) gives the number of genus g curves in the class β (interpreted as stable maps, and counted with multiplicities) which pass through the cycles $\Gamma_1, \ldots, \Gamma_k$. We will be mainly concerned with the case when all Γ_i's are points, which is the only relevant case when X is a surface.

(1.3) Terminology and conventions.. We always work over the field of complex numbers.

Let C be a curve. Its arithmetic genus, denoted by $p_a(C)$, is the integer $1 - \chi(\mathcal{O}_C)$. If C is reduced, its geometric genus is the arithmetic genus of its normalization, and is denoted by $p_g(C)$. When I write 'genus', this means 'geometric genus'.

A reduced curve C is *immersed* when the differential of its normalization map is everywhere non-degenerate. Concretely this means that C has no cuspidal points; it may have however points of any multiplicity, and non-ordinary singularities (e.g., a tacnode, i.e., a point at which there are two smooth local branches tangent one to another). A *node* is an ordinary double point.

A $K3$ surface S is a smooth surface with trivial canonical bundle and vanishing irregularity; we may occasionally qualify as $K3$ a surface with canonical singularities, the minimal smooth model of which is a smooth $K3$ surface. Let p be a positive integer. A $K3$ surface of genus p is a pair (S, L), where S is a $K3$ surface and L an effective line bundle on S, such that $L^2 = 2p-2$

[1] if one can find a stable $f : C \to X$ corresponding to a point of $\overline{M}_{g,k}(X,\beta)$ such that f is unramified on a dense open subset of X, this may be computed as $\chi(N_f) + k$ where N_f is the normal sheaf of f, i.e., the cokernel of the injective map $T_C \to f^*T_X$; see [38, § 3.4.2] or [29, § $1\frac{1}{2}$] for how to do this in general.

(in particular, the $K3$ surface S is algebraic). Under these assumptions, the complete linear system $|L|$ has dimension p, and its general member is a smooth curve of genus p. The pair (S, L) is *primitive* if the line bundle L is indivisible, i.e., there is no line bundle L' on S such that $L \cong (L')^{\otimes m}$ for some integer $m > 1$.

In the notation of (1.2), we write $\overline{M}_g(X, \beta)$ for $\overline{M}_{g,0}(X, \beta)$. If S is a surface equipped with an effective line bundle $L \to S$, we write $\overline{M}_{g,k}(X, L)$ for $\overline{M}_{g,k}(X, \beta)$ where β is the homology class of the members of $|L|$.

(1.4) Let (S, L) be a $K3$ surface of genus p. Members of $|L|$ with exactly δ nodes as singularities have geometric genus $p - \delta$, and are expected to fill up a locus of codimension δ in $|L|$. For this reason (and because $|L|$ has dimension p), the locus of genus g curves in $|L|$ has expected dimension g; note that this does *not* match with the virtual dimension (1.2.2) of $\overline{M}_g(S, L)$ (see subsection 3.1). One can actually prove that this is indeed the correct dimension, and that the locus of genus g curves is equidimensional (see [11, § 4.2]). This implies that for a general set of g points $x_1, \ldots, x_g \in S$, there is a finite number of genus g curves in $|L|$ passing through all points $x_1, \ldots, x_g$.

(1.5) Acknowledgments.. I thank Jim Bryan and Rahul Pandharipande for patiently answering my naive questions.

2 – Rational curves in a primitive class

In this Section we discuss the following result proved by Beauville [3], following a strategy proposed by Yau and Zaslow [28].

(2.1) Theorem ((Yau–Zaslow, Beauville))**.** *Let (S, L) be a smooth primitive $K3$ surface of genus p_0, and assume that* $\operatorname{Pic} S \cong \mathbf{Z} \cdot L$. *Then there is a finite number N^{p_0} of rational curves in the complete linear system $|L|$, and it is determined by the formula*

$$1 + \sum_{p=1}^{+\infty} N^p q^p = \prod_{n=1}^{+\infty} \frac{1}{(1-q^n)^{24}} = 1 + 24q + 324q^2 + 3200q^3 + \cdots \tag{2.1.1}$$

Of course, N^p has to be understood as the number of rational curves counted with multiplicities for formula (2.1.1) to hold without any further genericity assumption. As we shall see, the multiplicity with which a given integral rational curve counts is the topological Euler number of its compactified Jacobian $e(\bar{J}C)$, which depends only on its singularities, and may be explicitly computed; it is 1 whenever the curve is immersed. For a very general (S, L), all rational curves in $|L|$ are actually nodal by [7], hence (2.1.1) holds without multiplicities.

The assumption about (S, L) that is really used in the proof is that *all* members of $|L|$ are integral curves. Although it may possible to drop the assumption that all members of $|L|$ are irreducible, it seems unavoidable to require that they are all reduced (see however Section 4 for some hints on how to handle this situation).

Theorem (2.1) is a particular case of the more general result that we treat in Section 3.2. We will recall there the relevant facts from the theory of modular forms needed to explore the modular aspects of formula (2.1.1), and give more values of N^p for small p.

The strategy of Yau–Zaslow was inspired by physics; it is an elaboration of the elementary argumentation using Euler numbers presented in Subsec. 2.1. The BPS state counts for Calabi-Yau 3-folds introduced by Gopakumar and Vafa are conjecturally computable in a similar way, see [29, Sec. 2½] for an introduction. The corresponding invariants are considered in Section 4.

2.1 – An elementary topological counting formula

Let X be a complex variety, and $\mathfrak{f}$ a 1-dimensional family of divisors of X, the general member of which is smooth. It is possible to count the number of singular members of $\mathfrak{f}$ using the standard topological Lemma (2.3).

(2.2) Euler number.. Let X be a topological space. Recall that the (topological) Euler number of X is

$$e(X) := \sum_i (-1)^i \dim \mathrm{H}^i(X, \mathbf{Z}),$$

where it is understood that the cohomology groups $\mathrm{H}^i(X, \mathbf{Z})$ should be replaced by the cohomology groups with compact support $\mathrm{H}^i_\mathrm{c}(X, \mathbf{Z})$ whenever X is not compact.

If $F \subseteq X$ is a closed subset, there is a long exact sequence

$$\cdots \to \mathrm{H}^i_\mathrm{c}(X - F, \mathbf{Z}) \to \mathrm{H}^i(X, \mathbf{Z}) \to \mathrm{H}^i(F, \mathbf{Z}) \to \mathrm{H}^{i+1}_\mathrm{c}(X - F, \mathbf{Z}) \to \cdots$$

which implies the additivity formula

$$e(X) = e(X - F) + e(F).$$

(2.3) Lemma. *Let $f : X \to C$ be a surjective morphism from a projective manifold onto a smooth curve. One has*

$$e(X) = e(F_\mathrm{gen})\, e(C) + \sum_{y \in \mathrm{Disc} f} \big(e(F_y) - e(F_\mathrm{gen})\big), \tag{2.3.1}$$

where F_gen and F_y respectively denote the fibres of f over the generic point of B and a closed point $y \in C$, and $\mathrm{Disc} f$ *is the set of points above which f is not smooth.*

This may be applied to the situation described in the introduction of this subsection by replacing X by its blow-up at the base points of the family $\mathfrak{f}$.

Proof. Set $U := X - \bigcup_{y \in \mathrm{Disc} f} F_y$. The map $f : U \to C - \mathrm{Disc} f$ is a topological fibre bundle, hence

$$e(U) = e(C - \mathrm{Disc} f)\, e(F_\mathrm{gen}).$$

The formula then follows by additivity of the Euler number. □

(2.4) When X is a surface and the schematic fibre over y is reduced, the difference $e(F_y) - e(F_\mathrm{gen})$ is determined by the singularities of F_y.

Let D be a reduced projective curve, Σ its singular locus, $\nu : \bar{D} \to D$ its normalization, and $\bar{\Sigma} = \nu^{-1}(\Sigma)$. By additivity of the Euler number, one has

$$\begin{aligned} e(D) &= e(D - \Sigma) + e(\Sigma) \\ &= e(\bar{D} - \bar{\Sigma}) + e(\bar{\Sigma}) + e(\Sigma) - e(\bar{\Sigma}) \\ &= e(\bar{D}) - \big(\mathrm{Card}(\bar{\Sigma}) - \mathrm{Card}(\Sigma)\big). \end{aligned}$$

Let $f : S \to C$ be a surjective morphism from a smooth projective surface to a smooth curve, and consider a point $y \in C$ such that the schematic fibre F_y is reduced. Then the curve F_y has the same arithmetic genus as the general fibre F_{gen}, hence

$$e(\bar{F}_y) = e(F_{\mathrm{gen}}) + 2\delta,$$

where $\delta = p_a(F_y) - p_g(F_y)$ is the sum of the δ-invariants of all singularities of F_y, and

$$e(F_y) - e(F_{\mathrm{gen}}) = 2\delta - \big(\mathrm{Card}(\bar{\Sigma}_y) - \mathrm{Card}(\Sigma_y)\big).$$

This proves the following.

(2.4.1) Lemma.. *In the above notation, the multiplicity with which the fibre F_y is counted in formula* (2.3.1) *is a sum of local multiplicities computed at the singular points of F_y, namely*

$$e(F_y) - e(F_{\mathrm{gen}}) = \sum_{x \in \mathrm{Sing}\, F_y} \left(\delta(F_y, x) - \#\binom{\textit{local branches}}{\textit{of } F_y \textit{ at } x} + 1 \right).$$

This gives for example the local multiplicity 1 for a node, 2 for an ordinary cusp, 3 for a tacnode, and 4 for an ordinary triple point.

(2.5) Application to elliptically fibred $K3$ surfaces.. Let (S, L) be a primitive $K3$ surface of genus 1. Then $|L|$ is a base-point-free pencil of elliptic curves, and all its members are reduced since L is not divisible. Since S is a $K3$ surface, one has $e(S) = 24$; the Euler number of a smooth elliptic curve being 0, it follows from formula (2.3.1) that $|L|$ has 24 singular members, counted with the multiplicity given in Lemma (2.4.1). This agrees with Theorem (2.1).

2.2 – Proof of the Beauville–Yau–Zaslow formula

Let p be a positive integer, (S, L) a smooth primitive $K3$ surface of genus p, and call $\mathfrak{L}$ the complete linear system $|L|$. We assume that all members of $\mathfrak{L}$ are integral.

The relevant feature of the map $S \to \mathbf{P}^1$ considered in (2.5) is that its generic fibre is a complex torus, hence the only fibres with non-vanishing Euler number are those corresponding to a rational curve in the pencil. We let $\mathcal{C}$ be the universal curve over $\mathfrak{L}$, and consider

$$\pi : \bar{\mathcal{J}}^p\mathcal{C} \to \mathfrak{L}$$

the component of the compactified Picard scheme of the family $\mathcal{C} \to \mathfrak{L}$ parametrizing pairs (C, M) where C is any member of $\mathfrak{L}$ and M is a rank 1, torsion-free coherent sheaf of degree p on C. The total space $\bar{\mathcal{J}}^p\mathcal{C}$ is a projective variety of dimension $2p$.

(2.6) Beauville proves that the Euler number of a fibre $\pi^{-1}([C]) = \bar{J}^pC$ is zero if C is not rational, and positive if C is rational (see Propositions (2.8) and (2.11) below). Let us now explain how this shows that the Euler number $e(\bar{\mathcal{J}}^p\mathcal{C})$ is the number of rational curves in $\mathfrak{L}$ counted with multiplicities.

This is basically an elaboration of the proof of Lemma (2.3). There exists a stratification $\mathfrak{L} = \coprod_\alpha \Sigma_\alpha$ by locally closed subsets such that π is locally trivial above each stratum Σ_α [42]. For each α one has

$$e\big(\pi^{-1}(\Sigma_\alpha)\big) = e(\Sigma_\alpha) \times e(J_\alpha)$$

where J_α stands for the fibre of π over any point in the stratum Σ_α. Applying repeatedly the additivity of the Euler number, one then gets

$$e(\bar{\mathcal{J}}^p\mathcal{C}) = \sum_\alpha e\big(\pi^{-1}(\Sigma_\alpha)\big).$$

Eventually, the fact that $e(\bar{J}^pC) = 0$ if the curve C is not rational implies that

$$e(\bar{\mathcal{J}}^p\mathcal{C}) = \sum_{[C]\in\mathfrak{L}_{\mathrm{rat}}} e(\bar{J}^pC), \tag{2.6.1}$$

where $\mathfrak{L}_{\mathrm{rat}}$ is the union of those strata Σ_α, the points of which correspond to rational curves; it is necessarily a finite set (see, e.g., [11, Prop. (4.7)]).

Equation (2.6.1) says that $e(\bar{\mathcal{J}}^p\mathcal{C})$ is the number of rational curves in $\mathfrak{L}$, each rational curve C being counted with the multipicity $e(\bar{J}^pC)$ which is a positive integer; this proves the claim made at the beginning of the paragraph.

(2.7) On the other hand it is possible to identify the Euler number of $\bar{\mathcal{J}}^p\mathcal{C}$ with the knowledge at our disposal, and this together with (2.6) ends the proof of Theorem (2.1).

First, as noted in [28, Example 0.5] $\bar{\mathcal{J}}^p\mathcal{C}$ is a connected component of the moduli space of simple sheaves on the $K3$ surface S, and this shows that it is actually smooth and Hyperkähler.

Next, one proves as follows that $\bar{\mathcal{J}}^p\mathcal{C}$ is birational to $S^{[p]}$, the component of the Hilbert scheme of S parametrizing 0-dimensional subschemes of length p, which as well is a smooth Hyperkähler variety. There is an open subset $U \subseteq \bar{\mathcal{J}}^p\mathcal{C}$ whose points are pairs (C, M) with C a smooth curve and M a non-special line bundle. For such a pair one has $h^0(C, M) = 1$, and this associates to (S, M) the unique divisor D in the complete linear system $|M|$, which has degree p hence may be seen as a point of $S^{[p]}$. On the other hand, the fact that $h^0(C, \mathcal{O}_C(D)) = 1$ implies that D imposes p independent linear conditions to $\mathfrak{L}$, or in other words that C is the unique member of $\mathfrak{L}$ that contains D: this shows that the mapping $(C, M) \mapsto D$ is $1:1$, and ends the proof.

One concludes that $\bar{\mathcal{J}}^p$ and $S^{[p]}$ are actually deformation equivalent by a theorem of Huybrechts [19, p. 65], hence share the same Betti numbers, so that $e(\bar{\mathcal{J}}^p) = e(S^{[p]})$. This finally proves as required that $e(\bar{\mathcal{J}}^p)$ is the coefficient of q^p in the Fourier expansion of $\prod_{n=1}^{\infty}(1-q^n)^{-24}$ as in (2.1.1), thanks to the computation by Göttsche of the Betti and Euler numbers of $S^{[p]}$ for any complex smooth projective surface [17]. The latter computation is based on the by now rather widespread yet wonderful idea of using the Weil conjectures (proved by Deligne) to translate this into the problem of counting the points of $S^{[p]}$ over finite fields.

2.3 – Compactified Jacobian of an integral curve.

Let C be an integral curve. We now turn to the study of the Euler number of the compactified Jacobian $\bar{J}^dC$ of rank one torsion free coherent sheaves of degree d on C. This is required for Beauville's proof of the Yau–Zaslow formula, displayed in Subsection 2.2 above; in particular, we shall justify the assertions at the beginning of (2.6).

As is well-known, the choice of an invertible sheaf of degree d on C induces an isomorphism between $\bar{J}^dC$ and $\bar{J}^0C =: \bar{J}C$, so we will restrict our attention to the latter variety.

(2.8) Proposition. *If C is an integral curve of positive geometric genus, then $e(\bar{J}C) = 0$.*

Proof. There is an exact sequence

$$0 \to H \to JC \to J\tilde{C} \to 0$$

where $\tilde{C}$ is the normalization of C[2], J denotes the Jacobian Pic^0, and H is a product of copies of $(\mathbf{C}, +)$ and $(\mathbf{C}^*, \times)$ (this is standard; see, e.g., [23, Thm. 7.5.19]). It splits as an exact sequence of Abelian groups since H is divisible, so we may find for every positive integer n a subgroup

[2]exceptionally, I do not use the notation $\bar{C}$ in order to avoid unpleasant confusions between $\bar{J}C$ and $J\bar{C}$.

$G_n < JC$ of order n that injects in $J\tilde{C}$ (here we use the fact that $J\tilde{C}$ is not trivial, given by the assumption on the geometric genus of C).

Then [3, Lem. 2.1] tells us that for all $M \in G_n < JC$ and $\mathcal{F} \in \bar{J}C$ the two sheaves $\mathcal{F}$ and $\mathcal{F} \otimes M$ are *not* isomorphic. Thus G_n acts freely on $\bar{J}C$, which implies that n divides $e(\bar{J}C)$. This being true for any n, we conclude that $e(\bar{J}C) = 0$. □

We need the following local construction (cf. [3, §3.6] and the references therein) in order to explicit $e(\bar{J}C)$ for a rational curve C.

(2.9) Let (C, x) be a germ of curve which we assume to be unibranch (i.e., C is analitically locally irreducible at x), and $\tilde{C}$ the normalization of C; there is only one point in the preimage of x, which we also call x. Set $\delta_x := \dim \mathcal{O}_{\tilde{C},x}/\mathcal{O}_{C,x}$ (this is the number by which a singularity equivalent to (C, x) makes the geometric genus drop), and $\mathbf{c}'_x$ the ideal $\mathcal{O}_{\tilde{C}}(-2\delta \cdot x)$[3]. We then consider the two finite-dimensional algebras $A_x := \mathcal{O}_{C,x}/\mathbf{c}'$ and $\tilde{A}_x := \mathcal{O}_{\tilde{C},x}/\mathbf{c}'$.

Eventually, let $\mathbf{G}_x$ be the closed subvariety of the Grassmannian $\mathbf{G}(\delta_x, \tilde{A}_x)$ parametrizing codimension δ_x subspaces of $\tilde{A}_x$ with the additional property of being sub-A_x-modules of $\tilde{A}_x$; it may also be seen as the variety parametrizing codimension δ_x sub-$\mathcal{O}_{C,x}$-modules of $\mathcal{O}_{\tilde{C},x}$. It only depends on the completion $\hat{\mathcal{O}}_{C,x}$, hence only on the analytic type of the singularity (C, x).

(2.10) Let C be a curve. It is *unibranch* if its normalization is a homeomorphism, or equivalently if it is everywhere analytically locally irreducible. Any curve C has a "unibranchization" $\breve{\nu} : \breve{C} \to C$, i.e., there is a unique such partial normalization such that any other partial normalization $\nu' : C' \to C$ with C' unibranch factors through $\breve{\nu}$.

If C is a unibranch curve with singular locus $\Sigma \subseteq C$, the product $\prod_{x\in\Sigma} \mathbf{G}_x$ parametrizes sub-$\mathcal{O}_C$-modules $\mathcal{F} \subseteq \mathcal{O}_{\tilde{C}}$ such that $\dim \mathcal{O}_{\tilde{C},x}/\mathcal{F}_x = \delta_x$ for all x. Such an $\mathcal{F}$ enjoys the property that $\chi(\mathcal{F}) = \chi(\mathcal{O}_C)$, which implies $\mathcal{F} \in \bar{J}C$. This defines a morphism

$$\varepsilon : \textstyle\prod_{x\in\Sigma} \mathbf{G}_x \to \bar{J}C.$$

(2.11) Proposition. (i) [3, Prop. 3.3] *If C is an integral curve, then $e(\bar{J}C) = e(\bar{J}\breve{C})$.*
(ii) [3, Prop. 3.8] *If C is a unibranch rational curve with singular locus Σ, then $e(\bar{J}C) = \prod_{x\in\Sigma} e(\mathbf{G}_x)$.*

If C is not integral, it is certainly not true that $e(\bar{J}C) = e(\bar{J}\breve{C})$. Part (ii) in the above statement is proved by showing that the morphism $\varepsilon : \prod_{x\in\Sigma} \mathbf{G}_x \to \bar{J}C$ is a homeomorphism if C is rational, though in general not an isomorphism. Note that since $\mathbf{G}_x$ is a point when (C, x) is a smooth curve germ, one has $\prod_{x\in\Sigma} e(\mathbf{G}_x) = \prod_{x\in C} e(\mathbf{G}_x)$.

As a consequence of (i), one sees that $e(\bar{J}C) = 1$ for an immersed rational curve C. Part (ii) on the other hand shows that, for any rational curve C (unibranch or not, thanks to (i)), $e(\bar{J}C)$ only depends on the singularities of C. The fact that $e(\bar{J}C) > 0$ for any rational curve C is best seen as an immediate consequence of (2.15). Note moreover that whenever C has only planar singularities (a condition which obviously holds when C is contained in a surface), the satisfactory fact that $e(\bar{J}C)$ actually only depends on the topological type of the singularities of C has been proven by Maulik [25] (see (2.16) below).

[3] we reserve the notation $\mathbf{c}_x$ for the conductor ideal, which contains $\mathbf{c}'_x$.

(2.12) Examples.. [3, § 4] If (C, x) is the germ of curve given by the equation $u^p + v^q = 0$ at the origin in the affine plane, with p and q relatively prime, then

$$e(\mathbf{G}_x) = \frac{1}{p+q}\binom{p+q}{p}. \tag{2.12.1}$$

This is particularly meaningful if one takes into account the constancy of $e(\mathbf{G}_x)$ in topological equivalence classes of planar singularities. As a particular case, one gets $e(\mathbf{G}_x) = \ell+1$ for (C, x) the cuspidal singularity defined by the equation $u^2 + v^{2\ell+1} = 0$.

Using the fact (Proposition (2.11), (ii)) that the local contribution $e(\mathbf{G}_x)$ of a germ (C, x) is the product of the local contributions of all local irreducible branches of (C, x), (2.12.1) is enough to determine the local contribution of any simple curve singularity, see [3, Prop. 4.5].

(2.13) Remark.. The fact that any immersed rational curve counts with multiplicity 1 seems to disagree with the results of subsection. 2.1, see in particular Lemma (2.4.1). However if $|F|$ is a complete pencil of elliptic curves, the assumption that all curves in $|F|$ are integral readily implies that all rational curves in $|F|$ are curves of arithmetic genus 1 with either a node or an ordinary cusp as their unique singular point, in which cases the two multiplicities agree.

On the other hand, if $|F|$ has non-integral members, then Proposition (2.11) does not hold for them. Assume for instance there is a member C of $|F|$ that splits as a cycle of two rational curves, i.e., C is a degenerate fibre of Kodaira type I_2. Then there are two distinct partial normalizations of C with arithmetic genus 0, so from the point of view of stable maps — which seems to be the appropriate one, see (2.17.1) and Lemma (3.4) —, the curve C should count with multiplicity 2, in agreement with Lemma (2.4.1).

(2.14) Remark.. Returning to the case of a p-dimensional linear system $\mathfrak{L}$ with all members integral, Beauville makes a remark similar to (2.13), deeming "rather surprising" the fact that "some highly singular [immersed] curves count with multiplicity one", and considers the case $p = 2$ to provide a confirming example. I shall add some more details about this example in (2.14.2) below.

A good conceptual explanation of this fact is, as we have already mentioned, that the numbers N^p should be seen as counting stable maps rather than embedded curves, and stable maps don't make any difference between nodal and arbitrary immersed curves. Yet this does not give a satisfactory "embedded" explanation. I propose a particular instance of such an explanation in (2.14.1) below; ultimately, it relies on the smoothness of the equigeneric deformation space of an immersed singularity (see (2.18) in the next subsection).

(2.14.1) Let S be a non-degenerate surface in $\mathbf{P}^3$. The linear system $|L| := |\mathcal{O}_S(1)|$ identifies with the dual projective space $\check{\mathbf{P}}^3$, the locus of singular curves in $|L|$ with the dual surface $\check{S} \subseteq \check{\mathbf{P}}^3$ (which by definition parametrizes hyperplanes in $\mathbf{P}^3$ tangent to S), and the closure of the locus of 2-nodal (resp. 1-cuspidal) curves with the ordinary double curve D_b (resp. cuspidal double curve D_c) of $\check{S}$.

Of course, the $K3$ surfaces in $\mathbf{P}^3$ are quartic hypersurfaces, and their hyperplane sections have arithmetic genus 3, so that rational curves among them are expected to be 3-nodal (at any rate, they have δ-invariant 3). Still, I shall discuss the geometry of the locus of 2-nodal curves, as it gives in my opinion a clearer picture of what is going on.

It is classically known [36, § 612], see [34, 35] for more up-to-date treatments[4], that the locus of tacnodal curves in $|L|$ consists of those intersection points of D_b and D_c at which D_b is smooth and D_c has a cuspidal point. This implies that 1-tacnodal curves, as they correspond to

[4] beware that in [35, p. 391] the geometries of $T_xS \cap S$ in cases d) and e) have been mistakenly exchanged.

simple points of D_b, count for one co-genus 2 curve as do ordinary 2-nodal curves; they count however for two cuspidal curves.

The local description of the dual $\check{S}$ at a tacnodal curve reflects the geometry of various strata in the semi-universal deformation space of a tacnode. One may obtain with the same ingredients a local description of $\check{S}$ around a point corresponding to an immersed rational curve, e.g., a curve with one tacnode and one node, or one oscnode, and it would confirm that it counts for one rational curve only. I will not undertake this here.

(2.14.2) Let (S, L) be a general $K3$ surface of genus $p = 2$; then S is a double covering of the plane ramified over a smooth sextic curve B, and the members of $|L|$ are the pull-back of lines. Rational, 2-nodal, curves correspond to bitangent lines of B.

When B is Plücker general, i.e., when its dual curve $\check{B}$ has only nodes and cusps as singularities, the number of bitangents to B may be computed using the Plücker formulæ. It will be useful to unfold this explicitely, in order to handle more special cases later on. The dual curve $\check{B}$ as degree $6 \times (6-1) = 30$, and its cusps correspond to the inflection points of B; the latter are the intersection points of B with its Hessian hypersurface, which has degree $3 \times (6-2) = 12$; it follows that $\check{B}$ has $\check{\kappa} = 72$ cusps. The number $\check{\delta}$ of nodes of $\check{B}$ may then be derived arguing that the geometric genus of $\check{B}$ equals that of B, which is 10. This gives $\check{\delta} = p_a(\check{B}) - 10 - 72 = 324$, in accord with (2.1.1).

Now assume that B has a hyperflex o of order 4, i.e., the tangent line $\mathbf{T}_{B,o}$ has contact of order 4 with B at o; the pull-back of this line to S is an immersed rational curve, with one ordinary tacnode as only singularity. I shall now explain why it counts as one ordinary rational curve only. A local computation shows that the hyperflex o corresponds to a singularity on $\check{B}$ of the kind $y^4 = x^3$ at the point $\check{o} := (\mathbf{T}_{B,o})^{\perp}$. Such a singularity has δ-invariant 3, i.e., it makes the genus of $\check{B}$ drop by 3 with respect to the arithmetic genus $p_a(\check{B})$. On the other hand, B has a contact of order 2 with its Hessian at o, so it amounts for two ordinary flexes, and correspondingly $\check{o}$ amounts for two cusps of $\check{B}$. The fact that the δ-invariant of $(\check{B}, \check{o})$ be 3 then implies that $\check{o}$ amounts for one node of $\check{B}$, and correspondingly the line $\mathbf{T}_{B,o}$ amounts for one bitangent only, hence the pull-back of $\mathbf{T}_{B,o}$ amounts for one rational curve only. To sum up, the tangent line $\mathbf{T}_{B,o}$ amounts at the same time for one bitangent and two flex tangents, similar to what happened in (2.14.1).

This kind of elaboration on the Plücker formulæ has recently been formalized by Kulikov [21] for integral plane curves with arbitrary singularities.

In the next subsection we will see two results of Fantechi, Göttsche, and van Straten which extend and confirm the considerations of Remark (2.14) above.

2.4 – Two fundamental interpretations of the multiplicity

In this last subsection, I state two enlightening geometric interpretations of the local multiplicities $e(\mathbf{G}_x)$ defined in the previous subsection 2.3, and their global counterpart the product $\prod_{x \in C} e(\mathbf{G}_x)$. They have been obtained by Fantechi, Göttsche and van Straten [12].

(2.15) Theorem ([12, Thm. 1]). *Let (C, x) be a reduced plane curve singularity, and $\mathbf{G}_x$ be as in (2.9). Then the topological Euler number $e(\mathbf{G}_x)$ equals the multiplicity at the point $[(C, x)]$ of the equigeneric locus $\mathrm{EG}(C, x)$ in the semi-universal deformation space of the singularity (C, x).*

Recall that the *equigeneric locus* $\mathrm{EG}(C, x)$ is defined as the reduced subscheme of the semi-universal deformation space of the singularity (C, x) supported on those points corresponding to singularities with the same δ-invariant as (C, x); see, e.g., [II] for more details.

(2.16) Together with Proposition (2.11), this implies that for C an integral rational curve with only planar singularities, the topological Euler number $e(\bar{J}C)$ of the compactified Jacobian of C equals the multiplicity at the point $[C]$ of the equigeneric stratum $\mathrm{EG}(C)$ in a semi-universal deformation space of C. The latter result has been subsequently generalized by Shende [40] to (the closures of) all δ-constant strata in the semi-universal deformation space of C.

Given a reduced plane curve singularity (C, x), there exists a rational curve $\tilde{C}$ with (C, x) as its only singularity (this follows for instance from [27]), and one then has $\mathbf{G}_x \cong \bar{J}\tilde{C}$. This, in conjunction with Maulik's main theorem in [25] gives the aforementioned constancy of the invariant $e(\mathbf{G}_x)$ on topological equivalence classes of plane curve singularities. Similarly, Shende and Maulik results together give the constancy on topological equivalence classes of the multiplicities at $[(C, x)]$ of (the closures of) all δ-constant strata in the semi-universal deformation space of (C, x) (see [25, § 6.5]).

Let C be an integral curve of geometric genus g. Recall that $\overline{M}_g(C, [C])$ is the space of genus g stable maps with target C and realizing the class $[C] \in \mathrm{H}_2(C, \mathbf{Z})$. This is a 0-dimensional scheme, which contains a single closed point, corresponding to the normalization $[\nu : \bar{C} \to C]$ of C.

(2.17) Theorem ([12, Thm. 2])**.** *In the above notation, assume the curve C has only planar singularities. Then the length of the 0-dimensional scheme $\overline{M}_g(C, [C])$ equals the multiplicity at $[C]$ of the semi-universal deformation space of the curve C.*

Together with Theorem (2.15) above, this implies that the length of $\overline{M}_g(C, [C])$ equals $\prod_{x \in C} e(\mathbf{G}_x)$.

When C is rational, this is precisely $e(\bar{J}C)$. It is thus tempting to interpret Theorem (2.17) as telling us that what the Yau–Zaslow formula (2.1.1) really computes are the numbers of genus 0 stable maps realizing primitive classes on $K3$ surfaces.

Actually, if C is an isolated genus g curve in a smooth manifold X, then $\overline{M}_g(C, [C])$ is a subscheme of $\overline{M}(X, [C])$ and the length of the former scheme is a lower bound for the length of the latter at the normalization of C. For rational curves on $K3$ surfaces, Fantechi, Göttsche and van Straten show that this is in fact an equality.

(2.17.1) [12, Thm. 2] *Let C be an integral rational curve contained in a smooth $K3$ surface S. Then the topological Euler number $e(\bar{J}C)$ equals the length of the space of stable maps $\overline{M}_0(S, [C])$ at the closed point corresponding to the normalization of C.*

Lemma (3.4) in the next Section somehow deals with the same question for curves of any genus on a $K3$ surface. We refer to [11, § 2.2] for a general analysis, given an integral curve C on a smooth surface S, of the local relationship between $\overline{M}_g(S, [C])$ and the Severi variety of equigeneric deformations of C in S.

(2.18) As a corollary of Theorem (2.15) and Proposition (2.11), one obtains that if (C, x) is an immersed planar curve singularity, then the equigeneric locus $\mathrm{EG}(C, x)$ in the space of semi-universal deformations is smooth at the point $[(C, x)]$.

Certainly, this is merely a baroque way to prove a result otherwise accessible by a more straightforward argument. Still, I don't know wether the converse holds.

(2.18.1) Question.. *Let (C, x) be a unibranch non-immersed planar curve singularity. Is it true that the equigeneric locus $\mathrm{EG}(C, x)$ is singular at the point $[(C, x)]$? equivalently, is it true that $e(\mathbf{G}_x) > 1$?*

I believe this is related to the question asked in [11, (3.16)]: let (C, x) be a non-immersed planar singularity; is it true that the respective pull-back of the adjoint and equisingular ideals to the normalization $\bar{C}$ are different?

3 – Curves of any genus in a primitive class

3.1 – Reduced Gromov–Witten theories for $K3$ surfaces

(3.1) A vanishing phenomenon.. It happens that all Gromov–Witten invariants of $K3$ surfaces are trivial. The fundamental reason for this is that Gromov–Witten invariants are deformation invariant (and this is indeed a desirable feature of any well-behaved counting invariants), and there exist non-algebraic $K3$ surfaces, which in general do not contain any curve at all.

Somewhat more concretely, the explanation is that the virtual and the actual dimensions of the moduli spaces of stable maps on $K3$ surfaces do not match, as we already pointed out in (1.4). Let S be a $K3$ surface, and $C \subseteq S$ an integral curve of geometric genus g. Consider the stable map $f : \bar{C} \to S$ obtained by composing the normalization $\bar{C} \to C$ with the inclusion $C \subseteq S$. The normal sheaf N_f of f is isomorphic to the canonical bundle $\omega_{\bar{C}}$, and therefore $h^0(N_f) = g$ and $h^1(N_f) = 1$. It follows that the virtual dimension of $\overline{M}_g(S, [C])$ is $g - 1$, whereas the curve C actually moves in a g-dimensional family of curves of genus g on S (see [11, § 4.2] for more details). This implies that the Gromov–Witten invariants counting genus g curves on S passing through the appropriate number of points (namely g) vanishes for mere degree reasons.

The following two paths have been successfully followed to circumvent this phenomenon, and define modified invariants for algebraic $K3$ surfaces which capture the relevant enumerative information.

(3.2) Invariants of families of symplectic structures.. [1, § 2–3] This has been chronologically the first workaround to be proposed, and enabled the counting of curves of any genus in a primitive class on a $K3$ surface reported on in subsection 3.2 below.

Let S be a polarized $K3$ surface. The idea here is really to take into account the existence of non-algebraic deformations of S. To this effect, instead of counting curves directly on S, one counts curves in a family of Kähler surfaces defined over the 2-sphere $\mathbf{S}^2$, canonically attached to S, and in which roughly speaking S is the only one to be algebraic, so that all the curves we count are actually concentrated on S.

This family of Kähler surfaces is the twistor family of S (cf. [37, p. 124]): the polarization on S determines a Kähler class α, and Yau's celebrated theorem asserts that there is a unique Kählerian metric g in α with vanishing Ricci curvature. Then the holonomy defines an action of $\mathbf{H}$ (the field of quaternion numbers) on the holomorphic tangent bundle TS by parallel endomorphisms. The quaternions of square -1 define those complex structures on the differentiable manifold S for which the metric g remains Kählerian. There is a 2-sphere worth of such quaternions, and it parametrizes the family we are interested in.

(3.3) Reduced Gromov–Witten theory.. [26, 2.2] (see also [24]). In this second approach, the idea is to plug in the fact that, for a stable map f as in (3.1), the space $\mathrm{H}^1(\bar{C}, N_f)$ although non-trivial does not contain any actual obstruction to deform f as a map with target S, following for instance Ran's results on deformation theory and the semiregularity map. To this end, Maulik and Pandharipande define a suitable perfect obstruction theory which they dub *reduced*, and which provides, following the construction pioneered by Behrend and Fantechi, a reduced virtual fundamental class $[\overline{M}_g(S, \beta)]^{\mathrm{red}}$ for all integers $g \geqslant 0$ and algebraic class $\beta \in \mathrm{H}_2(S, \mathbf{Z})$, which has the appropriate (real) dimension $2g$.

This in turn gives reduced Gromov–Witten invariants, by replacing the virtual fundamental class by its reduced version in the integral (1.2.1). They are invariant under *algebraic* deformations of $K3$ surfaces.

3.2 – The Göttsche–Bryan–Leung formula

In this subsection, I discuss a result of Bryan and Leung [1] giving the number of curves in a primitive linear system on a $K3$ surface that have a given genus and pass through the appropriate number of base points. The formula was conjectured by Göttsche [18] as a particular case of a more general framework.

Let

$$N_g^p := \int_{[\overline{M}_{g,g}(S,L)]^{\mathrm{red}}} \mathrm{ev}_1^*(\mathrm{pt}) \cup \cdots \cup \mathrm{ev}_g^*(\mathrm{pt})$$

be the reduced Gromov–Witten invariant counting curves of genus g in the linear system $|L|$ on a primitive $K3$ surface (S, L) of genus p ($\mathrm{pt} \in \mathrm{H}^4(S, \mathbf{Z})$ is the point class, and the $\mathrm{ev}_1, \ldots, \mathrm{ev}_g$ are the evaluation maps $\overline{M}_{g,g}(S, L) \to S$). The following result tells us that these invariants do indeed count curves.

(3.4) Lemma. *The invariants N_g^p are strongly enumerative, in the following sense: let (S, L) be a very general primitive $K3$ surface of genus p; then N_g^p is the actual number of genus g curves in $|L|$ passing through a general set of g points, all counted with multiplicity* 1.

Proof. The key fact is that genus g curves on a $K3$ surface move in g-dimensional families, see e.g., [11, Prop. (4.7)]; for $g > 0$, this implies by a deformation argument that the general member of such a family is an immersed curve [11, Prop. (4.8)]. The same holds for $g = 0$ by the more difficult result of Chen [7], which requires the generality assumption and asserts that all rational curves in $|L|$ are nodal.

Let $\mathbf{x} = (x_1, \ldots, x_g)$ be a general (ordered) set of g points on S. The invariant N_g^p may be computed by integration against a virtual class on the cut-down moduli space $\overline{M}(S, \mathbf{x}) \subseteq \overline{M}_{g,g}(S, L)$ consisting of those genus g stable maps sending the i-th marked point to $x_i \in S$ for $i = 1, \ldots, g$ [1, Appendix A].

Let $[f : C \to S] \in \overline{M}(S, \mathbf{x})$. Thanks to the generality assumption on (S, L), we may and will assume that L generates the Picard group of S. The condition $f_*C \in |L|$ thus imposes that f_*C is an integral cycle, hence that f contracts all irreducible components of C but one, and restricts to a birational map on the latter component, which we will call C_1. Now the points $x_1, \ldots, x_g$, being general, impose g general independent linear conditions on $|L|$, which implies that the curve $f(C) = f(C_1)$ must have geometric genus at least g by the key fact mentioned at the beginning of the proof. Therefore C_1 as well must have geometric genus at least g, and because of the inequality of arithmetic genera

$$p_a(C_1) \leqslant p_a(C) = g,$$

this implies that C_1 is smooth of genus g; moreover, the stability conditions then imply that $C = C_1$, hence f is the normalization of the integral genus g curve $f(C_1)$.

This already tells us that the space $\overline{M}(S, \mathbf{x})$ is 0-dimensional, and isomorphic as a set to the space of genus g curves in $|L|$ passing through $x_1, \ldots, x_g$. Since $\overline{M}(S, \mathbf{x})$ has the expected dimension, the virtual class on it simply encodes its schematic structure. It follows that the number N_g^p is weakly enumerative, i.e., it gives the number of genus g curves passing through $x_1, \ldots, x_g$ counted with multiplicities.

The fact that these multiplicities all equal 1 follows from the fact that all the curves we consider are immersed, as follows. Let $[f : C \to S] \in \overline{M}(S, \mathbf{x})$ as above. Since f is an immersion, it has normal bundle

$$N_f = f^*\omega_S \otimes \omega_C = \omega_C$$

by [11, (2.3)], hence $\mathrm{h}^0(N_f) = g$ and the full moduli space $\overline{M}_{g,0}(S, L)$ is smooth of dimension g at the point $[f]$. By [11, Lemma (2.5)], there is a surjective map e from a neighbourhood of $[f]$ in $\overline{M}_{g,0}(S, L)$ onto a neighbourhood of $f(C)$ in the locally closed subset of $|L|$ parametrizing genus g curves. By generality of the points $x_1, \dots, x_g$, the latter space is smooth of dimension g at the point $[f(C)]$. Therefore e is a local isomorphism at $[f]$, and this implies that the scheme $\overline{M}(S, \mathbf{x})$ is reduced at $[f]$, which shows that the stable map f counts with multiplicity 1. $\square$

For all integers $g \geqslant 0$, set

$$F_g(q) := \sum_{p=g}^{+\infty} N_g^p \, q^p \tag{3.5.1}$$

as a formal power series in the variable q, where we set N_0^0 by convention so that F_0 equals the power series of (2.1.1). Beware the shift in degree between the definition (3.5.1) of the series F_g and that given in [1]. Note that $N_g^p = 0$ whenever $p < g$.

(3.5) Theorem ((Bryan–Leung))**.** *The power series F_g is the Fourier expansion of*

$$\left(\sum_{n=1}^{+\infty} n\sigma_1(n)\, q^n\right)^g \prod_{m=1}^{+\infty} \frac{1}{(1-q^m)^{24}}, \tag{3.5.2}$$

where $\sigma_1(n) := \sum_{d|n} d$ is the sum of all positive integer divisors of n.

This of course gives the possibility to explicitly compute as many numbers N_g^p as we want. Table 1 (p. 201) gives sample values for small p and g. Note that since columns are indexed by $\delta := p - g$, the Fourier coefficients of a given F_g are read along a diagonal; this gives for instance F_0 as in (2.1.1),

$$F_1(q) = q + 30q^2 + 480q^3 + 5460q^4 + \cdots$$

and so on.

We discuss the proof of Theorem (3.5) in subsection 3.3 below.

(3.6) Modularity.. There is a modular form theoretic aspect to formula (3.5.2), which I explicitly state in subparagraph (3.6.5) below. There is somehow a meaning to this, but I will not try to discuss it here. I will however make a couple of points, at least to set things right and introduce notation for further use (I follow [39]).

(3.6.1) For every integer $k > 1$, define the k-th *Eisenstein series* to be

$$G_k(z) := \sum_{\substack{(m,n)\in\mathbf{Z}^2:\\ (m,n)\neq(0,0)}} \frac{1}{(m+nz)^{2k}};$$

it is a modular form of weight $2k$ [39, Prop. VII.4], which means that it is holomorphic and $G_k(z)dz^k$ is invariant under the action of $\mathrm{PSL}_2(\mathbf{Z})$. Its Fourier expansion at infinity is

$$G_k(z) = \frac{2^{2k}}{(2k)!}B_k\pi^{2k} + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{+\infty}\sigma_{2k-1}(n)q^n,$$

p \ δ	1	2	3	4	5	6	7	8	9
1	24								
2	30	324							
3	36	480	3200						
4	42	672	5460	25650					
5	48	900	8728	49440	176256				
6	54	1164	13220	88830	378420	1073720			
7	60	1464	19152	150300	754992	2540160	5930496		
8	66	1800	26740	241626	1412676	5573456	15326880	30178575	
9	72	2172	36200	371880	2499648	11436560	36693360	84602400	143184000
10	78	2580	47748	551430	4213332	22116456	81993600	219548277	432841110
11		3024	61600	791940	6808176	40588544	172237344	531065070	1210781880
12			77972	1106370	10603428	71127680	342358560	1205336715	3154067950
13				1508976	15990912	119665872	647773200	2582847180	7698660544
14					23442804	194196632	1172896512	5255204625	17710394230
15						305225984	2041899840	10205262330	38607114200
16							3431986848	19002853575	80149394030
17								34070137272	159184435520
18									303705014550

Table 1: First values of N_g^p $(\delta = p - g)$

where $q = e^{2\pi i z}$, $\sigma_k(n) = \sum_{d|n} d^k$, and B_k is the k-th Bernoulli number, defined by the formula

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{+\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}$$

[39, Prop. VII.8]. We set

$$\begin{aligned} E_k(z) := \frac{(2k)!}{2^{2k} B_k \pi^{2k}} G_k(z) &= 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{+\infty} \sigma_{2k-1}(n) q^n \\ &= 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{+\infty} n^{2k-1} \frac{q^n}{1-q^n}. \end{aligned}$$

(3.6.2) Define

$$\Delta(z) := \big(60 G_2(z)\big)^3 - 27\big(140 G_3(z)\big)^2,$$

the discriminant of the cubic polynomial $4X^3 - 60G_2 X - 140 G_3$ divided by 16. It is a modular form of weight 12 vanishing at infinity, and it is a theorem of Jacobi [39, Thm. VII.6] that

$$\Delta(z) = (2\pi)^{12} q \prod_{n+1}^{+\infty} (1 - q^n)^{24}.$$

(3.6.3) In the $k = 1$ case, we set

$$G_1(z) := \sum_{n \in \mathbf{Z}} \sum_{\substack{m \in \mathbf{Z}: \\ (m,n) \neq (0,0)}} \frac{1}{m + nz^2}$$

(note that the order of summation is significant). It has the Fourier expansion at infinity

$$G_1(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{+\infty} \sigma_1(n) q^n$$

[39, § VII.4.4], and we set

$$E_1(z) := \frac{3}{\pi^2} G_1(z) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) q^n.$$

One has the identity [39, § VII.4.4]

$$\frac{d\Delta}{\Delta} = 2\pi i E_1(z) dz.$$

(3.6.4) The function G_1 is *not* a modular form, but still it does satisfy a functional equation close to that equivalent to the invariance of $G_1(z)dz$ under the action of $\mathrm{PSL}_2(\mathbf{Z})$ [4, Prop. 6 p. 19]. For this reason, it is called a *quasi-modular* form.

One may then define the ring of quasi-modular forms as the $\mathbf{C}$-algebra generated by G_1 and the algebra of modular forms (see [4] for a more intrinsic definition). Since the ring of modular forms is generated by G_2 and G_3 [39, Cor. VII.2], the ring of quasi-modular forms may be concretely described as $\mathbf{C}[G_1, G_2, G_3]$. The ring of quasi-modular forms is closed under differentiation by the operator

$$D := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$$

[4, Prop. 15 p. 49].

(3.6.5) Quasi-modularity of $qF_g(q)$.. Taking into account the various stunning formulæ above, (3.5.2) may be rewritten as

$$qF_g(q) = \left(-\frac{1}{24}q\frac{dE_1}{dq}\right)^g \left(\frac{\Delta}{(2\pi)^{12}}\right)^{-1},$$

from which it follows that $qF_g(q)$ is a quasi-modular form, with a simple pole at infinity (i.e., at $q = 0$) if $g = 0$.

3.3 – Proof of the Göttsche–Bryan–Leung formula

As Theorem (3.5) really is about counting actual curves, as Lemma (3.4) attests, one may prefer in the first place to avoid the complications of Gromov–Witten theory to prove it. It will yet be clear in a moment that this is not really possible as far as the proof proposed by Bryan and Leung goes, as the latter fundamentally relies on the agile possibilities featured by Gromov–Witten theory, precisely as a reward to the aforementioned complications.

(3.7) Degeneration to an elliptic $K3$.. Let $g \geqslant 0$, $p > 0$ be integers. By deformation invariance, we are free to compute the number N_g^p on our favourite primitively polarized $K3$ surface (S, L) of genus p. We let S be an elliptic $K3$ surface with a section E, and denote by F the class of the ellitic fibres; the intersection form (or its restriction to the subspace $\langle E, F\rangle$) is given in the basis (E, F) by the matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

We set $L := \mathcal{O}_S(E + pF)$. Then $L^2 = 2p - 2$, and L is a primitive polarization of genus p.

We shall compute the numbers N_g^p on the pair (S, L). Note that while in the proof of the Yau–Zaslow formula above we considered a construction generalizing the structure of Jacobian fibration of elliptic $K3$ surfaces, this time we really degenerate to an actual elliptic $K3$.

(3.8) The linear system $|E + pF|$ has dimension p and consists solely of reducible curves $E + F_1 + \cdots + F_p$, where the F_i's are (not necessarily distinct) in the class F. From this it readily follows that if we fix a general set of g points $\mathbf{y} = (y_1, \ldots, y_g)$ on S, then the moduli space

$$\overline{M}_{g,\mathbf{y}}(S, E + pF) := \overline{M}_{g,g}(S, E + pF) \cap \mathrm{ev}_1^*(y_1) \cap \ldots \cap \mathrm{ev}_g^*(y_g)$$

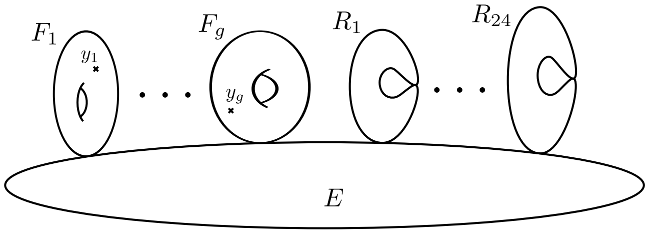
of genus g stable maps passing through the points $y_1, \ldots, y_g$ decomposes as the disjoint union

$$\coprod_{\mathbf{a},\mathbf{b}} \overline{M}_{\mathbf{a},\mathbf{b}},$$

where $\mathbf{a} = (a_1, \ldots, a_{24})$ and $\mathbf{b} = (b_1, \ldots, b_g)$ range through $\mathbf{Z}^{24}_{\geqslant 0}$ and $\mathbf{Z}^g_{>0}$ respectively subject to the condition that $\sum_i a_i + \sum_j b_j = p$, and $\overline{M}_{\mathbf{a},\mathbf{b}}$ is the moduli space of genus g stable maps $(f : C \to S, x_1, \ldots, x_g)$ such that $f(x_j) = y_j$ for $j = 1, \ldots, g$ and

$$f_*C = E + \sum_{i=1}^{24} a_i R_i + \sum_{j=1}^{g} b_j F_j,$$

$R_1, \ldots, R_{24}$ being the 24 rational members of the pencil $|F|$, and F_j being the unique member of $|F|$ containing the point y_j for $j = 1, \ldots, g$ (see figure below). We may and do assume that all members of $|F|$ are irreducible, and the R_i's are 1-nodal.



(3.9) Partition function.. For all positive integers n, we let $p(n)$ be the *number of partitions* of n, i.e., the number of ways to write $n = \lambda_1 + \cdots + \lambda_k$, $\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 1$ (k is not fixed). The numbers $p(n)$ may be computed using the generating series

$$\begin{aligned} 1 + \sum_{n=1}^{+\infty} p(n)\, t^n &= (1 + t^1 + t^{1+1} + t^{1+1+1} + \cdots) \times (1 + t^2 + t^{2+2} + t^{2+2+2} + \cdots) \\ &\quad \times (1 + t^3 + t^{3+3} + t^{3+3+3} + \cdots) \times \cdots \\ &= \prod_{n=1}^{+\infty} \frac{1}{1 - t^n}. \end{aligned} \tag{3.9.1}$$

See [13, Chap. 4] for much more about this.

The following result is the key to formula (3.5.2) for (S, L) an elliptic $K3$ surface as set-up in § (3.7)–(3.8).

(3.10) Proposition. *The contribution of $\overline{M}_{\mathbf{a},\mathbf{b}}$ to N_g^p is*

$$\left(\prod_{i=1}^{24} p(a_i) \right) \left(\prod_{j=1}^{g} b_j \sigma_1(b_j) \right). \tag{3.10.1}$$

The latter results yields Formula (3.5.2) after a series of elementary manipulations which I don't reproduce here (see [1, p. 383] for details). Note that the identity (3.9.1) comes into play. The rest of this subsection is dedicated to the proof of Proposition (3.10).

(3.11) Enumeration of elliptic multiple covers.. We first explain the factors $b_j \sigma_1(b_j)$ in (3.10.1). They are simple to understand, as they are of a combinatorial nature.

Let $f : C \to S$ be a member of $\overline{M}_{\mathbf{a},\mathbf{b}}$. It is necessarily shaped as follows: the curve C consists of (i) a smooth rational curve mapped isomorphically to the section $E \subseteq S$, which we will abusively call E as well, (ii) g smooth elliptic curves $G_1, \ldots, G_g$, pairwise disjoint and each attached at one point to E, and (iii) 24 trees of smooth rational curves $T_1, \ldots, T_{24}$, pairwise disjoint, each disjoint from the G_j's and attached to E at one point; for $j = 1, \ldots, g$, f maps G_j to the elliptic fibre F_j with degree b_j and there is a marked point $x_j \in G_j$ mapped to y_j, and for $i = 1, \ldots, 24$ one has $f_* T_i = a_i R_i$.

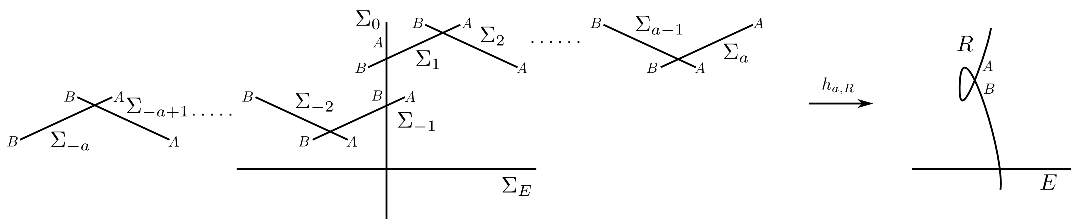
For all j, we may fix the intersection point with E as the origin of G_j and F_j respectively, which makes $f|_{G_j} : G_j \to F_j$ a degree b_j homomorphism of elliptic curves. Such homomorphisms are in $1:1$ correspondence with index b_j sublattices of the lattice defining F_j as a complex torus, and the number of such sublattices is $\sigma_1(b_j)$ [39, § VII.5.2]. Next, there are b_j possibilities to choose the marked point x_j in the preimage of y_j in G_j.

Once the data of the homomorphisms $G_j \to F_j$ and the marked points $x_j \in G_f$ is fixed, the corresponding sub-moduli space of $\overline{M}_{\mathbf{a},\mathbf{b}}$ decomposes as a product

$$\prod_{i=1}^{24} \overline{M}_{a_i\mathbf{e}_i,0},$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_{24} = (0, \ldots, 0, 1)$ denotes the canonical basis of $\mathbf{Z}^{24}$. The moduli space $\overline{M}_{\mathbf{a},\mathbf{b}}$ thus consists of $\prod_{j=1}^{g} b_j\sigma_1(b_j)$ disjoint copies of the space $\prod_{i=1}^{24} \overline{M}_{a_i\mathbf{e}_i,0}$, and the rest of the proof of Proposition (3.10) consists in showing that each of those contributes by $\prod_{i=1}^{24} p(a_i)$ to N_g^p. Before we can proceed, we need the following.

(3.12) Description of $\overline{M}_{a_i\mathbf{e}_i,0}$.. Let us start by defining a model stable map $h_{a,R} : \mathbf{\Sigma}_a \to S$ for all positive integers a and 1-nodal rational curves $R \in \{R_1, \ldots, R_{24}\}$. The curve $\mathbf{\Sigma}_a$ is a tree of $2a + 2$ smooth rational curves $\Sigma_E, \Sigma_{-a}, \ldots, \Sigma_0, \ldots \Sigma_{+a}$ as depicted on the figure below.



The map $h_{a,R}$ is chosen so that it restricts to an isomorphism $\Sigma_E \cong E$ (hence from now on we will denote Σ_E by E) and to $2a + 1$ copies $\Sigma_i \to R$ of the normalization of the 1-nodal rational curve R, in such a way that it is everywhere a local isomorphism between $\mathbf{\Sigma}_a$ and $E \cup R$. Concretely, the latter requirement is that locally at every node $\Sigma_i \cap \Sigma_{i+1}$, the map $h_{a,R}$ should send Σ_i to one of the two local branches of R at its node and Σ_{i+1} to the other.

There are basically two possible (indifferent) choices. We indicate one of them on the above figure by decorating each local branch at a node of $\Sigma_{-a} \cup \ldots \cup \Sigma_{+a}$ with a letter A or B, with the convention that A and B label the two local branches of R at its node.

The following lemma provides a basic description of the objects in $\overline{M}_{a_i\mathbf{e}_i,0}$.

(3.12.1) Lemma.. *For every $(f : C \to S) \in \overline{M}_{a\mathbf{e}_i,0}$, there is a unique lift of f to a stable map $\tilde{f} : C \to \mathbf{\Sigma}_a$, meaning that $f = h_{a,R_i} \circ \tilde{f}$.*

Proof. The curve C is necessarily made of a smooth rational component mapped isomorphically to E, which we denote by E as well, and a tree of smooth rational curves T attached at one point to E, as in (3.11). For each irreducible component C_s of T, one has $f_*C_s = k_sR_i$ for some non-negative integer k_s. We determine the lift $\tilde{f}$ by exploring the dual graph of T along all possible paths from its root to one of its leaves, as follows.

There is a unique irreducible component C_0 of T intersecting E; we call the corresponding vertex of the dual graph of T the *root* of the dual graph. The *leaves* are those vertices corresponding to irreducible components of T intersecting only one other irreducible component. Now the lift $\tilde{f}$, should it exist, necessarily maps C_0 to Σ_0, and there is a unique suitable map $C_0 \to \Sigma_0$ (possibly contracting C_0 to the point $\Sigma_0 \cap E$) by the universal property of normalization.

Suppose a putative lift $\tilde{f}$ is determined on an irreducible component C_s of T, and consider an arbitrary component C_{s+1} of T intersecting C_s at one point z_{s+1}. I claim that the behaviour of $\tilde{f}$ on C_{s+1} is uniquely determined by the already constructed piece $\tilde{f}\big|_{C_s}$. If C_{s+1} is contracted by f, this is clear; otherwise, it is enough by the universal property of normalization to determine which component of $\mathbf{\Sigma}_a$ the lift $\tilde{f}$ should map C_{s+1} to. If $\tilde{f}(z_{s+1})$ is a smooth point of $\mathbf{\Sigma}_a$, then $\tilde{f}$ has to map C_{s+1} to the same component it maps C_s to; if not, $\tilde{f}(z_{s+1})$ is a node $\Sigma_{t_s} \cap \Sigma_{t_s+\varepsilon}$, $\varepsilon \in \{\pm 1\}$, and $\tilde{f}$ has to map C_{s+1} to Σ_{t_s} or $\Sigma_{t_s+\varepsilon}$, depending on which of the local branches A and B of R_i at its node the local branch of C_{s+1} at z_{s+1} is mapped to by f.

This discussion shows that one may algorithmically construct an $\tilde{f}$ such that $h_{a,R_i} \circ \tilde{f} = f$, and that there is a unique such lift. (Note that the chain of rational curves $\Sigma_{-a} \cup \ldots \cup \Sigma_a$ is long enough for the construction to go through without any trouble: since $f_*C = E + aR_i$, if at some point during the algorithm we hit an irreducible component C_s of C that has to be mapped to Σ_{-a+1} or Σ_{a-1}, then the push-forward by f of the sum of all components already visited by the algorithm fills out the class aR_i, hence all components of C not yet touched by the algorithm are contracted by f, and we don't have to go beyond Σ_{-a+1} or Σ_{a-1} in $\mathbf{\Sigma}_a$). ☐

Using Lemma (3.12.1), one can associate to every stable map $(f : C \to S) \in \overline{M}_{a\mathbf{e}_i,0}$ a combinatorial datum called an *admissible sequence* of weight a: this is a sequence of $2a+1$ non-negative integers
$$\mathbf{k} = (0, \ldots, 0, k_{-m}, \ldots, k_0, \ldots, k_n, 0, \ldots, 0)$$
with $m, n \geqslant 0$, $k_{-m}, \ldots, k_n > 0$, and $k_{-m} + \cdots + k_n = a$.

The association goes as follows. Let $h_{a,R_i} \circ \tilde{f}$ be the factorization of f, and write
$$\tilde{f}_*C = E + \sum_{s=-a}^{a} k_s \Sigma_s.$$

It follows from the construction of $\tilde{f}$ in the proof of Lemma (3.12.1) that $(k_{-a}, \ldots, k_a)$ is an admissible sequence of weight a.

(3.12.2) The moduli space $\overline{M}_{a\mathbf{e}_i,0}$ thus decomposes as the disjoint union
$$\overline{M}_{a\mathbf{e}_i,0} = \coprod_{\mathbf{k}} \overline{M}_{\mathbf{k}},$$

where $\mathbf{k}$ ranges through all weight a admissible sequences, and $\overline{M}_{\mathbf{k}}$ is the sub-moduli space of $\overline{M}_{a\mathbf{e}_i,0}$ parametrizing those f with associated admissible sequence $\mathbf{k}$.

(3.13) Identification of the virtual class.. Recall that in (3.11) we saw the moduli space $\overline{M}_{\mathbf{a},\mathbf{b}}$ decomposes in a disjoint union of copies of the product $\prod_1^{24} \overline{M}_{a_i\mathbf{e}_i,0}$ (each corresponding to a given behaviour over the elliptic fibres $F_1, \ldots, F_g$); each $\overline{M}_{a_i\mathbf{e}_i,0}$ in turn decomposes as a disjoint union of moduli spaces $\overline{M}_{\mathbf{k}_i}$ of stable maps with target the curve $\mathbf{\Sigma}_{a_i}$, with $\mathbf{k}_i$ an admissible sequence of weight a_i, as we have described in (3.12). Eventually, $\overline{M}_{\mathbf{a},\mathbf{b}}$ is thus a disjoint union of various products $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$.

The heart of the proof of Bryan and Leung is the explicit identification of the restriction to the product $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$ of the virtual class giving rise to the invariant N_g^p [1, § 5.2]. This is by far the most demanding part of their article, and I will not attempt to give any idea of the proof.

The result is that (i) the virtual class on $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$ is a product of virtual classes on the various factors, and (ii) the virtual class on the factor $\overline{M}_{\mathbf{k}_i}$ is computed by means of a "virtual tangent bundle" T on the target curve $\mathbf{\Sigma}_{a_i}$. This virtual tangent bundle is the vector bundle T on $\mathbf{\Sigma}_{a_i}$ defined by the conditions that it is isomorphic to $h^*_{a_i,R_i} T_S$ on $\Sigma_{-a} \cup \ldots \cup \Sigma_a$ and to $T_E \oplus \mathcal{O}_E(-1)$ on E.

Note that $h^*_{a_i,R_i} T_S$ restricts to $T_E \oplus \mathcal{O}_E(-2)$ on E; the correction made to define T corresponds to the fact that we want to kill the obstruction space $\mathrm{H}^1(C, N_f)$ as we know the actual obstruction space is trivial although $\mathrm{H}^1(C, N_f)$ is not (see (3.3)).

(3.14) Planar model.. Let a be a non-negative integer. Thanks to the result of (3.13), it is possible to construct a model for $\mathbf{\Sigma}_a$ embedded in a familiar surface where its actual deformation theory is isomorphic to the virtual theory leading to the invariants N_g^p. This will eventually let us compute the contribution of the $\overline{M}_{\mathbf{k}_i}$'s (and hence of the $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$'s) to the invariant N_g^p.

Consider three distinct points p, p_{-1}, p_1 lying on a line in the projective plane $\mathbf{P}^2$. Let $P_1 \to \mathbf{P}^2$ be the blow-up at these three points, and call E the exceptional divisor over p, Σ_0 the proper transform of the line through p, p_{-1}, p_1, and Σ_{-1}, Σ_1 the exceptional divisors over p_{-1}, p_1 respectively. Next, recursively for all $s = 1, \dots, a$, we perform the blow-up $P_{s+1} \to P_s$ at two general points of Σ_{-s} and Σ_s respectively, and let Σ_{-s-1} and Σ_{s+1} be the two corresponding exceptional divisors; we call $E, \Sigma_{-s}, \dots, \Sigma_s$ respectively the proper transforms in P_{s+1} of the curves with the same name in P_s.

The curve $\Sigma_{-a} \cup \ldots \cup \Sigma_0 \cup \Sigma_a$ (note that this excludes the last two exceptional curves Σ_{-a-1} and Σ_{a+1}) is isomorphic as an abstract curve to $\mathbf{\Sigma}_a$. Moreover, the tangent bundle of P_{a+1} restricts to $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ on E and to $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$ on $\Sigma_{-a}, \dots, \Sigma_a$, and is therefore isomorphic to the "virtual tangent bundle" T introduced in (3.13). As a consequence, Bryan–Leung prove the following.

(3.14.1) Lemma.. *For all admissible sequences $\mathbf{k} = (k_{-a}, \dots, k_0, \dots, k_a)$, the "local" contribution $\int_{[\overline{M}_{\mathbf{k}}]^{\mathrm{vir}}} 1$ of $\overline{M}_{\mathbf{k}}$ to the invariant N_g^p equals the ordinary genus 0 Gromov–Witten integral $\int_{[\overline{M}_0(P_{a+1},\beta)]^{\mathrm{vir}}} 1$ for the class $\beta = E + \sum_{-a}^a k_s \Sigma_s$.*

This follows from (3.13) and the isomorphism between the restriction of $T_{P_{a+1}}$ and T, provided the two moduli spaces $\overline{M}_{\mathbf{k}}$ and the ordinary $\overline{M}_0(P_{a+1}, E + \sum_{-a}^a k_s \Sigma_s)$ are isomorphic as sets. Bryan and Leung are able to prove this by elementary arguments using a slightly more evolved set-up: they start with a linear $\mathbf{C}^*$ action (not $(\mathbf{C}^*)^2$) on $\mathbf{P}^2$ leaving the line $\langle p, p_{-1}, p_1 \rangle$ and a point q fixed, and this $\mathbf{C}^*$ action survives in P_{a+1}. We refer to [1, Lem. 5.7] for the proof.

Eventually, by deformation invariance of Gromov–Witten invariants we may transport the computation on the projective plane blown-up at $2a+3$ general points. This gives the following.

(3.14.2) Lemma.. *Let $\tilde{\mathbf{P}}^2$ be the blow-up of $\mathbf{P}^2$ at a general set of $2a+3$ points, with exceptional divisors $E, E_{-a-1}, \dots, E_{-1}, E_1, \dots, E_{a+1}$ (all (-1)-curves of course). We call H the pull-back of the line class. Then for all admissible sequences $\mathbf{k} = (k_{-a}, \dots, k_0, \dots, k_a)$, the "local" contribution $\int_{[\overline{M}_{\mathbf{k}}]^{\mathrm{vir}}} 1$ of $\overline{M}_{\mathbf{k}}$ to the invariant N_g^p equals the ordinary genus 0 Gromov–Witten integral $\int_{[\overline{M}_0(\tilde{\mathbf{P}}^2,\beta_{\mathbf{k}})]^{\mathrm{vir}}} 1$ for the class*

$$\begin{aligned}\beta_{\mathbf{k}} &= E + \sum_{s=1}^{a} k_{-s}(E_{-s} - E_{-s-1}) + k_0(H - E - E_1 - E_{-1}) + \sum_{s=1}^{a} k_s(E_s - E_{s+1}) \\ &= k_0 H + (1-k_0)E + \sum_{s=1}^{a}(k_s - k_{s-1})E_s - k_a E_{a+1} + \sum_{s=1}^{a}(k_{-s} - k_{-s+1})E_{-s} - k_{-a}E_{-a-1}.\end{aligned}$$

Note that

$$(k_0 - 1) + \sum_{s=1}^{a}(k_{s-1} - k_s) + k_a + \sum_{s=1}^{a}(k_{-s+1} - k_{-s}) + k_{-a} = 3k_0 - 1,$$

so the virtual class $[\overline{M}_0(\tilde{\mathbf{P}}^2, \beta_{\mathbf{k}})]^{\mathrm{vir}}$ has dimension 0 (see (3.15) below).

(3.15) The computation on the blown-up plane.. The Gromov–Witten invariants gotten in Lemma (3.14.2) are computable in practice thanks to the analysis of genus 0 Gromov–Witten invariants of blow-ups of $\mathbf{P}^2$ carried out by Göttsche and Pandharipande [16].

Let n be a non-negative integer, $d, \alpha_1, \dots, \alpha_n$ integers. We call $N(d; \alpha_1, \dots, \alpha_n)$ the genus 0 Gromov–Witten invariant for $\tilde{\mathbf{P}}^2$ and the class $dH - \sum_i \alpha_i E_i$ (mind the minus sign, introduced for obvious geometric reasons), where $\tilde{\mathbf{P}}^2$ is the projective plane blown-up at a general set of n points, H the pull-back of the line class, and $E_1, \dots, E_n$ the exceptional (-1)-curves. The corresponding moduli space of stable maps has virtual dimension $3d - 1 - \sum_i \alpha_i$; if this is positive, then we impose the appropriate number of point constraints, and if this is negative, then we set the invariant to 0.

These invariants enjoy the following properties (see [16]):
1. $N(1) = 1$;
2. $N(d; \alpha_1, \dots, \alpha_{n-1}, 1) = N(d; \alpha_1, \dots, \alpha_{n-1}, 0) = N(d; \alpha_1, \dots, \alpha_{n-1})$;
3. $N(d; \alpha_1, \dots, \alpha_n) = N(d; \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$ for any permutation $\sigma \in \mathfrak{S}_n$;
4. $N(d; \alpha_1, \dots, \alpha_n) = 0$ if there is an index i for which $\alpha_i < 0$, unless $dH - \sum_i \alpha_i E_i = E_{i_0}$ for some i_0 in which case the invariant is 1;
5. The invariant $N(d; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots, \alpha_n)$ is invariant under the isomorphism given by the quadratic Cremona transformation corresponding to the linear system $|2H - E_1 - E_2 - E_3|$, i.e.,

$$N(d; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots, \alpha_n) = \\ N(2d - \alpha_1 - \alpha_2 - \alpha_3, d - \alpha_2 - \alpha_3, d - \alpha_2 - \alpha_3, d - \alpha_2 - \alpha_3, \alpha_4, \dots, \alpha_n).$$

We need the following definition to state the result. An admissible sequence $(k_{-a}, \dots, k_0, \dots, k_a)$ is *1-pyramidal* if

$$k_s - 1 \leqslant k_{s+1} \leqslant k_s \qquad \text{and} \qquad k_{-s} - 1 \leqslant k_{-s-1} \leqslant k_{-s}$$

for $s = 0, \dots, a-1$.

(3.15.1) Lemma.. *Let $\mathbf{k} = (k_{-a}, \dots, k_0, \dots, k_a)$ be an admissible sequence of weight a. Then the genus 0 Gromov–Witten invariant*

$$N(\mathbf{k}) := N(k_0;\, k_0 - 1,\, k_{-a},\, k_{-a+1} - k_{-a},\, \dots,\, k_0 - k_{-1},\, k_0 - k_1,\, \dots,\, k_{a-1} - k_a,\, k_a)$$

equals 1 if $\mathbf{k}$ is 1-pyramidal, and 0 otherwise.

Proof. [5] We first show that $N(\mathbf{k}) = 0$ if $\mathbf{k}$ is not 1-pyramidal. Since $k_0 > 0$ by definition of an admissible sequence, it follows from Property (4) above that $N(\mathbf{k}) \neq 0$ implies

$$k_{-s} \leqslant k_{-s+1} \qquad \text{and} \qquad k_{s-1} \geqslant k_s \tag{3.15.2}$$

for all $s \in \{1, \dots, a\}$. Next, we apply the Cremona transformation defined by $|2H - E - E_1 - E_{s+1}|$ $(2 \leqslant s \leqslant a-1)$ in the notation of Lemma (3.14.2) and get

$$N(\mathbf{k}) = N(1 + k_1 + k_{s+1} - k_s;\, k_1 - k_s + k_{s+1},\, 1 - k_s + k_{s+1},\, 1 - k_0 + k_1,\, \dots)$$

by Property (5) above. If $N(\mathbf{k}) \neq 0$, we have $k_s \leqslant k_1$ by (3.15.2), hence $1 + k_1 + k_{s+1} - k_s > 0$, and Property (4) then implies that $k_1 \geqslant k_0 - 1$ and $k_{s+1} \geqslant k_s - 1$. An analogous move shows that $k_{-s-1} \geqslant k_s - 1$ for $s = 0, \dots, a-1$ if $N(\mathbf{k}) \neq 0$, so that eventually we see that the non-vanishing of $N(\mathbf{k})$ implies that $\mathbf{k}$ is 1-pyramidal.

Conversely, let's assume that $\mathbf{k}$ is 1-pyramidal and of weight a. Then $k_a = k_{-a} = 0$ (otherwise the weight exceeds a; we have somehow already made this observation in the course of the proof

[5] there is a transcription mistake in [1, p. 399] for the class $\beta_{\mathbf{k}}$ of our Lemma (3.14.2); this leads to a minor correction in the present proof.

of Lemma (3.12.1)), and all coefficients $k_s - k_{s-1}$ and $k_{-s} - k_{-s+1}$, $1 \leqslant s \leqslant a$, equal 0 or 1. It thus follows from Property (2) that $N(\mathbf{k}) = N(k_0; k_0 - 1)$, which is readily seen to equal 1: the moduli space of genus 0 stable maps $\overline{M}_0\big(\tilde{\mathbf{P}}^2, k_0 H - (k_0-1)E\big)$ has only one enumeratively meaningful irreducible component, isomorphic to the family of degree k_0 plane curves with multiplicity $k_0 - 1$ at a fixed point $x_E \in \mathbf{P}^2$, and this is a linear system. □

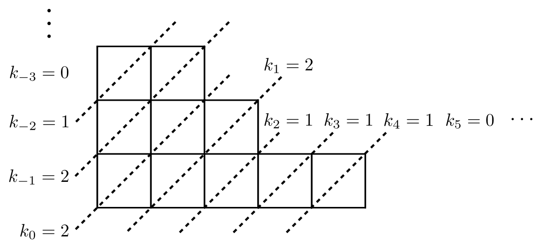
(3.16) Conclusion.. The proof of Theorem (3.5) will be completed once we show the following clever combinatorial result.

(3.16.1) Lemma.. *The number of* 1*-pyramidal admissible sequences* $(k_{-a}, \dots, k_0, \dots, k_a)$ *of weight* a *equals the partition number* $p(a)$ *(cf. (3.9)).*

Indeed, together with (3.12.2) and Lemmata (3.14.2) and (3.15.1), this shows that the "local" contribution of $\overline{M}_{a_i \mathbf{e}_i, 0}$ equals $p(a_i)$, hence the contribution of each copy of the product $\prod_1^{24} \overline{M}_{a_i \mathbf{e}_i, 0}$ equals $\prod_1^{24} p(a_i)$, which proves Proposition (3.10) thanks to the enumeration of elliptic multiple covers performed in (3.11); as we have seen in (3.8), the latter Proposition implies Theorem (3.5).

Proof of Lemma (3.16.1). Partitions of an integer a are in bijective correspondence with Young diagrams of size a [13, Chap. 4]; we exhibit a bijective correspondence between Young diagrams of size a and 1-pyramidal admissible sequences $(k_{-a}, \dots, k_0, \dots, k_a)$ of weight a as follows.

We see Young diagrams as embedded in the upper-right quadrant of a Cartesian plane, leaning on both the x and y axes, and with blocks squares of size 1. Given such a Young diagram, we let k_s be the number of blocks on the line $y - x = s$ for $s = -a, \dots, 0, \dots, a$. We give an example of the procedure in the figure below.



We leave it to the reader to check that this is indeed a bijection. □

4 – BPS state counts

In this Section, I discuss why and how curve counting in non-primitive classes imply the use of multiple covers formulæ. This features the generalization of the Yau–Zaslow formula of Theorem (2.1) to non-primitive classes.

4.1 – Rational curves on the quintic threefold

To describe the picture in its simplest form, let me first discuss a question slightly at the margin of the scope of these notes, that of counting rational curves on a general quintic hypersurface V of $\mathbf{P}^4$.

(4.1) The Clemens conjecture (see (4.5) below) predicts there are finitely many such curves of any given degree d; this is in keeping with the virtual dimension

(4.1.1) $$\operatorname{vdim} \overline{M}_g(V,\beta) = (\dim V - 3)(g-1) - K_V \cdot \beta$$

being 0 for any homology class on the Calabi–Yau threefold V. This suggests that the numbers

$$N_d^V := \int_{[\overline{M}_0(V,d\ell)]^{\mathrm{vir}}} 1$$

(where ℓ denotes the homology class of a line) may indeed give the actual number of rational curves of degree d in V. This would be particularly appealing, since the numbers N_d^V may in theory be rigorously computed using the predictions of mirror symmetry, and they are for small values of d (see [9] for a thorough discussion).

A first objection to such an ideal statement to hold is that there may be rational curves with non-trivial infinitesimal deformations, but the Clemens conjecture predicts as well that this does not happen.

(4.2) Multiple cover formula.. A somehow more serious grain of salt comes from the existence of components of $\overline{M}_0(V,d\ell)$ of dimension larger than expected: suppose we are given a smooth degree d rational curve $C \subseteq V$, and let $f : \mathbf{P}^1 \to V$ be the stable map induced by its normalization ; then for any positive integer k, the degree k covers

(4.2.1) $$\mathbf{P}^1 \xrightarrow{k:1} \mathbf{P}^1 \xrightarrow{f} V$$

constitute an irreducible variety M_{kC} of dimension $2k-2$.

(4.2.2) The Aspinwall-Morrison formula asserts that the corresponding irreducible component $\overline{M}_{kC}$ of $\overline{M}_0(V,kd\ell)$ contributes by $\frac{1}{k^3}$ to the integral N_{kd}^V. This has been mathematically proved by Kontsevich and Manin, and Voisin (see [43, § 5.6], [9, Thm. 7.4.4]).

To explain where the factor $\frac{1}{k^3}$ comes from, it is convenient to replace the integrals N_d^V by their close cousins

$$\langle I_{0,3,d\ell}^V \rangle(\omega_1,\omega_2,\omega_3) := \int_{[\overline{M}_{0,3}(V,d\ell)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\omega_1) \wedge \mathrm{ev}_2^*(\omega_2) \wedge \mathrm{ev}_3^*(\omega_3),$$

where $\overline{M}_{0,3}(V,d\ell)$ is the space of genus 0 stable maps with 3 marked points (which has the advantage of identifying locally with the Hilbert scheme $\mathrm{Hom}(\mathbf{P}^1,V)$ at stable maps with source $\mathbf{P}^1$), and the ω_i are Kähler forms on V. It follows from the divisorial axiom of Gromov–Witten theory that

$$\langle I_{0,3,d\ell}^V \rangle(\omega_1,\omega_2,\omega_3) = \int_{d\ell} \omega_1 \times \int_{d\ell} \omega_2 \times \int_{d\ell} \omega_3 \times N_d^V.$$

On the other hand each physical rational curve $C \subseteq V$ contributes through its normalization $f : \mathbf{P}^1 \to V$ by

$$\int_{\mathbf{P}^1} f^*\omega_1 \times \int_{\mathbf{P}^1} f^*\omega_2 \times \int_{\mathbf{P}^1} f^*\omega_3.$$

Assume we knew what the compactification $\overline{M}_{kC}$ of M_{kC} in $\overline{M}_{0,3}(V,kd\ell)$ looks like, and we had a vector bundle E on it with fibre over $g \in M_{kC}$ (as in (4.2.1)) the obstruction space $E_g = \mathrm{H}^1(\mathbf{P}^1, g^*T_V)$. Then we could compute the contribution of $\overline{M}_{kC}$ by the excess formula

(4.2.3) $$\int_{\overline{M}_{kC}} c_{2k-2}(E) \wedge \mathrm{ev}_1^*(\omega_1) \wedge \mathrm{ev}_2^*(\omega_2) \wedge \mathrm{ev}_3^*(\omega_3).$$

The heuristic computation of [43, p. 115–116] shows that a convenient model for $\overline{M}_{kC}$ leads to the excess contribution (4.2.3) being

$$\int_{\mathbf{P}^1} f^*\omega_1 \times \int_{\mathbf{P}^1} f^*\omega_2 \times \int_{\mathbf{P}^1} f^*\omega_3. = \left(\frac{1}{k}\int_{k[C]}\omega_1\right) \times \left(\frac{1}{k}\int_{k[C]}\omega_2\right) \times \left(\frac{1}{k}\int_{k[C]}\omega_3\right),$$

which would justify the contribution by $\frac{1}{k^3}$ of $\overline{M}_{kC}$ to the integral $\langle I^V_{0,3,kd\ell}\rangle(\omega_1,\omega_2,\omega_3)$.

(4.3) Instanton numbers.. We now wish to define new invariants n_d^V from the Gromov–Witten integrals N_d^V that better reflect the enumerative geometry of the Calabi–Yau threefold V, taking into account the multiple cover phenomenon described in paragraph (4.2) above. The relations that these numbers should satisfy are

$$N_d^V = \sum_{k|d} \frac{1}{k^3} n^V_{\frac{d}{k}} \tag{4.3.1}$$

for all positive integers d, where the sum runs over all positive integral divisors of d. This is an invertible triangular set of relations, and it follows that the number n_d^V is uniquely determined by the invariants $N_{d'}^V$ for all positive divisors d' of d.

These new numbers are traditionally called instanton numbers. The name obviously bears some physical meaning; I shall not discuss this here.

(4.4) Reducible covers of singular curves.. There is yet another phenomenon, first observed by Pandharipande, that prevents the instanton numbers n_d^V defined in (4.3) above to be the actual numbers of integral degree d rational curves on V. It is linked to the existence of singular integral rational curves.

To describe the simplest instance of this phenomenon, let $C \subseteq V$ be an integral rational curve with normalization $f : \mathbf{P}^1 \to V$, and assume it has an ordinary double point at $x \in C$. Let x_1 and x_2 be the two preimages of x in the normalization, and consider the nodal rational curve $\mathbf{P}^1 \cup_{x_1,x_2} \mathbf{P}^1$ obtained by glueing tranversaly two copies of $\mathbf{P}^1$ in such a way that x_1 in the first copy is identified with x_2 in the second copy. Then the map

$$f_x : \mathbf{P}^1 \cup_{x_1,x_2} \mathbf{P}^1 \to V,$$

the restriction of which to both components equals f, is a stable map of genus 0 realizing the homology class $2[C]$ on V, hence contributes to the Gromov–Witten invariant $N^V_{2\deg C}$.

The latter contribution is by 1 if C is infinitesimally rigid (cf. [9, § 9.2.3]). It follows that a δ-nodal rational curve of degree d (i.e., a curve with exactly δ ordinary double points as singularities) contributes in the above described fashion by δ to the Gromov–Witten integral N^V_{2d}.

Fortunately, the Clemens conjecture below predicts that the complications don't go beyond this in this particular situation. Note in particular that part (iii) of the conjecture implies that the numbers N_d^V (or n_d^V) count *irreducible* physical rational curves $C \subseteq V$, since by definition the source of a stable map is connected.

(4.5) Conjecture. *Let $V \subseteq \mathbf{P}^4$ be a general quintic hypersurface.*
(i) For each integer $d \geqslant 1$, there are only finitely many irreducible rational curves $C \subseteq V$ of degree d.
(ii) For every integral rational curve $C \subseteq V$ with normalisation $f : \mathbf{P}^1 \to V$, the normal bundle N_f of the map f is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ (i.e., C is infinitesimally rigid).
(iii) All *the integral rational curves on V (of any degree) are pairwise disjoint.*

It is completely proved in degree $\leqslant 11$ (cf. [8, 10] for the latest steps), and part (i) is known in degree 12 [1].

We end this prologue about quintic threefolds by an explicit example displaying all these phenomena together.

(4.6) Rational curves of degree 10 on a quintic threefold.. (cf. [9, § 9.2.3] for a thorough analysis). First note that by definition of the instanton numbers, the Gromov–Witten integral N_{10}^V decomposes as

$$N_{10} = \frac{1}{10^3}n_1 + \frac{1}{5^3}n_2 + \frac{1}{2^3}n_5 + n_{10}$$

(I dropped the superscript V to lighten notations), and one has

$$\begin{aligned} n_1 &= 2,875 \\ n_2 &= 609,250 \\ n_5 &= 229,305,888,887,625 \\ n_{10} &= 704,288,164,978,454,686,113,488,249,750, \end{aligned}$$

cf. [6]. While n_1 and n_2 are simply the numbers of lines and conics respectively on a general quintic threefold, n_5 counts two kinds of rational curves of degree 5. Indeed, the planes in $\mathbf{P}^4$ are parametrized by a 6-dimensional Grassmannian, and for a general quintic $V \subseteq \mathbf{P}^4$, finitely many of them are 6-tangent to V; the corresponding plane sections of V are 6-nodal plane quintic curves, and in particular they are rational curves. Vainsencher [41] has been able to compute their number

$$n_5' = 17,601,000.$$

Each such curve contributes to n_5 (or N_5) by 1 [9, Lem. 9.2.4], and

$$n_5'' := n_5 - n_5'$$

is indeed the number of *smooth* rational curves of degree 5 on V.

Whereas n_5 still is the number of rational curves of degree 5 on V, this is no longer true for n_{10} as the discussion in (4.4) indicates. The actual number of degree 10 integral rational curves on V is

$$n_{10}^\circ := n_{10} - 6n_5'$$

[9, Thm. 9.2.6][6].

4.2 – Degree 8 rational curves on a sextic double plane

In this subsection, I give the enumerative interpretation due to Gathmann [15] of the reduced Gromov–Witten invariant

$$N_{0,2}^2 := \int_{[\overline{M}_0(S,2L)]^{\mathrm{red}}} 1$$

of a general primitively polarized $K3$ surface (S, L) of genus 2 (i.e., S is a double covering of the plane $\pi : S \to \mathbf{P}^2$ branched over a general sextic curve B, and L is the pull-back of the line class).

(4.7) The analysis carried out in subsection 4.1 indicates that the integral $N_{0,2}^2$ is a sum of contributions corresponding to the following types of curves.

[6] there is a misprint there: $6\frac{1}{8}$ should be replaced by $6 + \frac{1}{8}$.

(i) 5-nodal integral curves in $|2L|$; these are the preimages of the conics in $\mathbf{P}^2$ tangent to the branch sextic B at 5 distinct points. There are $70,956$ of those, as Gathmann was able to compute using his theory of relative Gromov–Witten invariants for hypersurfaces.
(ii) Reducible rational curves made of two distinct rational curves in $|L|$; let C_1, C_2 be two such curves (each of these is the pull-backs of line bitangent to B), with normalizations $f_i : \mathbf{P}^1 \to S$; they intersect in two points x, x' where both of them are smooth. There are correspondingly two distinct stable maps

$$f : \mathbf{P}^1 \cup_{f_1^{-1}(x), f_2^{-1}(x)} \mathbf{P}^1 \to S \quad \text{and} \quad f' : \mathbf{P}^1 \cup_{f_1^{-1}(x'), f_2^{-1}(x')} \mathbf{P}^1 \to S$$

with source the union of two $\mathbf{P}^1$'s meeting transversely at one point, which realize the physical curve $C_1 + C_2 \in |2L|$.

Each of these contributes by 1 to $N^2_{0,2}$, so each of the $\binom{324}{2}$ pairs of distinct rational curves in $|L|$ contributes by 2, thus giving a total contribution of $104,652$ to $N^2_{0,2}$ (recall that there are 324 bitangent lines to B, as can be classicaly computed, or extracted from Thm. (2.1)).
(iii) Reducible double coverings of rational curves in $|L|$, as in (4.4). Since all rational curves in $|L|$ are 2-nodal, all 324 of them give 2 stable maps with reducible source contributing by 1 each to $N^2_{0,2}$, for a total contribution of 648.
(iv) Double covers of rational curves in $|L|$, as in (4.2). One may expect that each of the 324 corresponding irreducible components of $\overline{M}_0(S, 2L)$ gives a contribution to $N^2_{0,2}$ similar to that prescribed by the Aspinwall–Morrison formula, although there is at first sight no obvious reason for this to be the case. Gathmann [15, Lem. 4.1] proves that indeed each irreducible component contributes by $\frac{1}{8}$.

(4.8) Remark.. An important difference with the case of the quintic threefold, although rather innocent-looking, is that in the present situation integral rational curves do intersect each other, contrary to the prediction of part (iii) of Clemens' Conjecture. In the above discussion (4.7), this amounts to case (ii) needing to be added with respect to the discussion for quintic threefolds, which is still manageable. When looking at linear systems $|mL|$ with $m \geqslant 3$ however, this soon gets much more complicated, see (4.15).

(4.9) From the enumeration of (4.7), one may deduce the value of $N^2_{0,2}$, which was not known before. But the striking observation of [15] is that the sum of the contributions (i)–(iii) above, which would be the instanton number $n^2_{0,2}$ in the language of (4.3), actually equals

$$N^5_{0,1} := N^5_0 = 176,256,$$

the number of degree 8 rational curves in a primitively polarized $K3$ surface of genus 5, computed by formula (2.1.1). This suggests the amazing possibility that the number of rational curves of degree d on a $K3$ surface only depends on d, and not on the algebraic geometry of the $K3$ surface! We will come back to this in detail in subsection 4.4 below.

4.3 – Elliptic curves in a 2-divisible class on a $K3$ surface

Here, I report on the computation by Lee and Leung [22] of the reduced Gromov–Witten invariant

$$N^p_{1,2} := \int_{[\overline{M}_{1,1}(S,2L)]^{\mathrm{red}}} \mathrm{ev}_1^*(\mathrm{pt})$$

of a general primitively polarized $K3$ surface (S, L) of genus p, which "counts" genus 1 curves in $|2L|$ passing through 1 general point on S. Using a suitable version of topological recursion, they prove the following formula.

(4.10) Theorem. [22] *One has*

$$N^p_{1,2} = N^{4p-3}_{1,1} + 2N^p_{1,1}.$$

The numbers $N^q_{1,1} := N^q_1$ are those giving the number of elliptic curves through a general point in the primitive class of a $K3$ surface of genus q, as in formula (3.5). Note that $p' := 4p-3$ is the integer such that $(2L)^2 = 2p' - 2$.

(4.11) Lee and Leung propose the following interpretation of their formula (4.10), to put it in tune with (4.9) and more generally with the results of subsection 4.4 below.

Given a smooth elliptic curve E, there are $\sigma_1(2) = 1 + 2 = 3$ morphisms of elliptic curves $E' \to E$ of degree 2 (note that we require that the origin is respected), as we have already seen in (3.11). This implies that each of the $N^p_{1,1}$ elliptic curves C in the primitive linear system $|L|$ passing through a general point $x_1 \in S$ gives via double covers of its normalization 3 genus 1 stable maps realizing the homology class $2[C]$, each contributing by 1 to the number $N^p_{1,2}$.

Lee and Leung deduce from this that N^{4p-3}_1 is the actual number of physical elliptic curves in $|2L|$, meaning that it counts each integral elliptic curve $C \in |2L|$ for 1 and each $2C$ where $C \in |L|$ is an integral elliptic curve for 1 as well. This is indeed a striking interpretation, although arguably debatable.

There is at any rate a phenomenon that prevents this interpretation to be anything more than philosophical, namely that reducible curves $C_0 + C_1$, where C_0 (resp. C_1) is a rational (resp. elliptic) integral curve in $|L|$, also contribute to the invariant $N^p_{1,2}$. The two curves C_0 and C_1 intersect (transversely, say) at $2p-2$ points $y_1, \dots, y_{2p-2}$, and this gives $2p-2$ genus 1 stable maps

$$\mathbf{P}^1 \cup_{y_i} \bar{C}_1 \to S$$

realizing the class $2L$ and passing through the appropriate fixed point whenever C_1 does (the source is the transverse union of $\mathbf{P}^1$ and the normalization of C_1 attached at one point, the preimages of y_i in the normalizations of C_0 and C_1 respectively).

4.4 – The Yau–Zaslow formula for non-primitive classes

We now come to the general statement confirming (4.9) and recently proved by Klemm, Maulik, Pandharipande, and Scheidegger.

(4.12) Let S be a $K3$ surface, and $L \in \mathrm{Pic}S$. It follows from deformation invariance of reduced Gromov–Witten integrals and the global Torelli theorem for $K3$ surfaces that the integral $\int_{[\overline{M}_0(S,L)]^{\mathrm{red}}} 1$ only depends on the self-intersection L^2 and the divisibility index of L in $\mathrm{Pic}S$, i.e., the largest integer m for which there exists $L' \in \mathrm{Pic}S$ such that $L = mL'$. We may thus make the following definition.

For integers $p > 0$ and $m \geqslant 1$, we let

$$N^p_{0,m} := \int_{[\overline{M}_0(S,mL)]^{\mathrm{red}}} 1$$

where (S, L) is any primitively polarized $K3$ surface of genus p.

(4.13) BPS states.. Similar to what has been done in (4.3), and following the insight of (4.9), we now define new invariants from the $N^p_{0,m}$ of (4.12) above by applying the corrections indicated by the Aspinwall–Morrison formula.

Let us formulate this in terms of generating series as follows. Given a positive integer p, set

$$F^p(v) := \sum_{m\geqslant 1} N^p_{0,m}\, v^m \tag{4.13.1}$$

as a formal power series in the variable v. Then the new set of invariants $n^p_{0,m}$ is uniquely determined by the rewriting of the generating series as

$$F^p(v) := \sum_{m\geqslant 1} n^p_{0,m} \Big(\sum_{d>0} \frac{1}{d^3}\, v^{dm} \Big) \tag{4.13.2}$$

(note that this is exactly the same modification as that of (4.3.1)).

Note that this is not mere makeshift reformulation. The invariants $n^p_{0,m}$ are believed to count objects named BPS states by the physicists, after Bogomol'nyi, Prasad, and Sommerfield; the mathematical nature of these objects is however not clear yet. In particular, it should be possible to define the $n^p_{0,m}$ intrinsically, not relying on the $N^p_{0,m}$; the relation (4.13.2) would then tie together these two sets of indipendently defined invariants. See the enlightening survey [29, § $2\frac{1}{2}$] for more about this. There is moreover a physical meaning to the introduction of generating series, that I will not discuss.

(4.14) Theorem. [20] *The invariants $n^p_{0,m}$ do not depend upon the divisibility index, i.e., one has for all integers $m, p \geqslant 1$*

$$n^p_{0,m} = n^{m^2p-m^2+1}_{0,1} = N^{m^2p-m^2+1}_0 \tag{4.14.1}$$

(the integer $p' = m^2p - m^2 + 1$ is designed such that $(mL)^2 = 2p' - 2$ if $L^2 = 2p - 2$).

Recall that N^p_0 was defined in section 2; the second equality in (4.14.1) is by definition of $n^p_{0,1}$ and N^p_0. This statement was part of the Yau–Zaslow conjecture [28]. Together with Theorem (2.1), which was also part of the Yau–Zaslow conjecture, it implies that all the $n^p_{0,m}$'s may be computed by means of formula (2.1.1). The set of relations (4.13.1) being triangular invertible, this also gives all genus 0 reduced Gromov–Witten invariants of $K3$ surfaces. Section 6 below contains an overview of the proof given by Klemm, Maulik, Pandharipande, and Scheidegger [20] of Theorem (4.14).

As we already noted in (4.9), the truly remarkable feature of the invariants $n^p_{0,m}$ displayed by this statement is that the number of rational curves of prescribed degree in an algebraic $K3$ surface does not depend on the algebraic geometry of the surface.

(4.15) In spite of formula (4.13.2) taking into account the Aspinwall-Morrison multiple cover correction, the invariants $n^p_{0,m}$ do not in general count the actual number of rational curves in $|mL|$.

One reason for this is the existence of more non-reduced curves with rational support than those taken in consideration in the correction (4.13.2), namely curves with reducible support. For instance let $m = 3$ and consider two integral rational curves C_1, C_2. Then $2C_1 + C_2 \in |3L|$, and there are correspondingly finitely many positive-dimensional components of $\overline{M}_0(S, 3L)$, the general points of which correspond to stable maps

$$\mathbf{P}^1 \cup_x \mathbf{P}^1 \to S$$

with source a transverse union of two $\mathbf{P}^1$'s, consisting of a double cover of C_1 on the first component and the normalization of C_2 on the other. These certainly give an excess contribution to the invariant $N^p_{0,3}$, which is not taken into account in the definition of $n^p_{0,3}$.

Such problematic phenomena do not occur for $m \leqslant 2$, so that $n^p_{0,1}$ is directly enumerative as was already noted in Section 2, and $n^p_{0,2}$ counts reduced rational curves in the way described in subsection 4.2. It would be very interesting to relate $n^p_{0,m}$ to the number of integral rational curves in $|mL|$ for $m \geqslant 3$.

There were, at least conjecturally, no such phenomena at work in the case of the quintic threefold discussed in subsection 4.1, as part (iii) of the Clemens conjecture (4.5) asserts that two integral rational curves in a general quintic threefold never intersect. In a surface, there is of course not enough space for two curves to avoid each other, so we inevitably have to deal with the aforementoined degenerate contributions.

The philosophy, as R. Pandharipande communicated to me, is that what the BPS numbers for $K3$ surfaces are virtually counting, are rational curves in some perturbation of the twistor family of the $K3$ surface (a threefold, cf. (3.2)). We shall consider in more detail the close interplay between counting invariants for $K3$ surfaces and Calabi–Yau threefolds in the next Section 5.

5 – Relations with threefold invariants

It is already visible in the very foundation of the theory of Gromov–Witten invariants for algebraic $K3$ surfaces developped by Bryan–Leung [1], see (3.2), that these invariants are fundamentally attached to a threefold (even though the approach by Maulik–Pandharipande [26], see (3.3), enables one to bypass this). Another revealing evidence of the 3-dimensional nature of these invariants is the meaningful role played by the Aspinwall–Morrison in the Yau–Zaslow statement discussed in subsection 4.4 above, a tool specifically designed for Calabi–Yau threefolds.

In this section we will try to describe this relation in a more conceptual way. It it wise to keep in mind the symplectic nature of Gromov–Witten invariants throughout.

5.1 – Two obstruction theories

(5.1) A threefold degenerate contribution.. Let V be a Calabi–Yau threefold. It follows from formula (4.1.1) that the virtual dimension of any space of stable maps of any genus is always 0 (it is actually true even if the canonical class K_V is not trivial that the dimension only depends on the homology class β). This makes the following phenomenon happen.

Let $C_0 \subseteq V$ be a rational curve (smooth and infinitesimally rigid, say). Its normalization $f : \mathbf{P}^1 \to V$ contributes regularly by 1 to the integral $\int_{[\overline{M}_0(V,[C_0])]^{\mathrm{vir}}} 1$. But for any stable curve C' of genus $g \geqslant 1$ we may obtain a genus g stable map realizing the class $[C_0]$ by attaching C' to the normalization of C_0 over a smooth point x, and letting

$$f_{C',x} : \mathbf{P}^1 \cup_x C_0 \to V$$

equal to f along $\mathbf{P}^1$ and collapsing C' to x. This produces a positive dimensional moduli space of genus g stable maps all having the same image $C_0 \subseteq V$; its contribution to the Gromov–Witten invariant $\int_{[\overline{M}_g(V,[C_0])]^{\mathrm{vir}}} 1$ must be computed via Hodge integrals over the moduli space of stable curves of genus g. This has been studied by Faber and Pandharipande, see [29, § $1\frac{1}{2}$] and the references therein.

(5.2) Curves in the twistor space of a $K3$.. Let S be a $K3$ surface, together with an algebraic class $\beta \in \mathrm{H}_2(S, \mathbf{Z})$. We consider its twistor space $T \to \mathbf{S}^2$ described in (3.2) above (we emphasize that this is a real 6-dimensional variety), and let $\iota : S \hookrightarrow T$ be the canonical inclusion of S. Since curves in T can only appear in the fibre S, we have the equality of moduli spaces of stable maps

$$\overline{M}_g(T, \iota_*\beta) = \overline{M}_g(S, \beta),$$

a priori only as sets but in fact as Deligne–Mumford stacks. They come however with two different obstructions theories, hence also with two different virtual classes. Gromov–Witten invariants on T are related to those on S (within the reduced theory for $K3$ surfaces, cf. (3.3)) by the formula

$$\text{(5.2.1)} \qquad \int_{[\overline{M}_g(T,\iota_*\beta)]^{\mathrm{vir}}} 1 \;=\; \int_{[\overline{M}_g(S,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g,$$

where λ_g stands for the top Chern class $c_g(\mathbf{E}_g)$ of the Hodge bundle $\mathbf{E}_g \to \overline{M}_g(S, \beta)$, whose fibre over the stable map $f : C \to S$ is $\mathrm{H}^0(C, \omega_C)$.

(5.3) Hodge integrals.. It follows from the invariance of reduced Gromov–Witten invariants under algebraic deformation and the global Torelli theorem for $K3$ surfaces that the right-hand side of (5.2.1) depends only on the self-intersection β^2 and the divisibility index of β as an algebraic class. We may thus formulate the following definition.

For integers $g \geqslant 0$ and $p, m \geqslant 1$, let

$$\text{(5.3.1)} \qquad R^p_{g,m} := \int_{[\overline{M}_g(S,mL)]^{\mathrm{red}}} (-1)^g \lambda_g$$

where (S, L) is any primitively polarized $K3$ surface of genus p, and λ_g is the top Chern class of the Hodge bundle as in (5.2) above.

This extends the definition of the invariants $N^p_{0,m}$ in (4.12) above, in the sense that $N^p_{0,m} = R^p_{0,m}$ (note however that the invariants $N^p_{1,2}$ used in subsection 4.3 do not coincide with the $R^p_{1,2}$). For $g > 0$, these invariants are certainly not counting curves on S; rather, formula (5.2.1) tells us that they virtually give the excess contribution of S to the vertical Gromov–Witten theory of any $K3$-fibred threefold in which it appears as a fibre. This philosophy is put into concrete form by Theorem (6.9) below.

5.2 – The Katz–Klemm–Vafa formula

This is an extension of the Yau–Zaslow conjecture discussed in subsection 4.4 above to the invariants $R^p_{g,m}$. It has been proved by Pandharipande and Thomas [33], see also [32].

(5.4) BPS invariants.. It is admittedly better to organize the invariants $R^p_{g,m}$ in BPS form as in (4.13). We have now a clear justification for this, as we have seen in subsection 5.1 above that these invariants really count objects on threefolds.

We first let

$$F^p(u, v) := \sum_{g \geqslant 0} \sum_{m > 0} R^p_{g,m}\, u^{2g-2} v^m$$

as formal power series in the two variables u, v for all positive integers p. One then defines new invariants $r^p_{g,m}$ for all integers $p, m > 0$, $g \geqslant 0$, by setting

$$
\begin{aligned}
F^p(u,v) &= \sum_{m>0}\sum_{g\geqslant 0} r^p_{g,m} u^{2g-2}\left(\sum_{d>0}\frac{1}{d}\left(\frac{\sin d\frac{u}{2}}{\frac{u}{2}}\right)^{2g-2} v^{dm}\right)\\
&= \sum_{m>0}\Bigg(r^p_{0,m}u^{-2}\sum_{d>0}\Big(\frac{1}{d^3}+\frac{1}{12d}u^2+\frac{d}{240}u^4+\frac{d^3}{6048}u^6+\frac{d^5}{172800}u^8+\cdots\Big)v^{dm}\\
&\qquad + n^p_{1,m}\sum_{d>0}\frac{1}{d}v^{dm}\\
&\qquad + n^p_{2,m}u^2\sum_{d>0}\Big(d-\frac{d^3}{12}u^2+\frac{d^5}{360}u^4-\frac{d^7}{20160}u^6+\frac{d^9}{1814400}u^8+\cdots\Big)v^{dm}\\
&\qquad + n^p_{3,m}u^4\sum_{d>0}\Big(d^3-\frac{d^5}{6}u^2+\frac{d^7}{80}u^4-\frac{17d^9}{30240}u^6+\frac{31d^{11}}{1814400}u^8+\cdots\Big)v^{dm}\\
&\qquad + \cdots\Bigg).
\end{aligned}
$$

The modifications for genus $g > 0$ objects did not appear earlier in this text. Note that every object counted by $r^p_{g,m}$ contributes to the invariants $R^p_{g',m}$ for all $g' \geqslant g$ (except when $g = 0$), with alternated sign if $g \geqslant 2$. This is in accord with what the phenomenon described in (5.1) suggests.

(5.5) Theorem ((Katz–Klemm–Vafa formula, [33])). *The invariants $r^p_{g,m}$ do not depend on the divisibility index, meaning that one has*

$$r^p_{g,m} = r^{m^2p-m^2+1}_{g,1} = R^{m^2p-m^2+1}_g$$

for all integers $p, m > 0$, $g \geqslant 0$.

They are all determined by the formula

$$\sum_{p\geqslant 0}\sum_{g\geqslant 0}(-1)^g r^p_g (y^{\frac12}-y^{-\frac12})^{2g}q^p = \prod_{n\geqslant 1}\frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}, \tag{5.5.1}$$

where we set $r^p_g := r^p_{g,1}$ (and $r^0_0 = 1$, $r^0_g = 0$ if $g > 0$, for convenience).

Setting $y = 1$ in the formula restricts to the invariants r^p_0, and recovers the Yau–Zaslow formula of Theorem (2.1). As a first corollary, one gets that $r^p_g = 0$ if $g > p$, and $r^p_p = (-1)^p(p+1)$. The first values of r^p_g are tabulated below.

$g \backslash p$	0	1	2	3	4
0	1	24	324	3200	25650
1		−2	−54	−800	−8550
2			3	88	1401
3				−4	−126
4					5

Table 2: First values of r^p_g

5.3 – Further Gromov–Witten integrals

I close this section with a short discussion of further results about Gromov–Witten integrals on $K3$ surfaces. They all come from [24].

(5.6) Hodge integrals with point insertions.. As a direct generalization of the invariants (5.3.1), one may consider the integrals

$$R^p_{g,k,m} := \int_{[\overline{M}_{g,k}(S,mL)]^{\mathrm{red}}} (-1)^{g-k}\lambda_{g-k} \cup \mathrm{ev}_1^*(\mathrm{pt}) \cup \cdots \cup \mathrm{ev}_k^*(\mathrm{pt}),$$

where (S,L) is a primitive $K3$ surface of genus p, $\overline{M}_{g,k}(S,mL)$ is the moduli space of genus g stable maps with k marked points, and λ_i is the i-th Chern class of the Hodge bundle $\mathbf{E}_{g,k} \to \overline{M}_{g,k}(S,mL)$ as in (5.2).

For *primitive* classes on $K3$ surfaces, the following formula is proved by Maulik, Pandharipande and Thomas [24, Thm. 3]:

$$\text{(5.6.1)} \qquad \sum_{g=0}^{+\infty}\sum_{p=0}^{+\infty} R^p_{g,k,1}\, u^{2g-2}q^p \;=$$
$$q\,\frac{(2\pi)^{12}}{u^2\Delta(q)} \cdot \exp\left(\sum_{g=1}^{+\infty} u^{2g}\frac{B_{2g}}{g(2g)!}E_g(q)\right)\cdot\left(\sum_{m=1}^{+\infty} q^m \sum_{d|m}\frac{m}{d}\big(2\sin d\tfrac{u}{2}\big)^2\right)^k$$

(notation is as in (3.6)).

Note that in the $k=0$ case, this contains nothing new with respect to the formula given in Theorem (5.5), as the expression (5.6.1) may be deduced from (5.5.1) using known identities, see [24, § 5.4].

(5.7) Descendent Gromov–Witten invariants.. So far, we have been essentially concerned with the reduced Gromov–Witten invariants (see (3.3))

$$N_g(S,\beta) = \int_{[\overline{M}_{g,g}(S,\beta)]^{\mathrm{red}}} \mathrm{ev}_1^*(\mathrm{pt}) \cup \cdots \cup \mathrm{ev}_g^*(\mathrm{pt})$$

counting curves in the algebraic class $\beta \in \mathrm{H}_2(S,\mathbf{Z})$ on the $K3$ surface S and passing through g general fixed points ($\mathrm{pt} \in \mathrm{H}^4(S,\mathbf{Z})$ denotes the (co)homology class of a point). It is of course possible to pull-back more general cohomology classes $\gamma_i \in \mathrm{H}^*(S,\mathbf{Z})$ by the evaluation maps, thus encoding more general incidence conditions than the passing through a given point (although this is not of crucial interest for surfaces due to the divisor axiom of Gromov–Witten theory). Beware that when doing so one gets integrals that do depend on the class β itself, and not only on its self-intersection and divisibility index, as classes in $\mathrm{H}^2(S,\mathbf{Z})$ are not monodromy invariant.

A more sensible generalization is to integrate *descendent classes*. Let $\overline{M}_{g,k}(S,\beta)$ be the moduli space of genus g stable maps with k marked points realizing the class β, and $\mathrm{ev}_1,\dots,\mathrm{ev}_k$ the corresponding evaluation maps $\overline{M}_{g,k}(S,\beta) \to S$. For all $i=1,\dots,k$, define the *i-th cotangent line bundle* L_i to be the line bundle over $\overline{M}_{g,k}(S,\beta)$ the fibre of which over the point $(f : C \to S, p_1,\dots,p_k)$ is the $\mathbf{C}$-line Ω^1_{C,p_i}. The descendent classes on $\overline{M}_{g,k}(S,\beta)$ are those gotten from the Chern classes of these line bundles.

Let $\psi_i := c_1(L_i) \in \mathrm{H}^2\big(\overline{M}_{g,k}(S,\beta), \mathbf{Q}\big)$. For all cohomology classes $\gamma_1, \ldots, \gamma_k \in \mathrm{H}^*(S, \mathbf{Z})$ and non-negative integers $n_1, \ldots, n_k$ we define the reduced descendent Gromov–Witten invariants

$$\big\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \big\rangle_g^{S,\beta} := \int_{[\overline{M}_{g,k}(S,\beta)]^{\mathrm{red}}} \psi_1^{k_1} \cup \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \psi_k^{k_k} \cup \mathrm{ev}_k^*(\gamma_k) \tag{5.7.1}$$

whenever the degree of the integrand equals the (real) dimension $2g+2k$ of the reduced virtual class, and $\big\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \big\rangle_g^{S,\beta} := 0$ otherwise.

How to geometrically interpret the insertion of the classes ψ_i is not straightforward; I refer to [31] and [14] for some discussions about this. See however [14, Thm. 2.2.6], where descendent classes are used to define Gromov–Witten invariants of a projective manifold X *relative* to a smooth very ample hypersurface Y, i.e., invariants virtually counting curves in X with prescribed tangency conditions along Y.

(5.8) Quasi-modularity.. The integrals (5.7.1) for fixed integrand and fixed g and divisibility index of β are expected to fit together as the Fourier coefficients of a quasi-modular form, as in Theorem (3.5). Due to their dependency on the class β and not only on its numerical characters, this is formulated as follows.

Let S be an arbitrarily fixed $K3$ surface possessing an elliptic fibration $\pi : S \to \mathbf{P}^1$ and a section E of π. Call $\mathsf{e}, \mathsf{f} \in \mathrm{H}_2(S, \mathbf{Z})$ the classes of E and the fibres of π respectively. It follows from deformation invariance and the same standard degeneration argument as in the proof of Theorem (3.5) that any integral of the form (5.7.1) on any algebraic $K3$ surface equals an integral of the same kind on S with $\beta = a\mathsf{e} + b\mathsf{f}$, a, b non-negative integers.

For all integers $g \geqslant 0$ and $m > 0$, we set

$$F_{g,m}^S\big(\tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k)\big) := \sum_{n \geqslant 0} \big\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \big\rangle_g^{S, m\mathsf{e}+n\mathsf{f}} q^{m(n-m)}$$

as a formal power series in the variable q. Maulik and Pandharipande conjecture the following.

(5.8.1) Conjecture.. ([30, Conj. 3] and [24, § 7.5]) *The power series $F_{g,m}^S\big(\tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k)\big)$ is the Fourier expansion in q of a quasi-modular form of level m^2 with pole at $q = 0$ of order at most m^2.*

(A quasi-modular form of level N with possible pole at $q = 0$ is by definition an element of the $\mathbf{C}$-algebra generated by the Eisenstein series G_1 (see (3.6) and modular forms of level N; recall in addition that a modular form of level N is a form satisfying the modular equation for transformations in the congruence subgroup $\Gamma_0(N)$ consisting of elements of $\mathrm{PSL}_2(\mathbf{Z})$ congruent to the identity matrix modulo N).

For $m = 1$, i.e., for primitive classes, this has been proved by Maulik, Pandharipande and Thomas [24, Thm. 4]. Note however that, even in the primitive case, there is as far as I know no general explicit formula for the modular form in question. Theorem (3.5) provides particular instances of such a formula. At any rate, modularity strongly constrains the invariants and in favorable cases enables one to compute them all (see (6.11) for an example in a different context).

. Although I will say nothing about the proofs of the results presented in this section, I would like to point out that one fundamental ingredient for them is the use of other counting invariants than those coming from Gromov–Witten theory, together with correspondence theorems between the two. They are more algebraic in nature than Gromov–Witten invariants, and more agile to study the problems we have been discussing. These invariants virtually count *stable pairs*; they were defined by Pandharipande and Thomas, specifically for threefolds up to now. See [29] for a presentation.

6 – Noether–Lefschetz theory and applications

6.1 – Lattice polarized $K3$ surfaces and Noether–Lefschetz theory

In this subsection we define Noether–Lefschetz divisors in the moduli spaces of lattice polarized $K3$ surfaces. While the version we will use is the refined one of (6.4), the elementary version of (6.3) is needed to give a proper definition.

Let $\mathbf{L}_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ be the $K3$ lattice (see, e.g., [2]). Throughout this subsection, we consider a fixed lattice Λ of rank r and signature $(1, r-1)$, together with a primitive embedding $\iota : \Lambda \hookrightarrow \mathbf{L}_{K3}$ (the embedding is *primitive* if the corresponding quotient $\mathbf{L}_{K3}/i(\Lambda)$ is torsion-free).

(6.1) Definition. *Let S be a $K3$ surface. A Λ-polarization on S is a primitive embedding $j : \Lambda \hookrightarrow \mathrm{Pic}S$ such that*
1. there is a nef and big class in $j(\Lambda) \subseteq \mathrm{Pic}S$;
2. there exists an isometry $\phi : \mathrm{H}^2(S, \mathbf{Z}) \to \mathbf{L}_{K3}$ such that $\phi \circ j = \iota$.

A Λ-polarized $K3$ surface is a pair (S, j) where S is a $K3$ surface and j is a Λ-polarization on S.

There exists a moduli space $\mathcal{K}_\Lambda$ of Λ-polarized $K3$ surfaces, which may be constructed relying on the global Torelli theorem by adapting the method of [37, Exp. XIII, §3].

(6.2) Define the discriminant of a rank s lattice L to be the signed determinant

$$\mathrm{Disc}L := (-1)^{s-1} \det\big(\langle v_i, v_j\rangle\big)_{1 \leqslant i,j \leqslant s}$$

where $(v_1, \ldots, v_s)$ is an integral basis of L (the sign has been added to the usual definition so that $\mathrm{Disc}\Lambda > 0$); this does not depend on the choice of the basis.

Let $\mathbf{L}$ be a rank $r+1$ lattice with an even symmetric bilinear form, together with a primitive embedding $i : \Lambda \hookrightarrow \mathbf{L}$. There is an invariant of the pair $(\mathbf{L}, i)$ called the *coset*, which is defined as follows. Consider any vector $v \in \mathbf{L}$ such that $\mathbf{L} = i(\Lambda) \oplus v$; the pairing with v determines an element $\ell_v \in \Lambda^\vee$ in the lattice dual to Λ. On the other hand let $G_\Lambda := \Lambda^\vee/\Lambda$ be the quotient of the injection defined by the pairing on Λ; it is an abelian group of order $\mathrm{Disc}\Lambda$. Now the coset δ of $(\mathbf{L}, i)$ is the class of ℓ_v in $G/\pm$; it does not depend on the choice of v.

Two pairs $(\mathbf{L}, i)$ and $(\mathbf{L}', i')$ as above are isomorphic (i.e., there exists an isometry $\phi : \mathbf{L} \to \mathbf{L}'$ such that $\phi \circ i = i'$) if and only if the two following conditions both hold: (i) $\mathrm{Disc}(\mathbf{L}) = \mathrm{Disc}(\mathbf{L}')$, and (ii) $\delta(\mathbf{L}, i) = \delta(\mathbf{L}', i')$.

(6.3) Elementary Noether–Lefschetz divisors.. The Noether–Lefschetz divisor $P^\Lambda_{\Delta,\delta} \subseteq \mathcal{K}_\Lambda$ is defined as the closure of the locus of Λ-polarized $K3$ surfaces (S, j) such that $\mathrm{Pic}S$ has rank $r+1$ and discriminant Δ, and the coset $\delta(\mathrm{Pic}S, j)$ equals δ.

It follows from the Hodge index theorem that the divisor $P^\Lambda_{\Delta,\delta}$ is empty when $\Delta \leqslant 0$.

(6.4) Refined Noether–Lefschetz divisors.. We now fix an integral basis $\mathbf{v}_\Lambda = (v_1, \ldots, v_r)$ for Λ, and let $m \in \mathbf{Z}_{>0}$, $(p, \mathbf{d}) = (p, d_1, \ldots, d_r) \in \mathbf{Z}^{r+1}$. We want to define a Noether–Lefschetz divisor $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} \subseteq \mathcal{K}_\Lambda$ corresponding to Λ-polarized $K3$ surfaces (S, j) with an extra class $\beta \in \mathrm{Pic}S$ of divisibility index m, and such that $\langle\beta, \beta\rangle = 2p-2$ and $\langle\beta, v_i\rangle = d_i$ for $i = 1, \ldots, r$.

This goes as follows: let

$$\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} := (-1)^r \begin{vmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_r\rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r, v_1\rangle & \cdots & \langle v_r, v_r\rangle & d_r \\ d_1 & \cdots & d_r & 2p-2 \end{vmatrix};$$

- if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} > 0$, set

$$D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} := \sum_{\Delta,\delta} \mu^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}(\Delta,\delta) \cdot P^{\Lambda}_{\Delta,\delta}$$

where the sum runs over all Δ, δ such that there exists a pair $(\mathbf{L}, i)$ as in (6.2) with $\mathrm{Disc}\mathbf{L} = \Delta$ and $\delta(\mathbf{L}, i) = \delta$ (the pair $(\mathbf{L}, i)$ is then unique up to isomorphism), and $\mu^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}(\Delta,\delta)$ is the number of elements $\beta \in \mathbf{L}$ having divisibility index m and satisfying $\langle\beta,\beta\rangle = 2p-2$ and $\langle\beta, v_i\rangle = d_i$ for $i = 1, \dots, r$. Note that $\mu^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}(\Delta,\delta)$ may be 0; in particular its non-vanishing implies that Δ divides $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}}$, so the above sum has only finitely many terms. The condition $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} > 0$ implies that any β such that $\langle\beta,\beta\rangle = 2p-2$ and $\langle\beta, v_i\rangle = d_i$ for all i does not belong to $i(\Lambda)$;
- if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} < 0$, set $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} := 0$;
- if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} = 0$ and $m = \gcd(d_1, \dots, g_r)$, let $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}$ be the divisor associated to the dual of the Hodge line bundle $\mathcal{E} \to \mathcal{K}_\Lambda$ (the fibre of $\mathcal{E}$ over the point (S, i) is $\mathrm{H}^{2,0}(S)$);
- if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} = 0$ and $m \neq \gcd(d_1, \dots, g_r)$, set $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} := 0$.

6.2 – Invariants of families of lattice polarized $K3$ surfaces

(6.5) Families of lattice polarized $K3$ surfaces.. Let $\iota : \Lambda \hookrightarrow \mathbf{L}_{K3}$ be a primitive embedding of a lattice Λ of rank r and signature $(1, r-1)$. A 1-parameter family of Λ-polarized $K3$ surfaces is a smooth family $\pi : X \to C$ of $K3$ surfaces equipped with line bundles $L_1, \dots, L_r$ on X such that:
1. X is a compact 3-dimensional complex manifold (not necessarily algebraic), C is a complete smooth complex curve, and π is a holomorphic submersion;
2. for each $t \in C$, the fibre X_t of π over t is a (smooth) $K3$ surface;
3. there exists a linear combination L^π of the holomorphic line bundles L_i on X, the restriction of which to every fibre of π is nef and big;
4. there exists an integral basis $(v_1, \dots, v_r)$ of Λ such that for each $t \in C$, the map $j_t : \Lambda \to \mathrm{Pic}X_t$ defined by $v_i \mapsto L_{i,t}$ (the restriction of L_i to X_t) is a Λ-polarization of X_t.

For the remainder of this subsection, we consider $(\pi : X \to C, L_1, \dots, L_r)$ a 1-parameter family of Λ-polarized $K3$ surfaces as in Definition (6.5) above.

(6.6) Noether–Lefschetz numbers.. Let $m \in \mathbf{Z}_{>0}$ and $(p, \mathbf{d}) = (p, d_1, \dots, d_r) \in \mathbf{Z}^{r+1}$. The Noether–Lefschetz number $\mathrm{NL}^\pi_{m,p,\mathbf{d}}$ is defined as

$$\mathrm{NL}^\pi_{m,p,\mathbf{d}} := \int_C f_\pi^*\big(D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}\big),$$

where $f : C \to \mathcal{K}_\Lambda$ is the morphism induced from $(\pi : X \to C, L_1, \dots, L_r)$ by the universal property of $\mathcal{K}_\Lambda$, and $\mathbf{v}_\Lambda$ is the integral basis of Λ defined by $(L_1, \dots, L_r)$ through point (4) of Definition (6.5).

Note that this is a classical intersection product (i.e., there is no need to define a virtual class), although it may be given by an excess formula in case the image $f_\pi(C)$ is fully contained in the divisor $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}$.

(6.7) Gromov–Witten invariants for vertical curve classes.. Although it may not be a projective variety, the total space X carries a $(1,1)$-form ω_π which is Kähler on the fibres of π; this is sufficient to define Gromov–Witten theory for non-zero *vertical* classes $\gamma \in \mathrm{H}_2(X, \mathbf{Z})^\pi$, i.e., classes $\gamma \in \mathrm{H}_2(X, \mathbf{Z})$ such that $\pi_*(\gamma) = 0$ (see [26, §2.1] for details).

We thus have a set of invariants

$$N^X_{g,\gamma} := \int_{[\overline{M}_g(X,\gamma)]^{\mathrm{vir}}} 1$$

for non-zero *vertical* classes γ, where the moduli spaces of genus g stable maps $\overline{M}_g(X,\gamma)$ all have virtual dimension 0. We consider the invariants $n^X_{g,\gamma}$ obtained from the $N^X_{g,\gamma}$ by applying the BPS corrections packaged in the formula of (5.4): we let

$$F^X(u,v) := \sum_{g\geqslant 0} \sum_{0\neq\gamma\in \mathrm{H}_2(Z,\mathbf{Z})^\pi} N^X_{g,\gamma}\, u^{2g-2} v^\gamma$$

as a formal power series in the variables u, v, where the powers of v are indexed by $\mathrm{H}_2(Z,\mathbf{Z})^\pi$, and set

$$F^X(u,v) := \sum_{g\geqslant 0} \sum_{0\neq\gamma\in \mathrm{H}_2(Z,\mathbf{Z})^\pi} n^X_{g,\gamma}\, u^{2g-2} \left(\sum_{d>0} \frac{1}{d} \left(\frac{\sin d\frac{u}{2}}{\frac{u}{2}} \right)^{2g-2} v^{d\gamma} \right).$$

Eventually, for a non-zero multidegree $\mathbf{d} = (d_1,\dots,d_r) \in \mathbf{Z}^r$, we let $n^X_{g,\mathbf{d}}$ be the invariant counting genus g stable maps in vertical classes of degree $d_1,\dots,d_r$ with respect to $L_1,\dots,L_r$ respectively, i.e.,

$$n^X_{g,\mathbf{d}} := \sum_{\gamma\in\mathrm{H}_2(X,\mathbf{Z})^\pi:\ \int_\gamma L_i = d_i} n^X_{g,\gamma}. \tag{6.7.1}$$

(6.8) Reduced Gromov–Witten invariants of $K3$ fibres.. We also consider the invariants $r^p_{g,m}$ for $K3$ surfaces which have been defined in (5.4); recall they are the reduced Hodge integrals (5.3.1) put under BPS form.

We need to maintain the dependency on the divisibility index m, because Theorem (6.9) below is needed for the proof of the independence on m conjectured by Yau–Zaslow.

A multidegree $\mathbf{d} = (d_1,\dots,d_r) \in \mathbf{Z}^r$ is positive with respect to L^π if for any line bundle M on some fibre X_t of π, $(M, L_{i,t}) = d_i$ for all i implies $(M, L^\pi) > 0$; since L^π is a linear combination of the L_i this is an elementary linear algebraic condition.

(6.9) Theorem ([26])**.** *Let $\mathbf{d} = (d_1,\dots,d_r) \in \mathbf{Z}^r$ be a multidegree positive with respect to L^π. Then*

$$n^X_{g,\mathbf{d}} = \sum_{p=0}^{+\infty} \sum_{m=1}^{+\infty} r^p_{g,m} \cdot \mathrm{NL}^\pi_{m,p,\mathbf{d}}. \tag{6.9.1}$$

(This is stated in [26] in the $r = 1$ case (*i.e.*, $\mathbf{d} \in \mathbf{Z}$), but as noted in [20] the same proof goes through in general).

The philosophy behind this relation is rather natural, and ought to be compared to the discussion of subsection 5.1 above. Consider the genus $g = 0$ case for simplicity; then the invariant $n^X_{0,\mathbf{d}}$ counts vertical rational curves in X of prescribed degrees with respect to $L_1,\dots,L_r$, and these are virtually in finite number. There are on the other hand finitely many members of the family π with algebraic divisor classes of the prescribed degrees with respect $L_1,\dots,L_r$, and each of these provides a finite number of rational curves. The theorem morally says that the number of rational curves in X is the sum of these isolated contributions from the fibres.

Of course the actual story is more complicated than this, if only because of the existence of 1-dimensional families of rational curves on X, coming from finitely many rational curves in *all* $K3$ members of the family (which all have algebraic divisor classes, as they are Λ-polarized), in spite of the virtual dimension being 0. In other words, a Calabi-Yau threefold X as in Theorem (6.9) above is far from satisfying the same properties than the perturbations of the twistor families of algebraic $K3$ surfaces on which BPS numbers are supposed to count curves.

6.3 – Application to the Yau–Zaslow conjecture

In this subsection we give an outline of the proof by Klemm, Maulik, Pandharipande, and Scheidegger of the Yau–Zaslow conjecture (Theorem (4.14) above). Recall that the invariants $r^p_{g,m}$ being invariant under algebraic deformations of the $K3$ surface, it is enough to prove the result for our favourite $K3$ surface. These invariants for certain elliptic $K3$ surfaces are approached by means of the relation (6.9) for a particular family.

(6.10) The *STU* model.. The central character of the proof is a smooth projective Calabi–Yau 3-fold X, known as the STU model and coming from physics (quoting [20], the letter S stands for the dilaton and T and U label the torus moduli in the heterotic string). It is contructed as an anticanonical section of a smooth projective toric 4-fold Y defined by an explicit fan in $\mathbf{Z}^4$.

The variety X has the structure of a fibration $\pi : X \to \mathbf{P}^1$, the general fibre of which is a smooth $K3$ surface, itself with an elliptic fibration. It comes with two line bundles $L_1, L_2 \to X$, defining a Λ-polarization on the family $\pi : X \to \mathbf{P}^1$ (leaving aside the fact that there are inevitably singular members), where Λ is the lattice with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The family π has the shape of a Lefschetz pencil, in particular each of its singular members has a unique ordinary double point as its only singularity. One may thus build an actual family $\tilde{\pi} : \tilde{X} \to C$ of Λ-polarized $K3$ surfaces from π as follows. One first performs a base change by $t \mapsto t^2$ around each singular member; to do so, one considers the $2:1$ covering $\varepsilon : C \to \mathbf{P}^1$ with branch divisor $\mathrm{Disc}(\pi)$, the set of points above which π fails to be smooth, and let $\pi^\flat : X^\flat \to C$ be the family obtained from π by applying the base change $\varepsilon : C \to \mathbf{P}^1$. The new total space $X^\flat$ is singular, precisely it has an ordinary 3-fold double point at each singular point of a fibre (analytically locally around such a point, X is defined by the equation $x^2 + y^2 + z^2 = t$ in a 4-dimensional complex ball, hence $X^\flat$ is defined by $x^2 + y^2 + z^2 = t^2$). One then chooses for $\tilde{X}$ any small resolution of all these singularities: this may be understood as first blowing-up once all singular points, and then contracting one ruling of each exceptional divisors (they are all smooth quadric surfaces). This has the effect of replacing each fibre of $X^\flat$ by its minimal model.

One may determine the number of singular members of π by the same topological Euler characteristic computation as in subsection 2.1. The Euler number $e(X)$ is found to be -480 by toric intersection in the 4-fold Y, and then the number of singular fibres equals

$$e(K3) \cdot e(\mathbf{P}^1) - e(X) = 528.$$

(6.11) Modularity for Noether–Lefschetz numbers.. It is a stunning application of a theory developed by Borcherds and Kudla–Millson (see [26, 20] and the references therein) that the Noether–Lefschetz numbers of a family of Λ-polarized $K3$ surfaces fit into a vector valued modular form.

Let notation be as in (6.6) for a moment, in order to state this precisely (see [26, § 4] for a complete treatment). One may define divisors $D^{\mathbf{v}_\Lambda}_{p,\mathbf{d}}$ and subsequently numbers $\mathrm{NL}^{\pi}_{p,\mathbf{d}}$ by dropping the requirement on the divisibility index m in (6.4). It is an elementary result [20, Lemma 1] that the full set of the numbers $\mathrm{NL}^{\pi}_{p,\mathbf{d}}$ determine the refined Noether–Lefschetz numbers $\mathrm{NL}^{\pi}_{m,p,\mathbf{d}}$. Let $\mathrm{Mp}_2(\mathbf{Z})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbf{Z})$. There is a canonical representation

$$\rho^*_\Lambda : \mathrm{Mp}_2(\mathbf{Z}) \to \mathrm{End}\big(\mathbf{C}[G_\Lambda]\big)$$

associated to Λ (recall that $G_\Lambda = \Lambda^\vee/\Lambda$).

(6.11.1) Theorem.. (Borcherds, Kudla–Millson, Maulik–Pandharipande) *There exists a vector-valued modular form*

$$\Phi^\pi(q) = \sum_{\gamma\in G} \Phi^\pi_\gamma(q)\, u^\gamma \in \mathbf{C}[[q^{\frac{1}{2\mathrm{Disc}\Lambda}}]] \otimes \mathbf{C}[G]$$

of weight $\frac{22-r}{2}$ *and type* ρ^*_Λ, *such that the Noether–Lefschetz number* $\mathrm{NL}^{\pi}_{p,\mathbf{d}}$ *is the coefficient of* Φ^π_γ *in* q *to the power* $\frac{\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}}}{2\mathrm{Disc}\Lambda}$, *where* $\gamma \in G$ *is any of the two liftings of the coset* $\delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} \in G/\pm$ *represented by the linear functional* $v_i \mapsto d_i$.

Taking advantage of the strong structure results for modular forms, Maulik and Pandharipande are able to use this theorem to derive explicitly the Noether–Lefchetz numbers of classical families of $K3$ surfaces of genus $2 \leqslant p \leqslant 5$ (i.e., double planes and complete intersection $K3$'s).

A similar calculation is carried out in [20] for the *STU* family, as one of the key steps in the proof of the Yau–Zaslow conjecture. We now return to the notation of (6.10). Theorem (6.11.1) tells that the Noether–Lefschetz numbers of the family $\tilde{\pi} : \tilde{X} \to C$ are the Fourier coefficients of a scalar modular form of weight 10. The vector space of such forms has dimension 1 and is generated by the Eisenstein series

$$E_5(q) = E_2(q)E_3(q) = 1 - 264 \sum_{n=1}^{+\infty} \sigma_9(n) q^n$$

[39, § VII.3.2] (notation as in (3.6)). It follows that it is enough to know one Noether–Lefschetz number to determine the full modular form, and since we do know of them, given by the number 528 of singular members of the *STU* family, one obtains that the number $\mathrm{NL}^{\tilde{\pi}}_{p,d_1,d_2}$ is the coefficient in q to the power $\frac{1}{2}\Delta(p, d_1, d_2)$ of the modular form $-4E_2(q)E_3(q)$, where

$$\Delta(p, d_1, d_2) = \begin{vmatrix} 0 & 1 & d_1 \\ 1 & 0 & d_2 \\ d_1 & d_2 & 2p-2 \end{vmatrix}.$$

(6.12) Mirror symmetry.. The *STU* model X being an anticanonical section of a smooth semi-positive toric variety, its genus 0 Gromov–Witten invariants are known by mathematically proven mirror symmetry results. This gives the corresponding invariants of $\tilde{X}$, the latter being twice those of X [26, § 5.2].

Precisely, Givental has proven the relation of the genus 0 Gromov–Witten invariants of X by mirror transformation to hypergeometric solutions of the Picard–Fuchs equations of the Batyrev–Borisov mirror, see [26, 20] and the references therein. This gives the following formula of Klemm–Mayr–Lerche [20, Prop. 5]

$$\sum_{(d_1,d_2)\in\mathcal{P}} (d_2)^3 N^X_{0,(d_1,d_2)}\, q_1^{d_1} q_2^{d_2} = -2 + 2\, \frac{E_2(q_1)E_3(q_1)}{(2\pi)^{-12}\Delta(q_1)}\, \frac{E_2(q_2)}{j(q_1) - j(q_2)}, \tag{6.12.1}$$

where

$$j(q) := 1728\,\frac{\big(60G_2(q)\big)^3}{\Delta(q)} = (2\pi)^{12}\frac{E_2(q)^3}{\Delta(q)} = \frac{1}{q} + 744 + 196884q + \cdots$$

(notation as in (3.6)) is the normalized j function, $\mathcal{P} = \big\{(d_1,d_2) \neq (0,0) : d_1 \geqslant 0, d_1 \geqslant -d_2\big\}$, and $N^X_{0,(d_1,d_2)}$ is defined by formula (6.7.1) from the various $N^X_{0,\gamma}$, $\gamma \in \mathrm{H}_2(X,\mathbf{Z})^\pi$.

(6.13) Conclusion: the Harvey–Moore identity.. Using the fact that the lattice Λ has rank 2, Klemm–Maulik–Pandharipande–Scheidegger then show that the invariants $r^p_{0,m}$ are uniquely determined by the relations (6.9.1) for the family $\tilde\pi : \tilde X \to C$ and the numbers $n^{\tilde X}_{0,(d_1,d_2)}$ and $\mathrm{NL}^{\tilde\pi}_{m,p,(d_1,d_2)}$ [20, Prop. 3]. The latter two sets of numbers being known by the results of (6.11) and (6.12), it is therefore enough, in order to end the proof of Theorem (4.14), to show that the numbers $r^p_{0,m}$ predicted by the Yau–Zaslow conjecture (i.e., $r^p_{0,m} = r^{m^2p-m^2+1}_{0,1}$ together with the formula of (2.1) giving the $r^p_{0,1}$'s) indeed fit in the relations (6.9.1).

This takes the form of an identity between modular forms: let

$$f(z) := \frac{E_2(z)E_3(z)}{(2\pi)^{-12}\Delta(z)} = \sum_{n=-1}^{+\infty} c(n)q^n, \quad q = e^{2\pi i z};$$

what has to be proven is

$$\frac{f(z_1)E_2(z_2)}{j(z_1)-j(z_2)} = \frac{q_1}{q_1-q_2} + E_2(z_2) - \sum_{d,k,l>0} l^3 c(kl)\, q_1^{kd} q_2^{ld}.$$

This is the Harvey–Moore identity, which has been proven by Zagier, see [20, § 4.2].

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Lecture XII
[coming soon]Some classical formulæ for curves and surfaces

by Thomas Dedieu

Appendix C
Generalized weight properties of resultants and discriminants, and applications to projective enumerative geometry

by Laurent Busé and Thomas Dedieu

In his wonderful book [17], first published in 1862, Salmon casually gives the leading term of some Taylor expansion of the discriminant of a plane algebraic curve. Consider the curve $\mathrm{V}(F) \subseteq \mathbf{P}^2$; in a suitable system of homogeneous coordinates, its equation F has the form

$$F(X,Y,Z) = TZ^{d-1}Y + \frac{1}{2}Z^{d-2}(AX^2 + 2BXY + CY^2) + \sum_{k\geqslant 3} Z^{d-k}F_k(X,Y) = 0,$$

where each polynomial $F_k(X,Y)$ is homogeneous of degree k in X, Y. Salmon states, without any explanation or reference, that the discriminant of F is of the form

$$\mathrm{Disc}(F) = T^2A(B^2 - AC)^2\,\Phi + T^3\,\Psi,$$

where Φ is "the discriminant when T vanishes" [17, § 605]. This note arose as an attempt to understand and prove this formula.

We introduce the *reduced discriminant* of a hypersurface (or homogeneous polynomial), of which Φ is an instance, it turns out. It is deeply linked with the *reduced resultant*, introduced by Zariski in [20], in 1936 long after Salmon. Of course, when T is zero, $\mathrm{V}(F)$ is singular at the point $(0:0:1)$, no matter the other coefficients of F, and correspondingly $\mathrm{Disc}(F)$ vanishes identically. The polynomial Φ, in fact, vanishes at those values of the coefficients of F for which the curve $\mathrm{V}(F)$ has some additional singularity besides the point $(0:0:1)$. The reduced discriminant generalizes this to the case when F has a singular point of arbitrary multiplicity s. Somehow built in the theory is a general form of Formula (𓅓). We show how to get (𓅓) specifically (in fact with a normalization factor $-\frac{1}{2}$, which probably was not of any interest for Salmon), and generalize the formula for hypersurfaces of arbitrary dimension, see Section 2.3. This is certainly what Salmon had in mind, although arguably only for $s=2$, and possibly only empirically. It had since then been apparently completely forgotten; we have only found a faint trace of these ideas in Salmon's works, see Paragraph (2.18). We point out that the reduced discriminant is a particular instance of the theory of toric discriminants studied in [11], which is more general in terms of the family of hypersurfaces that can be considered. The methodology that we develop here in the specific setting of the reduced discriminant is different from the theory of toric discriminants, and we found it particularly well adapted, if not necessary, to tackle Salmon's formula.

Salmon then uses Formula (𓅓) to derive various enumerative quantities for surfaces $S \subseteq \mathbf{P}^3$. In particular, he computes the number of bitangent planes passing through a fixed general point $p \in \mathbf{P}^3$. We explain his method, which involves a remarkable trick in elimination theory, and generalize it for hypersurfaces of arbitrary dimension. Salmon's strategy is to consider a pencil of planes with center a line tangent to the surface S at some point p'; this pencil contains a finite number of planes tangent to S, among which the tangent plane at p' counts with multiplicity 2 in general, and with greater multiplicity if it has some special feature, e.g., if it is a bitangent plane. In an appropriate setting, this multiplicity is the valuation in T of the polynomial in (𓅓), and the game is to understand the conditions on the point p' that make this multiplicity jump. In the upshot we get, indirectly from the reduced discriminant Φ, a curve in S parametrizing those points p' such that the tangent plane at p' is a bitangent plane; following Salmon, we call it the *node-couple curve*. There are other well-known ways to compute the number of bitangent planes to a surface in $\mathbf{P}^3$ (see, e.g., [XII]). The present method has the advantage of being core level: it is only a study of homogeneous polynomials and their discriminants.

The present text includes two other projective enumerative computations taken from Salmon's book; they both fit in the framework of what we would call *reduced elimination theory*. The first one gives the number of bitangent lines to a surface in $\mathbf{P}^3$ through a general point. We generalize it in two ways: to the enumeration of lines in $\mathbf{P}^n$ having contact of the form $2p+(n-1)q+\cdots$ with a hypersurface on the one hand, and to the enumeration of bitangent codimension 2 linear spaces on the other hand. While Salmon's computation and the first generalization do not involve openly the reduced discriminant (thanks to the existence of a cheap substitute in the two indeterminates case), it is essential for the latter generalization. The second computation is that of the degree of the *flecnodal curve* of a surface $S \subseteq \mathbf{P}^3$, which is the locus of points p at which the tangent plane cuts out a curve with one of its two branches at p that has an inflection point. We present here Salmon's ideas, noting that this computation is carried out in modern standards and in arbitrary dimension in [2].

We also give a synthetic account of the theory of resultants and discriminants, in its modern form worked out by Jouanolou, see, e.g., [14]. On the one hand this is necessary for us to develop the theory of the reduced discriminant (the reduced resultant had already been treated by Ould Mohamdi in [15], under the direction of Jouanolou; note, however, that passing from the resultant to the discriminant is not straightforward, see, e.g., , the comments after Definition (2.10)). On the other hand, we believe this text is a good occasion to make Jouanolou's formalism accessible to the XXI$^{\text{st}}$ century classical algebraic geometer.

We put particular focus on the homogeneity properties of the resultants and discriminants, with respect to various gradings. They are indeed essential for the enumerative applications, and we give numerous illustrations, including a proof of Bezout's theorem and the computation of the degree of the dual of a smooth hypersurface. Beyond that, they are central in the theory of reduced resultants and discriminants, see Theorem (1.18) and Corollary (2.19), and indeed the key to Formula () and its generalizations.

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(0.1) Notation. Let $\mathbf{k}$ be a commutative ring, $n \geqslant 1$ be an integer, and $\mathbf{X} := (X_0, X_1, \ldots, X_n)$ be a sequence of indeterminates. Given a multi-index $\boldsymbol{\alpha} := (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, we denote by $\mathbf{X}^{\boldsymbol{\alpha}}$ the monomial $X_0^{\alpha_0} X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, and set $|\boldsymbol{\alpha}| = \sum_{i=0}^n \alpha_i$.

1 – Reduced resultant

Let $F_1, \ldots, F_{n+1}$ be $n+1$ homogeneous polynomials in the indeterminates $X_0, X_1, \ldots, X_n$. They define a collection of $n+1$ hypersurfaces in the projective space $\mathbf{P}^n$, and the intersection of these $n+1$ hypersurfaces is empty if $F_1, \ldots, F_{n+1}$ are general. In fact, the emptiness of this intersection is characterized by the non-vanishing of the resultant $\mathrm{Res}(F_1, \ldots, F_{n+1})$ of these polynomials (see §1.1). Thus, the resultant characterizes those collections of homogeneous polynomials that have a common non-trivial root. The purpose of the *reduced resultant* is similar: one still considers $n+1$ homogeneous polynomials $F_1, \ldots, F_{n+1}$, but this time one assumes that they have a common non-trivial root; then the reduced resultant will characterize those collections of polynomials having an additional common non-trivial root. It has been introduced by Zariski [20]; a more complete and modern treatment is given in [15].

In what follows, we use the resultant of multivariate homogeneous polynomials as developed in [14] (see also [6, 7, 11]). An introduction and a brief recap are given in Section 1.1 below.

1.1 – Inertia forms and the ordinary multivariate resultant

We follow the beautiful presentation in [19, Chapter XI] (beware that this Chapter on Elimination Theory has disappeared in later editions).

(1.1) Saturation of a homogeneous ideal. We recall the following for the convenience of the reader; see, e.g., [12, Lecture 5] or [13, Exercise II.5.10] for more details. The *saturation* of a homogeneous ideal $I \subseteq \mathbf{k}[\mathbf{X}]$ is the homogeneous ideal

$$\begin{aligned} \bar{I} &= \left\{ F \in \mathbf{k}[\mathbf{X}] : \forall i = 0, \ldots, n,\ \exists N_i \text{ such that } X_i^{N_i} F \in I \right\} \\ &= I : (X_0, \ldots, X_n)^\infty. \end{aligned} \tag{1.1.1}$$

For sufficiently large m, the graded pieces I_m and $\bar{I}_m$ are equal. Moreover, for two homogeneous ideals I and J, the following three propositions are equivalent:
(i) $\bar{I} = \bar{J}$;
(ii) $I_m = J_m$ for sufficiently large m;
(iii) $I \cdot \mathbf{k}[\mathbf{X}, \frac{1}{X_i}] = J \cdot \mathbf{k}[\mathbf{X}, \frac{1}{X_i}]$ for all $i = 0, \ldots, n$.
In other words, a subscheme $V \subseteq \mathbf{P}^n_{\mathbf{k}}$ is defined (scheme-theoretically) by a homogeneous ideal $I \subseteq \mathbf{k}[\mathbf{X}]$ if and only if the saturation $\bar{I}$ equals the homogeneous ideal I_V of V.

In particular, if $\mathbf{k}$ is a field, a subscheme $V \subseteq \mathbf{P}^n_{\mathbf{k}}$ defined by a homogeneous ideal I is empty if and only if the degree 0 piece $\bar{I}_0$ is non-zero: this is the homogeneous Nullstellensatz! (see also [5]). On the other hand, the non-emptiness of V is equivalent to the existence of a point in $V(\mathbf{k}')$ for some finite field extension $\mathbf{k}'$ of $\mathbf{k}$.

When $\mathbf{k}$ is an arbitrary commutative integral domain,[1] the vanishing of $\bar{I}_0$ is equivalent to the scheme-theoretic image of the map $V \to \operatorname{Spec}(\mathbf{k})$ being equal to the whole $\operatorname{Spec}(\mathbf{k})$. Indeed, the subscheme of $\operatorname{Spec}(\mathbf{k})$ defined by $\bar{I}_0$ coincides as a set with the image of $V \to \operatorname{Spec}(\mathbf{k})$ — this is the proof that projective morphisms are closed —, and moreover $\bar{I}_0$ defines the scheme-theoretic image of $V \to \operatorname{Spec}(\mathbf{k})$ (see, e.g., [14, §1] for more details).

(1.2) Let $d_1, \dots, d_r$ be positive integers. For all $j = 1, \dots, r$, we consider the *generic* homogeneous degree d_j polynomial in the indeterminates $\mathbf{X} = (X_0, \dots, X_n)$

$$F_j := \sum_{|\boldsymbol{\alpha}|=d_j} U_{j,\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}.$$

"Generic" refers to the fact that the coefficients of the F_j's are indeterminates; indeed, F_j is an incarnation of the generic point, in the sense of schemes, of the affine space parametrizing homogeneous degree d_j polynomials.

We let $A_{\mathbf{Z}}$ be the algebra generated by the indeterminates coefficients of the F_j's, i.e., we set $A_{\mathbf{Z}} := \mathbf{Z}\big[U_{j,\boldsymbol{\alpha}}\big]_{j=1,\dots,r,\,|\boldsymbol{\alpha}|=d_j}$. Thus, $F_j \in A_{\mathbf{Z}}[\mathbf{X}]$ for all $j = 1, \dots, r$.

Let $\mathbf{k}$ be a commutative ring, and let $u_{j,\boldsymbol{\alpha}} \in \mathbf{k}$ for for all $j = 1, \dots, r$ and $|\boldsymbol{\alpha}| = d_j$. For all $T \in A_{\mathbf{Z}}[\mathbf{X}]$, we denote by $T(u_{j,\boldsymbol{\alpha}}) \in \mathbf{k}[\mathbf{X}]$ the polynomial obtained by evaluating the indeterminates $U_{j,\boldsymbol{\alpha}}$ at $u_{j,\boldsymbol{\alpha}} \in \mathbf{k}$ for all $j = 1, \dots, r$ and $|\boldsymbol{\alpha}| = d_j$ in the polynomial T.

(1.3) Definition. *An* inertia form *for the polynomials $F_1, \dots, F_r$ is an element $T \in A_{\mathbf{Z}}[\mathbf{X}]$ such for all $i = 0, \dots, n$, there exists $N_i \in \mathbf{N}$ such that $X_i^{N_i} T \in (F_1, \dots, F_r)$.*

In other words, the inertia forms for $F_1, \dots, F_r$ are the elements of the saturation of the ideal $(F_1, \dots, F_r)$ in $A_{\mathbf{Z}}[\mathbf{X}]$. We denote by $\mathfrak{I}_{\mathbf{Z}}$ the ideal of all degree 0 inertia forms for $F_1, \dots, F_r$; thus, in the notation introduced in (1.1.1),

$$\mathfrak{I}_{\mathbf{Z}} = \big(I : (X_0, \dots, X_n)^\infty\big) \cap A_{\mathbf{Z}}.$$

Note that this is a homogeneous ideal with respect to the standard grading on $A_{\mathbf{Z}}$. In view of the above considerations, we have the following.

(1.4) Theorem. *Suppose that $\mathbf{k}$ is a field, and let $u_{j,\boldsymbol{\alpha}} \in \mathbf{k}$ for for all $j = 1, \dots, r$ and $|\boldsymbol{\alpha}| = d_j$. Consider the polynomials $F_1(u_{1,\boldsymbol{\alpha}}), \dots, F_r(u_{r,\boldsymbol{\alpha}}) \in \mathbf{k}[\mathbf{X}]$. The two following propositions are equivalent:*
(i) the ideal $\big(F_1(u_{1,\boldsymbol{\alpha}}), \dots, F_r(u_{r,\boldsymbol{\alpha}})\big)$ defines a non-empty subscheme of $\mathbf{P}^n_{\mathbf{k}}$;
(ii) for all $T \in \mathfrak{I}_{\mathbf{Z}}$, $T(u_{j,\boldsymbol{\alpha}}) = 0$.

This tells us that a given specialization to a field of the polynomials F_j defines a non-empty subscheme if and only if all the constants in the saturation of $(F_1, \dots, F_r)$ vanish in this specialization (see also [5]). Note that for all $T \in \mathfrak{I}_{\mathbf{Z}}$, one has $T(u_{j,\boldsymbol{\alpha}}) \in \mathbf{k}$, since T has degree 0 in $\mathbf{X}$.

We emphasize that in general the subscheme of $\operatorname{Spec}(\mathbf{k})$ defined by the specialization of $\mathfrak{I}_{\mathbf{Z}}$ coincides only set-theoretically with the scheme-theoretic image of $V \to \operatorname{Spec}(\mathbf{k})$ (see [7, §3, Remarque 1] and [14, §1]), which is the reason why we assume that $\mathbf{k}$ is a field in Theorem (1.4).

[1] this arguably unconventional notation is used ubiquitously by Jouanolou.

If $\mathbf{k}$ is an arbitrary commutative ring, what is indeed true is that the subscheme defined by the $F_j(u_{j,\alpha})$'s surjects onto $\mathrm{Spec}(\mathbf{k})$ as a set if and only if $\mathfrak{I}_{\mathbf{Z}} \otimes_{A_{\mathbf{Z}}} \mathbf{k}$ is contained in the nilradical $\sqrt{(0)}$ of $\mathbf{k}$, but this says nothing more than Theorem (1.4).

(1.5) Theorem (see [4, §2.1])**.** *The ideal of inertia forms for $F_1, \dots, F_r$ is prime, and so is the ideal $\mathfrak{I}_{\mathbf{Z}} \subseteq A_{\mathbf{Z}}$ of inertia forms of degree 0.*

The *resultant situation* is when we consider $n+1$ homogeneous equations in $\mathbf{P}^n$.

(1.6) Theorem. *If $r = n+1$, the ideal $\mathfrak{I}_{\mathbf{Z}}$ is principal. Up to sign it has a unique generator, which is an irreducible element of $A_{\mathbf{Z}}$; we denote it by $\mathrm{Res}_{d_1,\dots,d_{n+1}} \in A_{\mathbf{Z}}$.*

Moreover, for all $k \in [\![1, n+1]\!]$, $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ is homogeneous of degree $\prod_{j\neq k} d_j$ with respect to the coefficients of the polynomial F_k, i.e., with respect to the indeterminates $U_{k,\boldsymbol{\alpha}}$, $|\boldsymbol{\alpha}| = d_k$ (all assumed to have weight one).

Let $f_1, \dots, f_{n+1} \in \mathbf{k}[\mathbf{X}]$ be polynomials of respective degrees $d_1, \dots, d_{n+1}$. They are specializations $F_1(u_{1,\boldsymbol{\alpha}}), \dots, F_r(u_{r,\boldsymbol{\alpha}})$ of $F_1, \dots, F_{n+1} \in A_{\mathbf{Z}}[\mathbf{X}]$ respectively, for an appropriate canonical choice of $u_{j,\boldsymbol{\alpha}} \in \mathbf{k}$, for all $j = 1, \dots, n+1$ and $|\boldsymbol{\alpha}| = d_k$. We let $\mathrm{Res}(f_1, \dots, f_{n+1}) \in \mathbf{k}$ (or $\mathrm{Res}_{d_1,\dots,d_{n+1}}(f_1, \dots, f_{n+1}) \in \mathbf{k}$, if we want to emphasize the dependency on the degrees) be the corresponding specialization of $\mathrm{Res}_{d_1,\dots,d_{n+1}} \in A_{\mathbf{Z}}$. The multi-homogeneity property stated in the above theorem may then be rephrased as follows: for all $\lambda \in \mathbf{k}$,

$$\mathrm{Res}_{d_1,\dots,d_{n+1}}(f_1, \dots, \lambda f_k, \dots, f_{n+1}) = \lambda^{d_1 \cdots d_{k-1} d_{k+1} \cdots d_{n+1}} \mathrm{Res}_{d_1,\dots,d_{n+1}}(f_1, \dots, f_k, \dots, f_{n+1}).$$

The sign indeterminacy in the definition of $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ is usually removed by imposing the normalizing equality $\mathrm{Res}(X_0^{d_0}, \dots, X_n^{d_n}) = 1$.

(1.7) Divisibility property (see, e.g., [14, §5.6])**.** Let $F_1, \dots, F_{n+1}$ and $G_1, \dots, G_{n+1}$ be two sequences of homogeneous polynomials in $\mathbf{k}[\mathbf{X}]$ such that we have the inclusion of ideals of $\mathbf{k}[\mathbf{X}]$,

$$(G_1, \dots, G_{n+1}) \subseteq (F_1, \dots, F_{n+1}).$$

Then, $\mathrm{Res}(F_1, \dots, F_{n+1})$ divides $\mathrm{Res}(G_1, \dots, G_{n+1})$ in $\mathbf{k}$.

Besides its ordinary multi-homogeneity property given in Theorem (1.6), the resultant has other homogeneous structures that we call "weight properties" to emphasize that the grading of the coefficient ring $A_{\mathbf{Z}}$ is not the standard one.

(1.8) Proposition (see [14, §5.13.2])**.** *In the notation of Paragraph (1.2), let k be an integer in $[\![0, n]\!]$, and consider the grading on $A_{\mathbf{Z}} = \mathbf{Z}[U_{j,\boldsymbol{\alpha}}]_{j=1,\dots,n+1,\, |\boldsymbol{\alpha}|=d_j}$ defined by*

$$\mathrm{weight}(U_{j,\boldsymbol{\alpha}}) = \alpha_k. \tag{1.8.1}$$

In this grading, the resultant $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ is homogeneous of degree $d_1 d_2 \cdots d_{n+1}$.

(1.9) The Bezout Theorem, which counts the number of roots of a finite complete intersection scheme in a projective space, and thus is the mother of all statements in projective enumerative geometry, can be deduced from this property. This goes as follows.

Consider n homogeneous polynomials $F_1, \dots, F_n \in \mathbf{k}[X_0, \dots, X_n]$, of degrees $d_1, \dots, d_n$ respectively. The idea is to project $\mathbf{P}^n$ to $\mathbf{P}^1$ from a codimension 2 linear space, in order to reduce to plain polynomials in one indeterminate only, i.e., homogeneous polynomials in two indeterminates. We thus consider

$$\tilde{F}_j(T_0, T_1, \dots, T_{n-1}) = F_j\big(T_0X_0,\, T_0X_1,\, T_1, \dots, T_{n-1}\big) \in \mathbf{k}[X_0, X_1][T_0, \dots, T_{n-1}] \tag{1.9.1}$$

for all $j = 1, \dots, n$. A point $(x_0 : x_1) \in \mathbf{P}^1_{\mathbf{k}}$ sits in the projection of $\mathrm{V}(F_1, \dots, F_n) \subseteq \mathbf{P}^n$ if and only if the specializations of $\tilde{F}_1, \dots, \tilde{F}_n$, obtained by specializing X_0 and X_1 to x_0 and x_1 respectively, have a common zero in $\mathbf{P}^{n-1}$. We thus consider the resultant of $\tilde{F}_1, \dots, \tilde{F}_n$ with respect to the indeterminates $T_0, \dots, T_{n-1}$,

$$\mathrm{Res}\big(\tilde{F}_1, \dots, \tilde{F}_n\big) \in \mathbf{k}[X_0, X_1].$$

Writing $F_j = \sum_{|\boldsymbol{\alpha}|=d_j} u_{j,\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}$, one has

$$\tilde{F}_j = \sum_{|\boldsymbol{\alpha}|=d_j} \big(u_{j,\boldsymbol{\alpha}} X_0^{\alpha_0} X_1^{\alpha_1}\big)\, T_0^{\alpha_0+\alpha_1} T_1^{\alpha_2} \cdots T_{n-1}^{\alpha_n}.$$

Thus, $\tilde{F}_j$ has degree d_j in $T_0, \dots, T_{n-1}$, and its coefficient in the monomial $T_0^{\tilde{\alpha}_0} T_1^{\tilde{\alpha}_1} \cdots T_{n-1}^{\tilde{\alpha}_{n-1}}$ is a degree $\tilde{\alpha}_0$ homogeneous polynomial in X_0, X_1.

Then, it follows from the above Proposition (1.8) that

$$\mathrm{Res}_{d_1,\dots,d_n}\big(\tilde{F}_1, \dots, \tilde{F}_n\big) \in \mathbf{k}[X_0, X_1]$$

is homogeneous of degree $d_1 \cdots d_n$. Therefore, $\mathrm{V}\big(\mathrm{Res}(\tilde{F}_1, \dots, \tilde{F}_n)\big) \subseteq \mathbf{P}^1_{\mathbf{k}}$ consists of $d_1 \cdots d_n$ points, counted with multiplicities. One may then conclude that $\mathrm{V}(F_1, \dots, F_n) \subseteq \mathbf{P}^n_{\mathbf{k}}$ itself consists of $d_1 \cdots d_n$ points, by the classical arguments used to prove Bezout's theorem for the intersection of two plane curves, see e.g., [1, §4.5].

The following is a parent of the weight property given in Proposition (1.8) above. It is proven in [14, §5.13] as well.

(1.10) Proposition. *In the same situation as in Proposition (1.8) above, consider the grading on $A_{\mathbf{Z}}$ defined by*

$$\mathrm{weight}(U_{j,\boldsymbol{\alpha}}) = d_j - \alpha_k. \tag{1.10.1}$$

In this grading, the resultant $\mathrm{Res}_{d_1,\dots,d_{n+1}} \in A_{\mathbf{Z}}$ is homogeneous of degree $n d_1 d_2 \cdots d_{n+1}$.

One may also use the following corollary, based on the standard homogeneity of the resultant in Theorem (1.6), to obtain additional weight properties from (1.8) and (1.10).

(1.11) Corollary. *Assume that the resultant $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ is homogeneous of degree δ for the grading on $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{j,\boldsymbol{\alpha}}) = w_{j,\boldsymbol{\alpha}}$. Let $r_1, \dots, r_{n+1} \in \mathbf{Z}$. For the new grading on $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{j,\boldsymbol{\alpha}}) = w_{j,\boldsymbol{\alpha}} + r_j$, the resultant $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ is homogeneous of degree*

$$\delta + \sum_{1 \leqslant k \leqslant n+1} \Big(r_k \prod\nolimits_{j \neq k} d_j\Big).$$

Proof. Let $k \in [\![1, n+1]\!]$. Since the resultant is homogeneous of degree $\prod_{j \neq k} d_j$ with respect to the indeterminates $(U_{k,\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}|=d_k}$ (for the standard grading), a shift by r_k in the weights of all the indeterminates $(U_{k,\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}|=d_k}$ induces a shift by $r_k \prod_{j \neq k} d_j$ in the degree of the resultant. ☐

1.2 – The reduced resultant

We shall now explain how to adapt the ideas of the previous paragraph to develop the theory of the reduced resultant. We refer to [20] and [15] for the details and proofs. Somehow, this is a generalization of the following toy example.

(1.12) Example (projection of a complete intersection from one of its points)**.** Let $F, G \in \mathbf{k}[\mathbf{X}]$ be two homogeneous polynomials of degrees a and b, defining a complete intersection $V \subseteq \mathbf{P}^n$, and suppose one wants to project V from a point $p_0 \in \mathbf{P}^n$. Assume for simplicity that $\mathbf{k}$ is an algebraically closed field. We may take $p_0 = (1 : 0 : \ldots : 0)$. Then, we consider the two polynomials

$$\begin{aligned} & F(T, X_1, \ldots, X_n) = F_0 T^a + F_1 T^{a-1} + \cdots + F_a \\ \text{and} \quad & G(T, X_1, \ldots, X_n) = G_0 T^b + G_1 T^{b-1} + \cdots + G_b \end{aligned} \tag{1.12.1}$$

in $\mathbf{k}[X_1, \ldots, X_n][T]$. We are abusing notation here, as one should consider instead the two polynomials $F(T, S\,X_1, \ldots, S\,X_n)$ and $G(T, S\,X_1, \ldots, S\,X_n)$ that are homogeneous in the couple of indeterminates (S, T). Moreover, beware that (1.12.1) above is written "in reverse order" with respect to what we did when discussing the Bezout Theorem, see (1.9.1).

If $p_0 \notin V$, the point $(x_1 : \ldots : x_n) \in \mathbf{P}^{n-1}_{\mathbf{k}}$ belongs to the projection of V from p_0 if and only if the two polynomials in (1.12.1) have a common root in $\mathbf{P}^1$, hence the equation of the projection is given by

$$\mathrm{Res}_{a,b}(F, G) \in \mathbf{k}[X_1, \ldots, X_n],$$

which is homogeneous of degree ab in the indeterminates $(X_1, \ldots, X_n)$ by Proposition (1.10), as the polynomials F_i are homogeneous of degree i in $\mathbf{k}[X_1, \ldots, X_n]$.

On the other hand, if $p_0 \in V$ then, letting a' and b' be the respective multiplicities of p_0 in the hypersurfaces $\mathrm{V}(F)$ and $\mathrm{V}(G)$, one has

$$F_0 = \cdots = F_{a'-1} = G_0 = \cdots = G_{b'-1} = 0,$$

so that (1.12.1) becomes

$$\begin{aligned} & F(T, X_1, \ldots, X_n) = F_{a'} T^{a-a'} + \cdots + F_a = {}^\flat F \\ \text{and} \quad & G(T, X_1, \ldots, X_n) = G_{b'} T^{b-b'} + \cdots + G_b = {}^\flat G. \end{aligned} \tag{1.12.2}$$

It follows that the equation of the projection of V from p_0 is given by

$$\mathrm{Res}_{a-a',b-b'}({}^\flat F, {}^\flat G) \in \mathbf{k}[X_1, \ldots, X_n].$$

We shall see later on that this polynomial is the *reduced resultant* of F and G truncated at orders $a - a'$ and $b - b'$ respectively, as polynomials in the indeterminate T. It is a homogeneous polynomial of degree $ab - a'b'$ in $(X_1, \ldots, X_n)$: indeed, the coefficient of ${}^\flat F$ (respectively, ${}^\flat G$) in T^i is a homogeneous polynomial in $(X_1, \ldots, X_n)$ of degree $a - i = \deg_T({}^\flat F) - i + a'$ (respectively, $b - i = \deg_T({}^\flat G) - i + b'$). Therefore, Corollary (1.11) applied to the grading of Proposition (1.10) gives that $\mathrm{Res}_{a-a',b-b'}({}^\flat F, {}^\flat G)$ is homogeneous of degree

$$(a - a')(b - b') + a'(b - b') + b'(a - a') = ab - a'b',$$

as we had announced. This weight property is an instance of that given in Corollary (1.19), which applies to reduced resultants in general.

(1.13) Let $d_1, \ldots, d_{n+1}$ be positive integers. For all $j \in [\![1, n+1]\!]$, we consider the generic homogeneous degree d_j polynomial F_j, which we write as

$$F_j = \sum_{|\boldsymbol{\alpha}| = d_j} U_{j,\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}} = \sum_{k=0}^{d_j} X_0^{d_j - k} F_{j,k}(X_1, \ldots, X_n).$$

Thus, for all $k = 0, \dots, d_j$, the polynomial $F_{j,k}$ is a degree k homogeneous element of $A_{\mathbf{Z}}[X_1, \dots, X_n]$. We fix integers $s_j \in [\![1, d_j]\!]$ for all $j = 1, \dots, n+1$. The *truncation* of F_j at order $d_j - s_j$ with respect to X_0 is the polynomial

$$H_j = \sum_{k=s_j}^{d_j} X_0^{d_j-k} F_{j,k} = X_0^{d_j-s_j} F_{j,s_j} + \cdots + X_0 F_{j,d_j-1} + F_{j,d_j} \in A_{\mathbf{Z}}[\mathbf{x}].$$

This definition gives a special role to the indeterminate X_0, and to the point $(1:0:\ldots:0) \in \mathbf{P}^n$. One may thus think of the latter as the (chosen) origin in the affine space $\mathbf{P}^n - \mathrm{V}(X_0)$, and of $\mathrm{V}(X_0)$ as the hyperplane at infinity. We may occasionally use this terminology.

The purpose of reduced elimination theory is to study inertia forms of the truncations $H_1, \dots, H_{n+1}$ defined above. Note that, for all $j = 1, \dots, n+1$, the truncation H_j is the generic homogeneous polynomial of degree d_j with a multiplicity s_j zero in the origin $(1:0:\ldots:0)$. The wish for the reduced resultant is that it is a polynomial in the coefficients of $H_1, \dots, H_{n+1}$, which vanishes if and only if $H_1, \dots, H_{n+1}$ have a non-trivial common root in addition to that of multiplicity $s_1 \cdots s_{n+1}$ at the origin.

This can be done with essentially the same strategy as in the classical case, which we have reviewed in Section 1.1. We will only consider the "resultant situation", when the number of polynomials is $n+1$.

(1.14) Theorem ([20, Theorem 6 and §8] and [15, Theorem II.0.5 and §IV.0])**.** *Assume that $d_j > s_j$ for some $j \in [\![1, n+1]\!]$. The ideal of reduced inertia forms of degree 0*

$$\mathcal{Q}_{\mathbf{Z}} = \big((H_1, \dots, H_{n+1}) : (X_1, \dots, X_n)^\infty\big) \cap A_{\mathbf{Z}}$$

is a prime and principal ideal of $A_{\mathbf{Z}}$. The reduced resultant*, denoted*

$$\mathrm{redRes}_{d_1,\dots,d_{n+1}}^{s_1,\dots,s_{n+1}} \in A_{\mathbf{Z}},$$

is defined, up to sign, as the generator of $\mathcal{Q}_{\mathbf{Z}}$; it is therefore an irreducible element of $A_{\mathbf{Z}}$.

Moreover, if $d_j > s_j$ for at least two distinct integers $j, j' \in [\![1, n+1]\!]$, then for all $i \in [\![1, n+1]\!]$ the reduced resultant is a homogeneous polynomial of degree

$$\frac{d_1 d_2 \cdots d_{n+1}}{d_i} - \frac{s_1 s_2 \cdots s_{n+1}}{s_i}$$

with respect to the coefficients of the polynomial H_i, i.e., with respect to the indeterminates $U_{i,\boldsymbol{\alpha}}$ such that $|\boldsymbol{\alpha}| = d_i$ and $\alpha_0 \leqslant d_i - s_i$.

If there is only one integer $j \in [\![1, n+1]\!]$ such that $d_j > s_j$, then the reduced resultant is equal to the resultant of the polynomials $H_1, \dots, H_{j-1}, H_{j+1}, \dots, H_{n+1}$.

The sign indeterminacy in the definition of the reduced resultant can be removed by means of Theorem (1.18) below, once the sign of the ordinary resultant has been chosen.

Note that, in the above statement, the ideal $(H_1, \dots, H_{n+1})$ is saturated with respect to $(X_1, \dots, X_n)$, which is the defining ideal of the point $(1:0:\ldots:0)$, whereas for the plain resultant we considered instead the saturation with respect to the irrelevant ideal $(X_0, X_1, \dots, X_n)$. Beware moreover that the polynomials $H_1, \dots, H_{n+1}$ are not homogeneous in the set of indeterminates $(X_1, \dots, X_n)$.

The reduced resultant depends, of course, only on the coefficients of the generic truncated polynomials $H_1, \dots, H_{n+1}$, and not on all the coefficients of the polynomials $F_1, \dots, F_{n+1}$. We will often denote it by $\mathrm{redRes}(H_1, \dots, H_{n+1})$ without printing the integers d_i and s_i, that are implicitly given by the polynomials $H_1, \dots, H_{n+1}$. We may also use the notation $\operatorname{redRes}(F_1, \dots, F_{n+1})$, to avoid giving particular names to the truncation.

The reduced resultant of a collection of polynomials $h_1, \dots, h_{n+1} \in \mathbf{k}[\mathbf{X}]$ with zeros of respective multiplicities at least $s_1, \dots, s_{n+1}$ at the origin is defined as the corresponding specialization of the generic reduced resultant; it is an element in $\mathbf{k}$, denoted by $\operatorname{redRes}(h_1, \dots, h_{n+1})$, or possibly $\operatorname{redRes}(f_1, \dots, f_{n+1})$ depending on the context.

(1.15) Vanishing of the reduced resultant. The reduced resultant $\operatorname{redRes}(H_1, \dots, H_{n+1})$ is a polynomial in the coefficients of the polynomials H_j, $j = 1, \dots, n+1$, i.e., an element of the ring

$$\mathbf{Z}\big[U_{i,\boldsymbol{\alpha}}\big]_{i=1,\dots,n+1,\, |\boldsymbol{\alpha}|=d_i,\, \alpha_0 \leqslant d_i - s_i} \subseteq A_{\mathbf{Z}}.$$

Its vanishing on an algebraically closed field $\mathbf{k}$ characterizes those collections of hypersurfaces of $\mathbf{P}^n_{\mathbf{k}}$ defined by $h_1, \dots, h_{n+1}$ that have a further intersection point, infinitely near or not, besides the origin $(1:0:\ldots:0)$, i.e., those collections such that one of the two following conditions holds:

(a) the hypersurfaces defined by $h_1, \dots, h_{n+1}$ intersect at a point which is different from $(1:0:\ldots:0)$;

(b) the polynomials f_{j,s_j}, $j = 1, \dots, n+1$, have a common root in $\mathbf{P}^{n-1}_{\mathbf{k}}$, which means that the tangent cones of the hypersurfaces $\mathrm{V}(h_1), \dots, \mathrm{V}(h_{n+1})$ at the point $(1:0:\ldots:0)$ have a line in common.

This property of the reduced resultant is proved in [20, Theorem 3.1, 3.2 and 3.3], and in [15, Proposition I.1].

1.3 – Generalized weight properties

In [20] Zariski showed that the reduced resultant can be computed from the corresponding resultant. To obtain this property, he introduced a generalization of the grading (1.8.1) and, although the resultant is not homogeneous with respect to this new grading, he proved that its graded piece of smallest degree is connected to the reduced resultant.
We maintain the notation of Section 1.2.

(1.16) The Zariski grading. We define a grading on $A_{\mathbf{Z}} = \mathbf{Z}[U_{j,\boldsymbol{\alpha}}]$ by assigning for all j

$$\operatorname{weight}(U_{j,\boldsymbol{\alpha}}) = \begin{cases} 0 & \text{if } \alpha_0 < d_j - s_j \\ \alpha_0 - d_j + s_j & \text{otherwise,} \end{cases}$$

and weight 0 to the constants. We find it helpful to visualize this definition as follows:

$$(1.16.1) \quad F_j = \underbrace{X_0^{d_j} F_{j,0}}_{\text{coeffs have weight } s_j} + \cdots + \underbrace{X_0^{d_j - s_j + 1} F_{j,s_j - 1}}_{\text{coeffs have weight } 1} + \underbrace{X_0^{d_j - s_j} F_{j,s_j} + \cdots + F_{j,d_j}}_{\text{coeffs have weight } 0}.$$

Note in particular that the indeterminates $U_{j,\boldsymbol{\alpha}}$ whose weight equals 0 in this grading are exactly the coefficients of the truncation H_j of the polynomial F_j. The grading (1.8.1) introduced in Proposition (1.8) is a particular case of a Zariski grading (corresponding to $s_j = d_j$ for all j), which explains the terminology "generalized weight properties".

The main property of the Zariski grading is that it allows the computation of the reduced resultant of $H_1, \dots, H_{n+1}$ (the truncations of $F_1, \dots, F_{n+1}$ at the orders $d_1 - s_1, \dots, d_{n+1} - s_{n+1}$, respectively) from the resultant of $F_1, \dots, F_{n+1}$. We need one more piece of notation to see how this goes.

(1.17) For all $j = 1, \ldots, n+1$, we let G_j be the quotient of the Euclidean division of F_j by $X_0^{d_j - s_j}$ in $A_{\mathbf{Z}}[X_1, \ldots, X_n][X_0]$, i.e.,

$$G_j = \frac{1}{X_0^{d_j-s_j}} \sum_{k \leqslant s_j} X_0^{d_j-k} F_{j,k} = \sum_{k=0}^{s_j} X_0^{s_j-k} F_{j,k}$$
$$= X_0^{s_j} F_{j,0} + \cdots + X_0 F_{j,s_j-1} + F_{j,s_j}.$$

The polynomial G_j is a generic degree s_j homogeneous polynomial in the set of variables $\mathbf{X}$. Beware that F_j does not equal $H_j + X_0^{d_j-s_j} G_j$, as F_{j,s_j} appears in both H_j and G_j:

(1.17.1) $$F_j = \underbrace{X_0^{d_j} F_{j,0} + \cdots + X_0^{d_j-s_j} F_{j,s_j}}_{= X_0^{d_j-s_j} G_j} \quad \text{with} \quad \underbrace{X_0^{d_j-s_j} F_{j,s_j} + \cdots + F_{j,d_j}}_{= H_j}.$$

Also, we advise the reader to compare Displays (1.16.1) and (1.17.1).

(1.18) Theorem (Zariski Formula, [20, Theorem 5.1 and Theorem 5.2] and [15, Lemme IV.1.6]). *The nonzero homogeneous piece of lowest degree with respect to the Zariski grading of* $\operatorname{Res}(F_1, \ldots, F_{n+1})$ *has degree* $s_1 s_2 \cdots s_{n+1}$. *Denote it by* $\big[\operatorname{Res}(F_1, \ldots, F_{n+1})\big]_{s_1 s_2 \cdots s_{n+1}}$.
(a) If $s_j < d_j$ *for at least two distinct integers* $j = j_1, j_2 \in [\![1, n+1]\!]$, *then*

$$\big[\operatorname{Res}(F_1, \ldots, F_{n+1})\big]_{s_1 s_2 \cdots s_{n+1}} = \operatorname{Res}(G_1, \ldots, G_{n+1}) \cdot \operatorname{redRes}(H_1, \ldots, H_{n+1}).$$

(b) If there exists $j_0 \in [\![1, n+1]\!]$ *such that* $s_{j_0} < d_{j_0}$ *and* $s_j = d_j$ *for all* $j \neq j_0$, *then*

$$\big[\operatorname{Res}(F_1, \ldots, F_{n+1})\big]_{s_1 s_2 \cdots s_{n+1}} = \operatorname{Res}(G_1, \ldots, G_{n+1}) \cdot \operatorname{redRes}(H_1, \ldots, H_{n+1})^{d_{j_0} - s_{j_0}}.$$

Note that despite appearances, there is a Zariski Formula in all cases, because if none of the two conditions in (a) and (b) of the above statement is verified, then the reduced resultant is actually an ordinary resultant.

Observe that, in the Zariski grading: (i) the coefficients of $H_1, \ldots, H_{n+1}$ all have weight 0, hence $\operatorname{redRes}(H_1, \ldots, H_{n+1})$ is homogeneous of degree 0; (ii) for all $j = 1, \ldots, n+1$, the coefficient of G_j in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight α_0 hence, by Proposition (1.8), $\operatorname{Res}(G_1, \ldots, G_{n+1})$ is homogeneous of weight $s_1 s_2 \cdots s_{n+1}$. Thus, indeed,

$$\operatorname{Res}(G_1, \ldots, G_{n+1}) \cdot \operatorname{redRes}(H_1, \ldots, H_{n+1})^e$$

is homogeneous of weight $s_1 s_2 \cdots s_{n+1}$ for all e.

(1.19) Corollary. *Consider the grading of* $A_{\mathbf{Z}}$ *defined by*

$$\operatorname{weight}(U_{j,\boldsymbol{\alpha}}) = d_j - \alpha_0.$$

With respect to this grading, $\operatorname{redRes}(H_1, \ldots, H_{n+1})$ *is homogeneous of degree*

$$n(d_1 \cdots d_{n+1} - s_1 \cdots s_{n+1})$$

if there are at least two integers $j = j_1, j_2$ *such that* $s_j < d_j$, *and homogeneous of degree*

$$n\, d_1 \ldots d_{n+1} / d_{j_0}$$

if there exists j_0 *such that* $s_{j_0} < d_{j_0}$ *and* $s_j = d_j$ *for all* $j \neq j_0$.

This is the homogeneity property of the reduced resultant that particularizes to give the degree of $\mathrm{Res}({}^\flat F, {}^\flat G)$ in Example (1.12).

Proof. The grading of $A_{\mathbf{Z}}$ we are considering is that of Proposition (1.10), with $k = 0$; thus for all $j = 1, \ldots, n+1$ the coefficients of $F_{j,l}$ all have weight l, for $l = 0, \ldots, d_j$. Then we know by Proposition (1.10) that the resultant of $F_1, \ldots, F_{n+1}$ is homogeneous of degree $n\, d_1 \cdots d_{n+1}$, and by the same argument the resultant of $G_1, \ldots, G_{n+1}$ is homogeneous of degree $n\, s_1 \cdots s_{n+1}$. Therefore, we deduce from Theorem (1.18) that the reduced resultant of $H_1, \ldots, H_{n+1}$ is homogeneous, of degree

$$n\, d_1 \cdots d_{n+1} - n\, s_1 \cdots s_{n+1}$$

if $s_j < d_j$ for two distinct indices $j = j_1, j_2$, and of degree

$$\frac{n\, d_1 \cdots d_{n+1} - n\, s_1 \cdots s_{n+1}}{d_{j_0} - s_{j_0}} = n\, \frac{d_1 \cdots d_{n+1}}{d_{j_0}}\, \frac{d_{j_0} - s_{j_0}}{d_{j_0} - s_{j_0}}$$

if $s_j = d_j$ for all $j \neq j_0$ and $s_{j_0} < d_{j_0}$. $\square$

2 – Reduced discriminant and Salmon formula

In this Section we give a rigorous proof of formula (). This is done by introducing the concept of reduced discriminant. We begin with a quick recap on the ordinary discriminant of a hypersurface, following [4, §4]; see also [7] and [11, Chapter 13, §D].

2.1 – Discriminant of a homogeneous polynomial

(2.1) Let d be a positive integer, and consider the generic homogeneous degree d polynomial $F = \sum_{|\boldsymbol{\alpha}|=d} U_\alpha \mathbf{X}^{\boldsymbol{\alpha}}$ in $n+1$ indeterminates $\mathbf{X} = (X_0, \ldots, X_n)$. We set $A_{\mathbf{Z}} = \mathbf{Z}[U_{\boldsymbol{\alpha}}]_{|\boldsymbol{\alpha}|=d}$. For all $i = 0, \ldots, n$ we let ∂_i denote derivation with respect to the indeterminate X_i.

(2.2) Definition. *There is a unique element* $\mathrm{Disc}_d(F) \in A_{\mathbf{Z}}$ *(often simply denoted by* $\mathrm{Disc}(F)$*) such that*

$$d^{a(n,d)}\mathrm{Disc}_d(F) = \mathrm{Res}\big(\partial_0 F, \ldots, \partial_n F\big) \tag{2.2.1}$$

in $A_{\mathbf{Z}}$*, where* $a(n,d) = \frac{(d-1)^{n+1} - (-1)^{n+1}}{d} \in \mathbf{Z}$*. It is homogeneous of degree* $(n+1)(d-1)^n$ *with respect to the coefficients of the polynomial* F*, i.e., with respect to the indeterminates* $U_{\boldsymbol{\alpha}}$*,* $|\boldsymbol{\alpha}| = d$*.*

For a homogeneous degree d *polynomial* $f \in \mathbf{k}[\mathbf{X}]$*, we define the* discriminant $\mathrm{Disc}(f) \in \mathbf{k}$ *of* f *as the specialization* $\sigma(\mathrm{Disc}(F)) \in \mathbf{k}$*, where* $\sigma : A_{\mathbf{Z}} \to \mathbf{k}$ *is the unique specialization morphism mapping* F *to* f*.*

We emphasize that the factor $d^{a(n,d)}$ in Equation (2.2.1) is here to make the discriminant a universal object (i.e., to make it behave well under specialization) which yields the expected smoothness criterion. An alternative way of defining the discriminant is to use Formula (2.5.1) below. Either way there is an annoying parasitic factor. The issue comes from the fact that, while the resultant characterizes the existence of a non-trivial common zero in $\mathbf{P}^n$ for $n+1$ polynomials, the wish for the discriminant is that it will characterize the existence of a non-trivial common zero for the $n+2$ polynomials $F, \partial_0 F, \partial_1 F, \ldots, \partial_n F$ which are linked by the Euler Formula. The next result tells us that with Definition (2.2), the discriminant grants our wish.

(2.3) Proposition. *The ideal of inertia forms*

$$(\partial_0 F, \partial_1 F, \ldots, \partial_n F, F) : (X_0, \ldots, X_n)^\infty$$

is a prime and principal ideal in $A_{\mathbf{Z}}$. It is generated by the discriminant $\operatorname{Disc}(F)$*, which is therefore an irreducible polynomial in $A_{\mathbf{Z}}$.*

In fact, what we really want is that the (non-)vanishing of the discriminant characterizes smoothness. By the Jacobian criterion for smoothness, this is equivalent to the non-existence of a common root $F, \partial_0 F, \partial_1 F, \ldots, \partial_n F$. Thus, Proposition (2.3) gives the following.

(2.4) Theorem (smoothness criterion)**.** *Suppose* $\mathbf{k}$ *is an algebraically closed field, and consider a degree d homogeneous polynomial $f \in \mathbf{k}[\mathbf{X}]$. The following are equivalent:*
(i) the hypersurface $\mathrm{V}(f) \subseteq \mathbf{P}^n_{\mathbf{k}}$ *is smooth;*
(ii) $\operatorname{Disc}(f) \neq 0$.

The following formula gives an alternative way of defining the discriminant. This is the path we shall follow to define the reduced discriminant. As we shall see, however, there will be some additional technical difficulty to overcome, which did not occur when passing from the ordinary resultant to its reduced form either.

(2.5) Proposition. *Let $\bar{F}$ be the polynomial $F(0, X_1, \ldots, X_n) \in A_{\mathbf{Z}}[X_1, \ldots, X_n]$. We have the following identity in $A_{\mathbf{Z}}$:*

$$\operatorname{Res}\big(\partial_1 F, \ldots, \partial_n F, F\big) = \operatorname{Disc}(F) \cdot \operatorname{Disc}(\bar{F}). \tag{2.5.1}$$

The polynomial $\bar{F}$ is the equation of the hypersurface in $\mathbf{P}^{n-1}$ cut out by $\mathrm{V}(F)$ on the hyperplane $\mathrm{V}(X_0) \subseteq \mathbf{P}^n$, of which we think as the hyperplane at infinity. If $\mathbf{k}$ is an algebraically closed field and $f \in \mathbf{k}[\mathbf{X}]$ is a degree d homogeneous polynomial, then, by Theorem (2.4), the vanishing of $\operatorname{Disc}(\bar{f})$ is equivalent to the hyperplane section at infinity $\mathrm{V}(f) \cap \mathrm{V}(X_0)$ being singular. For a general f such that $\operatorname{Disc}(\bar{f}) = 0$, the hypersurface $\mathrm{V}(f)$ is non-singular and tangent to the hyperplane $\mathrm{V}(X_0)$.

Proof of Proposition (2.5). We shall prove the following identity in $A_{\mathbf{Z}}$:

$$d^{(d-1)^n} \operatorname{Res}\big(\partial_1 F, \ldots, \partial_n F, F\big) = \operatorname{Res}\big(\partial_0 F, \ldots, \partial_n F\big) \cdot \operatorname{Res}\big(\partial_1 \bar{F}, \ldots, \partial_n \bar{F}\big). \tag{2.5.2}$$

We claim that it is equivalent to Formula (2.5.1). To see why, first note that at the right-hand-side of Display (2.5.2), the first (respectively second) factor is the resultant of $n+1$ (respectively n) polynomials in $n+1$ (respectively n) indeterminates. Since $a(n,d) + a(n-1,d) = (d-1)^n$, it follows from Formula (2.2.1) that the identities (2.5.2) and (2.5.1) are indeed equivalent.

Now, the identity (2.5.2) may be derived from the Euler Formula, as follows. The latter formula says that $d \cdot F$ is congruent to $X_0\, \partial_0 F$ modulo the ideal $(\partial_1 F, \ldots, \partial_n F)$, hence

$$\operatorname{Res}\big(\partial_1 F, \ldots, \partial_n F,\ d \cdot F\big) = \operatorname{Res}\big(\partial_1 F, \ldots, \partial_n F,\ X_0\, \partial_0 F\big) \tag{2.5.3}$$

by [14, §5.9, Transformations élémentaires]. The left-hand-side of (2.5.3) equals that of (2.5.2) by the elementary homogeneity properties of the resultant, see Theorem (1.6). Its right-hand-side, on the other hand, equals

$$\operatorname{Res}\big(\partial_1 F, \ldots, \partial_n F,\ X_0\big) \cdot \operatorname{Res}\big(\partial_1 F, \ldots, \partial_n F,\ \partial_0 F\big)$$

by multiplicativity of the resultant, see [14, §5.7]. Eventually,

$$\operatorname{Res}\big(\partial_1 F, \ldots, \partial_n F,\ X_0\big) = (-1)^{n(d-1)^n} \operatorname{Res}\big(\, \partial_1 F|_{X_0=0}\,, \ldots,\, \partial_n F|_{X_0=0}\big)$$

by [14, (5.13.5)][2], and

$$\mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, \partial_0 F\big) = (-1)^{n(d-1)^{n+1}} \mathrm{Res}\big(\partial_0 F,\, \partial_1 F, \dots, \partial_n F\big)$$

by [14, §5.8, Effet d'une permutation des polynômes]. The upshot is that

$$\mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, X_0\, \partial_0 F\big) = (-1)^{nd(d-1)^n}\, \mathrm{Res}\big(\partial_0 F, \dots, \partial_n F\big) \cdot \mathrm{Res}\big(\partial_1 \bar{F}, \dots, \partial_n \bar{F}\big),$$

which gives the result since $d(d-1)$ is always even. □

Similarly to the resultant, the discriminant is homogeneous under the two gradings of the coefficient ring $A_{\mathbf{Z}}$ introduced in Propositions (1.8) and (1.10).

(2.6) Proposition. *Let k be an integer in $[\![0, n]\!]$.*
(2.6.1) In the grading of $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = \alpha_k$*, the discriminant* $\mathrm{Disc}(F)$ *is homogeneous of degree $d(d-1)^n$.*
(2.6.2) In the grading of $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = d - \alpha_k$*, the discriminant* $\mathrm{Disc}(F)$ *is homogeneous of degree $nd(d-1)^n$.*

One may use Proposition (2.6.1) to compute the degree of the dual to a smooth hypersurface in $\mathbf{P}^{n+1}$, see Paragraph (3.7).

Proof. The idea is to consider (2.2.1) and the corresponding weight properties of the resultant. We may assume $k \neq 0$, for otherwise we may write a formula analogous to (2.5.1) with respect to an indeterminate other than X_0 in order to reduce to this case.

First consider the grading of (2.6.1). For $j \neq k$ the coefficient of $\partial_j F$ in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight α_k, whereas for $j = k$ the coefficient of $\partial_k F$ in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight $\alpha_k + 1$. We may thus apply Corollary (1.11) to the homogeneity property (1.8), with $r_k = 1$ and $r_j = 0$ for all $j \neq k$, which gives that $\mathrm{Res}\big(\partial_0 F, \dots, \partial_n F\big)$ is homogeneous of degree

$$(d-1)^{n+1} + (d-1)^n.$$

Using (2.2.1), this gives (2.6.1).

Similarly, for the grading of (2.6.2): for $j \neq k$ the coefficient of $\partial_j F$ in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight $d - \alpha_k = (d-1) - \alpha_k + 1$, whereas for $j = k$ the coefficient of $\partial_k F$ in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight $d - (\alpha_k + 1) = (d-1) - \alpha_k$.

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[2] the $(-1)^{n(d-1)^n}$ factor is due to the fact that we need to exchange the roles of X_0 and X_n, which is not indifferent since by definition the resultant is normalized by imposing $\mathrm{Res}(X_0^{d_0}, \dots, X_n^{d_n}) = 1$, see right after Theorem (1.6).

To work out the claimed identity, one may first perform a base change by exchanging X_0 and X_n which, by [14, (5.13.1)], gives

$$\mathrm{Res}\Big(\partial_1 F, \dots, \partial_n F,\, X_0\Big) = (-1)^{(d-1)^n} \mathrm{Res}\Big(\partial_1 F(X_n, X_1, \dots, X_{n-1}, X_0),\, \dots,\, X_n\Big);$$

then, apply [14, (5.13.5)] to the resultant on the right-hand-side, and eventually perform another base change to go from $(X_1, \dots, X_{n-1}, X_0)$ to $(X_0, X_1, \dots, X_{n-1})$, which introduces the additional factor $(-1)^{(n-1)(d-1)^n}$, hence in total the factor $(-1)^{n(d-1)^n}$ indeed.

An easier way to get the correct sign in general is to consider the normalizing condition, once one knows that there exists $c \in \mathbf{Z}$ such that

$$\mathrm{Res}_{d_1, \dots, d_n, 1}(F_1, \dots, F_n, X_0) = c\, \mathrm{Res}_{d_1, \dots, d_n}\big(F_1|_{X_0=0}, \dots, F_n|_{X_0=0}\big).$$

Specializing $F_1, \dots, F_n$ to $X_1^{d_1}, \dots, X_n^{d_n}$ respectively, one gets

$$\mathrm{Res}_{d_1, \dots, d_n, 1}(X_1^{d_1}, \dots, X_n^{d_n}, X_0) = c\, \mathrm{Res}_{d_1, \dots, d_n}(X_1^{d_1}, \dots, X_n^{d_n}),$$

where the left-hand-side equals $(-1)^{nd_1 \cdots d_n}$ by [14, §5.8], and the right-hand-side equals c.

We thus apply Corollary (1.11) to the homogeneity property (1.10), with $r_j = 1$ for $j \neq k$ and $r_k = 0$, which gives that $\operatorname{Res}\big(\partial_0 F, \dots, \partial_n F\big)$ is homogeneous of degree

$$n(d-1)^{n+1} + n(d-1)^n.$$

Again this gives the wanted result by (2.2.1). □

In turn, one may reproduce the argument given in (1.11) to deduce further weight properties from the two latter results and the standard homogeneity property of the discriminant stated in Definition (2.2).

(2.7) Corollary. *Assume that the discriminant* $\operatorname{Disc}_d \in A_{\mathbf{Z}}$ *is homogeneous of degree* δ *for the grading on* $A_{\mathbf{Z}}$ *defined by* $\operatorname{weight}(U_{\boldsymbol{\alpha}}) = w_{\boldsymbol{\alpha}}$. *Let* $r \in \mathbf{Z}$. *For the new grading on* $A_{\mathbf{Z}}$ *defined by* $\operatorname{weight}(U_{\boldsymbol{\alpha}}) = w_{\boldsymbol{\alpha}} + r$, *the discriminant* Disc_d *is homogeneous of degree*

$$\delta + r(n+1)(d-1)^n.$$

2.2 – The reduced discriminant

(2.8) We write the generic homogeneous degree d polynomial as

$$F = X_0^d F_0 + \dots + X_0^{d-s} F_s + \dots + F_d$$

where each F_k is homogeneous of degree k in the indeterminates $X_1, \dots, X_n$. Given an integer $s \in [\![2, d-1]\!]$, we consider the truncation H of F at order $d-s$ with respect to X_0, defined as in Paragraph (1.13), and set G as in Paragraph (1.17); thus,

$$\begin{aligned} H &= \qquad\qquad\qquad\qquad X_0^{d-s} F_s + X_0^{d-s-1} F_{s+1} + \dots + F_d, \\ G &= X_0^s F_0 + X_0^{s-1} F_1 + \dots + F_s. \end{aligned}$$

The truncation H is the generic degree d polynomial with a multiplicity s zero at the point $(1 : 0 : \dots : 0)$. The wish for the reduced discriminant is that it is a polynomial in the coefficients of H, the vanishing of which characterizes the existence of an additional singular non-trivial zero of H.

The polynomial H has degree d, and has valuation s with respect to the indeterminates $X_1, \dots, X_n$; moreover its partial derivatives with respect to the indeterminates $X_1, \dots, X_n$ all have degree $d-1$, and valuation $s-1$, so that the reduced resultant

$$\operatorname{redRes}(\partial_1 H, \dots, \partial_n H, H) = \operatorname{redRes}_{d-1,\dots,d-1,d}^{s-1,\dots,s-1,s}(\partial_1 H, \dots, \partial_n H, H)$$

is well defined.

(2.9) Proposition. *With the above notation, the discriminants* $\operatorname{Disc}(F_s)$ *and* $\operatorname{Disc}(F_d)$ *both divide the reduced resultant* $\operatorname{redRes}(\partial_1 H, \dots, \partial_n H, H)$.

Proof. Both discriminants $\operatorname{Disc}(F_s)$ and $\operatorname{Disc}(F_d)$ are irreducible as elements of $A_{\mathbf{Z}}$. In addition, their vanishing implies the vanishing of the reduced resultant by (1.15). Indeed, the vanishing of $\operatorname{Disc}(F_d)$ implies the existence of a common root at infinity ($x_0 = 0$) of the polynomial system $\partial_1 H = \dots = \partial_n H = H = 0$. In the same way, the vanishing of $\operatorname{Disc}(F_s)$ implies the existence of a common root of the polynomial system $\partial_1 H = \dots = \partial_n H = H = 0$ infinitely near to the point $(1 : 0 : \dots : 0)$. We thus conclude that $\operatorname{Disc}(F_s)$ and $\operatorname{Disc}(F_d)$, which are coprime as they are irreducible and do not depend on the same coefficients of F, both divide $\operatorname{redRes}(\partial_1 H, \dots, \partial_n H, H)$, which ends the proof. □

Alternatively, this proposition can be proved by means of inertia forms, as follows. By Theorem (1.14), the reduced resultant $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$ belongs to the ideal of inertia forms $(\partial_1 H, \ldots, \partial_n H, H) : (X_1, \ldots, X_n)^\infty$. Therefore, for all integer $i = 1, \ldots, n$, there exists an integer N_i such that

$$X_i^{N_i}\, \mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H) \in (\partial_1 H, \ldots, \partial_n H, H); \tag{2.9.1}$$

specializing the variable X_0 to 0 in (2.9.1), we get that

$$X_i^{N_i}\, \mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H) \in (\partial_1 F_d, \ldots, \partial_n F_d, F_d).$$

It follows that $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$ belongs to the ideal of inertia forms $(\partial_1 F_d, \ldots, \partial_n F_d, F_d) : (X_1, \ldots, X_n)^\infty$. By Proposition (2.3), this ideal is generated by the discriminant of the polynomial F_d, hence $\mathrm{Disc}(F_d)$ divides $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$. A similar argument, albeit slightly more technical, can be used to show that $\mathrm{Disc}(F_s)$ divides $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$: see [15, Lemme I.1.3].

In the notation of (2.5), $F_d = \bar{F}$ and $F_s = \bar{G}$. Observe that F_d and F_s are generic homogeneous polynomials of degrees d and k respectively, in the indeterminates $(X_1, \ldots, X_n)$. Proposition (2.9) then leads to the following definition.

(2.10) Definition. *The* reduced discriminant *of F with respect to the truncation at order $d-s$ for the indeterminate X_0, denoted by* $\mathrm{redDisc}_d^s(H)$*, or simply* $\mathrm{redDisc}(H)$*, is defined by the equality*

$$\mathrm{Disc}(F_d)\mathrm{Disc}(F_s)\ \mathrm{redDisc}(H) = \mathrm{redRes}\big(\partial_1 H, \ldots, \partial_n H, H\big) \in A_{\mathbf{Z}}. \tag{2.10.1}$$

For a degree d polynomial $h \in \mathbf{k}[\mathbf{X}]$ with a singularity of order s at the origin, $\mathrm{redDisc}_d^s(h)$ is defined as the specialization of $\mathrm{redDisc}_d^s(H)$ with respect to the unique specialization $A_{\mathbf{Z}} \to \mathbf{k}$ mapping H to h.

Definition (2.10) implies that $\mathrm{redDisc}(H)$ is a primitive polynomial in $A_{\mathbf{Z}}$ (i.e., the greatest common divisor of its coefficients equals 1), because $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$ is primitive by [4, Proposition 4.24].

The identity (2.10.1) should be compared to (2.5.1). Beware that the reduced discriminant is not merely the reduced resultant of all the partial derivatives, because of the factor $\mathrm{Disc}(F_s)$ in (2.10.1). The factor $\mathrm{Disc}(F_d)$ is an artefact of our definition by considering $\mathrm{redRes}\big(\partial_1 H, \ldots, \partial_n H, H\big)$ and not plainly $\mathrm{redRes}\big(\partial_0 H, \partial_1 H, \ldots, \partial_n H\big)$, in order to define the reduced discriminant as a primitive polynomial with integer coefficients without dealing with possible constant factors, similar to $d^{a(n,d)}$ in (2.2.1). The factor $\mathrm{Disc}(F_s)$ on the other hand is geometrically meaningful, as we will see in (2.12) below.

According to (2.2.1), the classical discriminant of a homogeneous polynomial can be computed from the resultant of all its partial derivatives, up to the extraneous integer factor $d^{a(n,d)}$. It turns out that a similar formula holds for the reduced discriminant.

(2.11) Proposition. *The following identity in $A_{\mathbf{Z}}$ holds:*

$$d^{a(n,d)-(s-1)^n}\mathrm{Disc}(F_s)\,\mathrm{redDisc}(H) = \mathrm{redRes}(\partial_0 H, \ldots, \partial_n H). \tag{2.11.1}$$

Proof. To prove this formula, we consider the reduced resultant $\mathcal{R} := \mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, dH)$. First, from the homogeneity property (see Theorem (1.14)), we have

$$\mathcal{R} := d^{(d-1)^n-(s-1)^n}\ \mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H).$$

Next, the Euler Identity gives $\mathcal{R} := \operatorname{redRes}(\partial_1 H, \dots, \partial_n H, \sum_{i=0}^n X_i \partial_i H)$, and we claim that

$$\mathcal{R} = \operatorname{redRes}(\partial_1 H, \dots, \partial_n H, X_0 \partial_0 H) \tag{2.11.2}$$
$$= (-1)^{n(d-1)^n} \operatorname{Res}(\partial_1 F_d, \dots, \partial_n F_d) \operatorname{redRes}(\partial_1 H, \dots, \partial_n H, \partial_0 H) \tag{2.11.3}$$

(observe that $\partial_0 H$ is of degree $d-1$ and valuation s, and $H|_{X_0=0} = F_d$). We conclude the proof of (2.11.1) before justifying the two above equalities. The Zariski formula together with the property of resultants under permutation of the polynomials [14, §5.8] yields the equality:

$$\operatorname{redRes}(\partial_1 H, \dots, \partial_n H, \partial_0 H) = (-1)^{n(d-1)^{n+1} - ns(s-1)^n} \operatorname{redRes}(\partial_0 H, \dots, \partial_n H).$$

In addition, $\operatorname{Res}(\partial_1 F_d, \dots, \partial_n F_d) = d^{a(n-1,d)} \operatorname{Disc}(F_d)$, so putting everything together, including (2.10.1), we deduce that

$$d^{(d-1)^n - (s-1)^n - a(n-1,d)} \operatorname{Disc}(F_s) \operatorname{redDisc}(H) = (-1)^N \operatorname{redRes}(\partial_0 H, \dots, \partial_n H),$$

where $N = n(d-1)^n + n(d-1)^{n+1} - n(s-1)^n s = nd(d-1)^n - n(s-1)^n s$. Now, N is an even integer, and $(d-1)^n - a(n-1,d) = a(n,d)$, hence the claimed formula.

To justify (2.11.2), we prove the following invariance property of the reduced resultant under some elementary transformations. Let $F_1, \dots, F_{n+1}$ be generic polynomials of degree $d-1, \dots, d-1, d$ respectively, as well as generic linear forms $L_1, \dots, L_n$ in the variables $X_1, \dots, X_n$. Then, from the definition of the reduced resultant as a generator of the ideal of inertia forms in the generic setting, we deduce that

$$\operatorname{redRes}^{s-1,\dots,s-1,s}_{d-1,\dots,d-1,d}(H_1, \dots, H_{n+1}) \text{ divides } \operatorname{redRes}^{s-1,\dots,s-1,s}_{d-1,\dots,d-1,d}(H_1, \dots, H_{n+1} + \sum_{i=1}^n L_i H_i).$$

We notice that this divisibility property remains valid under any specialization, in particular under the specialization sending F_{n+1} to $F_{n+1} - \sum_{i=1}^n L_i F_i$, and leaving $F_1, \dots, F_n$ invariant. This implies that the above divisibility property also holds in the other direction, hence these two reduced resultant are equal, up to sign. Applying Zariski formula and the invariance of the resultant under elementary transformations [14, §5.9], we deduce that this sign equals 1.

The equality (2.11.3), on the other hand, is a consequence of the following multiplicativity property of the reduced resultant:

$$\begin{aligned} &\operatorname{redRes}^{s_1,\dots,s_{n+1}}_{d_1,\dots,d_{n+1}+1}(H_1, \dots, H_n, X_0 H_{n+1}) \\ &\qquad = (-1)^{d_1 \cdots d_n} \operatorname{Res}(\bar{F}_1, \dots, \bar{F}_n) \cdot \operatorname{redRes}^{s_1,\dots,s_{n+1}}_{d_1,\dots,d_{n+1}}(H_1, \dots, H_n, H_{n+1}), \end{aligned} \tag{2.11.4}$$

which follows from the Zariski formula. Indeed, let $F_1, \dots, F_{n+1}$ be generic polynomials as above, and denote by H_i and G_i their decompositions corresponding to the truncation at order $d_i - s_i$ for all i. On the one hand, we have (observe that F_{n+1} and $X_0 F_{n+1}$ have the same "G", namely G_{n+1}):

$$\begin{aligned} \operatorname{Res}(F_1, \dots, F_n, X_0 F_{n+1}) = \operatorname{Res}(G_1, \dots, G_n, G_{n+1}) \operatorname{redRes}(H_1, \dots, H_n, X_0 H_{n+1}) \\ + \text{ terms of higher weight.} \end{aligned}$$

On the other hand, by properties of the classical resultant,

$$\operatorname{Res}(F_1, \dots, F_n, X_0 F_{n+1}) = (-1)^{d_1 \cdots, d_n} \operatorname{Res}(\bar{F}_1, \dots, \bar{F}_n) \operatorname{Res}(F_1, \dots, F_n, F_{n+1}),$$

hence, by the Zariski formula,

$$\begin{aligned}&\operatorname{Res}(F_1,\dots,F_n,X_0F_{n+1}) = \\ &(-1)^{d_1\dots,d_n}\operatorname{Res}(\bar{F_1},\dots,\bar{F_n})\operatorname{Res}(G_1,\dots,G_{n+1})\operatorname{redRes}(H_1,\dots,H_{n+1}) + \text{ terms of higher weight.}\end{aligned}$$

The comparison of the two above expressions of $\operatorname{Res}(F_1,\dots,F_n,X_0F_{n+1})$ yields the expected equality. □

(2.12) Vanishing of the reduced discriminant. By definition of the truncation H, the generic hypersurface $\mathrm{V}(H) \subseteq \mathbf{P}^n$ has an ordinary s-fold point at the origin $(1:0:\dots:0)$. By Paragraph (1.15), the vanishing of $\operatorname{redRes}\big(\partial_1 h,\dots,\partial_n h, h\big)$ for some specialization h of H in an algebraically closed field $\mathbf{k}$ corresponds to one of the two following properties:
(a) the existence of a common zero for the polynomials $\partial_1 h,\dots,\partial_n h, h$ which is not the origin;
(b) the existence of a common zero for the polynomials $\partial_1 f_s,\dots,\partial_n f_s, f_s$, equivalently a common root for the polynomials $\partial_1 f_s,\dots,\partial_n f_s$ by Euler identity, assuming for simplicity that the degree d is non-zero in $\mathbf{k}$.

Property (b) is equivalent to the vanishing of $\operatorname{Disc}(f_s)$, and to the tangent cone to the hypersurface $\mathrm{V}(h)$ at the origin being a cone over a singular degree s hypersurface (with the convention that a hypersurface of degree $s' > s$ is a singular degree s hypersurface); in other words, in the blow-up of $\mathrm{V}(h)$ at the origin, the exceptional divisor is singular.

Property (a) is implied by the vanishing of $\operatorname{Disc}(f_d)$, as Proposition (2.9) tells us; this is equivalent to the hypersurface $\mathrm{V}(h)$ being tangent to the hyperplane at infinity $\mathrm{V}(X_0)$.

In Definition (2.10) we discard the two factors $\operatorname{Disc}(F_s)$ and $\operatorname{Disc}(F_d)$; the upshot is that the vanishing of the reduced discriminant $\operatorname{redDisc}(H)$ defines a divisor which has as a dense subset the locus of those hypersurfaces $\mathrm{V}(h)$ that have a singularity off the origin $(1:0:\dots:0)$. It turns out that this divisor is irreducible, and that the reduced discriminant is irreducible; the proof of this result will appear in [3], but we point out that this property also follows from the theory of toric discriminants (see [11, Chapter 9]) of which the reduced discriminants are specific instances.

(2.13) Remark. In the reduced resultant situation, the condition that the $n+1$ hypersurfaces $\mathrm{V}(h_1),\dots,\mathrm{V}(h_{n+1})$ have an infinitely near additional intersection point is not divisorial, i.e., it does not correspond to an irreducible element factoring out of $\operatorname{redRes}(H_1,\dots,H_{n+1})$. The reason is that this condition has codimension larger than one: indeed it amounts to the $n+1$ leading terms $f_{1,s_1},\dots,f_{n+1,s_{n+1}}$ having a non-trivial common zero, in other words the $n+1$ corresponding hypersurfaces in $\mathbf{P}^{n-1}$ have non-trivial intersection: this is $n+1$ hypersurfaces in $\mathbf{P}^{n-1}$, hence a condition of codimension larger than one.

In contrast, in the reduced discriminant situation the condition that the hypersurface $\mathrm{V}(h)$ has an infinitely near additional singularity, i.e., a singularity worse than an ordinary s-fold point, is indeed divisorial as we have seen above. It corresponds to the vanishing of $\operatorname{Disc}(F_s)$ at the leading term f_s of h.

(2.14) There is an interesting connection to Milnor number. Assume that h defines a hypersurface with isolated singularities, so that both its Milnor number at the origin $\mu_0(h)$ and its total Milnor number $\mu(h)$ are well-defined. If the only singularity of h is an ordinary s-fold point in the origin, then $\mu(h) = \mu_0(f_s) = (s-1)^n$. Thus, Property (b) in Paragraph (2.12) above is equivalent to the condition that $\mu_0(h) > (s-1)^n$, see [10, Theorem 1]. Therefore, it follows that if $\operatorname{Disc}(f_d) \neq 0$, then $\mu(h) > (s-1)^n$ if and only $\operatorname{Disc}(f_s)\operatorname{redDisc}(h) = 0$. Pushing further, we see that if $\operatorname{Disc}(f_d) \neq 0$ and $\operatorname{Disc}(f_s) \neq 0$, then $\operatorname{redDisc}(h)$ vanishes if and only if $\mu(h) > \mu_0(h) = (s-1)^n$.

(2.15) Lemma. *In the standard grading of $A_{\mathbf{Z}}$, given by* weight$(U_{\boldsymbol{\alpha}}) = 1$ *for all $\boldsymbol{\alpha}$, the reduced discriminant* $\mathrm{redDisc}_d^s$ *is homogeneous of degree*

$$(n+1)\big[(d-1)^n - (s-1)^n\big] - 2n(s-1)^{n-1}. \tag{2.15.1}$$

Proof. This follows from the definition by a direct computation, since the degrees of the other quantities in Display (2.10.1) are known. It is arguably safer, however, to carry this out explicitly.

The two plain discriminants $\mathrm{Disc}(F_d)$ and $\mathrm{Disc}(F_s)$ are homogeneous of respective degrees $n(d-1)^{n-1}$ and $n(s-1)^{n-1}$, see Definition (2.2). Moreover, it follows from Theorem (1.14) that

$$\mathrm{redRes}_{d-1,\ldots,d-1,d}^{s-1,\ldots,s-1,s}(\partial_1 H, \ldots, \partial_n H, H)$$

is homogeneous of degree

$$n\big(d(d-1)^{n-1} - s(s-1)^{n-1}\big) + \big((d-1)^n - (s-1)^n\big).$$

It thus follows from the equality in Display (2.10.1) that $\mathrm{redDisc}(H)$ is homogeneous, of degree

$$\begin{aligned}\big[nd(d-1)^{n-1} + (d-1)^n - n(d-1)^{n-1}\big] - \big[ns(s-1)^{n-1} + (s-1)^n + n(s-1)^{n-1}\big]\\ = (n+1)(d-1)^n - (s-1)^{n-1}\big((n+1)s + n - 1\big),\end{aligned}$$

and the result follows. □

We now turn to the generalized weight properties of the reduced discriminant.

(2.16) Example. As an illustrative example, we consider the case $n = 1$ and set

$$F = \sum\nolimits_{i=0}^{d} U_i X_0^{d-i} X_1^i = X_0^d F_0 + X_0^{d-1} F_1 + \cdots + X_0^{d-s} F_s + \cdots F_d,$$

with $F_i = U_i X_1^i$ for all $i = 0, \ldots, d$. We consider

$$H = \sum\nolimits_{i=s}^{d} U_i X_0^{d-i} X_1^i = \sum\nolimits_{i=s}^{d} X_0^{d-i} F_i = X_1^s\big(U_s X_0^{d-s} + U_{s+1} X_0^{d-s-1} X_1 + \cdots + U_d X_1^{d-s}\big),$$

the truncation of F at order $d - s > 0$ with respect to X_0. Then, setting $H = X_1^s \cdot {}^\flat H$ it is easy to check that $\mathrm{redDisc}_d^s(H) = \pm \mathrm{Disc}({}^\flat H)$; it is therefore an irreducible polynomial of degree $2(d - s - 1)$ in the coefficients of H (compare with Formula (2.15.1) for the degree).

Moreover, using the weight properties in Proposition (2.6), we can deduce weight properties of $\mathrm{redDisc}(H)$. Suppose that $A_{\mathbf{Z}}$ is graded with the rule $\mathrm{weight}(U_i) = \max(0, i - s)$; then the reduced discriminant $\mathrm{redDisc}_d^s(H)$ is homogeneous of degree

$$(d-s)(d-s-1) \tag{2.16.1}$$

by Proposition (2.6), as ${}^\flat H$ has degree $d-s$. Similarly, if $A_{\mathbf{Z}}$ is graded with the rule $\mathrm{weight}(U_i) = d - i$ then the same conclusion holds (note that $d - i = (d-s) - (i-s)$).

We can generalize this following Corollary (1.11). Let r be an integer and consider the grading of $A_{\mathbf{Z}}$ defined by the rule $\mathrm{weight}(U_i) = i - s + r$ if $i \geqslant s$, and $\mathrm{weight}(U_i) = 0$ otherwise, then $\mathrm{redDisc}_d^s(H)$ is homogeneous of degree $(d - s + 2r)(d - s - 1)$. In particular, if $r = s$, i.e., $\mathrm{weight}(U_i) = i$ if $i \geqslant s$ and $\mathrm{weight}(U_i) = 0$ otherwise, we get that $\mathrm{redDisc}_d^s(H)$ is homogeneous of degree

$$(d+s)(d-s-1) = d(d-1) - s(s+1). \tag{2.16.2}$$

In fact, since $\mathrm{redDisc}_d^s(H)$ a polynomial in $U_s, \ldots, U_d$ only, the weights of $U_0, \ldots, U_{s-1}$ don't matter. Similarly, in the grading of $A_{\mathbf{Z}}$ is defined by the rule $\mathrm{weight}(U_i) = d - i + r$ if $i \geqslant s$ and $\mathrm{weight}(U_i) = 0$ otherwise, $\mathrm{redDisc}_d^s(H)$ is homogeneous of degree $(d - s + 2r)(d - s - 1)$.

The following result is similar to Theorem (1.18). It is the key to the generalized Salmon formula for the discriminant.

(2.17) Theorem ([4, Theorem 4.25])**.** *Suppose that the ring $A_{\mathbf{Z}}$ is graded by means of the Zariski grading (1.16), i.e.,* $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = \max(\alpha_0 - d + s, 0)$*. Then* $\mathrm{Disc}(F)$ *has valuation $s(s-1)^n$, and its homogeneous part D_0 in this degree satisfies the following equality in $A_{\mathbf{Z}}$:*

$$D_0 \,\mathrm{Disc}(F_d) = \mathrm{Disc}(G)\,\mathrm{Disc}(F_s)\ \mathrm{redRes}\big(\partial_1 H, \ldots, \partial_n H, H\big). \tag{2.17.1}$$

Let us point out that the three elements $\mathrm{Disc}(F_d)$, $\mathrm{Disc}(F_s)$, and $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$ have degree 0 with respect to the Zariski grading, while $\mathrm{Disc}(G)$ is homogeneous of degree $s(s-1)^n$ by Proposition (2.6.1); note that F_d and F_s are generic homogeneous polynomials of respective degrees d and s in the indeterminates $(X_1, \ldots, X_n)$, while G is generic homogeneous of degree s in $(X_0, \ldots, X_n)$.

(2.18) Remark. The only trace towards the reduced discriminant in Salmon's work that we have found is in [18, §117], where he proves, in the above language, that for $s = 2$, $\mathrm{Disc}(F)$ has valuation at least 2.

He obtains this as a direct consequence of the fact that if f is a polynomial which is singular at a point x, then the tangent space of the discriminant hypersurface $\mathrm{V}(\mathrm{Disc})$ at $[f]$ contains the hyperplane of polynomials vanishing at x. Thus, for the polynomial

$$f = \sum_{a>1} F_a X_0^{d-a} + T \cdot \big(U'_1 X_1 + \cdots + U'_n X_n\big)\, X_0^{d-1},$$

one finds that $\mathrm{Disc}(f)$ is divisible by T^2 (here, F is as in (2.1) and (2.8), and $T, U'_1, \ldots, U'_n$ are new indeterminates). It follows that $\mathrm{Disc}(F)$ sits in the ideal

$$\big(U_{d,0,\ldots,0}\big) + \big(U_{d-1,1,\ldots,0},\ \ldots\ ,\ U_{d-1,0,\ldots,1}\big)^2 \subseteq A_{\mathbf{Z}},$$

which means that it has valuation at least 2 with respect to the Zariski grading for $s = 2$.

Proof of Theorem (2.17). The idea is to specialize Theorem (1.18), part (a), and in particular the formula

$$\big[\mathrm{Res}(F_1, \ldots, F_{n+1})\big]_{s_1 s_2 \cdots s_{n+1}} = \mathrm{Res}(G_1, \ldots, G_{n+1}) \cdot \mathrm{redRes}(H_1, \ldots, H_{n+1}),$$

to the discriminant situation, i.e., $F_1, \ldots, F_{n+1}$ specialize to $\partial_1 F, \ldots, \partial_n F, F$ respectively, and the G_j's and H_j's specialize according to the truncations corresponding to $s_1, \ldots, s_{n+1} = s-1, \ldots, s-1, s$, as indicated in Paragraph (2.8). Thus, we consider the following specializations:

$$\begin{aligned}
\mathrm{Res}(F_1, \ldots, F_{n+1}) &\rightsquigarrow \mathrm{Res}(\partial_1 F, \ldots, \partial_n F, F)\\
\mathrm{Res}(G_1, \ldots, G_{n+1}) &\rightsquigarrow \mathrm{Res}(\partial_1 G, \ldots, \partial_n G, G)\\
\mathrm{redRes}(H_1, \ldots, H_{n+1}) &\rightsquigarrow \mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)
\end{aligned}$$

On the other hand, we have

$$\mathrm{Res}(\partial_1 F, \ldots, \partial_n F, F) = \mathrm{Disc}(F)\,\mathrm{Disc}(\bar{F}) \tag{2.18.1}$$

$$\mathrm{Res}(\partial_1 G, \ldots, \partial_n G, G) = \mathrm{Disc}(G)\,\mathrm{Disc}(\bar{G}) \tag{2.18.2}$$

by Proposition (2.5). By definition, $\bar{F} = F_d$ and $\bar{G} = G_s = F_s$, and the result follows. We refer to [4] for more details. ∎

The following corollary is the general form of Salmon's formula. As we will see in the next section, Salmon's formula is a particular form of the corollary in the case $s = 2$.

(2.19) Corollary. *In the Zariski grading of $A_{\mathbf{Z}}$ as in Theorem (2.17), the discriminant* $\mathrm{Disc}(F)$ *has valuation $s(s-1)^n$, and can be written as*

(2.19.1) $$\mathrm{Disc}(F) = \mathrm{Disc}(G)\,\mathrm{Disc}(F_s)^2\ \mathrm{redDisc}(H) + \big(\text{terms of Zariski weight} > s(s-1)^n\big).$$

Proof. The only novelty with respect to the previous theorem is the expression for D_0 (in the notation of Theorem (2.17)) in terms of the reduced discriminant. By Definition (2.10), one has

$$\mathrm{redRes}\big(\partial_1 H, \ldots, \partial_n H, H\big) = \mathrm{Disc}(F_d)\,\mathrm{Disc}(F_s)\ \mathrm{redDisc}(H).$$

Thus, Equation (2.17.1) writes

$$D_0\,\mathrm{Disc}(F_d) = \mathrm{Disc}(G)\,\mathrm{Disc}(F_s)\,\mathrm{Disc}(F_d)\,\mathrm{Disc}(F_s)\ \mathrm{redDisc}(H)$$

which gives

$$D_0 = \mathrm{Disc}(G)\,\mathrm{Disc}(F_s)^2\,\mathrm{redDisc}(H)$$

as we wanted. □

We point out that, in the above, the factor $\mathrm{Disc}(F_s)$ comes once from Formula (2.17.1), where it came from Formula (2.18.2) as $\mathrm{Disc}(\bar{G})$, and once from the definition of $\mathrm{redDisc}(H)$, which explains the square in Formula (2.19.1). In the discussion following Definition (2.10) above, we explained that, in that definition, $\mathrm{Disc}(F_s)$ is a geometrically meaningful term, as F_s is the equation of the tangent cone of $\mathrm{V}(H)$ at the origin, whereas $\mathrm{Disc}(F_d)$ is a technical artefact. It is thus fortunate that the latter term does not appear in Formula (2.19.1). Note that, indeed, the factor $\mathrm{Disc}(F_d)$ in the definition of $\mathrm{redDisc}(H)$ cancels out with the same factor coming from Formula (2.18.1) as $\mathrm{Disc}(\bar{F})$.

Corollary (2.19) provides an interesting connection between ordinary and reduced discriminants. As a first illustration of its interest, we give the following generalized weight properties of the reduced discriminant, which generalize the computations of Example (2.16) to arbitrary n.

(2.20) Proposition.
(2.20.1) In the grading of $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = \alpha_0$, *the reduced discriminant* $\mathrm{redDisc}_d^s(F)$ *is homogeneous of degree*

$$d(d-1)^n - (s-1)^{n-1}\big[d\left((n+1)s + n - 1\right) - ns(s+1)\big].$$

(2.20.2) In the grading of $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = d - \alpha_0$, *the reduced discriminant* $\mathrm{redDisc}_d^s(F)$ *is homogeneous of degree*

$$n\big[d(d-1)^n - s(s+1)(s-1)^{n-1}\big].$$

Proof. In the two gradings of (2.20.1) and (2.20.2), the plain discriminant $\mathrm{Disc}(F)$ is homogeneous. Thus all its summands are homogeneous, and in particular its piece of lowest degree in the Zariski grading is homogeneous. We shall see that $\mathrm{Disc}(G)$ and $\mathrm{Disc}(F_s)$ are homogeneous as well. Therefore, it follows from Formula (2.19.1) that $\mathrm{redDisc}(H)$ is homogeneous. Then it is only a matter of computing its degree.

In the grading of (2.20.1), $\mathrm{Disc}_d(F)$ has weight $d(d-1)^n$ by by Proposition (2.6). One has $G = X_0^s F_0 + X_0^{s-1} F_1 + \cdots + F_s$ and the coefficients of F_i have weight $d - i = (s-i) + (d-s)$, hence $\mathrm{Disc}_d(G)$ has weight

$$s(s-1)^n + (n+1)(d-s)(s-1)^n$$

by Proposition (2.6) and Corollary (2.7). The coefficients of F_s all have weight $d-s$, hence $\mathrm{Disc}_s(F_s)$ has weight
$$n(d-s)(s-1)^{n-1}$$
by the homogeneity property of the discriminant in Definition (2.2) and Corollary (2.7) (note that $\mathrm{Disc}_s(F_s)$ is a discriminant for polynomials in n indeterminates, i.e., one less than $\mathrm{Disc}_d(F)$ and $\mathrm{Disc}_d(G)$). One thus finds, by the above Corollary (2.19), that $\mathrm{redDisc}_d^s(F)$ is homogeneous of degree

$$\begin{aligned} d(d-1)^n - s(s-1)^n - (n+1)(d-s)(s-1)^n - 2n(d-s)(s-1)^{n-1} \\ &= d(d-1)^n - (s-1)^{n-1}\big[s(s-1) + (n+1)(d-s)(s-1) + 2n(d-s)\big] \\ &= d(d-1)^n - (s-1)^{n-1}\big[d\left((n+1)s + (n-1)\right) - ns(s+1)\big]. \end{aligned}$$

In the grading of (2.20.2), $\mathrm{Disc}(F)$ is homogeneous of degree $nd(d-1)^n$ by Proposition (2.6), and $\mathrm{Disc}(G)$ is homogeneous of degree $ns(s-1)^n$ by the same result. In addition, all the coefficients of F_s have weight s in this grading hence, by the homogeneity property given in Definition (2.2), $\mathrm{Disc}(F_s)$ is homogeneous of degree $ns(s-1)^{n-1}$ (note that F is a homogeneous polynomial in n variables only). Then the conclusion follows by a direct computation. □

For $n = 1$, the degree in (2.20.1) is
$$d(d-1) - \big[2sd - s(s+1)\big] = d^2 - (2s+1)d + s(s+1),$$
which agrees with (2.16.1), and the degree in (2.20.2) agrees with (2.16.2).

In the above paragraphs we have introduced the reduced discriminant and provided some first properties that are sufficient for our purposes. A more detailed and complete study of this new eliminant polynomial, including for instance its irreducibility and the geometric meaning of its vanishing, in particular its connection to Milnor number, will appear in [3].

2.3 – Application to the Salmon Formula

We shall now see that Salmon's formula () is a particular case of the decomposition formula given in Corollary (2.19). We will then be able to generalize it to the case of a hypersurface in arbitrary dimension.

(2.21) Original Salmon Formula. Let $F(X,Y,Z)$ be the generic homogeneous polynomial of degree $d \geqslant 3$, and set
$$F = X^d F_0 + X^{d-1}F_1 + X^{d-2}F_2 + \cdots + XF_{d-1} + F_d,$$
$$F_0 = U, \qquad F_1(Y,Z) = SY + TZ, \qquad F_2(Y,Z) = \frac{1}{2}(AY^2 + 2BYZ + CZ^2);$$
(we introduce a rational number in the definition of F_2 to follow Salmon's notation; this is natural in terms of the Taylor–Newton Formula of [XII, Paragraph **??**]).

We consider the truncation of F at order $d-2$ with respect to X, and the corresponding Zariski grading. So, U has degree 2, S and T have degree 1, and the coefficients of the F_k's with $k \geqslant 2$ have degree 0 (this includes the coefficients A, B, C). We let
$$H = X^{d-2}F_2 + \cdots + XF_{d-1} + F_d \quad \text{and} \quad G = X^2F_0 + XF_1 + F_2.$$

Then, Corollary (2.19) tells us that

$$\operatorname{Disc}(F) = \operatorname{Disc}(G)\operatorname{Disc}(F_2)^2 \operatorname{redDisc}(H) + \bigl(\text{terms of Zariski weight} \geqslant 3\bigr). \tag{2.21.1}$$

Discriminants of quadratic forms are readily computed, see, e.g., [7, n^{o}5 Exemple 6, p. 363], which is based on Identity (2.2.1); one has

$$\operatorname{Disc}(F_2) = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2,$$

and

$$\operatorname{Disc}(G) = \frac{1}{2} \cdot \begin{vmatrix} 2U & S & T \\ S & A & B \\ T & B & C \end{vmatrix} = \frac{1}{2}\left(2ACU - AT^2 - 2B^2U + 2BST - CS^2\right).$$

Thus, Display (2.21.1) reads

$$\operatorname{Disc}(F) = \frac{1}{2}\left(2ACU - AT^2 - 2B^2U + 2BST - CS^2\right) \cdot \left(AC - B^2\right)^2 \cdot \operatorname{redDisc}(H) \mod \bigl((S,T)^3 + U(S,T) + (U^2)\bigr).$$

Then, the specialization $U = S = 0$ yields

$$\operatorname{Disc}(F) = -\frac{1}{2} A T^2 \cdot \left(AC - B^2\right)^2 \cdot \operatorname{redDisc}(H) \mod (T^3); \tag{2.21.2}$$

this is Salmon's Formula ( ), with the normalization factor $-\frac{1}{2}$, and with $\Phi = \operatorname{redDisc}(H)$.

(2.22) Salmon Formula in arbitrary dimension. The Salmon Formula for the discriminant of a plane curve can be generalized to the case of a hypersurface in a projective space of arbitrary dimension as follows. In a suitable system of homogeneous coordinates, any hypersurface $\mathrm{V}(F) \subseteq \mathbf{P}^n$ has an equation of the form

$$F(X_0, X_1, \ldots, X_n) = T X_0^{d-1} X_n + \sum_{k=2}^{d} X_0^{d-k} F_k(X_1, \ldots, X_n), \tag{2.22.1}$$

where for all $k = 2, \ldots, d$, the polynomial F_k is homogeneous of degree k in the indeterminates $X_1, \ldots, X_n$. This normal form merely imposes that the hypersurface $\mathrm{V}(F)$ goes through the point $(1 : 0 : \cdots : 0)$ and that its tangent hyperplane at this point is given by $X_n = 0$. Applying Corollary (2.19) as above, and setting $H = \sum_{k=2}^{d} X_0^{d-k} F_k$, we find that

$$\operatorname{Disc}(F) = \operatorname{Disc}(T X_0 X_n + F_2)\operatorname{Disc}(F_2)^2 \operatorname{redDisc}(H) \mod T^3.$$

Let $\bar{F}_2(X_1, \ldots, X_{n-1})$ be the homogeneous polynomial of degree 2 in the indeterminates $X_1, \ldots, X_{n-1}$ defined as

$$\bar{F}_2(X_1, \ldots, X_{n-1}) = F_2(X_1, \ldots, X_{n-1}, 0)$$

(beware the difference in notation with (2.5)). We have:

$$\begin{aligned}
& 2^{a(n,2)} \operatorname{Disc}(T X_0 X_n + F_2) \\
&\qquad = \operatorname{Res}\bigl(\partial_0(T X_0 X_n + F_2),\, \partial_1(T X_0 X_n + F_2),\, \ldots,\, \partial_n(T X_0 X_n + F_2)\bigr) \\
&\qquad = \operatorname{Res}\bigl(T X_n,\, \partial_1 F_2,\, \ldots,\, \partial_{n-1} F_2,\, T X_0 + \partial_n F_2\bigr) \\
&\qquad = -\operatorname{Res}\bigl(T X_0 + \partial_n F_2,\, \partial_1 F_2,\, \ldots,\, \partial_{n-1} F_2,\, T X_n\bigr) && (2.22.2) \\
&\qquad = -T \cdot \operatorname{Res}\bigl(T X_0 + \partial_n F_2|_{X_n=0},\, \partial_1 F_2|_{X_n=0},\, \ldots,\, \partial_{n-1} F_2|_{X_n=0}\bigr) && (2.22.3) \\
&\qquad = -T^2 \cdot \operatorname{Res}(\partial_1 \bar{F}_2,\, \ldots,\, \partial_{n-1} \bar{F}_2) = -2^{a(n-2,2)} T^2 \cdot \operatorname{Disc}(\bar{F}_2), && (2.22.4)
\end{aligned}$$

where:

— the exponents $a(n,2)$ and $a(n-2,2)$ are as defined in (2.2);
— (2.22.2) is obtained by [14, §5.8, Effet d'une permutation des polynômes];
— (2.22.3) is obtained by [14, (5.13.5)];
— (2.22.4) is obtained by [14, §5.10, Formule de Laplace].
Alternatively, since the resultants above are resultants of homogeneous polynomials of degree 1 they may be computed as determinants, see [14, §5.3, Cas des formes linéaires], to the effect that

$$\operatorname{Res}\big(TX_n,\,\partial_1 F_2,\,\ldots,\,\partial_{n-1}F_2,\,TX_0+\partial_n F_2\big) = \begin{vmatrix} 0 & 0 & \cdots & \cdots & 0 & T \\ 0 & [& & \partial_1 F_2 & &] \\ \vdots & & & \vdots & & \\ 0 & [& & \partial_{n-1}F_2 & &] \\ T & [& & \partial_n F_2 & &] \end{vmatrix}, \tag{2.22.5}$$

where each "[$\partial_i F_2$]" denotes the line containing the n coefficients of $\partial_i F_2$ in the indeterminates $X_1,\ldots,X_n$; from formula (2.22.5) one may easily retrace the previous computations.

Either way, the upshot is that $\operatorname{Disc}(TX_0X_n+F_2) = -T^2\operatorname{Disc}(\bar F_2)$ (note that $a(n,2) = a(n-2,2)$), and eventually we obtain the following generalized Salmon formula:

$$\operatorname{Disc}(F) = -T^2\operatorname{Disc}(\bar F_2)\operatorname{Disc}(F_2)^2\operatorname{redDisc}(H) \quad \bmod T^3. \tag{2.22.6}$$

In the situation of the original Salmon Formula above, one has $F_2(Y,Z) = \frac{1}{2}(AY^2+2BYZ+CZ^2)$ and $\bar F_2(Y) = \frac12 AY^2$, hence $\operatorname{Disc}(F_2) = AC-B^2$ and $\operatorname{Disc}(\bar F_2) = \frac12 A$, and thus the two formulas (2.21.2) and (2.22.6) are coherent.

3 – Computation of the node-couple degree by elimination

In [17, §605–607], Salmon sets up the following strategy to compute the number of 2-nodal curves in a general net of hyperplane sections of a smooth (hyper)surface $S\subseteq \mathbf{P}^3$. For $p'\in S$ and $p''\in \mathbf{T}_{p'}S-\{p'\}$, consider the pencil $\langle p',p''\rangle^\perp \subseteq \check{\mathbf{P}}^3$ of (hyper)planes containing p' and p''. It cuts out $d^\vee = \deg(S^\vee)$ points on the dual surface $S^\vee$, counted with multiplicities, which correspond to planes tangent to S. Among these, $\mathbf{T}_{p'}S$ counts doubly if it is a plain tangent plane, and triply if it is plainly bitangent.

Indeed, the line $\langle p',p''\rangle^\perp$ is contained in the plane $(p')^\perp$, hence tangent to $S^\vee$ at the point $(\mathbf{T}_{p'}S)^\perp\in S^\vee$. If the plane $\mathbf{T}_{p'}S$ is plainly bitangent to S, then $(\mathbf{T}_{p'}S)^\perp$ is a general point on the ordinary double curve of $S^\vee$, and the line $\langle p',p''\rangle^\perp$ is tangent to one of the two transverse sheets of $S^\vee$ at $(\mathbf{T}_{p'}S)^\perp$. The idea is then firstly to determine the conditions on p' for $\mathbf{T}_{p'}S$ to count with multiplicity greater than 2 in $\langle p',p''\rangle^\perp$, and secondly to sort out the various corresponding geometric situations. A key element to carry this out is the famous formula (𓄿); another one is the elimination procedure (3.2).

We work out Salmon's procedure in subsection 3.2, and in subsection 3.3 we show how it carries over for hypersurfaces in a projective space of arbitrary dimension. From now on, we work over an algebraically closed field $\mathbf{k}$ of characteristic 0. We use freely the theory of polarity, on which there is a recap in the previous chapter [XII, Appendix A].

3.1 – An elimination trick

This section is dedicated to an elimination trick due to Salmon, see [17, §606]. It is the main technical device that he uses to conduct his study of pencils of planes orthogonal to a tangent line, which eventually gives the node-couple degree. We advise the reader to skip this section in first reading and move on to Section 3.2 where the general picture of Salmon's approach is given, and to come back here only when needed.

(3.1) We let $\mathbf{k}$ be an algebraically closed field of characteristic 0, and consider a smooth, degree d, surface $S \subseteq \mathbf{P}^3$ defined by a homogeneous polynomial $F \in \mathbf{k}[X, Y, Z, W]$; we set

$$\mathcal{T}_S = \big\{(p', p'') : p' \in S \text{ and } p'' \in \mathbf{T}_{p'}S\big\} \subseteq \mathbf{P}^3 \times \mathbf{P}^3$$

Let $\Phi_{\hat{p}',\hat{p}''}(\hat{p})$ be a trihomogeneous polynomial in the sets of variables $(\hat{p}', \hat{p}'', \hat{p}) \in (\mathbf{k}^4)^3$, of tridegree (λ, μ, μ) for some non-negative integers λ, μ. Assume that for all $(p', p'') \in \mathcal{T}_S$, $p' \neq p''$, the hypersurface $\mathrm{V}(\Phi_{\hat{p}',\hat{p}''}) \subseteq \mathbf{P}^3$ consists of μ planes, counted with multiplicities, all containing the line $\langle p', p''\rangle$.

(3.2) Theorem. *Let p' be a point of S. The following assertions are equivalent:*
(i) there exists $p'' \in \mathbf{T}_{p'}S - \{p'\}$ such that the tangent plane $\mathbf{T}_{p'}S$ is a component of $\mathrm{V}(\Phi_{\hat{p}',\hat{p}''})$;
(ii) for all $p'' \in \mathbf{T}_{p'}S - \{p'\}$, $\mathbf{T}_{p'}S$ is a component of $\mathrm{V}(\Phi_{\hat{p}',\hat{p}''})$.
Moreover, there exists a homogeneous polynomial $G \in \mathbf{k}[X, Y, Z, W]$ of degree $\lambda + (d-2)\mu$ such that (i) and (ii) are equivalent to:
(iii) p' lies on the hypersurface $\mathrm{V}(G) \subseteq \mathbf{P}^3$.

We need some preparation for the proof. We begin with the following characterization of the non-emptiness of the intersection of two lines in $\mathbf{P}^3$, one given parametrically and the other defined by equations.

(3.3) Lemma. *Let $L', L'' \in \mathbf{k}[X, Y, Z, W]$ be two linear functionals, and $\hat{p}', \hat{p}'' \in \mathbf{k}^4$. The intersection of the two lines $\mathrm{V}(L', L'')$ and $\langle p', p''\rangle$ is non-empty if and only if*

$$D_{L',L''}(p', p'') \stackrel{def.}{=} \begin{vmatrix} L'(\hat{p}') & L''(\hat{p}') \\ L'(\hat{p}'') & L''(\hat{p}'') \end{vmatrix}$$

vanishes.

Proof. The line $\langle p', p''\rangle$ is the image of the map $\mathbf{P}^1 \to \mathbf{P}^3$ defined by the linear map $(u, v) \in \mathbf{k}^2 \mapsto u\hat{p}' + v\hat{p}'' \in \mathbf{k}^4$. Therefore, the lines $\mathrm{V}(L', L'')$ and $\langle p', p''\rangle$ intersect if and only if the two polynomials $L'(u\hat{p}' + v\hat{p}'')$ and $L''(u\hat{p}' + v\hat{p}'')$ share a common root in $\mathbf{P}^1$. These polynomials are linear forms in u, v, and the polynomial $D_{L',L''}(p', p'')$ is the determinant of the corresponding linear system. □

For all L', L'', $D_{L',L''}$ is a bihomogeneous polynomial in the two sets of variables (p', p''), of bidegree $(1, 1)$ and anti-symmetric; this implies that it is irreducible.

(3.4) Lemma. *For general linear forms L', L'', the intersection in $\mathbf{P}^3 \times \mathbf{P}^3$ of $\mathcal{T}_S$ and the hypersurface $\mathrm{V}(D_{L',L''})$ is integral.*

Proof. A natural idea is to try and prove the lemma as an application of Bertini's theorem. It is however delicate to proceed this way, as for instance the map defined on $\mathbf{P}^3 \times \mathbf{P}^3$ by the linear system of anti-symmetric forms of bidegree $(1, 1)$ is certainly not generically finite, and even less so the map defined by the system of all $D_{L',L''}$. We will thus rather prove the result by hand.

Let us first note that as L', L'' range through all linear forms, the bihomogeneous forms $D_{L',L''}$ cover all rank 2, bilinear, anti-symmetric forms. This is a direct calculation: for

$$L' = A'X + B'Y + C'Z + D'W \quad \text{and} \quad L'' = A''X + B''Y + C''Z + D''W,$$

$D_{L',L''}$ is the bilinear form

$$\left(\begin{pmatrix} X' \\ Y' \\ Z' \\ W' \end{pmatrix}, \begin{pmatrix} X'' \\ Y'' \\ Z'' \\ W'' \end{pmatrix}\right) \longmapsto (X', Y', Z', W') \begin{pmatrix} 0 & U_{01} & U_{02} & U_{03} \\ -U_{01} & 0 & U_{12} & U_{13} \\ -U_{02} & -U_{12} & 0 & U_{23} \\ -U_{03} & -U_{13} & -U_{23} & 0 \end{pmatrix} \begin{pmatrix} X'' \\ Y'' \\ Z'' \\ W'' \end{pmatrix}$$

where the coefficients U_{ij} are the six 2×2 minors of the matrix

$$\begin{pmatrix} A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \end{pmatrix}.$$

It follows that the only relation between the coefficients U_{ij} is the vanishing of the Pfaffian of the anti-symmetric matrix (U_{ij}), i.e.,

$$U_{01}U_{23} - U_{02}U_{13} + U_{03}U_{12} = 0$$

(this is the Plücker equation of the Grassmannian $\mathrm{Gr}(2,4)$ in $\mathbf{P}^5$).

Now to the lemma. Projection on the first factor in $\mathbf{P}^3 \times \mathbf{P}^3$ makes $\mathcal{T}_S$ a $\mathbf{P}^2$ bundle over S: for all $p' \in S$, the fibre over p' is the zero locus of the linear form $\mathrm{D}_{p'}F$ (the differential of F, a defining equation for S, at the point p'; it is non-zero as we assume that S is smooth). Similarly, for all $p' \in \mathbf{P}^3$ the fibre $\mathrm{pr}_1^{-1}(p') \cap \mathrm{V}(D_{L',L''})$ is the zero locus of the linear form $D_{L',L''}(p', \_)$, which is either a hyperplane or the whole $\mathbf{P}^3$. We claim that for general L', L'', the intersection $\mathcal{T}_S \cap \mathrm{V}(D_{L',L''})$ is the closure of a $\mathbf{P}^1$-bundle over an open subset of S, which implies at once that it is irreducible. Moreover, this implies that the intersection $\mathcal{T}_S \cap \mathrm{V}(D_{L',L''})$ is fibrewise over S a transverse intersection of hyperplanes, and therefore it is reduced. Thus the claim proves the lemma. To prove the claim it suffices to show that, for general L', L'', the set of points $p' \in S$ such that the linear form $D_{L',L''}(p', \_)$ is a multiple of $\mathrm{D}_{p'}F$ has dimension 0; in particular, the intersection $\mathcal{T}_S \cap \mathrm{V}(D_{L',L''})$ cannot have any irreducible component generically a $\mathbf{P}^2$-bundle over a curve in S.

The claim follows from explicit computations. The two linear forms $\mathrm{D}_{p'}(F)$ and $D_{L',L''}(p', \_)$ are, in coordinates,

$$\big(\partial_X F(p'),\, \partial_Y F(p'),\, \partial_Z F(p'),\, \partial_W F(p')\big) \quad \text{and} \quad \begin{pmatrix} X'' \\ Y'' \\ Z'' \\ W'' \end{pmatrix}^{\mathsf{T}} \cdot \begin{pmatrix} 0 & U_{01} & U_{02} & U_{03} \\ -U_{01} & 0 & U_{12} & U_{13} \\ -U_{02} & -U_{12} & 0 & U_{23} \\ -U_{03} & -U_{13} & -U_{23} & 0 \end{pmatrix}.$$

Both have the line $p' \subseteq \mathbf{k}^4$ in their kernel, so we consider the corresponding linear forms on the quotient $\mathbf{k}^4/p'$, given by line matrices with three entries; up to a linear change of coordinates we may assume that $p' = (0 : 0 : 0 : 1)$, in which case we are simply forgetting the last entry in the two above line matrices. The claim then boils down to the fact that the three 2×2 minors of the 2×3 matrix obtained by concatenating the two line matrices corresponding to $\mathrm{D}_{p'}(F)$ and $D_{L',L''}(p', \_)$ define a locus of codimension at least two on S, for general L', L''. Since everything is explicit, one may observe that for a general choice of L', L'', any two of these three minors have no common factor, which proves the claim. We believe it is best to let the reader write down this elementary verification on her or his own. ☐

(3.5) Proof of Theorem (3.2). The proof is based on the fact that $\mathbf{T}_{p'}S$ is a component of $\mathrm{V}(\Phi_{p',p''})$ if and only if, for all line $\Lambda \subseteq \mathbf{P}^3$ the intersection $L \cap \mathbf{T}_{p'}S \cap \mathrm{V}(\Phi_{p',p''})$ is non-empty; a moment of thought should convince the reader that this fact is indeed true. Then the idea is to consider a suitable resultant to express this fact, and to factor a power of $D_{L',L''}(p',p'')$ out of this resultant.

Consider $T_{p'} = \mathrm{D}^p F(p')$ as a trihomogeneous polynomial of tridegree $(d-1,0,1)$ in (p',p'',p). Let $L', L'' \in \check{\mathbf{P}}^3$ be two generic linear forms in p, and consider the line $\Lambda = \mathrm{V}(L',L'')$ in $\mathbf{P}^3$. It intersects $\mathbf{T}_{p'}S \cap \mathrm{V}(\Phi_{p',p''})$ if and only if the resultant

$$R = \mathrm{Res}(T_{p'}, \Phi_{p',p''}, L', L''),$$

with respect to the set of variables p, vanishes. It follows from the homogeneity properties of the resultant, see Theorem (1.6), that R is a multi-homogeneous polynomial in the sets of variables (L',L'',p',p''), of multi-degree $(\mu,\mu,\mu(d-1)+\lambda,\mu)$.

For a given $(p',p'') \in \mathcal{T}_S$, the polynomial $\Phi_{p',p''}$ splits as the product of μ linear forms $\Phi^i_{p',p''}$, $i=1,\dots,\mu$, that all vanish along the line $\langle p',p''\rangle$. Although this splitting may not exist globally over $\mathcal{T}_S$ (in other words the factors $\Phi^i_{p',p''}$ may not be polynomials in (p',p'')), there exists a global splitting over a finite cover $\tilde{\mathcal{T}_S}$ of $\mathcal{T}_S$[3] (in other words the factors $\Phi^i_{p',p''}$ have their coefficients in a finite extension of $\mathbf{k}[p',p'']$). Then, by the multiplicativity property, see [14, § 5.7, p. 154], the resultant R splits as the product of the μ resultants $R_i = \mathrm{Res}(T_{p'}, \Phi^i_{p',p''}, L', L'')$. By Lemma (3.3), for all i the resultant R_i vanishes wherever $D_{L',L''}$ vanishes on $\mathcal{T}_S$, or rather on $\tilde{\mathcal{T}_S}$, as both forms $T_{p'}$ and $\Phi^i_{p',p''}$ vanish along the line $\langle p',p''\rangle$. By Lemma (3.4), this implies that $D_{L',L''}$ divides R_i as a function of $(p',p'') \in \tilde{\mathcal{T}_S}$. Eventually, the upshot is that $(D_{L',L''})^\mu$ divides R, and since both $D_{L',L''}$ and R are well-defined as functions of $(p',p'') \in \mathcal{T}_S$, so is the quotient. Therefore, there exists a multi-homogeneous polynomial R' in the sets of variables (L',L'',p',p'') such that after restriction to $\check{\mathbf{P}}^3 \times \check{\mathbf{P}}^3 \times \mathcal{T}_S$ one has $R = (D_{L',L''})^\mu R'$. Computing degrees, one finds that R' has multi-degree $(0,0,\mu(d-2)+\lambda,0)$ in (L',L'',p',p''), which proves the theorem. $\square$

3.2 – Study of a pencil of planes orthogonal to a tangent line

(3.6) General setup. We recall that $\mathbf{k}$ is assumed to be an algebraically closed field of characteristic 0. We let $S \subseteq \mathbf{P}^3$ be a smooth surface of degree d, defined by a homogeneous polynomial $F(X,Y,Z,W)$. We consider generic 4-tuples $\hat{p}' = (X',Y',Z',W')$, $\hat{p}'' = (X'',Y'',Z'',W'')$, $\hat{p} = (X,Y,Z,W)$, and call p',p'',p the corresponding points in $\mathbf{P}^3$ (beware the unusual distribution of the prime decorations). The choice of $\hat{p}',\hat{p}'',\hat{p}$ defines a system of homogeneous

[3] let us briefly recap how such a finite cover may be constructed. The family of planes defined over $\mathcal{T}_S$ by $\Phi_{p',p''}$ corresponds to a subscheme $\mathcal{P}$ of $\mathcal{T}_S \times \check{\mathbf{P}}^3$ with the first projection inducing a degree μ map $\mathcal{P} \to \mathcal{T}_S$. Our goal is to show that there exists a base change $\tilde{\mathcal{T}_S} \to \mathcal{T}_S$ such that $\mathcal{P} \times_{\mathcal{T}_S} \tilde{\mathcal{T}_S}$ splits as μ copies of $\tilde{\mathcal{T}_S}$. As a first step, we consider the following (somewhat absurd non-sensical) base change.

$$\begin{tikzcd} \mathcal{P} \times_{\mathcal{T}_S} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\ \mathcal{P} \arrow[r] & \mathcal{T}_S \end{tikzcd}$$

The diagonal Δ provides a section of the degree μ map $\mathcal{P} \times_{\mathcal{T}_S} \mathcal{P} \to \mathcal{P}$, hence $\mathcal{P} \times_{\mathcal{T}_S} \mathcal{P}$ splits as $\Delta + \mathcal{P}'$ where $\mathcal{P}'$ comes with a degree $\mu - 1$ map to $\mathcal{P}$. Then, by induction on μ, one may construct a finite cover $\tilde{\mathcal{P}} \to \mathcal{P}$ such that $\mathcal{P}' \times_{\mathcal{P}} \tilde{\mathcal{P}}$ splits as $\mu - 1$ copies of $\tilde{\mathcal{P}}$, hence $\mathcal{P} \times_{\mathcal{T}_S} \tilde{\mathcal{P}}$ splits as μ copies of $\tilde{\mathcal{P}}$. Thus $\tilde{\mathcal{P}}$ is suitable as the $\tilde{\mathcal{T}_S}$ we were looking for. We point out the (non-accidental) similarity of this construction with that of a splitting field for a given polynomial P in one indeterminate X. The first base change in our construction is the analogue of the field extension $\mathbf{k}[X]/(P)$. Also note that, in general, the map $\tilde{\mathcal{T}_S} \to \mathcal{T}_S$ will have degree $\mu!$.

coordinates $(\alpha : \beta : \gamma)$ on the plane $\langle p', p'', p\rangle$ generated by p', p'' and p. We consider

$$
\begin{aligned}
f(\alpha,\beta,\gamma) &\overset{\text{def.}}{=} F(\alpha\hat{p}' + \beta\hat{p}'' + \gamma\hat{p}) \\
&= \alpha^d \cdot F(\hat{p}') + \alpha^{d-1}\cdot \mathrm{D}^{\beta\hat{p}''+\gamma\hat{p}}F(\hat{p}') + \frac{1}{2}\alpha^{d-2}\cdot \mathrm{D}^{(\beta\hat{p}''+\gamma\hat{p})^2}F(\hat{p}') \mod (\beta,\gamma)^3 \\
&= \alpha^d \cdot F(\hat{p}') + \alpha^{d-1}\big(\beta \mathrm{D}^{\hat{p}''}F(\hat{p}') + \gamma \mathrm{D}^{\hat{p}}F(\hat{p}')\big) \\
&\quad + \frac{1}{2}\alpha^{d-2}\big(\beta^2\mathrm{D}^{\hat{p}''^2}F(\hat{p}') + 2\beta\gamma \mathrm{D}^{\hat{p}''\hat{p}}F(\hat{p}') + \gamma^2\mathrm{D}^{\hat{p}^2}F(\hat{p}')\big) \mod (\beta,\gamma)^3.
\end{aligned}
\tag{3.6.1}
$$

Considered as a homogeneous polynomial in the variables (α,β,γ), this is the equation of the hyperplane section of S by $\langle p', p'', p\rangle$.

(3.7) Degree of the dual surface. As a warm-up, let us derive the degree of the dual surface $S^\vee \subseteq \check{\mathbf{P}}^3$ of S from the relevant homogeneity property of the discriminant. We specialize p'', p to two general points of $\mathbf{P}^3_{\mathbf{k}}$. Then the coefficients of $f(\alpha,\beta,\gamma)$ are polynomials in the set of indeterminates $\hat{p}' = (X', Y', Z', W')$, and so is the discriminant $\mathrm{Disc}(f)$ with respect to the variables α, β, γ. The latter vanishes at a point $p' \in \mathbf{P}^3_{\mathbf{k}}$ if and only if the plane $\langle p', p'', p\rangle$ is tangent to S. Thus the zero locus $\mathrm{V}\big(\mathrm{Disc}(f)\big) \subseteq \mathbf{P}^3_{\mathbf{k}}$ is the union of all planes tangent to S and containing the line $\langle p'', p\rangle$; in particular the degree of $\mathrm{Disc}(f)$ (as a polynomial in p', what else?) is the number of tangent planes to S containing the line $\langle p'', p\rangle$, which is the degree of the dual surface $S^\vee \subseteq \check{\mathbf{P}}^3$. This is also the number of points in the intersection of $S^\vee$ with the line $(p'', p)^\perp \subseteq \check{\mathbf{P}}^3$, the latter parametrizing planes passing through p'' and p.

The coefficient of $f(\alpha,\beta,\gamma)$ in $\alpha^a\beta^b\gamma^c$ is a polynomial of degree a in the set of indeterminates $\hat{p}'$, as can be seen on the Taylor–Newton expansion, the first terms of which are given in Display (3.6.1). Therefore, by Proposition (2.6.1), the discriminant $\mathrm{Disc}(f)$ is homogeneous of degree $d(d-1)^2$ in $\hat{p}'$, and the dual surface $S^\vee$ has degree $d^\vee = d(d-1)^2$ in $\check{\mathbf{P}}^3$.

(3.8) Remark. If we let all three points (or 4-tuples) p', p'', p be generic, then $\mathrm{Disc}(f(\alpha,\beta,\gamma))$ is tri-homogeneous with respect to the three sets of indeterminates p', p'', p, of tri-degree

$$(d^\vee, d^\vee, d^\vee) = (d(d-1)^2, d(d-1)^2, d(d-1)^2),$$

as follows from the above analysis. Note that obviously p', p'', p play symmetric roles in the definition of f.

In order to enumerate bitangent planes, we will take points p', p'' such that the line $\langle p', p''\rangle$ is tangent to S at p', and consider a reduced discriminant of the corresponding $f(\alpha,\beta,\gamma)$. Thus we need to specialize our setup a little.

(3.9) The "tangent line" setup. In the situation set-up in Paragraph (3.6) above, we specialize p' to the general point of S, and p'' to the general point of $\mathbf{T}_{p'}S$. Then $F(\hat{p}') = \mathrm{D}^{\hat{p}''}F(\hat{p}') = 0$, so (3.6.1) reduces to

$$f(\alpha,\beta,\gamma) = T\alpha^{d-1}\gamma + \frac{1}{2}\alpha^{d-2}(A\beta^2 + 2B\beta\gamma + C\gamma^2) \mod (\beta,\gamma)^3, \tag{3.9.1}$$

where $T = \mathrm{D}_{\hat{p}}f(\hat{p}')$, $A = \mathrm{D}_{\hat{p}''^2}f(\hat{p}')$, $B = \mathrm{D}_{\hat{p}''\hat{p}}f(\hat{p}')$, $C = \mathrm{D}_{\hat{p}^2}f(\hat{p}')$. Note in particular that, as homogeneous polynomials in the set of indeterminates $\hat{p} = (X, Y, Z, W)$, T and C are the equations of the tangent plane $\mathbf{T}_{p'}S$, and the polar quadric $\mathrm{D}^2S(p')$ of S at p', respectively.

We consider the discriminant of the plane curve $S \cap \langle p', p'', p\rangle$, i.e., the discriminant of $f \in A[\alpha,\beta,\gamma]$ with $A = \mathbf{k}[X, Y, Z, W]$; by Formula (𓄿), or rather Formula (2.21.2), it writes

$$\mathrm{Disc}(f) = -\frac{1}{2}T^2\big(A(B^2 - AC)^2\,\Phi + T\,\Psi\big), \quad \text{with} \quad \Phi = \mathrm{redDisc}^2_d(f). \tag{3.9.2}$$

As explained in the previous paragraphs, it vanishes if and only if the plane $\langle p', p'', p\rangle$ is tangent to S; It is a trihomogeneous polynomial in the variables $\hat{p}'$, $\hat{p}''$ and $\hat{p}$; as a polynomial in p, it vanishes along the $d^\vee$ planes (counted with multiplicities) tangent to S in the pencil $(p', p'')^\perp \subseteq \check{\mathbf{P}}^3$ of planes containing the line $\langle p', p''\rangle \subseteq \mathbf{P}^3$.

(3.10) Remark. The fact that T^2 factors out of $\mathrm{Disc}(f)$ gives an algebraic proof of the fact that the tangent plane $\mathbf{T}_{p'}S$ appears with multiplicity $\geqslant 2$ in the scheme $(p', p'')^\perp \cap S^\vee$; or rather, the point $(\mathbf{T}_{p'}S)^\perp \in \check{\mathbf{P}}^3$ appears with multiplicity $\geqslant 2$ in the scheme $(p', p'')^\perp \cap S^\vee$.

In other words, if a line $\Lambda \subseteq \mathbf{P}^3$ is tangent to S at some point p, then the line $\Lambda^\perp \subseteq \check{\mathbf{P}}^3$ is tangent to the dual surface $S^\vee$ at the point $(\mathbf{T}_pS)^\perp$.

Salmon's idea is that if the tangent plane $\mathbf{T}_{p'}S$ is also tangent to S at some point different from p', then it should appear with multiplicity > 2 in $(p', p'')^\perp \cap S^\vee$; equivalently, $\mathrm{Disc}(f)$ should be divisible by T^3 in the above setup.

Therefore, we shall now discuss the various possibilities for T to divide $A(B^2 - AC)^2\,\Phi$, in the notation of Equation (3.9.2).

(3.11) Vanishing of A and biduality for inflectional tangents. The polynomial A has degree 0 in p, therefore it is divisible by T if and only if it is identically zero. By definition $A = \mathrm{D}^{\hat{p}''^2} f(\hat{p}')$, so its vanishing is equivalent to the point p'' being on the polar quadric $\mathrm{D}_{p'^2}S$ of S at p'. Since $p'' \in \mathbf{T}_{p'}S$, this in turn is equivalent to the line $\langle p', p''\rangle$ being one of the two inflectional tangents of S at p' (i.e., the two lines Λ having intersection multiplicity with S at p' at least 3; in symbols, $i(\langle p', p''\rangle, S)_{p'} \geqslant 3$).

We thus find an algebraic proof of the fact that

$$\text{if} \quad i\big(\langle p', p''\rangle, S\big)_{p'} \geqslant 3, \quad \text{then} \quad i\big((p', p'')^\perp, S^\vee\big)_{(\mathbf{T}_{p'}S)^\perp} \geqslant 3,$$

which is a manifestation of biduality. Geometrically, if a line $\Lambda \subseteq \mathbf{P}^3$ is an inflectional tangent to S at p' (i.e., L is the tangent line to one of the two local branches of $\mathbf{T}_pS \cap S$ at p'), then its orthogonal $\Lambda^\perp \subseteq \check{\mathbf{P}}^3$ is an inflectional tangent to $S^\vee$ at the point $(\mathbf{T}_{p'}S)^\perp$.

As a side remark note that when $A = 0$, all the curves cut out on S by a member of the pencil $(p', p'')^\perp$ have an inflection point at p'.

(3.12) We shall analyze the divisibilities of $B^2 - AC$ and Φ by T using Theorem (3.2). To see that the latter result indeed applies, we note that for $(p', p'') \in \mathcal{T}_S$ (see the notation in Paragraph (3.1)), the discriminant in Display (3.9.2), $T^2A(B^2 - AC)^2\Phi + T^3\Psi$, defines as a homogeneous polynomial in the variable p a hypersurface consisting of $d^\vee$ planes, counted with multiplicities, all containing the line $\langle p', p''\rangle$. This implies that so does its homogeneous piece of lowest degree with respect to any grading, as the piece of lowest degree of a product is the product of the pieces of lowest degree. For the Zariski grading, see subsection 2.3, the piece of lowest degree of the discriminant is $T^2A(B^2 - AC)^2\Phi$. The upshot is that $T^2A(B^2 - AC)^2\Phi$, as a polynomial in p, defines a sum of planes, all containing the line $\langle p', p''\rangle$, which implies that so do all the factors $T, A, B^2 - AC, \Phi$.

Note that this is obvious for T, which defines the plane $\mathbf{T}_{p'}S$. The polynomial A, on the other hand, is independent on the variable p, hence defines either the whole space, or the empty set.

(3.13) Divisibility of $B^2 - AC$ by T and biduality at parabolic points. The polynomial $B^2 - AC$ is tri-homogeneous of tri-degree $(2(d-2), 2, 2)$ in the variables (p', p'', p). By Theorem (3.2), which have seen in Paragraph (3.12) above can be applied in our situation, there exists a homogeneous polynomial H of degree $4(d-2)$ in the variable p', with constant coefficients, such that for fixed $p' \in S$ and $p'' \in \mathbf{T}_{p'}S$, T divides $B^2 - AC$ as polynomials in p if and only if $H(p') = 0$.

The polynomial H is, in fact, the Hessian determinant. Indeed, it follows from (3.9.1) that the tangent cone of the section of S by its tangent hyperplane at p', $S \cap \mathbf{T}_{p'}S$, is defined by the quadratic form given by the symmetric matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}.$$

Thus, $B^2 - AC$ is zero modulo T if and only if the curve $S \cap \mathbf{T}_{p'}S$ has a degenerate tangent cone at p', i.e., p' is a parabolic point of S.

Geometrically this is explained as follows. If p' is a parabolic point of S, then the point $(\mathbf{T}_{p'}S)^\perp \in \check{\mathbf{P}}^3$ sits on the cuspidal double curve of $S^\vee$, and for $p'' \in \mathbf{T}_{p'}S - \{p'\}$, the line $(p', p'')^\perp \subseteq \check{\mathbf{P}}^3$ is contained in the tangent cone of $S^\vee$ at the point $(\mathbf{T}_{p'}S)^\perp \in \check{\mathbf{P}}^3$, hence $i((p',p'')^\perp, S^\vee)_{(\mathbf{T}_{p'}S)^\perp} \geqslant 3$.

(3.14) Divisibility of the reduced discriminant Φ by T. The discriminant in Display (3.9.2) is tri-homogeneous in the variables (p', p'', p) with respect to the standard grading, of degree $d^\vee = d(d-1)^2$ with respect to all three variables, as we have observed above. This implies that also the homogeneous piece of (3.9.2) of lowest degree with respect to the Zariski grading is tri-homogeneous of tri-degree $(d^\vee, d^\vee, d^\vee)$ for the standard grading. Moreover, T, A, and $B^2 - AC$ have respective tri-degrees $(d-1, 0, 1)$, $(d-2, 2, 0)$, and $(2(d-2), 2, 2)$ in (p', p'', p). Thus, computing degrees, one finds that Φ has tri-degree (λ, μ, μ), with

$$\lambda = (d-2)(d^2-6) \quad \text{and} \quad \mu = d^3 - 2d^2 + d - 6.$$

Since Φ is $\mathrm{redDisc}_d^2(f)$ with respect to the indeterminate α, and the coefficient of $f(\alpha, \beta, \gamma)$ in $\alpha^a \beta^b \gamma^c$ is a polynomial of degree a in the indeterminates p', the fact that Φ is homogeneous in p' and its degree follow directly from Proposition (2.20.1); to wit, in our situation the degree in Proposition (2.20.1) reads

$$d(d-1)^2 - \big[7d - 12\big] = d^3 - 2d^2 - 6d + 12 = (d-2)(d^2-6).$$

The homogeneity and degrees in p and p'' however do not directly follow from Proposition (2.20), as for instance the coefficients in $A\beta^2 + 2B\beta\gamma + C\gamma^2$ do not all have the same weights in p or p''. It is noticeable that $T^2 A (B^2 - AC)^2$ is homogeneous in p and p'' because the weights of T^2 and A balance each other; this is a very elementary reason why T "should" appear with a square in Formula (3.9.2). The degree of $\mathrm{redDisc}_d^2(f)$ with the weights as in Proposition (2.20.2) is

$$2\big[d(d-1)^2 - 6\big] = 2\big[d^3 - 2d^2 + d - 6\big],$$

and one observes that its degrees in p and p'' respectively are one half of this, thus this degree "distributes equally" between p and p''.

In conclusion, it follows as in Paragraph (3.13) from Theorem (3.2) that there exists a homogeneous polynomial K of degree $(d-2)(d^3 - d^2 + d - 12)$ in the variable p', with constant coefficients, such that for fixed $p' \in S$ and $p'' \in \mathbf{T}_{p'}S$, T divides Φ if and only if $K(p') = 0$.

We have thus arrived at the following statement.

(3.15) Theorem. *Let S be a smooth, degree d, surface in $\mathbf{P}^3$. There is a hypersurface $V(K)$ of degree $(d-2)(d^3 - d^2 + d - 12)$, the intersection of which with S is the locus of tangency points of planes bitangent to S.*

It has to be recognized that the proof given above of the latter statement rests on the not fully justified fact that T divides $\Phi = \operatorname{redDisc}(f)$ if and only if the tangent plane $\mathbf{T}_{p'}S$ is also tangent to S at some additional point. The "if" part is established, see, e.g., [XII, Theorem **??**] due to Dimca, but what is really needed is the "only if" part. Evidence in favour of that assertion is provided by the irreducibility of the reduced discriminant Φ (which, as already mentioned, follows from the fact that the reduced discriminant is a toric discriminant; see also [3]).

(3.16) Corollary. *The ordinary double curve of the dual surface $S^\vee$ has degree $\frac{1}{2}d(d-1)(d-2)(d^3-d^2+d-12)$.*

Proof. Let $p'' \in \mathbf{P}^3$ be a general point. The locus of those points $p' \in S$ such that there exists a plane through p'' tangent to S at p' is the apparent boundary $\mathrm{D}^{p''}S \cap S$. Therefore, by Theorem (3.15), the locus of points $p' \in S$ such that $\mathbf{T}_{p'}S$ is bitangent and passes through p'' is $\mathrm{D}_{p''}S \cap S \cap V(K)$. Now for each bitangent plane there are two tangency points p', so the number of bitangent planes passing through p'' is

$$\frac{1}{2} \cdot \deg(S) \cdot \deg(\mathrm{D}^{p''}S) \cdot \deg K.$$

□

3.3 – Generalization to hypersurfaces of arbitrary dimension

In fact, Salmon's procedure works in arbitrary dimension, using the generalization (2.22.6) of formula (𓄿). The arguments are direct generalizations, so we are going to be sketchy.

(3.17) Let $V = \mathrm{V}(F)$ be a smooth hypersurface of degree d in $\mathbf{P}^n$, and consider n points $p_1, \ldots, p_{n-1}, p$ (or rather tuples $\hat{p}_1, \ldots, \hat{p}_{n-1}, \hat{p} \in \mathbf{k}^{n+1}$). The points $p_1, \ldots, p_{n-1}$ (if in linear general position) define a pencil of hyperplanes in $\mathbf{P}^n$, and for all p (not in $\langle p_1, \ldots, p_{n-1}\rangle$), the tuples $\hat{p}_1, \ldots, \hat{p}_{n-1}, \hat{p}$ define a system of homogeneous coordinates $(\alpha_1 : \ldots : \alpha_{n-1} : \alpha)$ on the member H_p of the pencil determined by p, i.e., $\langle p_1, \ldots, p_{n-1}, p\rangle$. In this system of coordinates, the hyperplane section $H_p \cap V$ is the zero locus of the polynomial

$$\begin{aligned} f(\alpha_1, \ldots, \alpha_{n-1}, \alpha) &\overset{\text{def.}}{=} F(\alpha_1\hat{p}_1 + \cdots + \alpha_{n-1}\hat{p}_{n-1} + \alpha\hat{p}) \\ &= \sum_{k=0}^{d} \frac{1}{k!}\, \alpha_1^{d-k}\, \mathrm{D}^{(\alpha_2\hat{p}_2+\cdots+\alpha\hat{p})^k} F(\hat{p}_1). \end{aligned}$$

We shall assume that $p_1 \in V$ and $p_2, \ldots, p_{n-1} \in \mathbf{T}_{p_1}V$, equivalently

$$F(p_1) = \mathrm{D}^{p_2}F(p_1) = \cdots = \mathrm{D}^{p_{n-1}}F(p_1) = 0.$$

Then,

$$f(\alpha_1, \ldots, \alpha_{n-1}, \alpha) = T\alpha_1^{d-1}\alpha + \frac{1}{2}\alpha_1^{d-2} f_2(\alpha_2, \ldots, \alpha_n, \alpha) \mod (\alpha_2, \ldots, \alpha)^3$$

where $T = \mathrm{D}_{\hat{p}} f(\hat{p}_1)$ and $f_2(\alpha_2, \ldots, \alpha_n, \alpha) = \mathrm{D}^{(\alpha_2\hat{p}_2+\cdots+\alpha\hat{p})^2}F(\hat{p}_1)$. By Formula (2.22.6), the generalization of (𓄿), one has

$$\operatorname{Disc}(f) = T^2 \cdot \operatorname{Disc}(\bar{f}_2) \cdot \operatorname{Disc}(f_2)^2 \cdot \Phi \mod T^3,$$

with $\bar{f}_2(\alpha_2, \ldots, \alpha_n) = f_2(\alpha_2, \ldots, \alpha_n, 0)$ and $\Phi = \operatorname{redDisc}_d^2(f)$. Our task is to analyze the divisibilities of $\operatorname{Disc}(f_2)$ and Φ by T ($\bar{f}_2$ has degree 0 in p, hence T divides $\operatorname{Disc}(\bar{f}_2)$ if and only if $\operatorname{Disc}(\bar{f}_2) = 0$).

(3.18) Theorem (3.2) (Salmon's elimination trick) generalizes as follows. Given $n-1$ linear forms $L_1, \ldots, L_{n-1} \in \mathbf{P}^n$, the line $\mathrm{V}(L_1, \ldots, L_{n-1})$ intersects the $(n-2)$-dimensional linear space $\langle p_1, \ldots, p_{n-1}\rangle$ if and only if

$$D_{L_1,\ldots,L_{n-1}}(p_1, \ldots, p_{n-1}) \overset{\text{def.}}{=} \det\bigl(L_i(p_j)\bigr)_{1\leqslant i,j\leqslant n-1}$$

vanishes.

Let

$$\mathcal{T}_V := \{(p_1, \ldots, p_{n-1}) \in (\mathbf{P}^n)^{n-1} : p_1 \in V \text{ and } p_2, \ldots, p_{n-1} \in \mathbf{T}_{p_1} V\},$$

and consider a multihomogeneous polynomial $\Phi_{p_1,\ldots,p_{n-1}}(p)$ of multidegree $(\lambda, \mu, \ldots, \mu)$ in the sets of variables $p_1, \ldots, p_{n-1}, p$, such that for all $(p_1, \ldots, p_{n-1}) \in \mathcal{T}_V$, the hypersurface $\mathrm{V}(\Phi_{p_1,\ldots,p_{n-1}})$ consists of μ hyperplanes, counted with multiplicities, all containing the $(n-2)$-plane $\langle p_1, \ldots, p_{n-1}\rangle$.

Then, the locus of those $p_1 \in V$ such that there exist $p_2, \ldots, p_{n-1} \in \mathbf{T}_{p_1} V$ such that $\mathbf{T}_{p_1} X$ is a component of $\mathrm{V}(\Phi_{p_1,\ldots,p_{n-1}})$ is cut out on V by a hypersurface of degree $\lambda + \mu(d-2)$.

One may then argue as in (3.12) to analyze the divisibilities of $\mathrm{Disc}(f_2)$ and Φ by T.

(3.19) For $\mathrm{Disc}(f_2)$, we find as before that it is divisible by T if and only if p_1 lies on the Hessian hypersurface of V, since $f_2 \pmod T$ defines the tangent cone at p_1 of the tangential hyperplane section $\mathbf{T}_{p_1} V \cap V$, hence $\mathrm{Disc}(f_2)$ vanishes modulo T if and only if the latter tangent cone is singular, i.e., if and only if p_1 is a parabolic point of V.

And indeed $\mathrm{Disc}(f_2)$, being the determinant of the symmetric matrix

$$\begin{pmatrix} \mathrm{D}^{(p_2)^2} F(p_1) & \cdots & \mathrm{D}^{p_2 p_{n-1}} F(p_1) & \mathrm{D}^{p_2 p} F(p_1) \\ \vdots & \ddots & \vdots & \vdots \\ \mathrm{D}^{p_{n-1} p_2} F(p_1) & \cdots & \mathrm{D}^{(p_{n-1})^2} F(p_1) & \mathrm{D}^{p_{n-1} p} F(p_1) \\ \mathrm{D}^{p p_2} F(p_1) & \cdots & \mathrm{D}^{p p_{n-1}} F(p_1) & \mathrm{D}^{p^2} F(p_1) \end{pmatrix},$$

has multidegree $\bigl((n-1)(d-2), 2, \ldots, 2\bigr)$ in $p_1, \ldots, p_{n-1}, p$, so the argument of (3.18) produces a homogeneous polynomial H of degree

$$(n-1)(d-2) + 2(d-2) = (n+1)(d-2)$$

in the variable p_1.

(3.20) The analysis of the divisibility of the reduced discriminant Φ by T, on the other hand, gives rise to the couple-nodal polynomial K.

Let us first compute the multidegree of Φ in the variables $p_1, \ldots, p_{n-1}, p$. This goes as in Paragraph (3.14). First of all, $\mathrm{Disc}(f)$ is n-homogeneous of n-degree $(d^\vee, \ldots, d^\vee)$ in $p_1, \ldots, p_{n-1}, p$, with $d^\vee = d(d-1)^{n-1}$, hence so is its homogeneous piece of lowest degree with respect to the Zariski grading

$$T^2 \,\mathrm{Disc}(\bar{f}_2)\, \mathrm{Disc}(f_2)^2\, \Phi.$$

On the other hand, T has n-degree $(d-1, 0, \ldots, 0, 1)$, while $\mathrm{Disc}(f_2)$ has n-degree $((n-1)(d-2), 2, \ldots, 2)$ as we saw in Paragraph (3.19) above. The same computation gives the n-degree of $\mathrm{Disc}(\bar{f}_2)$, namely $((n-2)(d-2), 2, \ldots, 2, 0)$. Eventually, one finds that Φ has degrees

$$d\bigl[(d-1)^{n-1} - 1\bigr] - 3(n-1)(d-2) \quad \text{and} \quad d(d-1)^{n-1} - 6$$

in p_1, and $p_2, \ldots, p_{n-1}, p$, respectively (note that the former degree is divisible by $d-2$). As in Paragraph (3.14), the degree in p_1 of $\mathrm{redDisc}^2_d(f)$ is given by Proposition (2.20.1), while the degree with respect to the grading of Proposition (2.20.2) is equally distributed between the degrees in $p_2, \ldots, p_{n-1}, p$ respectively.

Eventually, by the result in Paragraph (3.18), there exists a polynomial K in p_1, homogeneous of degree

$$d\big[(d-1)^n - 1\big] - 3(n+1)(d-2) = (d-2)\Big(d \cdot \frac{(d-1)^n - 1}{d-2} - 3(n+1)\Big),$$

such that T divides $\Phi = \mathrm{redDisc}_d^2(f)$ if and only if $p_1 \in V$ lies on the hypersurface $\mathrm{V}(K)$. We call K the couple-nodal polynomial. One thus obtains the following result.

(3.21) Theorem. *Let V be a smooth degree d hypersurface in $\mathbf{P}^n$, $n > 1$. The number of hyperplanes bitangent to V passing through $n-2$ fixed general points in $\mathbf{P}^n$ is*

$$\frac{1}{2} d(d-1)^{n-2}(d-2)\Big(d \cdot \frac{(d-1)^n - 1}{d-2} - 3(n+1)\Big).$$

For $n = 3$ one recovers Theorem (3.15), and for $n = 2$ the number of bitangents to a smooth plane curve of degree d, viz.

$$\frac{1}{2}(d^\vee - 1)(d^\vee - 2) - 3d(d-2) - \frac{1}{2}(d-1)(d-2) = \frac{1}{2} d(d-2)(d-3)(d+3).$$

4 – Number of bitangent lines and generalizations

In this section, we elaborate on the computation by Salmon of the number of bitangent lines to a surface in $\mathbf{P}^3$ passing through a general point. We give his proof in Section 4.1, and a generalization to hypersurfaces in $\mathbf{P}^n$ in the next Section 4.2. These proofs only involve the reduced discriminant of polynomials in two indeterminates, which is easily expressed as a plain discriminant, see Example (2.16). We give however another generalization in Section 4.3, for which it is necessary to consider reduced discriminants of polynomials with an arbitrary number of indeterminates. The two statements of Sections 4.1 and 4.2 are instances of a vast generalization by Fehér and Juhász [9], which is described in [XIII, Theorem **??**].

4.1 – Number of bitangent lines

The Grassmannian of lines in $\mathbf{P}^3$ has dimension 4. Passing through a fixed point imposes 2 conditions to a line in $\mathbf{P}^3$, and being tangent to a surface (at an unprescribed point) imposes 1 condition, so one expects finitely many bitangent lines to a surface passing through a general point in $\mathbf{P}^3$. In this subsection we prove the following, along the lines of [16, §279].

(4.1) Theorem. *Let S be a smooth surface of degree d in $\mathbf{P}^3$, and $p \in \mathbf{P}^3$ a general point. The number of lines bitangent to S and passing through p is*

$$\frac{1}{2} d(d-1)(d-2)(d-3). \tag{4.1.1}$$

Salmon's strategy is similar in spirit to that exposed in subsection 3.2. The fundamental fact is the following. Consider as before

$$\mathcal{T}_S = \big\{(p', p'') \in \mathbf{P}^3 \times \mathbf{P}^3 : p' \in S \text{ and } p'' \in \mathbf{T}_{p'}S - \{p'\}\big\};$$

equivalently, (p', p'') is in $\mathcal{T}_S$ if and only if $\langle p', p'' \rangle$ is a line, tangent to S at p'.

(4.2) Proposition. *There exists a polynomial R, bihomogeneous in p', p'' with respective degrees $(d-2)(d-3)$ and $(d+2)(d-3)$, such that the locus of points $(p', p'') \in \mathcal{T}_S$ such that the line $\langle p', p''\rangle$ is bitangent to S is cut out on $\mathcal{T}_S$ by $V(R)$.*

Proof. Let F be an equation of $S \subseteq \mathbf{P}^3$. For $(p', p'') \in \mathcal{T}_S$, we consider the homogeneous polynomial in (α, β)

$$\begin{aligned} f(\alpha,\beta) &= F(\alpha p' + \beta p'') \\ &= \alpha^d F(p') + \alpha^{d-1}\beta \mathrm{D}^{p''}F(p') + \frac{1}{2}\alpha^{d-2}\beta^2\mathrm{D}^{(p'')^2}F(p') + \cdots + \beta^d F(p'') \\ &= \beta^2\Big(\frac{1}{2}\alpha^{d-2}\mathrm{D}^{(p'')^2}F(p') + \cdots + \beta^{d-2}F(p'')\Big) \end{aligned} \tag{4.2.1}$$

(this is an abuse of notation: actually one should consider two liftings $\hat{p}', \hat{p}'' \in \mathbf{k}^4$ of p' and p'' respectively, and $F(\alpha\hat{p}' + \beta\hat{p}'')$); the last equality comes from the fact that $\langle p', p''\rangle$ is tangent to S at p'. Let ${}^\flat f(\alpha,\beta)$ be the homogeneous polynomial of degree $d-2$ between parentheses at the right-hand-side of (4.2.1). The line $\langle p', p''\rangle$ is a bitangent to S if and only if ${}^\flat f$ has a multiple root, so the polynomial R we are looking for is merely the discriminant of ${}^\flat f$ which is none other, up to sign, than the reduced discriminant of f, as noticed in (2.16).

Now, the polynomial f is of the form

$$f(\alpha,\beta) = \beta^2\left(a_2\,\alpha^{d-2} + a_3\,\alpha^{d-3}\beta + \cdots a_d\,\beta^{d-2}\right)$$

where the coefficients a_i are homogeneous in p' and p'' of degrees $d-i$ and i respectively. Therefore, we deduce from Example (2.16) (see also Proposition (2.20)) that $\mathrm{redDisc}(f)$ has degrees $(d-2)(d-3)$ and $(d+2)(d-3)$ in p' and p'' respectively, which concludes the proof. □

(4.3) Remark. The degree of $\mathrm{redDisc}(f)$ in p'' may be computed alternatively as follows. The plane curve $C_{p'} := \mathbf{T}_{p'}S \cap S$ has in general a double point at p', and what we want is the number of lines in $\mathbf{T}_{p'}S$ passing through p' and tangent to $C_{p'}$ at some other point. This is the number of ramification points of the projection of $C_{p'}$ from p'; the latter is a $(d-2):1$ map $\bar{C}_{p'} \to \mathbf{P}^1$, where $\bar{C}_{p'}$ denotes the normalization of $C_{p'}$ at p', so it follows from the Riemann–Hurwitz formula that the number of ramification points equals

$$2g(\bar{C}_{p'}) - 2 + 2(d-2) = \big((d-1)(d-2) - 2\big) - 2 + 2(d-2) = (d+2)(d-3)$$

as required.

We were not able to find, on the other hand, a geometric argument to compute the degree in p' in a comparable fashion. We wonder wether there is an explanation to why this degree $(d-2)(d-3)$ is so nice, in particular in its role in (4.1.1). It is conceivable that it has something to do with the degree of the dual to a smooth plane curve of degree $d-2$.

Proof of Theorem (4.1). Let $p \in \mathbf{P}^3$ be a general point. The locus of those points $p' \in S$ such that $(p', p) \in \mathcal{T}_S$ (equivalently, $p \in \mathbf{T}_{p'}S$) is the apparent boundary $\mathrm{D}^pS \cap S$. Among these points p', the locus of those p' for which the line $\langle p, p'\rangle$ is a bitangent to S is cut out by $V(R)$, where R is the polynomial of Proposition (4.2); it is therefore a complete intersection in $\mathbf{P}^3$ of type $(d, d-1, (d-2)(d-3))$. One concludes by observing that there are two points p' for every bitangent line to S passing through p. □

4.2 – A generalization in arbitrary dimension

The following is a fairly direct generalization of Theorem (4.1); as we will see the proof is almost identical to that of Salmon for $n = 2$ described in the previous subsection.

(4.4) Theorem. *Consider an integer $n > 3$. Let V be a smooth hypersurface of degree $d \geqslant n+1$ in $\mathbf{P}^n$, and p be a general point of $\mathbf{P}^n$. The number of lines passing through p and with two contact points of orders 2 and $n-1$ respectively is*

$$\prod_{k=0}^{n}(d-k). \tag{4.4.1}$$

To be precise, the lines under consideration in the above statement are those lines L in $\mathbf{P}^n$ such that the intersection scheme of L with V has the form

$$L \cap V = 2p_1 + (n-1)p_2 + p_3 + \cdots + p_{d-n+1}.$$

For $n = 3$ this would be bitangent lines as in Theorem (4.1); the reason why we assume $n > 3$ above is because if $n = 3$ the number (4.4.1) actually counts ordered pairs of points (p_1, p_2) such that $\langle p_1, p_2\rangle$ is a bitangent line, so that the number of bitangent lines is only one half of the number (4.4.1), as we have seen indeed in Theorem (4.1).

The family of lines in $\mathbf{P}^n$ passing through p has dimension $n-1$: indeed, a line containing p is uniquely determined by the datum of a point in the projective quotient $\mathbf{P}^n/p$.[4] On the other hand an order 2 contact (i.e., an ordinary tangency) is 1 condition, and an order $n-1$ contact is $n-2$ conditions. It is thus expected that the number of lines as in the theorem be finite.

Proof of the theorem. Consider

$$\mathcal{T}_V = \{(p', p'') \in \mathbf{P}^n \times \mathbf{P}^n : \quad p' \in V, \quad p'' \in \mathrm{D}_{p'}V \cap \cdots \cap \mathrm{D}_{p'^{n-2}}V, \quad \text{and} \quad p'' \neq p'\} :$$

the pair (p', p'') is in $\mathcal{T}_V$ if and only if $\langle p', p''\rangle$ is a line, with contact of order $n-1$ with V at p', see [XII, Theorem **??**]. For $(p', p'') \in \mathcal{T}_S$, we consider the homogeneous polynomial in (α, β)

$$\begin{aligned} f(\alpha,\beta) &= F(\alpha p' + \beta p'') \\ &= \alpha^d F(p') + \alpha^{d-1}\beta \mathrm{D}^{p''}F(p') + \frac{1}{2}\alpha^{d-2}\beta^2\mathrm{D}^{p''^2}F(p') + \cdots + \beta^d F(p'') \\ &= \beta^{n-1}\Big(\frac{1}{(n-1)!}\alpha^{d-n+1}\mathrm{D}^{p''^{n-1}}F(p') + \cdots + \beta^{d-n+1}F(p'')\Big), \end{aligned} \tag{4.4.2}$$

and denote by ${}^\flat f(\alpha,\beta)$ the degree $d-n+1$ polynomial between parentheses in the last expression. The line $\langle p', p''\rangle$ has an additional tangent point with V if and only if the discriminant of ${}^\flat f$ vanishes. For all $i = n-1, \ldots, d$, $\mathrm{D}^{p''^i}F(p')$ has degree i in p'' and $d-i$ in p'. Therefore, by Example (2.16), $\mathrm{Disc}({}^\flat f)$ has degree $(d-n+1)(d-n)$ in p'.

Now, the locus of those points $p' \in V$ such that $(p', p) \in \mathcal{T}_V$, i.e., such that the line $\langle p, p'\rangle$ has contact of order $n-1$ with V at p', is the intersection $V \cap \mathrm{D}^p V \cap \cdots \cap \mathrm{D}^{p^{n-2}}$ (see [XII, Theorem **??**], or recall [XII, Display **??**]), which is the complete intersection of $n-1$ hypersurfaces of degrees $d, d-1, \ldots, d-n+2$ respectively. By the first part of the proof, those p' such that the line $\langle p, p'\rangle$ has an additional contact of order 2 are cut out in this complete intersection by an n-th equation, of degree $(d-n+1)(d-n)$. One thus gets a 0-dimensional complete intersection of degree

$$d\,(d-1)\cdots(d-n+2)\big[(d-n+1)(d-n)\big],$$

which is isomorphic to the family of lines we are interested in. ☐

[4] this abusive but suggestive notation has the following meaning: if $\mathbf{P}^n = \mathbf{P}(\mathbf{k}^{n+1})$, and $\hat{p} \in \mathbf{k}^{n+1}$ represents $p \in \mathbf{P}^n$, then $\mathbf{P}^n/p$ stands for the $(n-1)$-dimensional projective space $\mathbf{P}\big(\mathbf{k}^{n+1}/\langle\hat{p}\rangle\big)$.

4.3 – A dual generalization in arbitrary dimension

The family of codimension 2 linear subspaces $\Lambda \subseteq \mathbf{P}^n$ passing through $n-2$ points $p_1, \ldots, p_{n-2}$ in linear general position is isomorphic to $\mathbf{P}^2$: indeed, such a linear subspace Λ is uniquely determined by the datum of a point in the 2-dimensional projective quotient

$$\mathbf{P}^n / \langle p_1, \ldots, p_{n-2} \rangle$$

(i.e., $\mathbf{P}\big(\mathbf{k}^{n+1}/\langle \hat{p}_1, \ldots, \hat{p}_{n-2}\rangle$, see footnote 4). Besides, the Grassmannian of codimension 2 linear subspaces in $\mathbf{P}^n$ has dimension $2(n-1)$, and in this codimension passing through a given point is given by two independent linear conditions.

In $\mathbf{P}^n/\langle p_1, \ldots, p_{n-2}\rangle \cong \mathbf{P}^2$, those Λ that are tangent to V at one point are parametrized by a curve C, and those that are bitangent to V (i.e., tangent at two points) are finitely many. The degree of C is readily computed; the theorem below gives the number of bitangent codimension 2 linear subspaces.

In Paragraph (4.6) below, we explain how the degree of C may be found using homogeneity properties of the discriminant; we shall see that the number of bitangent codimension 2 subspaces involves analogously a homogeneity property of the reduced discriminant.

(4.5) Theorem. *Consider an integer $n \geqslant 3$. Let V be a general hypersurface of degree $d \geqslant 4$ in $\mathbf{P}^n$, and $p_1, \ldots, p_{n-2}$ be points in linear general position in $\mathbf{P}^n$. Then the number of codimension 2 linear subspaces $\Lambda \subseteq \mathbf{P}^n$ passing through $p_1, \ldots, p_{n-2}$ and bitangent to V is*

$$\frac{1}{2}\, d(d-1)^{n-2}(d-2)\, \frac{d(d-1)^{n-2} - (3n-5)d + 6(n-2)}{d-2}.$$

The peculiar way in which the formula is written is meant to emphasize that 2 is a root of the polynomial $d(d-1)^{n-2} - (3n-5)d + 6(n-2)$.

The transversality condition that must be satisfied by V for the formula to be valid is the following: V must be smooth and have finitely many bitangent codimension 2 subspaces through general points $p_1, \ldots, p_{n-2}$, each being tangent at only finitely many points. If it happens that some bitangent subspace has more than two ordinary tangency points, then it has to be counted with the appropriate multiplicity.

We will not insist on verifying that the open subset of the linear space of all degree d hypersurfaces defined by this transversality condition is indeed non-empty, but we will indicate where it is needed in the proof.

(4.6) The degree of the curve $C \subseteq \mathbf{P}^2 \cong \mathbf{P}^n/\langle p_1, \ldots, p_{n-2}\rangle$ of simply tangent codimension 2 subspaces is the number of its intersection points with a given line in $\mathbf{P}^2$; it is also the number of $(n-2)$-planes $\Lambda \subseteq \mathbf{P}^n$ tangent to V, passing through $p_1, \ldots, p_{n-2}$, and contained in a fixed hyperplane H which itself passes through $p_1, \ldots, p_{n-2}$.

The degree of C may be computed using the homogeneity properties of the (plain) discriminant. Let $F \in \mathbf{k}[X_0, \ldots, X_n]$ be an equation of V. We consider the polynomial $f \in \mathbf{k}[A, A_1, \ldots, A_{n-2}]$, depending on the (implicit) parameters $p, p_1, \ldots, p_{n-2} \in \mathbf{P}^n$, defined by

$$f(A, A_1, \ldots, A_{n-2}) = F(A.p + A_1.p_1 + \cdots + A_{n-2}.p_{n-2})$$

(with the usual abuse of notation that we do not distinguish between $p, p_1, \ldots, p_{n-2} \in \mathbf{P}^n$ and their representatives $\hat{p}, \hat{p}_1, \ldots, \hat{p}_{n-2} \in \mathbf{k}^{n+1}$). Then,

$$f(A, A_1, \ldots, A_{n-2}) =\\ A^d F(p) + A^{d-1} \mathrm{D}^{A_1.p_1 + \cdots + A_{n-2}.p_{n-2}} F(p) + \cdots + \frac{1}{d!} \mathrm{D}^{(A_1.p_1 + \cdots + A_{n-2}.p_{n-2})^d} F(p)$$

by the Taylor–Newton formula, and from this expression we see that the coefficient of f in the monomial $A^\alpha A_1^{\alpha_1} \cdots A_{n-2}^{\alpha_{n-2}}$ is homogeneous of degree α in the parameter p. Therefore, the discriminant $\mathrm{Disc}(f)$ is homogenous of degree $d(d-1)^{n-2}$ in the parameter p, by Proposition (2.6.1).

The hypersurface in $\mathbf{P}^n$ defined by this homogeneous polynomial is the cone of vertex $\langle p_1, \ldots, p_{n-2}\rangle$ swept out by those $(n-2)$-planes $\langle p, p_1, \ldots, p_{n-2}\rangle$ that are tangent to V. It projects from $\langle p_1, \ldots, p_{n-2}\rangle$ to the curve C in $\mathbf{P}^n/\langle p_1, \ldots, p_{n-2}\rangle \cong \mathbf{P}^2$, hence C has degree $d(d-1)^{n-2}$.

As we will see in the proof of Theorem (4.5), the tangency points with V of the $(n-2)$-planes parametrized by C form the complete intersection curve $V \cap \mathrm{D}^{p_1} V \cap \cdots \cap \mathrm{D}^{p_{n-2}} V$, which has degree $d(d-1)^{n-2}$ and projects birationally to C from $\langle p_1, \ldots, p_{n-2}\rangle$; this gives another way of computing the degree of C.

(4.7) Proof of Theorem (4.5). We continue with the setup introduced in (4.6) above, but now we assume that p sits on V and is such that the $(n-2)$-plane $\langle p, p_1, \ldots, p_{n-2}\rangle$ is tangent to V at p itself; equivalently, p sits on the complete intersection curve $V \cap \mathrm{D}^{p_1} V \cap \cdots \cap \mathrm{D}^{p_{n-2}} V$.[5] Under this assumption, the polynomial f takes the form

$$f(A, A_1, \ldots, A_{n-2}) = \\ \frac{1}{2} A^{d-2} \mathrm{D}^{(A_1 \cdot p_1 + \cdots + A_{n-2} \cdot p_{n-2})^2} F(p) + \cdots + \frac{1}{d!} \mathrm{D}^{(A_1 \cdot p_1 + \cdots + A_{n-2} \cdot p_{n-2})^d} F(p),$$

and its coefficient in the monomial $A^\alpha A_1^{\alpha_1} \cdots A_{n-2}^{\alpha_{n-2}}$ is still homogeneous of degree α in the parameter p, if non-zero.

The $(n-2)$-plane $\langle p, p_1, \ldots, p_{n-2}\rangle$ is bitangent to V if and only if the reduced discriminant $\mathrm{redDisc}_d^2(f)$ vanishes. Thus, the locus of points $p \in V$ such that $\langle p, p_1, \ldots, p_{n-2}\rangle$ is tangent to V in two points including p is the complete intersection of the curve $V \cap \mathrm{D}^{p_1} V \cap \cdots \cap \mathrm{D}^{p_{n-2}} V$ with the hypersurface defined by $\mathrm{redDisc}_d^2(f)$ as a polynomial in p.[6] The latter is homogeneous of degree $d(d-1)^{n-2} - (3n-5)d + 6(n-2)$ by Proposition (2.20.1). Therefore, the locus of tangency points of bitangent $(n-2)$-planes through $p_1, \ldots, p_{n-2}$ has degree

$$d(d-1)^{n-2}\big[d(d-1)^{n-2} - (3n-5)d + 6(n-2)\big].$$

The result follows, since each bitangent $(n-2)$-plane is tangent in two points. □

5 – The flecnodal polynomial

This is carried out by Salmon in [17, §588], with [17, §473] as a fundamental tool. This has already been revisited in modern standards in [2], and actually extended there to hypersurfaces in a projective space of arbitrary dimension, so we are going to be brief.

(5.1) The problem. Let S be a smooth surface in $\mathbf{P}^3$ of degree $d > 1$. For a general point $p \in S$, there are two lines having intersection multiplicity at p with S strictly greater than 2: these are the tangent lines to the two smooth branches at p of the curve $\mathbf{T}_p S \cap S$, which has an ordinary double point at p; thus these two lines intersect S with multiplicity 3 at p.

[5] this equivalence needs the smoothness of V; the fact that this is a complete intersection is equivalent to the fact that tangent subspaces through $p_1, \ldots, p_{n-2}$ form a curve.

[6] again, the fact that this is indeed a complete intersection is equivalent to the fact that bitangent subspaces through $p_1, \ldots, p_{n-2}$ are finitely many, and each of those is tangent to V at only finitely many points; multiple points in this complete intersection will occur if some bitangent subspace has more than two ordinary tangency points with V.

We shall see that those points $p \in S$ such that there is a line intersecting S with multiplicity strictly greater than 3 at p is a curve $Fl(S)$, cut out on S by a polynomial of degree $11d - 24$. We call this curve (respectively polynomial) the *flecnodal* curve (respectively polynomial) of S.

At a general point p of the flecnodal curve, the section of S by its tangent hyperplane $\mathbf{T}_p S$ is a curve with a non-degenerate double point at p (i.e., a double point with tangent cone of maximal rank), such that one of its two local branches has an inflection point at p. In general, the tangent line to the latter branch meets S with multiplicity 4 at p. Those points $p \in S$ such that the curve $\mathbf{T}_p S \cap S$ has a tacnode (i.e., a double point with local equation $y^2 = x^4$) also belong to the flecnodal curve: they are its intersection points with the Hessian of S, and they are cuspidal points of the cuspidal double curve of $S^\vee$ (the latter curve parametrizes those hyperplanes that cut out a cuspidal curve on S).

The following statement is definitely a result in reduced elimination theory, although it does not strictly fit in the framework of Section 1.

(5.2) Proposition ([17, §473])**.** *Let $F_q(p), G_q(p), H_q(p)$ be three bi-homogeneous polynomials in $p, q \in \mathbf{P}^3$, of bi-degrees (λ, μ), (λ', μ'), (λ'', μ'') respectively. We assume that for the general point $q \in \mathbf{P}^3$,*

$$\operatorname{mult}_q\big(\mathrm{V}(F_q, G_q, H_q)\big) = \lambda\lambda'\lambda''.$$

The locus of those $q \in \mathbf{P}^3$ such that $\mathrm{V}(F_q, G_q, H_q)$ contains a point in addition to q counted with multiplicity $\lambda\lambda'\lambda''$ is the zero locus of a homogeneous polynomial of degree

$$\lambda'\lambda''\mu + \lambda\lambda''\mu' + \lambda\lambda'\mu'' - \lambda\lambda'\lambda''.$$

Of course, the condition that the scheme $\mathrm{V}(F_q, G_q, H_q)$ contains a point in addition to q is equivalent to its having positive dimension. Salmon claims that it is equivalent to the fact that $\mathrm{V}(F_q, G_q, H_q)$ contains a line; we have not been able to prove it, but this is not needed for the application.

Proof. We want to characterize when the scheme $\mathrm{V}(F_q, G_q, H_q)$ has positive dimension. The idea is that this is equivalent to its having non-empty intersection with any hyperplane. So let L be a non-zero linear form in p, and consider the resultant $\operatorname{Res}(L, F_q, G_q, H_q)$. It follows from the Poisson formula (see, e.g., [2, Prop. 2.2] and the references therein) and our assumption on F, G, H that there exists a polynomial R such that

$$\operatorname{Res}(L, F_q, G_q, H_q) = L(q)^{\lambda\lambda'\lambda''} \cdot R(L, F_q, G_q, H_q). \tag{5.2.1}$$

Computing degrees, one sees that R is homogeneous of degree 0 in the coefficients of L, i.e., it does not depend on L. It follows that $V(F_q, G_q, H_q)$ has positive dimension if and only if $R(F_q, G_q, H_q) = 0$. Eventually, one computes the degrees of $R(F_q, G_q, H_q)$ using the identity (5.2.1). □

(5.3) Theorem. *Let S be a smooth surface in $\mathbf{P}^3$ of degree d. There exists a homogeneous polynomial Fl of degree $11d - 24$, such that the locus of points $p \in S$ such that there is a line intersecting S with multiplicity at least 4 in p is cut out on S by $\mathrm{V}(Fl)$.*

Proof. Let $p \in S$. It follows from [XII, Thm. **??**] that there is a line intersecting S with multiplicity at least 4 in p if and only if the three polar hypersurfaces $\mathrm{D}_p S, \mathrm{D}_{p^2} S, \mathrm{D}_{p^3} S$ (respectively, the tangent plane, the polar quadric, and the polar cubic of S at p) have a common point besides p, and that this in turn is equivalent to their having a whole line in common.

On the other hand, $\mathrm{D}_p S, \mathrm{D}_{p^2} S, \mathrm{D}_{p^3} S$, which have degrees $1, 2, 3$ respectively, intersect with multiplicity 6 at p by [XII, Cor. **??**]. We are therefore in a position to apply Proposition (5.2), with (λ, μ), (λ', μ'), (λ'', μ'') equal to $(1, d-1)$, $(2, d-2)$, $(3, d-3)$ respectively. The result follows. □

References

Chapters in this Volume

[XII] Th. Dedieu, *Some formulas for curves and surfaces in* $\mathbf{P}^3$.

[XIII] Th. Dedieu, *Node polynomials for curves on surfaces.*

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Lecture XIII
[coming soon]Node polynomials for curves on surfaces

by Thomas Dedieu

Lecture XIV
Göttsche conjecture and Göttsche–Yau–Zaslow formula

by Francesco Bastianelli and Thomas Dedieu

Abstract. This note collects and expands a couple of lectures given by the first author at University of Rome Tor Vergata on May 2017, concerning some important enumerative conjectures stated and discussed by Göttsche. The aim of the paper is to retrace and summarize the proof due to Kool, Shende and Thomas of Göttsche conjecture and the proof given by Tzeng of Göttsche–Yau–Zaslow formula.
In particular, given a smooth complex projective surface S and a positive integer δ, Göttsche conjecture predicts that if L is a sufficiently positive line bundle on S, then the number of δ–nodal curves in a general δ–dimensional linear subsystem $V \subseteq |L|$ is given by a universal polynomial of degree δ in the four numbers L^2, LK_S, K_S^2 and $c_2(S)$.
Besides, Göttsche–Yau–Zaslow formula expresses the generating function of the aforementioned numbers of δ–nodal curves in terms of three explicit quasimodular forms and two unknown universal power series, whose coefficients can be determined by recursion.

1 – Introduction

This note collects and expands a couple of lectures given by the first author at University of Rome Tor Vergata on May 2017, concerning the enumerative conjectures stated and discussed by Göttsche in [7]. In particular, we aim to retrace and summarize the proof due to Kool, Shende and Thomas [14] of Göttsche conjecture and the proof given by Tzeng [24] of Göttsche–Yau–Zaslow formula. Accordingly, [7], [14] and [24] shall be the main sources for this note, and proofs shall often overlap those included in the original papers.

Let us consider a smooth complex projective surface S and let δ be a positive integer. A δ*–nodal* curve on S is a reduced (possibly reducible) curve $C \subseteq S$ having exactly δ ordinary nodes and no other singularities. Given a line bundle L on S, we denote by $a_\delta(S,L)$ the number of δ–nodal curves contained in a general linear subsystem $V \subseteq |L|$ of dimension δ. When L is sufficiently ample, δ–nodal curves in $|L|$ occur indeed in codimension δ, so that $a_\delta(S,L)$ is a finite number and it coincides with the number of δ–nodal curves passing through $\dim|L| - \delta$ general points of S (cf. [12, p. 234]). The study of $a_\delta(S,L)$ is a very classical issue and it is in fact the main topic of this note. We refer the reader to [13, 14, 24] for a detailed bibliography and a treatise of the background underlying the results we are going to discuss.

The first result we would like to present is the so-called Göttsche conjecture. In the light of [25, 12] where the numbers $a_\delta(S,L)$ were described for $\delta \leqslant 8$, it predicts that if the line bundle L on S is sufficiently ample with respect to δ, the number $a_\delta(S,L)$ is computed by a universal polynomial of degree δ in the numbers $L^2, LK_S, K_S^2, c_2(L)$ (see [7, Conjecture 2.1]). Göttsche conjecture has been proved in terms of the following positivity notion for a line bundle L on S.

(1.1) Definition. *Let L be a line bundle on a smooth projective surface S, and let $k \geqslant 0$ be an integer. We say that L is k–*very ample *if for any 0–dimensional subscheme $Z \subseteq S$ of length $k+1$, the natural map $H^0(S,L) \longrightarrow H^0(Z, L \otimes \mathcal{O}_Z)$ is surjective.*

In particular, Kool, Shende and Thomas achieved the following (see [14, Theorem 4.1]).

(1.2) Theorem (Göttsche conjecture)**.** *For any positive integer δ, there exists a universal polynomial $T_\delta(x,y,z,t)$ of degree δ with the following property: for any smooth complex projective surface S and for any δ–very ample line bundle L on S, a general δ–dimensional linear subsystem $V \subseteq |L|$ contains exactly $T_\delta(L^2, LK_S, K_S^2, c_2(S))$ δ–nodal curves, i.e.*

$$a_\delta(S,L) = T_\delta(L^2, LK_S, K_S^2, c_2(S)). \tag{1.2.1}$$

It is worth noting that the coefficients of the polynomial $T_\delta(x,y,z,t)$ could be computed for any $\delta \geqslant 0$ (where we set $T_0(x,y,z,t) = 1$, which is the number $a_0(S,L)$ when L is sufficiently positive). Indeed, the numbers $a_\delta(S,L)$ can be determined by [2, Theorem 1.1] for all line bundles on $\mathbb{P}^2$ and by [1, Theorem 1.1] for all primitive line bundles on a K3 surface. In particular, the 4–tuples of topological invariants $(L^2, LK_S, K_S^2, c_2(L))$ are given by $(n^2, -3n, 9, 3)$ if $(S,L) = \left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)\right)$ and by $(L^2, 0, 0, 24)$ if L is a primitive line bundle on a K3 surface S, so that the coefficients of each $T_\delta(x,y,z,t)$ can be computed by solving a system of linear equations obtained from (1.2.1) using different values of n and L^2.

In Section 2 we shall retrace the proof of Theorem (1.2) included in [14], which relies on various techniques, such as deformation of singularities of curves (see e.g. [3]), BPS calculus interpreted in the setting of Hilbert schemes of points on singular curves (cf. [22, 23]), and the calculation of tautological integrals on Hilbert schemes of points on S by means of the recursion introduced in [5]. The main idea underlying the proof is to describe the numbers $a_\delta(S,L)$ of δ–nodal curves in a general linear subsystem $V \subseteq |L|$ in terms of the Euler characteristics of relative Hilbert schemes of points associated to the universal curve $\mathcal{C} \longrightarrow V$.

Moreover, in Section 3 we shall see in passing an alternative proof of Theorem (1.2) given by Tzeng under the stronger assumption on L of $(5\delta - 1)$–very ampleness (see Remark 3.4.18). We also mention that a different proof of Göttsche conjecture was given by Liu [19, 20], but we are not going to discuss it.

The other results we aim to discuss concern the generating function of the universal polynomials $T_\delta(L^2, LK_S, K_S^2, c_2(S))$, that is the universal power series

$$T(S,L) := \sum_{\delta \geqslant 0} T_\delta(L^2, LK_S, K_S^2, c_2(S)) x^\delta.$$

In [7], Göttsche proved that if the numbers $a_\delta(S,L)$ are computed by universal polynomials as in the assertion of Theorem (1.2), then the structure of the generating function $T(S,L)$ must satisfy rather strong conditions. In particular, in view of the validity of Göttsche conjecture, the generating function has the following multiplicative structure (cf. [7, Proposition 2.3]).

(1.3) Theorem. *There exist universal (invertible) power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[[x]]$ such that*

$$T(S,L) = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)}.$$

Göttsche's proof of this fact consists of considering a pair (S,L) obtained as the disjoint union of pairs of the form $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))$ and $(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(n,n))$, and using the additivity properties of the 4–tuples $(L^2, LK_S, K_S^2, c_2(S))$. We shall instead follow Tzeng's argument for achieving Theorem (1.3), which ultimately shall lead to the proof of Göttsche–Yau–Zaslow formula below.

We point out further that, as for the coefficients of the universal polynomials $T_\delta(x,y,z,t)$, the coefficients of the universal power series A_1, A_2, A_3, A_4 could be determined by the recursive formulas describing the numbers $a_\delta(S,L)$ for certain classes of pairs (S,L) (see e.g. the aforementioned [2, Theorem 1.1] for line bundles on $\mathbb{P}^2$ and [26, Theorem 6.7] for the case of Hirzebruch surfaces).

The most important result concerning the generating function $T(S,L)$ is Göttsche–Yau–Zaslow formula, which was conjectured by Göttsche [7] according to the formulas in [25, 12], as a generalization of Yau–Zaslow formula for rational curves on $K3$ surfaces [28]. In particular, Göttsche–Yau–Zaslow formula expresses the universal power series $T(S,L)$ in terms of five universal generating functions in one variable q: three explicit quasimodular forms and two unknown universal power series, whose coefficients could be however determined by the recursion in [2].

We refer the reader to [10] for some preliminary notions on modular and quasimodular forms. Let $\tau \in \mathbb{C}$ be a complex number with positive imaginary part, and set $q := e^{2\pi i\tau}$. Consider the *discriminant form*

$$\Delta(\tau) := q \prod_{n>0} (1-q^n)^{24}$$

and the *second Eisenstein series*

$$G_2(\tau) := -\frac{1}{24} + \sum_{n>0} \left(\sum_{k|n} k \right) q^n,$$

which are a modular form and a quasimodular form, respectively. Furthermore, given the differential operator $D := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$, we have that for any quasimodular form f, the derivative Df is again a quasimodular form. Then the following holds (see [7, Conjecture 2.4] and [24, Theorem 1.2]).

(1.4) Theorem (Göttsche-Yau–Zaslow formula)**.** *There exist universal power series $B_1(q)$ and $B_2(q)$ such that*

$$\sum_{\delta \geqslant 0} T_\delta(L^2, LK_S, K_S^2, c_2(S))\,(DG_2(\tau))^\delta = \frac{(DG_2(\tau)/q)^{\chi(L)}\, B_1(q)^{K_S^2} B_2(q)^{LK_S}}{(\Delta(\tau) D^2 G_2(\tau)/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

We note that, according to [7, Remark 2.6] and [24, Corollary 4.4], Göttsche–Yau–Zaslow formula can be reformulated as follows. For a smooth projective surface S and for any $l, m, r \in \mathbb{Z}$, we define

$$N_r^S(l,m) := T_{l+\chi(\mathcal{O}_S)-1-r}(2l+m, m, K_S^2, c_2(S)).$$

Therefore Theorem (1.4) gives that

$$\sum_{l\in\mathbb{Z}} N_r^S(l,m)q^l = B_1(q)^{K_S^2} B_2(q)^m (DG_2(\tau))^r \frac{D^2G_2(\tau)}{(\Delta(\tau)D^2G_2(\tau)/q^2)^{\chi(\mathcal{O}_S)/2}},$$

and Theorem (1.2) ensures that if L is a δ–very ample line bundle on S with $\delta = \chi(L) - 1 - r$, then the coefficient $N_r^S\left(\frac{L^2-LK_S}{2}, LK_S\right)$ counts the number of δ–nodal curves in a general linear subsystem $W \subseteq |L|$ of codimension r. Analogously, setting $g = \frac{L^2+LK_S}{2} + 1 - \delta$, the coefficient

$$M_g^S\left(\tfrac{L^2-LK_S}{2}, LK_S\right) := N_{g-m-2+\chi(\mathcal{O}_S)}\left(\tfrac{L^2-LK_S}{2}, LK_S\right)$$

computes the number of nodal curves of geometric genus g in a general linear subsystem of $|L|$ having codimension $r = g - LK_S - 2 + \chi(\mathcal{O}_S)$.

In Section 3 we shall summarize the proof of Theorem (1.4) due to Tzeng [24]. Her approach combines various arguments using algebraic cobordism theory (see [15, 16]), Göttsche's enumerative integrals introduced in [7, Section 5], and Li-Wu construction of a moduli stack of ideal sheaves (cf. [17]). A crucial idea in the proof is studying the generating function of the intersection numbers $d_\delta(S,L)$ arising from Göttsche's enumerative integrals which—unlike the numbers $a_\delta(S,L)$—are well-defined independently of the positivity of L, are well-behaved under degeneration of the pair (S,L), and do coincide with $a_\delta(S,L)$ when L is sufficiently ample.

(1.5) Notation. We shall work throughout over the field $\mathbb{C}$ of complex numbers. By *curve* we mean a connected complete reduced algebraic curve over the field of complex numbers, unless otherwise stated. When we speak of a *smooth* projective variety, we implicitly assume it to be irreducible.

Given a projective variety X, we say that a property holds for a *general* point $x \in X$ if it holds on an open non-empty subset of X. In particular, if L is a line bundle on a smooth surface S, by *general* linear subsystem $V \subseteq |L|$ of dimension δ we mean that V is parameterized over an open subset of the Grassmannian of δ–dimensional linear subspaces of $|L| \cong \mathbb{P}^{h^0(L)-1}$.

For a positive integer k, we denote by $X^{[k]}$ the k-fold Hilbert scheme of points on a variety X, parameterizing 0–dimensional subschemes of S of length k, whereas we denote by $\mathrm{Hilb}^k(\mathcal{X}/B)$ the k-fold relative Hilbert scheme of points on the fibers of a family $\mathcal{X} \longrightarrow B$ of varieties.

2 – Kool-Shende-Thomas' proof of Göttsche conjecture

In this section, we present the proof of Theorem (1.2) given by Kool, Shende and Thomas [14]. So, we consider a line bundle L on a smooth surface S and we assume that L is δ–very ample for some integer $\delta \geqslant 1$. In §2.1, we show that a general linear subsystem of $V \subseteq |L|$ of dimension δ contains a finite number of δ–nodal curves, each occurring with multiplicity 1, and all the other elements of V are reduced curves having higher geometric genus (i.e. δ–nodal curves are somehow the most singular curves in V).

In §2.2, we firstly review some facts concerning the generating functions of the Euler characteristics $e_k := e(C^{[k]})$ of Hilbert schemes of points on curves C contained in S. Then we express the number $a_\delta(S,L)$ of δ–nodal curves in a general $V \subseteq |L|$ as above by a linear combination of the Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/V))$ of the relative Hilbert schemes of points on the fibers of the universal curve $\mathcal{C} \longrightarrow V$, with $k = 0, \ldots, \delta$.

In §2.3, we finally conclude the proof of Göttsche conjecture by showing that each term appearing in such a linear combination is indeed a polynomial in L^2, LK_S, K_S^2 and $c_2(L)$. To this aim, we retrace Kool–Shende–Thomas' computation of the Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/V))$ in terms of certain tautological integrals, which relies on the recursion by Ellingsrud, Göttsche and Lehn included in [5, Sections 3 and 4].

We point out that the idea underlying the count of δ–nodal curves generalizes the following argument applying to the case $\delta = 1$. Assume that L is a very ample line bundle and $V \subseteq |L|$ is a general pencil, and denote by p the genus of the smooth fibers of the universal curve $\mathcal{C} \longrightarrow V \cong \mathbb{P}^1$. We note that the linear combination of Euler characteristics $e_1 - (2-2p)e_0$ equals 1 if C is 1–nodal, whereas it is 0 if C is smooth. Therefore, as we sum over $V \cong \mathbb{P}^1$ this quantity, we obtain that the number of 1–nodal curves satisfies

$$a_1(S,L) = e(\mathcal{C}) - (2-2p)e(\mathbb{P}^1), \tag{2.0.1}$$

i.e. we get a linear combination of $e(\mathrm{Hilb}^1(\mathcal{C}/V))$ and $e(\mathrm{Hilb}^0(\mathcal{C}/V))$. The argument of §2.2 is indeed analogous to the previous one, as we describe $a_\delta(S,L)$ by summing over $V \cong \mathbb{P}^\delta$ a linear combination of the Euler characteristics $e_0, \dots, e_\delta$, which is 1 for δ–nodal curves and it is 0 for curves having higher geometric genus. So, in order to prove Göttsche conjecture, it remains to express the right–hand side of (2.0.1) in terms of the topological invariants L^2, LK_S, K_S^2 and $c_2(S)$, which is the purpose of §2.3 (of course, the case $\delta = 1$ is much simpler: $e(\mathcal{C}) = c_2(S) + L^2$ as $\mathcal{C}$ is the blow–up of S at L^2 points, $2p-2 = L(L+K_S)$ by adjunction formula, and $e(\mathbb{P}^1) = 2$, so that we obtain

$$a_1(S,L) = c_2(S) + 3L^2 + 2LK_S,$$

which is a polynomial of degree 1 in the topological numbers $L^2, LK_S, K_S^2, c_2(S)$).

2.1 – Sufficiently ample line bundles

Let L be a line bundle on a smooth projective surface S. We follow [14, Section 2] and show how the δ-very ampleness condition on L gives enough control on the singularities of curves in a general dimension δ linear subsystem $\Lambda \subseteq |L|$ to make sense of the count of δ-nodal curves in Λ by means of the Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$ in subsection 2.2. Roughly speaking, this condition tells us that everything happens as expected up to codimension δ in $|L|$. The precise statement is the following.

(2.1) Theorem. *Assume L is δ–very ample for some positive integer δ. Let p denote the common arithmetic genus of all members of $|L|$, and consider a general linear subsystem $\Lambda \subseteq |L|$ of dimension δ. Then all members of Λ are reduced curves, and have geometric genus $g \geqslant p - \delta$; there are finitely many members of genus $g = p - \delta$, which are all δ-nodal curves and appear with multiplicity* 1.

The precise meaning of the last statement is the following: i) the *locally closed subset* of Λ consisting of those members that have genus $p - \delta$ consists only of δ-nodal curves; ii) the *subscheme* of Λ parametrizing δ-nodal members is reduced and of dimension 0.

Proof. We shall use the deformation theory of planar curve singularities, for which we refer to [II] in this volume, and to [3, 8] for extended treatises; the reader may also consult [4]. The idea is that the δ-very ampleness property implies that sufficiently many maps from $|L|$ to semi-universal deformation spaces of planar curve singularities are smooth.

2.1.2. Let us first show that the subscheme $V^\delta \subseteq |L|$ of δ-nodal curves in $|L|$ is smooth of codimension δ. By Bertini theorem this implies that δ-nodal curves in Λ are finitely many and appear with multiplicity 1.

Let $[C] \in |L|$ be a δ-nodal curve, and J be its Jacobian ideal. The vector space $B := H^0(C, \mathcal{O}_C/J)$ is the product of the semi-universal deformation spaces of the various singularities of C. We thus have a map $\varphi : (|L|, [C]) \to (B, 0)$, and its differential is the restriction map

$$T_{|L|,[C]} \cong H^0(S, L)/\langle s_C \rangle \longrightarrow H^0(Z, L \otimes \mathcal{O}_C/J), \tag{2.2.1}$$

where $Z \subseteq S$ is the singular subscheme of C, defined in C by the ideal J, and $s_C \in H^0(S, L)$ is a section vanishing along C. Since C is δ-nodal, its singular subscheme has length δ, hence the restriction map $H^0(S, L) \to H^0(Z, L \otimes \mathcal{O}_Z)$ is surjective by the positivity property of L. This implies that the map (2.2.1) is surjective, hence φ is smooth.

The semi-universal deformation space of a node is a line, in which the equisingular locus is only the origin. Thus the locus of δ-nodal curves in B is only the origin, which is smooth and has codimension δ in B. By smoothness of φ, the same holds for V^δ around $[C]$, which proves our claim.

2.1.3. Consider a non-negative integer $g' < p - \delta$, and let $V_{g'} \subseteq |L|$ be the locally closed subset of reduced curves having geometric genus g'. We now show that $V_{g'}$ has codimension $> \delta$ in $|L|$, so that Λ does not contain any curve of genus g'.

Let $[C] \in V_{g'}$. We consider its conductor ideal A, and let $Y \subseteq C$ be the subscheme defined by A. By [II, Proposition 3.1] the tangent cone of $V_{g'}$ at $[C]$ is contained in the kernel of the restriction map

$$\rho_Y : H^0(S, L)/\langle s_C \rangle \longrightarrow H^0(Y, L \otimes \mathcal{O}_C/A),$$

where again s_C is a section vanishing along C.

Since the co-genus of C in $|L|$ is $> \delta$, we may find a subscheme $Z \subseteq Y$ of length $\delta + 1$. Then the positivity property of L implies that the restriction map

$$\rho_Z : H^0(S, L)/\langle s_C \rangle \longrightarrow H^0(Z, L \otimes \mathcal{O}_Z)$$

is surjective, hence its kernel has codimension $\delta + 1$. Since $\rho_Z = \pi \circ \rho_Y$, where π is the restriction map $H^0(L \otimes \mathcal{O}_Y) \to H^0(L \otimes \mathcal{O}_Z)$, this implies that the kernel of ρ_Y has codimension $\geqslant \delta + 1$, hence $V_{g'}$ has codimension $\geqslant \delta + 1$ in $|L|$ as required.

2.1.4. Let τ be a collection of topological types of planar singularities, which does not consist of δ nodes, and such that a member of $|L|$ with singularities of type τ has genus $p - \delta$. We now show that the subscheme $V^\tau \subseteq |L|$ parametrizing curves with topological type τ has codimension $> \delta$ in $|L|$.

Let $[C] \in V^\tau$, and I be its equisingular ideal. By [II, Proposition 3.1], the tangent space to V^τ at $[C]$ is the kernel of the restriction map

$$\rho_Y : H^0(S, L)/\langle s_C \rangle \longrightarrow H^0(Y, L \otimes \mathcal{O}_C/I),$$

where Y is the subscheme of C defined by I. The assumptions on τ imply that we may find a subscheme $Z \subseteq Y$ of length $\delta + 1$, see [II, Lemma 2.4]. Then arguing as in 2.1.3 we see that the positivity assumption on L implies that the kernel of ρ_Y has codimension $\geqslant \delta + 1$, hence V^τ has codimension $\geqslant \delta + 1$ in $|L|$ as required.

2.1.5. Eventually, we proceed to show that the locus $V^{\mathrm{nr}} \subseteq |L|$ parametrizing non-reduced curves has codimension $> \delta$, which will end the proof of Theorem (2.1).

We claim that if C is a non-reduced member of $|L|$, then the tangent cone to V^{nr} at $[C]$ is contained in the kernel of the restriction map

$$\rho : H^0(S, L)/\langle s_C \rangle \longrightarrow H^0(C_{\mathrm{red}}, L \otimes \mathcal{O}_{C_{\mathrm{red}}}).$$

When we know this, we conclude that V^{nr} has codimension $> \delta$ always along the same lines: we pick $Z \subseteq C_{\mathrm{red}}$ a 0-dimensional subscheme of length $\delta+1$, then the kernel of ρ is contained in the kernel of the restriction to Z, which has codimension $\delta+1$ since L is δ-very ample.

To prove our claim, we argue as in [27, §2.1] and consider the incidence variety

$$\mathcal{D} := \big\{ \big([C], p\big) \in |L| \times S : C \text{ is singular at } p \big\}.$$

It is smooth: indeed since L is very ample and S is smooth, $\mathcal{D}$ is a projective bundle over S. On the other hand, if $[C] \in V^{\mathrm{nr}}$ then $([C],p) \in \mathcal{D}$ for all $p \in C_{\mathrm{red}}$. It follows that the tangent cone to V^{nr} at $[C]$ is contained in the image of the tangent space to $\mathcal{D}$ at $([C],p)$ by the projection $T_{|L|\times S} \to T_{|L|}$ for all $p \in C_{\mathrm{red}}$. We shall show that the latter image is contained in the kernel of the restriction

$$H^0(S,L)/\langle s_C \rangle \longrightarrow H^0\big(p, L|_p\big),$$

and this will prove our claim.

The equations of $\mathcal{D}$ in $|L| \times S$ are as follows. Let $([C],p) \in |L| \times S$; we choose a local system of coordinates (x,y) on S around p, and see $|L|$ around $[C]$ as a finite dimensional affine space of polynomials $f \in \mathbf{C}[x,y]$. In this representation, $\mathcal{D}$ is defined by the equations

$$f(x,y) = df(x,y) = 0.$$

By differentiation, the latter equations give the equations of the tangent space of $\mathcal{D}$ in $T_{|L|} \times T_S$. In particular, if $(h,v) \in T_{|L|,f} \times T_{S,(x,y)}$ sits in $T_{\mathcal{D}}$, then

$$df(x,y)(v) + h(x,y) = 0,$$

which is equivalent to $h(x,y) = 0$ since $df(x,y) = 0$. In other words, if $([C],p) \in \mathcal{D}$, then the projection of $T_{\mathcal{D},([C],p)}$ to $T_{|L|,[C]}$ is contained in the kernel of the restriction map

$$H^0(S,L)/\langle s_C \rangle \longrightarrow H^0\big(p, L|_p\big),$$

as we wanted to prove. We have thus proved our claim, hence that V^{nr} has codimension $> \delta$, and thus the proof of the theorem is complete. □

2.2 – Euler characteristics of Hilbert schemes of points over curves

According to the assumption of Theorem (2.1), we consider a smooth projective surface S endowed with a δ–very ample line bundle L on S. We follow [14, Section 3] in order to express the number $a_\delta(S,L)$ of δ–nodal curves in a general δ–dimensional linear subsystem $V \subseteq |L|$ in terms of Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/V))$ of relative Hilbert schemes, where $\mathcal{C} \longrightarrow V$ is the universal curve.

To this aim, we recall a couple of preliminary results describing the generating series of the Euler characteristics $e(C^{[k]})$ of k–fold Hilbert schemes of points on curves $C \subseteq S$ (see [14, Propositions 3.1 and 3.2]). Both the results were first proved in [22] in the setting of stable pairs, whereas [23, Section 2] includes proofs using only Hilbert schemes.

2.2.6 Proposition. *Let S be a smooth surface and let $C \subseteq S$ be a curve having arithmetic genus p and geometric genus g. Then there exists a unique sequence of integers $(n_{C,g}, n_{C,g+1}, \ldots, n_{C,p})$, depending only on the topological type of C, such that the generating series of Euler characteristics $e(C^{[k]})$ satisfies*

$$\sum_{k=0}^{\infty} e(C^{[k]}) q^k = \sum_{i=g}^{p} n_{C,i} q^{p-i} (1-q)^{2i-2}.$$

As far as the proof of the proposition is concerned, writing the generating series $\sum e(C^{[k]})q^k$ as a sum $\sum n_{C,i}q^{p-i}(1-q)^{2i-2}$, with i varying between $-\infty$ and p, descends from a formal property of power series (cf. [23, Proposition 1]). The geometry of C is instead involved for proving that $n_{C,i}=0$ for any $i<g$, which is deduced from [23, Proposition 8] (see [23, Corollary 11]).

When the curve C is δ–nodal, the generating series of Euler characteristics $e(C^{[k]})$ is governed by the following (cf. [23, Corollary 12]).

2.2.7 Proposition. *Let C be a δ–nodal curve with arithmetic genus p. Then the coefficients $n_{C,i}$ appearing in Proposition 2.2.6 are*

$$n_{C,i}=\tbinom{\delta}{p-i}$$

for any $p-\delta\leqslant i\leqslant p$.

When C is a smooth curve this was proved by Macdonald [21], in order to give a new proof of the de Jonquières formulas discussed in [XII]. In this case one finds

$$\sum_{k=0}^{\infty} e\big(C^{[k]}\big)q^k=(1-q)^{2p-2}=\sum_{k=0}^{2p-2}(-1)^k\tbinom{2p-2}{k}q^k.$$

While of course $e(C)=2-2p$, and also $e(C^{[k]})=0$ if $k>2p-2$ because then $C^{[k]}\to J^k(C)$ gives a structure of projective bundle over a torus, the values for $e(C^{[k]})$ between these cases don't seem obvious to us.

In the nodal case of Proposition 2.2.7, Kool, Shende, and Thomas interpret the number $\binom{\delta}{p-i}$ as the number of partial normalizations of C at $p-i$ of its δ nodes. This is the reason why Proposition 2.2.7 is considered a statement in BPS calculus, see [XI].

Now, the main result of this section follows from Propositions 2.2.6, 2.2.7 and Theorem (2.1) (see [14, Theorem 3.4]).

2.2.8 Theorem. *We consider a smooth projective surface S equipped with a line bundle L, and denote by p be the arithmetic genus of all members of $|L|$. Let δ be a positive integer δ such that L is δ-very ample, and consider a general linear subsystem $V\subseteq|L|$ of dimension δ. Then the number $a_\delta(S,L)$ of δ–nodal curves in V equals the coefficient $n_{V,p-\delta}$ of $q^\delta(1-q)^{2p-2\delta-2}$ in the generating series*

$$\sum_{k=0}^{\infty} e(\mathrm{Hilb}^k(\mathcal{C}/V))q^k=\sum_{i=p-\delta}^{p} n_{V,i}q^{p-i}(1-q)^{2i-2}, \tag{2.8.1}$$

where $\mathcal{C}\to V$ is the universal curve.

In particular, $n_{V,p-\delta}$ is a linear combination of the numbers $e(\mathrm{Hilb}^k(\mathcal{C}/V))$ with $0\leqslant k\leqslant\delta$, in which the coefficient of each $e(\mathrm{Hilb}^k(\mathcal{C}/V))$ is a polynomial of degree at most $\delta-k$ in p, and the coefficient of $e(\mathrm{Hilb}^\delta(\mathcal{C}/V))$ equals 1.

In applying the second part of this statement, one should bear in mind to keep ideas clear that $p=1+\frac{1}{2}(K_SL+L^2)$ by the adjunction formula.

Proof. Let T be the (finite) set indexing the topological types of the members of V. For any $\tau\in T$, let $V_\tau\subseteq V$ be the locus of curves having topological type τ, and let g_τ denote their common geometric genus. Note that $g_\tau\geqslant p-\delta$ by Theorem (2.1). Let $\mathcal{C}_\tau\to V_\tau$ be the universal curve. For all $k \geqslant 0$ the relative Hilbert scheme $\mathrm{Hilb}^k(\mathcal{C}_\tau/V_\tau)$ is a topological fibre bundle, so $e(\mathrm{Hilb}^k(\mathcal{C}_\tau/V_\tau)) = e(C_\tau^{[k]})e(V_\tau)$ with C_τ a curve of topological type τ.

Then, by additivity of the Euler number, and using Proposition 2.2.6, we have

$$\begin{aligned}\sum_{k=0}^{\infty} e(\mathrm{Hilb}^k(\mathcal{C}/V))q^k &= \sum_{k=0}^{\infty}\Big(\sum_{\tau\in T} e(C_\tau^{[k]})e(V_\tau)\Big)q^k = \sum_{\tau\in T}\Big(\sum_{k=0}^{\infty} e(C_\tau^{[k]})q^k\Big)e(V_\tau) = \\ &= \sum_{\tau\in T}\Big(\sum_{i=g_\tau}^{p} n_{\tau,i}q^{p-i}(1-q)^{2i-2}\Big)e(V_\tau) = \sum_{i=p-\delta}^{p} n_{V,i}q^{p-i}(1-q)^{2i-2}\end{aligned}$$

with $n_{V,i} := \sum_{\tau\in T} n_{\tau,i}e(V_\tau)$, and we set $n_{\tau,i} = 0$ whenever $i < g_\tau$. Theorem (2.1) ensures that δ–nodal curves are the only curves in V having geometric genus $p-\delta$, they are finitely many and they appear with multiplicity 1. Therefore $n_{V,p-\delta} = a_\delta(S,L)\cdot n_{\delta\text{-nodal},p-\delta}$, and since $n_{\delta\text{-nodal},p-\delta} = 1$ by Proposition 2.2.7, the first part of the statement is proved.

In order to prove the second part, we expand $(1-q)^{2i-2}$ in (2.8.1) and obtain

$$\sum_{k=0}^{\infty} e(\mathrm{Hilb}^k(\mathcal{C}/V))q^k = \sum_{i=p-\delta}^{p} n_{V,i}\sum_{m=0}^{2i-2}\tbinom{2i-2}{m}(-1)^m q^{p-i+m}.$$

Then, by comparing the coefficients of q^k for $k = 0,\dots,\delta$, one sees with standard algebraic manipulations that

$$e(\mathrm{Hilb}^k(\mathcal{C}/V)) = \sum_{m=0}^{k}(-1)^m\tbinom{2(p-k+m)-2}{m}n_{V,p-k+m}. \tag{2.8.2}$$

Therefore, for any $k = 0,\dots,\delta$, we can express recursively n_{p-k} as a linear combination of the Euler characteristics $e(\mathrm{Hilb}^0(\mathcal{C}/V)),\dots,e(\mathrm{Hilb}^k(\mathcal{C}/V))$, see Lemma 2.2.9 below. The case $k=\delta$ gives the wanted result. □

We end this subsection by carrying out explicitly the recursive process ending the proof of Theorem 2.2.8.

2.2.9 Lemma. *For all $k = 0,\dots,\delta$, one has*

$$n_{V,p-k} = P_{k,0}(p)\,e_{V,0} + \ldots + P_{k,k-1}(p)\,e_{V,k-1} + e_{V,k},$$

where each $P_{k,s}$ is a polynomial of degree $\leqslant k-s$ independent of V.

Proof. We argue by recursion on k. For $k=0$ one has $n_{V,p} = e_{V,0}$ by (2.8.2), as required. For $1 \leqslant m \leqslant k$ we set $Q_{k,m}(p) = (-1)^m\binom{2(p-k+m)-2}{m}$, a polynomial of degree m in p. For $k>0$, we have by (2.8.2) and our recursion hypothesis

$$e_{V,k} = n_{V,p-k} + \sum_{m=1}^{k} Q_{k,m}(p)\sum_{s=0}^{k-m} P_{k-m,s}(p)e_{V,s} = n_{V,p-k} + \sum_{s=0}^{k-1}\sum_{m=1}^{k-s} Q_{k,m}(p)\,P_{k-m,s}(p)e_{V,s}$$

(setting $P_{m,m} = 1$). This gives an explicit expression for $P_{k,s}$, which ends the proof since

$$\deg(Q_{k,m}P_{k-m,s}) \leqslant m + (k-m-s) = k-s.$$

□

2.3 – Proof of the Göttsche conjecture

In this section we prove Theorem (1.2), following [14, Section 4]. Now that we know Theorem 2.2.8, the main point is to compute the Euler characteristics $e(\mathrm{Hilb}^k(\mathcal{C}/V))$, and to express them as polynomials of degree k in the numbers $L^2, LK_S, K_S^2, c_2(L)$ using the recursion of [5].

Proof of Theorem (1.2). Consider a smooth projective surface S and a δ–very ample line bundle L on S. Let p be the arithmetic genus of the curves in $|L|$, and let $V \subseteq |L|$ be a general linear subsystem of dimension δ with universal curve $\mathcal{C} \longrightarrow V$. Then Theorem 2.2.8 ensures that the number $a_\delta(S,L)$ of δ–nodal curves in V is a linear combination of the Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/V))$ with $0 \leqslant k \leqslant \delta$, where the coefficient of each $e(\mathrm{Hilb}^k(\mathcal{C}/V))$ is a polynomial of degree at most $\delta - k$ in p, and the coefficient of $e(\mathrm{Hilb}^\delta(\mathcal{C}/V))$ equals 1. Moreover, such a polynomial is independent of the pair (S,L) as above and, since $p = \frac{1}{2}L^2 + \frac{1}{2}K_S L + 1$ by adjunction formula, the coefficient of $e(\mathrm{Hilb}^k(\mathcal{C}/V))$ is a polynomial of degree at most $\delta - k$ in L^2 and $K_S L$. The Göttsche conjecture therefore follows from Proposition 2.3.10 below. □

2.3.10 Proposition. *The Euler characteristics $e(\mathrm{Hilb}^k(\mathcal{C}/V))$ is a polynomial of degree exactly k in the numbers $L^2, LK_S, K_S^2, c_2(L)$.*

Proof. The case $k = 0$ is trivial, as $V \cong \mathbb{P}^\delta$ and $e(\mathrm{Hilb}^0(\mathcal{C}/V)) = e(V) = \delta + 1$. So we assume hereafter $1 \leqslant k \leqslant \delta$, and we consider the Hilbert scheme $S^{[k]}$ and the universal subscheme $Z_k \subseteq S \times S^{[k]}$, endowed with the projections $\pi_1 \colon Z_k \longrightarrow S$ and $\pi_2 \colon Z_k \longrightarrow S^{[k]}$. Then we define a vector bundle $L^{[k]}$ of rank k on $S^{[k]}$ as

$$L^{[k]} := (\pi_2)_*(\pi_1)^* L.$$

Letting $\mathbb{P} := \mathbb{P}\left(H^0(S,L)\right) \cong |L|$, we consider the line bundle $L \boxtimes \mathcal{O}_{\mathbb{P}}(1) := (\pi_S)^* L \otimes (\pi_{\mathbb{P}})^* \mathcal{O}_{\mathbb{P}}(1)$ on $S \times \mathbb{P}$, where π_S and $\pi_{\mathbb{P}}$ are the natural projections. In view of the standard identification $H^0(S,L) \otimes H^0(S,L)^* \cong \mathrm{Hom}\left(H^0(S,L), H^0(S,L)\right)$, we may consider the canonical section $\mathrm{id}_{H^0(S,L)} \in H^0(S,L) \otimes H^0(S,L)^*$ of the line bundle $L \boxtimes \mathcal{O}_{\mathbb{P}}(1)$. By taking the pullback of $\mathrm{id}_{H^0(S,L)}$ to $S \times S^{[k]} \times \mathbb{P}$, and pushing down to $S^{[k]} \times \mathbb{P}$ its restriction to $Z_k \times \mathbb{P} \subseteq S \times S^{[k]} \times \mathbb{P}$, we obtain a tautological section of the vector bundle on $S^{[k]} \times \mathbb{P}$ given by

$$M^{[k]} := (\pi_{S^{[k]}})^* L^{[k]} \otimes (\pi_{\mathbb{P}})^* \mathcal{O}_{\mathbb{P}}(1).$$

By abuse of notation, we still denote by $\mathcal{C} \longrightarrow \mathbb{P}$ the universal curve over the complete linear system $|L|$. Then the the k–fold relative Hilbert scheme $\mathrm{Hilb}^k(\mathcal{C}/\mathbb{P})$ is a smooth subvariety of $S^{[k]} \times \mathbb{P}$, which coincides with the vanishing locus of the tautological section at hand. Therefore, $\mathrm{Hilb}^k(\mathcal{C}/\mathbb{P}) \subseteq S^{[k]} \times \mathbb{P}$ has codimension $k = \mathrm{rk} M^{[k]}$, and its class $\left[\mathrm{Hilb}^k(\mathcal{C}/\mathbb{P})\right]$ in the Chow ring of $S^{[k]} \times \mathbb{P}$ is Poincaré dual to the Chern class $c_k(M^{[k]})$, that is $\left[\mathrm{Hilb}^k(\mathcal{C}/\mathbb{P})\right] = c_k(M^{[k]}) \cap \left[S^{[k]} \times \mathbb{P}\right]$ (see e.g. [6, Example 3.2.16]). We note further that the fibre of the projection $\mathrm{Hilb}^k(\mathcal{C}/\mathbb{P}) \longrightarrow S^{[k]}$ over any 0–dimensional scheme $Z \in S^{[k]}$ is given by pairs $(Z,[s]) \in S^{[k]} \times \mathbb{P}$ such that s vanishes at Z, i.e. $s \in \ker\left(H^0(S,L) \xrightarrow{\gamma_Z} H^0\left(Z, L \otimes \mathcal{O}_Z\right)\right)$. As L is $(k-1)$–very ample for any $1 \leqslant k \leqslant \delta + 1$, we deduce that γ_Z is surjective, so that the tautologial section of $M^{[k]}$ vanishes transversally with respect to the fibration $\mathrm{Hilb}^k(\mathcal{C}/\mathbb{P}) \longrightarrow S^{[k]}$, whose fibers have then constant dimension. Moreover, since the linear subsystem $V \subseteq |L|$ is assumed to be general, it correspond to a δ–dimensional linear subspace of $H \subseteq \mathbb{P}$ cut out by general hyperplanes. Hence the restriction $\mathrm{Hilb}^k(\mathcal{C}/V) \subseteq S^{[k]} \times H$ is smooth by Bertini Theorem. In particular, the class

$\left[\mathrm{Hilb}^k(\mathcal{C}/V)\right]$ is Poincaré dual in the Chow ring of $S^{[k]} \times H$ to the k–th Chern class of the restriction of $M^{[k]}$. Thus the Euler characteristic of $\mathrm{Hilb}^k(\mathcal{C}/V)$ satisfies

$$\begin{aligned} e(\mathrm{Hilb}^k(\mathcal{C}/V)) &= \int_{\mathrm{Hilb}^k(\mathcal{C}/V)} c_k\left(T(\mathrm{Hilb}^k(\mathcal{C}/V))\right) = \int_{\mathrm{Hilb}^k(\mathcal{C}/V)} c_\bullet\left(T(\mathrm{Hilb}^k(\mathcal{C}/V))\right) \\ &= \int_{\mathrm{Hilb}^k(\mathcal{C}/V)} \frac{c_\bullet\left(T(S^{[k]}\times H)\right)}{c_\bullet(M^{[k]})} = \int_{S^{[k]}\times H} c_k(M^{[k]}) \frac{c_\bullet\left(T(S^{[k]}\times H)\right)}{c_\bullet(M^{[k]})}. \end{aligned}$$

Denoting by $\omega := c_1(O_H(1))$ the hyperplane class on $H \cong \mathbb{P}^\delta$, we focus on each class appearing in the latter integral. Being $c_\bullet(T(H)) = \sum_{k=0}^{\delta}\binom{\delta+1}{k}\omega^k = (1+\omega)^{\delta+1}$ (see e.g. [9, p.414]), we deduce that $c_\bullet\left(T(S^{[k]}\times H)\right) = c_\bullet\left(T(S^{[k]})\right)c_\bullet\left(T(H)\right) = c_\bullet\left(T(S^{[k]})\right)(1+\omega)^{\delta+1}$. Besides, since $M^{[k]}$ is the tensor product of the rank k vector bundle $(\pi_{S^{[k]}})^*L^{[k]}$ with the line bundle $(\pi_{\mathbb{P}})^*\mathcal{O}_{\mathbb{P}}(1)$, its restriction to H satisfies $c_k(M^{[k]}) = \sum_{i=0}^{k}\omega^i c_{k-i}(L^{[k]})$ (see e.g. [6, p. 55]). Finally, [6, Example 3.2.2] yields

$$\begin{aligned} c_\bullet(M^{[k]}) &= \sum_{p=0}^{k} c_p(M^{[k]}) = \sum_{p=0}^{k}\left(\sum_{i=0}^{p}\binom{k-i}{p-i}c_i(L^{[k]})\omega^{p-i}\right) = \\ &= \sum_{i=0}^{k} c_i(L^{[k]})\sum_{p=i}^{k}\binom{k-i}{p-i}\omega^{p-i} = \sum_{j=0}^{k} c_{k-j}(L^{[k]})(1+\omega)^j. \end{aligned}$$

Thus

$$e(\mathrm{Hilb}^k(\mathcal{C}/V)) = \int_{S^{[k]}\times H} \frac{c_\bullet\left(T(S^{[k]})\right)(1+\omega)^{\delta+1}\sum_{i=0}^{k}\omega^i c_{k-i}(L^{[k]})}{\sum_{j=0}^{k}c_{k-j}(L^{[k]})(1+\omega)^j} \tag{2.10.1}$$

$$= \int_{S^{[k]}} \mathrm{Coeff}_{\omega^\delta}\Big(\frac{c_\bullet\left(T(S^{[k]})\right)(1+\omega)^{\delta+1}\sum_{i=0}^{k}\omega^i c_{k-i}(L^{[k]})}{\sum_{j=0}^{k}c_{k-j}(L^{[k]})(1+\omega)^j}\Big). \tag{2.10.2}$$

To end the proof we use the recursion procedure developed in [5], and stated below as Proposition 2.4.12. The latter, applied k times, turns our integral over over $S^{[k]}$ into the integral over S^k of a polynomial in the Chern classes of pr_i^*L, pr_iT_S, and $\mathrm{pr}_{ij}^*T_\Delta$ $(i \neq j)$, where $\Delta \subseteq S\times S$ is the diagonal, $\mathrm{pr}_i : S^k \to S$ is the i-th projection, and $\mathrm{pr}_{ij} = (\mathrm{pr}_i, \mathrm{pr}_j)$. The end-product is a polynomial in L^2, LK_S, K_S^2, and $c_2(S)$ and of degree $\leqslant k$.

It remains to prove that this degree is in fact exactly k. To do so we concentrate on the $c_2(S)^k$ term. One looks straight in the eyes of the explicit formulas for the recursion, viz. (2.12.2), (2.12.3), (2.12.4), (2.12.5), (2.12.6), and finds that the contribution to the $c_2(S)^k$ term can only come from the $c_0(L^{[k]})$ term. One may thus for our present purpose replace the fraction in (2.10.2) by

$$\frac{c_\bullet\left(T(S^{[k]})\right)(1+\omega)^{\delta+1}\omega^k}{(1+\omega)^k} = c_\bullet\big(T(S^{[k]})\big)\omega^k(1+\omega)^{\delta-k+1},$$

the coefficient of which in ω^δ is $(\delta-k+1).c_\bullet\big(T(S^{[k]})\big)$. The corresponding summand of the integral (2.10.2) is $(\delta-k+1).c_{2k}\big(T(S^{[k]})\big)$, which contributes by $(\delta-k+1)/k!$ to the $c_2(S)^k$ term, the factorial coming from the k successive applications of (2.12.1). □

2.3.11 Example. Let us spell out this computation for the number of 1-nodal curves in a pencil, and see how it relates with the standard way to do it, which we recalled in the introduction to the present Section 2. In this case we have $V \cong \mathbf{P}^1$. Equation (2.8.2) gives

$$e(\mathrm{Hilb}^1(\mathcal{C}/V)) = n_{V,p-1} - (2p-2)n_{V,p} \quad \text{and} \quad e(\mathrm{Hilb}^0(\mathcal{C}/V)) = n_{V,p}.$$

We have $e(\mathrm{Hilb}^0(\mathcal{C}/V)) = e(V) = 2$, and we concentrate on the computation of $e(\mathrm{Hilb}^1(\mathcal{C}/V))$ as the integral (2.10.2). The numerator of the integrand of (2.10.1) is

$$\begin{aligned}(1-K+c_2)(1+\omega)^2(L+\omega) &= (1-K+c_2)(1+2\omega)(L+\omega)\\ &= \big[L-KL\big] + \big[1-K+2L+c_2-2KL\big]\omega,\end{aligned}$$

and on the other hand

$$\begin{aligned}\frac{1}{1+L+\omega} &= 1-(L+\omega)+(L+\omega)^2-(L+\omega)^3\\ &= 1-(L+\omega)+(L^2+2L\omega)-3L^2\omega = \big[1-L+L^2\big]+\big[-1+2L-3L^2\big]\omega.\end{aligned}$$

The coefficient in ω of the integrand of (2.10.1) is therefore

$$(L-KL)(-1+2L-3L^2)+(1-K+2L+c_2-2KL)(1-L+L^2),$$

which when integrated against the fundamental class $[S]$ gives L^2+c_2 as required.

2.4 – The Ellingsrud–Göttsche–Lehn recursion

Here we explain how the inductive computation of [5] explicitly enough, so that i) one may make sense of the arguments at the end of the proof of Proposition 2.3.10, and ii) one should be able with some work to set it up in practice. We refer to the original article [5] for the proofs.

Let S be a smooth projective surface, equipped with a line bundle $L \to S$. We consider the Hilbert schemes $S^{[n]}$ and the vector bundles $L^{[n]} \to S^{[n]}$. We call T_n the tangent sheaf of $S^{[n]}$, and $\mathcal{I}_n$ the ideal sheaf of the universal subscheme $\Sigma_n \subseteq S^{[n]} \times S$. For any sequence I of indices in $\{0,\dots,m\}$ we let pr_I be the projection from $S^{[n]} \times S^m = S^{[n]} \times S \times \cdots \times S$ to the factors indexed by I, $S^{[n]}$ being the 0-th factor. Eventually, let Δ denote the diagonal in S^2.

2.4.12 Proposition. *Let f be a polynomial in the Chern classes of the following sheaves on $S^{[n+1]} \times S^m$:*

$$\mathrm{pr}_0^*T_{n+1},\ \mathrm{pr}_{0i}^*\mathcal{I}_{n+1},\ \mathrm{pr}_{ij}^*\mathcal{O}_\Delta,\ \mathrm{pr}_i^*T_S,\ \mathrm{pr}_0^*L^{[n+1]},\ \mathrm{pr}_i^*L \qquad (\textit{for } 1 \leqslant i < j \leqslant m).$$

There is a polynomial $f^\flat$ in the Chern classes of the analogously defined sheaves on $S^{[n]} \times S^{m+1}$, depending only on f and such that

$$\int_{S^{[n+1]}\times S^m} f = \int_{S^{[n]}\times S^{m+1}} f^\flat.$$

The keystone of the recursion procedure is the incidence variety $S^{[n,n+1]} \subseteq S^{[n]} \times S^{[n+1]}$ of all pairs (Z,Z') such that $Z \subseteq Z'$. There is an identification $S^{[n,n+1]} \cong \mathbf{P}(\mathcal{I}_n)$, and we let $\mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{I}_n)}(1)$. This identification follows from the observation that if Z' is obtained by extending Z at the closed point $x \in S$, then the ideal sheaf defining Z in S is an extension of $\mathbf{C}(x)$ by the ideal sheaf of Z'.

We consider the following maps: ϕ and ψ are the two projections of $\mathbf{P}(\mathcal{I}_n)$ to $S^{[n]}$ and $S^{[n+1]}$ respectively, and $\rho : \mathbf{P}(\mathcal{I}_n) \to S$ maps a pair (Z,Z') to the point $x \in S$ at which Z and Z' differ.

$$\begin{tikzcd} & \mathbf{P}(\mathcal{I}_n) = S^{[n,n+1]} \arrow[dl, "\phi=\mathrm{pr}_1"'] \arrow[d, "\rho"] \arrow[dr, "\psi=\mathrm{pr}_2"] & \\ S^{[n]} & S & S^{[n+1]} \end{tikzcd}$$

Moreover, let $j = (\mathrm{id}, \rho) : \mathbf{P}(\mathcal{I}_n) \to \mathbf{P}(\mathcal{I}_n) \times S$, $\phi_S = \phi \times \mathrm{id}_S : \mathbf{P}(\mathcal{I}_n) \times S \to S^{[n]} \times S$, and $\psi_S = \psi \times \mathrm{id}_S : \mathbf{P}(\mathcal{I}_n) \times S \to S^{[n+1]} \times S$. We have the following exact sequence of sheaves on $S^{[n,n+1]} \times S$:

$$0 \to \psi_S^* \mathcal{I}_{n+1} \to \phi_S^* \mathcal{I}_n \to j_* \mathcal{L} \to 0. \tag{5}$$

Note that $j_*\mathcal{L} = p^*\mathcal{L} \otimes \rho_S^* \mathcal{O}_\Delta$, with p the projection $S^{[n,n+1]} \times S \to S^{[n,n+1]}$, and $\rho_S = \rho \times \mathrm{id}_S : S^{[n,n+1]} \times S \to S^2$.

Proof of Proposition 2.4.12. We want to relate integrals to $S^{[n+1]} \times S^m$ to integrals on $S^{[n]} \times S^{m+1}$. To this end we consider the product $\Pi = S^{[n,n+1]} \times S^m$, together with the maps $\Psi = \psi \times \mathrm{id}_{S^m} : \Pi \to S^{[n+1]} \times S^m$ and $\Phi = (\phi, \rho) \times \mathrm{id}_{S^m} : \Pi \to S^{[n]} \times S \times S^m$. Moreover, we call $p_\Pi : \Pi = S^{[n,n+1]} \times S^m \to S^{[n,n+1]}$ the projection on the first factor.

$$\begin{tikzcd}
& \Pi = S^{[n,n+1]} \times S^m \arrow[dl, "\Psi=\psi\times\mathrm{id}"'] \arrow[d, "p_\Pi"] \arrow[dr, "\Phi=(\phi,\rho)\times\mathrm{id}"] & \\
S^{[n+1]} \times S^m & S^{[n,n+1]} & S^{[n]} \times S \times S^m
\end{tikzcd}$$

The first step is to lift our integral to Π, using the fact that Ψ is generically finite of degree $n+1$. This gives:

$$\int_{S^{[n+1]} \times S^m} f = \frac{1}{n+1} \int_\Pi \Psi^* f. \tag{2.12.1}$$

Our goal is then to transform $\Psi^* f$ into something of the form $\sum_{\nu \geqslant 0} \Phi^* f_\nu \cdot p_\Pi^*\big(-c_1(\mathcal{L})^\nu\big)$.

We first have the following two straightforward formulas, which display on simple instances the index shift resulting from the additional S factor induced by the ρ component of Φ.

$$\Psi^* \mathrm{pr}_i T_S = \Phi^* \mathrm{pr}_{i+1} T_S \tag{2.12.2}$$

$$\Psi^* \mathrm{pr}_{ij} \mathcal{O}_\Delta = \Phi^* \mathrm{pr}_{i+1,j+1} \mathcal{O}_\Delta. \tag{2.12.3}$$

Beware in particular that the 'pr' on the left-hand-side refer to projections from $S^{[n+1]} \times S^m$ whereas those on the right-hand-side refer to projections from $S^{[n]} \times S^{m+1}$ The first non-trivial formula in the induction procedure is the following, and comes from (5) together with the given expression for $j_*\mathcal{L}$.

$$\Psi^* \mathrm{pr}_{0i}^* \mathcal{I}_{n+1} = \Phi^* \mathrm{pr}_{0,i+1}^* \mathcal{I}_n - p_\Pi^* \mathcal{L} \cdot \Phi^* \mathrm{pr}_{1,i+1}^* \mathcal{O}_\Delta. \tag{2.12.4}$$

Next we have the following relation, which is a consequence of the computation in the Grothendieck group of $S^{[n,n+1]}$ carried out in [5, §2]:

(2.12.5)

$$\Psi^* \mathrm{pr}_0^* T_{n+1} = \Phi^* \mathrm{pr}_0^* T_n + p_\Pi^* \mathcal{L} \cdot \Phi^* \mathrm{pr}_{01}^* \mathcal{I}_n^\vee + p_\Pi^* \mathcal{L}^\vee \cdot \Phi^* \mathrm{pr}_{01}^* \mathcal{I}_n \cdot \Phi^* \mathrm{pr}_1^* \omega_S^\vee - \Phi^* \mathrm{pr}_1^* (\mathcal{O}_S - T_S + \omega_S^\vee).$$

The last relation follows from [5, Lemma 2.1]:

$$\Psi^* \mathrm{pr}_0^* L^{[n+1]} = \Phi^* \mathrm{pr}_0^* L^{[n]} + p_\Pi^* \mathcal{L} \cdot \Phi^* \mathrm{pr}_1^* L. \tag{2.12.6}$$

We thus end up with an expression of the form

$$\int_{S^{[n+1]} \times S^m} f = \int_\Pi \Big(\sum\nolimits_{\nu \geqslant 0} \Phi^* f_\nu \cdot p_\Pi^*\big(-c_1(\mathcal{L})^\nu\big) \Big)$$

as we wanted, and this in turn equals by the projection formula

$$\int_{S^{[n]}\times S^{m+1}} \Phi_*\Big(\sum\nolimits_{\nu\geqslant 0} \Phi^* f_\nu \cdot p_\Pi^*\big(-c_1(\mathcal{L})^\nu\big)\Big) = \int_{S^{[n]}\times S^{m+1}} \Big(\sum\nolimits_{\nu\geqslant 0} f_\nu \cdot c_\nu(-\mathrm{pr}_{0i}^*\mathcal{I}_n)\Big),$$

the last equality being given by the identity $(\phi,\rho)_*\big(c_1(\mathcal{L})^i\big) = (-1)^i c_i(-\mathcal{I}_n)$ in the Chow ring of $S^{[n]}\times S$, see [5, Lemma 1.1]. [1] □

3 – Tzeng's proof of Göttsche–Yau–Zaslow formula

In this section, we summarize the proof due to Tzeng [24] of Theorem (1.4). The plan of the proof is the following. In §3.1, we follow [16] in order to define the algebraic cobordism group $(\omega_{2,1},+)$ of equivalence classes of pairs $[S,L]$ of surfaces and line bundles with respect to extended double point relation (cf. Definition 3.1.3), endowed with the sum induced by disjoint union of surfaces. Then we recall the description of $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}$ as a four dimensional vector space over $\mathbb{Q}$ spanned by $[\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}]$, $[\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)]$, $[\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}]$, and $[\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)]$.

In §3.2 we introduce Göttsche's enumerative integrals $d_\delta(S,L)$ of a line bundle L on a smooth surface S, and we show that if L is sufficiently ample, then $d_\delta(S,L)$ equals the number $a_\delta(S,L)$ of δ–nodal curves in a general δ–dimensional linear subsystem of $|L|$.

In §3.3 we summarize Tzeng's fine description of the behaviour under degeneration of the generating function of the numbers $d_\delta(S,L)$. In particular, denoting by $\phi(S,L) := \sum d_\delta(S,L)x^\delta$ such a generating function, we obtain a homomorphism ϕ of $\mathbb{Q}$-vector spaces between the algebraic cobordism group $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}$ and the space of invertible power series $(\mathbb{Q}[[x]]^\times,\cdot)$.

Finally, §3.4 is concerned with the proofs of Theorems (1.3) and (1.4). Thanks to the results of §3.1 and §3.3, we prove the analogous of Theorem (1.3) for the generating function $\phi(S,L)$, by considering the decomposition of an arbitrary pair $[S,L]$ as a sum of copies of $[\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}]$, $[\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)]$, $[\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}]$, and $[\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)]$. Therefore the assertion of Theorem (1.3) for the generating function $T(S,L)$ of the numbers $a_\delta(S,L)$ descends from Theorem (1.2) and the equality between $d_\delta(S,L)$ and $a_\delta(S,L)$ presented in §3.2. Then Göttsche–Yau–Zaslow formula is proved by means of Theorem (1.3) and Brian–Leung formula (see [1, Theorem 1.1]).

3.1 – The algebraic cobordism group of surfaces and line bundles

We aim to recall the construction of the algebraic cobordism group $(\omega_{2,1},+)$ of surfaces and line bundles, together with the results in [16] describing the $\mathbb{Q}$–vector space $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}$.

Let $(\mathcal{M}_2,+)$ denote the free abelian group generated by isomorphism classes $[S]$ of smooth (irreducible) projective surfaces. Notice that the sum of two distinct isomorphism classes $[S_1]$ and $[S_2]$ in $\mathcal{M}_2$ can be thought as the isomorphism class of the disjoint union $S_1\sqcup S_2$.

3.1.1 Definition. *We say that four smooth projective surfaces $S_0,\dots,S_3$ satisfy a* double point relation $[S_0]-[S_1]-[S_2]+[S_3]$ *if there exists a family $\mathcal{X}\xrightarrow{\pi}\mathbb{P}^1$ of surfaces such that*

(i) the total space $\mathcal{X}$ is smooth of pure dimension 3, and the morphism π is smooth away from the fibre $\pi^{-1}(\infty)$;

(ii) the fibre over 0 is the smooth surface $\pi^{-1}(0)=S_0$;

[1] By definition the Chern classes $c_\nu(-\mathcal{F})$ are the coefficients of the multiplicative inverse of the Chern polynomial $c_t(\mathcal{F})$, i.e., the Chern classes $c_\nu(-\mathcal{F})$ are the Segre classes $s_\nu(\mathcal{F}$.

(iii) the fibre $\pi^{-1}(\infty) = S_1 \cup_D S_2$ is the union of the smooth surfaces S_1, S_2 meeting transversally along a smooth divisor D;

(iv) $S_3 \xrightarrow{\eta} D$ is the $\mathbb{P}^1$–bundle given by $S_3 \cong \mathbb{P}(\mathcal{O}_D \oplus N_{S_1/D}) \cong \mathbb{P}(N_{S_2/D} \oplus \mathcal{O}_D)$, where $N_{S_i/D}$ denotes the normal bundle of D in S_i.

3.1.2 Remark. We note that the isomorphism $\mathbb{P}(\mathcal{O}_D \oplus N_{S_1/D}) \cong \mathbb{P}(N_{S_2/D} \oplus \mathcal{O}_D)$ in Definition 3.1.1.(iv) depends on the fact that $\mathcal{O}_D(S_1 + S_2) \cong \mathcal{O}_D$, which indeed implies $N_{S_1/D} \otimes N_{S_2/D} \cong \mathcal{O}_D$ and hence $\mathcal{O}_D \oplus N_{S_1/D} \cong (N_{S_1/D} \otimes N_{S_2/D}) \oplus (N_{S_1/D} \otimes \mathcal{O}_D) \cong N_{S_1/D} \otimes (N_{S_2/D} \oplus \mathcal{O}_D)$.

Analogously, we can consider the free abelian group $(\mathcal{M}_{2,1}, +)$ generated by isomorphism classes $[S, L]$, where S is a smooth projective surfaces and L is a line bundle on S. Then Definition 3.1.1 may be extended as follows.

3.1.3 Definition. *For $i = 0, \dots, 3$, let $[S_i, L_i]$ be the class of a pair of a smooth projective surface S_i and a line bundle L_i on it. We say that*

$$[S_0, L_0] - [S_1, L_1] - [S_2, L_2] + [S_3, L_3] \tag{3.3.1}$$

is an extended double point relation *if $[S_0] - [S_1] - [S_2] + [S_3]$ is a double point relation via a family $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1$ of surfaces as above, and there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $L_i = \mathcal{L}|_{S_i}$ for $i = 0, 1, 2$ and $L_3 = \eta^*(\mathcal{L}|_D)$.*

3.1.4 Example. As in [24, Lemma 2.4], let us consider a smooth projective surface S and a line bundle L on S. Given a smooth curve $C \subseteq S$, let $\mathcal{X} \xrightarrow{\mathrm{bl}} S \times \mathbb{P}^1$ be the blow–up of the product $S \times \mathbb{P}^1$ along the curve $C \times \{\infty\}$, and let $E \cong \mathbb{P}(\mathcal{O}_C \oplus N_{S/C})$ be the exceptional divisor endowed with the $\mathbb{P}^1$–bundle map $E \xrightarrow{\eta} C$. Then for any integer k, the following extended double point formula holds

$$[S, L] - [S, L \otimes \mathcal{O}_S(-kC)] - [E, \eta^*(L|_C) \otimes \mathcal{O}_E(kC)] + [E, \eta^*((\mathcal{O}_S(-kC) \otimes L)|_C)].$$

To see this fact, let $\mathrm{pr}_1 \colon S \times \mathbb{P}^1 \longrightarrow S$ and $\mathrm{pr}_2 \colon S \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ denote the natural projections. So we may consider the family $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1$ given by $\pi := \mathrm{pr}_2 \circ \mathrm{bl}$, and we may define a line bundle $\mathcal{L}$ on $\mathcal{X}$ as

$$\mathcal{L} := (\mathrm{bl}^* \mathrm{pr}_1^* L) \otimes \mathcal{O}_{\mathcal{X}}(-kC).$$

Therefore the morphism π is a smooth away from ∞, and its fibres satisfy $\pi^{-1}(0) = S \times \{0\} \cong S$ and $\pi^{-1}(\infty) = S \cup_C E$. Finally, defining $S_0, S_1 \cong S$ and $S_2, S_3 \cong E$ as in Definition 3.1.1, the line bundle $\mathcal{L}$ is such that $\mathcal{L}|_{S_0} = L$, $\mathcal{L}|_{S_1} = L \otimes \mathcal{O}_S(-kC)$, $\mathcal{L}|_{S_2} = \eta^*(L|_C) \otimes \mathcal{O}_E(kC)$, and $\mathcal{L}|_{S_3} = \eta^*((\mathcal{O}_S(-kC) \otimes L)|_C)$.

We can now define the algebraic cobordism group of surfaces and line bundles.

3.1.5 Definition. *Let $\mathcal{R}$ denote the subgroup of $\mathcal{M}_{2,1}$ generated by all extended double point relations. The* algebraic cobordism group *of surfaces and line bundles is defined as the quotient*

$$\omega_{2,1} := \mathcal{M}_{2,1}/\mathcal{R}.$$

It is worth noticing that the algebraic cobordism group can be similarly defined in the general setting $\omega_{n,r}$ of isomorphism classes of pairs of n-dimensional varieties and rank r-vector bundles. In this direction, Lee and Pandharipande [16] described the structure of the $\mathbb{Q}$–vector space $\omega_{n,r} \otimes_{\mathbb{Z}} \mathbb{Q}$ by exhibiting a basis and showing that Chern invariants respect algebraic cobordism. In the case of surfaces and line bundles, [16, Theorems 1 and 4] lead to the following result, which have been obtained by Tzeng independently (see [24, Theorem 2.3]).

3.1.6 Theorem. *There is an isomorphism* $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q} \cong \mathbb{Q}^4$ *given by* $[S,L] \longmapsto (L^2, LK_S, K_S^2, c_2(S))$. *Moreover, the set*

$$\{[\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}], [\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)], [\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}], [\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)]\} \tag{3.6.1}$$

is a basis of the vector space $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}$, *and for any pair* $[S,L]$, *we have*

$$[S,L] = a_1[\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}] + a_2[\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)] + a_3[\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}] + a_4[\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)],$$

where $a_1 = -L^2 + \frac{K_S^2-2c_2(S)}{3}$, $a_2 = L^2$, $a_3 = \frac{3L^2+LK_s}{2} - \frac{K_S^2-3c_2(S)}{4}$ *and* $a_4 = -\frac{3L^2+LK_S}{2}$.

Proof. The first assertion is included in [16, Theorem 4]. In order to prove that (3.6.1) is a basis for $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}$, it is enough to check that the corresponding 4–tuples $(L^2, LK_S, K_S^2, c_2(S)) \in \mathbb{Q}^4$ are the linearly independent vectors $(0,0,9,3)$, $(1,-3,9,3)$, $(0,0,8,4)$ and $(0,-2,8,4)$. Finally, the values of $a_1,\dots,a_4$ can be deduced from the equality $(L^2, LK_S, K_S^2, c_2(S)) = a_1(0,0,9,3) + \cdots + a_4(0,-2,8,4)$. $\square$

3.1.7 Remark. According to Theorem 3.1.6, we point out that a set $\{[S_i, L_i] | i = 1,\dots,4\} \subseteq \omega_{2,1}$ gives a basis for $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}$ if and only if the corresponding vectors $(L_i^2, L_iK_{S_i}, K_{S_i}^2, c_2(S_i)) \in \mathbb{Q}^4$ are linearly independent over $\mathbb{Q}$.

3.1.8 Remark. We recall that the 4–tuples $(L^2, LK_S, K_S^2, c_2(S))$ and $(\chi(L), LK_S, K_S^2, \chi(\mathcal{O}_S))$ determine each other by Noether's formula and Riemann–Roch theorem, i.e.

$$\chi(\mathcal{O}_S) = \frac{1}{12}\left(K_S^2 + c_2(S)\right) \quad \text{and} \quad \chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2}\left(L^2 - LK_S\right).$$

Thus we may define another isomorphism $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q} \cong \mathbb{Q}^4$ by $[S,L] \longmapsto (\chi(L), LK_S, K_S^2, \chi(\mathcal{O}_S))$. Besides, a set $\{[S_i, L_i] | i = 1,\dots,4\} \subseteq \omega_{2,1}$ is a basis for $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}$ if and only if the corresponding vectors $(\chi(L_i), L_iK_{S_i}, K_{S_i}^2, \chi(\mathcal{O}_{S_i})) \in \mathbb{Q}^4$ are linearly independent.
We note further that the above formulas assure that $L^2 - LK_S$ is even and $K_S^2 + c_2(S)$ is divisible by 3 and 4, so that the coefficients $a_1,\dots,a_4$ in Theorem 3.1.6 turn out to be integers, as expected.

3.1.9 Example. Let $[S_1, L_1]$ and $[S_2, L_2]$ be two classes consisting of a K3 surface endowed with primitive line bundle. Then the 4–tuples $((\chi(L), LK_S, K_S^2, \chi(\mathcal{O}_S))$ corresponding to the set

$$\{[\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}], [\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)], [S_1, L_1], [S_2, L_2]\} \tag{3.9.1}$$

are $(1,0,9,1)$, $(3,-3,9,1)$, $\left(2+\frac{L_1^2}{2}, 0, 0, 2\right)$ and $\left(2+\frac{L_2^2}{2}, 0, 0, 2\right)$. Therefore the latter remark yields that (3.9.1) is a basis of $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}$ if and only if $L_1^2 \neq L_2^2$.

3.2 – Göttsche's enumerative integrals

Let δ be a positive integer, and consider a smooth surface S and a line bundle L on S. Following [7, Section 5], we present Göttsche's idea of interpreting the number $a_\delta(S,L)$ of δ–nodal curves in a general δ–dimensional linear subsystem of $|L|$ in terms of intersection numbers on Hilbert schemes of points.

For an integer $k > 0$, let us consider the Hilbert scheme $S^{[k]}$ parameterizing 0–dimensional subschemes of S of length k, and let $Z_k \subseteq S \times S^{[k]}$ be the universal subscheme with projections $\pi_1\colon Z_k \longrightarrow S$ and $\pi_2\colon Z_k \longrightarrow S^{[k]}$. As in §2.3, we denote by $L^{[k]}$ the vector bundle of rank k on $S^{[k]}$ given by $L^{[k]} := (\pi_2)_*(\pi_1)^*L$. Then Göttsche's enumerative integrals are defined as follows (cf. [7, Definition 5.1]).

3.2.10 Definition. *Let $W_0^{3\delta} \subseteq S^{[3\delta]}$ be the locally closed subset*

$$\left\{ \coprod_{i=1}^{\delta} \operatorname{Spec}\left(\mathcal{O}_{S,x_i}/\mathfrak{m}_{S,x_i}^2\right) \middle| \, x_1,\dots,x_\delta \in S \text{ are distinct} \right\}$$

and let $W^{3\delta} \subseteq S^{[3\delta]}$ be its closure (with the reduced induced structure), which turns out to be birational to $S^{[\delta]}$. Then we define the intersection number $d_\delta(S,L)$ as

$$d_\delta(S,L) := \int_{W^{3\delta}} c_{2\delta}(L^{[3\delta]})$$

and we denote by

$$\phi(S,L) := \sum_{\delta \geqslant 0} d_\delta(S,L) x^\delta$$

the associated generating function.

In [7, Proposition 5.2], Göttsche proved that if L is sufficiently ample with respect to δ, then the number $d_\delta(S,L)$ does coincide with the number $a_\delta(S,L)$ of δ–nodal curves in a general δ–dimensional linear subsystem of $|L|$. Namely,

3.2.11 Theorem. *Let S be a smooth projective surface and for $\delta \geqslant 2$ (resp. $\delta = 1$), let L be a $(5\delta-1)$–very ample (resp. 5–very ample) line bundle on S. Then a general linear subsystem $V \subseteq |L|$ of dimension δ contains exactly $d_\delta(S,L)$ curves having precisely δ nodes as singularities, i.e. $a_\delta(S,L) = d_\delta(S,L)$.*

Proof. We note that a curve $C \subseteq S$ is singular at some point $p \in S$ if and only if C contains the 0–dimensional scheme $\operatorname{Spec}\left(\mathcal{O}_{S,p}/\mathfrak{m}_{S,p}^2\right)$. Initially, we want to prove that a general linear subsystem $V \subseteq |L|$ of dimension δ contains exactly $d_\delta(S,L)$ curves having at least δ singularities. To this aim, let $\{s_1,\dots,s_{\delta+1}\} \subseteq H^0(S,L)$ be a basis of the linear subspace corresponding to $V \subseteq |L|$, and let us consider the set of global sections $\{(\pi_2)_*(\pi_1)^*s_1,\dots,(\pi_2)_*(\pi_1)^*s_{\delta+1}\}$ of the vector bundle $L^{[3\delta]}$. Moreover, let $Y \subseteq S^{[3\delta]}$ be the locus of points $Z \in S^{[3\delta]}$ where the vectors $(\pi_2)_*(\pi_1)^*s_1(Z),\dots,(\pi_2)_*(\pi_1)^*s_{\delta+1}(Z)$ are linearly dependent. Of course, the line bundle L is by assumption $(3\delta-1)$–very ample, i.e. the restriction map $H^0(S,L) \longrightarrow H^0(Z, L \otimes \mathcal{O}_Z)$ is surjective for any 0–dimensional subscheme $Z \in S^{[3\delta]}$, and hence the evaluation map $H^0(S,L) \otimes \mathcal{O}_{S^{[3\delta]}} \longrightarrow L^{[3\delta]}$ is surjective as well. Therefore, by considering sections $(\pi_2)_*(\pi_1)^*s \in H^0(S^{[3\delta]}, L^{[3\delta]})$ as above, we deduce that the restriction to $W^{3\delta}$ of the rank 3δ vector bundle $L^{[3\delta]}$ is generated by its global section. Thus the locus $Y \cap W^{3\delta}$ has codimension 2δ in $W^{3\delta}$ and it is Poincaré dual to $c_{2\delta}(L^{[3\delta]})$ by Thom–Porteous formula, i.e. its class is $[Y \cap W^{3\delta}] = c_{2\delta}(L^{[3\delta]}) \cap [W^{3\delta}]$ (cf. [6, Examples 14.3.2 and 14.4.2]). Since the dimension of $W^{3\delta} \smallsetminus W_0^{3\delta}$ is smaller than $\dim W^{3\delta} = 2\delta$ and $V \subseteq |L|$ is general, we conclude that $Y \cap W^{3\delta}$ lies into $W_0^{3\delta}$. Furthermore, [11] and the generality of V assure that $Y \cap W^{3\delta}$ is smooth. Hence the number of curves in V containing a point of $W_0^{3\delta}$ coincides with the degree of $[Y \cap W^{3\delta}]$, which is $d_\delta(S,L) := \int_{W^{3\delta}} c_{2\delta}(L^{[3\delta]})$. Since a curve $C \subseteq S$ containing a scheme $Z \in W_0^{3\delta}$ is singular at each of the points $x_1,\dots,x_\delta \in S$ in the support of Z, we have that $d_\delta(S,L)$ is the number of curves in V having at least δ singularities.

So we need to prove that the curves at hand possess exactly δ nodes as singularities. Firstly, we point out that V does not contain curves admitting more than δ singular points. To see this fact, we note that by assumption the line bundle L is $(3\delta+2)$–very ample for any $\delta \geqslant 1$. Hence we could repeat the argument above for the restriction to $W^{3(\delta+1)}$ of the rank $3\delta+3$ vector bundle $L^{[3\delta+3]}$, deducing that the analogous locus $Y \cap W^{3(\delta+1)}$ should have codimension $2\delta+3$

in the $(2\delta+2)$–dimensional variety $W^{3(\delta+1)}$. Then V does not contain curves having at least $\delta+1$ singularities.

In order to prove that any curves having δ singular points is a nodal curve, we consider the locally closed subset $H_0^{5\delta} \subseteq S^{[5\delta]}$ defined as

$$\left\{ \coprod_{i=1}^{\delta} \operatorname{Spec}\left(\mathcal{O}_{S,x_i}/\left\langle \mathfrak{m}_{S,x_i}^3, xy\right\rangle\right) \,\middle|\, x_1,\dots,x_\delta \in S \text{ are distinct and } x,y \text{ are local parameters at } x_i \right\}$$

and we denote by $H^{5\delta} \subseteq S^{[5\delta]}$ its closure. We note that if $C \subseteq S$ is a curve having δ singular points such that it contains no subschemes $Z \subseteq H^{5\delta} \smallsetminus H_0^{5\delta}$, then C has only nodes as singularities.

We claim that $H_0^{5\delta}$ is smooth of dimension 4δ. To see this fact, we initially consider the case $\delta = 1$. Fixing $x_1 \in S$, ideals of the form $\langle \mathfrak{m}_{S,x_1}^3, xy\rangle$ are parameterized by 1–dimensional vector subspaces of $\mathfrak{m}_{S,x_1}^2/\mathfrak{m}_{S,x_1}^3 \cong \mathbb{C}^3$, such that their generators are not squares of elements of $\mathfrak{m}_{S,x_1}/\mathfrak{m}_{S,x_1}^2$ (because any generator is an element in $\mathfrak{m}_{S,x_1}^2/\mathfrak{m}_{S,x_1}^3$ which is a product of two local parameters). Therefore, ideals of the form $\langle \mathfrak{m}_{S,x_1}^3, xy\rangle$ are parameterized by a point of a projective plane outside a conic (which corresponds to 1–dimensional vector subspaces of $\mathfrak{m}_{S,x_1}^2/\mathfrak{m}_{S,x_1}^3$ whose generators are squares of elements of $\mathfrak{m}_{S,x_1}/\mathfrak{m}_{S,x_1}^2$). Equivalently, an ideal of the form $\langle \mathfrak{m}_{S,x_1}^3, xy\rangle$ corresponds to the choice of two distinct tangent directions of S at x_1, i.e. a point outside the diagonal of the second symmetric product $\operatorname{Sym}^2(\mathbb{P}^1)$, where $\mathbb{P}^1 \cong \mathbb{P}(T_{x_1}S)$ parameterizes tangent directions of S at x_1. Then the conic at hand is the image of the diagonal under the standard isomorphism $\operatorname{Sym}^2(\mathbb{P}^1) \cong \mathbb{P}^2$ given by $(a_0 : a_1) + (b_0 : b_1) \longmapsto (a_0b_0 : a_0b_1 + a_1b_0 : a_1b_1)$. Thus H_0^5 is an open subset in the $\mathbb{P}^2$–bundle $\operatorname{Sym}^2(\mathbb{P}(T(S)))$ on S, so that $H_0^5 \subseteq S^{[5]}$ is smooth of dimension 4. For $\delta > 1$, we consider the sublocus of $\left(H_0^5\right)^\delta$ parameterizing δ–tuples $(Z_1,\dots,Z_\delta)$ such that the schemes $Z_1,\dots,Z_\delta \in H_0^5$ are supported at distinct points $x_1,\dots,x_\delta$ of S. Of course, this locus is smooth of dimension 4δ, and the symmetric group S_δ acts freely on it by permuting factors. Since the quotient under this action is $H_0^{5\delta}$, we conclude that $H_0^{5\delta}$ is smooth of dimension 4δ, as claimed.

Since the line bundle L is $(5\delta-1)$–very ample for any $\delta \geqslant 1$, we can consider the restriction to $H^{5\delta} \smallsetminus H_0^{5\delta}$ of the rank 5δ vector bundle $L^{[5\delta]}$ and we can argue as above. Namely, we consider the global section $\{(\pi_2)_*(\pi_1)^*s_1,\dots,(\pi_2)_*(\pi_1)^*s_{\delta+1}\}$ of $L^{[5\delta]}$ and we define the locus Y of points $Z \in S^{[5\delta]}$ where the vectors $(\pi_2)_*(\pi_1)^*s_1(Z),\dots,(\pi_2)_*(\pi_1)^*s_{\delta+1}(Z)$ are linearly dependent. The locus $Y \cap \left(H^{5\delta} \smallsetminus H_0^{5\delta}\right)$ turns out to be empty as it would have codimension $4\delta > \dim\left(H^{5\delta} \smallsetminus H_0^{5\delta}\right)$. Thus we conclude that every curve contained in V and having δ singular points cannot have singularities other than nodes. ☐

3.3 – Degeneration of enumerative integrals

We follow [24, Section 3] in order to show how the generating function $\phi(S,L) = \sum d_\delta(S,L)x^\delta$ of Göttsche's enumerative integrals of a pair (S,L) can be determined from generating functions $\phi(S_1,L_1)$, $\phi(S_2,L_2)$ and $\phi(S_3,L_3)$, provided that $[S,L]-[S_1,L_1]-[S_2,L_2]+[S_3,L_3]$ is an extended double point relation in $\mathcal{M}_{2,1}$. More precisely, we outline the proof of the following (see [24, Theorem 3.2]).

3.3.12 Theorem. *If $[S_0,L_0]-[S_1,L_1]-[S_2,L_2]+[S_3,L_3]$ is an extended double point relation in $\mathcal{M}_{2,1}$, then*

$$\phi(S_0,L_0) = \frac{\phi(S_1,L_1)\phi(S_2,L_2)}{\phi(S_3,L_3)}.$$

Equivalently, the map $\phi\colon \omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow \mathbb{Q}[[x]]^\times$ induced by $[S,L] \longmapsto \phi(S,L)$ is a homomorphism of $\mathbb{Q}$-vector spaces between the algebraic cobordism group $(\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q},+)$ and the multiplicative group $(\mathbb{Q}[[x]]^\times,\cdot)$ of invertible power series.

According to Definitions 3.1.1 and 3.1.3, we consider a family $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1$ of surfaces inducing the extended double point relation $[S_0, L_0] - [S_1, L_1] - [S_2, L_2] + [S_3, L_3]$ appearing in Theorem 3.3.12. In particular, the fibre $\mathcal{X}_0 := \pi^{-1}(0)$ is S_0, and the fibre $\mathcal{X}_\infty := \pi^{-1}(\infty)$ is the union of S_1 and S_2 meeting transversally along a smooth divisor D. Let $U \subseteq \mathbb{P}^1$ be the largest Zariski open subset such that $\infty \in U$ and the fibre $\mathcal{X}_t := \pi^{-1}(t)$ is smooth for any $t \neq \infty$, and let us still denote by $\mathcal{X} \xrightarrow{\pi} U$ the pullback of the original family over $\mathbb{P}^1$. For an integer $\delta \geqslant 1$, let $W^{3\delta}_{S_0}$ (resp. $W^{3\delta}_{S_1/D}$, $W^{3\delta}_{S_2/D}$) denote the locally closed subset $W^{3\delta} \subseteq S^{[3\delta]}$ introduced in Definition 3.2.10 when $S = S_0$ (resp. S_1/D, S_2/D). The main idea in the proof of Theorem 3.3.12 is to study how $W^{3\delta}_{S_0}$ degenerates by using the construction of Li and Wu [17] of a family $\pi^{[3\delta]} \colon \mathcal{X}^{[3\delta]} \longrightarrow U$ such that for any $t \neq \infty$, the fibre over t is the Hilbert scheme $\mathcal{X}_t^{[3\delta]}$ of 3δ points on the smooth surface $\mathcal{X}_t$.

In [18], such a construction is summarized as follows. Given an integer $n \geqslant 2$, the first step is to replace $\mathcal{X} \xrightarrow{\pi} U$ by an alternative family $\pi[n] \colon \mathcal{X}[n] \longrightarrow U$ of surfaces having the same smooth fibres of π when $t \neq \infty$, whereas the fibre over ∞ is a semistable model

$$\mathcal{X}[n]_\infty := S_1 \cup \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_{n-1} \cup S_2,$$

where for any $1 \leqslant i \leqslant n-1$, the surface $\Delta_i \cong \mathbb{P}(\mathcal{O}_D \oplus N_{S_1/D})$ is a $\mathbb{P}^1$–bundle over D having two distinguished sections D_i^- and D_i^+, and $\mathcal{X}[n]_\infty$ is obtained by identifying $D_i^+ \subseteq \Delta_i$ with $D_{i+1}^- \subseteq \Delta_{i+1}$, $D \subseteq S_1$ with $D_1^- \subseteq \Delta_1$, and $D_{n-1}^+ \subseteq \Delta_{n-1}$ with $D \subseteq S_2$. Then it is possible to introduce a family $\pi^{[n]} \colon \mathcal{X}^{[n]} \longrightarrow U$ of Hilbert schemes such that for any $t \neq \infty$, the fibre $\left(\pi^{[n]}\right)^{-1}(t)$ is the Hilbert scheme $\mathcal{X}_t^{[n]}$ parameterizing 0–dimensional subschemes of $\mathcal{X}_t$ of length n, and the fibre $\left(\pi^{[n]}\right)^{-1}(\infty)$ is the union of products

$$\mathcal{X}_\infty^{[n]} := \bigcup_{k=0}^{n} \left((S_1/D)^{[k]} \times (S_2/D)^{[n-k]} \right) \tag{3.12.1}$$

such that $(S_1/D)^{[0]} \cong (S_2/D)^{[0]} \cong \{pt\}$, whereas if $k \geqslant 1$, the points of $(S_1/D)^{[k]}$ are 0–dimensional schemes of length k supported on the smooth locus of $S_1 \cup \Delta_1 \cup \cdots \cup \Delta_i$ and the points of $(S_2/D)^{[n-k]}$ are 0–dimensional schemes of length $n-k$ supported on the smooth locus of $\Delta_j \cup \cdots \cup \Delta_{n-1} \cup S_2$.[2] Furthermore, the family $\pi^{[n]} \colon \mathcal{X}^{[n]} \longrightarrow U$ turns out to be a moduli space satisfying many good properties, as e.g. being a proper and separated Deligne–Mumford stack of finite type over U.

We are intereted in the case $n = 3\delta$. Therefore, we consider the family $\pi[3\delta] \colon \mathcal{X}[3\delta] \longrightarrow U$ of surfaces and the family $\pi^{[3\delta]} \colon \mathcal{X}^{[3\delta]} \longrightarrow U$ of Hilbert schemes defined above, and we define a closed subset $\mathcal{W}^{3\delta} \subseteq \mathcal{X}^{3\delta}$ as the closure in $\mathcal{X}^{3\delta}$ of the set

$$\mathcal{V} := \left\{ \coprod_{i=1}^{\delta} \operatorname{Spec}\left(\mathcal{O}_{\mathcal{X}_t, x_i}/\mathfrak{m}^2_{\mathcal{X}_t, x_i}\right) \middle| \, t \in U - \{\infty\} \text{ and } x_1, \ldots, x_\delta \in \mathcal{X}_t \text{ are distinct} \right\}.$$

We note, in particular, that the restriction of $\mathcal{V}$ to a Hilbert scheme $\mathcal{X}_t^{[3\delta]}$ with $t \neq \infty$ corresponds to the subset $W_0^{3\delta}$ given by Definition 3.2.10 in the case $S = \mathcal{X}_t$. Using the properties of the family $\pi^{[3\delta]} \colon \mathcal{X}^{[3\delta]} \longrightarrow U$, it is then possible to prove the following result which describes the restrictions of $\mathcal{W}^{3\delta}$ to the fibers of $\pi^{[3\delta]} \colon \mathcal{X}^{[3\delta]} \longrightarrow U$ over 0 and ∞ (see [24, Lemma 3.8]).

[2] Given a point of $Z \in (S_1/D)^{[k]}$ supported on $S_1 \cup \Delta_1 \cup \cdots \cup \Delta_i$, any component of type Δ_l must contain a point of the support of Z, otherwise such a component could be contracted. The same holds for points $Z \in (S_2/D)^{[n-k]}$.

3.3.13 Lemma. *The restriction $\mathcal{W}^{3\delta} \longrightarrow U$ of $\pi^{[3\delta]}$ to $\mathcal{W}^{3\delta}$ is a flat family such that*

$$\mathcal{W}^{3\delta} \cap S_0^{[3\delta]} = W_{S_0}^{3\delta}$$

and

$$\mathcal{W}^{3\delta} \cap \Big((S_1/D)^{[m]} \times (S_2/D)^{[3\delta-m]}\Big) = \begin{cases} \varnothing & \textit{if } m \textit{ is not divisible by 3} \\ W_{S_1/D}^{3k} \times W_{S_2/D}^{3(\delta-k)} & \textit{if } m = 3k \textit{ for some } k \in \mathbb{N}. \end{cases}$$

Now, in order to describe how line bundles enter into the picture, let us consider the line bundle $\mathcal{L} \longrightarrow \mathcal{X}$ on $\mathcal{X}$ whose existence follows from Definition 3.1.3. As in [24, Definition-Proposition 3.4], we recall the following facts included in [17]. Given an integer $n \geqslant 2$, we consider the universal closed subscheme $Z_0^{[n]} \subseteq S_0 \times S_0^{[n]}$. For $i = 1, 2$ and $0 \leqslant k \leqslant n$, we may also introduce universal closed subschemes $Z_i^{[k]} \subseteq (S_i/D) \times (S_i/D)^{[k]}$. Let

$$\begin{tikzcd} Z_0^{[n]} \arrow[r, "q_0^{[n]}"] \arrow[d, "p_0^{[n]}"'] & S_0^{[n]} \\ S_0 & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z_i^{[k]} \arrow[r, "q_i^{[k]}"] \arrow[d, "p_i^{[k]}"'] & (S_i/D)^{[k]} \\ (S_i/D) & \end{tikzcd}$$

be the corresponding projections, and for any $i = 0, 1, 2$, we define the vector bundle of rank $3k$

$$L_i^{[k]} := (q_i^{[k]})_* (p_i^{[k]})^* L.$$

We denote further by $\pi_1^{[k,n]}$ and $\pi_2^{[k,n]}$ the natural projections

$$\begin{tikzcd} (S_1/D)^{[k]} \times (S_2/D)^{[n-k]} \arrow[r, "\pi_2^{[k,n]}"] \arrow[d, "\pi_1^{[k,n]}"'] & (S_2/D)^{[n-k]} \\ (S_1/D)^{[k]}. & \end{tikzcd}$$

With these pieces of notation, there exists a universal closed subscheme $\mathcal{Z} \subseteq \mathcal{X} \times \mathcal{X}^{[n]}$ with projections

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "Q"] \arrow[d, "P"'] & \mathcal{X}^{[n]} \\ \mathcal{X} & \end{tikzcd}$$

such that the the fibers over 0 and ∞ of the restriction $\mathcal{Z} \longrightarrow U$ are $Z_0^{[n]}$ and

$$\mathcal{Z}_\infty := \left(\bigcup_{k=0}^{n} \Big(Z_1^{[k]} \times (S_2/D)^{[n-k]} \Big) \right) \cup \left(\bigcup_{k=0}^{n} \Big((S_1/D)^{[k]} \times Z_2^{[n-k]} \Big) \right),$$

respectively. Setting $\mathcal{L}^{[n]} := Q_* P^* \mathcal{L}$, the following holds (see [24, Lemmas 3.6 and 3.7]).

3.3.14 Lemma. *The sheaf $\mathcal{L}^{[n]}$ is a vector bundle of rank $3n$ over $\mathcal{X}^{[n]}$ such that*

$$\mathcal{L}^{[n]}|_{S_0} \quad \textit{and} \quad \mathcal{L}^{[n]}|_{(S_1/D)^{[k]} \times (S_2/D)^{[n-k]}} = (\pi_1^{[k,n]})^* L_1^{[k]} \oplus (\pi_2^{[k,n]})^* L_2^{[n-k]}.$$

In order to study degenerations of Göttsche's enumerative numbers, we define for $i = 1, 2$ the *relative enumerative numbers*

$$d_k(S_i/D, L_i) = \begin{cases} 1 & \text{for } k = 0 \\ \int_{W^{3k}} c_{2k}(L_i^{[3k]}) & \text{for } k \geqslant 1, \end{cases}$$

and their *relative generating function*

$$\phi(S_i/D, L_i) = \sum_{k \geqslant 0} d_k(S_i/D, L_i) x^k,$$

which are related to Göttsche's enumerative numbers $d_\delta(S_0, L_0)$ and to the corresponding generating function $\phi(S_0, L_0) = \sum_{\delta \geqslant 0} d_\delta(S_0, L_0) x^\delta$ by the following result (see [24, Proposition 3.9]).

3.3.15 Lemma. *If $[S_0, L_0] - [S_1, L_1] - [S_2, L_2] + [S_3, L_3]$ is an extended double point relation in $\mathcal{M}_{2,1}$, then*

$$d_\delta\left(X_0, L_0\right) = \sum_{k=0}^{\delta} d_k\left(S_1/D, L_1\right) d_{\delta-k}\left(S_2/D, L_2\right),$$

that is $\phi(S_0, L_0) = \phi(S_1/D, L_1) \cdot \phi(S_2/D, L_2)$.

Proof. Given $\delta \geqslant 1$, we recall that $S_0^{[3\delta]}$ is the fibre of $\pi^{[3\delta]} \colon \mathcal{X}^{[3\delta]} \longrightarrow U$ over 0. Moreover, Lemmas 3.3.13 and 3.3.14 ensure that $\mathcal{W}^{3\delta} \cap S_0^{[3\delta]} = W_{S_0}^{3\delta}$ and $\mathcal{L}^{[3\delta]}|_{S_0^{[3\delta]}} = L_0^{[3\delta]}$. Analogously, the fibre of $\pi^{[3\delta]}$ over ∞ is given by (3.12.1) and the aforementioned results yield

$$\mathcal{W}^{3\delta} \cap \mathcal{X}_\infty^{[3\delta]} = \bigcup_{k=0}^{\delta} W_{S_1/D}^{3k} \times W_{S_2/D}^{3(\delta-k)}.$$

and

$$\mathcal{L}^{[3\delta]}|_{(S_1/D)^{[m]} \times (S_2/D)^{[3\delta-m]}} = (\pi_1^{[m,3\delta]})^* L_1^{[m]} \oplus (\pi_2^{[m,3\delta]})^* L_2^{[3\delta-m]}$$

Since the family $\pi^{[3\delta]} \colon \mathcal{X}^{[3\delta]} \longrightarrow U$ and the restriction $\mathcal{W}^{[3\delta]} \longrightarrow U$ are flat over the open subset $U \subseteq \mathbb{P}^1$, we deduce that the classes of the fibres $\mathcal{W}_0^{[3\delta]}$ and $\mathcal{W}_\infty^{[3\delta]}$ satisfy

(3.15.1) $$c_{2\delta}(\mathcal{L}^{[3\delta]}) \left[\mathcal{W}_0^{[3\delta]}\right] = c_{2\delta}(\mathcal{L}^{[3\delta]}) \left[\mathcal{W}_\infty^{[3\delta]}\right].$$

In the light of the description of the restrictions of $\mathcal{L}^{[3\delta]}$ and $\mathcal{W}^{[3\delta]}$ to those fibers, we obtain that

(3.15.2) $$c_{2\delta}(\mathcal{L}^{[3\delta]}) \left[\mathcal{W}_0^{[3\delta]}\right] = c_{2\delta}(\mathcal{L}^{[3\delta]}) \left[W_{S_0}^{[3\delta]}\right] = \int_{W_{S_0}^{3\delta}} c_{2\delta}(L_0^{[3\delta]}) = d_\delta(S_0, L_0)$$

and

(3.15.3)
$$c_{2\delta}(\mathcal{L}^{[3\delta]}) \left[\mathcal{W}_\infty^{[3\delta]}\right] = c_{2\delta}(\mathcal{L}^{[3\delta]}) \left[\bigcup_{k=0}^{\delta} W_{S_1/D}^{3k} \times W_{S_2/D}^{3(\delta-k)}\right]$$

(3.15.4) $$= \sum_{k=0}^{\delta} c_{2\delta}\left((\pi_1^{[3k,3\delta]})^* L_1^{[3k]} \oplus (\pi_2^{[3k,3\delta]})^* L_2^{[3(\delta-k)]}\right) \left[W_{S_1/D}^{3k} \times W_{S_2/D}^{3(\delta-k)}\right],$$

where

$$c_{2\delta}\left((\pi_1^{[3k,3\delta]})^* L_1^{[3k]} \oplus (\pi_2^{[3k,3\delta]})^* L_2^{[3(\delta-k)]}\right) = \sum_{l=0}^{2\delta} c_l\left((\pi_1^{[3k,3\delta]})^* L_1^{[3k]}\right) c_{2\delta-l}\left((\pi_2^{[3k,3\delta]})^* L_2^{[3(\delta-k)]}\right).$$

Thus the only non-zero term in the k–th summand of (3.15.4) is obtained for $l = \dim W^{3k}_{S_1/D} = 2k$, so that it is given by $c_{2k}\left((\pi_1^{[3k,3\delta]})^* L_1^{[3k]}\right)\left[W^{3k}_{S_1/D}\right] c_{2(\delta-k)}\left((\pi_2^{[3k,3\delta]})^* L_2^{[3(\delta-k)]}\right)\left[W^{3k}_{S_1/D} \times W^{3(\delta-k)}_{S_2/D}\right]$. Therefore

$$\begin{aligned} c_{2\delta}(\mathcal{L}^{[3\delta]})\left[\mathcal{W}^{[3\delta]}_\infty\right] &= \sum_{k=0}^{\delta}\int_{W^{3k}_{S_1/D}} c_{2k}(L_1^{[3k]})\int_{W^{3(\delta-k)}_{S_2/D}} c_{2(\delta-k)}(L_1^{[3(\delta-k)]}) \\ &= \sum_{k=0}^{\delta} d_k\left(S_1/D, L_1\right) d_{\delta-k}\left(S_2/D, L_2\right), \end{aligned}$$

and the assertion follows as we combine the latter equation with (3.15.1) and (3.15.2). □

Applying repeatedly Lemma 3.3.15 to the construction of Example 3.1.4, we can now conclude the proof of Theorem 3.3.12.

Proof of Theorem 3.3.12. Let $[S, L]$ be a pair in $\omega_{2,1}$, and let $C \subseteq S$ be a smooth curve with normal bundle $N_{S/C}$. Let $\mathcal{S} \xrightarrow{\text{bl}} S \times \mathbb{P}^1$ be the blow–up of $S \times \mathbb{P}^1$ along the curve $C \times \{\infty\}$ and let $E \cong \mathbb{P}(\mathcal{O}_C \oplus N_{S/C})$ be the exceptional divisor endowed with the $\mathbb{P}^1$–bundle map $E \xrightarrow{\eta} C$. Then Example 3.1.4 assures that for any integer k, the following extended double point relation holds

$$[S, L] - [S, L \otimes \mathcal{O}_S(-kC)] - [E, \eta^*(L|_C) \otimes \mathcal{O}_E(kC)] + [E, \eta^*((\mathcal{O}_S(-kC) \otimes L)|_C)].$$

In the case $k = 0$, Lemma 3.3.15 yields

$$\phi(S, L) = \phi(S/C, L) \cdot \phi(E/C, \eta^*(L|_C)). \tag{3.15.5}$$

Let $[S_1, L_1]$ and $[S_2, L_2]$ be the pairs appearing in the extended double point relation in the assumption of Theorem 3.3.12. Therefore the fibre over ∞ of the family $\mathcal{X} \xrightarrow{\pi} U$ is the union $\pi^{-1}(\infty) = S_1 \cup_D S_2$ of the two smooth surfaces S_1, S_2 meeting transversally along a smooth curve $D \subseteq S_1 \cap S_2$. For $i = 1, 2$, consider the $\mathbb{P}^1$–bundle $E_i := \mathbb{P}(\mathcal{O}_D \oplus N_{X_i/D}) \xrightarrow{\eta_i} D$, together with the two distinguished sections $D_i^- := \mathbb{P}(N_{X_i/D}) \subseteq E_i$ and $D_i^+ := \mathbb{P}(\mathcal{O}_D) \subseteq E_i$. By blowing up the product $S_i \times \mathbb{P}^1$ along the curve $D \times \{\infty\}$, formula (3.15.5) gives

$$\phi(S_i, L_i) = \phi(S_i/D, L_i) \cdot \phi(E_i/D_i^-, \eta_i^*(L_i|_D)) \quad \text{for } i = 1, 2. \tag{3.15.6}$$

We recall that the pair $[S_3, L_3]$ in the assertion of Theorem 3.3.12 satisfies $S_3 \cong \mathbb{P}(\mathcal{O}_D \oplus N_{X_1/D}) = E_1$ and $L_3 \cong \eta_1^*(\mathcal{L}|_D)$, where $\mathcal{L} \longrightarrow \mathcal{X}$ is the line bundle of Definition 3.1.3. Moreover, being $L_1 = \mathcal{L}|_{S_1}$, we obtain that $L_3 \cong \eta_1^*(\mathcal{L}|_D) = \eta_1^*(L_1|_D)$. Analogously, in the light of the isomorphism $S_3 \cong \mathbb{P}(\mathcal{O}_D \oplus N_{X_2/D}) = E_2$ and of the equality $L_2 = \mathcal{L}|_{S_2}$, we deduce that $L_3 \cong \eta_2^*(L_2|_D)$. Thus we may rewrite equations (3.15.6) as

$$\phi(S_i, L_i) = \phi(S_i/D, L_i) \cdot \phi(E_i/D_i^-, L_3) \quad \text{for } i = 1, 2. \tag{3.15.7}$$

We further consider the blow–up $\mathcal{S} \xrightarrow{\text{bl}} E_1 \times \mathbb{P}^1$ of $E_1 \times \mathbb{P}^1$ along the curve $D_1^- \times \{\infty\}$. Letting $\mathcal{S} \xrightarrow{\sigma} \mathbb{P}^1$ be the composition of the blow–up morphism with the projection $E_1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, we have that the general fibre of σ is E_1 and $\sigma^{-1}(\infty) = E_1 \cup_D E_1$, where the curve D is embeds in the first E_1 as D_1^- and embeds in the secon E_1 as D_1^-. Hence formula (3.15.5) gives

$$\phi(E_1, L_3) = \phi(E_1/D_1^-, L_3) \cdot \phi(E_1/D_1^+, L_3). \tag{3.15.8}$$

Since $E_1 \cong S_3$ and there exists a canonical isomorphism $(E_1, \eta_1^*(L_1|_D)) \cong (E_2, \eta_2^*(L_2|_D))$ sending D_1^+ to D_2^-, equation (3.15.8) becomes

$$\phi(S_3, L_3) = \phi(E_1/D_1^-, L_3) \cdot \phi(E_2/D_2^-, L_3). \tag{3.15.9}$$

Recalling that $\phi(S_0, L_0) = \phi(S_1/D, L_1)\cdot\phi(S_2/D, L_2)$ by Lemma 3.3.15, and pulling together equations (3.15.7) and (3.15.9), we obtain

$$\begin{aligned}\phi(S_0, L_0) \cdot \phi(S_3, L_3) &= \phi(S_1/D, L_1) \cdot \phi(S_2/D, L_2) \cdot \phi(E_1/D_1^-, L_3) \cdot \phi(E_2/D_2^-, L_3)\\ &= \phi(S_1, L_1) \cdot \phi(S_2, L_2),\end{aligned}$$

which gives the assertion. □

3.4 – Proofs

In this final section we are concerned with the proofs of Theorems (1.3) and (1.4). To start, we prove the analogous of Theorem (1.3) for the generating function $\phi(S, L) = \sum d_\delta(S, L)x^\delta$, and we describe explicitly the universal power series involved.

3.4.16 Proposition. *There exist universal (invertible) power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[[x]]$ such that*

$$\phi(S, L) = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)},$$

where

$$\begin{aligned}A_1 &= \phi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})^{-1}\phi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))\phi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})^{\frac{3}{2}}\phi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0))^{-\frac{3}{2}},\\ A_2 &= \phi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})^{\frac{1}{2}}\phi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0))^{-\frac{1}{2}},\\ A_3 &= \phi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})^{-\frac{1}{3}}\phi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})^{-\frac{1}{4}},\\ A_4 &= \phi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})^{-\frac{2}{3}}\phi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})^{\frac{3}{4}}.\end{aligned}$$

Proof. Given a class $[S, L] \in \omega_{2,1}$, Theorem 3.1.6 assures that

$$[S, L] = a_1[\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}] + a_2[\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)] + a_3[\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}] + a_4[\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)],$$

where $a_1 = -L^2 + \frac{K_S^2 - 2c_2(S)}{3}$, $a_2 = L^2$, $a_3 = \frac{3L^2 + LK_s}{2} - \frac{K_S^2 - 3c_2(S)}{4}$ and $a_4 = -\frac{3L^2+LK_S}{2}$. Therefore Theorem 3.3.12 yields

$$\phi(S, L) = \phi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})^{a_1}\phi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))^{a_2}\phi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})^{a_3}\phi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0))^{a_4},$$

so that the assertion follows easily by replacing the values of $a_1, \dots, a_4$ in the latter equality. □

3.4.17 Remark. According to Remark 3.1.7, we may easily determine an alternative basis of $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}$, and the corresponding coefficients of an arbitrary class $[S, L] \in \omega_{2,1}$ may be otained as in the proof of Theorem 3.1.6. Consequently, the universal power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[[x]]$ could be described in terms of any such a basis.

3.4.18 Remark. Tzeng proved Göttsche conjecture under the assumption that L is a $(5\delta-1)$–very ample line bundle on a smooth projective surface S, as follows (see [24, Theorem 1.1]). She observed that by expanding the series A_1, A_2, A_3, A_4 in Proposition 3.4.16, the coefficient $d_\delta(S,L)$ of x^δ in the generating function $\phi(S,L) = \sum d_\delta(S,L)x^\delta$ turns out to be a universal polynomial T_δ of degree δ in the variables $L^2, LK_S, K_S^2, c_2(S)$ (cf. [24, Corollary 4.2]), i.e. the analogous of Göttsche conjecture holds for $\phi(S,L)$ (without any assumption of positivity for L). Besides, Theorem 3.2.11 asserts that if L is a $(5\delta-1)$–very ample line bundle on a smooth surface S, then $a_\delta(S,L) = d_\delta(S,L)$. Thus she concludes that $a_\delta(S,L)$ is computed by the universal polynomial $T_\delta(L^2, LK_S, K_S^2, c_2(S))$, as well.

In the light of Proposition 3.4.16, Theorems (1.2) and 3.2.11, the proof of Theorem (1.3) is now straightforward.

Proof of Theorem (1.3). Consider the generating function $T(S,L) = \sum T_\delta(L^2, LK_S, K_S^2, c_2(S))x^\delta$ of the universal polynomials $T_\delta(L^2, LK_S, K_S^2, c_2(S))$, whose existence is guaranteed by Theorem (1.2). If L is a $(5\delta-1)$–very ample line bundle on a smooth surface S, then Theorems (1.2) and 3.2.11 ensure that $T_\delta(L^2, LK_S, K_S^2, c_2(S)) = a_\delta(S,L) = d_\delta(S,L)$. Thus the generating functions $T(S,L)$ and $\phi(S,L)$ do coincide, and we deduce from Proposition 3.4.16 that $T(S,L) = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)}$. □

Finally, we prove Göttsche–Yau–Zaslow formula.

Proof of Theorem (1.4). Let us consider the generating function

$$\gamma(S,L)(q) := \sum_{\delta \geqslant 0} T_\delta(L^2, LK_S, K_S^2, c_2(S))\,(DG_2(\tau))^\delta$$

appearing in the right–hand side of Göttsche–Yau–Zaslow formula. Given two integers $g \geqslant \delta \geqslant 0$, let (S,L) consists of a generic K3 surface S endowed with a primitive line bundle L on S such that $L^2 = 2(g+\delta-1)$, and let $M_g(\delta)$ denote the number of curves in $|L|$ having geometric genus g and passing through g general points of S (or, equivalently, contained in a general linear subsystem of $V \subseteq |L|$ of codimension g). Bryan and Leung proved that the generating function of the numbers $M_g(\delta)$ satisfies the formula (see [1, Theorem 1.1])

$$\sum_{\delta \geqslant 0} M_g(\delta) q^{g+\delta-1} = \frac{DG_2(\tau)^g}{\Delta(\tau)}$$

which, according to [7, Remark 2.6], can be reformulated as

$$\gamma(S,L)(q) = \frac{(DG_2(\tau)/q)^{\chi(L)}}{\Delta(\tau)D^2G_2(\tau)/q^2}. \tag{3.18.1}$$

On the other hand, we recall that the 4–tuples $(L^2, LK_S, K_S^2, c_2(S))$ and $(K_S^2, LK_S, \chi(L), \chi(\mathcal{O}_S))$ determine each other through linear relations (cf. Remark 3.1.8). Since $\gamma(S,L)(q)$ coincides with the generating function $\phi(S,L)$ computed at $x = DG_2(\tau)$, we deduce from Proposition 3.4.16 that there exist four universal power series $B_1(q), \ldots, B_4(q) \in \mathbb{Q}[[q]]$ such that

$$\gamma(S,L)(q) = B_1(q)^{K_S^2} B_2(q)^{LK_S} B_3(q)^{\chi(L)} B_4(q)^{\chi(\mathcal{O}_S)}. \tag{3.18.2}$$

When S is a K3 surface, equation (3.18.2) becomes $\gamma(S,L)(q) = B_3(q)^{\chi(L)}B_4(q)^2$. Moreover, as we consider different pairs (S,L) such that $L^2 = 2(g+\delta-1)$, the latter equality must be satisfied for any positive integer $\chi(L) = g+\delta+1$. Comparing $\gamma(S,L)(q) = B_3(q)^{\chi(L)}B_4(q)^2$

and (3.18.1), we then conclude that $B_3(q) = DG_2(\tau)/q$ and $B_4(q) = \big(\Delta(\tau)D^2G_2(\tau)/q^2\big)^{-1/2}$. Therefore (3.18.2) leads to Göttsche–Yau–Zaslow formula

$$\gamma(S,L)(q) = B_1(q)^{K_S^2} B_2(q)^{LK_S} \frac{(DG_2(\tau)/q)^{\chi(L)}}{(\Delta(\tau)D^2G_2(\tau)/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

□

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