Degenerations and enumerations of curves on surfaces

Seminar held in Roma Tor Vergata 2015–2017

edited by Th. Dedieu

version October 8, 2026

Tout ce qu'on savait, c'est que, lorsqu'il revint d'Italie, il était prêtre.

Victor Hugo, *Les misérables*

Contents

	Short Presentation	5
	List of Talks	7
I	**Limits of nodal curves: generalities and examples** by Ciro Ciliberto and Thomas Dedieu	9
II	**Deformations of curves on surfaces** by Thomas Dedieu and Edoardo Sernesi	39
III	**Geometry of logarithmic Severi varieties at a general point** by Thomas Dedieu	57
IV	**Deformations of an m-tacnode into $m-1$ nodes** by Margherita Lelli-Chiesa	83
V	**Products of deformation spaces of tacnodes** by Francesco Bastianelli	91
A	**Geometry of the deformation space of an m-tacnode into $m-1$ nodes** by Thomas Dedieu	97
VI	**Degenerations and deformations of generalized Severi varieties** by Thomas Dedieu and Concettina Galati	107
VII	**The Caporaso–Harris recursion within a degeneration of the plane** by Thomas Dedieu	141
VIII	**Proving the existence of singular curves on smooth surfaces by smoothing tacnodes on reducible surfaces** by Concettina Galati	153
B	**Flag semi-stable reduction of tacnodal curves at the limit of nodal curves** by Ciro Ciliberto, Thomas Dedieu, and Concettina Galati	171
IX	**Enumeration of curves by floor diagrams, via degenerations of the projective plane** by Thomas Dedieu	189
X	**Enumeration of curves in toric surfaces, tropical geometry, and floor diagrams** by Thomas Dedieu	211

XI **Enumerative geometry of $K3$ surfaces**
by Thomas Dedieu 267

XII **Some classical formulæ for curves and surfaces**
by Thomas Dedieu 309

C **Generalized weight properties of resultants and discriminants, and applications to projective enumerative geometry**
by Laurent Busé and Thomas Dedieu 369

XIII **Node polynomials for curves on surfaces**
by Thomas Dedieu 409

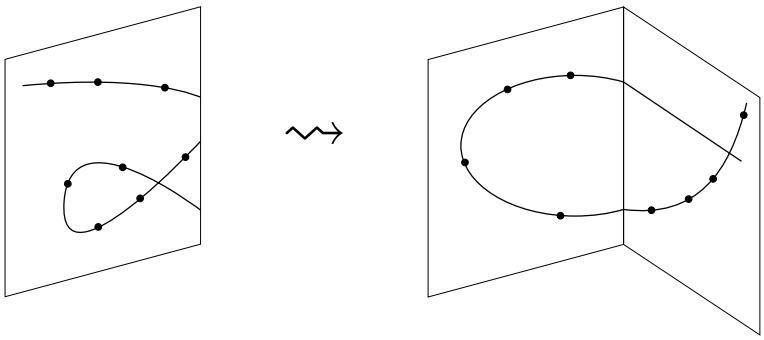
XIV **The Göttsche conjecture and the Göttsche–Yau–Zaslow formula**
by Francesco Bastianelli and Thomas Dedieu 467

Short Presentation

These notes originate in a workshop held in Rome, University "Tor Vergata", from October 2015 to June 2017, in the framework of the Marie Skłodowska Curie project "Families of Subvarieties in Complex Algebraic Varieties".[1] We would meet on a weekly basis, on Tuesday afternoons. The talks were scheduled to last two hours, but it happened frequently that we discussed over the whole afternoon. Let me take the occasion to thank all colleagues and friends in Rome for the lively, enthusiastic, and dedicated atmosphere in the workshop, which hopefully reflects on these notes.

The main theme of these notes is the enumeration of singular curves on surfaces. The first instance of this problem is arguably the enumeration of plane curves: consider the family V_g^d of degree d and genus g curves in the projective plane; it has dimension dimension $D_g^d = 3d+g-1$; now, the question is: given D_g^d fixed general points in the projective plane, how many curves are there in the family V_g^d that pass through these D_g^d points? For example, take $d = 3$ and $g = 0$, which is the first non-trivial case. Smooth plane cubics are elliptic curves, hence rational plane cubics have one singular point, either an ordinary double point or an ordinary cusp. The question is to find how many of these rational cubics pass through 8 general points; the answer is 12, as we will prove in many different ways in the course of the notes.

We give a particular focus to the approach to this problem by degeneration. The idea is to let the surface degenerate to, say, the transverse union of two surfaces, and in the same time let some of the fixed points go on one side and some on the other side, as indicated in the figure below.



The initial enumerative problem is thus reduced to a collection of simpler enumerative problems (the degrees and genera of these auxiliary problems are smaller than those of the initial problem).

There are other guises of approaching enumerative problems by degeneration. In these notes, we also consider such approaches using tropical geometry, and Gromov–Witten invariants. We also study solutions coming from intersection theory, both in a very classical form involving calculus on polynomials, as well as in a nowadays more standard form involving Chern classes and the likes.

The problem of counting curves on the projective plane is now fairly well understood, and its solution is discussed at length in this volume. The current research is more concerned with counting curves on varieties with trivial canonical bundle; we give particular attention to the

[1]Project FOSICAV, which has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 652782.

problem of counting curves on $K3$ surfaces, which indeed was the goal of the workshop these notes arose from.

The chapters in this volume are of different natures: some offer a detailed account with complete proofs of existing results (e.g., Lecture VI), while others propose a more streamlined presentation (e.g., Lecture XI); some chapters make the most of by now classical articles by pushing their ideas as far as possible (typically, Lecture III), and the appendices contain original material (Lectures B and C). All along we have striven to provide a comprehensive treatment, and as a result we haven't hesitated to work out numerous detailed examples.

It is in principle possible to read each chapter independently, although some make use of others.

List of Talks

- Seminario panoramico
  (Ciro Ciliberto, 13 ottobre 2015)
- Deformazioni di curve su superficie
  (Edoardo Sernesi, 27 ottobre 2015)
- Dimostrazione della formula di Caporaso–Harris: sezioni iperpiane delle varietà di Severi, I
  (Concettina Galati, 3 novembre 2015)
- Dimostrazione della formula di Caporaso–Harris: sezioni iperpiane delle varietà di Severi, II
  (Concettina Galati, 17 novembre 2015)
- Dimostrazione della formula di Caporaso–Harris: geometria delle varietà di Severi
  (Thomas Dedieu, 24 novembre 2015)
- Deformazioni d'un m-tacnodo in $m-1$ nodi
  (Margherita Lelli-Chiesa, 1° dicembre 2015)
- Prodotti di spazi di deformazioni di tacnodi e applicazione allo studio delle sezioni iperpiane delle varietà di Severi
  (Francesco Bastianelli, 15 dicembre 2015)
- Degenerazioni di curve nodali in una degenerazione normal crossings di superficie e applicazioni
  (Concettina Galati, 19 gennaio 2016)
- Enumerazione di curve piane mediante diagrammi a gradini, I
  (Thomas Dedieu, 2 febbraio 2016)
- Enumerazione di curve piane mediante diagrammi a gradini, II
  (Thomas Dedieu, 9 febbraio 2016)
- Enumerazione di curve piane: esempi!
  (Thomas Dedieu, 16 febbraio 2016)
- Introduzione agli invarianti di Gromov–Witten, I
  (Filippo Viviani, 5 aprile 2016)
- Introduzione agli invarianti di Gromov–Witten, II
  (Filippo Viviani, 12 aprile 2016)
- La formula di degenerazione di Jun Li, I
  (Margarida Melo, 26 aprile 2016)
- La formula di degenerazione di Jun Li, II
  (Margarida Melo, 3 maggio 2016)
- Geometria enumerativa delle superficie K3, I: la formula di Yau–Zaslow/Beauville
  (Thomas Dedieu, 10 maggio 2016)

- Geometria enumerativa delle superficie K3, II: la formula di Göttsche–Yau–Zaslow/Bryan–Leung
  (Thomas Dedieu, 24 maggio 2016)
- Geometria enumerativa delle superficie K3, III: risultati di Maulik–Pandharipande, Maulik–Pandharipande–Thomas, Klemm–Maulik–Pandharipande–Scheidegger, ecc.
  (Thomas Dedieu, 31 maggio 2016)
- Geometria enumerativa delle superficie K3, IV: numeri di Noether–Lefschetz e grande linee della dimostrazione della congettura di Yau–Zaslow secondo Maulik–Pandharipande e Klemm–Maulik–Pandharipande–Scheidegger
  (Thomas Dedieu, 14 giugno 2016)
- Enumerazione di curve in superficie toriche mediante geometria tropicale secondo Mikhalkin e Brugallé, I
  (Thomas Dedieu, 22 novembre 2016)
- Enumerazione di curve in superficie toriche mediante geometria tropicale secondo Mikhalkin e Brugallé, II
  (Thomas Dedieu, 30 novembre 2016)
- Enumerazione di curve in superficie toriche mediante geometria tropicale secondo Mikhalkin e Brugallé, III
  (Thomas Dedieu, 13 dicembre 2016)
- La congettura di Göttsche e le sue dimostrazioni, I
  (Francesco Bastianelli, 2 maggio 2017)
- Alcune formule classiche per superficie in $\mathbf{P}^3$
  (Thomas Dedieu, 9 maggio 2017)
- La congettura di Göttsche e le sue dimostrazioni, II
  (Francesco Bastianelli, 16 maggio 2017)
- "Polinomi nodi" per curve in superficie, secondo Vainsencher, Kleiman–Piene, Qviller
  (Thomas Dedieu, 30 maggio 2017)
- La congettura di Göttsche generalizzata
  (Filippo Viviani, 6 giugno 2017)
- "Polinomi nodi" per curve in superficie, II; conlusione.
  (Thomas Dedieu, 27 giugno 2017)

Lecture I
Limits of nodal curves: generalities and examples

by Ciro Ciliberto and Thomas Dedieu

1 The basic question 9
2 Classical results 10
3 Semistable degenerations 10
4 Limit linear systems 14
5 Severi varieties and their limits 15
6 Example: Limits of 1–nodal plane sections for surfaces in $\mathbf{P}^3$ 20
7 Example: Limits of 3–nodal plane sections for quartic surfaces in $\mathbf{P}^3$ . . . 21
7.1 Degeneration to two quadrics 21
7.2 Degeneration to a cubic and a plane 24
8 A (well-behaved) non-good case 31
8.1 Degeneration of cubics to three planes 31
8.2 Degeneration of quartics to four planes 35
References 35

1 – The basic question

The objective of this lecture is to introduce the main subjects of this seminars series. This exposition is mainly based on [4]. We will work over the field of complex numbers.

Let $f : S \to \mathbf{D}$ be a flat, projective family of surfaces of degree d in $\mathbf{P}^r$ (with $r \geqslant 3$), with S a smooth threefold, and $\mathbf{D}$ the complex unit disc. We assume that the fibre $S_t := f^{-1}(t)$ is smooth and irreducible if $t \neq 0$; the fibre S_0 on the other hand may be singular, and even reducible. We call this situation either a *degeneration* of the *general fibres*, i.e., the S_t with $t \in \mathbf{D}^* := \mathbf{D} - \{0\}$, to the *central fibre* S_0, or a *smooth deformation* of S_0. One has $S \subseteq \mathbf{D} \times \mathbf{P}^r$, so that S is endowed with the line bundle $\mathcal{L} = p^*(\mathcal{O}_{\mathbf{P}^r}(1))$, where $p : S \to \mathbf{P}^r$ is the projection to the second factor: we call $\mathcal{L}$ the *polarizing (hyperplane) bundle* on S.

(1.1) Question. *What are the limits of tangent, or in general of multitangent, hyperplanes to S_t, for $t \neq 0$, as t tends to 0?*

The same question makes sense, more generally, for varieties of any dimension. For the case of plane curves, see [12, pp. 134–135] for a brief discussion. In dimension higher than 2 the problem is quite complicated and few results are known (however, B. Segre's paper [27] is an interesting classical source of basic information on the subject).

One of the main interests in Question (1.1) arises form enumerative geometry, since, by degenerating a surface to a reducible one, whose components are much simpler than the original surface (e.g., a degeneration into a union of planes), we may hope that the configuration of limits of pluritangent hyperplanes is easier to understand and their number (if finite), or in general the degree of the variety parameterising them, may be computed. This is indeed the subject of the foundational work by L. Caporaso and J. Harris [2, 3], and independently by Z.

Ran [21, 22, 23], both of which aimed to study *Severi varieties*, i.e., the families of irreducible nodal plane curves of a given degree, via degeneration techniques.

Section 2 contains classical enumerative formulae for surfaces. Sections 3–5 present general facts about Question (1.1), following the presentation given in [4]. Finally, Sections 6–8 present several examples in detail in order to illustrate the general theory.

2 – Classical results

Enumerative results on the number of (pluri)tangent planes to a *general* surface S of degree $d > 1$ in $\mathbf{P}^3$ are quite classical. The first probably go back to G. Salmon, see [24, 25].

The set $\check{S} \subseteq \check{\mathbf{P}}^3$ of tangent planes to S is the *dual* of S, which is birational to S. One has

$$\deg(\check{S}) = d(d-1)^2.$$

As soon as $d > 2$, $\check{S}$ is singular. How do the singularities of $\check{S}$ look like if S is general enough? There is a *nodal curve* D_b, whose general points correspond to *bitangent planes* to S. Salmon computes

$$\deg(D_b) = \frac{1}{2}d(d-1)(d-2)(d^3-d^2+d-12).$$

There is also a *cuspidal curve* D_s, whose points correspond to *stationary tangent planes* to S, i.e., tangent planes to S in *parabolic points*, namely points $x \in S$ such that the tangent plane to S at x cuts S in a curve with a cusp at x. Salmon computes

$$\deg(D_s) = 4d(d-1)(d-2),$$

whereas the curve Γ_p of parabolic points on S is the complete intersection of S with its *Hessian surface*, hence

$$\deg(\Gamma_p) = 4d(d-2).$$

Finally, there are finitely many triple points of $\check{S}$, which are triple points of D_b too. Salmon computes their number

$$t = \frac{1}{6}d(d-2)(d^7-4d^6+7d^5-45d^4+114d^3-111d^2+548d-960). \tag{2.0.1}$$

For all these classical formulae, the reader is invited to consult [XII, C].

All these numbers have been computed in recent times by I. Vainsencher [29, 30] and by S. Kleiman and R. Piene [15, 16], via Chern class computations, see [XIII]. However several questions are still open, like:

(2.1) Question. *What are the singularities of D_b and of D_s? Are these curves irreducible? How do they intersect? Is Γ_p smooth? What is the monodromy of the covering $T \to U$, given by the triple points of $\check{S}$, when S moves in the open subset $U \subseteq |\mathcal{O}_{\mathbf{P}^3}(d)|$ of smooth surfaces?*

Partial answers to some of these question have been given for $d = 4$ in [4].

3 – Semistable degenerations

A degeneration $f : S \to \mathbf{D}$ as in §1 is said to be *semistable* if the central fibre S_0 is reduced and with *local normal crossing* singularities. We will often assume that all irreducible components

of S_0 are smooth, so the singularities of S_0 are: *double curves*, along which only two irreducible components of S_0 intersect transversally, and *triple points*, where only three irreducible components of S_0 intersect transversally.

As in §1, we will assume there is a *polarising line bundle* $\mathcal{L}$ on S (although, in principle, it could be neither ample nor relatively ample). Denote by L_t the restriction of $\mathcal{L}$ to S_t for all $t \in \mathbf{D}$. Then we will say that (S_0, L_0) is *a limit* of (S_t, L_t) (with $t \in \mathbf{D}^*$), when $t \to 0$ (or that (S_t, L_t) with $t \in \mathbf{D}^*$ is a deformation of (S_0, L_0)).

The limit (S_0, L_0) *is not unique* if S_0 is reducible. Indeed, if W is an effective divisor of S supported on the central fibre S_0, consider the line bundle $\mathcal{L}(-W)$, which is said to be obtained from $\mathcal{L}$ by *twisting* by W. For $t \in \mathbf{D}^*$, its restriction to S_t is the same as L_t, but in general this is not the case for S_0; any such line bundle $\mathcal{L}(-W)|_{S_0}$ is called a *limit line bundle* of $\mathcal{L}_t$ for $t \in \mathbf{D}^*$. If Q is an irreducible component of S_0, the restriction $\mathcal{L}_{W,Q} := \mathcal{L}(-W)_{|Q}$ is called the *aspect* of $\mathcal{L}(-W)$ on Q.

(3.1) Remark. Since $\mathrm{Pic}(\mathbf{D})$ is trivial, the divisor $S_0 \subseteq S$ is linearly equivalent to 0. So if W is a divisor supported on S_0, one has $\mathcal{L}(-W) \cong \mathcal{L}(mS_0 - W)$ for all integers m. In particular, if $W + W' = S_0$, then $\mathcal{L}(-W) \cong \mathcal{L}(W')$.

The non-unicity of the limit (S_0, L_0) is a fundamental phenomenon in the theory of degenerations and limit linear series, and it is essential in general, and for our purposes in particular, to understand what the possible alternative limits are. We shall see several illustrations in this text, e.g., in Paragraph (3.3.2) below and in Section 8.

Another family of surfaces $f' : S' \to \mathbf{D}$ as above is said to be a *model of* $f : S \to \mathbf{D}$ if there is a commutative diagram

$$\begin{array}{ccccccc}
S' & \longleftarrow & \bar{S}' & \overset{p}{\dashrightarrow} & \bar{S} & \longrightarrow & S \\
{\scriptstyle f'}\downarrow & \square & \downarrow & & \downarrow & \square & \downarrow{\scriptstyle f} \\
\mathbf{D} & \underset{t^{d'} \hookleftarrow t}{\longleftarrow} & \mathbf{D} & = & \mathbf{D} & \underset{t \mapsto t^d}{\longrightarrow} & \mathbf{D}
\end{array}$$

where the two squares marked with a $\square$ are Cartesian, and p is a birational map, which is an isomorphism over $\mathbf{D}^*$. The family $f' : S' \to \mathbf{D}$, if semistable, is a *semistable model of* $f : S \to \mathbf{D}$ if in addition $d' = 1$ and p is a morphism. The *semistable reduction theorem* of [13] asserts that any family $f : S \to \mathbf{D}$ as in §1 has a semistable model.

The following formula is useful to analyze the self-intersections of the double curves of the central fibre of a semistable degeneration in the various components of the central fibre (see, e.g., Section 7.2.2, Section 8, and [B]). We refer to [1, 6] for a more general formulation. In particular, Formula (3.2.1) is enhanced to a more subtle constraint in [6], which is not merely topological, and involves the first cotangent sheaf $T^1_{S_0}$ of the central fibre.

(3.2) Lemma (Triple Point Formula)**.** *Assume $f : S \to \mathbf{D}$ is semistable. Let Q, Q' be smooth irreducible components of S_0 intersecting along the* double curve *R. Then*

$$\deg(\mathbf{N}_{R/Q}) + \deg(\mathbf{N}_{R/Q'}) + \mathrm{Card}\left\{ \begin{array}{c} \textit{triple points of } S_0 \\ \textit{along } R \end{array} \right\} = 0, \tag{3.2.1}$$

where a triple point *is the intersection $R \cap Q''$ with a component Q'' of S_0 different from Q, Q'.*

(3.3) Example (Families of surfaces in $\mathbf{P}^3$)**.** Consider a *linear pencil* of degree d surfaces in $\mathbf{P}^3$, generated by a *general* surface S_∞ and a *special* one S_0, and assume that S_0 has local normal crossing singularities, and that S_∞ is smooth, intersects transversally the double curve of S_0, and avoids all triple points of S_0. This pencil gives rise to a flat, proper, family $f : \mathcal{S} \to \mathbf{P}^1$, with S_0, S_∞ as fibres over $0, \infty \in \mathbf{P}^1$, respectively, and with total space isomorphic to the

blow–up of $\mathbf{P}^3$ along the *base locus scheme* $S_0 \cap S_\infty$ of the pencil. In fact, $\mathcal{S}$ may be realized as the bidegree $(d,1)$ hypersurface in $\mathbf{P}^3 \times \mathbf{P}^1$ defined by the equation $sg_0 + tg_\infty = 0$, where g_0 and g_∞ are equations of S_0 and S_∞ in $\mathbf{P}^3$, and $(s:t)$ are homogeneous coordinates in $\mathbf{P}^1$; then $f : \mathcal{S} \to \mathbf{P}^1$ is the restriction of the projection of $\mathbf{P}^3 \times \mathbf{P}^1$ to its second factor.

We will study the family obtained by restricting $\mathcal{S}$ to a disk $\mathbf{D} \subseteq \mathbf{P}^1$ centered at 0, such that $\mathcal{S}_t$ is smooth for all $t \in \mathbf{D}^*$; by abuse of notation we will denote it by $f : \mathcal{S} \to \mathbf{D}$. In order to construct a semistable model of $f : \mathcal{S} \to \mathbf{D}$, we resolve the singularities of $\mathcal{S}$. Under our assumptions, they occur only in the central fibre of f, at the points mapped by $\mathcal{S}_0 \to S_0 \subseteq \mathbf{P}^3$ to the intersection points of S_∞ with the double curves of S_0: these are the singular points of the curve $S_0 \cap S_\infty$, and they are *ordinary double points* of $\mathcal{S}$, i.e., singularities analytically equivalent to the one at the origin of the hypersurface $xy = zt$ in $\mathbf{A}^4$. This may be seen from the equation $g_0 + tg_\infty = 0$ of $\mathcal{S}$ in $\mathbf{P}^3 \times \mathbf{D}$: in suitable local analytic coordinates (x,y,z) around such a point in $\mathbf{P}^3$, g_0 and g_∞ are xy and z plus higher order terms, respectively. Such a singularity is resolved by the blow–up of the singular point, which produces an exceptional divisor $F \cong \mathbf{P}^1 \times \mathbf{P}^1$. It turns out that it is then possible to contract F in the direction of either one of its rulings, without introducing any singularity: the result is called a *small resolution* of the ordinary double point. The exceptional locus of the small resolution is a curve isomorphic to $\mathbf{P}^1$, with normal bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$.

Let $\tilde{f} : \tilde{\mathcal{S}} \to \mathbf{D}$ be a semistable model obtained by taking a small resolution of all the singular points of $\mathcal{S}$, as above. One has $\tilde{\mathcal{S}}_t \cong \mathcal{S}_t$ for all $t \in \mathbf{D}^*$. If S_0 has irreducible components $Q_1, \ldots, Q_r$, then $\tilde{\mathcal{S}}_0$ has irreducible components $\tilde{Q}_1, \ldots, \tilde{Q}_r$, where each $\tilde{Q}_i$ is either a blow-up of Q_i or Q_i itself. Each singular point of $\mathcal{S}$ lies on two components Q' and Q'' of S_0, and when taking a small resolution we may choose to put the exceptional $\mathbf{P}^1$ either on Q' or Q'', depending on which ruling of the exceptional $\mathbf{P}^1 \times \mathbf{P}^1$ we choose to contract; thus, locally, either $\tilde{Q}'$ is the blow-up of Q' at the singular point and $\tilde{Q}''$ is isomorphic to Q'', or vice versa. We let q be the number of ordinary double points of the original total space $\mathcal{S}$, denote the ordinary double points by $e_1, \ldots, e_q$, and the exceptional curves on $\tilde{Q}_1, \ldots, \tilde{Q}_r$ arising from the small resolution process by $E_1, \ldots, E_q$.

(3.3.1) The simplest example of this situation is when S_0 is the union of a general surface S' of degree h, and a general surface S'' of degree $d-h$, intersecting along a smooth curve R of degree $h(d-h)$ (we will assume that $1 \leqslant h \leqslant \left\lfloor \frac{d}{2} \right\rfloor$). In this case $\tilde{\mathcal{S}}_0$ consists of two components $\tilde{Q}_1, \tilde{Q}_2$, and we may assume that $\tilde{Q}_2 = S''$, whereas $\tilde{Q}_1$ is the blow–up of S' at the $q = dh(d-h)$ intersection points of R with S_∞. Then $\tilde{Q}_1$ and $\tilde{Q}_2$ intersect along a curve which can be identified with R, and the exceptional curves $E_1, \ldots, E_q$ all lie on $\tilde{Q}_1$. In $\mathcal{S}$, the curve R has self-intersection $S''^2 \cdot S' = (d-h)^2 h$ in S' and $S'^2 \cdot S'' = h^2(d-h)$ in S''; the two sum up to $dh(d-h)$, hence Formula (3.2.1) does not hold (note that there are no triple points at all), illustrating the fact that $\mathcal{S} \to \mathbf{D}$ is not semistable, as $\mathcal{S}$ is singular at the points of $R \cap S_\infty$. In $\tilde{\mathcal{S}}$ however, R has self-intersection $(d-h)^2 h - q$ in $\tilde{Q}_1$, and $h^2(d-h)$ in $\tilde{Q}_2$, which indeed sum up to 0, and Formula (3.2.1) holds.

The restriction map

$$H^0(\mathcal{O}_{S_0}(d)) \cong H^0(\mathcal{O}_{\mathbf{P}^3}(d)) \big/ \langle g_0 \rangle \to H^0(\mathcal{O}_R(d))$$

is surjective, and has kernel $H^0(\mathcal{I}_R(d)) \big/ \langle g_0 \rangle$. The latter space identifies with the tangent space at $[S_0]$ of the locus in $|\mathcal{O}_{\mathbf{P}^3}(d)|$ of hypersurfaces with double points along a deformation of R.[1] Thus

$$\big[e_1 + \cdots + e_q\big] = \big[S_\infty \cap R\big] \in |\mathcal{O}_R(d)|$$

[1] This is a generalization of the basic fact that if S_0 is a hypersurface in $\mathbf{P}^N$ with one ordinary double point at p_0 and no further singularities, then the tangent space at the point $[S_0]$ to the discriminant locus in $|\mathcal{O}_{\mathbf{P}^N}(d)|$ identifies with $H^0(\mathcal{I}_{p_0}(d)) \big/ \langle g_0 \rangle$, where g_0 is a defining equation for S_0. This is explained in [XIV, (2.7)].

encodes the normal direction at the point $[S_0]$ of the pencil generated by S_0 and S_∞ with respect to the locus along which the singularity of S_0 along R is preserved (note that the tangent direction of the pencil corresponds to the class of S_∞ in $\mathbf{P}H^0(\mathcal{O}_{S_0}(d))$). In fact, it holds more generally that $\mathcal{O}_R(d)$ canonically identifies with the first cotangent sheaf $T^1_{S_0}$ (see [6, Proposition (2.3)]), the global sections of which parametrize deformations of S_0 which are not locally trivial, see [6, Remark (2.1)].

(3.3.2) The case $h = 1$, in which $S' = P$ is a general plane, is particularly interesting since it shows the geometric significance of twisting the polarising line bundle $\mathcal{L} = p^*(\mathcal{O}_{\mathbf{P}^3}(1))$ on $\tilde{\mathcal{S}}$. The polarising line bundle L_0 maps $\tilde{\mathcal{S}}_0$ to the reducible surface $S'' \cup P$. If we twist by $Q_1 = \tilde{P}$ and consider $\mathcal{L}' = \mathcal{L}(-\tilde{P})$, then its aspects are:
• the trivial bundle on $Q_2 = S''$;
• $|\mathcal{O}_{\tilde{P}}(dH - \sum_{i=1}^{d(d-1)} E_i)|$ on $Q_1 = \tilde{P}$.
Hence $L'_0 = \mathcal{L}'|_{\tilde{\mathcal{S}}_0}$ contracts S'' to a point x. Moreover $\dim(|\mathcal{O}_{\tilde{P}}(dH - \sum_{i=1}^{d(d-1)} E_i|) = 3$ and this linear system maps $\tilde{P}$ to a *monoid* Σ, i.e., to a rational surface of degree d with a point of multiplicity $d-1$ at x (see Remark (7.3) for the case $d = 4$).

In conclusion, by twisting, we see that the degeneration of $(S_t, \mathcal{O}_{S_t}(1))$ to $(S'' \cup \tilde{P}, L_0)$ is also a semistable model of the degeneration of $(S_t, \mathcal{O}_{S_t}(1))$ to a general monoid $(\Sigma, \mathcal{O}_\Sigma(1))$.

(3.4) Example (Families of polarised K3 surfaces)**.** A special case of the previous example is that of a general quartic surface in $\mathbf{P}^3$ degenerating to the general union of two quadrics. This is also a special case of degenerations of polarised K3 surfaces to a union of two scrolls (see [5]), which we will now describe. The latter are type II degenerations according to the Kulikov-Persson-Pinkham classification of semistable degenerations of $K3$ surfaces (see [19, 20]).

Let $\mathcal{K}_p$ be the moduli space of primitively polarized $K3$ surfaces (S, L) of genus $p \geqslant 3$, i.e., the dualising sheaf ω_S of S is trivial, $h^1(S, \mathcal{O}_S) = 0$, and L is big, nef and indivisible in $\mathrm{Pic}(S)$, with $L^2 = 2p - 2$.

Write $p = 2l + \varepsilon$, with $\varepsilon = 0, 1$ and $l \in \mathbf{N}$. Let $R \subseteq \mathbf{P}^p$ be an elliptic normal curve of degree $p+1$, and set $L_R = \mathcal{O}_R(1)$. Consider two *general* line bundles $L_1, L_2 \in \mathrm{Pic}^2(R)$ with $L_1 \neq L_2$. Denote by Q_i the rational normal scroll of degree $p-1$ in $\mathbf{P}^p$ described by the secant lines to R spanned by the divisors in $|L_i|$, for $1 \leqslant i \leqslant 2$. One has

$$Q_i \cong \begin{cases} \mathbf{P}^1 \times \mathbf{P}^1 & \text{if } \varepsilon = 1, \text{ i.e., } p \text{ is odd,} \\ \mathbf{F}_1 & \text{if } \varepsilon = 0, \text{ i.e., } p \text{ is even.} \end{cases}$$

The surfaces Q_1 and Q_2 are $\mathbf{P}^1$–bundles over $\mathbf{P}^1$. By [5, Thm. 1], they intersect transversally along R, so $S_0 = Q_1 \cup Q_2$ has normal crossings. Moreover, R is anticanonical on Q_1 and Q_2, so S_0 has trivial dualising sheaf ω_{S_0}. The first cotangent sheaf $T^1_{S_0}$ (cf. [6, § 1]) identifies with a line bundle on the double curve of S_0, namely

$$T^1_{S_0} \cong \mathbf{N}_{R/Q_1} \otimes \mathbf{N}_{R/Q_2} \tag{3.4.1}$$

where $\mathbf{N}_{R/Q_i}$ is the normal sheaf of R in Q_i for $i = 1, 2$. The identity (3.4.1) is analogous to the identification of $T^1_{S_0}$ with $\mathcal{O}_R(d)$ in Example (3.3.1), and follows from [6, Proposition (2.3)] as well. The sheaf $T^1_{S_0}$ has degree 16 as a line bundle on R, since $(-K_{Q_i})^2 = 8$ for all $i = 1, 2$, as Q_i is either $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$ blown-up at one point. Note that in (3.3.1) one indeed has $q = 16$ for $d = 4$ and $h = 2$.

For all $i = 1, 2$, denote by σ_i and f_i a minimal section and a fiber of the ruling of Q_i, respectively; one has $\sigma_i^2 = \varepsilon - 1$, $f_i^2 = 0$, and $\sigma_i \cdot f_i = 1$. The surface Q_i is a degree $p-1$ rational normal scroll in $\mathbf{P}^{p-1}$, so we may assume

$$L_{Q_i} := \mathcal{O}_{Q_i}(1) \simeq \mathcal{O}_{Q_i}(\sigma_i + l f_i). \tag{3.4.2}$$

Moreover,

$$R \sim -K_{Q_i} \sim 2\sigma_i + (3-\varepsilon)f_i, \tag{3.4.3}$$

where $\sim$ denotes linear equivalence. Noting that $\mathcal{O}_{Q_i}(f_i)|_R \simeq L_i$, one finds

$$\begin{aligned} \mathbf{N}_{R/Q_i} = \mathcal{O}_{Q_i}(-K_{Q_i})|_R &= \mathcal{O}_R\big(2(\sigma_i + lf_i)\big) \otimes \mathcal{O}_R\big((3-\varepsilon-2l)f_i\big) \\ &= \mathcal{O}_R(2) \otimes L_i^{\otimes(3-2l-\varepsilon)}, \end{aligned} \tag{3.4.4}$$

hence finally

$$T^1_{S_0} \cong \mathbf{N}_{R/Q_1} \otimes \mathbf{N}_{R/Q_2} \cong L_R^{\otimes 4} \otimes (L_1 \otimes L_2)^{\otimes(3-2l-\varepsilon)}. \tag{3.4.5}$$

It is proved in [5] that the surface S_0 is a flat limit of smooth $K3$ surfaces in $\mathbf{P}^p$, i.e., S_0 is a member of the unique irreducible component $\mathcal{H}_p$ of the Hilbert scheme of surfaces in $\mathbf{P}^p$ which contains $K3$ surfaces S such that $(S, \mathcal{O}_S(1)) \in \mathcal{K}_p$, and moreover, for general choices of R, L_1, L_2, the Hilbert scheme is smooth at S_0. However, the fact that $T^1_{S_0}$ is non-trivial implies that S_0 is not a semistable limit of $K3$ surfaces: indeed, the total space $\mathcal{S}$ of every flat deformation of S_0 to $K3$ surfaces in $\mathcal{H}_p$ is singular along a divisor $T \in |T^1_{S_0}|$ (cf. [6, Prop. 1.11 and § 2]); this is analogous to the fact that, in (3.3.1), $\mathcal{S}$ is singular at the points $e_1, \dots, e_q$. More precisely (see again [5] for details), if

$$\begin{array}{ccccc} S_0 & \hookrightarrow & \mathcal{S} & \hookrightarrow & \mathbf{D} \times \mathbf{P}^p \\ \downarrow & & \downarrow{\scriptstyle \pi'} & \searrow^{\mathrm{pr}_2} & \downarrow \\ 0 & \hookrightarrow & \mathbf{D} & & \mathbf{P}^p \end{array} \tag{3.4.6}$$

is a deformation of S_0 in $\mathbf{P}^p$ whose general member is a smooth $K3$ surface, then $\mathcal{S}$ has double points at the support of the divisor $T \in |T^1_{S_0}|$ associated to the tangent direction to $\mathcal{H}_p$ at $[S_0]$ of the deformation (3.4.6), via the map

$$T_{\mathcal{H}_p, S_0} \cong H^0(S_0, \mathbf{N}_{S_0/\mathbf{P}^p}) \to H^0(T^1_{S_0}) \tag{3.4.7}$$

induced by the surjective sheaf map $\mathbf{N}_{S_0/\mathbf{P}^p} \to T^1_{S_0}$; this is analogous to the fact that, in (3.3.1), the divisor $e_1 + \cdots + e_q$ is determined by the tangent direction at $[S_0]$ of the pencil generated by S_∞ and S_0. If T is reduced (this is the case if (3.4.6) is general enough), then the tangent cone to $\mathcal{S}$ at each of the 16 points of T is a rank 4 quadric, i.e., the singularities of $\mathcal{S}$ are ordinary double points. We may thus consider a small resolution of singularities, $u : \tilde{\mathcal{S}} \to \mathcal{S}$, and then $\tilde{f} := f \circ u : \tilde{\mathcal{S}} \to \mathbf{D}$ is a semistable degeneration of $K3$ surfaces, with central fiber $\tilde{\mathcal{S}}_0 := \tilde{Q}_1 \cup \tilde{Q}_2$, where $\tilde{Q}_i = u^{-1}(Q_i)$ for $i = 1, 2$, and still $\omega_{\tilde{\mathcal{S}}} \cong \mathcal{O}_{\tilde{\mathcal{S}}}$. The polarising line bundle on $\tilde{\mathcal{S}}$ is $\mathcal{L} := u^*(\mathrm{pr}_2^*(\mathcal{O}_{\mathbf{P}^p}(1)))$. Setting $\tilde{R} = \tilde{Q}_1 \cap \tilde{Q}_2$, one has $\tilde{R} \cong R$ and

$$T^1_{\tilde{\mathcal{S}}_0} \simeq \mathcal{O}_R. \tag{3.4.8}$$

Equation (3.4.8) gives finer information than the triple point formula (3.2.1), which only says that $T^1_{\tilde{\mathcal{S}}_0}$ has degree 0.

4 – Limit linear systems

Consider a semistable degeneration $f : \mathcal{S} \to \mathbf{D}$ as in §3. Suppose there is a polarising, ample, fixed-components-free line bundle $\mathcal{L}$ on the total space $\mathcal{S}$, such that $h^0(S_t, L_t)$ is constant for $t \in \mathbf{D}$.

Consider the subscheme $\mathrm{Hilb}(\mathcal{L})$ of the *relative Hilbert scheme* of curves of $\mathcal{S}$ over $\mathbf{D}$, which is the Zariski closure of the set of all curves $C \in |L_t|$, for $t \in \mathbf{D}^*$. Assume that $\mathrm{Hilb}(\mathcal{L})$ is a component of the relative Hilbert scheme, a condition satisfied if $\mathrm{Pic}(S_t)$ has no torsion for $t \in \mathbf{D}^*$. One has a natural projection morphism $\varphi : \mathrm{Hilb}(\mathcal{L}) \to \mathbf{D}$, which is a projective bundle over $\mathbf{D}^*$, because $\mathrm{Hilb}(\mathcal{L})$ is isomorphic to $\mathfrak{P} := \mathbf{P}(f_*(\mathcal{L}))$ over $\mathbf{D}^*$. We call the fibre of φ over 0 the *limit linear system* of $|L_t|$ as $t \in \mathbf{D}^*$ tends to 0, and we will denote it by $\mathfrak{L}$.

(4.1) Remark. In general, *the limit linear system is not a linear system.* One would be tempted to say that $\mathfrak{L}$ coincides with $|L_0|$. This is the case if S_0 is irreducible, but it is in general no longer true when S_0 is reducible. Indeed in this case, there may be non–zero sections of L_0 whose zero–locus contains some irreducible component of S_0, and accordingly points of $|L_0|$ which do not correspond to points in the Hilbert scheme of curves. This is related to the non–uniqueness of the limit line bundle and to the twisting procedure mentioned in § 3 (see, e.g., Example (4.2) below). We will come back to this in a while.

The variety $\mathrm{Hilb}(\mathcal{L})$ is birational to $\mathfrak{P}$, and $\mathfrak{L}$ is a suitable degeneration of the projective space $|L_t|$, $t \in \Delta^*$.

As the following example shows, in passing from $\mathfrak{P}$ to $\mathrm{Hilb}(\mathcal{L})$, one has to perform a series of blow–ups along smooth centres contained in the central fibre, which correspond to spaces of non–trivial sections of some (twisted) line bundles which vanish on divisors contained in the central fibre. The exceptional divisors one gets in this way give rise to components of $\mathfrak{L}$, and may be identified with birational modifications of sublinear systems of twisted linear systems restricted to S_0.

(4.2) Example (See [10])**.** Consider a family of degree d surfaces $f : \mathcal{S} \to \mathbf{D}$ arising, as in Example (3.3), Paragraph (3.3.2), from a linear pencil generated by a general surface S_∞ and by $S_0 = S'' \cup P$, where P is a plane and S'' a general surface of degree $d-1$. One has a semistable model $\tilde{f} : \tilde{S} \to \mathbf{D}$ as described in Example (3.3), from which we keep the notation.

Let $\mathcal{L}$ be the polarising hyperplane bundle. The component $\mathrm{Hilb}(\mathcal{L})$ of the Hilbert scheme is gotten from the projective bundle $\mathfrak{P} = \mathbf{P}(\tilde{f}_*\mathcal{L})$, by blowing up the point $[P_0]$ of the central fibre $|\mathcal{O}_{S_0}(1)|$ of $\mathfrak{P} \to \mathbf{D}$ corresponding to the unique section (up to multiplicative constant) vanishing on the plane P. The limit linear system $\mathfrak{L}$ is the union of $\mathfrak{L}_1$, the blow–up of $|\mathcal{O}_{S_0}(1)|$ at $[P_0]$, and the exceptional divisor $\mathfrak{L}_2 \cong \mathbf{P}^3$, which identifies with the twisted linear system $|\mathcal{O}_{\tilde{S}_0}(1) \otimes \mathcal{O}_{\tilde{S}}(-\tilde{P})|$, whose aspects have been described in Paragraph (3.3.2).

The components $\mathfrak{L}_1$ and $\mathfrak{L}_2$ of $\mathfrak{L}$ meet along the exceptional divisor $\mathfrak{E} \cong \mathbf{P}^2$ of the morphism $\mathfrak{L}_1 \to |\mathcal{O}_{S_0}(1)|$. The elements of $\mathfrak{E} \subseteq \mathfrak{L}_1$ identify as the points of $|\mathcal{O}_R(1)| \cong |\mathcal{O}_P(1)|$ (recall that R is the curve $S'' \cap P$), whereas the plane $\mathfrak{E} \subseteq \mathfrak{L}_2$ is the set of elements $C \in |\mathcal{O}_{\tilde{P}}(d) \otimes \mathcal{O}_{\tilde{P}}(-\sum_{i=1}^{d(d-1)} E_i)|$ containing the proper transform $\hat{R} \cong R$ of R on $\tilde{P}$ (recall that the E_i's are the exceptional divisors of the blow-up $\tilde{P} \to P$ at the base points of the pencil generated by S_∞ and S_0). The element of $|\mathcal{O}_R(1)|$ corresponding to such a curve $C \in \mathfrak{E} \subseteq \mathfrak{L}_2$ is cut out on $\hat{R}$ by the residual component of C, which is the pull–back to $\tilde{P}$ of a line in P.

This limit linear system is pictured in Figure 1. We shall come back to this in more detail in the case $d = 4$ in Section 7.2.

5 – Severi varieties and their limits

Let $f : \mathcal{S} \to \mathbf{D}$ be a (not necessarily semistable) family as in §3, polarised by a line bundle $\mathcal{L}$ on $\mathcal{S}$. Fix a non–negative integer δ, and consider the locally closed subset $\mathring{V}_\delta(\mathcal{S}, \mathcal{L})$ of $\mathrm{Hilb}(\mathcal{L})$ formed by all curves $D \in |L_t|$, for $t \in \mathbf{D}^*$, such that D is irreducible, nodal, and has exactly δ nodes. Define

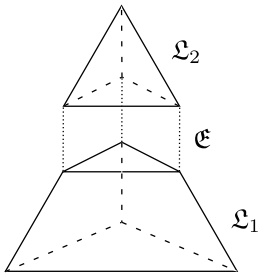


Figure 1: Limit hyperplane linear system for a degeneration into an irreducible $(d-1)$-ic and a plane

$V_\delta(\mathcal{S},\mathcal{L})$ (resp. $V^{\rm cr}_\delta(\mathcal{S},\mathcal{L})$) as the Zariski closure of $\mathring{V}_\delta(\mathcal{S},\mathcal{L})$ in $\mathrm{Hilb}(\mathcal{L})$ (resp. in $\mathfrak{P}=\mathbf{P}(f_*(\mathcal{L}))$). This is the *relative Severi variety* (resp. the *crude relative Severi variety*). Sometimes we may write $\mathring{V}_\delta$, V_δ, and $V^{\rm cr}_\delta$, rather than $\mathring{V}_\delta(\mathcal{S},\mathcal{L})$, $V_\delta(\mathcal{S},\mathcal{L})$ and $V^{\rm cr}_\delta(\mathcal{S},\mathcal{L})$, respectively.

There is a natural map $f_\delta : V_\delta \to \mathbf{D}$. If $t\in\mathbf{D}^*$, the fibre $V_{\delta,t}$ of f_δ over t is the *Severi variety* $V_\delta(S_t,L_t)$ of δ–nodal curves in the linear system $|L_t|$ on S_t. The degree of $V_{\delta,t}$ as a subvariety of $|L_t|$ is independent on $t\in\mathbf{D}^*$, and will be denoted by $d_\delta(\mathcal{L})$ (or simply by d_δ). Let $\mathfrak{V}_\delta(\mathcal{S},\mathcal{L})$ (or simply $\mathfrak{V}_\delta$) be the central fibre of $f_\delta : V_\delta\to\mathbf{D}$; it is the *limit Severi variety* of $V_\delta(S_t,\mathcal{L}_t)$ as $t\in\mathbf{D}^*$ tends to 0. This is a subscheme of the limit linear system $\mathfrak{L}$, which has been studied by various authors (in the present setting by Z. Ran [21, 22, 23], then by L. Caporaso, J. Harris [2, 3], more recently by C. Galati and C. Galati and A. Knutsen [8, 9] following Z. Ran; this theme is however quite classical as B. Segre's paper [27] shows, and it appears in disguise in work of other authors, like S. Kleiman [14]).

In a similar way, one defines the *crude limit Severi variety* $\mathfrak{V}^{\rm cr}_\delta(\mathcal{S},\mathcal{L})$ (or $\mathfrak{V}^{\rm cr}_\delta$), sitting in $|L_0|$.

(5.1) Remark. For $t\in\mathbf{D}^*$, the *expected dimension* of the Severi variety $V_\delta(S_t,\mathcal{L}_t)$ is $\dim(|L_t|)-\delta$ (see [28, Thm. 6.3]). We will always assume that the dimensions of (all components of) $V_\delta(S_t,L_t)$ equal the expected dimension for all $t\in\mathbf{D}^*$. This is a strong assumption, implying that $\mathring{V}_\delta$ is smooth; it will be satisfied in our applications.

Let us come now to the description of the limit Severi variety, under the assumption that the family $f:\mathcal{S}\to\mathbf{D}$ is semistable. We give a basic example in Section 6 below, after all the definitions and results have been given. We suggest to read the present section in parallel with this example.

We will freely use the notation introduced above and follow the approach in [8, 9]. We will suppose that the central fibre S_0 has smooth irreducible components $Q_1,\dots,Q_r$, with double curves $R_1,\dots,R_s$. We will assume, in addition, that there are q exceptional curves $E_1,\dots,E_q$ on S_0, arising from the small resolution of ordinary double points $e_1,\dots,e_q$ in an original family with singular total space, as discussed in §3. Beware that what we call here $\mathcal{S}$ is called $\tilde{\mathcal{S}}$ there.

The idea behind Definition (5.3) below and the rest of this section is that, under reasonable assumptions and after suitable reduction, the limits in S_0 of nodal curves in the general fibres S_t are (or at least should be) so that the limit of a node can essentially be of two types: either (a) a node on one of the components Q_i (off the double curves), or (b) a tacnode along one of the double curves R_i (off the triple points), i.e., the limit curve on S_0 consists locally of two smooth branches, both tangent to R_i at the same point. In fact, it is also possible to have several nodes tending to the same point, which is then a higher order tacnode along a double curve R_i.

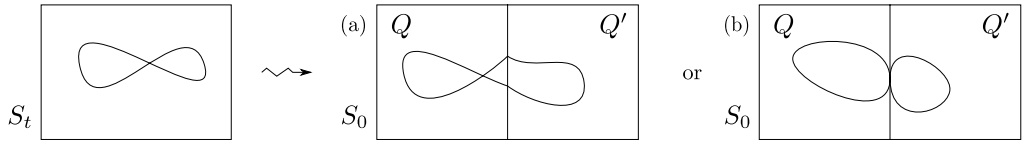


Figure 2: The two types of limits of nodes

It is arguably not immediate to understand how nodes may degenerate as in type (b) above, but we will not discuss this in this lecture (still, see Remark (6.1)); we refer instead to [B], as well as [VIII]; here we accept it, and discuss how to take it in consideration for our enumerative problem. Finally, it turns out to be convenient in practice to allow a third type of limits of nodes, concentrated at the ordinary double points of the singular model of the total space, but limits of this type can be reduced to limits of type (a).

(5.2) Notation. Let $\underline{\mathbf{N}}$ be the set of sequences $\underline{\tau} = (\tau_m)_{m\geqslant 2}$ of non–negative integers with only finitely many non–vanishing terms. Define two maps $\nu,\ \mu : \underline{\mathbf{N}} \to \mathbf{N}$ as follows

$$\nu\,(\underline{\tau}) = \sum\nolimits_{m\geqslant 2} \tau_m \cdot (m-1), \quad \text{and} \quad \mu\,(\underline{\tau}) = \prod\nolimits_{m\geqslant 2} m^{\tau_m}$$

(thus, in the notation of [III, (1.2)] and [VI, (1.2)], $\nu(\underline{\tau}) = I_{\underline{\tau}} - |\underline{\tau}|$, and $\mu(\underline{\tau}) = I^{\underline{\tau}}$). Given an s-tuple $\underline{\boldsymbol{\tau}} = (\underline{\tau}_1, \ldots, \underline{\tau}_s) \in \underline{\mathbf{N}}^s$, set

$$\nu(\underline{\boldsymbol{\tau}}) = \nu(\underline{\tau}_1) + \cdots + \nu(\underline{\tau}_s), \quad \text{and} \quad \mu(\underline{\boldsymbol{\tau}}) = \mu(\underline{\tau}_1) \cdots \mu(\underline{\tau}_s),$$

defining two maps $\nu,\ \mu : \underline{\mathbf{N}}^s \to \mathbf{N}$. Given $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_r) \in \mathbf{N}^r$ and $\mathbf{m} = (m_1, \ldots, m_q) \in \mathbf{N}^q$, set

$$|\boldsymbol{\delta}| = \delta_1 + \cdots + \delta_r \quad \text{and} \quad |\mathbf{m}^2| = m_1^2 + \cdots + m_q^2.$$

(5.3) Definition. *Consider a divisor W on $\mathcal{S}$, supported on the central fibre S_0, such that the twist $\mathcal{L}(-W)$ is* centrally effective*, i.e., all the aspects of $\mathcal{L}(-W)$ on the components $Q_1, \ldots, Q_r$ of S_0 are effective. Fix*

$$\boldsymbol{\delta} \in \mathbf{N}^r, \quad \underline{\boldsymbol{\tau}} \in \underline{\mathbf{N}}^p, \quad \textit{and} \quad \mathbf{m} \in \mathbf{N}^q.$$

Let $\mathring{V}(W, \boldsymbol{\delta}, \mathbf{m}, \underline{\boldsymbol{\tau}})$ be the Zariski locally closed subset in $|\mathcal{L}(-W) \otimes \mathcal{O}_{S_0}|$ parametrizing curves D such that:
(i) *D contains no double curve R_l, with $l \in \{1, \ldots, s\}$, and no triple point of S_0;*
(ii) *for every $i \in \{1, \ldots, q\}$, the contraction of the exceptional curve E_i in $\mathcal{S}$ maps D to a curve with an ordinary m_i-fold point at e_i on both components of S_0 that contain e_i; in particular, D contains E_i with multiplicity m_i (see below for a discussion of this condition);*
(iii) *off the singular locus of S_0 the curve $D - \sum_{i=1}^q m_i E_i$ has only nodes as singularities, and their number is δ_i on Q_i for all $i \in \{1, \ldots, r\}$;*
(iv) *for every $l \in \{1, \ldots, s\}$ and $m \geqslant 2$, there are exactly $\tau_{l,m}$ points on R_l, off the intersections with $\sum_{i=1}^q E_i$ and off the triple points of S_0 along R_l, at which D has an m-*tacnode *(see below for the definition), with reduced tangent cone equal to the tangent line of R_l there.*
We let $V(W, \boldsymbol{\delta}, \mathbf{m}, \underline{\boldsymbol{\tau}})$ be the Zariski closure of $\mathring{V}(W, \boldsymbol{\delta}, \mathbf{m}, \underline{\boldsymbol{\tau}})$ in $|\mathcal{L}(-W) \otimes \mathcal{O}_{S_0}|$.

Recall that an *m-tacnode* is an A_{2m-1}-double point, i.e., a plane curve singularity locally analytically isomorphic to the curve of $\mathbf{C}^2$ defined by the equation $y^2 = x^{2m}$ at the origin. Condition (iv) in Definition (5.3) requires that D is a Cartier divisor on S_0, having $\tau_{l,m}$ m–th order tangency points with the curve R_l, at points of R_l which are neither points on $\sum_{i\in I} E_i$ nor triple points of S_0.

When $m_i = 1$, condition (ii) simply means that E_i is a component of D, and $D - E_i$ and E_i intersect transversely in two points, one on the double curve of S_0 and the other off it. It

is the latter point which is a limit of nodes of curves on the surfaces S_t, $t \neq 0$. When $m_i > 1$, condition (ii) is arguably best understood on the large resolution of the ordinary double point e_i, which can be obtained as the blow-up $\varepsilon_i : \tilde{\mathcal{S}} \to \mathcal{S}$ along the exceptional curve E_i. The exceptional divisor F_i is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and has E_i as one of its rulings; let us call E_i' the other. The curves verifying condition (ii) correspond to curves in the linear system $|\varepsilon_i^*\mathcal{L}(-\varepsilon_i^* W - m_i F_i) \otimes \mathcal{O}_{\tilde{S}_0}|$, which has aspect $|\mathcal{O}_{F_i}(m_i E_i + m_i E_i')|$ on F_i, and for every curve on S_0 there is a unique corresponding curve in $\tilde{S}_0$ having m_i^2 nodes on F_i, off the singular locus of $\tilde{S}_0$, which is as indicated on the figure below.

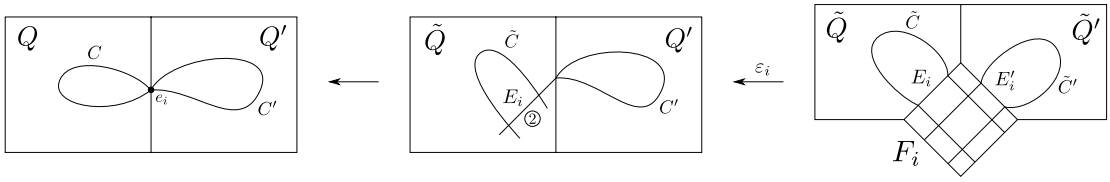


Figure 3: Limit of 4 nodes concentrated at an ordinary double point ($m_i = 2$)

(The curve in the central fibre of the singular total space is $C + C'$ (and both C and C' have a node at e_i), that in the small resolution is $\tilde{C} + 2E_i + C'$, and that in the large resolution is $\tilde{C}$ plus $\tilde{C}'$ plus two fibres of F_i linearly equivalent to E_i' plus two fibres of F_i linearly equivalent to E_i).

We will only need to consider condition (ii) with $m_i = 1$ for the examples that we shall consider in this text. Beware that the possibility to have $m_i > 1$ in condition (ii) has been overlooked in [4] (where it was harmless for the applications). There we used the notation $V(W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$, where I is a subset of $\{1, \dots, r\}$, to denote $V(W, \boldsymbol{\delta}, \mathbf{m}, \underline{\boldsymbol{\tau}})$ with $\mathbf{m} = (m_1, \dots, m_r)$ such that $m_i = 1$ if $i \in I$ and $m_i = 0$ otherwise. We will use this notation here as well, for compatibility reasons, and because it is convenient if all m_i are 0 or 1.

(5.4) Notation. Rather than using the notation $V(W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$ (see above), we may sometimes use a more expressive one like, e.g., $V(W, \delta_{Q_1} = 2, E_1, \tau_{R_1,2} = 1)$ for the variety parametrizing curves in $|\mathcal{L}(-W) \otimes \mathcal{O}_{S_0}|$, with two nodes on Q_1, one ordinary tacnode along R_1, and containing the exceptional curve E_1. Similarly $V(\delta_{Q_1} = 1)$ is the variety parametrizing curves in $|\mathcal{L} \otimes \mathcal{O}_{S_0}|$, with only one node on Q_1, etc.

(5.5) Proposition ([8, 9], see [VIII])**.** *Let $W, \boldsymbol{\delta}, \mathbf{m}, \underline{\boldsymbol{\tau}}$ be as above, and set*

$$\delta = |\boldsymbol{\delta}| + |\mathbf{m}^2| + \nu(\underline{\boldsymbol{\tau}}).$$

Let V be an irreducible component of $V(W, \boldsymbol{\delta}, \mathbf{m}, \underline{\boldsymbol{\tau}})$. If
(i) the linear system $|\mathcal{L}(-W) \otimes \mathcal{O}_{S_0}|$ has the same dimension as $|\mathcal{L}_t|$ for $t \in \mathbf{D}^*$, and
(ii) V has (the expected) codimension δ in $|\mathcal{L}(-W) \otimes \mathcal{O}_{S_0}|$,
then V is an irreducible component of multiplicity $\mu(V) := \mu(\underline{\boldsymbol{\tau}})$ of the limit Severi variety $\mathfrak{V}_\delta(S, \mathcal{L})$.

We emphasize that this proposition tells us in particular that a limit component in which nodes have degenerated to tacnodes has to be counted with a non-trivial multiplicity. For instance, if one node has degenerated to an ordinary tacnode and all the other nodes have stayed plain nodes, then the component has to be counted with multiplicity two. Again we will not try here to explain why (see [B] and [IV] instead), but only show how to take it in consideration for the applications.

(5.6) Remark. Same assumptions as in Proposition (5.5). If there is at most one tacnode (i.e. all $\tau_{l,m}$ but possibly one vanish, and this is equal to 1), the relative Severi variety V_δ is smooth at the general point of V (see [8, 9]), and thus V belongs to only one irreducible component of V_δ. There are other cases in which such a smoothness property holds (see [2] or, in this volume, [VI]).

If V_δ is smooth at the general point $D \in V$, the multiplicity of V in the limit Severi variety $\mathfrak{V}_\delta$ is the minimal integer m such that there are local analytic *m–multisections* of $V_\delta \to \mathbf{D}$, i.e. analytic smooth curves in V_δ, passing through D and intersecting the general fibre $V_{\delta,t}$, $t \in \mathbf{D}^*$, at m distinct points.

Proposition (5.5) does not provide a complete picture of the limit Severi variety. For instance, curves passing through a triple point of S_0 could (and in fact do; see [4], and §8.1 below) play a role in this limit. It would be desirable to know that one can always obtain a semistable model of the original family, where *every* irreducible component of the limit Severi variety is realized as a family of curves of the kind stated in Definition (5.3).

(5.7) Definition. *Let $f : \mathcal{S} \to \mathbf{D}$ be a semistable family as in §3, with a polarising line bundle $\mathcal{L}$, and δ a positive integer. The* regular part of the limit Severi variety $\mathfrak{V}_\delta(\mathcal{S}, \mathcal{L})$ *is the cycle in the limit linear system $\mathfrak{L} \subseteq \mathrm{Hilb}(\mathcal{L})$ defined as*

$$\mathfrak{V}_\delta^{\mathrm{reg}}(\mathcal{S}, \mathcal{L}) := \sum_W \sum_{|\boldsymbol{\delta}|+|\mathbf{m}^2|+\nu(\underline{\boldsymbol{\tau}})=\delta} \mu(\underline{\boldsymbol{\tau}}) \cdot \left(\sum_{V \in \mathrm{Irr}^\delta(V(W,\boldsymbol{\delta},\mathbf{m},\underline{\boldsymbol{\tau}}))} V \right), \tag{5.7.1}$$

(sometimes simply denoted by $\mathfrak{V}_\delta^{\mathrm{reg}}$) where:
(i) W varies among all effective divisors on S supported on the central fibre S_0, such that $\mathcal{L}(-W)$ is centrally effective and $h^0(\mathcal{L}_0(-W)) = h^0(L_t)$ for $t \in \mathbf{D}^*$;
(ii) $\mathrm{Irr}^\delta(Z)$ denotes the set of all codimension δ irreducible components of a scheme Z.

Proposition (5.5) asserts that the cycle $Z(\mathfrak{V}_\delta) - \mathfrak{V}_\delta^{\mathrm{reg}}$ is effective where $Z(\mathfrak{V}_\delta)$ is the cycle associated to $\mathfrak{V}_\delta$. We call the irreducible components of the support of $\mathfrak{V}_\delta^{\mathrm{reg}}$ the *regular components* of the limit Severi variety.

Let now $\tilde{f} : \tilde{\mathcal{S}} \to \mathbf{D}$ be a semistable model of a (not necessarily semistable) degeneration $f : \mathcal{S} \to \mathbf{D}$, and $\tilde{\mathcal{L}}$ the pull–back on $\tilde{\mathcal{S}}$ of a polarising line bundle $\mathcal{L}$ on $\mathcal{S}$. There is a natural map $\mathrm{Hilb}(\tilde{\mathcal{L}}) \to \mathrm{Hilb}(\mathcal{L})$, which induces a morphism $\phi : \tilde{\mathfrak{L}} \to |L_0|$.

(5.8) Definition. *The semistable model $\tilde{f} : \tilde{\mathcal{S}} \to \mathbf{D}$ is a* δ–good model *of $f : \mathcal{S} \to \mathbf{D}$ (or simply a* good model*, if it is clear which δ we are considering), if the following equality of cycles holds*

$$\phi_*\big(\mathfrak{V}_\delta^{\mathrm{reg}}(\tilde{\mathcal{S}}, \tilde{\mathcal{L}})\big) = \mathfrak{V}_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L}).$$

Note that the cycle $\mathfrak{V}_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L}) - \phi_*\big(\mathfrak{V}_\delta^{\mathrm{reg}}(\tilde{\mathcal{S}}, \tilde{\mathcal{L}})\big)$ is effective. The family $f : \mathcal{S} \to \mathbf{D}$ is said to be *δ–well behaved* (or simply *well behaved* if δ is understood) if it has a δ-good model.

(5.9) Remark. Suppose that $f : \mathcal{S} \to \mathbf{D}$ is δ–well behaved, with δ–good model $\tilde{f} : \tilde{\mathcal{S}} \to \mathbf{D}$. It is possible that some components in $\mathfrak{V}_\delta^{\mathrm{reg}}(\tilde{\mathcal{S}}, \tilde{\mathcal{L}})$ are contracted by ϕ to varieties of smaller dimension, and therefore their push–forwards to $\mathfrak{V}_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L})$ are zero. Hence these components of $\mathfrak{V}_\delta(\tilde{\mathcal{S}}, \tilde{\mathcal{L}})$ are *not visible* in $\mathfrak{V}_\delta^{\mathrm{cr}}(\mathcal{S}, \mathcal{L})$. They are however usually visible in the crude limit Severi variety of some other model $f' : \mathcal{S}' \to \mathbf{D}$, obtained from $\tilde{\mathcal{S}}$ via an appropriate twist of $\mathcal{L}$. The central fibre S_0' is then a flat limit of S_t, as $t \in \mathbf{D}^*$ tends to 0, different from S_0 (a situation met in Example (3.3)).

(5.10) Conjecture (See [4])**.** *Let $f : \mathcal{S} \to \mathbf{D}$ be a degeneration of surfaces, endowed with a line bundle $\mathcal{L}$ as above, and δ a positive integer. Then $f : \mathcal{S} \to \mathbf{D}$ is δ–well behaved.*

The local computations in [8, 9] provide a criterion for being well behaved:

(5.11) Proposition. *Assume there is a semistable model $\tilde{f} : \tilde{\mathcal{S}} \to \mathbf{D}$ of $f : \mathcal{S} \to \mathbf{D}$, with a limit linear system $\tilde{\mathfrak{L}}$ which does not contain, in codimension $\delta + 1$, curves of the following types:*
(i) curves containing double curves of $\tilde{S}_0$;
(ii) curves passing through a triple point of $\tilde{S}_0$;
(iii) non–reduced curves.
If in addition, for $W, \boldsymbol{\delta}, \mathbf{m}, \underline{\boldsymbol{\tau}}$ as in Definition (5.3), every irreducible component of $V(W, \boldsymbol{\delta}, \mathbf{m}, \underline{\boldsymbol{\tau}})$ has the expected codimension in $|L_0(-W)|$, then $\tilde{f} : \tilde{\mathcal{S}} \to \mathbf{D}$ is δ–well behaved.

(5.12) Remark. While there are good reasons to believe that Conjecture (5.10) has an affirmative answer, one cannot expect that, in the algebraic category, there is a good model of a given family, which *universally* works for all polarising line bundles $\mathcal{L}$. This in fact could require infinitely many birational modifications of the total space.

On the other hand, it is hopeless to ask for a semistable model on which all irreducible components of the limit Severi variety would parametrize nodal curves (i.e., such that there are no tacnodes as in Item (iv) of Definition (5.3)): we explain this in [B], together with C. Galati.

The three final sections are devoted to examples which will hopefully clarify the considerations and results that we have presented here.

6 – Example: Limits of 1–nodal plane sections for surfaces in $\mathbf{P}^3$

Consider a degeneration of a general degree d surface S in $\mathbf{P}^3$ to a reducible surface $S' \cup S''$, as in Example (3.3), from which we keep the notation. Here $S_0 = S' \cup S''$, and $R = S' \cap S''$ is a smooth curve of degree $h(d-h)$. Then in the semistable model $\tilde{f} : \tilde{\mathcal{S}} \to \mathbf{D}$ constructed in Example (3.3), $\tilde{S}_0 = Q_1 \cup Q_2$ where Q_2 is isomorphic to S'' and Q_1 is the blow-up of S' at the $q = dh(d-h)$ points $e_1, \ldots, e_q$ of $R \cap S_\infty$, with exceptional curves $E_1, \ldots, E_q$, and Q_1 and Q_2 intersect transversally along a curve which may be identified with R. This degeneration is described more explicitly in Sections 7.1 and 7.2 for $d = 4$ and $h = 2$ and 1, respectively.

We want to describe the regular part of the limit Severi variety of 1-nodal curves of $(\mathcal{S}, \mathcal{L})$, where $\mathcal{L}$ is the pull-back of $\mathcal{O}_{\mathbf{P}^3}(1)$. Thus we need to list all the types of limits for one node that are considered in Definition (5.3): the limit of a node may be either a node at a general point of Q_1 or Q_2, or a tacnode along the double curve R, or a node on Q_1 due to the fact that the limiting curve on Q_1 contains one exceptional curve E_i with multiplicity one.

In the case of the trivial twist $\mathcal{L}$, we are simply considering hyperplane sections of $S' \cup S''$. Then, the hyperplanes defining a curve with a node at a general point of Q_1 or Q_2 are those that are tangent to S' or S'', respectively; those defining a curve with a tacnode along R are those that are tangent to R; finally, those defining a curve with a node at a general point of an exceptional curve E_i are those passing through the corresponding point of $S' \cap S'' \cap S_\infty$.

We describe $\mathfrak{V}_1^{\mathrm{reg}}$ along these lines in the next paragraphs. Before we do so, let us pause to make the following observation.

(6.1) Remark. While it is arguably not obvious that a curve with a node on S_∞ may degenerate to a curve with a tacnode along $R = S' \cap S''$ on $S_0 = S' \cup S''$, it is nevertheless fairly natural that a tangent plane to S_∞ may degenerate to a plane that is tangent to the curve $R = S' \cap S''$, or in other words that planes tangent to R count as tangent planes to $S' \cup S''$.

(6.2) Assume $h \geqslant 2$. Then the only twist which is centrally effective is the trivial one, and $\mathfrak{V}_1^{\mathrm{reg}}$ consists of the following $3+q$ components:
(1) $V(\delta_{Q_i} = 1)$, with $i = 1, 2$, with multiplicity 1;
(2) $V(\tau_{R,2} = 1)$, with multiplicity 2;
(3) $V(E_i)$, for $i = 1, \ldots, q$, with multiplicity 1.

This tells us that in the degeneration $\check{\mathcal{S}} \to \mathbf{D}$, the limit of the dual variety $\check{S}$, which sits in $|\mathcal{L}_0| = \check{\mathbf{P}}^3$ and coincides with the crude limit of the Severi variety $\mathfrak{V}_1^{\mathrm{cr}}$, contains:
(i) the dual varieties $\check{S}'$ and $\check{S}''$ with multiplicity 1 (corresponding to (1));
(ii) the dual variety $\check{R}$ with multiplicity 2 (corresponding to (2));
(iii) the q planes $e_i^{\perp} \subseteq \check{\mathbf{P}}^3$, orthogonal to the points e_i, for $i = 1, \ldots, q$, with multiplicitiy 1 (corresponding to (3)).

One has:
(a) $\deg(\check{S}') = h(h-1)^2$ and $\deg(\check{S}'') = (d-h)(d-h-1)^2$, see Section 2;
(b) $\deg(\check{R}) = h(d-h)(d-2)$, see [XII];
(c) $\deg(e_i^{\perp}) = 1$ for all $i = 1, \ldots, q$.
So, adding up, we see that the sum of the degrees of the components in (i)–(iii) above, counted with the appropriate multiplicities, is $d(d-1)^2$, hence the components in (1)–(3) exhaust the limit Severi variety $\mathfrak{V}_1$, and the degeneration is 1–well behaved.

(6.3) If $h = 1$, the analysis is different. First off, S' is a plane, hence $\check{S}'$ is a point and cannot contribute directly to the limit Severi variety. On the other hand, we ought to consider the non–trivial twist $\mathcal{L}(-Q_1)$ which is now centrally effective. Thus, the 1-nodal curves in the linear system of plane curves $|\mathcal{O}_{Q_1}(dH - \sum_{i=1}^{d(d-1)} E_i)|$ (see Example (3.3)) will contribute to the regular part of the limit Severi variety: this is case (β) below. The upshot is that $\mathfrak{V}_1^{\mathrm{reg}}$ consists of the following components when $h = 1$:
(α) $V(\delta_{Q_2} = 1)$, with multiplicity 1 (note that $V(\delta_{Q_1} = 1)$ is zero);
(β) $V(Q_1, \delta_{Q_1} = 1)$, with multiplicity 1;
(γ) $V(\tau_{R,2} = 1)$, with multiplicity 2;
(δ) $V(E_i)$, for $i = 1, \ldots, q$, with multiplicity 1.

However, only the components in $(\alpha), (\gamma)$ and (δ) contribute to the crude limit of the Severi variety $\mathfrak{V}_1^{\mathrm{cr}}$ because the component in (β) is not visible in the crude limit, since it is contracted to the point $(S')^{\perp}$ in $|\mathcal{L}_0| = \check{\mathbf{P}}^3$ (see Remark (5.9)). More generally, δ-nodal curves in $|\mathcal{O}_{Q_1}(dH - \sum_{i=1}^{d(d-1)} E_i)|$ contribute to $\mathfrak{V}_\delta^{\mathrm{cr}}$ only if $\delta = 3$; we will see an example in Section 7.2.2.

The components in $(\alpha), (\gamma)$ and (δ) nevertheless exhaust $\mathfrak{V}_1^{\mathrm{cr}}$, by the exact same computation as above (check this for yourself!). In conclusion the degeneration is 1–well behaved when $h = 1$ as well, and in this case the limit of $\check{S}$ is the union of $\check{S}''$, plus $d(d-1)$ planes, plus the dual of the plane curve R counted with multiplicity 2.

7 – Example: Limits of 3–nodal plane sections for quartic surfaces in $\mathbf{P}^3$

The number of 3–nodal plane sections of a general quartic surface S in $\mathbf{P}^3$ (or of triple points of the dual $\check{S}$) given by Salmon's formula (2.0.1) is 3200.

7.1 – Degeneration to two quadrics

Let us recover this first by considering a degeneration to two quadrics: thus we will look at the case $h = d - h = 2$ in Example (3.3) (from which we keep the notation; this is the same as the

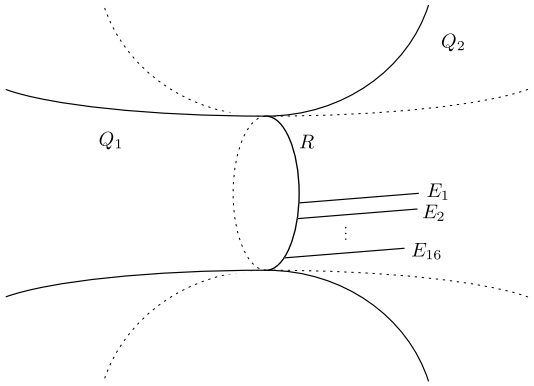


Figure 4: Degeneration into two quadrics

degeneration in Example (3.4) for $p = 3$), see also Section 6.

In this case R is an elliptic normal quartic curve, the semistable degeneration has central fibre $Q_1 \cup Q_2$, with $R = Q_1 \cap Q_2$, Q_2 is a quadric, and Q_1 a quadric blown up at the complete intersection points $e_1, \ldots, e_{16}$ of R with a general quartic surface S (see Figure 4).

For the computation, one has to consider the various cases for $V(W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$ as in Definition (5.3), see also Notation (5.4). No non–trivial twist is centrally effective, so $W = 0$. If $|\boldsymbol{\delta}| = 3$, then $V(W, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$ is empty, because one cannot have two (or more) singular points on a plane section of a smooth quadric. Hence:

- $0 \leqslant |\boldsymbol{\delta}| \leqslant 2$, and if $|\boldsymbol{\delta}| = 2$ then $\delta_1 = \delta_2 = 1$;
- $\tau = (\tau_2, \tau_3, \tau_4)$ and $|I| + 2\tau_2 + 3\tau_3 + 4\tau_4 \leqslant 4$;
- $|\boldsymbol{\delta}| + |I| + \tau_2 + 2\tau_3 + 3\tau_4 = 3$.

Therefore the cases to be discussed are the following.

Case 1: $\delta_1 = \delta_2 = 1$, $|I| = 1$.

For each $i = 1, \ldots, 16$, a plane tangent to both S' and S'', and passing through e_i in $\mathbf{P}^3$ is entirely determined by the choice of a line passing through e_i in both Q_1 and Q_2. There are exactly $2 \times 2 = 4$ such choices (two rulings on each quadric). We get $16 \times 4 = 64$ curves.

Case 2: $\delta_1 = \delta_2 = 1$, $\tau_2 = 1$.

The intersection $\check{S}' \cap \check{S}'' \cap \check{R}$ consists of $2 \times 2 \times 8 = 32$ points. From this, we want to remove the points corresponding to planes tangent to one quadric at some point lying on R. Each of them has multiplicity 2 in the intersection $\check{S}' \cap \check{S}'' \cap \check{R}$ by the:

(7.1) Lemma (Lemma 3.5, [4])**.** *Let R be a smooth, irreducible curve contained in a smooth surface S in $\mathbf{P}^3$. Let $\check{R}_S$ be the irreducible curve in $\check{\mathbf{P}}^3$ parametrizing planes tangent to S along R. Then the dual varieties $\check{S}$ and $\check{R}$ both contain $\check{R}_S$, and do not intersect transversally at its general point.*

Proof. Clearly $\check{R}_S$ is contained in $\check{S} \cap \check{R}$. If either $\check{S}$ or $\check{R}$ are singular at the general point of $\check{R}_S$, there is nothing to prove. Assume that $\check{S}$ and $\check{R}$ are both smooth at the general point of $\check{R}_S$. One has to show that they are tangent there. Let $x \in R$ be general. Let H be the tangent plane to S at x. Then $H \in \check{R}_S$ is the general point. The biduality theorem (see , e.g., [11, Example 16.20]) says that the tangent planes to $\check{S}$ and $\check{R}$ at H both coincide with the set of planes in $\mathbf{P}^3$ containing x, hence the assertion. □

The curve parametrizing planes tangent to S' at some point of R has degree 4 (to see this, intersect R with a general polar plane of S', cf. [XII, Appendix A.1]). Hence this curve intersects

$\check{S}''$ at 8 distinct points (without multiplicities). Eventually, we have to remove $2 \times 2 \times 8$ points from $\check{S}' \cap \check{S}'' \cap \check{R}$, and there does not remain anything.

Case 3: $\delta_i = 1$ *for an* $i = 1, 2$, $|I| = 2$.

For each $i = 1, 2$ and each set $\{r, s\} \subseteq I$, $r \neq s$, we count points of intersection of $\check{Q}_i$ with the line of $\check{\mathbf{P}}^3$ orthogonal to the line $\langle e_r, e_s \rangle$. There are $2 \times \binom{16}{2} \times 2 = 480$ such points.

Case 4: $\delta_i = 1$ *for an* $i = 1, 2$, $|I| = 1$, $\tau_2 = 1$.

For each $i = 1, 2$ and $r = 1, \ldots, 16$, there are two lines on Q_i through the point e_r. They determine two pencils of planes tangent to Q_i and passing through e_r.

The planes of each of these pencils cut out a g^1_2 on R: the intersection of such a plane with R consists of 4 points, two of which are fixed (the two intersection points of the chosen line through e_r with R).

Since a g^1_2 on an elliptic curve has 4 double points, we get 2×4 planes satisfying our conditions for each i and r. These give a contribution of $2 \times 16 \times 2 \times 8 = 512$ to our computation.

Case 5: $\delta_i = 1$ *for an* $i = 1, 2$, $\tau_2 = 2$.

Planes satisfying these conditions are spanned by two lines belonging respectively to the two rulings of Q_i, both tangent to R. By genericity, two such lines do not meet on R.

Each ruling of Q_i cuts out a g^1_2 on R, and thus contains exactly 4 tangent lines to R (since a g^1_2 on R has 4 ramification points). We thus get 16 planes satisfying our conditions for each i. This contributes $2 \times 4 \times 16 = 128$ to our computation.

Case 6: $\delta_i = 1$ *for an* $i = 1, 2$, $\tau_3 = 1$.

This requires planes spanned by two lines on Q_i, one of which is tangent to R, and meets the other line on R at this tangency point. Such a plane is tangent to Q_i at a point lying on R, and therefore cuts a curve with worse singularity than required. Hence they do not contribute to our computation.

Case 7: $\tau_4 = 1$.

Since R is a smooth degree 4 elliptic curve in $\mathbf{P}^3$, the set of its hyperplane sections is

$$\{(a, b, c, d) \in \mathrm{Sym}^4(R) \mid a + b + c + d = 0 \in (R, +)\}.$$

Hence, the planes we are looking for are in 1-1 correspondence with 4-torsion points of R and so there are 16 of them, each to be counted with multiplicity 4, so we get a contribution of 64.

Case 8: $|I| = 1$, $\tau_3 = 1$.

Fix $r = 1, \ldots, 16$, and consider the projection π from e_r. It maps R to a smooth cubic plane curve. Since this has 9 flexes, this gives us 9 planes through e_r satisfying the required condition. Each of them has to be counted with multiplicity 3, and this gives a contribution of $16 \times 3 \times 9 = 432$ to our computation.

Case 9: $|I| = 2$, $\tau_2 = 1$.

For given r, s distinct in $\{1, \ldots, 16\}$, the pencil of planes through e_r and e_s cuts out (off of $e_r + e_s$) a g^1_2 on R. It has 4 double points, hence the pencil contains 4 tangent planes to R. By genericity, the tangency points are distinct from e_r and e_s. This gives a contribution of $\binom{16}{2} \times 2 \times 4 = 960$ to the computation.

Case 10: $|I| = 3$.

For each choice of three different indices r, s and l in $\{1, \ldots, 16\}$, there is one plane through e_r, e_s and e_l. So we get a contribution of $\binom{16}{3} = 560$.

Adding up all the non–trivial contributions as in the following table, we get exactly 3200, which is the number predicted by Salmon's formula. This shows that this degeneration is 3–good.

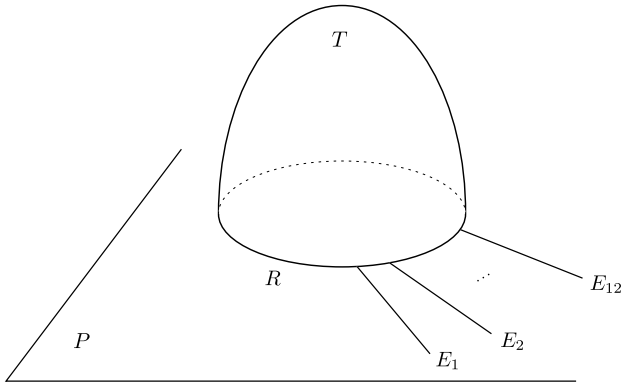


Figure 5: Degeneration into a cubic plus a plane

Cases	Contribution
Case 1	64
Case 3	480
Case 4	512
Case 5	128
Case 7	64
Case 8	432
Case 9	960
Case 10	560
Total	**3200**

7.2 – Degeneration to a cubic and a plane

We will now do the same enumeration as in Section 7.1, this time by considering a degeneration to a cubic and a plane. Thus, we want to count 3–nodal plane sections of a general quartic surface S in $\mathbf{P}^3$, and we will look at the degeneration in Example (3.3), with $d = 4$ and $h = 1$. This will be instructive, because it will require computing the degree of a Severi variety of plane nodal curves in one of the first cases for which this cannot be done with elementary means.

We simplify a bit the notation of Example (3.3). We denote by $T + P_0$ the reducible surface S_0 in Example (3.3) (so T is a general cubic, P_0 a general plane, with $P_0 \cap T = R$ a general plane cubic), and we denote by $T + P$ the central fibre of the semistable degeneration, where P is the plane P_0 blown–up at 12 points $e_1, \ldots, e_{12}$ forming a general divisor in $|\mathcal{O}_R(4)|$ (see Figure 5).

Now we explain in some detail in this case what we stated in Example (4.2) about the limit linear system $\mathfrak{L}$ of the plane sections of S_t.

(7.2) Proposition. *Let $\tilde{\mathcal{H}}_{\mathbf{P}^3}$ be the blow up of $|\mathcal{O}_{\mathbf{P}^3}(1)| = \check{\mathbf{P}}^3$ at the point corresponding to the plane $P_0 \subseteq \mathbf{P}^3$. Let $D_{P_0} \cong \mathbf{P}^2$ be the exceptional divisor of this blow up.*

Set $\mathcal{H}_P = |\mathcal{O}_P(4H - \sum_{i=1}^{12} E_i)| \cong \mathbf{P}^3$ and denote by $D_R \subseteq \mathcal{H}_P$ the plane parametrizing strict transforms of plane quartic curves that contain R (H denotes the pull back to P of a general line of P_0).

Then the limit of the limit linear system $\mathfrak{L}$ of $|\mathcal{O}_{S_t}(1)|$ as t tends to 0 is obtained by glueing $\tilde{\mathcal{H}}_{\mathbf{P}^3}$ and $\mathcal{H}_P$ along D_{P_0} and D_R.

Proof. One has $\dim(|4H - \sum_{i=1}^{12} E_i|) = 3$ (see Example (3.3) and Remark (7.3) below).

There is only one non–trivial twist of the polarising hyperplane bundle $\mathcal{L}$, i.e., the twist by P, which is centrally effective. As we saw in Example (3.3), the twist $\mathcal{L}(-P)$ is trivial on T, whereas on P it restricts to $\mathcal{O}_P(H + R) = \mathcal{O}_P(4H - \sum_{i=1}^{12} E_i)$.

Up to a multiplicative constant, there is only one non–zero section of $\mathcal{L}_{|T+P}$ vanishing on P, hence it gives no element in $\mathfrak{L}$. To see the corresponding curves in $\mathfrak{L}$ we have to blow up $|\mathcal{L}_{|T+P}| = |\mathcal{O}_{\mathbf{P}^3}(1)|$ at P_0. One has

$$T_{\check{\mathbf{P}}^3,[P_0]} = H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))/H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}) \cong H^0(\mathbf{P}^3, \mathcal{O}_{P_0}(1)).$$

This identifies $D_{P_0} \cong \mathbf{P}(T_{\check{\mathbf{P}}^3,[P_0]})$ with the 2-dimensional linear system $|\mathcal{O}_{P_0}(1)|$. To give a geometric interpretation of this, let us look at the limits of the curves cut out on $T + P_0$ by a plane Π of $\mathbf{P}^3$ tending to P_0. The limit curve $\Pi \cap T$ tends to R, while the limit curve of $\Pi \cap P_0$ is a line moving freely in $|\mathcal{O}_{P_0}(1)|$.

On the other hand, the plane D_R is

$$|4H - R| \cong |\mathcal{O}_P(H)| \subseteq |4H - \textstyle\sum_{i=1}^{12} E_i|.$$

There is therefore a natural identification between D_{P_0} and D_R, with the glueing as in the statement. □

(7.3) Remark. As mentioned in Example (3.3), Paragraph (3.3.2), the surface P is mapped to a quartic with a triple point in $\mathbf{P}^3$ by the linear system $|4H - \sum_{i=1}^{12} E_i|$. This mapping is an isomorphism outside R, and contracts R to the triple point.

Indeed, one has an exact sequence

$$0 \to \mathcal{O}_P(H) \to \mathcal{O}_P(4H - \sum_{i=1}^{12} E_i) \to \mathcal{O}_R \to 0.$$

Since $h^1(P, \mathcal{O}_P(H)) = h^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) = 0$, the sequence is exact on global sections, which implies $h^0(P, \mathcal{O}_P(4H - \sum_{i=1}^{12} E_i)) = 4$.

The linear system $|4H - \sum_{i=1}^{12} E_i|$ has no base points and the morphism determined by it is birational onto its image (we leave this to the reader to check). The image of P via this map has degree 4, since $(4H - \sum_{i=1}^{12} E_i)^2 = 4$. The map contracts $R \equiv 3H - \sum_{i=1}^{12} E_i$ because $(4H - \sum_{i=1}^{12} E_i) \cdot (3H - \sum_{i=1}^{12} E_i) = 0$, and its image is a triple point because $R^2 = (3H - \sum_{i=1}^{12} E_i)^2 = -3$.

7.2.1 Curves in the hyperplane bundle

According to the structure of the limit linear system $\mathfrak{L}$, one has to compute the limit of trinodal curves appearing in both $\tilde{\mathcal{H}}_{\mathbf{P}^3}$ and in $\mathcal{H}_P$. First we perform the computation for $\tilde{\mathcal{H}}_{\mathbf{P}^3}$.

Here one has to consider the various cases for $V(0, \boldsymbol{\delta}, I, \underline{\boldsymbol{\tau}})$ as in Definition (5.3) and Notation (5.4), with $I \subseteq \{1, \ldots, 12\}$. Since there cannot be any node on P in this linear system, one has $|\boldsymbol{\delta}| = \delta$, the number of nodes on T. Then:

- $0 \leqslant \delta \leqslant 3$ and $\tau = (\tau_2, \tau_3)$;
- $|I| + 2\tau_2 + 3\tau_3 \leqslant 3$;
- $\delta + |I| + \tau_2 + 2\tau_3 = 3$.

Hence the cases to be discussed are the following.

Case 1: $\delta = 3$.

Tritangent planes to the general cubic surface T correspond to the triangles contained in T. These are 45: the reader may apply Salmon's formula (2.0.1) or, better, directly compute this number as an exercise.

Case 2: $\delta = 2$, $|I| = 1$.

Binodal hyperplane sections of T must contain a line. Since T contains 27 lines, there are 27 pencils of such curves. In each of these pencils, and for every $r \in \{1, \dots, 12\}$, there is exactly one curve passing in addition through the point e_r. This gives a contribution of $27 \times 12 = 324$ to the computation.

Case 3: $\delta = 2$, $\tau_2 = 1$.

Each of the 27 pencils of bitangent planes to T cuts out a g^1_2 on R, off the fixed intersection of the line with R.

A g^1_2 on an elliptic curve possesses 4 ramification points. Each of this contributes to the computation, with multiplicity 2, hence the total contribution is $27 \times 4 \times 2 = 216$.

Case 4: $\delta = 1$, $|I| = 2$.

For any pair of distinct indices r and s in $\{1, \dots, 12\}$, the pencil of planes passing through e_r and e_s contains $12 = \deg(\check{T})$ tangent planes to T. The contribution is then $\binom{12}{2} \times 12 = 792$.

Case 5: $\delta = 1$, $|I| = 1$, $\tau_2 = 1$.

For each $r = 1, \dots, 12$, consider the projection π_r from the point e_r. It gives a double cover $\tilde{T} \to \mathbf{P}^2$ branched along a plane quartic curve B, where $\tilde{T}$ is the blow up of T at e_r (the curve B is the projection from e_r of the curve $T \cap \mathrm{D}^{e_r}T$, which has degree 6 and multiplicity 2 at e_r, see [XII, Section 2.2]; by $\mathrm{D}^{e_r}T$ we denote the first polar of T with respect to the point e_r). Being contained in the plane P_0, which passes through e_r, R is mapped 2-1 to a line $\ell_R \subseteq \mathbf{P}^2$, which is nowhere tangent to B.

Tangent planes to T containing e_r in $\mathbf{P}^3$ map via π_r to tangent lines to B. Tangent planes to R containing e_r map to lines passing through one of the 4 points of $\ell_R \cap B$.

Let p be a point in $\ell_R \cap B$. The pencil of lines through p in $\mathbf{P}^2$ contains $12 = \deg(\check{B})$ tangent lines to B. Among them there is the tangent line to B at q, with multiplicity 2 (this can be seen as in Lemma (7.1)), which corresponds to a plane tangent to T at a point lying on R in $\mathbf{P}^3$. Such a plane gives a section of $T + P$ with a singularity worse than a tacnode, so it has to be discarded from the computation.

Finally, the contribution in this case is $12 \times 4 \times 2 \times 10 = 960$.

Case 6: $\delta = 1$, $\tau_3 = 1$.

The plane cubic R possesses 9 flexes. For each of them, there is the pencil of planes containing the tangent line to R at that flex. Such a pencil is a line in $\check{\mathbf{P}}^3$, meeting $\check{T}$ at 12 points. One needs to remove from these 12 points the one corresponding to the tangent plane to T at the flex of R. It is of multiplicity 3 (the proof of this fact is similar to the one of Lemma (7.1) and we leave it to the reader).

Planes satisfying the given conditions have to be counted with multiplicity 3. So we get a contribution of $3 \times 9 \times 9 = 243$.

The table below shows the result of the total summation.

Cases	Contribution
Case 1	45
Case 2	324
Case 3	216
Case 4	792
Case 5	960
Case 6	243
Total	**2580**

7.2.2 Curves in the twisted hyperplane bundle: plane quartics

Since Salmon's formula (2.0.1) gives a total of 3200 tritangent planes to a general quartic, we are missing a total of $3200 - 2580 = 620$ limit trinodal curves. These have to be seen in $\mathcal{H}_P = |4H - \sum_{i=1}^{12} E_i|$. This system however contains the 2–dimensional system D_R of trinodal curves, which is the plane along which $\mathcal{H}_P$ and $\tilde{\mathcal{H}}_{\mathbf{P}^3}$ glue. So these curves do not contribute to the computation. Hence, we have to compute the number of trinodal curves in $|4H - \sum_{i=1}^{12} E_i|$ off D_R. All such curves will be irreducible, since, by the genericity assumptions, the only reducible curves in $|4H - \sum_{i=1}^{12} E_i|$ are the ones in D_R (the reader is invited to prove this).

The linear system $|4H - \sum_{i=1}^{12} E_i|$ is mapped to $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_D|$ on the plane P_0, where $D = \{e_1, \ldots, e_{12}\}$ and $e_1 + \cdots + e_{12} \in |\mathcal{O}_R(4)|$ is a general divisor. The varieties of δ-nodal curves not containing R in this linear system are logarithmic Severi varieties of the pair $(\mathbf{P}^2, R)$: on logarithmic Severi varieties, see [III], and in particular [III, Section 5.3] about the particular kind we are considering right now.

Since $\dim(|\mathcal{O}_{\mathbf{P}^2}(4)|) = 14$, D imposes only 11 conditions to $|\mathcal{O}_{\mathbf{P}^2}(4)|$. Since passing through a point imposes at most one condition, there is a set Y of 11 points in D that imposes the 11 conditions (in fact, by genericity of $e_1 + \cdots + e_{12} \in |\mathcal{O}_R(4)|$, any subset of 11 points does). Hence

$$|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_D| = |\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Y|.$$

Finally, Y can be seen as the limit of $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Z|$, where Z is a general set of 11 points in the plane.

We will compute the number of trinodal curves in $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Z|$. Their limits, when Z tends to Y, will be trinodal curves in $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Y| = |\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_D|$, and this gives the required information on the number we want to compute.

Finally what we have to compute is the number of trinodal quartics passing through 11 general points in the plane, which is the degree of the codimension 3 family of trinodal curves in $|\mathcal{O}_{\mathbf{P}^2}(4)| \cong \mathbf{P}^{14}$, i.e., the degree of the appropriate Severi variety. This has been done, in general, by Caporaso and Harris in [2, 3] (for rational curves this is also given by the earlier (and different) Kontsevich formula, see [7], or [17]). Their result in this case reads as follows:

(7.4) Proposition. *If Z is a general set of 11 points in $\mathbf{P}^2$, then the 3–dimensional linear system $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_Z|$ contains 675 trinodal curves:*
(i) 620 of them are irreducible;
(ii) $55 = \binom{11}{2}$ are reducible in a line joining 2 of the points of Z and the cubic through the remaining 9 points of Z.

When Z tends to Y as above, the 55 reducible quartics as in (ii) of Proposition (7.4) tend to curves in D_R, which ought to be discarded. The limit of the 620 curve in (i) is the missing contribution we need.

It would be however unsatisfactory at this point to rely on the (quite difficult) general Caporaso–Harris' result to finish the computation. And in fact we can perform it *our way*, by using a new degeneration.

The degeneration we want to use is the following. Consider the blow up $\mathcal{S}$ of $\mathbf{D} \times \mathbf{P}^2$ along a general line E of $\mathbf{P}^2$ in the *central fibre* over $0 \in \mathbf{D}$. Then there is a morphism $f : \mathcal{S} \to \mathbf{D}$ which is a semistable degeneration of $S_t = \mathbf{P}^2$ for $t \neq 0$ to $S_0 = P \cup F$, with $P \cong \mathbf{P}^2$ the proper transform of the original central fibre and F the exceptional divisor of the blow up (see Figure 6).

Note that $F \cong \mathbf{F}_1$, i.e., the plane blown up at a point p, and $P \cap F = E$, with $E^2 = -1$ on F (i.e., the exceptional divisor corresponding to p), according to the Triple Point Formula in Lemma (3.2).

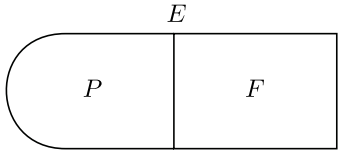


Figure 6: the degeneration of the plane

There is an obvious polarizing bundle $\mathcal{L}$ on $\mathcal{S}$, which restrict to $\mathcal{O}_{\mathbf{P}^2}(1)$ on the general fibre and to L_0 on the central fibre, with the aspects

$$\mathcal{L}_P = \mathcal{O}_{\mathbf{P}^2}(1), \quad \mathcal{L}_F = \mathcal{O}_F(f),$$

where f is a fibre of the morphism $\pi : F \cong \mathbf{F}_1 \to \mathbf{P}^1$.

Now pick a set $A = \{\alpha_1, \ldots, \alpha_6\}$ of six general points on P and a set $B = \{\beta_1, \ldots, \beta_5\}$ of five general points on F. Then $Z_0 := A \cup B$ is the limit of a set Z_t of 11 general points on the general fibre S_t. Denote by $\mathcal{Z}$ the curve $\{Z_t\}_{t\in\mathbf{D}}$ in $\mathcal{S}$. Consider $\mathcal{L}^{\otimes 4} \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$, which restricts to $\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_{Z_t}$ on the general fibre S_t. In order to have information about trinodal curves in $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_{Z_t}|$, we have to look at their limits in the limit linear system $\mathfrak{L}$ of $\mathcal{L}^{\otimes 4} \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$.

(7.5) Lemma. *There is only one twist of $\mathcal{L}^{\otimes 4} \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$ which is centrally effective, namely $\mathcal{L}^{\otimes 4} \otimes \mathcal{O}_S(-F) \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$.*

Proof. Any twist is of the form $\mathcal{L}^{\otimes 4} \otimes \mathcal{O}_S(-aF) \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}}$ and on S_0 this resticts to:
(i) $\mathcal{O}_{\mathbf{P}^2}(4-a) \otimes \mathcal{I}_A$ on $P \cong \mathbf{P}^2$;
(ii) $\mathcal{O}_{\mathbf{F}_1}(4f + aE) \otimes \mathcal{I}_B$ on $F \cong \mathbf{F}_1$. As $\mathbf{F}_1$ is the blow up of $\mathbf{P}^2$ at p, then (with a slight abuse of notation) one has $|\mathcal{O}_{\mathbf{F}_1}(4f + aE) \otimes \mathcal{I}_B| \cong |\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_B \otimes \mathcal{I}_p^{\otimes 4-a}|$.

Since B is a general set of 5 points on F, for the effectivity on F one needs $a \geqslant 1$, because $\dim(|\mathcal{O}_{\mathbf{F}_1}(4f)|) = 4$. Similarly, for the effectivity on P we need $a \leqslant 1$, because $|\mathcal{O}_{\mathbf{P}^2}(h) \otimes \mathcal{I}_A|$ is empty for $h \leqslant 2$. This proves the assertion. □

In conclusion, we have to look at limits of trinodal curves only in $|\mathcal{L}^{\otimes 4} \otimes \mathcal{O}_{\mathcal{S}}(-F) \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}} \otimes \mathcal{O}_{S_0}|$, which is:
(i) $\mathcal{L}_P := |\mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_A|$, the aspect on P, of dimension 3;
(ii) $\mathcal{L}_F := |\mathcal{O}_{\mathbf{F}_1}(4f + E) \otimes \mathcal{I}_B|$, the aspect on F, which is the same as $|\mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I}_p^{\otimes 3} \otimes \mathcal{I}_B|$ in the identification of F with the blow-up of $\mathbf{P}^2$ at a point p, also of dimension 3;
(iii) the two curves on P and F in the two aspects have to *match* along E, i.e., they have to cut out on E the same divisor. This *matching condition* implies dimension 3 for $|\mathcal{L}^{\otimes 4} \otimes \mathcal{O}_{\mathcal{S}}(-F) \otimes \mathcal{I}_{\mathcal{Z}|\mathcal{S}} \otimes \mathcal{O}_{S_0}|$: indeed, given a curve in $|\mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_A|$, which depends on 3 parameters, this fixes the degree 3 divisor it cuts on E and there is only one curve in $|\mathcal{O}_{\mathbf{F}_1}(4f - E) \otimes \mathcal{I}_B|$ matching it.

Next we introduce the self-explanatory notation $\boldsymbol{\delta} = (\delta_P, \delta_F)$ and $\tau = (\tau_2, \tau_3)$ (in this case there is no I to be considered) and

$$|\boldsymbol{\delta}| + \tau_2 + 2\tau_3 = 3.$$

It is useful to notice that a curve in $|\mathcal{O}_{\mathbf{F}_1}(4f + E)| \otimes \mathcal{I}_B$ with a node at a point $x \in F$, splits in the unique element f_x of $|f|$ through x, and in a curve of $|\mathcal{O}_{\mathbf{F}_1}(3f + E) \otimes \mathcal{I}_B|$ (which is the same as $|\mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_B \otimes \mathcal{I}_p^{\otimes 2}|$) passing through x.

Now the cases to be analyzed are the following. We will be sketchy, leaving the details to the reader. The corresponding analysis for the significantly simpler enumeration of 2-nodal plane quartics is carried out in [VII, Section 4], and the reader may wish to consult that first. Also, we encourage her or him to compute in the same way the number of 1-nodal plane quartics through

13 general points (the result is $27 = 3 \cdot 3^2$), and even the number of 1-nodal cubics through 8 general points (the result is $12 = 3 \cdot 2^2$).

Case 1: $\delta_F = 3$.

A trinodal curve on F contains three curves in $|f|$ with residual in $|\mathcal{O}_{\mathbf{F}_1}(f+E)|$, which is the same as $|\mathcal{O}_{\mathbf{P}^2}(1)|$ (with the curves in $|f|$ mapping to lines through p). This implies that the three splitting curves in $|f|$ have to contain three points of B, and the residual in $|\mathcal{O}_{\mathbf{F}_1}(f+E)|$ contains the remaining 2. This fixes the curve on F and accordingly also the curve on P. This shows that the contribution in this case is $\binom{5}{2} = 10$.

Case 2: $\delta_F = 2, \delta_P = 1$.

Two curves in $|f|$ split from the curve on F and either only one or exactly two points of B lie on splitting elements of $|f|$. If two points β_1, β_2 of B lie on splitting elements of $|f|$, these curves cut E each in one point b_1, b_2. Then the matching curves on P lie in the pencil of cubics $|\mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_{A\cup\{b_1,b_2\}}|$ which, as well known, contains exactly 12 nodal curves (the reader may be asked to verify this as an exercise). Each such curve cuts a divisor on E (which contains $b_1 + b_2$), and the matching uniquely determines the remaining component of the curve on F, which lies in the pencil $|\mathcal{O}_{\mathbf{F}_1}(2f+E) \otimes \mathcal{I}_{B-\{\beta_1,\beta_2\}}| = |\mathcal{O}_{\mathbf{P}^2}(2) \otimes \mathcal{I}_{B-\{\beta_1,\beta_2\}\cup\{p\}}|$. This gives a contribution of $\binom{5}{2} \times 12 = 120$ to our computation.

The reader may check, with similar arguments, that if only one point of B lies on a splitting element of $|f|$ (and there are 5 possibilities for this), then one also has a well determined pencil of curves on P with 12 nodal curves, hence we have $5 \times 12 = 60$ more limit trinodal curves of this type, for a total of 180.

Case 3: $\delta_F = 2, |\tau| \neq 0$.

Since, as above, two curves in $|f|$ split from the curve on F, it is not possible to have a tangency to E, so this case gives no contribution to the computation.

Case 4: $\delta_F = 1, \delta_P = 2$.

Only one curve in $|f|$ splits from the curve on F, and there are two possible cases:
(a) this curve does not contain any point of B;
(b) this curve contains one point of B (this depends on 5 choices).

Also the curve on P is reducible in a conic and a line, and there are two possible cases:
(a') 5 of the points in A lie on the conic, and one lies on the line (this depends of 6 choices);
(b') 4 of the points in A lie on the conic, and two lie on the line (this depends of 15 choices).

One has that:
Case (a,a') contributes with 12 curves,
Case (a,b') contributes with 30 curves,
Case (b,a') contributes with 30 curves,
Case (b,b') contributes with 75 curves, for a total contribution of 147.
Let us check case (b,b'), and leave the others to the reader. Each of the 15 choices of 2 points in A determines a unique line, and then the residual conic through the 4 remaining points moves in a pencil. Each choice of one point in B determines a unique curve in $|f|$, and then the residual curve in $|3f+E|$ through the 4 remaining points moves in a 2-dimensional linear system. Finally, the residual conic on P is determined by its matching with the fixed curve in $|f|$ on F, and then the residual curve in $|3f+E|$ on F is fixed with its matching with the conic on P at its remaining point of intersection with E and with the fixed line on P. Hence the contribution is of $5 \times 15 = 75$.

Case 5: $\delta_F = 1, \delta_P = 1, \tau_2 = 1$.

Again one has the two alternatives (a) and (b) as in Case 4. Case (a) contributes with 40 curves, case (b) with 160. Let us check case (b), and leave case (a) to the reader.

Once the splitting curve of $|f|$ has been fixed (which depends on 5 choices, i.e., the choice of the point in B), it intersects E at a point q. The matching condition on P gives the net of

cubics through A and q. This net maps P to $\mathbf{P}^2$ as a double cover, branched along a quartic D which, by genericity, is smooth. The curve E is mapped to a conic C. We have to count the number of lines tangent to both, D and C, which is the intersection number 24 of $\check{C}$ (a conic) and $\check{D}$ (a curve of degree 12), minus the number of points where D and C are tangent to each other (which is 4, the number of branch points of the double cover $E \to C$), counted with multiplicity 2 (this is similar to Lemma (7.1)). Since each of these curves contributes with multiplicity 2 to the total, we have a total of $2 \times 5 \times 16 = 160$.

Case 6: $\delta_P = 1, \tau_3 = 1$.

Each of the curves in question counts with multiplicity 3. The linear system $\mathcal{L}_P$ maps P to a smooth cubic surface Φ in $\mathbf{P}^3$, and E is mapped to a rational normal cubic $\Gamma \subseteq \Phi$. Consider the duals:
• $\check{\Phi}$, which is a surface of degree 12;
• $\check{\Gamma}$, which is a scroll of degree 4, with a cuspidal curve Γ^* of degree 3, which is the rational normal cubic of $\check{\mathbf{P}}^3$ described by all osculating planes to Γ in $\mathbf{P}^3$ (see [XII]).

The surface $\check{\Gamma}$ is the projection of a smooth rational normal scroll Σ of degree 4 in $\mathbf{P}^5$, and we denote by H the hyperplane class in this embedding and $|\mathfrak{f}|$ the pencil of lines on Σ. One has $H^2 = 4, \mathfrak{f}^2 = 0, H \cdot \mathfrak{f} = 1$. [2]

The strict transform of Γ^* on Σ is clearly unisecant with the curves in $|\mathfrak{f}|$ and therefore $\Gamma^* \equiv H - \mathfrak{f}$.

The curve Γ' described by the tangent planes to Φ at the points of Γ sits in $\check{\Phi} \cap \check{\Gamma}$. It is a rational curve of degree 6 (to see this, intersect Γ with a general polar quadric of S'), meeting the curves of $|\mathfrak{f}|$ in 1 point. Hence on Σ one has $\Gamma' \equiv H + 2\mathfrak{f}$, and therefore $\Gamma' \cdot \Gamma^* = 5$.

We have to compute the intersection number of $\check{\Phi}$ and Γ^*, which is 36, and subtract the number of osculating planes to Γ tangent to Φ at a point of Γ, which is $\Gamma' \cdot \Gamma^* = 5$, counted with multiplicity 3 (to see this, imitate the proof of Lemma (7.1)). In conclusion we get a contribution of $3 \times 21 = 63$.

Case 7: $\delta_P = 2, \tau_2 = 1$.

Each of the curves in question counts with multiplicity 2. The curve on P splits, and we have the two alternatives (a') and (b') as in Case 4. However, case (a') gives no contribution to the computation, because in a pencil of lines there is no line tangent to E. Each of the situations in case (b') (which depend on 15 choices, i.e., the pairs of points of the splitting line ℓ on P) gives rise to 2 curves, corresponding to the conics of the residual pencil to ℓ tangent to E. In conclusion, one has a contribution of $2 \times 15 \times 2 = 60$.

Case 8: $\delta_P = 3$.

The curve on P splits in 3 lines, which have to contain the points in A. The reader will readily check that their number is 15.

The following table shows the result of the total summation of non–trivial contributions.

[2] One has that $\Sigma \cong \mathbf{F}_0$ embedded in $\mathbf{P}^5$ via the morphism determined by the linear system $|H|$ of curves of type $(2, 1)$. Indeed it cannot be $\Sigma \cong \mathbf{F}_2$ because otherwise there would be a pencil of tangent planes to Γ along moving points, which is not possible.

Cases	Contribution
Case 1	10
Case 2	180
Case 4	147
Case 5	200
Case 6	63
Case 7	60
Case 8	15
Total	**675**

From this total we have to subtract the limit of reducible quartic curves passing through 11 general points of the plane. Their number is $55 = \binom{11}{2}$, since they are the ones appearing in (ii) of Proposition (7.4). We leave it to the reader to check that the limits of these 55 curves are the ones in Case 1 and Case 4 (b,a') and (a,b').

(7.6) Remark. In conclusion, also the degeneration to a cubic plus a plane is 1–good. However the reader will have noticed that, both in this case and in the previous one, we do not have an a priori proof of this (as in the intentions of Conjecture (5.10)); rather, the proof relies on Salmon's formula (2.0.1) instead of proving it.

8 – A (well-behaved) non-good case

8.1 – Degeneration of cubics to three planes

Consider a degeneration as in Example (3.3), where $d = 3$, with S_0 consisting of the union of three distinct planes P_0, P_1, P_2. Let us denote by ℓ_i the intersection line of P_i and P_{i+1} (for $i = 0, 1, 2$; the indices will now be considered to vary in $\mathbf{Z}/3\mathbf{Z}$), and by p the intersection point of P_0, P_1, P_2. The central fibre of the corresponding semistable degeneration constructed in Example (3.3) is $\tilde{S}_0 = Q_0 \cup Q_1 \cup Q_2$ as in Figure 7, and we may assume that each of the surfaces Q_i is the blow up of P_i at three points along ℓ_i.

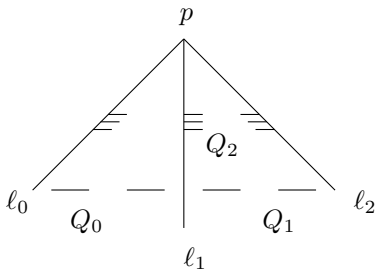


Figure 7: the degeneration of a cubic to the union of three planes

If we look at limits of nodal curves in the polarising hyperplane bundle, according to Proposition (5.5) we have to consider only planes passing by one of the aforementioned blown up points. This gives a contribution of 9 to the degree of the dual of a general cubic surface in $\mathbf{P}^3$, whereas we know that this degree is 12. So we are missing something of degree 3, which shows that this degeneration is not 1–good. In what follows we will construct a 1–good model, which will show that the degeneration under consideration is 1–well behaved.

(8.1) Remark. This degeneration is however 3–good. To see this, notice that the limit of the 27 lines on a general cubic can be seen here as the lines different from the ℓ_i's joining pairs of

blown up points. Hence we see all triangles formed by these lines, which are 45, so we see also the limit of all 3–nodal planes sections of a general cubic surface.

For the same reason this degeneration is also 2–good.

The obvious guess is that the limit of the dual of the general cubic consists of the 9 aforementioned planes in $\check{\mathbf{P}}^3$ plus the plane $p^\perp$ counted with multiplicity 3, i.e., *there is something of degree 3 hidden at p*. This is indeed the case and to see it, one has to look at another model of the degeneration. This is explained in detail in [4] in the more complicated situation of a general quartic degenerating to a tetrahedron (i.e., the union of four linearly independent planes in $\mathbf{P}^3$). Let us now see how this 1–good model is gotten in the present case.

(8.2) Construction of a good model. Let $\bar{f} : \bar{S} \to \mathbf{D}$ be the family obtained from $\tilde{f} : \tilde{S} \to \mathbf{D}$ (notation as in Example (3.3)) by the base change $t \in \mathbf{D} \mapsto t^3 \in \mathbf{D}$. The central fibre $\bar{S}_0$ is isomorphic to $\tilde{S}_0$, so we will keep the above notation for it.

Analytically locally around p, the total space $\bar{S}$ is isomorphic to the hypersurface of $\mathbf{C}^4$ defined by the equation $xyz = t^3$ at the origin. Blow-up $\bar{S}$ at p. The blown–up total space locally sits in $\mathbf{C^4} \times \mathbf{P^3}$. Let $[\xi : \eta : \zeta : \vartheta]$ be the homogeneous coordinates in $\mathbf{P^3}$. Then the new total space is locally defined in $\mathbf{C^4} \times \mathbf{P^3}$ by the equations

$$\operatorname{rk} \begin{pmatrix} x & y & z & t \\ \xi & \eta & \zeta & \vartheta \end{pmatrix} \leqslant 1 \qquad \text{and} \qquad \xi\eta\zeta = \vartheta^3. \tag{8.2.1}$$

Denote the exceptional divisor by T; it is isomorphic to the cubic surface with equation $\xi\eta\zeta = \vartheta^3$ in the $\mathbf{P}^3$ with coordinates $[\xi : \eta : \zeta : \vartheta]$. This cubic contains three lines, along which it intersects the proper transforms $\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$ of Q_0, Q_1, Q_2, respectively; thus, each line on T identifies with the exceptional (-1)-curve on one of $\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$. The cubic T has three A_2 double points, located at the intersections of T with the proper transforms $\tilde{\ell}_0, \tilde{\ell}_1, \tilde{\ell}_2$ of ℓ_0, ℓ_1, ℓ_2, respectively. See Remark (8.3) below for more about this cubic. The new central fibre is shown in Figure 8.

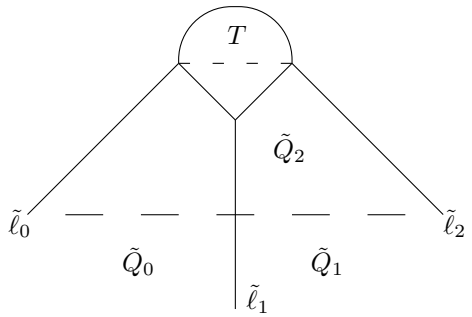


Figure 8: the central fibre after base change and blow–up of the vertex p

The model we have arrived at at this point is good enough to see all limits of 1-nodal curves, but it is not yet a good model because it is not semistable. Indeed, the total space is singular along the proper transforms $\tilde{\ell}_0, \tilde{\ell}_1, \tilde{\ell}_2$ of ℓ_0, ℓ_1, ℓ_2: analytically locally around the general point of one of these curves, it is isomorphic to a neighbourhood of the origin in the hypersurface defined by $xy = t^3$ in $\mathbf{C}^4$; thus, the total space is locally the product of an A_2 surface singularity with a smooth curve.

To resolve these singularities, we blow up along the three (disjoint) curves $\tilde{\ell}_0, \tilde{\ell}_1, \tilde{\ell}_2$. This has the effect of replacing each of these curves by a chain of two ruled surfaces; correspondingly, all three A_2 double points of T are resolved, each being replaced by a chain of two (-2)-curves. Thus, T is replaced by its minimal resolution, which we will denote by $\tilde{T}$. The three surfaces

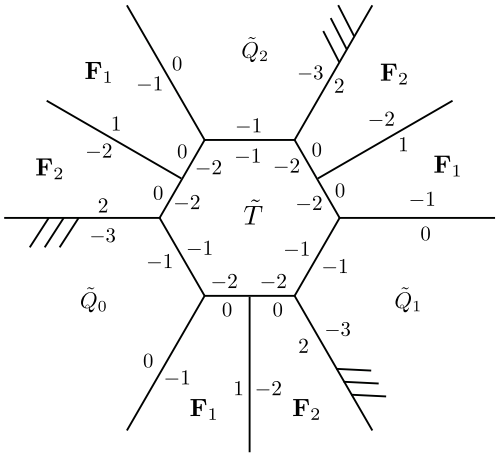


Figure 9: the central fibre of the good model (seen from above)

$\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$, on the other hand, remain unchanged, and we will denote their proper transforms by the same symbols.

The new central fibre is shown in Figure 9. We leave to the reader the verification of the self-intersections of the double curves of the central fibre indicated on the figure, as an exercise in the triple point formula. With our choice of distribution of the (-1)-curves in the small resolution at the beginning, the chains of rational ruled surfaces are made of an $\mathbf{F}_1$ and an $\mathbf{F}_2$ surface, as indicated on the figure.

(8.3) Remark. The cubic surface T is the image of the plane via the linear system $\mathcal{C}$ of cubic curves with two base points e_1, e_2, and infinitely near base points such that the cubics $C \in \mathcal{C}$ have two common flexes along lines L_1 and L_2. One has to blow up the plane at e_1 and e_2 three consecutive times to make the proper transform $\tilde{\mathcal{C}}$ of $\mathcal{C}$ base point free. This is shown in Figure 10, where we denote by $\tilde{C}$ the proper transform of the general curve $C \in \mathcal{C}$, by $\tilde{L}_i$ the proper transforms of the lines L_i, for $i = 1, 2$, and by E_{ij}, for $i = 1, 2$, $j = 1, 2, 3$, the exceptional curves of the blow–ups, and one has $E_{i3}^2 = -1$ for $i = 1, 2$ and $E_{ij}^2 = -2$, for $i, j = 1, 2$. [3]

The cubic T has three A_2–double points, at the vertices of the triangle $\vartheta = 0$, $\xi\eta\zeta = 0$. They are the images of $\tilde{L}_1 + \tilde{L}_2$ and of $E_{i1} + E_{i2}$, for $i = 1, 2$. The three lines pairwise joining the A_2–double points, lie on T and are easily seen to be the only lines on T. They correspond to the two exceptional divisors E_{13} and E_{23} and to the line L_3 joining e_1 and e_2.

The cubic T is *self–dual*, i.e., $\check{T}$ is projectively equivalent to T, hence it has degree 3. This (heuristically) explains the contribution by 3 to the degree of the dual hidden at p.

(8.4) Goodness of the new model. The degeneration we have just constructed is 1–good. Indeed now there is a new centrally effective twist, which corresponds to a degeneration of the cubics S_t, $t \in \mathbf{D} \setminus \{0\}$, to the cubic $T \subseteq \mathbf{P}^3$. It provides the missing contribution by 3, as we will now briefly explain.

Let $\mathcal{L}$ be the pull-back to the new degeneration of the polarising hyperplane bundle on the initial degeneration. On the central fibre, it restricts to the pull-back H of the line class on the

[3] In [4] we gave a different plane linear system of cubics representing the same surface T. It is the linear system of cubics that pass through three independent points y_1, y_2, y_3 in the plane and are tangent there to the lines $\langle y_i, y_{i+1} \rangle$, with $i =\in \mathbf{Z}/3\mathbf{Z}$. To pass from $\mathcal{C}$ to this linear system, just make a quadratic transformation based at e_1 and e_2 and at the point infinitely near to e_1 along L_1.

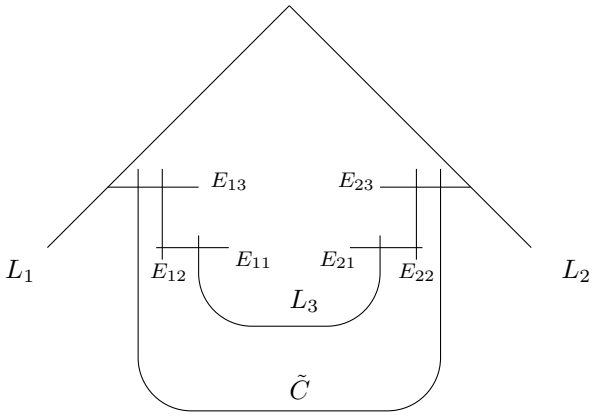


Figure 10: the linear system of curves $\tilde{C}$

surfaces $\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$, to the trivial class on $\tilde{T}$, and to the class of the ruling on the chains of ruled surfaces.

The new relevant twist is $\mathcal{L}(-\tilde{T})$: it restricts to the class $H - E_i$ on $\tilde{Q}_i$ for all $i = 0, 1, 2$ (recall the notation from (8.2)), to the pull-back of the hyperplane class of $T \subseteq \mathbf{P}^3$ on $\tilde{T}$, and it is trivial on the chains of ruled surface; thus, it maps $\tilde{T}$ to T, contracts $\tilde{Q}_i$ to the line E_i on T for all $i = 0, 1, 2$, and contracts the three chains of ruled surfaces to the three A_2 double points of T.

Now, let us fix a general pencil of planes in the $\mathbf{P}^3$ spanned by S_0, and let us count the "tangent planes to S_0" in this pencil. Nine of them are the planes in the pencil passing through each of the special points on the lines ℓ_i, $0 \leqslant i \leqslant 2$. As for the remaining three, consider the plane in the pencil passing through the triple point; it cuts out three coplanar lines meeting at the point p, the triple point of S_0. Thus its proper transform determines three points on T, one on each line of T, and these three points are collinear. The planes in the $\mathbf{P}^3$ spanned by T which contain these three points form a pencil, and in this pencil there are three planes which are tangent to T in simple points. This provides us with the remaining three "tangent planes to S_0". For further details, see [4].

(8.5) Remark. It follows from our analysis that, essentially, the two cubic surfaces $S_0 = P_0 \cup P_1 \cup P_2$ and $T = (xyz = w^3)$ are limits of the same family $(S_t)_{t \in \mathbf{D}^*}$. The parenthood of S_0 and T has also been observed in the study of moduli spaces of cubic surfaces and their compactifications. The interested reader may consult [26] (in particular, see Remark 1.3) and the references therein.

In the Kollár–Shepherd-Barron–Alexeev theory (see [18]), moduli spaces parametrize pairs consisting of a surface and a divisor. For smooth cubic surfaces there is a canonical choice for the divisor, namely the sum of its 27 lines, and in order to compactify the moduli space one must understand the limits of the 27 lines along degenerations of cubic surfaces. For a smooth cubic $(S, \mathcal{O}_S(1))$, the Severi variety $V_2(S, \mathcal{O}_S(1))$ of 2-nodal hyperplane sections is the union of the lines $\Lambda^\perp \subseteq \check{\mathbf{P}}^3$ for Λ ranging through all 27 lines of S; indeed, every 2-nodal hyperplane section of S is the union of a line and a conic, and therefore it is cut out by a member of one of the 27 pencils $\Lambda^\perp$. In fact, the Severi variety $V_3(S, \mathcal{O}_S(1))$ as well is determined by the lines of S, because its members are the 45 triangles made of three lines of S. With this in mind, it is clear that the two problems of compactifying the moduli space of smooth cubic surfaces and of studying limit Severi varieties of smooth cubic surfaces are related.

For smooth quartic surfaces, which are $K3$, a plausible canonical choice for the divisor is the

sum of all 3200 rational hyperplane sections. More generally, for degree d smooth surfaces, one may take the sum of all 3-nodal hyperplane sections.

8.2 – Degeneration of quartics to four planes

We conclude this text by a brief discussion of degenerations, as in Example (3.3) and Section 8.1 above, of smooth quartics S_t to the union S_0 of four planes P_0, P_1, P_2, P_3 in linear general position, which we call a *tetrahedron*. This is one of the main topics in [4], and many examples in this volume are inspired by this situation. An interesting feature is that this is a degeneration of $K3$ surfaces.

There are four different kinds of alternative degenerations of the S_t's to be considered in order to reveal all limits of δ-nodal hyperplane sections, $1 \leqslant \delta \leqslant 3$. Each corresponds to a kind of twist in some suitable good model of the initial degeneration.

— The contributions of the *faces* of the tetrahedron (i.e., the planes P_i themselves) are visible in degenerations to monoid quartic surfaces, similarly to what we have seen for the degeneration to a cubic and a plane in Section 7.2.

— The contributions of the sections by a hyperplane containing a *vertex* (i.e., a triple point $P_i \cap P_j \cap P_k$) are visible in degenerations to the union of a cubic $T \subseteq \mathbf{P}^3$ as in the previous Section 8.1 and the plane spanned by the three lines of T; the plane corresponds to the face of the tetrahedron opposite to the vertex in consideration.

— The contributions of the sections by a hyperplane containing an *edge* (i.e., a double line $P_i \cap P_j$) are visible in degenerations to a rational quartic surface with a double line and two triple points on the latter; the double line is the image of the opposite edge, and the two triple points are the respective images of the two faces adjacent to the edge.

More can be found on the rational surfaces appearing in these degenerations in [X, Section 3.3].

The construction of a good model follows the same lines as in Section 8.1; it is rather complicated in practice, but can be organized so as to be manageable, see [4]. Starting from the degeneration to $S_0 = P_0 + P_1 + P_2 + P_3$ as in Example (3.3), one performs the base change $t \mapsto t^6$ and then resolve the singularities. Morally, a degree 3 base change is needed to unravel the contributions involving the vertices, as in Section 8.1, and a degree 2 base change is needed for those involving the edges. In fact, an additional technical step is necessary after the resolution of singularities, namely one must flop certain double curves of the central fibre. The central fibre of the good model thus obtained is represented in Figure 11; the reader may also see [4, p. 143] for another picture, with a different design. The dotted curve represents (in a tropically inspired fashion, see [X, Section 9], which also includes explanations for this particular drawing) the pull-back of a hyperplane section of the initial tetrahedron.

References

Chapters in this Volume

[III] Th. Dedieu, *Geometry of logarithmic Severi varieties at a general point.*

[IV] M. Lelli-Chiesa, *Deformations of an m-tacnode into $m-1$ nodes.*

[VI] Th. Dedieu and C. Galati, *Degenerations and deformations of generalized Severi varieties.*

[VII] Th. Dedieu, *The Caporaso–Harris recursion within a degeneration of the plane.*

[VIII] C. Galati, *Proving the existence of curves on smooth surfaces by smoothing tacnodes on reducible surfaces.*

[X] Th. Dedieu, *Enumeration of curves in toric surfaces, tropical geometry, and floor diagrams.*

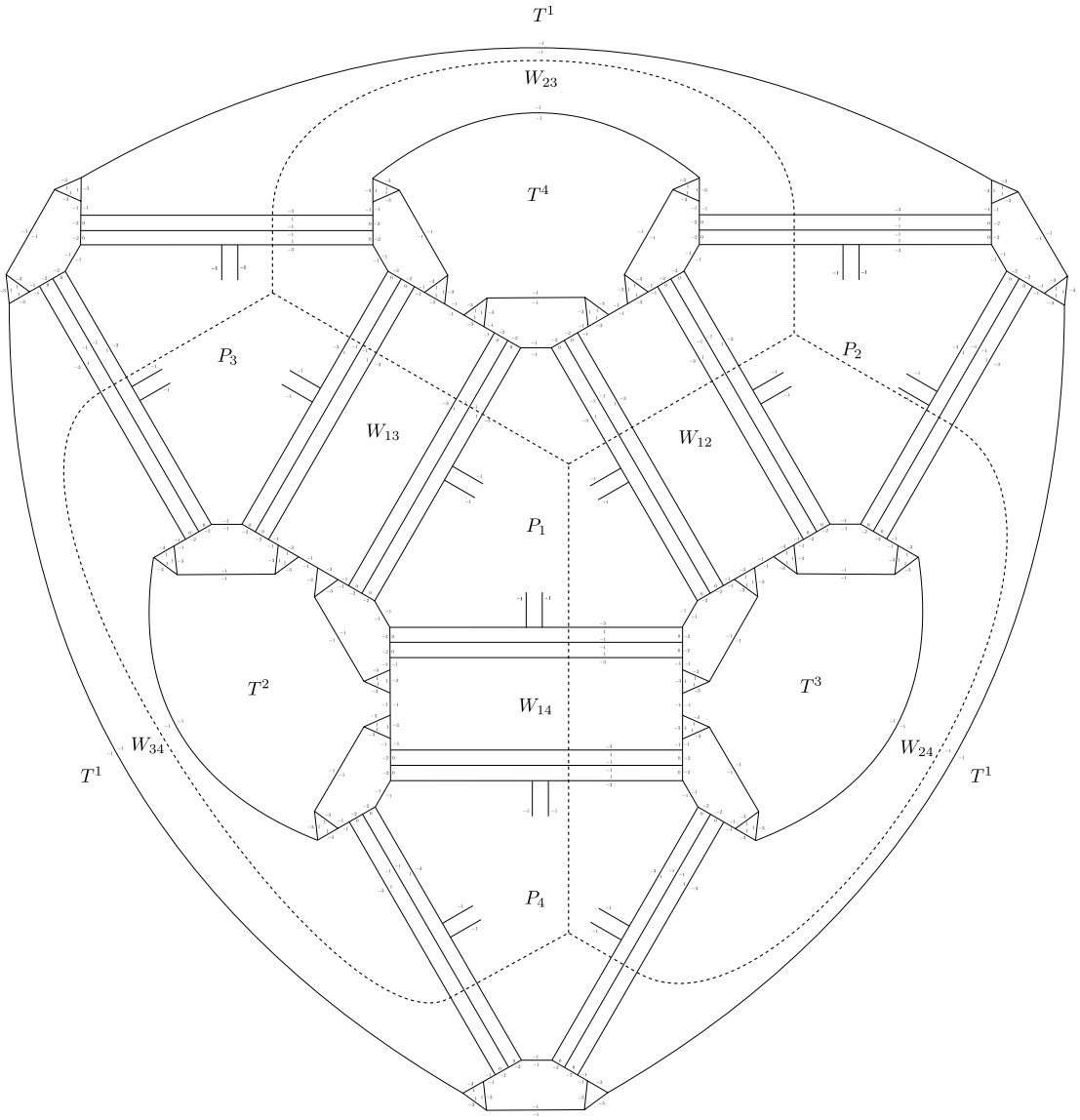


Figure 11: Central fibre of a good model of the degeneration of quartic surfaces to a tetrahedron

[B] C Ciliberto, Th. Dedieu, and C. Galati, *Flag semi-stable reduction of tacnodal curves at the limit of nodal curves*.

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces*.

[XIII] Th. Dedieu, *Node polynomials for curves on surfaces*.

[C] L. Busé and Th. Dedieu, *Generalized weight properties of resultants and discriminants, and applications to projective enumerative geometry*.

[XIV] F. Bastianelli and Th. Dedieu, *The Göttsche conjecture and the Göttsche–Yau–Zaslow formula*.

External References

[1] A. Calabri, C. Ciliberto, F. Flamini, R. Miranda: *On the K^2 of degenerations of surfaces and the multiple point formula*, Ann. of Math., **165** (2007), 335–395.

[2] L. Caporaso, J. Harris: *Counting plane curves of any genus*, Invent. Math. **131** (1998), no.2, 345–392.

[3] L. Caporaso, J. Harris: *Parameter spaces for curves on surfaces and enumeration of rational curves*, Compositio Math. **113** (1998), no.2, 155–208.

[4] C. Ciliberto, Th. Dedieu: *Limits of pluri–tangent planes to quartic surfaces*, in "Algebraic and Complex Geometry", Springer Proceedings in Math. & Statistics, **71**, 2014, pp. 123–200.

[5] C. Ciliberto, A. F. Lopez, R. Miranda: *Projective degenerations of $K3$ surfaces, Gaussian maps and Fano threefolds*, Invent. Math. **114** (1993), 641–667.

[6] R. Friedman: *Global smoothings of varieties with normal crossings*, Ann. of Math. **118** (1983), 75–114.

[7] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, in *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math., vol. 62, Part 2, Amer. Math. Soc., Providence, RI, 1997, 45–96.

[8] C. Galati: *Degenerating curves and surfaces: first results*, Rend. Circ. Mat. Palermo **58** (2009), no. 2, 211–243.

[9] C. Galati, A.L. Knutsen: *On the existence of curves with A_k–singularities on $K3$ surfaces*, Mathematical Research Letters **21**, (2014), 1069–1109.

[10] P. Griffiths, J. Harris: *On the Noether-Lefschetz theorem and some remarks on codimension-two cycles*, Math. Ann. **271** (1985), 31–51.

[11] J. Harris: *Algebraic Geometry, A First Course*, Graduate Texts in Math. 133, Springer–Verlag, 1992.

[12] J. Harris, I. Morrison: *Moduli of Curves*, Graduate Texts in Mathematics 187, Springer-Verlag, 1998.

[13] G. Kempf, F. F. Knudsen, D. Mumford, B. Saint–Donat: *Toroidal embeddings I*, Lecture Notes in Mathematics 339, Springer–Verlag, Berlin-New York, 1973.

[14] S. Kleiman: *About the conormal scheme*, in Complete intersections, Springer Berlin Heidelberg, 1984. 161–197.

[15] S. Kleiman, R. Piene: *Enumerating singular curves on surfaces*, in: Algebraic Geometry – Hirzebruch 70, Cont. Math. **241** (1999) 209–238 (corrections and revision in math.AG/9903192).

[16] S. Kleiman, R. Piene: *Node polynomials for families: methods and applications*, Math. Nachr. 271 (2004), 69 – 90.

[17] J. Kock and I. Vainsencher, *An invitation to quantum cohomology*, Kontsevich's formula for rational plane curves, Progress in Mathematics, vol. 249, Birkhäuser Boston, Inc., Boston, MA, 2007.

[18] J. Kollár, *Families of varieties of general type*, Cambridge Tracts in Mathematics **231**, Cambridge University Press, 2023. With the collaboration of K. Altmann and S. Kovács.

[19] V. Kulikov: *Degenerations of K3 surfaces and Enriques surfaces*, Math. USSR Izvestiya **11** (1977), 957–989.

[20] U. Persson, H. Pinkham: *Degenerations of surfaces with trivial canonical bundle*, Ann. of Math. **113** (1981), 45–66.

[21] Z. Ran: *On nodal plane curves*, Invent. Math. **86** (1986), no. 3, 529–534.

[22] Z. Ran: *Families of plane curves and their limits: Enriques' conjecture and beyond*, Ann. of Math. **130** (1989), no. 2, 121–157.

[23] Z. Ran: *Enumerative geometry of singular plane curves*, Invent. Math. **97** (1989), no. 3, 447–465.

[24] G. Salmon: *On the degree of a surface reciprocal to a given one*, Camb. and Dubl. Math. J. **2** (1847), 65–73.

[25] G. Salmon: *On the degree of the surface reciprocal to a given one*, Trans. Royal Irish Acad., **23** (1859), 461–488.

[26] N. Schock: Moduli of weighted stable marked cubic surfaces, preprint arXiv:2305.06922.

[27] B. Segre: *On limits of algebraic varieties, in particular of their intersections and tangential forms*, Proc. London Math. Soc., (2), **47** (1942), 351–403.

[28] E. Sernesi, *A smoothing criterion for families of curves*, `http://www.mat.uniroma3.it/users/sernesi/papers.html`

[29] I. Vainsencher: *Counting divisors with prescribed singularities*, Trans. Amer. Math. Soc. **267** (1981), no. 2, 399–422.

[30] I. Vainsencher, Israel: *Enumeration of n–fold tangent hyperplanes to a surface*, Journal of Algebraic Geometry **4** (1995), 503–526.

Lecture II
Deformations of curves on surfaces

by Thomas Dedieu and Edoardo Sernesi

1 Algebroid plane curves 39
2 The conductor 41
3 Equivalence of singularities 45
4 Generalities on deformations 46
5 The semiuniversal deformation of an isolated singularity 48
6 Equigenericity and equisingularity 51
7 Families of curves on surfaces 51
8 Deformations of morphisms 52
9 A criterion for the density of nodal curves 54
References 55

This text surveys some foundational material from the deformation theory of singular curves on surfaces, of ubiquitous use and necessity in this whole collection. The included topics are best described by the above table of contents.

1 – Algebroid plane curves

A *curve singularity* is a scheme of the form $\mathrm{Spec}(B)$, where $(B, \mathfrak{m})$ is the local ring at a singular point of a reduced algebraic curve. If the curve is contained in a nonsingular algebraic surface then we speak of a *planar curve singularity*. We will be mostly interested in this second case and in some of our considerations it will be convenient to replace such a B by its $\mathfrak{m}$-adic completion. In this section we will focus on this case and therefore we will consider local rings of the form

$$B = \mathbb{C}[[X, Y]]/(f)$$

for some power series f without constant term and without multiple factors. The scheme $C := \mathrm{Spec}(B)$ for such a ring is called an *algebroid plane curve*. If f is irreducible then C is called an *analytic branch*.

Assume that f is irreducible. Then the normalization $\widetilde{B}$ of B is a complete DVR and therefore it is isomorphic to a formal power series ring in one variable. Thus there exists $t \in K(B)$, the field of fractions of B, such that $\widetilde{B} = \mathbb{C}[[t]]$ and $K(B) = \mathbb{C}((t))$, the fraction field of $\mathbb{C}[[t]]$, which coincides with the field of formal Laurent power series in t.

We can write:

$$f(X, Y) = f_n(X, Y) + f_{n+1}(X, Y) + \cdots$$

for some $n \geq 1$, where f_i is a homogeneous polynomial of degree i. The integer n is the *multiplicity* of f (or of C) at $0 = [\mathfrak{m}]$. The polynomial $f_n(X, Y)$ is called the *initial form* of f.

We have two ideals associated to the algebroid plane curve C:

$$\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right) = (f_X, f_Y)$$

is called the *Milnor ideal* or *gradient ideal* of f. The corresponding algebra:

$$M_f := \mathbb{C}[[X,Y]]/\,(f_X, f_Y)$$

is the *Milnor algebra* of f.

The ideal

$$(f, f_X, f_Y)$$

is called the *Tjurina ideal* of f and

$$T_f := \mathbb{C}[[X,Y]]/\,(f, f_X, f_Y)$$

is the *Tjurina algebra* of f. It will be convenient to describe the Tjurina algebra as follows:

$$T_f = B/J$$

where $J = (f, f_X, f_Y)/(f) \subseteq B$ is the so-called *jacobian ideal*. Viewed as a B-module T_f is also called the *first cotangent module* of B, and denoted by T^1_B. Since C is either nonsingular or has an isolated singularity at $[\mathfrak{m}]$, the Tjurina ideal is either (1) or supported at $[\mathfrak{m}]$, and therefore T_f is a finite dimensional $\mathbb{C}$-vector space. Then

$$\tau(f) := \dim(T_f) < \infty$$

is called the *Tjurina number* of f (or of B). On the other hand it is also true that M_f is finite dimensional because of the following:

(1.1) Lemma (Risler [14])**.** *If $f \in (X,Y)^2 \subseteq \mathbb{C}[[X,Y]]$ has no multiple factors then f_X, f_Y form a regular sequence.*

Proof. It suffices to prove that f_X and f_Y have no common factors because they are contained in (X,Y). Assume that they have a non-constant irreducible factor p and consider the algebra $A = \mathbb{C}[[X,Y]]/(p)$. Let $\overline{f} \in A$ be the image of f. Then $\overline{f}$ is annihilated by every $\mathbb{C}$-derivation $D : K(A) \to K(A)$. Let $t \in K(A)$ be such that $K(A) = \mathbb{C}((t))$ and consider the derivation $D = \frac{d}{dt}$. Then $\ker(D) = \mathbb{C}$. Therefore $\overline{f} \in \mathbb{C}$ and therefore $\overline{f} = 0$, which means that $f = pq$ for some $q \in (X,Y)$. Therefore, since $f_X = p_X q + p q_X$ is divisible by p but p_X is not, it follows that q is divisible by p, and therefore f is divisible by p^2, a contradiction. □

For a proof of Lemma (1.1) in the case of convergent power series see [9], Lemma 2.3 p. 113. The dimension

$$\mu(f) = \dim(M_f)$$

is called the *Milnor number* of f (or of B). By construction we have:

$$\tau(f) \leq \mu(f)$$

and equality holds if and only if $f \in (f_X, f_Y)$. Notice also that

$$\mu(f) = i(f_X, f_Y)$$

where $i(-,-)$ denotes intersection multiplicity.

(1.2) Definition. *A polynomial* $P(X,Y) = \sum_\alpha a_\alpha X^{\alpha_1}Y^{\alpha_2}$ *is called* quasi-homogeneous *(qh) of weights* $w = (w_1, w_2) \in \mathbb{N}^2$ *and degree* d *if*

$$\alpha_1 w_1 + \alpha_2 w_2 = d$$

for all α *such that* $a_\alpha \neq 0$. *An algebroid plane curve* $\mathrm{Spec}\,(\mathbb{C}[[X,Y]]/(f))$ *is quasi homogeneous if there is a q.h. polynomial* P *such that* $\mathbb{C}[[X,Y]]/(f) \cong \mathbb{C}[[X,Y]]/(P)$.

If $P(X,Y)$ is qh then we have the generalized *Euler relation* in $\mathbb{C}[X,Y]$:

$$dP = w_1 X P_X + w_2 Y P_Y$$

and therefore $P \in (P_X, P_Y)$. Therefore an algebroid qh plane curve satisfies $\mu(f) = \tau(f)$. The converse is true, by the following:

(1.3) Theorem (Saito [16])**.** *The algebroid plane curve* $\mathrm{Spec}\,(\mathbb{C}[[X,Y]]/(f))$ *is qh if and only if* $\mu(f) = \tau(f)$.

(1.4) Example ([9, p. 111])**.** The polynomial $P(X,Y) = X^5 + Y^5 + X^2Y^2$ is not qh. One can compute that a $\mathbb{C}$-basis of T_f is defined by the monomials $1, X, \ldots, X^4, XY, Y, \ldots, Y^4$ and a $\mathbb{C}$-basis of M_f has the additional monomial Y^5. Thus $\tau(P) = 10 < \mu(P) = 11$.

(1.5) Example. The polynomial $f = X^a \pm Y^b$ is qh of weights (b,a) and degree ab. Therefore:

$$\tau(f) = \mu(f) = (a-1)(b-1)$$

(1.6) Example. The *simple singularities* (also called *ADE singularities*) are qh. They have the following equations:

- A_k: $x^{k+1} + y^2 = 0, \quad k \geq 1$.
- D_k: $x(y^2 + x^{k-2}) = 0, \quad k \geq 4$.
- E_6: $x^3 + y^4 = 0$.
- E_7: $x(x^2 + y^3) = 0$.
- E_8: $x^3 + y^5 = 0$.

We have:

$$\mu(A_k) = \mu(D_k) = \mu(E_k) = k.$$

2 – The conductor

Let B be a ring and let $\overline{B} \subseteq \mathrm{Frac}(B)$ be a subring of the total ring of fractions containing B. The homomorphism

$$\mathrm{Hom}_B(\overline{B}, B) \to B, \quad \varphi \mapsto \varphi(1)$$

induces an isomorphism

$$\mathrm{Hom}_B(\overline{B}, B) \cong \mathrm{Ann}_B(\overline{B}/B).$$

Then $\mathrm{Ann}_B(\overline{B}/B)$ is an ideal both in B and in $\overline{B}$. It is called the *conductor* of $\overline{B}$ in B. We define the *δ-invariant* of $B \subseteq \overline{B}$ as

$$\delta(\overline{B}/B) := \dim_{\mathbb{C}}(\overline{B}/B).$$

If $\overline{B}$ is the integral closure of B in $\mathrm{Frac}(B)$ then $\mathrm{Ann}_B(\overline{B}/B)$ is simply called the *conductor of* B and denoted by $\mathcal{A}(B)$; $\delta(\overline{B}/B)$ is denoted by $\delta(B)$ and called *δ-invariant of* B.

In case $C = \mathrm{Spec}(B)$ is either a curve singularity or an algebroid plane curve, $\delta(B)$ is denoted by $\delta(C)$ or $\delta(f)$ in case $B = \mathbb{C}[[X,Y]]/(f)$. It is called the *$\delta$-invariant* of B (of C).

Notice that if $C = \mathrm{Spec}(B)$ is any curve singularity then $\delta(B) = \delta(\widehat{B})$, by the flatness of $\widehat{B}$ over B.

(2.1) Definition. *Let $\varphi : \widetilde{C} \longrightarrow C$ be a birational morphism of reduced curves. The* conductor *of φ is the sheaf of $\mathcal{O}_C$-ideals:*

$$\mathcal{A}(\varphi) := \mathcal{A}nn(\varphi_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C)$$

or, equivalently:

$$\mathcal{A}(\varphi) = \mathcal{H}om(\varphi_*\mathcal{O}_{\widetilde{C}}, \mathcal{O}_C).$$

For each $x \in C$, the finite nonnegative integer

$$\delta(\varphi, x) := \delta\big((\varphi_*\mathcal{O}_{\widetilde{C}})_x/\mathcal{O}_{C,x}\big) = \dim_{\mathbb{C}}\Big((\varphi_*\mathcal{O}_{\widetilde{C}})_x/\mathcal{O}_{C,x}\Big)$$

is called the (local) δ-invariant of φ at x, *and*

$$\delta(\varphi) := h^0(\varphi_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) = \sum_{x\in C}\delta(\varphi, x)$$

is the (global) δ-invariant of φ. *In case φ is the normalization of C (i.e., $\widetilde{C}$ is nonsingular) we write $\mathcal{A}(\varphi) = \mathcal{A}(C)$ and $\delta(\varphi) = \delta(C)$ and call them the* conductor *of C and the* δ-invariant of C *respectively.*

The exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_*\mathcal{O}_{\widetilde{C}} \longrightarrow \nu_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C \longrightarrow 0$$

gives

$$\chi(\mathcal{O}_{\widetilde{C}}) = \chi(\nu_*\mathcal{O}_{\widetilde{C}}) = \chi(\mathcal{O}_C) + \delta(\varphi),$$

which can be rephrased as:

$$\delta(\varphi) = p_a(C) - p_a(\widetilde{C}) \tag{2.1.1}$$

where as usual $p_a(-)$ denotes *arithmetic genus* of $-$. In case φ is the normalization of C, the above identity takes the form:

$$\delta(C) = p_a(C) - p_g(C) \tag{2.1.2}$$

which shows that the δ-invariant of C measures the difference between the arithmetic genus and the geometric genus of the curve. Formula (2.1.2) is classical, and has been stated in this form by Rosenlicht [15] and Hironaka [12].

For the next lemma, observe that $\mathcal{A}(\varphi)$ is naturally a sheaf of $\varphi_*\mathcal{O}_{\widetilde{C}}$-modules. Therefore, since φ is affine, we have $\widetilde{C} = \mathrm{Spec}(\varphi_*\mathcal{O}_{\widetilde{C}})$ and $\mathcal{A}(\varphi)$ corresponds to a sheaf of $\mathcal{O}_{\widetilde{C}}$-modules which we denote by $\widetilde{\mathcal{A}}(\varphi)$.

(2.2) Lemma. *Let $\varphi : \widetilde{C} \to C$ be a birational morphism of reduced projective curves. Then there is a canonical isomorphism:*

$$\varphi_*\omega_{\widetilde{C}} \cong \mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \omega_C) \tag{2.2.1}$$

If moreover C is Gorenstein, then

$$\varphi_*\omega_{\widetilde{C}} \cong \mathcal{A}(\varphi) \otimes_{\mathcal{O}_C} \omega_C \tag{2.2.2}$$

and

$$\omega_{\widetilde{C}} = \widetilde{\mathcal{A}}(\varphi) \otimes_{\mathcal{O}_{\widetilde{C}}} \varphi^*\omega_C. \tag{2.2.3}$$

Proof. The proof of (2.2.1) follows from standard facts and is valid for finite morphisms of projective schemes. Recall the two following facts from [10, III, Ex. 6.10, p. 239]. For each quasi-coherent sheaf $\mathcal{G}$ on C, the module $\mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \mathcal{G})$ has a natural structure of quasi-coherent sheaf on $\widetilde{C}$, and as such it is denoted by $\varphi^!\mathcal{G}$. For each quasi-coherent sheaf $\mathcal{F}$ on $\widetilde{C}$, there is a natural isomorphism:

$$\varphi_*\mathcal{H}om_{\mathcal{O}_{\widetilde{C}}}(\mathcal{F}, \varphi^!\mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{F}, \mathcal{G}).$$

Taking $\mathcal{G} = \omega_C$ and $\mathcal{F} = \mathcal{O}_{\widetilde{C}}$ gives

$$\varphi_*(\varphi^!\omega_C) \cong Hom_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \omega_C).$$

Therefore it suffices to prove that $\varphi^!\omega_C = \omega_{\widetilde{C}}$ (which is [10, III, Ex. 7.2(a)]). For all coherent $\mathcal{F}$ on $\widetilde{C}$ we have:

$$H^1(\widetilde{C}, \mathcal{F})^\vee = H^1(C, \varphi_*\mathcal{F})^\vee = \mathrm{Hom}_{\mathcal{O}_C}(\varphi_*\mathcal{F}, \omega_C) \cong \mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\mathcal{F}, \varphi^!\omega_C),$$

and this isomorphism is functorial in $\mathcal{F}$. This implies that $\varphi^!\omega_C$ is a dualizing sheaf for $\widetilde{C}$ in the sense of [10, Definition p. 241] (see [21, Exercise 29.1.A]) , and thus (2.2.1) is proved.

If ω_C is invertible, then we have:

$$\varphi_*\omega_{\widetilde{C}} = \mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \omega_C) \cong \mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \mathcal{O}_C) \otimes \omega_C = \mathcal{A}(\varphi) \otimes \omega_C,$$

which is (2.2.2). Then, by the Projection Formula we have

$$\varphi_*\big(\widetilde{\mathcal{A}}(\varphi) \otimes \varphi^*\omega_C\big) = \varphi_*\widetilde{\mathcal{A}}(\varphi) \otimes \omega_C = \mathcal{A}(\varphi) \otimes \omega_C,$$

where the equality $\varphi_*\widetilde{\mathcal{A}}(\varphi) = \mathcal{A}(\varphi)$ follows from [10, II, Ex. 5.17e]. Thus

$$\varphi_*\omega_{\widetilde{C}} = \varphi_*\big(\widetilde{\mathcal{A}}(\varphi) \otimes \varphi^*\omega_C\big),$$

hence $\omega_{\widetilde{C}} = \widetilde{\mathcal{A}}(\varphi) \otimes \varphi^*\omega_C$, again by [10, II, Ex. 5.17e]. □

We refer the reader to [21, §29.3–4] for versions of the formula $\varphi^!\omega_C = \omega_{\widetilde{C}}$ valid in a more general framework.

(2.3) Corollary. *(a) Let $C = \mathrm{Spec}(B)$ be a Gorenstein reduced curve singularity. Then:*

$$\delta(B) = \dim_{\mathbb{C}}(B/\mathcal{A}(B)).$$

(b) Let C be a Gorenstein reduced projective curve and $\varphi : \widetilde{C} \to C$ a birational morphism. Then

$$\delta(\varphi) = h^0(\mathcal{O}_C/\mathcal{A}(\varphi)).$$

Proof. It suffices to prove (b). Let $\mathcal{A} = \mathcal{A}(\varphi)$. By definition we have an exact sequence:

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_C}(\varphi_*\mathcal{O}_{\widetilde{C}}, \mathcal{O}_C) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C/\mathcal{A} \longrightarrow 0\ .$$

Tensoring by ω_C, using the fact that C is Gorenstein and recalling (2.2.1), we obtain:

$$0 \longrightarrow \varphi_*\omega_{\widetilde{C}} \longrightarrow \omega_C \longrightarrow [\mathcal{O}_C/\mathcal{A}] \otimes \omega_C \longrightarrow 0\ .$$

From this exact sequence we deduce:

$$\begin{aligned} h^0(\mathcal{O}_C/\mathcal{A}) &= \chi([\mathcal{O}_C/\mathcal{A}]\otimes\omega_C) \\ &= \chi(\omega_C) - \chi(\varphi_*\omega_{\widetilde{C}}) \\ &= \chi(\omega_C) - \chi(\omega_{\widetilde{C}}) \\ &= p_a(C) - p_a(\widetilde{C}) \\ &= \delta(\varphi) \end{aligned}$$

and comparing with (2.1.1) we conclude. □

(2.4) Corollary. *Let $\varphi : \widetilde{C} \to C$ be a birational morphism of projective reduced curves and assume that $C \subseteq Y$, a projective nonsingular surface. Then*

$$\mathcal{A}(\varphi)\otimes\mathcal{O}_C(C) = \varphi_*\omega_{\widetilde{C}}\otimes\omega_Y^{-1}$$

and

$$\widetilde{\mathcal{A}}(\varphi)\otimes\varphi^*\mathcal{O}_C(C) = \omega_{\widetilde{C}}\otimes\varphi^*\omega_Y^{-1}.$$

Proof. The assumptions imply that C is Gorenstein. Then the corollary is an easy consequence of Lemma (2.2). □

(2.5) Lemma. *Let C be a reduced curve contained in a nonsingular algebraic surface Y. Let $p \in C$ be a point of multiplicity m and $b : Z \to Y$ the blow-up of Y at p. Then, letting $\overline{C} \subseteq Z$ be the proper transform of C and $E \subseteq Z$ the exceptional curve we have:*

$$b_*\omega_{\overline{C}} = \omega_C \otimes b_*\mathcal{O}_Z\big(-(m-1)E\big).$$

Proof. Left to the reader. □

The Milnor number and the δ-invariant are related as follows:

(2.6) Proposition (Milnor formula)**.** *Let $f = f_1\cdots f_r \in \mathbb{C}[[X,Y]]$, where $f_1,\dots,f_r$ are pairwise distinct analytic branches. Then:*

$$\delta(f) = \frac{1}{2}\big[\mu(f) + r - 1\big].$$

Proof. See [9, Prop. 3.35, p. 208]. □

(2.7) Remark. Let $\varphi : \widetilde{C} \longrightarrow C$ be a birational morphism of reduced curves, and denote by $\Delta \subseteq C$ and $\widetilde{\Delta} \subseteq \widetilde{C}$ the two subschemes defined by $\mathcal{A}(\varphi)$ and $\widetilde{\mathcal{A}}(\varphi)$ respectively. If both C and $\widetilde{C}$ are Gorenstein, then it follows from (2.2.3) that $\widetilde{\Delta}$ is Cartier. However it is not true in general that Δ is a Cartier divisor (for instance, at a node $xy = 0$ the conductor is the ideal generated by x and y).

(2.8) Remark. In general the two sheaves of $\mathcal{O}_{\widetilde{C}}$-modules $\widetilde{\mathcal{A}}(\varphi)$ and $\varphi^*\mathcal{A}(\varphi)$ are different, even though, in the words of [13, (1.6), p. 350], "many writers who should know better write $\varphi^*\mathcal{A}(\varphi)$ for the ideal $\mathcal{A}(\varphi)\cdot\mathcal{O}_{\widetilde{C}}$" (the latter being $\widetilde{\mathcal{A}}(\varphi)$ in our situation).

For the remainder of this remark, we write $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ for $\mathcal{A}(\varphi)$ and $\widetilde{\mathcal{A}}(\varphi)$ respectively. By definition, $\varphi^*\mathcal{A} = \varphi^{-1}\mathcal{A}\otimes_{\varphi^{-1}\mathcal{O}_C}\mathcal{O}_{\widetilde{C}}$, and then the multiplication map induces an exact sequence of $\mathcal{O}_{\widetilde{C}}$-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \varphi^*\mathcal{A} \longrightarrow \widetilde{\mathcal{A}} \longrightarrow 0.$$

We shall see in the two following examples that in general $\mathcal{K}$ is non-trivial and $\varphi^*\mathcal{A}$ has a non-trivial torsion part, which is a manifestation of the fact that φ is not flat. The conclusion is that the object we want to consider is indeed $\widetilde{\mathcal{A}}$ and not $\varphi^*\mathcal{A}$.

Our first example is the normalization of the node $xy = 0$. The adjoint ideal is (x, y). Locally on the branch $y = 0$ in the normalization, the multiplication map $\varphi^{-1}\mathcal{A} \otimes_{\varphi^{-1}\mathcal{O}_C} \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{C}}$ corresponds to the multiplication map

$$(x, y) \otimes_{\mathbb{C}[x,y]/(xy)} \mathbb{C}[x] \longrightarrow \mathbb{C}[x],$$

which has non-trivial kernel, generated by $y \otimes 1$ and thus killed by multiplication by x. The upshot is that the torsion part of $\varphi^*\mathcal{A}$ is $\mathcal{O}_{\widetilde{\Delta}}$, where $\widetilde{\Delta}$ consists of the two points in the preimage of the node.

Our second example is the normalization of the cusp $y^2 - x^3 = 0$, which we see as $t \mapsto (t^2, t^3)$. The adjoint ideal is again (x, y), equivalently (t^2, t^3). The multiplication map $\varphi^{-1}\mathcal{A} \otimes_{\varphi^{-1}\mathcal{O}_C} \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{C}}$ corresponds to the multiplication map

$$(t^2, t^3) \otimes_{\mathbb{C}[t^2,t^3]} \mathbb{C}[t] \longrightarrow \mathbb{C}[t],$$

which has non-trivial kernel generated by $t^2 \otimes t - t^3 \otimes 1$ and thus killed by multiplication by t^2. Again the torsion part of $\varphi^*\mathcal{A}$ is $\mathcal{O}_{\widetilde{\Delta}}$, where this time $\widetilde{\Delta}$ is defined by the ideal (t^2) of $\mathbb{C}[t]$.

3 – Equivalence of singularities

We say that two algebroid plane curves:

$$C = \mathrm{Spec}(\mathbb{C}[[X, Y]]/(f), \quad D = \mathrm{Spec}(\mathbb{C}[[X, Y]]/(g)$$

are *isomorphic* or *analytically equivalent* if there is an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[[X, Y]]/(f) \cong \mathbb{C}[[X, Y]]/(g).$$

A theorem of Mather and Yau (see [9, Theorem I.2.26]) asserts that two algebroid plane curves are analytically equivalent if and only if their Tjurina algebras are isomorphic.

Because of the following result it is not restrictive to assume that f is a polynomial when studying isomorphism classes of algebroid plane curves.

(3.1) Theorem (Samuel [17])**.** *Given a reduced non-constant $f \in \mathbb{C}[[X, Y]]$ there is a polynomial g such that $\mathbb{C}[[X, Y]]/(f) \cong \mathbb{C}[[X, Y]]/(g)$.*

A weaker equivalence relation has been introduced by Zariski as follows. Consider and analytic branch $C = \mathrm{Spec}(\mathbb{C}[[X, Y]]/(f)$ and let $n = e(C)$ be its multiplicity (at 0). The blowing up of $\mathrm{Spec}(\mathbb{C}[[X, Y]])$ at 0 is defined by the substitution

$$X = X', \quad Y = X'Y'$$

Then

$$f(X, Y) = f(X', X'Y') = X'^n f'(X', Y')$$

and f' is the *proper transform* of f. One easily shows (see [23], p. 3) that f' is irreducible, i.e., defines an analytic branch C'. Let $e(C')$ its multiplicity. Then $e(C') \leq e(C)$. The theorem of resolution of singularities guarantees that after a finite number of iterations of this operation we obtain a nonsingular branch $\overline{C}$, i.e., such that $e(\overline{C}) = 1$. Let

$$C, C', \ldots, C^{(i)}, \ldots$$

be the successive proper transforms of C and $e_i = e(C^{(i)})$. Let $N \geq 1$ be such that $e_N = 1$ but $e_{N-1} \neq 1$ and let

$$e^*(C) = \{e_0, e_1, \ldots, e_{N-1}\}$$

The sequence $e^*(C)$ is uniquely determined by C.

(3.2) Definition. *Two analytic branches C, D are said to be* equivalent *or* equisingular*, if*

$$e^*(C) = e^*(D)$$

It is possible to show that two simple singularities are equisingular if and only if they are analytically equivalent. The simplest examples for which the two notions differ are ordinary n-fold points, with $n \geq 4$. For example, any two ordinary 4-fold points are equisingular, but they are isomorphic if and only if their respective 4-tuples of principal tangent lines have equivalent cross ratios. For more details about equisingularity we refer to [23].

4 – Generalities on deformations

Deformation theory studies local deformations of a given algebro-geometric object X (a projective scheme, a singularity, etc.). Under good hypotheses it produces a family containing all sufficiently small deformations of X, up to isomorphism, and having certain functorial properties. Here we are interested in deformations of planar curve singularities, a case where the theory simplifies consistently. Before adventuring into the details we need a general preamble about what we mean by (local) deformations.

Consider the category $\mathfrak{Loc}$ of local $\mathbb{C}$-algebras with residue field $\mathbb{C}$. A *deformation* of X parametrized by $\mathrm{Spec}(R)$, where $R \in \mathrm{Ob}(\mathfrak{Loc})$ is a pullback diagram:

$$\begin{array}{rccc} \zeta: & X & \hookrightarrow & \mathcal{X} \\ & \downarrow & & \downarrow{\scriptstyle \pi} \\ & \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

where π is flat. Given another deformation of X parametrized by $\mathrm{Spec}(R)$:

$$\begin{array}{rccc} \eta: & X & \hookrightarrow & \mathcal{Y} \\ & \downarrow & & \downarrow{\scriptstyle \rho} \\ & \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

we say that ζ and η are *isomorphic* if there is an isomorphism of R-schemes

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\cong} & \mathcal{Y} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & \mathrm{Spec}(R) & \end{array}$$

compatible with the identifications of X with the fibres over $\mathrm{Spec}(\mathbb{C})$.

Deformation theory associates to X a covariant functor $M_X : \mathfrak{Loc} \to \mathfrak{Sets}$ taking values in the category of sets. It is defined by:

$$M_X(R) = \{\text{isom. classes of def.s of } X \text{ parametrized by } \mathrm{Spec}(R)\}$$

for all $R \in \mathrm{Ob}(\mathfrak{Loc})$, and $M_X(f : R \to S)$ is defined by pullback. This functor can be extended to the category of pointed $\mathbb{C}$-schemes in an obvious way. Since deformation theory is unable to give a satisfactory information on either of these functors what can be done is to restrict M_X to more amenable subcategories of $\mathfrak{Loc}$. Two full subcategories are relevant in deformation theory. The first one is $\widehat{\mathfrak{Loc}}$, the category of complete local $\mathbb{C}$-algebras with residue field $\mathbb{C}$. The second one is $\mathfrak{Art}$, whose objects are the local artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$.

Covariant functors $F : \mathfrak{Art} \to \mathfrak{Sets}$ are called *functors of Artin rings*. The restriction of M_X to $\mathfrak{Art}$ is such a functor. If $A \in \mathrm{Ob}(\mathfrak{Art})$ then the elements of $M_X(A)$ are called *infinitesimal deformations* of X parametrized by A.

A *formal deformation* of X is a pair $(R, \widehat{\xi})$ where $(R, \mathfrak{m})$ is a complete local $\mathbb{C}$-algebra, i.e., $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$, and $\widehat{\xi} = \{\xi_n \in M_X(R/\mathfrak{m}^{n+1})\}_{n\geq 1}$ is a sequence of infinitesimal deformations which are compatible, i.e., such that $\xi_n \mapsto \xi_{n-1}$ under the map

$$M_X(R/\mathfrak{m}^{n+1}) \to M_X(R/\mathfrak{m}^n)$$

for each $n \geq 1$. Associating to every $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$ the set of formal deformations $(R, \widehat{\xi})$ we define a new functor

$$\widehat{M}_X : \widehat{\mathfrak{Loc}} \to \mathfrak{Sets}.$$

A formal deformation $(R, \widehat{\xi})$ is called *effective* if there is a deformation $\xi \in M_X(R)$ such that $\xi \mapsto \xi_n$ under the map:

$$M_X(\pi_n) : M_X(R) \to M_X(R/\mathfrak{m}^{n+1})$$

induced by the projection $\pi_n : R \to R/\mathfrak{m}^{n+1}$, for each $n \geq 1$.

Important point: Not every formal deformation is effective, i.e., the functor $\widehat{M}_X$ is not isomorphic to the restriction of M_X to $\widehat{\mathfrak{Loc}}$. Namely in general $\widehat{M}_X(R) \neq M_X(R)$.

A formal deformation $(R, \widehat{\xi})$ is called *algebraizable* if there is a deformation ζ of X parametrized by a pointed algebraic scheme (S, s_0) and an isomorphism $u : R \cong \widehat{\mathcal{O}_{S,s_0}}$ and such that the sequence of deformations

$$\widehat{\zeta} = \{\zeta_n \in M_X(\mathcal{O}_{S,s_0}/\mathfrak{m}^{n+1})\}$$

obtained by pulling back ζ is mapped to $\widehat{\xi}$ under u. Every algebraizable formal deformation is effective because ζ can be pulled back to $\widehat{\mathcal{O}_{S,s_0}}$, but the converse is not true.

To every $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$ one can associate the functor of Artin rings:

$$h_R : \mathfrak{Art} \to \mathfrak{Sets}, \quad A \mapsto \mathrm{Hom}(R, A).$$

Functors of this form are called *prorepresentable*. Consider a formal deformation $\widehat{\xi} \in \widehat{M}_X(R)$, for some $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$. Let $A \in \mathrm{Ob}(\mathfrak{Art})$ and $f \in h_R(A) = \mathrm{Hom}(R, A)$. Since A is artinian there is an $n \gg 0$ such that f factors as

$$\begin{array}{ccc} R & \longrightarrow & R/\mathfrak{m}^{n+1} \\ & \searrow^{f} & \downarrow{\scriptstyle f_n} \\ & & A \end{array}$$

One can then associate to f the element $M_X(f_n)(\xi_n) \in M_X(A)$. Since this is independent of n we have defined a morphism of functors of Artin rings that we denote with the same symbol:

$$\widehat{\xi} : h_R \to M_X.$$

(4.1) Definition. *If a formal deformation $\widehat{\xi} \in \widehat{M}_X(R)$ is such that $\widehat{\xi}$ is an isomorphism of functors then we say that the pair $(R, \widehat{\xi})$* prorepresents *M_X. In this case M_X is prorepresentable.*

In practice if a pair $(R, \widehat{\xi})$ prorepresents M_X it means that for every $A \in \mathrm{Ob}(\mathfrak{Art})$ every $\xi \in M_X(A)$ is induced as above by a unique $f : R \to A$.

The prorepresentability of M_X is a strong property which is often not satisfied even by very simple objects X. For example, if X is the singularity $XY = 0$ then M_X is not prorepresentable (see [18, Example 2.6.8, p. 95]). The condition can be weakened in two ways as follows.

(4.2) Definition. *A formal deformation $\widehat{\xi} \in \widehat{M}_X(R)$ is called* versal *if for every $A \in \mathrm{Ob}(\mathfrak{Art})$ the induced map*

$$\widehat{\xi} : h_R(A) \to M_X(A)$$

is surjective.

A formal deformation $\widehat{\xi} \in \widehat{M}_X(R)$ is called miniversal *(or* semiuniversal*) if it is versal and in addition*

$$\widehat{\xi}(\mathbb{C}[\varepsilon]) : h_R(\mathbb{C}[\varepsilon]) \to M_X(\mathbb{C}[\varepsilon])$$

is bijective.

The map $\widehat{\xi}(\mathbb{C}[\varepsilon])$ is called the *differential* of $\widehat{\xi}$. Versality and miniversality are satisfied in most natural geometrical situations. Finally we give the following:

(4.3) Definition. *An algebraic deformation ζ of X parametrized by a pointed algebraic scheme (S, s_0) is called* formally versal *(resp.* formally semiuniversal*) if the associated algebraizable formal deformation $(\widehat{\mathcal{O}}_{S,s_0}, \widehat{\zeta})$ is versal (resp. semiuniversal).*

For more details on the contents of this section we refer to [18] and [11].

5 – The semiuniversal deformation of an isolated singularity

The case we will consider is $X = \mathrm{Spec}(B)$ where $(B, \mathfrak{m})$ is a noetherian local $\mathbb{C}$-algebra with residue field $B/\mathfrak{m} = \mathbb{C}$. Assume that B is either essentially of finite type (eft), i.e., a localization of a $\mathbb{C}$-algebra of finite type, or complete, and that $\mathrm{Spec}(B)$ has an isolated singularity at $[\mathfrak{m}]$. Under these conditions deformation theory ensures the existence of a formal semiuniversal family of deformations of B, or of $\mathrm{Spec}(B)$ (see [18]). In the cases of interest for us such a deformation can be easily constructed explicitly. Note firstly the following:

(5.1) Proposition. *If B and B' are local $\mathbb{C}$-algebras as above such that $\widehat{B} = \widehat{B}'$, then their deformation functors are isomorphic.*

Proof. [11, ex. 18.6, p. 127]. □

This means that it is not restrictive to study deformations of a complete local $\mathbb{C}$-algebra. Since we are interested in planar curve singularities, i.e., in the local ring B of a reduced curve C contained in a nonsingular surface Y at a point $p \in C$, we can replace B by its completion and therefore we may assume that $B = \mathbb{C}[[X, Y]]/(f)$, i.e., that C is an algebroid plane curve.

Consider indeterminates $t_1, \ldots, t_N$ where $N = \tau(f)$ and let

$$p_1(X, Y), \ldots, p_N(X, Y) \in \mathbb{C}[[X, Y]]$$

be polynomials inducing a basis of the Tjurina algebra T_f. Consider the following family:

(5.1.1)
$$\begin{array}{ccc} C & \hookrightarrow & \mathrm{Spec}(\mathcal{B}) =: \mathcal{C} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}[[t_1,\dots,t_N]]) =: M(B) \end{array}$$

where

$$\mathcal{B} := \mathbb{C}[[t_1,\dots,t_N,X,Y]]/(f(Y,Y)+\sum_1^N t_i p_i(X,Y)).$$

(5.2) Proposition. *The family* (5.1.1) *is a formally semiuniversal deformation of* C.

Replacing $\mathbb{C}[[t_1,\dots,t_N]]$ *by* $\mathbb{C}[t_1,\dots,t_N]$ *we obtain an algebraic formally semiuniversal family of deformations of* C*:*

(5.2.1)
$$\begin{array}{ccc} C & \hookrightarrow & \mathrm{Spec}\left(\mathbb{C}[t_1,\dots,t_N][[X,Y]])/(f(Y,Y)+\sum_1^N t_i p_i(X,Y))\right) \\ \downarrow & & \downarrow \pi_a \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}[t_1,\dots,t_N]) \end{array}$$

Therefore, letting

$$S_n := \mathrm{Spec}(\mathbb{C}[[t_1,\dots,t_N]]/(t_1,\dots,t_N)^{n+1}), \quad n \geq 1,$$

the restrictions of (5.1.1) *(or, what is the same, of* (5.2.1)*) to* S_n *define a sequence of (infinitesimal) deformations of* C*:*

$$\begin{array}{ccc} C & \hookrightarrow & \mathcal{C}_n \\ \downarrow & & \downarrow \pi_n \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & S_n \end{array}$$

which is an algebraizable formal semiuniversal deformation $\widehat{\pi} = \{\pi_n\}$ *of* C.

Proof. See [11], §14, for a detailed explicit proof. □

The support of $\mathcal{C}$ is different from the support of π_n, which is C, and a priori there is no information on how to recover $\mathcal{C}$ starting from $\widehat{\pi}$. Nevertheless from general properties of formal schemes it follows that π is uniquely determined by $\widehat{\pi} := \{\pi_n\}$ (see e.g. [18], Theorem 2.5.11, p. 81). This fact implies that Proposition (5.2) can be strengthened as follows.

(5.3) Corollary. *a) For every deformation*

$$\begin{array}{ccc} C & \hookrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

of C *where* $R \in \mathrm{Ob}(\widehat{\mathfrak{Loc}})$*, there is a morphism* $\varphi : \mathrm{Spec}(R) \to M(B)$ *(not unique, but whose differential is unique) inducing an isomorphism*

$$\mathcal{D} \cong \mathrm{Spec}(R) \times_{M(B)} \mathcal{C}$$

compatible with the isomorphisms of the closed fibres with C.

b) Given a family of deformations of C:

$$\begin{array}{ccc} C & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{s_0} & S \end{array}$$

parametrized by an algebraic *scheme S (or by the spectrum of an eft local $\mathbb{C}$-algebra) there is a morphism (not unique, but whose differential is unique)*

$$\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0}) \to \mathrm{Spec}(\mathbb{C}[[t_1,\dots,t_N]]) = M(B)$$

which induces an isomorphism of deformations over $\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0})$*:*

$$\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0}) \times_S \mathcal{X} \cong \mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0}) \times_{M(B)} \mathcal{C}.$$

For another discussion of this property see [11], Prop. 15.2, p. 108.

(5.4) Remarks. (1) If C is nonsingular then $T^1_B = 0$ and the semiuniversal deformation is:

$$\begin{array}{ccc} C & = & C \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & = & \mathrm{Spec}(\mathbb{C}) \end{array}$$

(see §1 for the definition of T^1_B).

(2) If $C : f = 0$ is singular then it is immediate to verify that $f + \varepsilon = 0$ is nonsingular for small values of ε. This implies that C has nonsingular deformations, i.e., that it is *smoothable*. The same proof shows that any germ of hypersurface in $\mathbb{C}^n$ with an isolated singularity at the origin is smoothable. An obvious modification of the argument shows that a germ of complete intersection isolated singularity is smoothable.

(3) Versality is an open property, but not semiuniversality. In other words, the semiuniversal family (5.1.1) is versal in a neighborhood of 0. Obviously it is not semiuniversal, e.g. where the fibres are smooth, because there the fibres are rigid and their semiuniversal deformation just consists of the fibre itself (see Remark (1) above). The standard reference for openness of versality is [8].

(5.5) Example. (i) The semiuniversal deformation of $A_1 : X^2 - Y^2 = 0$ is $X^2 - Y^2 + t = 0$. The only singular fibre is for $t = 0$.

(ii) Consider $A_2 : X^2 - Y^3 = 0$. Then the semiuniversal deformation is

$$F(t,u,X,Y) = X^2 - Y^3 + tY + u = 0$$

The singular fibres are determined by the conditions $F = F_X = F_Y = 0$ which are equivalent to $t = 3Y^2$, $u = -2Y^3$. We obtain the locus $\Delta : 27u^2 - 4t^3 = 0$ inside $M(B)$. The fibres are the cusp at $(0,0)$ and one node over the other points of Δ. The fibres over $M(B) \setminus \Delta$ are nonsingular.

6 – Equigenericity and equisingularity

It is possible to define two interesting loci inside $M(B)$, the equigeneric locus and the equisingular locus.

The *equigeneric locus* $EG \subseteq M(B)$ is supported on the points of $M(B)$ whose fibre is a singularity imposing the same number of conditions to adjoints as C does, i.e., on the points having the same δ-invariant as C. The locus $EG \subseteq M(B)$ is only defined set-theoretically, it does not have a functorial definition. As a locally closed subset of $M(B)$ it is analytically irreducible near $0 \in M(B)$ and its tangent cone is a vector subspace of T_f. Identifying $T_f = B/J$ the tangent cone to EG at 0 is identified with A/J. In particular A contains J and EG has codimension $\delta(C)$ in $M(B)$. In Example (5.5)(ii) we have $EG = \Delta$.

The *equisingular locus* $ES \subseteq M(B)$ is roughly the locus where the topological type of the singularity is the same as that of C. In particular $ES \subseteq EG$. In example (5.5)(ii) we have $ES = \{0\}$.

The locus ES has a scheme structure because it has a (complicated) functorial definition that we do not need to recall here. ES is nonsingular and its tangent space at $0 \in M(B)$ is the vector space $I/J \subseteq B/J = T_f$ where $I \subseteq A \subseteq B$ is an ideal called the *equisingular ideal*.

(6.1) Lemma. *(i) $I = A$ if and only if B is an A_1-singularity (a node).*

(ii) All the simple singularities satisfy $I = J$, i.e., they have no nontrivial equisingular deformations. Equivalently $ES = \{0\}$.

(iii) The simplest example such that $\dim(ES) > 0$, i.e., $J \neq I$, is an ordinary 4-ple point.

For detailed information on equisingularity and many examples we refer to [9]. The following results due to Teissier are important in this context.

(6.2) Theorem (Teissier [20])**.** *Let $\mathcal{C} \to S$ be an equigeneric family of reduced curves, S reduced. Then there is a Zariski dense open set $U \subseteq S$ such that $\mathcal{C} \times_S U \to U$ is equisingular.*

(6.3) Theorem (Teissier [20], see also [4])**.** *Let $p : \mathcal{C} \to S$ be a flat family of reduced curves with $\mathcal{C}, S$ reduced of finite type. If S is normal then the following are equivalent:*

(a) $p : \mathcal{C} \to S$ is equigeneric.

(b) The composition

$$\mathcal{C}' \xrightarrow{\nu} \mathcal{C} \xrightarrow{p} S$$

where ν is the normalization, is smooth and for all $s \in S$ the induced morphism on the fibres $\nu_s : \mathcal{C}'_s \to \mathcal{C}_s$ is the normalization.

Note: if S is not assumed to be normal the theorem is false: see [6], Ex. (2.6).

7 – Families of curves on surfaces

We now fix a projective nonsingular algebraic surface Y, a class $\xi \in \mathrm{NS}(Y)$ and the scheme of curves of class ξ in Y, denoted Curves_Y^ξ. We have a universal family:

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathrm{Curves}_Y^\xi \times Y \\ \downarrow & & \\ \mathrm{Curves}_Y^\xi & & \end{array}$$

whose fibres have arithmetic genus $p_a(\xi)$ depending only of ξ. If $[C] \in \mathrm{Curves}_Y^\xi$ then

$$p_a(\xi) = \frac{(C + K_Y) \cdot C}{2} + 1.$$

For a given $0 \leq g \leq p_a(\xi)$ one can consider the (possibly empty) locally closed locus $V_g^\xi \subseteq \mathrm{Curves}_Y^\xi$ parametrizing reduced curves having geometric genus g. Inside V_g^ξ one has the locus $V^{\xi,\delta}$ parametrizing reduced curves having $\delta = p_a(\xi) - g$ nodes and no other singularities. We want to compare these two loci, exploring the conditions under which we have $\overline{V^{\xi,\delta}} = \overline{V_g^\xi}$.

Let $C \subseteq Y$ be a reduced curve parametrized by a point of V_g^ξ. Dualizing the conormal sequence we obtain an exact sequence of sheaves on C:

$$0 \longrightarrow T_C \longrightarrow T_{Y|C} \longrightarrow \mathcal{O}_C(C) \longrightarrow T_C^1 \longrightarrow 0$$

where T_C^1 is the first cotangent sheaf of C, which has been described locally in §1. Note that in §1 we considered the case of algebroid plane curves. The module T_B^1 has an analogous definition for eft local rings and, being a torsion module in our case (isolated singularities), it is invariant under completion.

Locally at the point $v \in V_g^\xi$ parametrizing C the restricted universal family

$$p : \mathcal{C}_g \to V_g^\xi$$

induces a family of deformations of the singular points of C. The semiuniversal families induce a morphism from $\mathrm{Spec}(\widehat{\mathcal{O}}_{V_g^\xi, v})$ to the product of the bases of the semiuniversal families of deformations of the singularities of C. The differential of this morphism turns out to be precisely the map:

$$H^0(C, \mathcal{O}_C(C)) \to H^0(T_C^1)$$

appearing in the above exact sequence. Let's denote, as before, by $\mathcal{J} \subseteq \mathcal{I} \subseteq \mathcal{A} \subseteq \mathcal{O}_C$ respectively the jacobian, equisingular, and conductor ideal sheaves. Then this observation implies the following:

(7.1) Proposition. *If we identify $T_v(\mathrm{Curves}_Y^\xi) = H^0(C, \mathcal{O}_C(C))$ then:*

(i) $H^0(C, \mathcal{J} \otimes \mathcal{O}_C(C))$ is the tangent space to the subscheme of formally locally trivial deformations of C in Y.

(ii) $H^0(C, \mathcal{I} \otimes \mathcal{O}_C(C))$ is the tangent space to the subscheme of equisingular deformations of C in Y.

(iii) $H^0(C, \mathcal{A} \otimes \mathcal{O}_C(C))$ contains the reduced tangent cone to the subscheme of equigeneric deformations of C in Y.

8 – Deformations of morphisms

Given $C \subseteq Y$ as in the previous section we may consider a partial normalization morphism:

$$\varphi : \widetilde{C} \to C \subseteq Y$$

where we assume that $\widetilde{C}$ has at worst local complete intersection singularities. There is a well defined deformation theory for such a morphism, which produces a semiuniversal deformation

of φ:

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \xrightarrow{\Phi} & M(\varphi)\times Y \\ \downarrow & & \\ M(\varphi) & & \end{array}$$

where $M(\varphi) = \mathrm{Spec}(R)$ for some complete local $\mathbb{C}$-algebra. Semiuniversality here means a property analogous to the one considered in §5 for deformations of local rings; in particular φ is the fibre of Φ over the the closed point $0 := [\mathfrak{m}_R] \in M(\varphi)$. The properties of $M(\varphi)$ are determined by the deformation functor Def_φ of φ. For the infinitesimal study of $M(\varphi)$ one considers the complex:

$$\Omega^\bullet_\varphi := [\varphi^*\Omega^1_Y \to \Omega^1_{\widetilde{C}}].$$

The hyperext spaces $\mathrm{Ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}})$ and $\mathrm{Ext}^2_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}})$ are respectively the tangent space and the obstruction space of Def_φ. Then deformation theory implies the following:

(8.1) Proposition. *There is a canonical identification*

$$T_0M(\varphi) = \mathrm{Ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}})$$

and inequalities:

$$\mathrm{ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}) - \mathrm{ext}^2_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}) \leq \dim(M(\varphi)) \leq \mathrm{ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}).$$

For details we refer to [7, Sec. 3.2]. Our assumptions imply that φ is unramified at the general point of every component of $\widetilde{C}$. Therefore a simple computation (see [3, Lemma 11]) implies that there is a sheaf N_φ which fits into an exact sequence:

$$0 \to \mathcal{H}om(\Omega^1_{\widetilde{C}}, \mathcal{O}_{\widetilde{C}}) \to \mathcal{H}om(f^*\Omega^1_Y, \mathcal{O}_{\widetilde{C}}) \to N_\varphi \to \mathcal{E}xt^1(\Omega^1_{\widetilde{C}}, \mathcal{O}_{\widetilde{C}}) \to 0 \tag{8.1.1}$$

and such that

$$H^0(C, N_\varphi) = \mathrm{Ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}), \quad H^1(C, N_\varphi) = \mathrm{Ext}^2_{\mathcal{O}_{\widetilde{C}}}(\Omega^\bullet_\varphi, \mathcal{O}_{\widetilde{C}}).$$

The sheaf N_φ is called *normal sheaf* of φ.

We will be interested in two special cases:

(a) $\widetilde{C}$ is nonsingular, i.e., φ is the normalization map.

(b) φ is unramified.

In case (a) the sheaf $\Omega^1_{\widetilde{C}}$ is locally free and therefore the sequence (8.1.1) becomes:

$$0 \longrightarrow T_{\widetilde{C}} \longrightarrow \varphi^*T_Y \longrightarrow N_\varphi \longrightarrow 0. \tag{8.1.2}$$

The sheaf N_φ has torsion in general. More precisely we have an exact sequence:

$$0 \longrightarrow \mathcal{H}_\varphi \longrightarrow N_\varphi \longrightarrow \overline{N}_\varphi \longrightarrow 0 \tag{8.1.3}$$

where $\mathcal{H}_\varphi = \mathcal{O}_Z$ for some effective divisor Z supported on the locus where the differential $d\varphi$ degenerates. Therefore $\varphi(Z) \subseteq C$ is the set of points where C has a non-linear branch. The exact sequences (8.1.2) and (8.1.3) imply that:

$$c_1(N_\varphi) = \omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}$$

and

$$\overline{N}_\varphi \cong \omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}(-Z).$$

We have the following useful:

(8.2) Lemma (Arbarello-Cornalba [2])**.** *If $\theta \in H^0(\mathcal{H}_\varphi)$ then the corresponding infinitesimal (first order) deformation of φ leaves the image fixed.*

In case (b) we have an exact sequence

$$0 \to \mathcal{K} \longrightarrow \varphi^*\Omega^1_Y \longrightarrow \Omega^1_{\widetilde{C}} \to 0 \tag{8.2.1}$$

where $\mathcal{K}$ is invertible because $\Omega^1_{\widetilde{C}}$ has homological dimension 1 since $\widetilde{C}$ has l.c.i. singularities (which implies that $\Omega^1_{\widetilde{C}}$ is resolved by the conormal exact sequence of a local embedding). Then in the sequence (8.1.1) we have an identification $N_\varphi = \mathcal{K}^\vee$.

(8.3) Lemma. *In case (b) we have*

$$N_\varphi \cong \omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}.$$

Proof. Since $\widetilde{C}$ is a local complete intersection, we have

$$\omega_{\widetilde{C}} = \det(\Omega^1_{\widetilde{C}})$$

by [10, Thm. III.7.11]. On the other hand the exact sequence (8.2.1) yields

$$\det(\Omega^1_{\widetilde{C}}) = \det(\varphi^*\Omega^1_Y) \otimes \mathcal{K}^{-1} = \varphi^*\omega_Y \otimes N_\varphi.$$

The result follows since $\varphi^*\omega_Y$ is invertible. □

(8.4) Corollary. *In case (b) we have, in the notation of Section 2 for the conductor:*

$$N_\varphi \cong \varphi^* N_{C/Y} \otimes \widetilde{\mathcal{A}}(\varphi)$$

and

$$\varphi_* N_\varphi \cong N_{C/Y} \otimes \mathcal{A}(\varphi).$$

Proof. It is a direct consequence of the previous lemma and Corollary (2.4). □

9 – A criterion for the density of nodal curves

We keep the same notations of the previous section.

(9.1) Theorem. *Let $V \subseteq V^\xi_g$ be an irreducible component and let $[C] \in V$ be a general point with normalization $\varphi : \widetilde{C} \to Y$. Consider the following conditions:*

(a) $\omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}$ *is globally generated.*

(b) $\dim(V) \geq h^0(\widetilde{C}, \omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1})$.

(c) $h^0(\omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}(-\mathfrak{a})) = h^0(\omega_{\widetilde{C}} \otimes \varphi^*\omega_Y^{-1}) - 3$ *for every effective divisor $\mathfrak{a}$ of degree 3 on $\widetilde{C}$.*

If (a) and (b) are satisfied then C is immersed, i.e., all its local branches are linear. If also (c) is satisfied then C is nodal.

For the proof we refer to [5]. Let's see an application.

(9.2) Corollary (Arbarello-Cornalba [1], Zariski [22])**.** *The general element of any component of the Severi variety $V_{d,g}$ of integral plane curve of degree d and genus g is a nodal curve.*

Proof (outline). Let $[C] \in V_{d,g}$ be a general element of a component. We have

$$\deg(\omega_{\widetilde{C}} \otimes \varphi^* \omega_{\mathbb{P}^2}^{-1}) = 2g - 2 + 3d$$

Since $d \geq 3$ (otherwise there is nothing to prove) conditions (a) and (c) of the Theorem are clearly satisfied. One can construct a family of curves in $V_{d,g}$ as the image of the semiuniversal deformation of φ. Using (8.1.3) one can easily deduce that this family satisfies (b) as well. □

References

[1] Arbarello E., Cornalba M., *Su una proprietà notevole dei morfismi di una curva a moduli generali in uno spazio proiettivo*, Rend. Sem. Mat. Univ. Politec. Torino **38** (1980), no. 2, 87–99 (1981).

[2] ———, *Su una congettura di Petri*, Comment. Math. Helv. **56** (1981), no. 1, 1–38.

[3] Bogomolov F., Hassett B., Tschinkel Y., *Constructing rational curves on K3 surfaces*, Duke Math. J. **157** (2011), 535- 550.

[4] Chiang-Hsieh H.-J. and Lipman J., *A numerical criterion for simultaneous normalization*, Duke Math. J. **133** (2006), no. 2, 347–390.

[5] Dedieu Th., Sernesi E., *Equigeneric and equisingular families of curves on surfaces*, Pub. Mat. **62** (2017), 175–212.

[6] Diaz S., Harris J., *Ideals associated to deformations of singular plane curves*, Trans. AMS **309** (1988), no. 2, 433–468.

[7] Druel S., *Courbes sur les variétés et leurs espaces de modules – Quelques applications*, lecture notes available at `http://druel.perso.math.cnrs.fr/textes/ecole2010.pdf`.

[8] Flenner H., *Ein Kriterium für die Offenheit der Versalität*, Math. Z. **178** (1981), 449-473.

[9] Greuel M., Lossen C., Shustin E., *Introduction to Singularities and Deformations,* Springer Monographs in Mathematics (2007).

[10] Hartshorne R., *Algebraic Geometry*, Springer GTM n. 52 (1977).

[11] ———, *Deformation Theory*, Springer GTM n. 257 (2010).

[12] Hironaka H., *On the arithmetic genera and the effective genera of algebraic curves*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **30** (1957), 177–195.

[13] Reid M.: *Young Person's Guide to Canonical Singularities*, in Algebraic geometry, Proc. Summer Res. Inst., Brunswick/Maine 1985, part 1, Proc. Symp. Pure Math. 46, 345–414 (1987).

[14] Risler J.J., *Sur l'idéal jacobien d'une courbe plane*, Bull. Soc. Math. de France **99** (1971), 305-311.

[15] Rosenlicht M., *Equivalence relations on algebraic curves*, Ann. of Math. (2) **56** (1952), 169–191.

[16] Saito K., *Quasihomogene isolierte Singularitäten von Hyperflächen*, Inventiones Math. **14** (1971), 123-142.

[17] Samuel P., *Algebraicité de certains points singuliers algébroides*, J. de Mathématiques Pures et Appliquées **35** (1956), p. 1-6.

[18] Sernesi E., *Deformations of Algebraic Schemes*, Springer Grundlehren b. 334 (2006).

[19] Teissier B., *The hunting of invariants in the geometry of discriminants*, in *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, 565–678.

[20] ———, *Résolution simultanée I, II*, in *Séminaire sur les singularités des surfaces, Cent. Math. Éc. Polytech., Palaiseau 1976–77*, Lecture Notes in Mathematics, vol. 777, Springer-Verlag, Berlin, 1980, 71–146.

[21] Vakil R., *The rising sea — Foundations of Algebraic Geometry*, February 21, 2024 version, available at `http://math.stanford.edu/~vakil/216blog/FOAGfeb2124public.pdf`.

[22] Zariski O., *Dimension-theoretic characterization of maximal irreducible algebraic systems of plane nodal curves of a given order n and with a given number d of nodes*, Amer. J. Math. **104** (1982), no. 1, 209–226.

[23] ———, *Le problème des modules pour les branches planes*, Hermann (Paris), 1986.

Lecture III
Geometry of logarithmic Severi varieties at a general point

by Thomas Dedieu

1 Definitions and main results 58
2 Background from deformation theory 61
2.1 Deformations of maps with fixed target 61
2.2 Comparison of the spaces of maps and curves 62
2.3 Tangency conditions with respect to a fixed curve 64
3 Proof of the Main Theorem 66
3.1 Applications of the Riemann–Roch formula 66
3.2 Proof of the main theorem 67
4 A dimensional characterization of logarithmic Severi varieties 71
5 Examples 75
5.1 An example with contact points in special position 75
5.2 Superabundant log Severi varieties coming from double covers . . . 76
5.3 Logarithmic $K3$ surfaces 77
References 80

This lecture is devoted to the study of logarithmic Severi varieties of a pair (S, R), where S is a surface and R is a curve on S: these are the families of curves on S with prescribed homology class and geometric genus, and prescribed contact pattern with R, meaning that contact orders with R are prescribed both at fixed assigned points and at unassigned, a priori mobile, points. Logarithmic Severi varieties are often referred to as relative Severi varieties.

Logarithmic Severi varieties naturally appear as irreducible components of limit Severi varieties in degenerations of smooth surfaces, see [I]. The emblematic example is that of logarithmic Severi varieties of the pair $(\mathbf{P}^2, \text{line})$ appearing in degenerations of Severi varieties of plane curves of a given degree and genus, see [VI] and [VII] in this collection. On the other hand, in degenerations of $K3$ surfaces we will rather encounter logarithmic Severi varieties of pairs (S, R) such that $K_S + R = 0$, see Section 5.3.

The central result of this text is Theorem (1.7). On the one hand it gives the expected dimension of logarithmic Severi varieties, together with a condition ensuring that this is indeed the actual dimension. On the other hand it gives various regularity properties of the general members of logarithmic Severi varieties, under suitable assumptions on the positivity of $-K_S$ with respect to the prescribed homology class and contact conditions with R.

Another important result is Proposition (4.3), which characterizes logarithmic Severi varieties as essentially the families of curves of maximal dimension with respect to the homology class, the genus, and the number of moving contact points with R, under a positivity assumption on the logarithmic canonical divisor $-(K_S + R)$. It will be used in subsequent lectures to prove (in specific situations) that the irreducible components of a degeneration of logarithmic Severi varieties are again logarithmic Severi varieties (or rather, irreducible components thereof).

The main source for this lecture is Caporaso and Harris' [6, §2]; slightly earlier important works are [19] and [4]. I have adapted the presentation given in [6], which is restricted to the pair consisting of the projective plane and a line, in order to make it fit to the more general situation studied here, and needed further in this volume. My treatment is very much inspired by [10], which I worked out together with Edoardo Sernesi, see also [II] earlier in this volume. I have included many examples throughout, with the two purposes of clarifying the subtleties of the various definitions and statements, and of putting these results in perspective. There are also many (other) examples in [10]. Kleiman and Shende's [14], as well as the appendix by Tyomkin, is an excellent reference on the same topic.

The organization of the text is as follows. Section 1 is devoted to the definition of logarithmic Severi varieties and the statement of the main theorem (1.7), and Section 2 contains the material in deformation theory necessary for the proof of the main theorem, which is given in Section 3. Section 4 is devoted to the aforementioned Proposition (4.3) characterizing logarithmic Severi varieties by the maximality of their dimensions, and its proof. Finally, Section 5 is devoted to expanded examples, featuring many examples of superabundant logarithmic Severi varieties, and highlighting some specificities of the case when the logarithmic canonical divisor $K_S + R$ is trivial; we also point out some interesting applications of logarithmic Severi varieties (e.g., to the tropical vertex group, and to $\mathbf{A}^1$-curves on open surfaces obtained as the complement of an anticanonical curve).

1 – Definitions and main results

(1.1) Let S be a nonsingular, projective, connected, algebraic surface over the field $\mathbf{C}$ of complex numbers, and let $R \subseteq S$ be a reduced curve.

By a curve on S, we always mean a closed subscheme of S of pure dimension 1. The *geometric genus* of a reduced curve C is the (arithmetic) genus of its normalization $\bar{C}$, namely

$$1 - \chi(\mathcal{O}_{\bar{C}}) = \sum_{i=1}^{n} g_i - n + 1$$

where $g_1, \ldots, g_n$ are the respective genera of the connected components $\bar{C}_1, \ldots, \bar{C}_n$ of $\bar{C}$. Note that with this definition, denoting by g the geometric genus of C, the canonical bundle of the normalization $\bar{C}$ has degree $2g - 2$, and one has the usual Riemann–Roch Formula on $\bar{C}$, i.e.,

$$\chi(L) = 1 - g + \deg(L)$$

for all line bundles on $\bar{C}$.

For every $\xi \in \mathrm{NS}(S)$ (thus, ξ is a homology class on S that can be represented by a divisor) and integer g, we consider the space $M_g^{\xi,\mathrm{bir}}(S)$ parametrizing morphisms

$$\phi : C \to S$$

from a smooth genus g curve C (projective, but possibly disconnected) that are birational to their image, and such that $\phi_*[C] = \xi$. The space $M_g^{\xi,\mathrm{bir}}(S)$ is defined more precisely and studied in Subsection 2.1.

We also consider the locally closed subscheme $V_g^{\xi}(S)$ of $\mathrm{Curve}(S)$, consisting of those points $[C]$ such that C is a reduced curve having geometric genus g and homology class ξ; by $\mathrm{Curve}(S)$, we denote the Hilbert scheme of curves on S.

(1.2) We denote by $\underline{\mathbf{N}}$ the set of sequences $\alpha = [\alpha_1, \alpha_2, \ldots]$ of non-negative integers with only finitely many α_i non-zero. In practice, we may omit the infinite sequence of zeros at the end. We may also drop the brackets if only α_1 is non-zero.

For all $\alpha \in \underline{\mathbf{N}}$, we let

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots ;$$
$$I\alpha = \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n + \cdots .$$

For all $\alpha, \alpha' \in \underline{\mathbf{N}}$, we say that $\alpha \geqslant \alpha'$ if $\alpha_i \geqslant \alpha'_i$ for all $i \geqslant 1$.

By a set Ω of cardinality $\alpha \in \underline{\mathbf{N}}$, we mean a sequence of sets $\Omega = (\Omega_1, \Omega_2, \ldots)$ such that each Ω_i has cardinality α_i.

(1.3) Definition. *Let $g \in \mathbf{Z}$, $\xi \in \mathrm{NS}(S)$, and $\alpha, \beta \in \underline{\mathbf{N}}$ be such that*

$$I\alpha + I\beta = \xi \cdot R,$$

and consider a set $\Omega = \left(\{p_{i,j}\}_{1 \leqslant j \leqslant \alpha_i}\right)_{i \geqslant 1}$ of α points on R.[1]

We define $M_g^\xi(\alpha, \beta)(\Omega)$ as the locally closed subset of $M_g^{\xi, bir}(S)$ consisting of those $[\phi : C \to S]$ such that the intersection $\phi(C) \cap R$ is proper and contained in the smooth locus of R, and there exist α points $q_{i,j} \in C$, $1 \leqslant j \leqslant \alpha_i$, and β points $r_{i,j} \in C$, $1 \leqslant j \leqslant \beta_i$, such that

$$\forall\, 1 \leqslant j \leqslant \alpha_i : \quad \phi(q_{i,j}) = p_{i,j} \qquad \text{and} \tag{1.3.1}$$
$$\phi^* R = \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} i\, r_{i,j}. \tag{1.3.2}$$

Note that this definition is functorial. In other words, $M_g^\xi(\alpha, \beta)(\Omega)$ represents a certain functor, see (2.10).

(1.4) Remark. We emphasize that we do not assume the points in Ω to be general, and neither do we in the Main Theorem (1.7) below. It will be important for some applications to have this flexibility.

(1.5) Definition. *Continuing with the situation of Definition (1.3), we let $V_g^\xi(\alpha, \beta)(\Omega)$ be the locally closed subscheme of $V_g^\xi(S)$ consisting of those points $[C]$ such that the normalization $\nu : \bar{C} \to C \subseteq S$ belongs to $M_g^\xi(\alpha, \beta)(\Omega)$. We call it a* logarithmic[2] Severi variety *of the pair (S, R).*

(1.6) Notation, examples, and comments. In practice we will try to find a balance between rigorous and decipherable notation. For instance we will frequently drop the Ω, and replace ξ with an adequate shorthand.

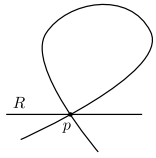
Let us consider the emblematic case when $S = \mathbf{P}^2$ and R is a line. Let $\xi = d[H]$ with $[H]$ the hyperplane class; we shall simply denote ξ by d. For example, for all g, we may write $V_g^d\big([0,1], d-2\big)$ for the family of plane d-ics of genus g, tangent to the line R at some prescribed general point $p \in R$ (in this case, $\Omega = \{p\}$).

[1] If R is reducible one should consider more precise data, specifying the distribution of the contact points with R on its various components: let $R_1, \ldots, R_n$ be the irreducible components of R; one should consider $\alpha_1, \ldots, \alpha_n \in \underline{\mathbf{N}}$ and $\beta_1, \ldots, \beta_n \in \underline{\mathbf{N}}$ such that $I\alpha_i + I\beta_i = \xi \cdot R_i$ for all $i = 1, \ldots, n$, and $\Omega_1, \ldots, \Omega_n$ sets of points on $R_1, \ldots, R_n$ respectively, with cardinalities $\alpha_1, \ldots, \alpha_n$. For the purposes of this chapter we can safely ignore this, and we will do so for obvious reasons of simplicity. (There will be, however, some examples with reducible R, in Section 5 most notably, and then of course we will consider data with the necessary precision).

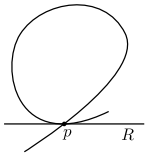
[2] in [6] it is a called a *generalized* Severi variety; the terminology *relative* Severi variety is nowadays quite popular; I prefer the term *logarithmic* which refers to the fact that we consider pairs instead of "absolute" surfaces, whereas *relative* often refers to the consideration of a map (or a family) instead of a surface.

Let $\mathrm{p}(d)$ be the arithmetic genus of plane curves of degree d. The logarithmic Severi variety $V^d_{\mathrm{p}(d)}\big([0,1], d-2\big)$ parametrizes smooth d-ics tangent to R at p; it has codimension 2 in the linear system $|dH|$. The logarithmic Severi variety $V^d_{\mathrm{p}(d)-1}\big([0,1], d-2\big)$ parametrizes plane d-ics of cogenus 1, tangent to R at p; it has codimension 3 in the linear system $|dH|$ (hence $V^d_{\mathrm{p}(d)-1}\big([0,1], d-2\big)$ is a divisor in $V^d_{\mathrm{p}(d)}\big([0,1], d-2\big)$), and its general member is a curve with one node at a general point of $\mathbf{P}^2$, as we will see.

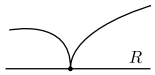
We emphasize that Definition (1.3) requires that curves in $V^d_{\mathrm{p}(d)-1}\big([0,1], d-2\big)$ have one local branch with the prescribed order two of contact with R at p; we illustrate this below.



not in $V^d_{p(d)-1}\big([0,1]\big)$



in $V^d_{p(d)-1}\big([0,1]\big)$



in $V^d_{p(d)-1}\big([0,1]\big)$

Another subtlety of Definition (1.3) worth noting is the following. Consider the logarithmic Severi variety $V = V^d_{\mathrm{p}(d)}\big(1, [d-3, 1]\big)$: loosely speaking, it parametrizes curves passing through an assigned point on R, call it p, and tangent to R. However, looking closely at the definition, one sees that a curve that is simply tangent to R at p and otherwise intersects it transversely does not belong V; a curve having an order 3 contact at p and otherwise transverse does belong to V, on the other hand.

The following is the main result of this chapter. It is a vast generalization of the result proved independently by Arbarello and Cornalba [3] and Zariski [19] for plane curves of fixed degree and genus.

(1.7) Theorem. *Let $g \in \mathbf{Z}$, $\xi \in \mathrm{NS}(S)$, $\alpha, \beta \in \underline{\mathbf{N}}$, and $\Omega = \{p_{i,j}\}_{1 \leqslant j \leqslant \alpha_i} \subseteq R$ be as in Definition (1.3), and consider an irreducible component V of $V^{\xi}_g(\alpha, \beta)(\Omega)$. Let $[C]$ be a general member of V, $\phi : \bar{C} \to C \subseteq S$ its normalization, $q_{i,j}$ $(1 \leqslant j \leqslant \alpha_i)$, $r_{i,j}$ $(1 \leqslant j \leqslant \beta_i)$ points on $\bar{C}$ such that* (1.3.1) *and* (1.3.2) *hold. Set*

$$D = \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} (i-1)\, r_{i,j}.$$

(1.7.00) One has

$$\dim V \geqslant -(K_S + R) \cdot \xi + g - 1 + |\beta|.$$

(1.7.0) If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 1$ for every irreducible component C_i of C, then the inequality in (1.7.00) above is an equality.

(1.7.1) If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 2$ for every irreducible component C_i of C, then:

(a$^\natural$) *the normalization map ϕ is an immersion, except possibly at the points $r_{i,j}$;*

(b) *the points $q_{i,j}$ and $r_{i,j}$ of $\bar{C}$ are altogether pairwise distinct;*

(c$^\flat$) *none of the points $s_{i,j} := \phi(r_{i,j})$ belongs to Ω;*

(d) *for every curve $G \subseteq S$ and finite set $\Gamma \subseteq S$ such that $(G \cup \Gamma) \cap \Omega = \varnothing$, if $[C]$ is general with respect to G and Γ, then C intersects G transversely and does not intersect Γ.*

(1.7.2) If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 3$ for every irreducible component C_i of C, then:

(a$^\flat$) *the normalization map ϕ is an immersion;*

(c) *the points $p_{i,j}$ and $s_{i,j} = \phi(r_{i,j})$ on C are altogether pairwise distinct;*

(e) *the curve C is smooth at its intersection points with R.*

(1.7.3) If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 4$ *for every irreducible component* C_i *of* C*, then*
(a) *the curve* C *is nodal.*

The tangent space of $V_g^\xi(\alpha,\beta)(\Omega)$ at the general point $[C]$ of V is described in Remark (3.9) under the assumptions of (1.7.2), and in Paragraph (3.6) under the weaker assumptions of (1.7.0).

We emphasize once again that it is not required in the above statement that the set Ω be general. The only requirement is that the points in Ω be smooth points of R, which in turn is imposed by the requirement, in Definitions (1.3) and (1.5), that the intersection with R be proper and contained in the smooth locus of R.

We will see examples in Section 5 of the effect that imposing the passing through a set Ω of non-general points may have. We will also see examples of components of Severi varieties parametrizing reducible curves, for which the assumption of (1.7.0) fails for one irreducible component C_i, and having dimension strictly larger than expected (i.e., the inequality in (1.7.00) is strict). We will also hint at the construction of examples (parametrizing irreducible curves) for which the assumption of (1.7.1) does not hold and the assertions (b), ($c^\flat$), and (d) do not hold.

2 – Background from deformation theory

Throughout this section, we consider a nonsingular, connected, projective surface S, defined over $\mathbf{C}$.

2.1 – Deformations of maps with fixed target

(2.1) Let $\phi : C \to S$ be a non-constant morphism from a smooth projective curve C. A *deformation of* ϕ *with fixed target* over a pointed base $(B,0)$ is the data of a deformation $\mathcal{C} \xrightarrow{\pi} B$ of C over $(B,0)$ together with a morphism $\Phi : \mathcal{C} \to S \times B$ of B-schemes, such that the restriction of Φ over 0 equals ϕ.

This defines the *deformation functor* $\mathrm{Def}_{\phi/S}$ of ϕ with fixed target S. It is prorepresented by a complete local $\mathbf{C}$-algebra R_ϕ, see [16, Thm. 3.4.8].

(2.2) The deformations of ϕ with fixed target S are controlled by the *normal sheaf of* ϕ, i.e., the sheaf N_ϕ of $\mathcal{O}_C$-modules defined by the exact sequence on C

$$0 \to T_C \xrightarrow{d\phi} \phi^* T_S \to N_\phi \to 0 \ : \tag{2.2.1}$$

the spaces $H^0(C,N_\phi)$ and $H^1(C,N_\phi)$ are respectively the Zariski tangent space and an obstruction space for the deformations of ϕ with fixed target S. In particular, we have

$$\chi(N_\phi) \leqslant \dim R_\phi \leqslant h^0(N_\phi). \tag{2.2.2}$$

(2.3) The rank 1 sheaf N_ϕ may have torsion. We denote by $\mathcal{H}_\phi$ its torsion part and by $\bar{N}_\phi$ its maximal torsion-free quotient; they fit in an exact sequence

$$0 \to \mathcal{H}_\phi \to N_\phi \to \bar{N}_\phi \to 0. \tag{2.3.1}$$

The torsion sheaf $\mathcal{H}_\phi$ is supported on the divisor Z of zeroes of the differential $d\phi$, and it is zero if and only if $Z = 0$, i.e., if and only if the differential $d\phi$ is everywhere non-vanishing; in this

case, we say that ϕ is an immersion. Moreover, there is an exact sequence of locally free sheaves on C

$$0 \to T_C(Z) \to \phi^* T_S \to \bar{N}_\phi \to 0, \tag{2.3.2}$$

which readily implies the identification of line bundles on C

$$\bar{N}_\phi \cong \omega_C \otimes \phi^* \omega_S^{-1}(-Z). \tag{2.3.3}$$

(2.4) The scheme of morphisms. We will pretend in this text that the global deformation functor of maps from smooth genus g curves to the surface S is represented by a scheme, which we will denote by $M_g(S)$. This comports the existence of a family $\mathcal{U} \to M_g(S)$ of smooth genus g curves defined over $M_g(S)$, and of a universal morphism $\Phi : \mathcal{U} \to S \times M_g(S)$.

It is not strictly true that the global deformation functor of genus g maps to S is representable in the category of schemes, but it will be harmless for our purposes to pretend it is: one possible way to make things rigorously work out is to use Hartshorne's modular families [12, Def. p.171], as in [10, Subsec. 2.1]. A modular family of curves of genus g is a surrogate of a universal family of curves of genus g. Let us consider such a modular family $\mathcal{C}_g/\mathcal{M}_g$, with the abuse of notation of denoting its base $\mathcal{M}_g$ although it is not the moduli space of genus g curves. Then we may work with the scheme $\mathrm{Hom}_{\mathcal{M}_g}(\mathcal{C}_g, S \times \mathcal{M}_g)$ as a surrogate for $M_g(S)$, with Hom defined as in [15, Thm. I.1.10]. We will not enter in these technicalities here, and refer the reader to [10, Subsec. 2.1] for the complete details.

In this text, we will be mostly interested in the following subspace of $M_g(S)$. For all $\xi \in \mathrm{NS}(S)$, we consider $M_g^{\xi,\mathrm{bir}}(S)$ the subspace of $M_g(S)$ parametrizing morphisms ϕ that are birational on their image, and such that $\phi_*[C] = \xi$.

2.2 – Comparison of the spaces of maps and curves

Let $\xi \in \mathrm{NS}(S)$, and g be a non-negative integer.

(2.5) From maps to curves. Consider the universal morphism $\Phi : \mathcal{U}_M \to S \times M_g^{\xi,\mathrm{bir}}$ defined over $M_g^{\xi,\mathrm{bir}}$. Let $\bar{M}$ be the semi-normalization (see [15, Section I.7.2]) of the reduced scheme underlying $M_g^{\xi,\mathrm{bir}}$, $\bar{\mathcal{U}}_M := \mathcal{U}_M \times_{M_g^{\xi,\mathrm{bir}}} \bar{M}$ the base change of the universal family $\mathcal{U}_M$, and $\bar{\Phi} : \bar{\mathcal{U}}_M \to S \times \bar{M}$ the induced morphism of $\bar{M}$-schemes. I claim that the scheme-theoretic image $\bar{\Phi}(\bar{\mathcal{U}}_M)$ is flat over $\bar{M}$.

Indeed, the morphism $\varpi := \mathrm{pr}_2 : \bar{\Phi}(\bar{\mathcal{U}}_M) \to \bar{M}$ is a well-defined family of codimension 1 algebraic cycles of S in the sense of [15, I.3.11]. Since $\bar{M}$ is semi-normal, the claim follows from [15, I.3.23.2].

It follows that there is a morphism from $\bar{M}$ to the Hilbert scheme of curves on S. It factorizes through $V_g^\xi(S)$, and actually through the semi-normalization $\bar{V}_g^\xi \to V_g^\xi$ by the universal property of the semi-normalization. Since by definition the semi-normalization morphism $\bar{M} \to M_g^{\xi,\mathrm{bir}}$ is $1:1$, two points $[\phi : C \to S], [\phi' : C' \to S] \in \bar{M}$ are mapped to the same point $[\Gamma] \in V_g^\xi$ if and only if there exists an isomorphism $\iota : C \cong C'$ such that $\phi = \phi' \circ \iota$.

(2.6) From curves to maps. On the other hand, consider the universal family $\mathcal{U}_V \to V_g^\xi$ of curves gotten from the universal family over the Hilbert scheme of curves on S. Let $\bar{V}$ be the normalization of V_g^ξ, and $\bar{\mathcal{U}}_V$ the normalization of $\mathcal{U}_V \times_{V_g^\xi} \bar{V}$. Teissier's Résolution Simultanée Theorem [18] asserts that $\bar{\mathcal{U}}_V \to \bar{V}$ is a family of smooth genus g curves; it comes with a morphism of $\bar{V}$-schemes

$$\bar{\mathcal{U}}_V \to \mathcal{U}_V \times_{V_g^\xi} \bar{V} \subseteq S \times \bar{V}.$$

It follows that there is a morphism from $\bar{V}$ to the space $M_g^{\xi,\mathrm{bir}}$. It is generically injective, because the universal family of curves over V_g^ξ is nowhere isotrivial.

From the considerations in (2.5) and (2.6) above, one deduces the following.

(2.7) Proposition. *Let $[\phi : C \to S] \in M_g^{\xi,bir}(S)$ be a general point (i.e., a general point of any irreducible component of $M_g^{\xi,bir}(S)$). Let $\Gamma := \phi(C)$, $\xi \in \mathrm{NS}(S)$ the homology class of Γ, and g its geometric genus. Then $[\Gamma]$ belongs to a unique irreducible component of V_g^ξ and*

$$\dim_{[\Gamma]} V_g^\xi = \dim R_\phi.$$

(Recall that R_{ϕ_0} is the complete local $\mathbf{C}$-algebra that prorepresents $\mathrm{Def}_{\phi/S}$).

The next result provides a sharper upper bound on the dimension of the Severi varieties than that given by the inequality $\dim R_\phi \leqslant h^0(N_\phi)$ in (2.2.2).

Let $\phi : C \to S$ be a morphism from a smooth projective curve C, birational onto its image Γ. Let $\xi \in \mathrm{NS}(S)$ be the homology class of Γ, and g its geometric genus. We consider $\Phi : \mathcal{C} \to S \times B$ a deformation of ϕ over a pointed normal connected scheme $(B, 0)$. Then $\Phi(\mathcal{C}) \subseteq S \times B$ is a deformation of Γ over $(B, 0)$, see (2.5). There are thus two classifying morphisms κ and γ from $(B, 0)$ to $M_g^{\xi,\mathrm{bir}}(S)$ (or R_ϕ) and $\mathrm{Curve}(S)$ respectively, with differentials

$$d\kappa : T_{B,0} \to H^0(C, N_\phi) \quad \text{and} \quad d\gamma : T_{B,0} \to H^0(\Gamma, N_{\Gamma/X}).$$

(2.8) Lemma. *The inverse image by $d\kappa$ of the torsion $H^0(C, \mathcal{H}_\phi) \subseteq H^0(C, N_\phi)$ is contained in the kernel of $d\gamma$.*

Proof. Given a non-zero section $\sigma \in H^0(C, N_\phi)$, the first order deformation of ϕ defined by σ can be described in the following way: consider an affine open cover $\{U_i\}_{i\in I}$ of C, and for each $i \in I$ consider a lifting $\theta_i \in C(U_i, \phi^* T_X)$ of the restriction $\sigma_{|U_i}$. Each θ_i defines a morphism

$$\tilde{\phi}_i : U_i \times \mathrm{Spec}(\mathbf{C}[\varepsilon]) \to S$$

extending $\phi_{|U_i} : U_i \to X$. The morphisms $\tilde{\phi}_i$ are then made compatible after gluing the trivial deformations $U_i \times \mathrm{Spec}(\mathbf{C}[\varepsilon])$ into the first order deformation of C defined by the coboundary $\partial(\sigma) \in H^1(C, T_C)$ of the exact sequence (2.2.1). In case $\sigma \in \mathrm{H}^0(C, \mathcal{H}_\phi)$, everyone of the maps $\tilde{\phi}_i$ is the trivial deformation of $\sigma_{|U_i}$ over an open subset. This implies that the corresponding first order deformation of ϕ leaves the image fixed, hence the vanishing of $dq_0(\sigma)$. □

(2.9) Corollary. *Let $g \in \mathbf{Z}$, $\xi \in \mathrm{NS}(S)$. Let $[C]$ be a general point of V_g^ξ, and $\phi : \bar{C} \to C \subseteq S$ its normalization. Then*

$$\dim V_g^\xi \leqslant h^0(\bar{C}, \bar{N}_\phi).$$

Proof. By generality we may assume that $[C]$ is a smooth point of V_g^ξ. Then $\dim V_g^\xi = \dim T_{[C]} V_g^\xi$, and by (2.6) there is a map

$$d\kappa_{[\phi]} : T_{[C]} V_g^\xi \to H^0(\bar{C}, N_\phi).$$

It is injective because to every tangent vector $\theta \in T_{[C]} V$ corresponds a non-trivial deformation of C. On the other hand, it follows from Lemma (2.8) that $\mathrm{Im}\, d\kappa_{[\phi]} \subseteq H^0(\bar{C}, \bar{N}_\phi)$. □

Lemma (2.8) (see also [II, Lemma (8.2)]) is a crucial observation (and indeed one of the cornerstones of the proof of Theorem (1.7)) that was made by Arbarello and Cornalba [4, p. 26], who deemed it a *fenomeno assai curioso*. They write: « nel caso in cui ϕ sia una birazionalità tra C e Γ, la presenza di "cuspidi" su Γ, comporta l'esistenza, dal punto di vista infinitesimo, di più di un modello liscio della curva Γ, se così ci possiamo esprimere. »[3]

Next, paraphrasing them, in order to use this phenomenon constructively they establish [4, Cor. 6.11]: in the above notation, if B is the complex unit disk and if the family of curves is *equisingular*, then $d\kappa(\partial/\partial t)$ belongs to $H^0(C, \mathcal{H}_\phi)$ if and only if it is zero. Later, Caporaso and Harris (together with J. de Jong, they write) state and prove [6, Lem. 2.3]. They add the remark that this is linked to the notion of equisingularity, even though they make absolutely no use of that notion, neither in their statement nor in its proof.

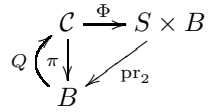
The treatment I give above is that of Sernesi and myself in [10]. Although essentially equivalent to that of [6], it slightly differs in its formulation. This formulation, I hope, sheds some light on what is actually going on, and in particular shows that it is not necessary to invoke equisingularity in order to prove Corollary (2.9).

The standard application of Corollary (2.9) is to the proof that curves in a given family are immersed, i.e., they have no cusps: we will use it in Paragraph (3.7) below to prove assertion ($\mathrm{a}^\flat$) of Theorem (1.7).

2.3 – Tangency conditions with respect to a fixed curve

We consider R a fixed reduced curve on S. In this subsection we study deformations of curves on S satisfying some tangency conditions with R; it follows from our Definition (1.3) that it suffices to treat the case when R is smooth.

(2.10) Let m be a non-negative integer. Let $\phi : C \to S$ be a non-constant morphism from a smooth projective curve C. A deformation of ϕ with fixed target *preserving a tangency of order m with R* over a pointed base $(B, 0)$ is a deformation $\Phi : \mathcal{C} \to S \times B$ of ϕ with fixed target as in (2.1), such that there exists a section Q of $\mathcal{C} \to B$ such that the pulled-back divisor $\Phi^* R$ contains Q with multiplicity m (i.e., $\Phi^* R - mQ \geqslant 0$).



The tangency is said to be respectively at a *fixed point* $p \in R$ if $\Phi(Q) = \{p\} \times B$, and at a *variable point* if $\mathrm{pr}_1 \circ \Phi(Q)$ is a curve.

We say that a family of maps $\Phi : \mathcal{C} \to S \times B$ preserves a tangency of order m with R if for all $b \in B$ it is locally around b a deformation of maps preserving a tangency of order m.

The following result displays the additional conditions the class of a deformation of maps has to meet for this deformation to preserve a tangency with R. It is [6, Lem. 2.6]. Let B be a base which is smooth at its general point (e.g., B is an integral scheme of finite type), and $\Phi : \mathcal{C} \to S \times B$ be a family of maps preserving a tangency with R of order exactly m. Let $b \in B$ be a general point, and $\phi : C \to S$ be the corresponding map. This comes with a classifying map κ, with differential $d\kappa : T_0 B \to H^0(C, N_\phi)$; we call $\bar{d}\kappa$ its composition with the projection $H^0(C, N_\phi) \to H^0(C, \bar{N}_\phi)$.

[3] a very curious phenomenon; in the case when ϕ is birational between C and Γ, the presence of "cusps" on Γ comports the existence, at the infinitesimal level, of more than one smooth model of the curve Γ, if we may say so.

Let $q := Q \cap C$, and $l-1$ be the order of vanishing of the differential $d\phi$ at q (i.e., l is the multiplicity of the point $p := \phi(q)$ in the local branch of $\phi(C)$ corresponding to q). Note that necessarily $l \leqslant m$.

(2.11) Lemma. *Let $\sigma \in \mathrm{Im}(\bar{d}\kappa) \subseteq H^0(C, \bar{N}_\phi)$ be a non-zero section, and denote by $v_q(\sigma)$ its order of vanishing at $q = Q \cap C$.*
(a) *One has $v_q(\sigma) \in \{m-l\} \cup [\![m, +\infty[\![$.*[4]
(b) *If the tangency is at a fixed point of R, then actually $v_q(\sigma) \in [\![m, +\infty[\![$.*

Proof. This is a local computation. Let (x, y) be (analytic) local coordinates on S at $p = \phi(q)$, such that R is defined by the equation $y = 0$. Then the vector fields $\partial/\partial x$ and $\partial/\partial y$ generate T_S near p, and their pull-backs generate $\phi^* T_S$ near q; by abuse of notation we shall denote them by $\partial/\partial x$ and $\partial/\partial y$ as well.

We may assume that B is a curve. Let ε be a local coordinate on B centered at b, and t be a local equation of the section Q near q. Thus (t, ε) are local coordinates on $\mathcal{C}$ at q. We may assume that t gives a local coordinate on C at q, in such a way that ϕ is given locally by

$$\phi(t) = \begin{cases} \big(t^l + a_{l+1}t^{l+1} + \cdots, t^m\big), & \text{if } l < m \\ \big(t^n + a_{n+1}t^{n+1} + \cdots, t^m\big) \text{ for some } n \geqslant m, & \text{if } l = m. \end{cases}$$

From now on we assume $l < m$ and leave the other, similar, case to the reader. Then the differential of ϕ at q is

$$\begin{aligned} \tfrac{\partial}{\partial t} \longmapsto & (lt^{l-1} + (l+1)a_{l+1}t^l + \cdots) \cdot \tfrac{\partial}{\partial x} + mt^{m-1} \cdot \tfrac{\partial}{\partial y} \\ & = t^{l-1}\big((l + (l+1)a_{l+1}t + \cdots) \cdot \tfrac{\partial}{\partial x} + mt^{m-l} \cdot \tfrac{\partial}{\partial y}\big). \end{aligned}$$

We see that around q on C, the torsion part $\mathcal{H}_\phi$ of N_ϕ is a skycraper sheaf of length $l-1$ concentrated at q, generated by the image in N_ϕ of the local section

$$\tau : t \mapsto \big(l + (l+1)a_{l+1}t + \cdots\big) \cdot \tfrac{\partial}{\partial x} + mt^{m-l} \cdot \tfrac{\partial}{\partial y}$$

of $\phi^* T_S$; thus the torsion-free quotient $\bar{N}_\phi$ is generated by the class of $\partial/\partial y$ modulo τ. Observe that modulo τ, $\partial/\partial x$ equals $t^{m-l} \cdot \partial/\partial y$ times an invertible, hence the image of $\partial/\partial x$ in $\bar{N}_\phi$ vanishes to the order exactly $m-l$ at q.

In turn the family Φ may be written locally as

$$\Phi(t, \varepsilon) = \big((t^l + a_{l+1}t^{l+1} + \cdots) + \varepsilon(u_0 + u_1 t + \cdots) + O(\varepsilon^2), t^m, \varepsilon\big),$$

since $\Phi^* R$ contains Q with multiplicity m. By definition, the corresponding section $\bar{d}\kappa(\partial/\partial\varepsilon)$ of $\bar{N}_\phi$ is

$$\bar{d}\kappa(\tfrac{\partial}{\partial\varepsilon}) = (u_0 + u_1 t + \cdots) \cdot \tfrac{\partial}{\partial x} \bmod \tau.$$

Since $\partial/\partial x$ itself vanishes modulo τ to the order $m-l$ as we have seen, one has $v_q\big(\bar{d}\kappa(\partial/\partial\varepsilon)\big) \geqslant m-l$ in any event.

Moreover, by generality of $b \in B$ we may assume that all $\Phi(\cdot, \varepsilon)$ have their differential vanishing to the order l at $Q \cap \mathcal{C}_\varepsilon$, which translates into the fact that $u_1 = \cdots = u_{l-1} = 0$. Then $\bar{d}\kappa(\partial/\partial\varepsilon)$ vanishes either to the order $m-l$, if $u_0 \neq 0$, or to some order larger than m, if $u_0 = 0$. When the tangency is maintained at a fixed point we are necessarily in the latter case. The lemma is proved. □

[4] I use the notation $[\![a, b]\!] = [a, b] \cap \mathbf{Z}$ for $a, b \in \mathbf{R} \cup \{\pm\infty\}$; albeit being standard probably only in France, it is convenient.

Let $\mathcal{C} \subseteq S \times B$ be a family of a reduced curves over a reduced base B. It is said to preserve a tangency of order m with R if the corresponding family of normalization maps over the normalization $\bar{B}$ of B (see (2.6)) does.

(2.12) Corollary. *Let $V \subseteq \mathrm{Curve}(S)$ be a family of curves of genus g having a tangency of order m with the divisor R. Let $[C]$ be a general member of V, $\phi : \bar{C} \to C \subseteq S$ be the normalization of C, $q \in \bar{C}$ the tangency point, and $l-1$ the order of vanishing of the differential $d\phi$ at q. Then*

$$\dim V \leqslant h^0\big(\bar{C}, \bar{N}_\phi(-(m-l)_+ \cdot q)\big),$$

where $(m-l)_+$ denotes $\max(m-l, 0)$. If the tangency is at a fixed prescribed point on R, then actually

$$\dim V \leqslant h^0\big(\bar{C}, \bar{N}_\phi(-m \cdot q)\big).$$

Proof. As in the proof of Corollary (2.9), we have $\dim V = \dim T_{[C]}V$ by generality of $[C]$, and there is an injective map $T_{[C]}V \to H^0(\bar{C}, \bar{N}_\phi)$. By Lemma (2.11) the image of this map is contained in $H^0\big(\bar{C}, \bar{N}_\phi(-a \cdot q)\big)$ with $a = (m-l)_+$ if the tangency is mobile, and $a = m$ if the tangency is fixed. This ends the proof. □

3 – Proof of the Main Theorem

In this section we prove Theorem (1.7). The proof itself is in Subsection 3.2, after we give some lemmas in Subsection 3.1.

3.1 – Applications of the Riemann–Roch formula

Lemma (3.1) below is standard, but we will also use the more clever Lemma (3.2), which amounts to [6, Observation 2.5].

(3.1) Lemma. *Let X be a smooth (possibly disconnected) projective curve, and L a line bundle on X. Let $k \in \mathbf{N}^*$. If $\deg\big(L \otimes \omega_X^{-1}\big|_{X_i}\big) > k$ for all irreducible component X_i of X, then the linear system $|L|$ separates any k points on X.*

Proof. Let Z be a subscheme of X of length k, and let Z' be another subscheme of X such that $Z' \subsetneq Z$. The assumption on the degree of L ensures that both $L(-Z)$ and $L(-Z')$ are non-special, hence $h^0\big(L(-Z)\big) < h^0\big(L(-Z')\big)$ by the Riemann–Roch formula. □

(3.2) Lemma. *Let X be a smooth (possibly disconnected) projective curve of genus $g = 1 - \chi(\mathcal{O}_X)$, and L, M two line bundles on X such that*

$$\forall\, X_i \text{ component of } X: \quad \deg(M|_{X_i}) \leqslant \deg(L|_{X_i}).$$

(3.2.1) If $\deg\big(L \otimes \omega_X^{-1}\big|_{X_i}\big) > 0$ for every component X_i of X, then

$$(\star) \qquad h^0(X, M) \leqslant h^0(X, L) = \deg(L) - g + 1.$$

(3.2.2) If $\deg\big(L \otimes \omega_X^{-1}\big|_{X_i}\big) > 1$ for every component X_i of X, then equality holds in $(\star)$ if and only if $\deg M = \deg L$.

Proof. For every component X_i of X, let g_i be the genus of X_i, and set $L_i = L|_{X_i}$, $M_i = M|_{X_i}$, $d_i = \deg(L_i)$, and $e_i = \deg(M_i)$. Note that $g = \sum g_i - n + 1$, where n is the number of components of X.

Assume as in (3.2.1) that $d_i > 2g_i - 2$ for all i. Then L is non-special, hence the right-hand-side equality in $(\star)$, by the Riemann–Roch formula. Somewhat more precisely, each L_i is non-special, hence $h^0(L_i) = d_i - g_i + 1$, and thus

$$h^0(L) = \sum h^0(L_i) = \sum d_i - \sum g_i + n = \deg(L) - g + 1.$$

Moreover, $h^0(L_i) \geqslant g_i$ for all i.

Consider an irreducible component X_i of X. If M_i is non-special, then $h^0(M_i) = h^0(L_i) + e_i - d_i$, again by the Riemann–Roch formula, hence $h^0(M_i) \leqslant h^0(L_i)$, and equality holds if and only if $e_i = d_i$. If M_i is special, then $h^0(M_i) \leqslant g_i$, which implies in our situation that $h^0(M_i) \leqslant h^0(L_i)$. In the upshot,

$$h^0(M) = \sum h^0(M_i) \leqslant \sum h^0(L_i) = h^0(L),$$

and (3.2.1) is proven.

Let us now make the stronger assumption that $d_i > 2g_i - 1$ for all i. Then, $h^0(L_i) > g_i$ for all i. Therefore, if M_i is special for some i, then $h^0(M_i) < h^0(L_i)$. Since it still holds that $h^0(M_i) \leqslant h^0(L_i)$ for all i, this implies that $h^0(M) < h^0(L)$. On the other hand, if M_i is special for some i, then $e_i \leqslant 2g_i - 2$, and therefore $e_i < d_i$. This implies that $\deg(M) < \deg(L)$, and thus the conclusion of (3.2.2) holds if M is special. If on the contrary M is non-special then, as we have seen above, $h^0(M) = h^0(L) + \deg(M) - \deg(L)$, and the conclusion of (3.2.2) holds yet again. Thus, the conclusion of (3.2.2) holds unconditionally. □

3.2 – Proof of the main theorem

This section is dedicated to the proof of Theorem (1.7). We consider an irreducible component V of $V_g^\xi(\alpha,\beta)(\Omega)$ and a general point $[C]$ of V, as in the theorem. We use freely the notation introduced in the statement.

(3.3) *We start by proving that V has expected dimension*

$$\operatorname{expdim} V_g^\xi(\alpha,\beta)(\Omega) = -(K_S + R)\cdot\xi + g - 1 + |\beta|. \tag{3.3.1}$$

By (2.2.2) the expected dimension of V_g^ξ is $\chi(N_\phi)$, which by the Riemann–Roch formula and the exact sequence (2.2.1) equals

$$\operatorname{expdim} V_g^\xi = \deg \omega_{\bar{C}} - \deg \phi^*\omega_S + 1 - g = -K_S\cdot C + g - 1. \tag{3.3.2}$$

Now requiring that a curve C have tangency of order m with R at a specified point p is m linear conditions on the coefficients of the equation of C, and if we let the point p vary along R the expected codimension of the corresponding locus of curves C is one less, i.e. $m-1$. We thus end up with

$$\begin{aligned}\operatorname{expdim}\big(V_g^\xi(\alpha,\beta)(\Omega)\big) &= \operatorname{expdim}\big(V_g^\xi\big) - \sum_i \sum_{1\leqslant j\leqslant \alpha_i} i - \sum_i \sum_{1\leqslant j\leqslant \beta_i} (i-1)\\ &= \operatorname{expdim}\big(V_g^\xi\big) - I\alpha - (I\beta - |\beta|),\end{aligned}$$

which together with (3.3.2) gives (3.3.1) after one remarks that $I\alpha + I\beta = R \cdot \xi$.

Note that this proves that, in any event,

$$\dim(V) \geqslant -(K_S + R) \cdot \xi + g - 1 + |\beta|, \tag{3.3.3}$$

which is assertion (1.7.00). □

(3.4) *We now turn to the proof that the dimension of V equals its expected dimension under assumption (1.7.0).*

Note that the points $q_{i,j}$ are necessarily pairwise distinct because they have distinct images $p_{i,j} \in R$. Let us first assume in addition that the points $q_{i,j}$ and $r_{i,j}$ are altogether pairwise distinct; the case when this does not hold will be dealt with in (3.5).

We set

$$D := \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} (i-1)\, r_{i,j}, \tag{3.4.1}$$

the divisor on $\bar{C}$ of "infinitesimal tangency conditions with R" (compare (3.3)), and

$$D_0 := \sum_{1 \leqslant j \leqslant \beta_i} (l_{i,j} - 1)\, r_{i,j}, \tag{3.4.2}$$

where $l_{i,j} := v_{r_{i,j}}(d\phi)$ is the order of vanishing of the differential $d\phi$ at the point $r_{i,j}$, i.e., D_0 is the "ramification divisor of ϕ in the points $r_{i,j}$". We then decompose the difference of these two divisors as

$$D - D_0 = D_1 - D_1' \tag{3.4.3}$$

where D_1 and D_1' are non-negative divisors on $\bar{C}$ with disjoint supports; note that D_1' may be nonzero only at the points $r_{i,j}$, and that it is so if and only if $l_{i,j} > i$.

It follows from Corollary (2.12) that

$$\dim(V) \leqslant h^0\big(\bar{C}, \bar{N}_\phi(-D_1)\big). \tag{3.4.4}$$

Let Z_0 be the non-negative divisor on $\bar{C}$ such that the ramification divisor of ϕ is $D_0 + Z_0$. Then by (2.3.3) we have $\bar{N}_\phi \cong \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D_0 - Z_0)$, and therefore (3.4.4) above reads

$$\dim(V) \leqslant h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D_0 - Z_0 - D_1)\big) \tag{3.4.5}$$

$$= h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D - D_1' - Z_0)\big) \tag{3.4.6}$$

$$\leqslant h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big). \tag{3.4.7}$$

Now by assumption (1.7.0) the line bundle $\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)$ is non-special, hence

$$h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big) = h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S(R)^{-1}(\phi^*R - D)\big) \tag{3.4.8}$$

$$= 2g - 2 - (K_S + R) \cdot \xi + |\beta| + 1 - g \tag{3.4.9}$$

$$= \operatorname{expdim} V_g^\xi(\alpha, \beta)(\Omega). \tag{3.4.10}$$

We thus have $\dim V \leqslant \operatorname{expdim} V_g^\xi(\alpha, \beta)(\Omega)$ which, together with (3.3.3) implies that V has the expected dimension, if indeed the points $q_{i,j}$ and $r_{i,j}$ are all together pairwise distinct.

(3.5) Now if it is not true that the points $q_{i,j}$ and $r_{i,j}$ are all together pairwise distinct, then V is actually a component of some Severi variety $V_g^\xi(\alpha',\beta')(\Omega')$ with $|\beta'|<|\beta|$ for which the corresponding points $q'_{i,j}$ and $r'_{i,j}$ are indeed pairwise disjoint (as sets $\Omega=\Omega'$, i.e., $\bigcup_i \Omega_i=\bigcup_i \Omega'_i$, and $\Omega_i\subseteq\bigcup_{k\geqslant i}\Omega'_k$).

Then, setting correspondingly D' as in (3.4.1), one gets $\dim V\leqslant h^0(\bar{C},\omega_{\bar{C}}\otimes\phi^*\omega_S^{-1}(-D'))$ exactly as in (3.4). Now $\deg D'>\deg D$ because $|\beta'|<|\beta|$, and it therefore follows from Lemma (3.2), part (3.2.1) that

$$h^0\big(\bar{C},\omega_{\bar{C}}\otimes\phi^*\omega_S^{-1}(-D')\big)\leqslant h^0\big(\bar{C},\omega_{\bar{C}}\otimes\phi^*\omega_S^{-1}(-D)\big), \tag{3.5.1}$$

so that it still holds that $\dim V\leqslant \operatorname{expdim} V_g^\xi(\alpha,\beta)(\Omega)$, hence V has the expected dimension. □

(3.6) *Note that the above proof gives the additional fact that, under the assumption of (1.7.0), the tangent space of $V_g^\xi(\alpha,\beta)(\Omega)$ at the general point $[C]$ of V identifies with*

$$H^0\big(\bar{C},\omega_{\bar{C}}\otimes\phi^*\omega_S^{-1}(-D)\big)\cong H^0\big(\bar{C},\bar{N}_\phi(-D_1)\big)\cong H^0\big(\bar{C},\bar{N}_\phi(D_0-D)\big)\subseteq H^0\big(\bar{C},\bar{N}_\phi\big).$$

(3.7) *We now prove that under Assumption (1.7.1) the assertions* (a$^\natural$)*,* (b)*,* (c$^\flat$)*, and* (d) *hold.*

Suppose by contradiction that (b) doesn't hold. Then we argue as in (3.5). In this case, part (3.2.2) of Lemma (3.2) applies thanks to Assumption (1.7.1), and we get that the inequality (3.5.1) is strict, which is in contradiction with (3.3.3).

The same argument shows that none of the points $\phi(r_{i,j})$ can be fixed on R. This implies in particular that (c$^\flat$) holds.

The proof of (d) is similar: if C were tangent to G, then it would belong to an irreducible component W of some Severi variety of the pair $(S,R+G)$. Assumption (1.7.1) implies that W is liable for part (1.7.0) of Theorem (1.7), hence $\dim(W)<\dim(V)$, in contradiction with the fact that $[C]$ is a general member of V. The same argument shows that C avoids Γ (pick some random curve on S containing Γ).

Finally, we note that equality holds in (3.4.7) if and only if $D'_1=Z_0=0$ since the line bundle $\omega_{\bar{C}}\otimes\phi^*\omega_S^{-1}(-D)$ is globally generated by assumption (1.7.1). Now it is indeed the case that equality holds in (3.4.7) since we have proved that $\dim(V)=\operatorname{expdim}\big(V_g^\xi(\alpha,\beta)\big)$. We conclude that $D'_1=Z_0=0$, which means that (i) ϕ is an immersion outside of the points $r_{i,j}$ (this is assertion (a$^\natural$)) and (ii) $l_{i,j}\leqslant i$ for $1\leqslant j\leqslant\beta_i$. □

Remark. It is not enough to assume that $\omega_{\bar{C}}\otimes\phi^*\omega_S^{-1}(-D)$ is non-special and globally generated because of the argument we made to assume that the points $q_{i,j}$ and $r_{i,j}$ are pairwise distinct. We actually need to know something about every possible $\omega_{\bar{C}}\otimes\phi^*\omega_S^{-1}(-D')$, where $D'=\sum iq'_{i,j}+\sum(i-1)r'_{i,j}$ in the notation used for this argument.

(3.8) *We now prove that, under the assumption (1.7.2), ϕ is an immersion also at the points $r_{i,j}$, i.e. that $l_{i,j}=1$ for $1\leqslant j\leqslant\beta_i$, thus completing the proof of assertion* (a$^\flat$).

Let $i\geqslant 1$ and $1\leqslant j\leqslant\beta_i$. It follows from the assumption that the linear series $\big|\omega_{\bar{C}}\otimes\phi^*\omega_S^{-1}(-D)\big|$ separates any two points, so there exists a section $\sigma\in H^0\big(\bar{C},\omega_{\bar{C}}\otimes\phi^*\omega_S^{-1}(-D)\big)$ with vanishing order $v_{r_{i,j}}(\sigma)=1$ at the point $r_{i,j}$. Seen as a section $\tilde{\sigma}\in H^0(\bar{C},\bar{N}_\phi)$, it vanishes at $r_{i,j}$ with order $v_{r_{i,j}}(\tilde{\sigma})=1+(i-l_{i,j})$ (see (2.3.3)). By Lemma (2.11) this implies

$$1+i-l_{i,j}\in\{i-l_{i,j}\}\cup[\![i,+\infty[\![$$

and therefore $l_{i,j}=1$ as required. □

(3.9) Remark. Under the conditions of (1.7.2) the map ϕ is an immersion, hence $N_\phi = \bar{N}_\phi \cong \omega_C \otimes \phi^*\omega_S^{-1}$, see Paragraph (2.2). Therefore, the tangent space of $V_g^\xi(\alpha,\beta)(\Omega)$ at the general point $[C]$ of V identifies with

$$H^0\big(\bar{C}, N_\phi(-D)\big) \cong H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big)$$

(the definition of D is in (3.4.1) above).

(3.10) *Let us prove that Assumption (1.7.2) implies Assertion* (c)*, i.e., the points $p_{i,j}$ and $s_{i,j} = \phi(r_{i,j})$ are pairwise distinct.*

By (3.7), we already know that ($\mathrm{c}^\flat$) holds, i.e., none of the $s_{i,j} = \phi(r_{i,j})$ belongs to $\Omega = \{p_{i,j}\}$. We thus only need to show that no two of the points $s_{i,j}$ coincide.

Suppose there exist (i,j) and (i',j') distinct such that $\phi(r_{i,j}) = \phi(r_{i',j'})$. Assumption (1.7.2) implies that the series $|\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)| = |T_{[C]}V|$ separates any two points. Therefore there exists a section $\sigma \in T_{[C]}V \cong H^0(\bar{N}_\phi(-D))$ such that

$$v_{r_{i,j}}(\sigma) = 1 \quad \text{and} \quad v_{r_{i',j'}}(\sigma) = 0.$$

This implies the existence of a deformation of C in which the points $\phi(r_{i,j})$ and $\phi(r_{i',j'})$ no longer coincide, a contradiction to the generality of $[C]$ in V. □

(3.11) *Let us prove that Assumption (1.7.2) implies Assertion* (e)*, i.e., C is smooth at its intersection points with R.*

At this point we know that ($\mathrm{a}^\flat$) and (c) hold under Assumption (1.7.2), i.e., the curve C is immersed and the points $p_{i,j}$ and $s_{i,j}$ are pairwise distinct. Because the intersection $C \cap R$ is set-theoretically the union of all the points $p_{i,j}$ and $s_{i,j}$, this implies that C is smooth at its intersection points with R. □

(3.12) *We eventually prove that under the Assumption (1.7.3) the curve C is nodal, which is Assertion* (a) *of Theorem (1.7).*

Since we already know that the curve C is immersed, it is enough to show that for all point $p \in C$, C has neither three or more local branches, nor two or more tangent local branches.

To exclude the former possibility, suppose by contradiction that there exist $a, b, c \in \bar{C}$ pairwise distinct such that $\phi(a) = \phi(b) = \phi(c)$. The assumption (1.7.3) implies that the linear series $\big|\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big|$ separates any three points, so there exists a section $\sigma \in H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big)$ such that

$$\sigma(a) = \sigma(b) = 0 \quad \text{and} \quad \sigma(c) \neq 0;$$

it corresponds to a first-order deformation of ϕ leaving both $\phi(a)$ and $\phi(b)$ fixed while moving $\phi(c)$. By generality of $[C]$, there is correspondingly an actual deformation of the curve C for which the three local branches under consideration are no longer intersecting in a common point, a contradiction to the generality of $[C]$ in V (see, e.g., [10, Prop. 1.4]).

We exclude the second possibility in a similar fashion. Suppose by contradiction that there exist $a, b \in \bar{C}$ distinct such that $\phi(a) = \phi(b)$ and $\operatorname{Im} d\phi_a = \operatorname{Im} d\phi_b$. Again since $\big|\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big|$ separates any three points, there exists $\sigma \in H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big)$ such that $\sigma(a) = 0$ and $\sigma(b) \notin \operatorname{Im} d\phi_a$. It corresponds to a first-order deformation of ϕ leaving $\phi(a)$ fixed while moving $\phi(b)$ in a direction transverse to the common tangent to the two local branches of C under consideration. This contradicts the generality of $[C]$ as before. □

This ends the proof of Theorem (1.7).

4 – A dimensional characterization of logarithmic Severi varieties

We consider a pair (S, R) consisting of a smooth surface S and a reduced boundary divisor R, as described in Section 1. In this section we give an upper bound for the dimension of families V of (not necessarily reduced) curves in S with prescribed homology class and genus, together with a dimensional characterization of logarithmic Severi varieties: it turns out that the irreducible components of logarithmic Severi varieties are essentially those families for which the upper bound is reached. The content of this section is basically [6, Cor. 2.7]. The main result, Proposition (4.3), is a generalization of Zariski's dimensional characterization of (classical) Severi varieties [19].

(4.1) Let V be an irreducible locally closed subset of the Hilbert scheme of curves on S, and suppose that it parametrizes genus g curves in the following sense: for a general member X of V, there exists a smooth curve C of genus g, and a morphism $\phi : C \to X$, not constant on any component of C, and such that the push-forward in the sense of cycles $\phi_*[C]$ equals the fundamental cycle of X.

(4.2) Let $\xi \in \mathrm{NS}(S)$ be the homology class of members of V. Letting the general member X of V have irreducible components $X_1, \ldots, X_n$, with respective classes $\xi_1, \ldots, \xi_n$, we assume furthermore:

$$\forall i = 1, \ldots, n : \quad -(K_S + R) \cdot \xi_i > 0, \tag{4.2.1}$$

so that we may apply (1.7.0) without any restriction on the contact conditions with R.

(4.3) Proposition. *Consider a finite subset $\Omega \subseteq R$. Under the above assumptions in (4.1) and (4.2), for a general member X of V, one has*

$$\dim(V) \leqslant -(K_S + R) \cdot \xi + g - 1 + \mathrm{Card}\big((X \cap R) \setminus \Omega\big), \tag{4.3.1}$$

where the last number is defined set-theoretically (i.e., we do not count with multiplicities).

Moreover, if equality holds in (4.3.1)*, and if*

$$\mathrm{Card}\big(\phi^{-1}(R)\big) = \mathrm{Card}(X \cap R), \tag{4.3.2}$$

for some genus g morphism $\phi : C \to X$ as in (4.1), then there exist α, β such that V is a dense subset of a component of a log-Severi variety $V_g^{\xi}(\alpha, \beta)(\Omega)$.[5]

Note that α and β are of course uniquely determined by the contacts of X and R. Even if V is indeed dense in a component of some log-Severi variety, Condition (4.3.2) does not always hold, but it does hold if assertions (b) and (c) of Theorem (1.7) hold.

Before proving the proposition we make a few observations, in the hope that they will clarify some of its subtleties.

(4.4) Remark. Proposition (4.3) is really a result about families of embedded curves in S, not families of maps.

Indeed, if X is not reduced and the genus g map $\phi : C \to X$ involves multiple covers, then there is in general a positive dimensional family of maps giving the same X, so that in general the family of maps corresponding to V has dimension larger than V.

[5] If X is non-reduced, we need to twist the definition of log-Severi variety a little for this to be true, see (4.9).

When equality holds in (4.3.1), the map ϕ restricts to a birational map on each component of C, as we shall see in the proof of the proposition. If moreover $g \geqslant 0$, then the general member of V is actually reduced. See (4.9) for precisions about this.

(4.5) Remark. A straightforward corollary of Proposition (4.3) which may be useful for the applications is that the inequality (4.3.1) still holds if we only assume C to be reduced and replace g by the *arithmetic* genus of C; this alternative inequality is always strict when C is not smooth.

(4.6) Remark. Assumption (4.3.2) ensures that the normalization of X is unibranch over the points in $X \cap R$.
(4.6.1) Example. Set $(S, R) = (\mathbf{P}^2, L)$ whith L a line, and $\xi = 3[H]$ with $[H]$ the hyperplane class. The family V of plane cubics with one node on L and otherwise smooth has dimension 7, which equals

$$\underbrace{-(K_S + R)\cdot\xi}_{=2[H]\cdot 3[H]=6} + \underbrace{g}_{=0} - 1 + \underbrace{\mathrm{Card}\big((X \cap R) \setminus \Omega\big)}_{=2} \quad \text{with } \Omega = \varnothing.$$

It is a divisor in the log-Severi variety $V_0^{3[H]}(0, 3)(\varnothing)$, but it is not a component of the family $V_0^{3[H]}(0, [1, 1])(\varnothing)$ of rational plane cubics with one variable tangency along L, which has dimension 7 as well.

(4.7) Remark. In the equality case for (4.3.1), if one replaces (4.3.2) with the weaker condition that

$$\mathrm{Card}\big(\phi^{-1}(R \setminus \Omega)\big) = \mathrm{Card}\big((X \cap R) \setminus \Omega\big), \tag{4.7.1}$$

the families that we get that are not log-Severi varieties may be considered as "log-Severi varieties with Ω containing repeated points", see Example (4.7.2) below. (These are not log-Severi varieties according to Definition (1.3)).

Beware that in [6, Cor. 2.7], the weaker (4.7.1) is used instead of (4.3.2), which is slightly inaccurate.

(4.7.2) Example. Set $(S, R) = (\mathbf{P}^2, L)$ as above, and $\xi = 4[H]$, and fix $p \in L$. The family of plane quartics with one triple point at p has dimension $\binom{6}{2} - 1 - 6 = 8$, which equals

$$\underbrace{-(K_S + R)\cdot\xi}_{=2[H]\cdot 4[H]=8} + \underbrace{g}_{=0} - 1 + \underbrace{\mathrm{Card}\big((X \cap L) \setminus \Omega\big)}_{=1} \quad \text{with } \Omega = \{p\}.$$

One may wish to consider it as $V_0^{4[H]}(3, 1)(p, p, p)$.

Proof of Proposition (4.3). We divide the proof into several steps, which correspond to paragraphs (4.8)–(4.10). We first treat the case when X is irreducible; the general case is taken care of by induction in (4.10).

(4.8) We first prove the proposition under the assumption that X is reduced and irreducible; in this case ϕ is the normalization of X and $C \cong \bar{X}$.

If $e := \mathrm{Card}\big((X \cap R) \setminus \Omega\big) = 0$, then the statement is a slight variant of part (1.7.0) of the main Theorem (1.7): we get the required inequality (4.3.1) exactly as in paragraph (3.4), with $D = \phi^* R$ and $D_0 = 0$ in (3.4.1) and (3.4.2) respectively. The only difference with the setting of

(3.4) is that here two distinct points of the support of $\phi^*R \subseteq \bar{X}$ may have the same image by ϕ in X; this makes absolutely no difference in the argument.

Now if (4.3.2) holds, then $\bar{X}$ is unibranch over the points of $X \cap R$. This implies that V is contained in a certain log-Severi variety $V_g^{\xi}(\alpha, 0)(\Omega)$. If moreover equality holds in (4.3.1), then V is dense in an irreducible component of this same log-Severi variety by (1.7.0).

In the case when $e > 0$, we consider the map

$$\rho : V \to \mathrm{Sym}^e R$$

sending a curve to its reduced intersection scheme with $R \setminus \Omega$; this may not be well-defined everywhere since e may drop along certain closed subschemes of V, but it is in a neighbourhood of $[X]$.

Then we can apply the $e = 0$ case of the proposition to the fibres of ρ; for a general $\Sigma \in \mathrm{Sym}^e R$, setting $\Omega' = \Omega \cup \mathrm{Supp}\,\Sigma$ one gets that the fibre $\rho^{-1}(\Sigma)$ has dimension at most $-(K_S + R) \cdot \xi + g - 1$. Inequality (4.3.1) follows, and the equality case of the proposition as well, again applying the $e = 0$ case to the fibres of ρ.

Remark. Note that the above reasoning also proves that, if V is dense in a suitable irreducible component of a log-Severi variety, then the map $\rho : V \to \mathrm{Sym}^e R$ is dominant.

(4.9) Let us now consider the case when X is non-reduced, but still irreducible. We shall show that inequality (4.3.1) holds, and that it is almost always strict; one needs to be a little bit careful when equality holds in (4.3.1).

We have to consider $\phi : C \to X$ where X is non-reduced but irreducible, and C may be reducible (but nevertheless smooth). We let $m > 1$ be the degree of ϕ, i.e. the sum of the degrees of the various $\phi_i : C_i \to X$ for all components C_i of C, and let h be the geometric genus of X_{red}. We claim that

$$g - 1 \geqslant m(h - 1).$$

Indeed, let C_i be a component of C. Since C_i is smooth, by the universal property of the normalization, the map $\phi_i : C_i \to X$ factorizes through a map $\bar{\phi}_i : C_i \to \bar{X}$, where $\bar{X}$ is the normalization of X_{red}. Note that ϕ_i and $\bar{\phi}_i$ have the same degree. Then, by the Riemann–Hurwitz formula, we have $g_i - 1 \geqslant m_i(h - 1)$, where g_i is the genus of C_i and m_i is the degree of ϕ_i and $\bar{\phi}_i$. Summing all these inequalities we find $\sum(g_i - 1) \geqslant \sum m_i(h - 1)$, which is the inequality we want, since $g = \sum(g_i - 1) + 1$ and $m = \sum m_i$.

Next, we apply Proposition (4.3) in the reduced case, which we have already proven above, to the reduced curve X_{red} underlying X: it has class $\frac{1}{m}\xi$, so we get

$$\dim(V) \leqslant -(K_S + R) \cdot \frac{1}{m}\xi + h - 1 + \mathrm{Card}\big((X \cap R) \setminus \Omega\big) \tag{4.9.1}$$

$$\leqslant -(K_S + R) \cdot \frac{1}{m}\xi + \frac{g-1}{m} + \mathrm{Card}\big((X \cap R) \setminus \Omega\big); \tag{4.9.2}$$

recall that $X \cap R$ is considered as a set, so that

$$\mathrm{Card}\big((X \cap R) \setminus \Omega\big) = \mathrm{Card}\big((X_{\mathrm{red}} \cap R) \setminus \Omega\big).$$

Now the required inequality (4.3.1) follows from the basic inequality:

$$-(K_S + R) \cdot \frac{1}{m}\xi + \frac{g-1}{m} \leqslant -(K_S + R) \cdot \xi + g - 1. \tag{4.9.3}$$

If $g \geqslant 1$, the latter inequality (4.9.3) is straightforward, and always strict, as $-(K_S + R) \cdot \xi > 0$ and $m > 1$. If on the contrary $g \leqslant 0$, then on the one hand $-(K_S + R) \cdot \xi \geqslant m$ by our hypothesis

(4.2.1), because $\frac{1}{m}\xi$ is an integral class, and on the other hand $g \geqslant -m+1$, with equality holding if and only if C is the disjoint union of m smooth rational curves; this implies that inequality (4.9.3) holds also in this case, and that it is an equality if and only if $-(K_S + R) \cdot \frac{1}{m}\xi = 1$ and C is the disjoint union of m smooth rational curves.

Assume now that equality holds in (4.3.1), equivalently that equality holds in (4.9.1), (4.9.2), and (4.9.3). Then C is the disjoint union of $m > 1$ smooth rational curves, and ϕ restricts on each of those to the normalization of X_{red}. Thus, condition (4.3.2) holds if and only if $R \cdot X = 0$. In this case V is dense in a component of a log-Severi variety, by (1.7.0) ($-(K_S + R) \cdot \frac{1}{m}\xi = 1$ since equality holds in (4.9.3)) and using the same argument as in (4.8), provided we relax a little the definition of log-Severi varieties as follows: (i) in the definition of $M_g^{\xi,\mathrm{bir}}(S)$ (see (1.1) and (2.4)), we replace the condition that $\phi : C \to X$ is birational to its image by the condition that the restriction of ϕ to each component of C is birational to its image; (ii) instead of Definition (1.5), we let $V_g^{\xi}(\alpha, \beta)(\Omega)$ be the set of those $[X] \in \mathrm{Curve}(S)$ such that there exists $(\phi : C \to X) \in M_g^{\xi}(\alpha, \beta)(\Omega)$ with $\phi_*[C] = [X]$. Note that for reduced curves on S, this does not change anything. We have chosen not to take this relaxed definition in the text because we wanted our Severi varieties to be families of reduced subschemes of S and not families of maps, but other than that this would not have changed much.

(4.9.4) Example. Let S be the projective plane blown-up at eight general points, with exceptional divisors $E_1, \ldots, E_8$, and take $R = 0$. Consider the class

$$\xi_0 = 3H - \textstyle\sum_{i=1}^{8} E_i,$$

where H as usual denotes the pull-back to S of the line class on $\mathbf{P}^2$, and let $\xi = m\xi_0$. Note that $-K_S \cdot \xi_0 = 1$. Let X_0 be an irreducible rational curve in the one-dimensional linear system $|3H - \sum_{i=1}^{8} E_i|$, and consider the family V consisting solely of $X = mX_0$ as a 0-dimensional family of curves of genus $-m+1$, by means of the morphism $\phi : C \to X$ consisting of m disjoint copies of the normalisation of X_0. Equality holds for V in (4.3.1), condition (4.3.2) holds, and V is a component of the Severi variety V^{ξ}_{-m+1}, in the relaxed sense explained above.

(4.9.5) Example. Let S be the projective plane blown-up at seven general points, with exceptional divisors $E_1, \ldots, E_7$, and take

$$R = H - E_1 - E_2 \quad \text{and} \quad \xi_0 = 3H - \textstyle\sum_{i=1}^{7} E_i.$$

Note that $-K_S \cdot \xi_0 = 2$ and $R \cdot \xi_0 = 1$. Let $\xi = m\xi_0$, and let V be the one-dimensional family of all genus $-m+1$ curves mX_0 suported on an irreducible rational member X_0 of the two-dimensional linear system $|3H - \sum_{i=1}^{7} E_i|$. Equality holds for V in (4.3.1), condition (4.3.2) does not hold, and V is not dense in a component of the log-Severi variety $V^{\xi}_{-m+1}(0, m)$, because its members deform equigenerically to curves of the form $X_1 + \cdots + X_m$ with all X_i rational members of $|3H - \sum_{i=1}^{7} E_i|$.

(4.10) It remains to consider the case when X is reducible. Proceeding by induction on the number of irreducible components, we may assume that $X = X_1 \cup X_2$ where X_1 and X_2 move in two families V_1 and V_2 of curves of genera g_1 and g_2 respectively, such that $V \subseteq V_1 \times V_2$ and the Proposition holds for V_1 and V_2. Adding the two corresponding inequalities gives

(4.10.1)

$$\begin{aligned}\dim(V) &\leqslant \dim(V_1) + \dim(V_2)\\ &= -(K_S + R) \cdot \xi + (g_1 + g_2 - 1) - 1 + \mathrm{Card}\big((X_1 \cap R) \setminus \Omega\big) + \mathrm{Card}\big((X_2 \cap R) \setminus \Omega\big).\end{aligned}$$

If $(X_1 \cap X_2 \cap R) \setminus \Omega$ is empty then this readily gives the required inequality, and the second part of the proposition in the equality case also follows.

So let us consider the case when $(X_1 \cap X_2 \cap R) \setminus \Omega$ is non-empty, and let us assume for simplicity that this set consists of only one point p, the general case being strictly similar. In this case condition (4.3.2) does not hold because p has at least two preimages.

If p is a fixed point of the intersection with R for either one of the two families, say V_1, then it is also a fixed point for the other family: indeed, otherwise, by the generality of X we could perturb X_2 a little so that it does not pass through p, and then p is no longer a point in $X_1 \cap X_2 \cap R$. In this case, applying the proposition to V_1 and V_2 with $\Omega' := \Omega \cup \{p\}$, one gets

$$\dim(V) \leqslant -(K_S + R) \cdot \xi + (g_1 + g_2 - 1) - 1 + \underbrace{\mathrm{Card}\big((X_1 \cap R) \setminus \Omega'\big) + \mathrm{Card}\big((X_2 \cap R) \setminus \Omega'\big)}_{=\mathrm{Card}\left((X \cap R) \setminus \Omega\right) - 1},$$

and the result follows; note that inequality (4.3.1) is strict in this case.

Otherwise p is variable for both V_1 and V_2; in this case V necessarily has codimension at least 1 in $V_1 \times V_2$ (this may be proved for instance as in (4.8) by applying the Proposition to the fibres of the projection $V \to V_1$), and therefore (4.10.1) gives the required inequality (4.3.1). Equality may hold in (4.3.1), but V cannot be dense in a component of a log-Severi variety, since it already has positive codimension in $V_1 \times V_2$.

(4.10.2) Example. Take $S = \mathbf{P}^2$ and R a line, let $V_1 = V_2 = V_0^{[H]}(0,1)(\varnothing)$, and consider V the sub-family of $V_1 \times V_2$ parametrizing the sums of two lines, the intersection point of which lies on R. Then V has dimension 3, and parametrizes curves of genus -1, so that equality holds in (4.3.1) because $\mathrm{Card}(X \cap R) = 1$ for a general membre X of V. It has codimension 1 in $V_1 \times V_2$, which is an irreducible component of $V_{-1}^{[2H]}(0,2)(\varnothing)$. Concretely, on V we are artificially decreasing $\mathrm{Card}(X \cap R)$ by imposing that two contact points with R coincide, and the latter prescription is not of Severi type.

The proof of Proposition (4.3) is now over. □

5 – Examples

5.1 – An example with contact points in special position

In this short subsection I give an example illustrating (i) the effect of having the points in Ω in special position (which is also a theme in Subsection 5.3 below), and (ii) the importance of considering the positivity of $-K_S \cdot C - \deg \phi_* D$ (in the notation of Theorem (1.7)) on all irreducible components of C separately.

(5.1) Example. Let S be the projective plane, R be a smooth cubic, and Ω be cut out on R by another cubic R'. We consider the Severi variety $V_0^4(9,3)(\Omega)$ of rational quartics passing through the 9 intersection points of R and R' (and thus with 3 unassigned additional contact points with R); it has expected dimension

$$-(\underbrace{K_S + R}_{=0}) \cdot \xi + \underbrace{g}_{=0} - 1 + \underbrace{|\beta|}_{=3} = 2.$$

It has a 3-dimensional irreducible component V_{ni} parametrizing reducible quartics $C_1 + C_2$ where C_1 is a cubic in the pencil generated by R and R', and C_2 is a line. The component V_{ni} thus has dimension larger than expected, and indeed (1.7.0) does not apply in this case because, in the notation of Theorem (1.7),

$$-K_S \cdot C_1 - \deg \phi_* D|_{C_1} = 0.$$

The Severi variety $V_0^4(9,3)(\Omega)$ has another component V_{irr}, which parametrizes irreducible quartics C. This time,

$$-K_S \cdot C - \deg \phi_* D = 3,$$

so (1.7.0) applies and all irreducible components of V_{irr} have dimension 2 as expected. It is however not obvious that there indeed exist irreducible rational quartics passing through Ω (so that V_{irr} is non-empty): we give an argument in the next paragraph.

(5.2) Le me now show that V_{irr} introduced above is non-empty, following a suggestion of Edoardo Sernesi. Consider the family W of reducible quartics $C_1 + C_2$, where C_1 and C_2 are two conics, passing respectively through the first five points of Ω, and through the remaining four points. Since no four points of Ω are aligned, C_1 is fixed and C_2 moves in a pencil, hence W has dimension 1 as expected. Besides,

$$-K_S \cdot C_1 - \deg \phi_* D|_{C_1} = 1 \quad \text{and} \quad -K_S \cdot C_2 - \deg \phi_* D|_{C_2} = 2,$$

so (1.7.0) indeed applies to W.

The fact that the dimension of W equals the expected dimension implies that the nodes of a general member of W may be smoothed independently, i.e., a general member of W may be deformed in such a way that an arbitrary subset of its nodes are preserved while the other are smoothed. Indeed the assumption implies that W is smooth at a general point $[C]$, and the conditions defining its tangent space as a linear subspace of $H^0(C, \mathcal{O}_C(C))$ are independent, hence these conditions may be relaxed independently. This is a standard argument going back to Severi himself, of which a modern account may be found in [17].

The upshot is that one may smooth one and only one node of a general member $C_1 + C_2$ of W, while preserving the condition of passing through the points in Ω: the result is a necessarily irreducible quartic curve with three nodes, passing through Ω, as required.

5.2 – Superabundant log Severi varieties coming from double covers

In this subsection I give examples of superabundant logarithmic Severi varieties, i.e., such that the dimension exceeds the expected dimension: the inequality in (1.7.00) is strict, and (1.7.0) does not apply. These examples live on the projective plane, and come from linear systems on double covers. This is taken from [8]. The examples are given in (5.5) below, after a few recaps on double covers.

(5.3) We shall use the following elementary facts about double covers. Let d be a positive integer, and B be a degree $2d$ curve in $\mathbf{P}^2$. We consider the double cover $\pi : S \to \mathbf{P}^2$ branched over B. Let H be the line class on $\mathbf{P}^2$, and L be its pull-back to S. For all $k \in \mathbf{N}$ we have

$$H^0(S, kL) = \pi^* H^0(\mathbf{P}^2, kH) \oplus \pi^* H^0(\mathbf{P}^2, kH - \tfrac{1}{2}B),$$

which is the isotypical decomposition of $H^0(S, kL)$ as a representation of $\mathbf{Z}/2\mathbf{Z}$. The first summand corresponds to divisors that are double covers of degree k curves in $\mathbf{P}^2$, and the second to divisors that decompose as B (seen as the ramification divisor in S) plus the double cover of a degree $k - d$ curve in $\mathbf{P}^2$.

(5.4) Proposition. *For $k \geqslant d$, the general member C of $|kL|$ is not a double cover of some hypersurface in $\mathbf{P}^2$, the restriction $\pi|_C$ is birational on its image, a degree $2k$ hypersurface $C^\flat$ in $\mathbf{P}^2$ everywhere tangent to B, with a node at every point of a complete intersection Z of type $(k, k-d)$.*

Proof. The divisor C belongs to a unique pencil $\langle A', B+D'\rangle$, with A' and D' the double covers of curves A and D in $\mathbf{P}^2$ of respective degrees k and $k-d$. Thus $C^\flat := \pi(C)$ belongs to the pencil $\langle 2A, B+2D\rangle$, from which it follows that $C^\flat$ is double along $Z := A\cap D$, and touches B doubly along $A\cap B$, which accounts for the whole intersection scheme of $C^\flat$ and B. The base locus of this pencil is the scheme defined by the ideal sheaf $\mathcal{I}_Z^2(\mathcal{I}_A^2+\mathcal{I}_B)$.

The pull-back $\pi^*C^\flat \in |2kL|$ splits as $C+i(C)$, with i the involution on V associated to π ; it has a double singularity along $Z' := \pi^{-1}(Z)$ and $\pi^{-1}(B\cap A)$, with at each point one local sheet belonging to C and another to $i(C)$. The union $Z'\cup\pi^{-1}(B\cap A)$ is the base locus of the pencil $\langle A', B+D'\rangle$. □

(5.5) Example. We consider the image $V_{B,k}$ in $|2kH|$ of the linear system $|kL|$ on S. It has dimension

$$h^0(S,kL)-1 = h^0(\mathbf{P}^2,kH)+h^0(\mathbf{P}^2,(k-d)H)-1,$$

and parametrizes curves of geometric genus

$$g_{k,d} = \tfrac{1}{2}(2k-1)(2k-2)-k(k-d),$$

everywhere tangent to B; the number of contact points is thus $2kd$.

The family of curves $V_{B,k}$ is therefore contained in the log-Severi variety $V^{2k}_{g_{k,d}}(0,[0,2kd])$ of the pair $(\mathbf{P}^2,B)$, for which the expected dimension is

$$-(K_{\mathbf{P}^2}+B)\cdot 2kH + g_{k,d} - 1 + 2kd = k(k+3-d).$$

By (1.7.0) a component of the Severi variety has the expected dimension if it has an irreducible member and

$$-K_{\mathbf{P}^2}\cdot 2kH - 2kd \geqslant 1 \quad\Longleftrightarrow\quad 2k(3-d)\geqslant 1;$$

we note that the latter inequality holds if and only if $d\leqslant 2$.

It turns out that the dimension of our family $V_{B,k}$ exceeds the expected dimension of the log-Severi variety. Indeed a direct computation shows that

$$\begin{aligned}\dim(V_{B,k}) - \operatorname{expdim}\big(V^{2kH}_{g_{k,d}}(0,[0,2kd])\big) &= \tfrac{(d-1)(d-2)}{2}\\ &= p_g(S)\end{aligned}$$

(cf. [5, V.22 p.237] for the last equality).

5.3 – Logarithmic $K3$ surfaces

In this subsection I discuss logarithmic Severi varieties of pairs (S,R) with $K_S+R=0$, in relation with Severi varieties of $K3$ surfaces. This is a central theme of this volume, and as such it will be revisited in various other chapters.

(5.6) $K3$ surfaces. Let us first consider the case of "absolute" $K3$ surfaces: let S be a $K3$ surface, and $R=0$. In this case Theorem (1.7) is not quite accurate, a (well-known) prominent issue being that the condition in order for (1.7.0) to apply is not verified, and indeed the expected dimension given in (1.7.0) is not the actual dimension.

Suppose S is equipped with a polarisation L of genus p (i.e., $L^2=2p-2$). The expected dimension of V_g^L as defined in (3.3) is $g-1$, whereas its actual dimension is g, if $0\leqslant g\leqslant p$ and S is very general, say. Technically, the deformations of $[C]\in V_g^L$ are governed by the invertible sheaf $L|_C\cong\omega_C$, hence the obstruction space is $H^1(C,\omega_C)$ which is 1-dimensional; it turns out however that the equigeneric deformations of C are unobstructed, even though the obstruction

space is non-trivial. A conceptual explanation for this is that there exist non-algebraic $K3$ surfaces, which do not carry any curve at all; the expected dimension for Severi varieties on $K3$ surfaces thus becomes accurate if considered in family over all $K3$ surfaces, including the non-algebraic ones. I refer to [10, §4.2] for a detailed account; in the present volume, this phenomenon will be considered again in [XI, subsection 3.1], in the context of Gromov–Witten theory.

We shall now describe some analogous phenomena for $K3$ pairs, by looking at typical examples.

(5.7) Surfaces with canonical curve sections. Let S be $\mathbf{P}^2$, and R be a smooth cubic; note that in this case one has $K_S + R = 0$. Let C be a smooth curve of degree $d \geqslant 4$ on S, and set $\Omega = C \cap R$ (for simplicity we may assume that C and R intersect transversely). The blow-up S' of $\mathbf{P}^2$ at Ω is a smooth surface having a unique anticanonical divisor, namely the proper transform R' of R. The linear system $|C'|$ of the proper transform C' of C gives a birational model of S' in $\mathbf{P}^p$, in which the proper transform of R is contracted, and whose hyperplane sections are the canonical models of the degree d plane curves passing through Ω (we have let p equal $\mathrm{p}(d)$, the genus of smooth plane d-ics).

In many aspects the surface S' behaves like a $K3$ surface. Beware, however, that it may not always be deformed to a $K3$ surface: this is the case if and only if $d \leqslant 6$; if $d > 6$ the elliptic singularity on S' gotten from the contraction of the proper transform of R is not smoothable, see [2].

Surfaces of this kind naturally occur in degenerations of $K3$ surfaces. For instance, surfaces S' as above with $d = 4$ appear in degenerations of smooth quartics in $\mathbf{P}^3$ to the union of a smooth cubic and a plane, see [I, Section 7.2]; there we have seen that the logarithmic Severi variety $V^4_g(12, 0)(\Omega)$ of the pair $(\mathbf{P}^2, R)$ appears in the limit of the Severi varieties of genus g hyperplane sections of smooth quartics.

There are of course many variants of this situation. For instance, Du Val surfaces[6] have been brought in the foreground recently in [1]. In this volume we shall often consider the case when S is a toric surface, and R is a cycle of smooth rational curves, the sum of all toric divisors of S. An emblematic instance of this situation is that of the pair consisting of $\mathbf{P}^2$ and a triangle, endowed with the linear system of plane quartics passing through twelve points cut out as above on the triangle by a quartic; it occurs in the degeneration of quartic $K3$ surfaces in $\mathbf{P}^3$ to a tetrahedron, see [I, 8.2], in which it corresponds to the contribution of a face (see ibid.). Other examples are given in [X, Section 3.3].

(5.8) Logarithmic $K3$ surfaces. Let $S' \subseteq \mathbf{P}^p$ be as in (5.7). For all $g = 0, \ldots, p$, we may view the Severi variety of genus g hyperplane sections of $S' \subseteq \mathbf{P}^p$ as the logarithmic Severi variety $V^d_p(3d, 0)(\Omega)$ of the pair $(\mathbf{P}^2, R)$. Its expected dimension, as defined in (3.3), is $g - 1$, whereas its actual dimension is g; note that once again (1.7.0) does not apply, because the required inequality does not hold. In this case the discrepancy between the actual and expected dimensions comes from the fact that the points in Ω are not general points on R. This is illustrated in Examples (5.8.1) and (5.8.2) below.

It is pleasant to think of the pair $(\mathbf{P}^2, R)$ endowed with a set Ω as a *logarithmic $K3$ surface*, *algebraic* when Ω is cut out on R by a plane curve as above, and *non-algebraic* otherwise, when the points in Ω are in general position. One may prefer to consider that the logarithmic $K3$

[6] a genus p Du Val surface is the projective plane blown-up at nine points on a smooth cubic R, endowed with the linear system of the proper transforms of plane curves of degree $3g$ with a g-tuple ordinary point at the first eight blown-up points, and a $(g-1)$-tuple ordinary point at the remaining blown-up point; these curves have geometric genus g, and the linear system has dimension g (note that it has a tenth base point lying on R).

surface is the equivalent data of the pair (S', R'), where S' is the blow-up of $\mathbf{P}^2$ at Ω, and R' is the proper transform of R. For a more conceptual definition, as well as a classification, I refer to [13].

In the logarithmic context, the manoeuver of taking non-algebraic $K3$'s into account to adjust the expected dimension takes the following guise: one chooses an arbitrary point $a \in \Omega$, and considers $V_p^d(3d-1,1)(\Omega - a)$ instead of $V_p^d(3d,0)(\Omega)$. Statement (1.7.0) applies to the former, which thus has both expected and actual dimension equal to p, and it turns out that the mobile contact point with R remains in fact immobile, so that all members of $V_p^d(3d-1,1)(\Omega-a)$ automatically pass through the point a as well.

Note that Statement (1.7.1) does not apply to $V_p^d(3d-1,1)(\Omega - a)$ either, and indeed one can cook up examples (e.g., by letting C and R be tangent) in which the assertions (b), ($\mathrm{c}^\flat$), and (d) of (1.7.1) do not hold.

(5.8.1) Example. Let us illustrate this in the concrete case $d = 4$ (then, $p = 3$). The linear system of plane quartics has dimension 14. If we take a set Ω of 12 general points on R, then the Severi variety $V_3^4(12,0)(\Omega)$ is empty: Ω imposes 12 independent conditions on quartics, the linear system of quartics through Ω is 2-dimensional, and all its members are made of R plus a line. On the other hand if we take Ω the complete intersection of R with a smooth quartic C, then one sees using the restriction exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(1) \to \mathcal{O}_{\mathbf{P}^2}(4) \to \mathcal{O}_R(4) \to 0$$

that the linear system of quartics through Ω is 3-dimensional, generated by C and the net of reducible quartics containing R (see also [X, Lemma (3.1)]). If we take a set Ω of 11 general points on R, it imposes 11 independent conditions on quartics, and the linear system of quartics through Ω is 3-dimensional with a 12-th base point on R.

(5.8.2) Example. We may consider the linear system $|2C'|$ in a similar fashion. Seen on $\mathbf{P}^2$, it is the system of plane $(2d)$-ics with a node at each of the $3d$ points of $\Omega = C \cap R$.

Again we shall work this out in the case $d = 4$. One has $(2C')^2 = 4 \cdot (C')^2 = 16 = 2 \cdot 9 - 2$, so the adjunction formula on S', which is essentially the same as on a $K3$ surface, tells us that curves in $|2C'|$ have genus 9. Moreover the Riemann–Roch Formula, which works as on a $K3$ as well, tells us that $|2C'|$ has dimension 9.

On the other hand plane octics have arithmetic genus 21, so an octic with 12 nodes has geometric genus 9, confirming the above computations carried out on S'. The linear system of plane octics has dimension 44. Since a node at a prescribed point is 3 conditions, the expected dimension of a linear system of octics with 12 nodes at prescribed points is $44 - 36 = 8$. When the nodes are imposed at a general set of points Ω on R, this is indeed the correct dimension, and the linear system consists only of curves made of R plus a quintic curve passing (in general simply) through Ω. When Ω is cut out on R by a quartic curve C_1, the linear system has an extra generator (namely, the curve $2C_1$), and thus has one extra dimension.

This may also be verified using a resolution of the ideal sheaf $\mathcal{I}_\Omega^2$, where $\mathcal{I}_\Omega$ is the ideal sheaf of $\Omega \subseteq \mathbf{P}^2$. Let r and f be homogeneous equations of the curves R and C respectively. While for $\mathcal{I}_\Omega$ there is the Koszul resolution, for its square we have the exact sequence

$$0 \to \mathcal{O}(-10) \oplus \mathcal{O}(-11) \xrightarrow{\begin{pmatrix} -f & 0 \\ r & -f \\ 0 & r \end{pmatrix}} \mathcal{O}(-6) \oplus \mathcal{O}(-7) \oplus \mathcal{O}(-8) \xrightarrow{(r^2, rf, f^2)} \mathcal{I}_\Omega^2 \to 0,$$

which gives $h^0(\mathcal{O}_{\mathbf{P}^2}(8) \otimes \mathcal{I}_\Omega^2) = 10$ as required.

This carries over to all systems $|kC'|$, to the effect that the condition of having a k-uple point at all $3d$ points of $\Omega = C \cap R$ imposes one less condition on plane d-ics than if the $3d$ points of Ω were in general position. We leave this to the reader.

(5.9) Severi varieties of curves with full contact. Logarithmic Severi varieties of logarithmic $K3$ surfaces (S, R) parametrizing curves having one point of contact with high order with R have attracted the attention for several reasons. A first reason is the possibility, if (S, R) comes from a degeneration of $K3$ surfaces, to deform such a logarithmic Severi variety (roughly speaking) to a Severi variety on the nearby $K3$ surfaces, the contact point of high order m deforming to $m-1$ nodes: this is the central theme of the present volume, in particular see [VIII]. Besides, this possibility is not specific to $K3$ surfaces.

Another reason is the search for $\mathbf{A}^1$ curves (i.e., curves isomorphic to the affine line, in other words affine rational curves) on logarithmic $K3$ surfaces, which is the analogue of the search for rational curves on $K3$ surfaces. Let me use an example to clarify this problem. Consider the projective plane together with a smooth cubic curve R (and an empty set of points Ω). The problem is to find $\mathbf{A}^1$ curves lying in the open surface $\mathbf{P}^2 \setminus R$. To this effect it is relevant to consider the logarithmic Severi varieties $V_0^d(0, [0, \dots, 0, 3d])$ of the pair $(\mathbf{P}^2, R)$. They have expected dimension 0, and of course for any such curve the contact point with R must be one of the $(3d)^2$ order $3d$ torsion points of R. The problem of the existence of $\mathbf{A}^1$ curves thus corresponds to that of the non-emptiness of these logarithmic Severi varieties. The characterization of the log-$K3$ surfaces for which there exists infinitely many $\mathbf{A}^1$ curves has been carried out in [7]. The interested reader may also consult [9] for other non-emptiness results, in particular [9, Prop. 3.12].

Logarithmic Severi varieties with one point of full contact have also been related in [11] with the tropical vertex group. Elements of this group are formal families of symplectomorphisms of the 2-dimensional algebraic torus $(\mathbf{C}^*)^2$. The relevant Severi varieties are the following: let S be a toric surface, and let R be the sum of all toric divisors $R_1, \dots, R_n, R_{\text{out}}$ (R_{out} is simply one of the toric divisors, to which we want to give a special role); one considers logarithmic Severi varieties of the pair (S, R) of the form

$$V_0^{\xi}\big((\alpha_1, \dots, \alpha_n, 0), (0, \dots, 0, [0, \dots, 0, m])\big)(\Omega_1, \dots, \Omega_n, \varnothing),$$

which parametrize rational curves having fixed intersection schemes with $R_1, \dots, R_n$, and one contact point of maximal order $m = \xi \cdot R_{\text{out}}$ with R_{out}. These logarithmic Severi varieties have expected and actual dimension 0, and the number of points they are made of may be interpreted as Gromov–Witten invariants. The main result of [11] is that these numbers are structure constants of the tropical vertex group; they are in fact determined by ordered product factorizations of commutators in the tropical vertex group, namely they appear as the coefficients of the logarithms of the factors.

References

Chapters in this Volume

[I] C. Ciliberto and Th. Dedieu, *Limits of nodal curves: generalities and examples.*

[II] E. Sernesi, *Deformations of curves on surfaces.*

[VI] Th. Dedieu and C. Galati, *Degenerations and deformations of generalized Severi varieties.*

[VII] Th. Dedieu, *The Caporaso–Harris recursion within a degeneration of the plane.*

[VIII] C. Galati, *Proving the existence of curves on smooth surfaces by smoothing tacnodes on reducible surfaces.*

[X] Th. Dedieu, *Enumeration of curves in toric surfaces, tropical geometry, and floor diagrams.*

[XI] Th. Dedieu, *Enumerative geometry of $K3$ surfaces.*

External References

[1] E. Arbarello, A. Bruno, G. Farkas and G. Saccà, *Explicit Brill-Noether-Petri general curves*, Comment. Math. Helv. **91** (2016), no. 3, 477–491.

[2] E. Arbarello, A. Bruno and E. Sernesi, *On hyperplane sections of K3 surfaces*, Algebr. Geom. **4** (2017), no. 5, 562–596.

[3] E. Arbarello and M. Cornalba, *Su una proprietà notevole dei morfismi di una curva a moduli generali in uno spazio proiettivo*, Rend. Sem. Mat. Univ. Politec. Torino **38** (1980), no. 2, 87–99 (1981).

[4] E. Arbarello and M. Cornalba, *Su una congettura di Petri*, Comment. Math. Helv. **56** (1981), no. 1, 1–38.

[5] W. P. Barth, K. Hulek, C. A. M. Peters and A. Van de Ven, *Compact complex surfaces*, 2nd enlarged ed., Ergeb. Math. Grenzgeb., 3. Folge, vol. 4, Berlin: Springer, 2004.

[6] L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392.

[7] X. Chen and Y. Zhu, $\mathbb{A}^1$ *curves on log K3 surfaces*, Adv. Math. **313** (2017), 718–745.

[8] C. Ciliberto and Th. Dedieu, *Double covers and extensions*, to appear in Kyoto J. Math.

[9] C. Ciliberto, Th. Dedieu, C. Galati and A. L. Knutsen, *Nonemptiness of Severi varieties on Enriques surfaces*, Forum Math. Sigma **11** (2023), e52.

[10] Th. Dedieu and E. Sernesi, *Equigeneric and equisingular families of curves on surfaces*, Pub. Mat. **62** (2017), 175–212.

[11] M. Gross, R. Pandharipande and B. Siebert, *The tropical vertex*, Duke Math. J. **153** (2010), no. 2, 297–362.

[12] R. Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010.

[13] S. Iitaka, *On logarithmic K*3 *surfaces*, Osaka Math. J. **16** (1979), no. 3, 675–705.

[14] S. L. Kleiman and V. V. Shende, *On the Göttsche threshold*, in *A celebration of algebraic geometry*, Clay Math. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 2013, with an appendix by I. Tyomkin, 429–449.

[15] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 32, Springer-Verlag, Berlin, 1996.

[16] E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften, vol. 334, Springer-Verlag, Berlin, 2006.

[17] A. Tannenbaum, *Families of algebraic curves with nodes*, Compositio Math. **41** (1980), no. 1, 107–126.

[18] B. Teissier, *Résolution simultanée I, II*, in *Séminaire sur les singularités des surfaces, Cent. Math. Éc. Polytech., Palaiseau 1976–77*, Lecture Notes in Mathematics, vol. 777, Springer-Verlag, Berlin, 1980, 71–146.

[19] O. Zariski, *Dimension-theoretic characterization of maximal irreducible algebraic systems of plane nodal curves of a given order n and with a given number d of nodes*, Am. J. Math. **104** (1982), no. 1, 209–226.

Lecture IV
Deformations of an m-tacnode into $m-1$ nodes

by Margherita Lelli-Chiesa

1 Statement of results 83
2 Proof of the main theorem in a model case 84
3 Proof of the main theorem in general 86
A Appendix: Chebyshev polynomials 87
References 89

This text is dedicated to a somewhat technical-looking statement (Theorem (1.2)) which is the cornerstone of many results in this volume: it is the key to understand the phenomenon of, loosely speaking, $(m-1)$-nodal curves degenerating to m-tacnodal curves within a degeneration of smooth surfaces to the transverse union of two surfaces. About this phenomenon see, in particular, [I, VI, B].

1 – Statement of results

Let C be a planar complex curve. A point $P \in C$ is called an m-tacnode if the local equation of C at P has the form $y(y+x^m)=0$, where (x,y) are local coordinates centered at P. The Jacobian ideal of C at P is $J=(2y+x^m, yx^{m-1})$ and

$$\mathcal{O}_{\mathbf{A}^2}/J = \langle y, yx, \ldots, yx^{m-2}, 1, x, \ldots, x^{m-1}\rangle.$$

We denote by $\pi : S \to \Delta$ the versal deformation space of $P \in C$, where $\Delta \simeq \mathbf{A}^{2m-1}_{\alpha_0,\ldots,\alpha_{m-2},\beta_0,\ldots,\beta_{m-1}}$, and $S \subseteq \Delta \times \mathbf{A}^2_{x,y}$ is defined by the equation

$$y^2 + yx^m + \alpha_{m-2}yx^{m-2} + \ldots + \alpha_1 yx + \alpha_0 y + \beta_{m-1}x^{m-1} + \ldots + \beta_1 x + \beta_0 = 0, \tag{1.0.1}$$

see [II, Section 5]. We set

$$\Delta_m := \overline{\{(\underline{\alpha},\underline{\beta}) \in \Delta \,|\, \pi^{-1}(\underline{\alpha},\underline{\beta}) \text{ has } m \text{ nodes}\}}, \tag{1.0.2}$$

$$\Delta_{m-1} := \overline{\{(\underline{\alpha},\underline{\beta}) \in \Delta \,|\, \pi^{-1}(\underline{\alpha},\underline{\beta}) \text{ has } m-1 \text{ nodes}\}}. \tag{1.0.3}$$

(1.1) Proposition. (1.1.1) *The locus Δ_m is smooth of dimension $m-1$, and it is defined in Δ by the equations $\beta_0=\beta_1=\ldots=\beta_{m-1}=0$.*
(1.1.2) *The locus Δ_{m-1} has dimension m, and has m local sheets at a general point of Δ_m.*

Proof. The equation of $\pi^{-1}(\underline{\alpha},\underline{\beta})$ has degree two in y and its discriminant is given by

$$\delta_{\underline{\alpha},\underline{\beta}}(x) = (x^m + \alpha_{m-2}x^{m-2} + \ldots + \alpha_1 x + \alpha_0)^2 - 4(\beta_{m-1}x^{m-1} + \ldots + \beta_1 x + \beta_0).$$

If V denotes the space of monic polynomials of degree $2m$ in x with no degree $2m-1$ term, the regular map $\delta : \Delta \to V$ mapping a point $(\underline{\alpha},\underline{\beta})$ to its discriminant $\delta_{\underline{\alpha},\underline{\beta}}(x)$ is an isomorphism;

indeed, $\underline{\alpha}$ and $\underline{\beta}$ can be easily expressed in terms of the coefficients of $\delta_{\underline{\alpha},\underline{\beta}}(x)$. We remark that, for fixed $(\underline{\alpha},\underline{\beta})$, the locus $\pi^{-1}(\underline{\alpha},\underline{\beta})$ is a double cover of the x-axis branched along the vanishing locus of $\delta_{\underline{\alpha},\underline{\beta}}(x)$, which is a divisor of degree $2m$.

A point $(\underline{\alpha},\underline{\beta})$ lies in Δ_m whenever $\delta_{\underline{\alpha},\underline{\beta}}(x)$ has m double roots, that is, if and only if $\delta_{\underline{\alpha},\underline{\beta}}(x)$ is a square, equivalently if and only if $\beta_0=\beta_1=\ldots=\beta_{m-1}=0$. This proves (1.1.1).

Analogously, $(\underline{\alpha},\underline{\beta})$ lies in Δ_{m-1} exactly when $\delta_{\underline{\alpha},\underline{\beta}}(x)$ has $m-1$ double roots, that is, we can rewrite the discriminant as

$$\delta_{\underline{\alpha},\underline{\beta}}(x)=\left(\prod_{i=1}^{m-1}(x-a_i)^2\right)\cdot(x^2+bx+c), \tag{1.1.3}$$

where in fact $b=2\sum_{i=1}^{m-1}a_i$ because the degree $2m-1$ term of $\delta_{\underline{\alpha},\underline{\beta}}(x)$ vanishes. Hence, Δ_{m-1} has dimension m, and the m local sheets at a general point of Δ_m as in (1.1.2) reflect the possibilities for choosing $m-1$ double roots out of m. □

Our main goal is to prove the following theorem.

(1.2) Theorem ([1, Lemma 2.10])**.** *Let $\Lambda\subseteq\Delta$ be a smooth m-dimensional variety such that:*
(i) Λ contains Δ_m;
(ii) the tangent space $T_{\underline{0}}\Lambda$ of Λ at $\underline{0}$ is not contained in the hyperplane H defined by the equation $\beta_0=0$.
Then, we have $\Lambda\cap\Delta_{m-1}=\Delta_m\cup\Psi$, where Ψ is a smooth curve intersecting Δ_m at $\underline{0}$ with multiplicity m.

Note that Δ_{m-1} and Λ both have codimension $m-1$ in Δ, which has dimension $2m-1$, so that the expected dimension for their intersection is 1. The theorem thus states that if Λ contains Δ_m, under a suitable transversality assumption, namely (ii), the intersection $\Lambda\cap\Delta_{m-1}$ residual to the obvious superabundant component Δ_m has the expected dimension.

The reader may also consult [A] for complements and detailed examples.

2 – Proof of the main theorem in a model case

We first prove Theorem (1.2) in the special case where $\Lambda=\Lambda_0=\{\beta_1=\ldots=\beta_{m-1}=0\}$.

Equation (1.0.1) of S restricted to Λ is:

$$y^2+yx^m+\alpha_{m-2}yx^{m-2}+\ldots+\alpha_1yx+\alpha_0y+\beta_0=0, \tag{2.0.1}$$

and the discriminant of a point $(\underline{\alpha},\underline{\beta})\in\Lambda$ has the form:

$$\delta_{\underline{\alpha},\underline{\beta}}(x)=(x^m+\alpha_{m-2}x^{m-2}+\ldots+\alpha_1x+\alpha_0)^2-4\beta_0. \tag{2.0.2}$$

If $\beta_0=0$, then $(\underline{\alpha},\underline{\beta})$ lies in Δ_m. Therefore, $\Lambda\cap\Delta_{m-1}$ contains Δ_m with multiplicity m, since by Proposition (1.1), Δ_{m-1} has multiplicity m along Δ_m.

From now on, we assume $\beta_0\neq0$, set $\nu(x):=x^m+\alpha_{m-2}x^{m-2}+\ldots+\alpha_1x+\alpha_0$ and rewrite (2.0.2) as:

$$\delta_{\underline{\alpha},\underline{\beta}}(x)=(\nu(x)-2\sqrt{\beta_0})(\nu(x)+2\sqrt{\beta_0}). \tag{2.0.3}$$

Note that the polynomials $\nu(x)-2\sqrt{\beta_0}$ and $\nu(x)+2\sqrt{\beta_0}$ have no common roots since $\beta_0\neq0$. We impose that $\delta_{\underline{\alpha},\underline{\beta}}(x)$ has $m-1$ double roots, taking advantage of the following lemma.

(2.1) Lemma. *Let $\gamma := 2\sqrt{\beta_0}$ with $\beta_0 \neq 0$. Then, for every integer $m \geqslant 2$ there exists a polynomial*

$$\nu_\gamma(x) := x^m + c_{m-2}(\gamma)x^{m-2} + \ldots + c_1(\gamma)x + c_0(\gamma)$$

in x with complex coefficients $c_i(\gamma)$ such that $c_{m-2}(\gamma) \neq 0$ and the following are satisfied:

(a) if $m = 2l+1$ is odd, then both $\nu_\gamma(x) + \gamma$ and $\nu_\gamma(x) - \gamma$ have l double roots;

(b) if $m = 2l$ is even, then $\nu_\gamma(x) + \gamma$ ($\nu_\gamma(x) - \gamma$, respectively) has l ($l-1$, respectively) double roots.

Furthermore, $\nu_\gamma(x)$ is unique up to replacing it with $\nu_\gamma(\xi x)$ for some m-th root ξ of unity, and it is odd (even, respectively) if m is.

Taking the lemma for granted, we set $m := 2l + \varepsilon$ with $\varepsilon \in \{0, 1\}$ depending on the parity of m and consider the polynomial

$$\nu_1(x) = x^m + c_{m-2}(1)x^{m-2} + c_{m-4}(1)x^{m-4} + \ldots + c_\varepsilon(1)x^\varepsilon$$

obtained for $\gamma = 1$ (note that only monomials of degree congruent to m modulo 2 appear in $\nu_1(x)$ because $\nu_1(x)$ and m have the same parity). For any other $\gamma \in \mathbf{C}^*$, by setting $\gamma := u^m$, we get $\nu_\gamma(x) = u^m \nu_1\left(\frac{x}{u}\right)$. The coefficients of $\nu_\gamma(x)$ are thus obtained from those of $\nu_1(x)$ in the following way:

$$c_{2h+\varepsilon}(\gamma) = c_{2h+\varepsilon}(u) = u^{2l-2h} c_{2h+\varepsilon}(1), \text{ for } 0 \leqslant h \leqslant l-1.$$

By the change of variable $t := u^2$, the equations

$$c_{2h+\varepsilon}(t) = t^{l-h} c_{2h+\varepsilon}(1), \text{ for } 0 \leqslant h \leqslant l-1$$

parametrize a curve Ψ_0 in $\Lambda \cap \Delta_{m-1}$ (defined by setting $\beta_0(t) = \frac{t^m}{4}$, $\alpha_i(t) = c_i(t)$ for $i = 2h+\varepsilon$ with $0 \leqslant h \leqslant l-1$, and $\alpha_i(t) = 0$ otherwise). This curve is smooth because $c_{m-2}(1) \neq 0$, and its contact order with Δ_m at $\underline{0}$ is m because $\beta_0(t) = \frac{t^m}{4}$. Therefore, it only remains to prove the lemma.

Proof of Lemma (2.1). We first prove the existence of a degree m covering $f : \mathbf{P}^1 \to \mathbf{P}^1$ mapping ∞ to ∞ and with $\{\infty, \gamma, -\gamma\}$ as branch locus; we also require f to be totally ramified at ∞ and the number of ramification points in the fiber $f^{-1}(-\gamma)$ (respectively, in $f^{-1}(\gamma)$) to coincide with the number of double roots we are requiring for the polynomial $\nu_\gamma(x) + \gamma$ (respectively, $\nu_\gamma(x) - \gamma$).

Assume that $m = 2l + 1$ is odd. By Riemann's Existence Theorem, the existence of f is equivalent to the existence of two permutations τ, σ in the symmetric group S_m, each one the product of l disjoint transpositions, such that $\tau \cdot \sigma$ is cyclic of order m in S_m. The permutations $\tau := (1\,2)(3\,4)\cdots(2l-1\,2l)$ and $\sigma := (2\,3)(4\,5)\cdots(2l\,2l+1)$ satisfy these requirements and they are the only elements of S_m doing that, up to conjugation. As a consequence, f exists and is unique up to automorphisms of the domain fixing ∞. The even case is analogous.

Since $f(\infty) = \infty$, the covering f defines a polynomial $\nu_\gamma(x)$ satisfying (a) or (b) respectively. By acting with an automorphism of $\mathbf{P}^1$ fixing ∞ (i.e., composing $\nu_\gamma(x)$ with a polynomial of degree 1), we can assume $\nu_\gamma(x)$ to be monic and with no degree $m-1$ term. The polynomial $\nu_\gamma(x)$ is then unique up to replacing it with $\nu_\gamma(\xi x)$ for some m-th root ξ of unity. By Proposition (A.1) in the Appendix, we can therefore assume that $\nu_1(x)$ coincides with the Chebyshev polynomial of the first kind $T_m(x)$; in particular, it is even or odd depending on the parity of m, and the coefficient $c_{m-2}(1)$ is nonzero, as one can easily check using (A.0.1) in the Appendix. The same properties hold for $\nu_\gamma(x)$ for any $\gamma = u^m \in \mathbf{C}^*$ because $\nu_\gamma(x) = u^m \nu_1\left(\frac{x}{u}\right)$.

□

3 – Proof of the main theorem in general

We now prove Theorem (1.2) in full generality.

We consider the blow-up $\tau : \widetilde{\Delta} \to \Delta$ of Δ along Δ_m and denote by $Z := \tau^{-1}(\Delta_m)$ the exceptional divisor and by $\Phi := \tau^{-1}(\underline{0})$ the inverse image of $\underline{0}$. Then $\Phi \simeq \mathbf{P}^{m-1}_{\beta_0,\ldots,\beta_{m-1}}$ can be identified with the space of polynomials of degree at most $m-1$ in x modulo scalars. The open subset $\Phi \supset \Phi_0 := \{\beta_0 \neq 0\}$ parametrizes polynomials that do not vanish in $x = 0$ and the point $Q = [1 : 0 : \ldots : 0] \in \Phi_0$ corresponds to constants. The proof of Theorem (1.2) in general is based on the description of the strict transform $\widetilde{\Delta_{m-1}}$ of Δ_{m-1} in $\widetilde{\Delta}$ and proceeds by steps.

Step 1. *The point Q lies in $\widetilde{\Delta_{m-1}}$.*

Let $\widetilde{\Lambda_0}$ be the strict transform of the linear space $\Lambda_0 = \{\beta_1 = \ldots = \beta_{m-1} = 0\}$ under τ. We have already shown that $\Lambda_0 \cap \Delta_{m-1} = \Delta_m \cup \Psi_0$ as in Theorem (1.2). Since $\widetilde{\Lambda_0} \cap \Phi = \{Q\}$, then Q lies in the strict transform $\widetilde{\Psi_0}$ of the curve Ψ_0 and hence also in $\widetilde{\Delta_{m-1}}$.

Step 2. *The locus Φ is an irreducible component of $\widetilde{\Delta_{m-1}} \cap Z$ and is the only component containing Q.*

We identify Δ_m with the space of monic polynomials of degree m in x with no degree $m-1$ term. Let $\alpha(t) \subseteq \Delta_m$ be any arc such that $\alpha(0) = \underline{0}$. As t goes to 0, all the roots of the polynomial corresponding to $\alpha(t)$ tend to $x = 0$ and hence the limit of $\tau^{-1}(\alpha(t)) \cap \widetilde{\Delta_{m-1}}$ is not contained in Φ_0. As a consequence, any component of $\widetilde{\Delta_{m-1}} \cap Z$ containing Q is contained in Φ, and thus coincides with it by dimensional reason.

Step 3. *The strict transform $\widetilde{\Delta_{m-1}}$ is smooth at Q.*

The previous step implies that the only irreducible component of $\widetilde{\Lambda_0} \cap \widetilde{\Delta_{m-1}}$ containing Q is the curve $\widetilde{\Psi_0}$. In particular, the intersection is proper in a neighborhood of Q (indeed, $\dim \widetilde{\Lambda_0} = \dim \widetilde{\Delta_{m-1}} = m$, and $\dim \widetilde{\Delta} = 2m-1$); smoothness of $\widetilde{\Psi_0}$ thus yields smoothness of $\widetilde{\Delta_{m-1}}$ at Q.

Step 4. *The strict transform $\widetilde{\Delta_{m-1}}$ is smooth at every point of Φ_0.*

For every $c \in \mathbf{C}^*$, let σ_c be the automorphism of Δ such that $\sigma_c(\alpha_i) = c^{m-i}\alpha_i$ for $0 \leqslant i \leqslant m-2$ and $\sigma_c(\beta_j) = c^{2m-j}\beta_j$ for $0 \leqslant j \leqslant m-1$. This defines an action of the multiplicative group $\mathbf{C}^*$ on Δ, under which Δ_m is clearly invariant. Since $\delta_{\sigma_c(\underline{\alpha}),\sigma_c(\underline{\beta})}(x) = c^{2m}\delta_{\underline{\alpha},\underline{\beta}}\left(\frac{x}{c}\right)$, then Δ_{m-1} is invariant, too. As a consequence, the action of $\mathbf{C}^*$ on Δ lifts to an action on $\widetilde{\Delta}$ preserving $\widetilde{\Delta_{m-1}}$. Furthermore, $\underline{0}$ is a fixed point lying in the closure of any orbit and hence the lifted action also preserves Φ. In particular, given $c \in \mathbf{C}^*$, we obtain an automorphism $\widetilde{\sigma_c}$ of Φ mapping a point P of homogeneous coordinates $[\beta_0 : \beta_1 : \ldots : \beta_{m-1}]$ to

$$\widetilde{\sigma_c}(P) = [c^{2m}\beta_0 : c^{2m-1}\beta_1 : \ldots : c^{m+1}\beta_{m-1}] = [\beta_0 : c^{-1}\beta_1 : \ldots : c^{-(m-1)}\beta_{m-1}].$$

Since the limit for c^{-1} going to 0 of $\widetilde{\sigma_c}(P)$ is Q as soon as $\beta_0 \neq 0$, this shows that Q lies in the closure of any orbit of $\widetilde{\Delta_{m-1}}$ intersecting Φ_0. The statement thus follows from Step 3.

Step 5. *The intersection multiplicity of $\widetilde{\Delta_{m-1}}$ and Z along Φ is m.*

Since the intersection multiplicity of Ψ_0 and Δ_m at $\underline{0}$ is m, then the same holds for that of $\widetilde{\Psi_0}$ and Z at Q. On the other hand, the intersection $\widetilde{\Psi_0} \cap Z$ is contained in $\widetilde{\Delta_{m-1}} \cap Z \cap \widetilde{\Lambda_0}$, and the intersection multiplicity at Q of Φ (which is the only component of $\widetilde{\Delta_{m-1}} \cap Z$ containing Q by Step 2) with $\widetilde{\Lambda_0}$ is 1, because $\widetilde{\Lambda_0} \cap Z$ is a section of $\tau|_Z : Z \to \Delta_m$. The statement follows.

Step 6. *Conclusion.*

Let $\Delta_m \subseteq \Lambda \subsetneq \Delta$ be as in the hypotheses. Since $\widetilde{\Lambda} \cap Z$ is a section of $\tau|_Z : Z \to \Delta_m$, then the intersection $\widetilde{\Lambda} \cap \Phi$ is a point R with multiplicity one. The assumption $T_{\underline{0}}\Lambda \not\subseteq H = \{\beta_0 = 0\}$ ensures that R lies in Φ_0 and hence $\widetilde{\Delta_{m-1}}$ is smooth at R by Step 4. The m-dimensional tangent space $T_R\widetilde{\Delta_{m-1}}$ contains $T_R\Phi$, that has dimension $m-1$. Since $T_R\widetilde{\Lambda} \cap T_R\Phi = \{0\}$ and $\dim T_R\widetilde{\Lambda} = m$, then $T_R\widetilde{\Delta_{m-1}}$ and $T_R\widetilde{\Lambda}$ generate the $(2m-1)$-dimensional tangent space $T_R\widetilde{\Delta}$. Equivalently, $\widetilde{\Delta_{m-1}}$ and $\widetilde{\Lambda}$ intersect transversally along a smooth curve $\widetilde{\Psi}$ in a neighborhood of R. Moreover, the curve $\widetilde{\Psi}$ is not tangent at Φ in R (again because $T_R\widetilde{\Lambda} \cap T_R\Phi = \{0\}$) and its intersection multiplicity with Z at Q is m by Step 5. Therefore, $\Psi := \tau(\widetilde{\Psi})$ is a smooth curve having intersection multiplicity m with Δ_m at $\underline{0}$.

A – Appendix: Chebyshev polynomials

There exist four families of Chebyshev polynomials, T_m, U_m, V_m, W_m — called of the first, second, third and fourth kind — each satisfying

$$P_0(x) = 1, \quad \text{and} \quad \forall m \geqslant 1: \quad P_{m+1}(x) = 2xP_m(x) - P_{m-1}(x). \tag{A.0.1}$$

They may be defined by the following choice of initial conditions:

$$T_1(x) = x, \tag{A.0.2}$$
$$U_1(x) = 2x, \tag{A.0.3}$$
$$V_1(x) = 2x - 1, \tag{A.0.4}$$
$$W_1(x) = 2x + 1. \tag{A.0.5}$$

They are the unique polynomials satisfying

$$T_m(\cos\theta) = \cos(m\theta), \tag{A.0.6}$$
$$U_m(\cos\theta) = \frac{\sin((m+1)\theta)}{\sin\theta}, \tag{A.0.7}$$
$$V_m(\cos\theta) = \frac{\cos((m+1/2)\theta)}{\cos(\theta/2)}, \tag{A.0.8}$$
$$W_m(\cos\theta) = \frac{\sin((m+1/2)\theta)}{\sin(\theta/2)} \tag{A.0.9}$$

for all $\theta \in \mathbf{R}$. The Chebyshev polynomials of the first kind can be explicitly written down as

$$T_m(x) = \frac{1}{2}\Big[\big(x + \sqrt{x^2-1}\big)^m + \big(x - \sqrt{x^2-1}\big)^m\Big],$$

and are even (respectively, odd) whenever m is even (respectively, odd). Polynomials of the third and fourth kind satisfy $W_m(x) = (-1)^m V_m(-x)$.

These are the first few values of Chebyshev polynomials:

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x,\\
T_2(x) &= 2x^2-1,\\
T_3(x) &= 4x^3-3x,\\
T_4(x) &= 8x^4-8x^2+1,\dots
\end{aligned}$$

$$\begin{aligned}
U_0(x) &= 1,\\
U_1(x) &= 2x,\\
U_2(x) &= 4x^2-1,\\
U_3(x) &= 8x^3-4x,\\
U_4(x) &= 16x^4-12x^2+1,\dots
\end{aligned}$$

$$\begin{aligned}
V_0(x) &= 1,\\
V_1(x) &= 2x-1,\\
V_2(x) &= 4x^2-2x-1,\\
V_3(x) &= 8x^3-4x^2-4x+1,\\
V_4(x) &= 16x^4-8x^3-12x^2+4x+1,\dots
\end{aligned}$$

$$\begin{aligned}
W_0(x) &= 1,\\
W_1(x) &= 2x+1,\\
W_2(x) &= 4x^2+2x-1,\\
W_3(x) &= 8x^3+4x^2-4x-1,\\
W_4(x) &= 16x^4+8x^3-12x^2-4x+1\dots
\end{aligned}$$

The next result summarizes some properties of Chebyshev polynomials.

(A.1) Proposition. *If $m=2l$ is even, then*

(a1) $T_m(x)+1=2(T_l(x))^2$,

(a2) $T_m(x)-1=2(x^2-1)(U_{l-1}(x))^2$;

in particular, the polynomial $T_m(x)+1$ is a perfect square, while $T_m(x)-1$ is the product of a perfect square by a degree two polynomial.

If instead $m=2l+1$ is odd, then:

(b1) $T_m(x)+1=(x+1)V_l(x)^2$,

(b2) $T_m(x)-1=(x-1)W_l(x)^2$;

in particular, both $T_m(x)+1$ and $T_m(x)-1$ are the product of a perfect square by a linear polynomial.

Proof. Set $x=\cos\theta$. If $m=2l$ is even, then $T_m(x)+1=\cos(2l\theta)+1=2\cos^2(l\theta)=2(T_l(x))^2$. Similarly,

$$T_m(x)-1=\cos(2l\theta)-1=-2\sin^2(l\theta)=-2\sin^2\theta\cdot\frac{\sin^2(l\theta)}{\sin^2\theta}=2(x^2-1)(U_{l-1}(x))^2.$$

If instead $m = 2l + 1$ is odd, one has

$$T_m(x) + 1 = \cos(m\theta) + 1 = 2\cos^2(m\theta/2) = 2\cos^2(\theta/2) \cdot \frac{\cos^2(m\theta/2)}{\cos^2(\theta/2)} = (x+1)(V_l(x))^2,$$

and

$$T_m(x) - 1 = \cos(m\theta) - 1 = -2\sin^2(m\theta/2) = -2\sin^2(\theta/2) \cdot \frac{\sin^2(m\theta/2)}{\sin^2(\theta/2)} = (x-1)(W_l(x))^2.$$

□

References

Chapters in this Volume

[I] C. Ciliberto, *Limits of nodal curves: generalities and examples.*

[II] E. Sernesi, *Deformations of curves on surfaces.*

[A] Th. Dedieu, *Geometry of the deformation space of an m-tacnode into $m-1$ nodes.*

[VI] Th. Dedieu, C. Galati, *Degenerations and deformations of generalized Severi varieties.*

[B] C. Ciliberto, Th. Dedieu, C. Galati, *Flag semi-stable reduction of tacnodal curves at the limit of nodal curves.*

External References

[1] L. Caporaso, J. Harris, *Parameter spaces for curves on surfaces and enumeration of rational curves*, Compositio Math. **113** (1998), no. 2, 155–208.

Lecture V
Products of deformation spaces of tacnodes

by Francesco Bastianelli

1 Introduction 91
2 Proofs 93
References 96

1 – Introduction

This lecture retraces [1, Section 4.2], and concerns Caporaso and Harris' description of products of deformation spaces of higher-order tacnodes. Of course, this description relies on the analysis included in [IV] about deformations of a single tacnode. The main result of this lecture shall be involved in the proof of [VI, (1.4)], which is [1, Theorem 1.3]. Our aim is to describe the local geometry of hyperplane sections of generalized Severi varieties $V^{d,\delta}(\alpha,\beta)$ around a point $[X_0]$, which parameterizes a reducible curve $X_0 = X \cup L$ having some tacnodes of orders m_j, each deforming to $m_j - 1$ nodes on the curves parameterized nearby.

We consider a sequence $\underline{m} := (m_1, m_2, \ldots, m_n)$ of integers $m_j \geqslant 2$, and we set

$$\lambda := \operatorname{lcm}\{m_j\}, \qquad \mu := \prod_j m_j, \qquad \kappa := \frac{\mu}{\lambda},$$

where lcm denotes the least common multiple.
For any $1 \leqslant j \leqslant n$, let (C_j, p_j) be a tacnode of order m_j, and let $\pi_j \colon \mathcal{S}_j \longrightarrow \Delta_j$ be the corresponding versal deformation family described in [IV]. Therefore, $\Delta_j \cong \mathbb{A}^{2m_j-1}$ is an affine space with coordinates $\left(\underline{\alpha}_j, \underline{\beta}_j\right) = \left(\alpha_{j,m_j-2}, \ldots, \alpha_{j,0}, \beta_{j,m_j-1}, \ldots, \beta_{j,0}\right)$, the subscheme $\mathcal{S}_j \subseteq \mathbb{A}^2 \times \Delta_j$ has equation

$$y^2 + \left(x^{m_j} + \alpha_{j,m_j-2}x^{m_j-2} + \cdots + \alpha_{j,0}\right) y + \beta_{j,m_j-1}x^{m_j-1} + \cdots + \beta_{j,0} = 0,$$

and $\pi_j \colon \mathcal{S}_j \longrightarrow \Delta_j$ is the second projection. Moreover, we introduce the subloci Δ_{j,m_j} and Δ_{j,m_j-1} of Δ_j as

$$\Delta_{j,m_j} := \overline{\left\{ \left(\underline{\alpha}_j, \underline{\beta}_j\right) \in \Delta_j \,\middle|\, \pi_j^{-1}\left(\underline{\alpha}_j, \underline{\beta}_j\right) \text{ is a curve having } m_j \text{ nodes} \right\}}$$

and

$$\Delta_{j,m_j-1} := \overline{\left\{ \left(\underline{\alpha}_j, \underline{\beta}_j\right) \in \Delta_j \,\middle|\, \pi_j^{-1}\left(\underline{\alpha}_j, \underline{\beta}_j\right) \text{ is a curve having } m_j - 1 \text{ nodes} \right\}}.$$

We recall that $\Delta_{j,m_j} \cong \mathbb{A}^{m_j-1}$ is the linear (m_j-1)-dimensional subspace given by the equations $\beta_{j,m_j-1} = \cdots = \beta_{j,0} = 0$, and it is the locus over which the fibers $\pi_j^{-1}\left(\underline{\alpha}_j, \underline{\beta}_j\right)$ are reducible plane curves, whereas Δ_{j,m_j-1} is an m_j-dimensional subvariety containing Δ_{m_j}.

Then we define

$$\Delta := \Delta_1 \times \cdots \times \Delta_n,$$

$$\Delta_{\underline{m}} := \Delta_{1,m_1} \times \cdots \times \Delta_{n,m_n},$$

and

$$\Delta_{\underline{m}-1} := \Delta_{1,m_1-1} \times \cdots \times \Delta_{n,m_n-1},$$

so that $\Delta \cong \mathbb{A}^{\Sigma(2m_j-1)}$ is an affine space endowed with coordinates $\left(\underline{\alpha}_1, \underline{\beta}_1, \ldots, \underline{\alpha}_n, \underline{\beta}_n\right)$, the linear subspace $\Delta_{\underline{m}} \cong \mathbb{A}^{\Sigma(m_j-1)}$ is given by the equations $\{\beta_{j,i} = 0,\ 1 \leqslant j \leqslant n,\ 0 \leqslant i \leqslant m_j - 1\}$, and $\Delta_{\underline{m}-1}$ is a subvariety of dimension $\sum m_j$ containing $\Delta_{\underline{m}}$. Finally, we set

$$H := \bigcup_j \{\beta_{j,0} = 0\},$$

which is a union of hyperplanes of Δ.

We now state the main result of this lecture (cf. [1, Lemma 4.3]). Along the same lines as [IV, (1.2)] (cf. [1, Lemma 4.1]), it describes the local intersection at the origin $\underline{0} \in \Delta$ between $\Delta_{\underline{m}-1}$ and varieties $W \subseteq \Delta$ containing $\Delta_{\underline{m}}$ as a subvariety of codimension 1 and satisfying a suitable transversality assumption.

(1.1) Lemma. *Let $W \subseteq \Delta$ be a smooth subvariety of dimension $\sum(m_j - 1) + 1$ containing $\Delta_{\underline{m}}$, and such that the tangent space $T_{\underline{0}}W$ at the origin $\underline{0} \in \Delta$ is not contained in H. Then, in an étale neighborhood of the origin,*

$$W \cap \Delta_{\underline{m}-1} = \Delta_{\underline{m}} \cup \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_\kappa,$$

where $\Gamma_1, \ldots, \Gamma_\kappa \subseteq W$ are distinct reduced unibranched curves such that each Γ_α has intersection multiplicity $\left(\Gamma_\alpha \cdot \Delta_{\underline{m}}\right)_{\underline{0}} = \lambda$ with $\Delta_{\underline{m}}$ at the origin, and the origin is a point of multiplicity $\operatorname{mult}_{\underline{0}}(\Gamma_\alpha) = \frac{\lambda}{\max_j\{m_j\}}$ of Γ_α.

Note that $\Delta_{\underline{m}-1}$ has codimension $\sum(m_j - 1)$ in Δ, so the expected dimension of the intersection $W \cap \Delta_{\underline{m}-1}$ is 1.

(1.2) Remark. (i) The intersection multiplicity at $\underline{0} \in \Delta$ between $\Delta_{\underline{m}}$ and the reducible curve $\Gamma := \Gamma_1 \cup \cdots \cup \Gamma_\kappa$ is $\left(\Gamma \cdot \Delta_{\underline{m}}\right)_{\underline{0}} = \kappa\lambda = \mu$, in analogy with the case of a single tacnode.

(ii) If $\lambda = m_j$ for some j, then all the curves Γ_α are smooth.

Before proving Lemma (1.1), we state another result which shall turn out to be equivalent to it. We denote by

$$\widetilde{\Delta} := \operatorname{Bl}_{\Delta_{\underline{m}}}\Delta \xrightarrow{\pi} \Delta$$

the blow-up of Δ along $\Delta_{\underline{m}}$. Let $E \subseteq \widetilde{\Delta}$ be the exceptional divisor, and let $F := \pi^{-1}(\underline{0}) \subseteq E$ be the fibre over the origin $\underline{0} \in \Delta$. Finally, let $\widetilde{\Delta}_{\underline{m}-1}$ and $\widetilde{H}$ denote the proper transforms of $\Delta_{\underline{m}-1}$ and H, respectively. Then the following holds (cf. [1, Lemma 4.4]).

(1.3) Lemma. *The intersection $\widetilde{\Delta}_{\underline{m}-1} \cap E$ contains F as a component of multiplicity μ. Moreover, in an étale neighborhood of any point $p \in F$ not contained in $\widetilde{H}$, the variety $\widetilde{\Delta}_{\underline{m}-1}$ consists of κ reduced branches, each having multiplicity $\frac{\lambda}{\max_j\{m_j\}}$ along F, intersection number λ with E along F, and tangent cone at p supported on a linear space contained in E.*

2 – Proofs

In analogy with the argument used to deduce [1, Lemma 4.1], the proof of Lemma (1.1) proceeds in three steps. The first one consists of proving the assertion when W is assumed to be a linear space. Then we use it to achieve Lemma (1.3). Finally, we conclude by deducing Lemma (1.1) for an arbitrary subvariety W satisfying the hypothesis of the statement.

(2.1) Proof of Lemma (1.1) with linear $W \subseteq \Delta$. We assume that $W \cong \mathbb{A}^{\Sigma(m_j-1)+1}$ is a linear subspace of Δ such that $\Delta_{\underline{m}} \subseteq W$ and $W = T_{\underline{0}}W \not\subseteq H$. Thus $\Delta_{\underline{m}} \subseteq H$ is a hyperplane in W, and we may choose a point $v \in \Delta \smallsetminus H$ such that $W = \langle \Delta_{\underline{m}}, v \rangle$, i.e., W is the linear span of $\Delta_{\underline{m}}$ and v.

For any $1 \leqslant j \leqslant n$, let $\rho_j \colon \Delta \longrightarrow \Delta_j$ be the jth projection, and let $t \colon W \longrightarrow \mathbb{C}$ be a non-zero linear function vanishing along the hyperplane $\Delta_{\underline{m}}$. We note that $\rho_j\left(\Delta_{\underline{m}-1}\right) = \Delta_{m_j-1}$ and $\rho_j\left(\Delta_{\underline{m}}\right) = \Delta_{m_j}$. Moreover, setting $W_j := \rho_j(W)$ and $v_j := \rho_j(v)$, we have that $v_j \in W_j \smallsetminus \Delta_{m_j}$ as $v \notin H$. Thus $W_j = \langle \Delta_{m_j}, v_j \rangle \subseteq \Delta_j$ is an m_j-dimensional plane not contained in the hyperplane $\{\beta_{j,0} = 0\}$. In particular W_j satisfies the hypothesis of [IV, (1.2)], so the intersection between Δ_{m_j-1} and W_j in Δ_j consists of the union of Δ_{m_j} and a smooth curve Γ_j having intersection multiplicity $\left(\Gamma_j \cdot \Delta_{m_j}\right)_{\underline{0}} = m_j$ with Δ_{m_j} at the origin $\underline{0} \in \Delta_j$. Therefore, in some étale neighborhood of the origin $\underline{0} \in W_j$, we may choose local coordinates $\left(x_{j,0}, \ldots, x_{j,m_j-2}, t_j\right)$ such that $\rho_j^*\left(t_j\right) = t$,[1] Δ_{m_j} is the hyperplane with equation $t_j = 0$, and the curve Γ_j is defined by the equations

$$x_{j,1} = \cdots = x_{j,m_j-2} = 0 \quad \text{and} \quad t_j = \left(x_{j,0}\right)^{m_j}.$$

Then the functions $t = \rho_j^*\left(t_j\right)$ and $y_{j,i} := \rho_j^*\left(x_{j,i}\right)$, with $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant m_j - 2$, define a system of local coordinates on some étale neighborhood of W at the origin. It follows from their very definition that the intersection $W \cap \Delta_{\underline{m}-1}$ consists of the plane $\Delta_{\underline{m}} = \{t = 0\}$ and of the subvariety Γ defined by the equations

$$t = \left(y_{j,0}\right)^{m_j} \quad \text{and} \quad y_{j,i} = 0, \qquad \text{for } 1 \leqslant j \leqslant n \text{ and } 1 \leqslant i \leqslant m_j - 2. \tag{2.1.1}$$

We point out that Γ is a curve, as for any fixed value of $t \in \mathbb{C}$, there exist finitely many $\left(\left(\sum m_j - 1\right) + 1\right)$-tuples $(y_{1,0}, \ldots, y_{n,m_n-2}, t)$ satisfying (2.1.1), which depend only on the choice of an m_j^{th} root of t for each coordinate $y_{j,0}$.

In order to parameterize any branch of Γ, let us consider a sequence $\underline{\zeta} := (\zeta_1, \zeta_2, \ldots, \zeta_n)$ such that each ζ_j is a m_j^{th} root of unity. Then we define a local parameterization $\varphi_{\underline{\zeta}} \colon \mathbb{C} \longrightarrow W$ as

$$z \longmapsto \begin{cases} t &= z^{\lambda} \\ y_{j,0} &= \dfrac{z^{\frac{\lambda}{m_j}}}{\zeta_j} \quad \text{for } 1 \leqslant j \leqslant n \\ y_{j,i} &= 0 \quad \text{for } 1 \leqslant j \leqslant n \text{ and } 1 \leqslant i \leqslant m_j - 2, \end{cases} \tag{2.1.2}$$

and we denote by $\Gamma_{\underline{\zeta}}$ its image. We note that the map $\varphi_{\underline{\zeta}}$ is injective.[2] Furthermore, every branch of Γ can be parameterized in this way: for any point $\underline{y} \in \Gamma$ with coordinates $(y_{1,0}, \ldots, y_{n,m_n-2}, t)$, it is enough to choose a λth root z of t and set $\zeta_j = z^{\frac{\lambda}{m_j}}/y_{j,0}$, which is an m_j^{th} root of unity by (2.1.1). We also point out that the multiplicity of $\Gamma_{\underline{\zeta}}$ at the origin is the least power of

[1] It suffices to choose $t_j \colon W_j \longrightarrow \mathbb{C}$ to be a linear function such that $t_j(\Delta_{m_j}) = 0$ and $t_j(v_j) = t(v)$.

[2] Suppose that $\varphi_{\underline{\zeta}}(z) = \varphi_{\underline{\zeta}}(w)$ for some $z, w \in \mathbb{C}$. Then $(z/w)^{\frac{\lambda}{m_j}} = 1$ for any j, hence z/w is a νth root of unity, where ν divides any $\frac{\lambda}{m_j}$. Thus $\nu = 1$ as λ is the least common multiple of the integers m_j, hence $z = w$.

z appearing in the parameterization, i.e., $\mathrm{mult}_{\underline{0}}(\Gamma_{\underline{\zeta}}) = \frac{\lambda}{\max_j\{m_j\}}$. On the other hand, the intersection multiplicity at the origin between $\Gamma_{\underline{\zeta}}$ and the hyperplane $\Delta_{\underline{m}} = \{t = 0\}$ is $\left(\Gamma_{\underline{\zeta}} \cdot \Delta_{\underline{m}}\right)_{\underline{0}} = \lambda$, as $t = z^{\lambda}$ on $\Gamma_{\underline{\zeta}}$. Therefore the behavior of the branches of Γ satisfies the statement, and it remains to enumerate them.

To this aim, we note that two distinct sequences $\underline{\zeta} := (\zeta_1, \zeta_2, \ldots, \zeta_n)$ and $\underline{\eta} := (\eta_1, \eta_2, \ldots, \eta_n)$ parameterize the same branch if and only if $\zeta_j = \varepsilon^{\frac{\lambda}{m_j}}\eta_j$ for all j, where ε is some λ^{th} root of unity.[3]

Moreover, $\zeta_j = \varepsilon^{\frac{\lambda}{m_j}}\zeta_j$ for any j if and only if $\varepsilon = 1$ [4]. We conclude that the number of branches of Γ equals the number of sequences $\underline{\zeta} := (\zeta_1, \zeta_2, \ldots, \zeta_n)$ divided by the number of λth roots of unity, that is $\kappa = \frac{\mu}{\lambda}$. Thus we achieved Lemma (1.1) when W is assumed to be a linear subspace of Δ. □

We now turn to prove Lemma (1.3).

(2.2) Proof of Lemma (1.3). Let G be the Grassmannian of vector subspaces of $\Delta \cong \mathbb{A}^{\Sigma(2m_j-1)}$ of dimension $\sum(m_j - 1) + 1$, and let $B \cong \mathbb{P}^{\Sigma m_j - 1}$ be the subvariety of G parameterizing those subspaces containing $\Delta_{\underline{m}} \cong \mathbb{A}^{\Sigma(m_j-1)}$. Let $W_{\underline{\beta}}$ denote the subspace parameterized by $\underline{\beta} \in B$, and consider the incidence variety $\Lambda := \left\{ (\underline{y}, \underline{\beta}) \in \Delta \times B \middle| \underline{y} \in W_{\underline{\beta}} \right\} \subseteq \Delta \times B$. Then Λ is the restriction to B of the universal family over G, and it fits in the following diagram

(2.2.1)
$$\begin{array}{ccc} \Lambda & & \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \searrow^{\mathrm{pr}_2} & \\ \Delta & \overset{\psi}{- \to} & B, \end{array}$$

where pr_1 and pr_2 are the natural projections, while ψ is the map sending any $\underline{y} \in \Delta \smallsetminus \Delta_{\underline{m}}$ to the point parameterizing the linear span $\langle \underline{y}, \Delta_{\underline{m}} \rangle \subseteq \Delta$ [5]. We point out that $\mathrm{pr}_1 \colon \Lambda \longrightarrow \Delta$ is an isomorphism outside $(\mathrm{pr}_1)^{-1}\left(\Delta_{\underline{m}}\right)$, whereas the fibre $(\mathrm{pr}_1)^{-1}(\underline{y})$ over any point $\underline{y} \in \Delta_{\underline{m}}$ is a section of pr_2, that is a projective space of dimension $\sum m_j - 1$. Therefore $\mathrm{pr}_1 \colon \Lambda \longrightarrow \Delta$ coincides with the blow-up $\widetilde{\Delta} := \mathrm{Bl}_{\Delta_{\underline{m}}}\Delta \xrightarrow{\pi} \Delta$ of Δ along $\Delta_{\underline{m}}$ (see, e.g., [2, p.604]).[6]

Consider the proper transforms $\widetilde{\Delta}_{\underline{m}-1}, \widetilde{H} \subseteq \widetilde{\Delta}$ of $\Delta_{\underline{m}-1}$ and H. Let $\widetilde{\Delta}^0_{\underline{m}-1} \subseteq \widetilde{\Delta}_{\underline{m}-1}$ and $\Delta^0_{\underline{m}-1} \subseteq \Delta_{\underline{m}-1}$ denote the open subsets of points not lying on $\widetilde{H}$ and H, respectively, and let $B_0 \subseteq B$ be the open subvariety parameterizing subspaces not contained in H. Then the diagram

---

[3] Indeed, $\varphi_{\underline{\zeta}}(z) = \varphi_{\underline{\eta}}(w)$ for some $z, w \in \mathbb{C}$ if and only if $(z/w)^{\lambda} = 1$ and $\eta_j z^{\frac{\lambda}{m_j}} = \zeta_j w^{\frac{\lambda}{m_j}}$, that is $z = \varepsilon w$ for some λ^{th} root ε of unity and $\zeta_j = \varepsilon^{\frac{\lambda}{m_j}}\eta_j$.

[4] We note that $\varepsilon^{\frac{\lambda}{m_j}} = 1$ for any j if and only if ε is a νth root of unity, where ν divides any $\frac{\lambda}{m_j}$. Hence $\nu = 1$.

[5] We note that B is naturally identified with the projectivization $\mathbb{P}\left(\Delta/\Delta_{\underline{m}}\right) \cong \mathbb{P}^{\Sigma m_j - 1}$ of the quotient $\Delta/\Delta_{\underline{m}}$. Under this identification, we may view ψ as the map sending a point $\underline{y} = \left(\underline{\alpha}_1, \underline{\beta}_1, \ldots \underline{\alpha}_n, \underline{\beta}_n\right) \in \Delta \smallsetminus \Delta_{\underline{m}}$ to the point $\underline{\beta} \in \mathbb{P}\left(\Delta/\Delta_{\underline{m}}\right)$ having homogeneous coordinates $\underline{\beta} = \left[\underline{\beta}_1, \ldots, \underline{\beta}_n\right]$.

[6] In other terms, the dominant rational map $\psi \colon \Delta \dashrightarrow \mathbb{P}\left(\Delta/\Delta_{\underline{m}}\right)$ can be resolved to a morphism by blowing up Δ along the indeterminacy locus $\Delta_{\underline{m}}$.

(2.2.1) restricts to

(2.2.2)
$$\begin{array}{ccc} \widetilde{\Delta}^0_{\underline{m}-1} & & \\ \downarrow{\scriptstyle \pi_0} & \searrow{\scriptstyle \tau_0} & \\ \Delta^0_{\underline{m}-1} & \xrightarrow{\psi_0} & B_0. \end{array}$$

We note that the closure $\overline{(\psi_0)^{-1}(\underline{b})} \subseteq \Delta_{\underline{m}-1}$ of the fibre over a point $\underline{b} \in B_0$ is the residual intersection of $\Delta_{\underline{m}-1}$ and the subspace $W_{\underline{b}}$ when we remove $\Delta_{\underline{m}}$. By Lemma (1.1) applied to $W_{\underline{b}} \cong \mathbb{A}^{\Sigma(m_j-1)+1}$, in some étale neighborhood of the origin $\underline{0} \in \Delta$, the latter intersection consists of κ distinct reduced unibranched curves $\Gamma_1, \dots, \Gamma_\kappa$, each satisfying $\big(\Gamma_\alpha \cdot \Delta_{\underline{m}}\big)_{\underline{0}} = \lambda$ and $\mathrm{mult}_{\underline{0}}(\Gamma_\alpha) = \frac{\lambda}{\max_j\{m_j\}}$. We note that the proper transform $\widetilde{W}_b = (\mathrm{pr}_2)^{-1}(\underline{b})$ of W_b meets the fibre $F := \pi^{-1}(\underline{0})$ at a single point $p \in \widetilde{\Delta} \smallsetminus \widetilde{H}$ (recall that $\pi\colon \widetilde{\Delta} \longrightarrow \Delta$ is the blow-up along $\Delta_{\underline{m}}$, which identifies with $\mathrm{pr}_1\colon \Lambda \longrightarrow \Delta$). It follows that in some étale neighborhood of p, the closure of the fibre $(\tau_0)^{-1}(\underline{b}) \subseteq \widetilde{\Delta}_{\underline{m}-1}$ consists of the proper transforms $\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_\kappa$ of the curves $\Gamma_\alpha \subseteq \Delta_{\underline{m}-1}$. Moreover, recalling that E is the exceptional divisor of the blow-up $\widetilde{\Delta} \xrightarrow{\pi} \Delta$ along $\Delta_{\underline{m}}$, any $\widetilde{\Gamma}_\alpha$ is a reduced unibranched curve such that

(2.2.3)
$$\big(\widetilde{\Gamma}_\alpha \cdot E\big)_p = \big(\Gamma_\alpha \cdot \Delta_{\underline{m}}\big)_{\underline{0}} = \lambda$$

and

(2.2.4)
$$\big(\widetilde{\Gamma}_\alpha \cdot F\big)_p = \mathrm{mult}_{\underline{0}}\big(\Gamma_\alpha\big) = \frac{\lambda}{\max_j\{m_j\}},$$

by the Projection Formula and [3, Lemma 1.40 p.28].

Finally, as we vary $\underline{\beta} \in B_0$, each $\widetilde{\Gamma}_\alpha$ describes a reduced branch $\widetilde{\Delta}^\alpha$ of $\widetilde{\Delta}_{\underline{m}-1}$ having multiplicity $\frac{\lambda}{\max_j\{m_j\}}$ along F, intersection number λ with E along F, and since the curves $\widetilde{\Gamma}_\alpha$ are unibranched, the tangent cone to $\widetilde{\Delta}^\alpha$ at p is supported on a linear space contained in E. In particular, the intersection $\widetilde{\Delta}_{\underline{m}-1} \cap E$ contains the whole fibre F, with multiplicity given by the sum of the intersection multiplicities along F between E and each branch of $\widetilde{\Delta}_{\underline{m}-1}$, that is $\kappa\lambda = \mu$. □

Finally, we conclude this lecture by proving Lemma (1.1) for an arbitrary subvariety $W \subseteq \Delta$ satisfying the hypothesis of the statement.

(2.3) Proof of Lemma (1.1). Let $W \subseteq \Delta$ be a smooth $(\sum(m_j-1)+1)$-dimensional variety containing $\Delta_{\underline{m}}$ and such that $T_{\underline{0}}W \not\subseteq H$. By the smoothness of W, its proper transform $\widetilde{W} \subseteq \widetilde{\Delta}$ intersects the exceptional divisor E transversally, and cuts out on the latter a section over $\widetilde{\Delta}_{\underline{m}}$. Moreover, since $T_{\underline{0}}W \not\subseteq H$, we have that $\widetilde{W}$ meets $F := \pi^{-1}(\underline{0})$ at a single point $p \in \widetilde{\Delta}$ which lies off $\widetilde{H}$, hence $T_p\widetilde{W} \cap T_pF = \{0\}$. By Lemma (1.3), in some étale neighborhood of p, the variety $\widetilde{\Delta}_{\underline{m}-1}$ consists of κ reduced sheets $\widetilde{\Delta}^1, \dots, \widetilde{\Delta}^\kappa$, and the tangent cone to a branch $\widetilde{\Delta}^\alpha$ at p is supported on a linear space $L \subseteq E$. In particular, $T_pF \subseteq L \subseteq E$. We note that $\dim L = \dim \widetilde{\Delta}^\alpha = \sum m_j$, and $\dim T_pF = \dim F = \sum m_j - 1 = \dim \widetilde{\Delta}^\alpha - 1$. Thus $\dim\big(T_p\widetilde{W} \cap L\big) \leqslant 1$, so that the linear span of $T_p\widetilde{W}$ and L satisfies

$$\begin{aligned} \dim\langle T_p\widetilde{W}, L\rangle &\geqslant \dim T_p\widetilde{W} + \dim L - 1 = \sum(m_j-1) + 1 + \sum m_j - 1 \\ &= \sum(2m_j - 1) = \dim\widetilde{\Delta}, \end{aligned}$$

that is $\langle T_p\widetilde{W}, L\rangle = T_p\widetilde{\Delta}$. Therefore, the intersection around p between $\widetilde{W}$ and $\widetilde{\Delta}^\alpha$ is transverse, and it consists of a unibranched reduced curve $\widetilde{\Gamma}_\alpha$ such that $\big(\widetilde{\Gamma}_\alpha \cdot F\big)_p$ equals the multiplicity $\frac{\lambda}{\max_j\{m_j\}}$ of $\widetilde{\Delta}^\alpha$ along F, and $\big(\widetilde{\Gamma}_\alpha \cdot E\big)_p$ is the intersection number λ between $\widetilde{\Delta}^\alpha$ and E along F.

For any $1 \leqslant \alpha \leqslant \kappa$, we consider the reduced curve $\Gamma_\alpha := \pi\big(\widetilde{\Gamma}_\alpha\big)$, and in some étale neighborhood of the origin, we have

$$\Delta_{\underline{m}-1} \cap W = \Delta_{\underline{m}} \cup \Gamma_1 \cup \cdots \cup \Gamma_\kappa.$$

Since each $\widetilde{\Gamma}_\alpha$ meets E only at the point p, the curve Γ_α is unibranched as well. Furthermore, by arguing as in (2.2.3) and (2.2.4), we deduce that each Γ_α satisfies $\mathrm{mult}_{\underline{0}}(\Gamma_\alpha) = \big(\widetilde{\Gamma}_\alpha \cdot F\big)_p = \frac{\lambda}{\max_j\{m_j\}}$ and $\big(\Gamma_\alpha \cdot \Delta_{\underline{m}}\big)_{\underline{0}} = \big(\widetilde{\Gamma}_\alpha \cdot E\big)_p = \lambda$. Thus Lemma (1.1) follows. □

References

Chapters in this Volume

[IV] M. Lelli-Chiesa, *Deformations of an m-tacnode into $m-1$ nodes*.

[VI] Th. Dedieu and C. Galati, *Degenerations and deformations of generalized Severi varieties*.

External References

[1] L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392.

[2] P. A. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics, Wiley Interscience, New York, 1978.

[3] J. Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies 166, Princeton University Press, Princeton, NJ, 2007.

Appendix A
Geometry of the deformation space of an m-tacnode into $m-1$ nodes

by Thomas Dedieu

1 Introduction 97
2 General results 98
3 Complete description in the case $m = 2$ 99
4 The case $m = 3$ and beyond 102
4.1 The case $m = 3$ 102
4.2 General conclusions 104
References 104

1 – Introduction

This appendix complements the previous lecture [IV], of which we shall keep the notation: let $\pi : S \to \Delta$ be the versal deformation space of the m-tacnode defined by the equation $y(y + x^m) = 0$ in the affine plane; one has $\Delta \cong \mathbf{A}^{2m-1}_{\alpha_0,\dots,\alpha_{m-2},\beta_0,\dots,\beta_{m-1}}$, and S is defined in $\mathbf{A}^2_{x,y} \times \Delta$ by the equation

$$y^2 + y(x^m + \alpha_{m-2}x^{m-2} + \cdots + \alpha_0) + \beta_{m-1}x^{m-1} + \cdots + \beta_0 = 0.$$

Inside Δ we consider the two subvarieties Δ_{m-1} and Δ_m, the closures of the two loci of those $t \in \Delta$ such that the curves $\pi^{-1}(t)$ have $m-1$ nodes and m nodes, respectively.

The goal of [IV] was to establish some properties of Δ_{m-1} around the origin in Δ, of fundamental importance in order to carry out the program of [1], see [VI]. Here we shall provide complementary results, and work out detailed examples for $m = 2$ and 3. The first motivation is that this will hopefully give more insight into the main result of [IV], as well as help to follow the arguments given there. On the other hand, the variety Δ_{m-1} for $m = 2$ will show up under another guise in this volume, see [XII], namely as the local model for the dual variety $X^\vee \subseteq \check{\mathbf{P}}^3$ of a surface $X \subseteq \mathbf{P}^3$, around a point corresponding to a tacnodal hyperplane section; this makes Δ_{m-1} one of the main characters of this volume. We are therefore interested in more properties of Δ_{m-1} than those strictly necessary for the Caporaso–Harris Program.

Finally, I would like to point out before I start that the singularity of Δ_{m-1} at the origin is known for $m = 2$ as a swallowtail singularity, *queue d'aronde* in french, one of seven elementary situations in René Thom's *théorie des catastrophes*.

2 – General results

Let $\tau : \tilde{\Delta} \to \Delta$ be the blow-up along Δ_m; recall that $\Delta_m \subseteq \Delta$ is the affine $(m-1)$-plane with equations

$$\beta_0 = \cdots = \beta_{m-1} = 0,$$

and it is contained in Δ_{m-1}. Let $Z \subseteq \tilde{\Delta}$ be the exceptional divisor, a $\mathbf{P}^{m-1}$-bundle over Δ_m, and let Φ be the fibre of Z over the origin. Let $\tilde{\Delta}_{m-1}$ be the proper transform of Δ_{m-1}. Let H be the hyperplane $\beta_0 = 0$ in Δ, and let $\tilde{H}$ be its proper transform in $\tilde{\Delta}$.

The main result of [IV] may be reformulated as follows, cf. [1, Lemma 4.2], where the equivalence of the two statements is proved (all the ingredients for this proof are present in [IV]).

(2.1) Proposition. *Let Z_{m-1} be the intersection $\tilde{\Delta}_{m-1} \cap Z$ (this is the exceptional locus of $\tau : \tilde{\Delta}_{m-1} \to \Delta_{m-1}$). Then Z_{m-1} contains Φ as a component of multiplicity m. Moreover, $\tilde{\Delta}_{m-1}$ is smooth at all points of Φ not in $\tilde{H}$.*

We shall now prove the result below, following [2, §2.4.4]. It is interesting in itself, but also as a pretext to give a local description of Δ_{m-1} at an arbitrary point of Δ_m.

(2.2) Proposition. *The fibres of Z_{m-1} over Δ_m are unions of linear spaces.*

Before we prove this, we introduce a natural stratification of Δ_m. Each stratum is the locus in which the m-tacnode deforms into k non-infinitely near tacnodes of orders $m_1, \ldots, m_k$ respectively, with $m_1 + \cdots + m_k = m$.

The space Δ_m identifies with the space of monic degree m polynomials in x with no x^{m-1}-term, and we shall consider in this space the loci of polynomials having roots with given multiplicities. For all partition $m = m_1 + \cdots + m_k$, we let

$$\Delta\{m_1, \ldots, m_k\} \subseteq \Delta_m \subseteq \Delta$$

be the locus corresponding to monic degree m polynomials with no x^{m-1}-term of the form $(x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}$, with $\lambda_1, \ldots, \lambda_k$ pairwise distinct. Note that $\Delta\{m_1, \ldots, m_k\}$ has dimension $k-1$ (because we impose that there is no x^{m-1}-term, equivalently the λ_i's sum up to zero), so its codimension in Δ_m is $m-k = \sum_{i=1}^{k}(m_i - 1)$.

For $\alpha \in \Delta\{m_1, \ldots, m_k\}$, the curve $S_\alpha = \pi^{-1}(\alpha) \subseteq \mathbf{A}^2_{x,y}$ is a reducible curve with two branches, defined by the two equations $y = 0$ and $y = (x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}$, respectively; these two branches intersect at the k points $r_1 = (\lambda_1, 0), \ldots, r_k = (\lambda_k, 0)$ with multiplicities $m_1, \ldots, m_k$, respectively.

For all $i = 1, \ldots, k$ we consider the versal deformation space $\Delta(S_\alpha, r_i)$ of the singular point $r_i \in S_\alpha$; note that this is an m_i-tacnode. There is then a natural map

$$\sigma : U \to \prod_{i=1}^{k} \Delta(S_\alpha, r_i),$$

where U is an open neighbourhood U of α in Δ, and by the openness of versality it has surjective differential at α; the fibre of σ over the origin parametrizes the equisingular deformations of S_α, for which only the location of the points r_i varies along the x-axis.

For all $i = 1, \ldots, k$ we consider the subvarieties $\Delta_{m_i-1}(r_i)$ and $\Delta_{m_i}(r_i)$ of $\Delta(S_\alpha, r_i)$, defined as the closures of the loci where r_i deforms to $m_i - 1$ nodes and m_i nodes, respectively. Then, in the neighbourhood U of α in Δ, we have

$$\Delta_m = \sigma^{-1}\big(\Delta_{m_1}(r_1) \times \cdots \Delta_{m_k}(r_k)\big)$$

and

$$\Delta_{m-1} = \bigcup_{l=1}^{k} \sigma^{-1}\big(\Delta_{m_1}(r_1) \times \cdots \times \Delta_{m_l-1}(r_l) \times \cdots \Delta_{m_k}(r_k)\big).$$

This shows that Δ_{m-1} has k local sheets in a neighbourhood of α, each containing Δ_m. The l-th sheet looks like the product of $\Delta_{m_l-1} \subseteq \Delta_{m_l}$ with some smooth factor, and along this sheet the fibres of $\pi : S \to \Delta$ have m_i nodes tending to r_i for $i \neq l$, and $m_l - 1$ nodes tending to r_l.

We are now ready to prove the statement.

Proof of Proposition (2.2). In the above setup, applying Proposition (2.1) to the spaces $\Delta(S_\alpha, r_i)$, we will be able to describe the fibre of Z_{m-1} over $\alpha \in \Delta_m$. Note that Φ, the fibre of Z over the origin, identifies with the projectivization of the space of degree $m-1$ polynomials in x (with coefficients $\beta_0, \ldots, \beta_{m-1}$), equivalently with the projectivization of the space of polynomials modulo those vanishing to order m at the origin (i.e., $\mathbf{C}[x]/(x^m)$).

Then, for all $l = 1, \ldots, k$, by Proposition (2.1) applied to $\Delta_{m_l-1}(r_l)$, the proper transform of the l-th sheet of Δ_{m-1} intersects the fibre Φ_α of Z over α in the linear subspace of Φ_α corresponding to polynomials in x vanishing to order m_h at r_h for all $h \neq l$; in other words, if we identify Φ_α with the projectivization of

$$\mathbf{C}[x]/\big((x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}\big) \;\cong\; \prod_{h=1}^{k} \mathbf{C}[x]/\big((x-\lambda_h)^{m_h}\big),$$

then the proper transform of the l-th sheet of Δ_{m-1} intersects Φ_α in its linear subspace which is the projectivization of $\mathbf{C}[x]/\big((x-\lambda_l)^{m_l}\big)$ (with multiplicity m_l).

In turn, the intersection of $\tilde{\Delta}_{m-1}$ with the fibre Φ_α of Z over α is the union of these linear spaces for l ranging from 1 to k. □

(2.3) A complement. Consider a small arc $t \in \mathbf{D} \mapsto \alpha(t) \in \Delta_m$ tending to the origin as t tends to 0, where $\mathbf{D}$ stands for the unit disc in $\mathbf{C}$. If this arc is small enough, then all points $\alpha(t)$ with $t \neq 0$ lie in the same stratum $\Delta\{m_1, \ldots, m_k\}$. Moreover, as t goes to $0 \in \mathbf{D}$ the singular points $r_i(t)$ of the curve $S_{\alpha(t)} \subseteq \mathbf{A}^2_{x,y}$ tend to the origin. It follows that the limiting position in Φ of the intersection with $\Phi_{\alpha(t)}$ of the l-th sheet of Δ_{m-1} along $\Delta\{m_1, \ldots, m_k\}$ is the space of polynomials vanishing to order $m - m_l$ at 0 (in the identification of Φ with polynomials in x modulo those vanishing to order m at the origin).

This implies, as in [IV, 2], that Φ is an irreducible component of Z_{m-1}, the general point of which corresponds to the tangent direction of an arc in Δ_{m-1} which intersects Δ_m only in the origin.

3 – Complete description in the case $m = 2$

When $m = 2$, we are considering the versal deformation space of the ordinary tacnode, defined in $\mathbf{A}^2_{x,y}$ by the equation $y(y + x^2) = 0$; this is $S \to \Delta \cong \mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$, with S defined in $\mathbf{A}^2_{x,y} \times \Delta$ by the equation

$$y^2 + y(x^2 + \alpha_0) + \beta_1 x + \beta_0 = 0.$$

We are interested in the surface $\Delta_1 \subseteq \Delta$ of deformations of cogenus at least one of the tacnode, especially along the curve $\Delta_2 \subseteq \Delta$ of deformations of cogenus two, which is the line $\beta_1 = \beta_0 = 0$, along wich the tacnode deforms to two nodes. The picture looks as follows, as we shall see.

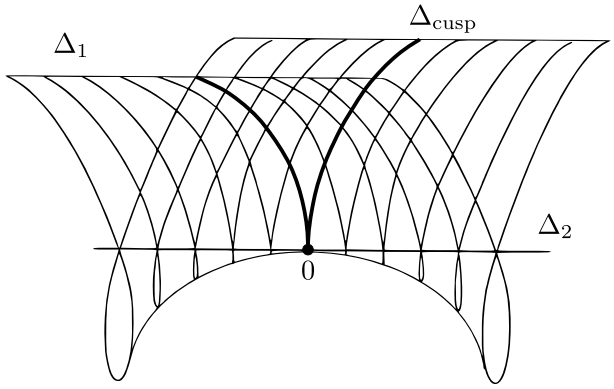


Figure 1: Geometry of Δ_1 along Δ_2 (beware the singularity along Δ_{cusp})
(reproduction from [2, p.181])

The surface Δ_1 has two local sheets along the line Δ_2; they intersect transversely along the complement of the origin in Δ_1, but they have the same tangent plane at the origin. Also, it is important to keep in mind that the surface Δ_1 is singular along the curve Δ_{cusp}, which is the locus along which the tacnode deforms to an ordinary cusp: the surface Δ_1 has a cuspidal singularity at the generic point of Δ_{cusp}, although this not pictured on the above figure.

The proper transform of Δ_1 under the blow-up of Δ along the line Δ_2 looks as follows.

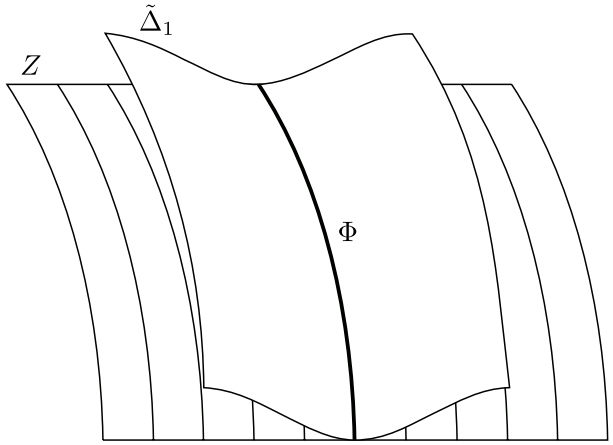


Figure 2: Intersection of $\tilde{\Delta}_1$ with the exceptional divisor
(reproduction from [1, p.379])

(3.1) Equation of Δ_1. The surface Δ_1 is defined in $\mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$ by the condition that the polynomial

$$\delta_{\alpha_0,\beta_0,\beta_1}(x) = (x^2+\alpha_0)^2 - 4(\beta_1 x + \beta_0) = x^4 + 2\alpha_0 x^2 - 4\beta_1 x + \alpha_0^2 - 4\beta_0$$

has at least one non-simple root (see [IV, Section 1]), equivalently its discriminant vanishes. Using the formula for the discriminant of monic degree 4 polynomials with no x^3-term, namely

$$\text{Disc}(x^4 + ax^2 + bx + c) = -4a^3b^2 + 16a^4c - 27b^4 + 144ab^2c - 128a^2c^2 + 256c^3,$$

we find the following equation for Δ_1 in $\mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$,

$$256 \cdot (16\alpha_0^3\beta_1^2 + 16\alpha_0^2\beta_0^2 - 72\alpha_0\beta_0\beta_1^2 - 27\beta_1^4 - 64\beta_0^3) = 0. \tag{3.1.1}$$

Note that it is quasi-homogeneous of degree 12 for the weights $2, 4, 3$ of $\alpha_0, \beta_0, \beta_1$ respectively.

(3.2) Equations of the curve Δ_{cusp}. We consider the locus Δ_{cusp} of those $t \in \Delta$ such that the curve S_t has (at least) a cuspidal singularity. The locus Δ_{cusp} is defined by the condition that the polynomial $\delta_{\alpha_0,\beta_0,\beta_1}$ has at least one triple root, in other words that there exists $u \in \mathbf{A}^1$ such that

$$\begin{aligned}\delta_{\alpha_0,\beta_0,\beta_1}(x) &= (x-u)^3 \cdot (x+3u) \\ &= x^4 - 6u^2x^2 + 8u^3x - 3u^4\end{aligned}$$

(recall that the roots of $\delta_{\alpha_0,\beta_0,\beta_1}$ sum up to zero as it has no x^3-term). Therefore, Δ_{cusp} is the projection on the second factor of the complete intersection in $\mathbf{A}^1_u \times \mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$ defined by the equations

$$\begin{cases} 2\alpha_0 = -6u^2 \\ -4\beta_1 = 8u^3 \\ \alpha_0^2 - 4\beta_0 = -3u^4. \end{cases}$$

Eliminating u from these equations (note that this can be done algorithmically), we find that Δ_{cusp} is defined in $\mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$ by the following equations,

$$\begin{cases} \alpha_0^2 - 3\beta_0 = 0 \\ 4\alpha_0^3 + 27\beta_1^2 = 0. \end{cases}$$

(3.3) Singularities of the surface Δ_1. To compute the equations of the singularities of Δ_1, we consider the Jacobian ideal $(\partial_{\alpha_0}F, \partial_{\beta_0}F, \partial_{\beta_1}F)$, where F is the equation of Δ_1 given in (3.1.1). It readily defines the singular subscheme of Δ_1, since F is quasi-homogeneous hence liable to the generalized Euler formula, cf. [II, Section 1]. Using a software for symbolic computation with polynomials, e.g., Macaulay2 [3] and its command `primaryDecomposition`, we compute that the ideal defining the singular subscheme of Δ_1 is the intersection of the two following primary ideals,

$$(\beta_0, \beta_1), \quad \text{and} \quad (9\alpha_0\beta_0^2 - 4\alpha_0^2\beta_1 + 24\beta_1^2,\ 8\alpha_0^3 - 27\beta_0^2 - 36\alpha_0\beta_1,\ 27\beta_0^4 + 16\alpha_0^2\beta_1^2 - 64\beta_1^3).$$

The former is the defining ideal of Δ_2, while the latter is not prime, with radical the defining ideal of the curve Δ_{cusp}.

In addition, it is directly visible from the equation of Δ_1, viz. (3.1.1), that Δ_1 has a triple point at the origin, with tangent cone the triple plane at the origin $\beta_0^3 = 0$.

(3.4) Blow-up along the line Δ_2. The blow-up of $\mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$ along the line Δ_2 is the hypersurface in $\mathbf{P}^1_{(u_0:u_1)} \times \mathbf{A}^3_{\alpha_0,\beta_0,\beta_1}$ defined by the equation $u_0\beta_1 - u_1\beta_0 = 0$. Pulling back the equation of Δ_1, (3.1.1), we find

$$\beta_1^2 \cdot \left(16\alpha_0^2u_0^2 - 64\beta_1u_0^3 + 16\alpha_0^3 - 72\alpha_0\beta_1u_0 - 27\beta_1^2\right) = 0$$

in the affine chart $u_1 \neq 0$, and

$$\beta_0^2 \cdot \left(16\alpha_0^2 - 64\beta_0 + 16\alpha_0^3u_1^2 - 72\alpha_0\beta_0u_1^2 - 27\beta_0^2u_1^4\right) = 0$$

in the affine chart $u_0 \neq 0$; the factors between parentheses are thus the defining equations of the proper transform $\widetilde{\Delta}_1$ in $\mathbf{A}^1_{u_0} \times \mathbf{A}^3_{\alpha_0,\beta_1}$ and $\mathbf{A}^1_{u_1} \times \mathbf{A}^3_{\alpha_0,\beta_0}$, respectively. From these equations

we see that $\tilde{\Delta}_1$ is smooth in the latter affine chart, and has a double point at the origin of the former, with tangent cone there given by $\beta_1^2 = 0$.

To obtain the intersection with the exceptional divisor Z, we plug $\beta_1 = 0$ and $\beta_0 = 0$, respectively, in these two equations; we obtain

$$16\alpha_0^2(u_0^2 + \alpha_0) = 0 \quad \text{and} \quad 16\alpha_0^2(1 + \alpha_0 u_1^2) = 0$$

for the two equations of $Z_1 = \tilde{\Delta}_1 \cap Z$ in the two charts $\mathbf{A}^1_{u_0} \times \mathbf{A}^3_{\alpha_0,\beta_1}$ and $\mathbf{A}^1_{u_1} \times \mathbf{A}^3_{\alpha_0,\beta_0}$, respectively. The exceptional divisor Z_1 thus has two irreducible components, the fibre Φ_0, and the hypersurface $u_0^2 + \alpha_0 u_1^2 = 0$; the latter maps $2:1$ onto the line Δ_2, and corresponds to the normal directions of the two sheets of Δ_1 along Δ_2.

4 – The case $m = 3$ and beyond

From this point on it is no longer advisable to do the computations by hand, and I have used Macaulay2 [3]. In fact, even with a computer it is not realistic to go much further. Still we can make some interesting observations, that are summed up in Subsection 4.2.

4.1 – The case $m = 3$

When $m = 3$, we are considering the versal deformation space of the 3-tacnode, defined in $\mathbf{A}^2_{x,y}$ by the equation $y(y + x^3) = 0$; this is $S \to \Delta \cong \mathbf{A}^5_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}$, with S defined in $\mathbf{A}^2_{x,y} \times \Delta$ by the equation

$$y^2 + y(x^3 + \alpha_1 x + \alpha_0) + \beta_2 x^2 + \beta_1 x + \beta_0 = 0.$$

We are interested in the solid $\Delta_2 \subseteq \Delta$ of deformations of cogenus at least two of the 3-tacnode, especially along the locus $\Delta_3 \subseteq \Delta$ of deformations of cogenus three, which is the plane $\beta_2 = \beta_1 = \beta_0 = 0$ along wich the 3-tacnode deforms to three (possibly infinitely near) nodes.

The singularities of Δ_2 are more complicated than those of Δ_1 in the case $m = 2$, and we will content ourselves with the description of the exceptional divisor of $\tilde{\Delta}_2$.

(4.1) Equations of Δ_2. We shall proceed by elimination, as in (3.2). The solid Δ_2 is defined by the condition that the polynomial

$$\begin{aligned}(4.1.1)\quad \delta_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}(x) &= (x^3 + \alpha_1 x + \alpha_0)^2 - 4(\beta_2 x^2 + \beta_1 x + \beta_0) \\ &= x^6 + 2\alpha_1 x^4 + 2\alpha_0 x^3 + (\alpha_1^2 - 4\beta_2)x^2 + (2\alpha_0\alpha_1 - 4\beta_1)x + (\alpha_0^2 - 4\beta_0)\end{aligned}$$

has at least two non-simple roots, equivalently that there exist $(u, v, w) \in \mathbf{A}^3$ such that

$$(4.1.2)\qquad \delta_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}(x) = (x-u)^2(x-v)^2(x-w)(x+2u+2v+w).$$

It is thus the projection to the second factor of the complete intersection in $\mathbf{A}^3_{u,v,w} \times \mathbf{A}^5_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}$ defined by the four non-trivial equations obtained by identifying the x-terms in (4.1.1) and (4.1.2). Eliminating u, v, w from these equations, for instance by using the function `eliminate` from Macaulay2 [3], we determine that the ideal of Δ_2 in Δ is the following:

$$\begin{aligned}\big(&48\alpha_0\alpha_1^2\beta_2^2 + 3\alpha_1^2\beta_0\beta_1 + 45\alpha_0\alpha_1\beta_1^2 - 72\alpha_0\alpha_1\beta_0\beta_2 + 27\alpha_0^2\beta_1\beta_2 \\ &\quad - 128\alpha_1\beta_1\beta_2^2 + 64\alpha_0\beta_2^3 + 27\alpha_0\beta_0^2 - 125\beta_1^3 + 180\beta_0\beta_1\beta_2, \\ &9\alpha_1^3\beta_0\beta_1 - 9\alpha_0\alpha_1^2\beta_1^2 + 63\alpha_0\alpha_1^2\beta_0\beta_2 - 54\alpha_0^2\alpha_1\beta_1\beta_2 - 81\alpha_0^3\beta_2^2 + 16\alpha_1^2\beta_1\beta_2^2 - 128\alpha_0\alpha_1\beta_2^3 - 54\alpha_0\alpha_1\beta_0^2 \\ &\quad - 81\alpha_0^2\beta_0\beta_1 + 25\alpha_1\beta_1^3 - 252\alpha_1\beta_0\beta_1\beta_2 + 135\alpha_0\beta_1^2\beta_2 + 324\alpha_0\beta_0\beta_2^2 + 192\beta_1\beta_2^3 + 405\beta_0^2\beta_1,\end{aligned}$$

$$16\alpha_1^4\beta_2^2 + 16\alpha_1^3\beta_1^2 - 16\alpha_1^3\beta_0\beta_2 - 216\alpha_0^2\alpha_1\beta_2^2 - 128\alpha_1^2\beta_2^3 - 81\alpha_0\alpha_1\beta_0\beta_1 - 135\alpha_0^2\beta_1^2 + 135\alpha_0^2\beta_0\beta_2$$
$$- 120\alpha_1\beta_1^2\beta_2 + 576\alpha_1\beta_0\beta_2^2 + 288\alpha_0\beta_1\beta_2^2 + 256\beta_2^4 + 675\beta_0\beta_1^2 - 432\beta_0^2\beta_2,$$
$$16\alpha_1^4\beta_0\beta_2 - 16\alpha_0\alpha_1^3\beta_1\beta_2 - 16\alpha_1^3\beta_0^2 - 81\alpha_0\alpha_1^2\beta_0\beta_1 + 81\alpha_0^2\alpha_1\beta_1^2 - 216\alpha_0^2\alpha_1\beta_0\beta_2 + 135\alpha_0^3\beta_1\beta_2$$
$$+ 16\alpha_1^2\beta_1^2\beta_2 - 128\alpha_1^2\beta_0\beta_2^2 + 192\alpha_0\alpha_1\beta_1\beta_2^2 + 135\alpha_0^2\beta_0^2 + 240\alpha_1\beta_0\beta_1^2 - 225\alpha_0\beta_1^3 + 576\alpha_1\beta_0^2\beta_2$$
$$- 252\alpha_0\beta_0\beta_1\beta_2 - 320\beta_1^2\beta_2^2 + 256\beta_0\beta_2^3 - 432\beta_0^3,$$
$$81\alpha_0\alpha_1^3\beta_0\beta_2 - 81\alpha_0^2\alpha_1^2\beta_1\beta_2 - 1215\alpha_0^3\alpha_1\beta_2^2 + 240\alpha_1^3\beta_1\beta_2^2 - 81\alpha_0\alpha_1^2\beta_0^2 - 486\alpha_0^2\alpha_1\beta_0\beta_1 - 729\alpha_0^3\beta_1^2$$
$$+ 240\alpha_1^2\beta_1^3 + 729\alpha_0^3\beta_0\beta_2 - 420\alpha_1^2\beta_0\beta_1\beta_2 + 261\alpha_0\alpha_1\beta_1^2\beta_2 + 1980\alpha_0\alpha_1\beta_0\beta_2^2 + 2052\alpha_0^2\beta_1\beta_2^2$$
$$- 2240\alpha_1\beta_1\beta_2^3 + 2560\alpha_0\beta_2^4 + 243\alpha_1\beta_0^2\beta_1 + 3645\alpha_0\beta_0\beta_1^2 - 1836\alpha_0\beta_0^2\beta_2 - 2300\beta_1^3\beta_2 + 3312\beta_0\beta_1\beta_2^2\big).$$

It has five generators, quasi-homogeneous of degrees $15, 17, 16, 18, 19$, respectively, for the weights $3, 2$ for α_0, α_1 and $6, 5, 4$ for $\beta_0, \beta_1, \beta_2$.

(4.2) Tangent cone at the origin. Using the ideal of Δ_2 in Δ, we compute (e.g., with the function `tangentCone` of Macaulay2) that its tangent cone at the origin is defined by the primary ideal

$$\big(25\beta_0\beta_1^2 - 16\beta_0^2\beta_2, \beta_0^2\beta_1, \beta_0^3, 27\alpha_0\beta_0^2 - 125\beta_1^3 + 180\beta_0\beta_1\beta_2\big).$$

It is thus a degree 6 projective scheme, supported over the 3-plane $\beta_1 = \beta_2 = 0$.

(4.3) Blow-up along the plane Δ_3. We consider the blow-up of Δ along Δ_3 as the subvariety of $\mathbf{P}^2_{(u_0:u_1:u_2)} \times \mathbf{A}^5_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2}$ defined by the equations $u_j\beta_i - u_i\beta_j = 0$ for all distinct i and j in $\{0, 1, 2\}$.

Let us consider the affine chart $u_2 = 1$ to fix ideas; it may be seen in $\mathbf{A}^2_{u_0,u_1} \times \mathbf{A}^3_{\alpha_0,\alpha_1,\beta_2}$ by eliminating β_0 and β_1, since $\beta_0 = u_0\beta_2$ and $\beta_1 = u_1\beta_2$ in this chart. The pull-back of the ideal of Δ_2 in Δ is the intersection of the two primary ideals (β_2^2) and

$$\big(3\alpha_1^2u_0u_1 + 45\alpha_0\alpha_1u_1^2 - 125\beta_2u_1^3 + 48\alpha_0\alpha_1^2 - 72\alpha_0\alpha_1u_0 + 27\alpha_0u_0^2 + 27\alpha_0^2u_1 - 128\alpha_1\beta_2u_1 + 180\beta_2u_0u_1 + 64\alpha_0\beta_2,$$
$$16\alpha_1^3u_1^2 + 16\alpha_1^4 - 16\alpha_1^3u_0 - 81\alpha_0\alpha_1u_0u_1 - 135\alpha_0^2u_1^2 - 120\alpha_1\beta_2u_1^2 + 675\beta_2u_0u_1^2 - 216\alpha_0^2\alpha_1 - 128\alpha_1^2\beta_2$$
$$+ 135\alpha_0^2u_0 + 576\alpha_1\beta_2u_0 - 432\beta_2u_0^2 + 288\alpha_0\beta_2u_1 + 256\beta_2^2,$$
$$144\alpha_0\alpha_1^2u_1^2 - 400\alpha_1\beta_2u_1^3 + 144\alpha_0\alpha_1^3 - 279\alpha_0\alpha_1^2u_0 + 135\alpha_0\alpha_1u_0^2 + 135\alpha_0^2\alpha_1u_1 - 400\alpha_1^2\beta_2u_1 + 81\alpha_0^2u_0u_1$$
$$+ 792\alpha_1\beta_2u_0u_1 - 405\beta_2u_0^2u_1 - 135\alpha_0\beta_2u_1^2 + 81\alpha_0^3 + 320\alpha_0\alpha_1\beta_2 - 324\alpha_0\beta_2u_0 - 192\beta_2^2u_1,$$
$$\alpha_1^4u_0 - \alpha_1^3u_0^2 - \alpha_0\alpha_1^3u_1 + 81\alpha_0^2\alpha_1u_1^2 + \alpha_1^2\beta_2u_1^2 + 15\alpha_1\beta_2u_0u_1^2 - 225\alpha_0\beta_2u_1^3 + 81\alpha_0^2\alpha_1^2 - 135\alpha_0^2\alpha_1u_0$$
$$- 8\alpha_1^2\beta_2u_0 + 54\alpha_0^2u_0^2 + 36\alpha_1\beta_2u_0^2 - 27\beta_2u_0^3 + 54\alpha_0^3u_1 - 204\alpha_0\alpha_1\beta_2u_1$$
$$+ 288\alpha_0\beta_2u_0u_1 - 20\beta_2^2u_1^2 + 108\alpha_0^2\beta_2 + 16\beta_2^2u_0\big).$$

The latter has four generators, and defines the proper transform $\tilde{\Delta}_2$ of Δ_2 in this affine chart. To obtain the intersection Z_2 of $\tilde{\Delta}_2$ with the exceptional divisor, we plug in the equation $\beta_2 = 0$. We obtain that the ideal of Z_2 in $\mathbf{A}^2_{u_0,u_1} \times \mathbf{A}^3_{\alpha_0,\alpha_1,\beta_2}$ is the intersection of the three primary ideals

$$\big(\alpha_0\alpha_1, \alpha_0^2, \alpha_1^2u_1 + 9\alpha_0u_0, \alpha_1^3\big),$$
$$\big(u_1^2 + \alpha_1 - u_0, u_0u_1 + \alpha_0, \alpha_1u_0 - u_0^2 - \alpha_0u_1\big), \quad \text{and}$$
$$\big(\alpha_1u_0u_1^2 - \alpha_0u_1^3 + \alpha_1^2u_0 - 2\alpha_1u_0^2 + u_0^3 - \alpha_0\alpha_1u_1 + 3\alpha_0u_0u_1 + \alpha_0^2,$$
$$16\alpha_1^2u_1^2 + 16\alpha_1^3 - 31\alpha_1^2u_0 + 15\alpha_1u_0^2 + 15\alpha_0\alpha_1u_1 + 9\alpha_0u_0u_1 + 9\alpha_0^2,$$
$$\alpha_1^2u_0u_1 + 15\alpha_0\alpha_1u_1^2 + 16\alpha_0\alpha_1^2 - 24\alpha_0\alpha_1u_0 + 9\alpha_0u_0^2 + 9\alpha_0^2u_1,$$
$$\alpha_1^3u_0 - \alpha_1^2u_0^2 - \alpha_0\alpha_1^2u_1 - 6\alpha_0\alpha_1u_0u_1 - 9\alpha_0^2u_1^2 - 15\alpha_0^2\alpha_1 + 9\alpha_0^2u_0,$$
$$16\alpha_1^4 - 15\alpha_1^2u_0^2 - 96\alpha_0\alpha_1^2u_1 - 90\alpha_0\alpha_1u_0u_1 - 135\alpha_0^2u_1^2 - 360\alpha_0^2\alpha_1 + 216\alpha_0^2u_0\big).$$

The first one has radical (α_0, α_1), and defines the fibre Φ_0 of Z over the origin with multiplicity 3. The second one is radical, and defines a locus that dominates Δ_3 (this can be seen by

eliminating the variables u_0 and u_1, which gives the zero ideal); moreover, we observe that there are three points in this locus over the generic point of Δ_3. As for the third one, if we eliminate u_0 and u_1, we find the ideal

$$\big((4\alpha_1^3 + 27\alpha_0^2)^2\big),$$

which is generated by the square of the discriminant of the polynomial $x^3 + \alpha_1 x + \alpha_0$; over the generic point of the locus defined by this discriminant in Δ_3, we find that the third ideal defines a line with multiplicity 2.

Computations in the two other affine charts are similar, although they take a little more time to the computer. In conclusion we find that the exceptional divisor Z_2 of $\tilde{\Delta}_2 \to \Delta_2$ has three irreducible components: (i) the fibre Φ_0 of Z over the origin; (ii) a surface dominating Δ_3 and corresponding to the normal directions of the three sheets of Δ_2 along Δ_3; (iii) a surface fibered in lines over the cuspidal curve $\Delta\{2,1\} \subseteq \Delta_3$ (in the notation of Section 2).

4.2 – General conclusions

(4.4) Some computations in the case $m = 4$. For $m = 4$ it is still possible to compute the defining ideal of the fourfold Δ_3 in $\mathbf{A}^7_{\alpha_0,\ldots,\alpha_2,\beta_0,\ldots,\beta_3}$, although it took about half an hour on my personal computer.

This ideal has 20 generators, all quasihomogenous if we assign the weights $4, 3, 2$ to $\alpha_2, \alpha_1, \alpha_0$ and $8, \ldots, 5$ to $\beta_3, \ldots, \beta_0$ respectively. For what it's worth, the degrees of the generators are $18, 19, 20, 20, 21, 21, 21, 22, 22, 22, 23, 23, 24, 24, 28, 28, 29, 30, 30, 31$.

The tangent cone to Δ_2 at the origin is supported on the 4-plane $\beta_0 = \beta_1 = \beta_2 = 0$, and has degree 10.

I have not been able to compute a primary decomposition of the intersection of $\tilde{\Delta}_3$ with the exceptional divisor Z of the blow-up of Δ along Δ_4.

(4.5) General conjectures. From the above computations one may conjecture that the defining ideal of Δ_{m-1} in Δ will always be quasihomogeneous for the weights $m, \ldots, 2$ for $\alpha_0, \ldots, \alpha_{m-2}$ and $2m, \ldots, m+1$ for $\beta_0, \ldots, \beta_{m-1}$, respectively.

The origin will be a point of multiplicity $\sum_{k=1}^{m} k$, with tangent cone there supported on the m-plane $\beta_0 = \cdots = \beta_{m-2} = 0$.

The exceptional divisor of $\tilde{\Delta}_{m-1} \to \Delta_{m-1}$ will have $m-1$ irreducible components, respectively dominating the following loci in Δ_m : $\Delta\{1, \ldots, 1\} = \Delta_m$, $\Delta\{2, 1, \ldots, 1\}$, $\ldots$, $\Delta\{m-1, 1\}$.

The latter two facts are probably not very hard to prove using the setup of Section 2. It is plausible that the first conjecture may be approached using reduced discriminant theory, see [C].

References

Chapters in this Volume

[II] E. Sernesi, *Deformations of curves on surfaces.*

[IV] M. Lelli-Chiesa, *Deformations of an m-tacnode into $m-1$ nodes.*

[VI] Th. Dedieu and C. Galati, *Degenerations and deformations of generalized severi varieties.*

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces.*

[C] L. Busé and Th. Dedieu, *Generalized weight properties of resultants and discriminants, and applications to projective enumerative geometry.*

External References

[1] L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392.

[2] ———, *Parameter spaces for curves on surfaces and enumeration of rational curves*, Compositio Math. **113** (1998), no. 2, 155–208.

[3] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

Lecture VI
Degenerations and deformations of generalized Severi varieties

by Thomas Dedieu and Concettina Galati

1 The counting problem and its solution 108
2 Recap on Severi varieties 110
3 Degenerations of generalized Severi varieties 111
3.1 General picture of the degeneration 113
3.2 Simplified proof of Proposition (3.15) 115
3.3 Complete proof of Proposition (3.15) 117
3.4 Local picture of the degeneration of curves 119
4 Deformations of generalized Severi varieties 122
4.1 Fixing an unassigned contact point 123
4.2 Merging a line 125
5 The formula for irreducible curves 135
References 139

This text is dedicated to the recursive formula obtained by Caporaso and Harris [1] as a solution to the problem of "counting plane curves of any genus", see Corollary (1.5) and Theorem (5.4). Fix a degree $d \in \mathbf{N}^*$, and denote by $p_a(d) = \frac{1}{2}(d-1)(d-2)$ the genus of smooth plane curves of degree d. For all $g = 0, \ldots, p_a(d)$, (reduced) plane curves of degree d and geometric genus g form a family of pure dimension $\nu_g^d = 3d + g - 1$. The problem is to find the number of plane curves of degree d and geometric genus g passing through ν_g^d general points in the plane. It turns out that this number equals the number of δ-nodal plane curves of degree d which pass through ν_g^d general points in the plane, with $\delta = p_a(d) - g$ (a curve is δ-nodal if its singularities are exactly δ nodes, i.e., δ ordinary double points).

Caporaso and Harris obtain their inductive formula by specializing one after another the aforementioned ν_g^d imposed crossing points to general points on a fixed line L. This may be conveniently formulated in terms of a degeneration of the projective plane $\mathbf{P}^2$ to the transverse union along a line of $\mathbf{P}^2$ and a minimal rational ruled surface $\mathbf{F}_1$, using the formalism presented in [I]. We will do it on an example in [VII], but here we stick to the presentation given in [1]. A similar result for curves on minimal rational ruled surfaces has been obtained by Vakil [5], following a similar degeneration procedure, and around the same period of the time. We shall nevertheless concentrate on the case of plane curves.

The above described degeneration procedure makes it necessary to generalize the problem. Thus, one is led to count curves of given degree and genus with, in addition, a prescribed intersection pattern with the line L, both at assigned and unassigned points. We shall call *generalized Severi varieties* the families formed by these curves: they are a particular case of the logarithmic Severi varieties studied in [III].

We give precise definitions and statements in Section 1 below. There, we shall also provide more details about the general strategy of the proof, which will explain the title of this text.

1 – The counting problem and its solution

Let us fix once and for all a line $L \subseteq \mathbf{P}^2$. We shall use freely the notation and definitions introduced in [III, Section 1], in the special case when the pair (S, R) is $(\mathbf{P}^2, L)$; every class $\xi \in \mathrm{NS}(\mathbf{P}^2)$ is d times the linear equivalence class of a line for some integer d, and therefore it shall be denoted simply by d.

Let $d \in \mathbf{N}$ and $g \in \mathbf{Z}$, let $\alpha = (\alpha_i)_{i\geqslant 1}$ and $\beta = (\beta_i)_{i\geqslant 1}$ be two sequences in $\underline{\mathbf{N}}$ such that $I\alpha + I\beta = d$, and let $\Omega = \big(\{p_{i,j}\}_{1\leqslant j\leqslant \alpha_i}\big)_{i\geqslant 1}$ be a set of α points on the line L. The *logarithmic Severi variety* $V_g^d(\alpha,\beta)(\Omega)$ is the locally closed subset of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ (the projective space of dimension $\frac{1}{2}d(d+3)$ parametrizing all plane curves of degree d) parametrizing those reduced plane curves of degree d and geometric genus g that intersect L at the points of Ω with the multiplicities prescribed by α and at further unassigned points with the multiplicities prescribed by β; for the rigorous definition, see [III, Definition (1.5)]. The points of Ω shall be referred to as the *fixed*, or *assigned*, contact points, and the other intersection points with L as *mobile*, or *unassigned*, contact points (as it turns out, in the present situation, the unassigned contact points indeed move as the curves moves in the Severi variety, see Proposition (2.2) below). The relevant notation for these points and their relatives is explained in Paragraph (2.1). Forgetting about the location of assigned intersection points at Ω, we may also loosely say that the curves in $V_g^d(\alpha,\beta)(\Omega)$ have intersection pattern with L prescribed by α and β.

As it turns out (see Proposition (2.2) below), the general element of every irreducible component of $V_g^d(\alpha,\beta)(\Omega)$ has δ nodes and no further singularities, with

$$\delta = p_a(d) - g = \frac{1}{2}(d-1)(d-2) - g.$$

The upshot is that it is indifferent to require either that the geometric genus be g or that the singularities be exactly δ nodes.

(1.1) Definition. *For all natural integers d and δ, and all $\alpha, \beta \in \underline{\mathbf{N}}$ and Ω as above, the* generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ *is the Zariski closure in the projective space $|\mathcal{O}_{\mathbf{P}^2}(d)|$ of the logarithmic Severi variety $V^d_{p_a(d)-\delta}(\alpha,\beta)(\Omega)$ of the pair $(\mathbf{P}^2, L)$.*

In the particular case when $\alpha = 0$, and thus $\Omega = \varnothing$, and $\beta = (d, 0, \ldots)$ (i.e., when no conditions are imposed on the intersection with L), the Severi varieties $V_g^d(\alpha,\beta)(\Omega)$ and $V^{d,\delta}(\alpha,\beta)(\Omega)$ will simply be denoted by V_g^d and $V^{d,\delta}$, respectively, and called *plain* Severi varieties.

Although this should not create any problem, beware that in our notation V_g^d and $V^{d,\delta}$ (with any additional decorations) are respectively locally closed and closed in $|\mathcal{O}_{\mathbf{P}^2}(d)|$.

For any point $p \in \mathbf{P}^2$, the degree d plane curves passing through the point p form a hyperplane $p^\perp$ in the projective space $|\mathcal{O}_{\mathbf{P}^2}(d)|$. Therefore, the problem of counting curves of degree d and genus g (and no further conditions) mentioned in the introduction amounts to the problem of computing the degree of the plain Severi variety $V^{d,\delta} = \overline{V}{}_g^d$, with $\delta = p_a(d) - g$. We shall denote this number by $N^{d,\delta}$. More generally, for all $\alpha, \beta \in \underline{\mathbf{N}}$, we shall denote by $N^{d,\delta}(\alpha,\beta)$ the common degree of all $V_g^d(\alpha,\beta)(\Omega)$, where Ω is a general, cardinality α, set of points of L.

The recursion procedure of Caporaso and Harris consists in computing the number $N^{d,\delta}$ by cutting $V^{d,\delta}$ by hyperplanes $p^\perp$ with $p \in L$. Their main result is Theorem (1.4) below. We need some additional notation to state it.

(1.2) Notation. For all $k \in \mathbf{N}^*$, we let $e_k = (0, \ldots, 0, 1) \in \underline{\mathbf{N}}$: it is the sequence with only one non-zero entry, in k-th position and equal to 1 (recall the convention made in [III] that we may

omit the infinite sequence of zeros at the end of elements of $\underline{\mathbf{N}}$). For all $\beta, \beta' \in \underline{\mathbf{N}}$, we let

$$I^\beta = \prod_{k\geqslant 1} k^{\beta_k} \qquad \text{and} \qquad \binom{\beta'}{\beta} = \prod_{k\geqslant 1} \binom{\beta'_k}{\beta_k}.$$

If $\Omega = (\Omega_1, \Omega_2, \ldots)$ is a set of cardinality $\alpha \in \underline{\mathbf{N}}$, a subset of Ω of cardinality $\alpha' \leqslant \alpha$ is simply a set $\Omega' = (\Omega'_1, \Omega'_2, \ldots)$ of cardinality $\alpha' \in \underline{\mathbf{N}}$ such that $\Omega'_k \subseteq \Omega_k$ for all $k \geqslant 1$.

The following notation is not needed to state the main theorem below, but we shall use it for the proof and in the more precise Theorem (4.1). For all $\beta \in \underline{\mathbf{N}}$, we let

$$\operatorname{lcm}(\beta) = \operatorname{lcm}\{i : \beta_i \neq 0\} \qquad \text{and} \qquad \max(\beta) = \max\{i : \beta_i \neq 0\}.$$

(1.3) Convention. In the statement below, generalized Severi varieties $V^{d-1,\delta'}(\alpha', \beta')(\Omega') \subseteq |\mathcal{O}_{\mathbf{P}^2}(d-1)|$ are considered as the projective subvarieties of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ parametrizing all curves of the form $C + L$ with $C \in V^{d-1,\delta'}(\alpha', \beta')(\Omega')$. We will make this identification freely all along the text, where needed.

(1.4) Theorem (Caporaso–Harris)**.** *Let d and δ be natural integers, and let $\alpha, \beta \in \underline{\mathbf{N}}$ be sequences such that $I\alpha + I\beta = d$, and Ω be a set of α general points on the line L. Let p be a general point of L. Then one has the following equality of cycles:*

$$V^{d,\delta}(\alpha,\beta)(\Omega) \cap p^\perp = \sum_{k\geqslant 1:\, \beta_k>0} k \cdot V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p\}) + \sum_{\substack{\Omega'\subseteq\Omega,\, \operatorname{Card}(\Omega')=\alpha' \\ \beta'\geqslant\beta:\, I\alpha'+I\beta'=d-1 \\ \delta'=\delta+|\beta'-\beta|-d+1}} I^{\beta'-\beta}\binom{\beta'}{\beta} \cdot V^{d-1,\delta'}(\alpha',\beta')(\Omega') \tag{1.4.1}$$

where, in the first sum, $\Omega \cup \{p\}$ means that we add the point p to the k-th term of $\Omega = (\Omega_1, \Omega_2, \ldots)$.

Note that, in the above theorem, if we start with a plain Severi variety for $V^{d,\delta}(\alpha, \beta)(\Omega)$ (i.e., if $\alpha = 0$ and thus $\Omega = \varnothing$, and $\beta = (d)$), then the intersection with $p^\perp$ decomposes as a sum of genuinely generalized (i.e., non-plain) Severi varieties. This is why one needs to generalize the problem of counting curves of fixed degree and genus for the inductive solution proposed by Caporaso and Harris.

It follows from Proposition (2.2) below that if p is outside of Ω, then no irreducible component of $V^{d,\delta}(\alpha, \beta)(\Omega)$ is contained in the hyperplane $p^\perp$. Therefore, all irreducible varieties appearing on the right-hand-side of (1.4.1) above have codimension one in $V^{d,\delta}(\alpha, \beta)(\Omega)$, and indeed we shall verify this by hand later in the text. The number $N^{d,\delta}(\alpha, \beta)$ we are looking for is the degree of $V^{d,\delta}(\alpha, \beta)(\Omega)$ as well as that of the cycle on the right-hand-side of (1.4.1). Thus, Theorem (1.4) has the following direct corollary.

(1.5) Corollary. *Let d and δ be natural integers, and let $\alpha, \beta \in \underline{\mathbf{N}}$ be sequences such that $I\alpha + I\beta = d$. Then one has the following equality:*

$$N^{d,\delta}(\alpha,\beta) = \sum_{k\geqslant 1:\, \beta_k>0} k \cdot N^{d,\delta}(\alpha + e_k, \beta - e_k) + \sum_{\substack{\alpha'\leqslant\alpha \\ \beta'\geqslant\beta:\, I\alpha'+I\beta'=d-1 \\ \delta'=\delta+|\beta'-\beta|-d+1}} I^{\beta'-\beta}\binom{\beta'}{\beta}\binom{\alpha'}{\alpha} \cdot N^{d-1,\delta'}(\alpha',\beta').$$

Since the numbers $N^{1,\delta}(\alpha, \beta)$ are readily computed, the above recursive formula enables one to compute all numbers $N^{d,\delta}(\alpha, \beta)$, and thus to completely solve our counting problem. Note

that the recursion indeed terminates, as those terms appearing in the first sum are all of the form $N^{d,\delta}(\alpha'', \beta'')$ with $\beta'' < \beta$, so that unassigned intersection with L are exhausted after finitely many applications of the formula, and then only terms of the form $N^{d-1,\delta'}(\alpha', \beta')$ appear. It is of course helpful to watch the inductive process work on a concrete example. For this we refer to [VII], or [1, p. 349]; see also Example (5.1).

Although, arguably, the relevant enumerative problem is to count irreducible plane curves of any genus, there is no irreducibility requirement in our definition of Severi varieties, and therefore the numbers $N^{d,\delta}(\alpha, \beta)$ count curves regardless of their being irreducible or not. It is yet possible to derive a recursion formula counting irreducible curves using the inductive process of Caporaso and Harris (and so do they in the original article). We postpone this until Section 5, as the formula is yet more complicated than the above one, and having the geometry of the inductive process in mind helps in understanding it.

The proof of Theorem (1.4) is in two parts. The first one consists in using stable reduction in order to limit the possible irreducible components of the intersection of $V^{d,\delta}(\alpha, \beta)(\Omega)$ and $p^{\perp}$. This is the degeneration part; it is carried out in Section 3 and the result is Theorem (3.1). The dimensional characterization of Severi varieties (Proposition (2.3)) is fundamental in identifying the components of the intersection as generalized Severi varieties.

The second part consists in showing that all the generalized Severi varieties found in the first part indeed occur as irreducible components of the intersection of $V^{d,\delta}(\alpha, \beta)(\Omega)$ and $p^{\perp}$, and to compute their multiplicites in this intersection. This is the deformation part; it is carried out in Section 4 and the result is Theorem (4.1). A fundamental result for this part is the study of the deformation space of an m-tacnode into $m-1$ nodes, also due to Caporaso and Harris, and described in [IV]. The latter result is certainly the central result of this whole volume of notes.

Theorems (3.1) and (4.1) together directly imply Theorem (1.4).

2 – Recap on Severi varieties

(2.1) We shall use consistent notation to denote assigned and unassigned contact points with the line L. Consider a generalized Severi variety $V^{d,\delta}(\alpha, \beta)(\Omega)$ as in Definition (1.1). The assigned contact points with L are the elements of Ω, which we write as $\Omega = \big(\{p_{i,j}\}_{1 \leqslant j \leqslant \alpha_i}\big)_{i \geqslant 1}$. Let $[C]$ be a general member of any irreducible component of $V^{d,\delta}(\alpha, \beta)(\Omega)$. Denote by $\bar{C} \to C$ the normalization of C, and by $\phi : \bar{C} \to \mathbf{P}^2$ its composition with the inclusion $C \subseteq \mathbf{P}^2$. By definition of the Severi varieties, there exist α points $q_{i,j} \in \bar{C}$, for all $i \geqslant 1$ and $1 \leqslant j \leqslant \alpha_i$, and β points $r_{i,j} \in \bar{C}$, for all $i \geqslant 1$ and $1 \leqslant j \leqslant \beta_i$, such that

$$\forall\, i \geqslant 1,\ \forall\, j = 1, \ldots, \alpha_i : \quad \phi(q_{i,j}) = p_{i,j}$$

and

$$\phi^* L = \sum_{i \geqslant 1} \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{i \geqslant 1} \sum_{1 \leqslant j \leqslant \beta_i} i\, r_{i,j}.$$

Moreover, we let $s_{i,j} = \phi(r_{i,j})$ for all $i \geqslant 1$ and $1 \leqslant j \leqslant \beta_i$.

Thus, the assigned and unassigned contact points of C with L are, respectively, the $p_{i,j}$'s and the $s_{i,j}$'s, and their respective counterparts on the normalization $\bar{C}$ are the $q_{i,j}$'s and the $r_{i,j}$'s.[1]

The two following results are key ingredients in the proof of both Theorems (3.1) and (4.1). They are direct applications of Theorem (1.7) and Proposition (4.3) in [III], respectively.

[1] The reader may wish to keep in mind that $\phi(q_{i,j}) = p_{i,j}$ whereas $\phi(r_{i,j}) = s_{i,j}$; thus, beware, ϕ goes backwards with the alphabetical order for the assigned contact points, but forward for the unassigned ones.

(2.2) Proposition. *Let the notation be as in (2.1) above. The generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ has pure dimension $2d+g-1+|\beta|$, with $g=p_a(d)-\delta$. Morever:*
— *the curve $C\subseteq\mathbf{P}^2$ is δ-nodal, and smooth at its intersection points with the line L;*
— *the contact points of C with L, i.e., all the points $p_{i,j}$ and $s_{i,j}$, are altogether pairwise distinct;*
— *the counterparts on the normalization $\bar{C}$ of the contact points of C with L, i.e., all the points $q_{i,j}$ and $r_{i,j}$, are altogether pairwise distinct;*
— *for any curve G and finite set Γ in $\mathbf{P}^2$, if $(G\cup\Gamma)\cap\Omega=\varnothing$ and $[C]$ is general with respect to G and Γ, then C intersects G transversely and does not intersect Γ.*

(2.3) Proposition. *Let $V\subseteq|\mathcal{O}_{\mathbf{P}^2}(d)|$ be an irreducible variety, parametrizing genus g curves in the following sense: for a general member $[X]$ of V, there exists a smooth curve $\tilde{X}$ of genus g, and a morphism $f:\tilde{X}\to X$, not constant on any component of $\tilde{X}$, and such that the push-forward in the sense of cycles $f_*\tilde{X}$ equals the fundamental cycle of X.*

Let $\Omega\subseteq L$ be a finite set of points. For any general member $[X]$ of V, one has

$$\dim(V)\leqslant 2d+g-1+\mathrm{Card}\big((X\cap L)\setminus\Omega\big),$$

where the last number is defined set-theoretically (i.e., multiplicities do not count).

Moreover, if equality holds, then there exist $\alpha,\beta\in\underline{\mathbf{N}}$ such that V is a dense subset of a component of the generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$, $\delta=p_a(d)-g$, if and only if

$$\mathrm{Card}\big(f^{-1}(X\cap L)\big)=\mathrm{Card}(X\cap L),$$

with $f:\tilde{X}\to X$ a genus g morphism as above.

Lastly, let us mention that, although we pretend not to know it in this text, all generalized Severi varieties $V^{d,\delta}(\alpha,\beta)(\Omega)$ have their irreducible components in bijection with the possible splittings of degree d, δ-nodal curves, into several irreducible components. Thus, for instance, there is at most one irreducible component of $V^{d,\delta}(\alpha,\beta)(\Omega)$ parametrizing irreducible curves. This is proved in [3] for plain Severi varieties, and in [6] for generalized Severi varieties.

3 – Degenerations of generalized Severi varieties

This section is dedicated to the analysis of how curves in a generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ degenerate when one imposes the passing through a general point of the line L. The main result of this section is Theorem (3.1) below, but we will also provide in Section 3.4 a geometric decription of the degeneration at play, of fundamental importance for the proof of the theorem in the other direction, i.e., Theorem (4.1) on deformations of generalized Severi varieties, which is the object of the next section.

In the following statement, and as in Convention (1.3), for all $\delta',\alpha',\beta',\Omega'$, the generalized Severi variety $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$ is identified with a variety parametrizing degree d curves containing the line L, namely

$$\big\{[C\cup L]:\ [C]\in V^{d-1,\delta'}(\alpha',\beta')(\Omega')\big\}\ \subseteq\ |\mathcal{O}_{\mathbf{P}^2}(d)|.$$

(3.1) Theorem ([1, Theorem 1.2])**.** *Let $p\in L$ be a general point, and let V be an irreducible component of the intersection $V^{d,\delta}(\alpha,\beta)(\Omega)\cap p^{\perp}$. Then V is an irreducible component of one of the following varieties:*
a) for all k such that $\beta_k>0$, the variety

$$V^{d,\delta}(\alpha+e_k,\beta-e_k)(\Omega\cup\{p_{k,\alpha_k+1}\})$$

with $p_{k,\alpha_k+1}=p$;

b) for all $\alpha' \leqslant \alpha$, $\beta' \geqslant \beta$, *and* $\delta' \leqslant \delta$ *such that (i)* $I\alpha'+I\beta' = d-1$, *and (ii)* $\delta-\delta'+|\beta'-\beta| = d-1$, *the variety*

$$V^{d-1,\delta'}(\alpha',\beta')(\Omega'),$$

where Ω' *is an arbitrary cardinality* α' *subset of* Ω.

(3.2) Remark. The condition $\delta-\delta'+|\beta'-\beta| = d-1$ in case b), is equivalent to the condition $\dim(V^{d-1,\delta'}(\alpha',\beta')(\Omega')) = \dim(V^{d,\delta}(\alpha,\beta)(\Omega)) - 1$. Indeed,

$$\begin{aligned}\dim(V^{d,\delta}(\alpha,\beta)) - \dim(V^{d-1,\delta'}(\alpha',\beta')) &= (2d+g-1+|\beta|) - (2(d-1)+g'-1+|\beta'|)\\ &= g-g'+|\beta-\beta'|-2,\end{aligned}$$

where we let g and g' be the genera of the members of $V^{d,\delta}(\alpha,\beta)$ and $V^{d-1,\delta'}(\alpha',\beta')$ respectively, and thus

$$g-g' = \binom{d-1}{2} - \delta - \binom{d-2}{2} + \delta' = d-2-\delta+\delta'$$

by the Pascal Formula.

Later on we will be able to explain geometrically where the $\delta-\delta'$ nodes degenerate, see Subsection 3.4.

(3.3) Remark. Note moreover that, still in case b), the conditions $\alpha' \leqslant \alpha$, $\beta' \geqslant \beta$, and $I\alpha'+I\beta' = d-1$ altogether imply that $\alpha' < \alpha$, i.e., there exists an i_0 such that $\alpha'_{i_0} < \alpha_{i_0}$.

It is however possible that $\beta' = \beta$, but in that case all the irreducible components of $V^{d,\delta}(\alpha,\beta)$ having V in their intersection with $p^\perp$ parametrize reducible curves, see Remark (4.21).

Proof of Theorem (3.1). Let $[X_0]$ be a general point of V. There are two cases to be considered. The first case is when the curve X_0 does not contain L. Then, in order for $[X_0]$ to sit in $p^\perp$, we must have one of the mobile contact points (which we may loosely refer to as points of type β) be at the prescribed point p (note that by generality, p is off Ω). Thus one mobile contact point must be turned into an assigned contact point (i.e., a point of type β is turned into one of type α), hence V must be contained in a variety

$$V^{d,\delta}(\alpha+e_k,\beta-e_k)(\Omega\cup\{p_{k,\alpha_k+1}=p\}).$$

Since the latter varieties all have pure dimension $\dim(V^{d,\delta}(\alpha,\beta)(\Omega))-1 = \dim(V)$, this implies that V is an irreducible component of a generalized Severi variety as in case a) of the theorem.

The remaining case is when X_0 has L as an irreducible component; then we shall loosely say that $X_0 = L\cup C$. In this case the content of Theorem (3.1) is Proposition (3.15) below. □

The proof of Proposition (3.15) occupies most of this section. The general idea is to apply (a suitable version of) semistable reduction to a family of general members of $V^{d,\delta}(\alpha,\beta)(\Omega)$ degenerating to X_0, in order to derive some necessary conditions on X_0 which will limit the number of dimensions in which it can move. Comparing this with the fact that X_0 has to move in dimension $\dim(V^{d,\delta}(\alpha,\beta)(\Omega))-1$, and taking advantage of the dimensional characterization of generalized Severi varieties (Corollary (2.3)), we will then be able to conclude.

(3.4) Setup. Let V be an irreducible component of $V^{d,\delta}(\alpha,\beta)\cap p^\perp$, the general member of which has L as an irreducible component. Let X_0 be a general member of V, and let C be the sum of the components of X_0 supported on curves different from L; we shall loosely write $X_0 = L\cup C$.

(3.5) Semistable reduction. Let Γ be a general analytic curve in $V^{d,\delta}(\alpha,\beta)$ passing through the point $[X_0]$, and let $\mathcal{X}\to\Gamma$ be the corresponding family of curves. By the generality of Γ, its general member $[X_\gamma]$ is general in an irreducible component of $V^{d,\delta}(\alpha,\beta)$, hence it corresponds to a reduced curve X_γ with δ nodes as its only singularities.

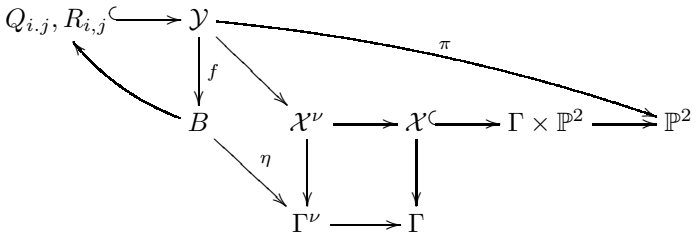
Let $\nu:\Gamma^\nu\to\Gamma$ be the normalization of Γ, and let $b_0\in\Gamma^\nu$ be a point with $\nu(b_0)=[X_0]$. We let $\mathcal{X}^\nu$ be the normalization of the total space $\mathcal{X}\times_\Gamma\Gamma^\nu$, obtained from $\mathcal{X}$ by the base change $\Gamma^\nu\to\Gamma$. The general fibre of $\mathcal{X}^\nu\to\Gamma^\nu$ is smooth of genus $g=p_a(d)-\delta$, by the Teissier Simultaneous Resolution Theorem, see [III].

Applying a suitable version of stable reduction to the family $\mathcal{X}^\nu\to\Gamma^\nu$, we arrive at a family of curves $f:\mathcal{Y}\to B$, with special fibre $Y_0=f^{-1}(b_0)$, satisfying the following conditions:

- $\mathcal{Y}\to B$ is a nodal reduction of $\mathcal{X}^\nu\to\Gamma^\nu$, i.e., all fibres of f are reduced, nodal curves, and B is a branched cover of Γ^ν;
- the total space $\mathcal{Y}$ is smooth;
- there are $|\alpha|$ sections $Q_{i,j}$ $(1\leqslant j\leqslant\alpha_i)$ and $|\beta|$ sections $R_{i,j}$ $(1\leqslant j\leqslant\beta_i)$ of $\mathcal{Y}\to B$, altogether pairwise disjoint, such that $\pi(Q_{i,j})=p_{i,j}$ (the points of Ω), and for all $b\neq b_0$ one has
$$(\pi|_{Y_b})^*L=\sum i\,Q_{i,j}|_{Y_b}+\sum i\,R_{i,j}|_{Y_b}\,;$$
- the map sending the general fibre Y_b of $\mathcal{Y}$ to the corresponding plane curve X_γ extends to a regular map $\pi:\mathcal{Y}\to\mathbb{P}^2$ (here $\gamma=\nu(\eta(b))$, see diagram (3.5.1) below);
- the family $\mathcal{Y}\to B$ is minimal with respect to the above properties.

Note that, since $\mathcal{Y}$ is smooth and the $Q_{i,j}$'s and $R_{i,j}$'s are sections, they do not pass through any singular point of the central fibre Y_0.

The family $\mathcal{Y}\to B$ is obtained from $\mathcal{X}^\nu\to\Gamma^\nu$ by applying a finite number of base changes and blow-ups. A number of additional such operations may be required with respect to the usual nodal reduction in order to first organize the points $q_{i,j}$ and $p_{i,j}$ on the fibres into sections (a priori they may be permuted by the monodromy), and then separate these sections. The situation is summarized in the diagram below.

(3.5.1)


3.1 – General picture of the degeneration

(3.6) Components of the central fibre of $\mathcal{Y}$. The irreducible components of the central fibre Y_0 of $\mathcal{Y}$ come in three types. Namely, we let:

- $\tilde{L}$ be the union of the components of Y_0 on which π is nonconstant and maps to L;
- $\tilde{C}$ be the union of the components of Y_0 on which π is nonconstant and maps to C;
- Z be the union of the components of Y_0 on which π is constant.

(3.7) Degenerations of assigned and mobile contact points. We shall now relabel the sections $(Q_{i,j})_{1\leqslant j\leqslant\alpha_i}$ and $(R_{i,j})_{1\leqslant j\leqslant\beta_i}$ according to the type of the irreducible component of the central fibre they pass through. We advise the reader to read this paragraph with Figure 1 at hand.

For all i, we let α_i^C and α_i^L be the number of the sections $Q_{i,j}$, $1 \leqslant j \leqslant \alpha_i$, passing through $\tilde{C}$ and $\tilde{L}$, respectively. We package all these numbers as sequences $\alpha^C = (\alpha_i^C)_{i\geqslant 1} \in \underline{\mathbf{N}}$ and $\alpha^L = (\alpha_i^L)_{i\geqslant 1} \in \underline{\mathbf{N}}$, and label the corresponding sections as $(Q_{i,j}^C)_{1\leqslant j\leqslant \alpha_i^C}$ and $(Q_{i,j}^L)_{1\leqslant j\leqslant \alpha_i^L}$, their respective intersection points with the central fibre as $(q_{i,j}^C)_{1\leqslant j\leqslant \alpha_i^C}$ and $(q_{i,j}^L)_{1\leqslant j\leqslant \alpha_i^L}$, and the images by π in L of the latter as $(p_{i,j}^C)_{1\leqslant j\leqslant \alpha_i^C}$ and $(p_{i,j}^L)_{1\leqslant j\leqslant \alpha_i^L}$. (Note that, by definition, also the $q_{i,j}^C \in \tilde{C}$ are mapped to L by π).

Similarly, for all i we let β_i^C be the number of the sections $R_{i,j}$, $1 \leqslant j \leqslant \beta_i$, passing through $\tilde{C}$. We also let $\beta^C = (\beta_i^C)_{i\geqslant 1} \in \underline{\mathbf{N}}$, label the corresponding sections $(R_{i,j}^C)_{1\leqslant j\leqslant \beta_i^C}$, and name $(r_{i,j}^C)_{1\leqslant j\leqslant \beta_i^C}$ their respective points on the central fibre Y_0, and $(s_{i,j}^C)_{1\leqslant j\leqslant \beta_i^C}$ their images on L by π.

The aware reader will have noticed that we have not considered all possibilities, as there are neither α^Z, nor β^Z or β^L. The reason for this is that these turn out not to exist, as we will see in Claims (3.9) and (3.10) below.

(3.8) New mobile contact points. We now identify some new points on $\tilde{C}$ which, as we shall see, induce new mobile contact points with L as C moves in the family V. These are the intersection points of $\tilde{C}$ with the vertical part of π^*L, i.e., the part of the divisor $\pi^*L \subseteq \mathcal{Y}$ with support contained in the central fibre Y_0, which is

$$(\pi^*L)^{\mathrm{vert}} = \pi^*L - \sum iQ_{i,j} - \sum iR_{i,j}.$$

Thus, for all i, we let $\beta_i^{C\cap L}$ be the number of points in $\tilde{C}\cap(\pi^*L)^{\mathrm{vert}}$ appearing with multiplicity i in the divisor $(\pi^*L)^{\mathrm{vert}}$, and label these points as $(r_{i,j}^{C\cap L})_{1\leqslant j\leqslant \beta_i^{C\cap L}}$. We also let $(s_{i,j}^{C\cap L})_{1\leqslant j\leqslant \beta_i^{C\cap L}}$ be the images of these points by π, and $\beta^{C\cap L} = (\beta_i^{C\cap L})_{i\geqslant 1} \in \underline{\mathbf{N}}$.

The reason for the notation is that it will turn out that the support of $(\pi^*L)^{\mathrm{vert}}$ is the union of $\tilde{L}$ and those connected components of Z which intersect $\tilde{L}$ (this follows from Claim (3.9) below and the argument given in (3.19)). Therefore, on the surface $\mathcal{Y}^\flat$ obtained from $\mathcal{Y}$ by contracting Z, we can see the points $r_{i,j}^{C\cap L}$ as the intersection points of (the images in $\mathcal{Y}^\flat$ of) $\tilde{C}$ and $\tilde{L}$.

The situation is summarized in Figure 1 below. In fact, we want this figure to picture the situation as it occurs in reality, and therefore we shall make a number of claims before we draw the picture. The proof of these claims occupies a substantial part of the remainder of this section.

The final interpretation in terms of degenerations of Severi varieties of the data introduced above is given in Proposition (3.15) below.

(3.9) Claim. The sections $(Q_{i,j})_{1\leqslant j\leqslant \alpha_i}$ and $(R_{i,j})_{1\leqslant j\leqslant \beta_i}$ are disjoint from Z.

(3.10) Claim. All sections $(R_{i,j})_{1\leqslant j\leqslant \beta_i}$ pass through $\tilde{C}$.

Note that Claims (3.9) and (3.10) imply that

$$\{Q_{i,j}\}_{1\leqslant i\leqslant \alpha_i} = \{Q_{i,j}^C\}_{1\leqslant i\leqslant \alpha_i'} \cup \{Q_{i,j}^L\}_{1\leqslant i\leqslant \alpha_i''} \quad \text{and} \quad \{R_{i,j}\}_{1\leqslant i\leqslant \alpha_i} = \{R_{i,j}^C\}_{1\leqslant i\leqslant \alpha_i'}.$$

(3.11) Claim. The curve $\tilde{L}$ consists of a unique irreducible component, on which π is an isomorphism.

This claim implies that the line L appears with multiplicity 1 in X_0, so $X_0 = C \cup L$ with C a degree $d-1$ curve.

(3.12) Claim. The curve Z consists of a disjoint union of chains of rational curves joining $\tilde{L}$ to $\tilde{C}$.

This Claim implies that the surface $\mathcal{Y}^\flat$ obtained from the smooth surface $\mathcal{Y}$ by contracting Z has only rational double points as singularities.

(3.13) Claim. The map π is a birational isomorphism on each irreducible component of $\tilde{C}$.

This Claim implies that the curve C is reduced.

(3.14) Claim. The curve $\tilde{C}$ is smooth (though not connected, in general).

We may now draw a picture of the situation.

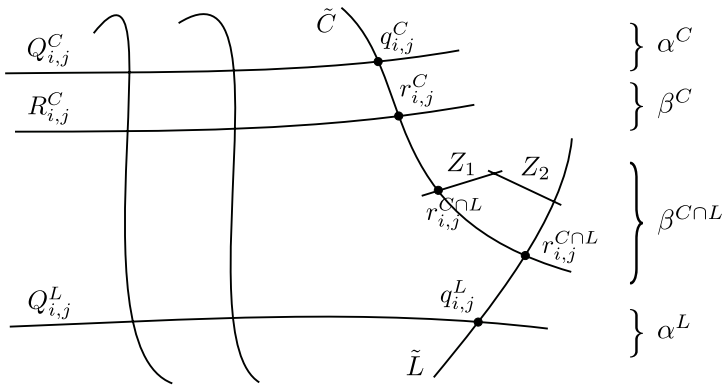


Figure 1: General picture of the degeneration

We shall prove the following result, which encapsulates Theorem (3.1) in the case of an irreducible component having general member of the form $X_0 = C \cup L$.

(3.15) Proposition. *In the situation of (3.4), the curve C is a general member of the generalized Severi variety $V^{d-1,\delta'}(\alpha', \beta')(\Omega')$, where:*
— $\alpha' = \alpha^C$, *and* $\Omega' = (p^C_{i,j})_{1\leqslant j\leqslant \alpha^C_i}$;
— $\beta' = \beta^C + \beta^{C\cap L}$;
— $\delta' = \delta - d + 1 + |\beta' - \beta|$.

In other words, we will prove that the assigned contact points for the curves X_γ (recall the notation from (3.5)) that end up on $\tilde{C}$ transfer to assigned contact points for C, whereas those that end up on $\tilde{L}$ vanish, being accounted for by the component L of $X_0 = C \cup L$; moreover, all mobile contact points of X_γ end up on $\tilde{C}$ and thus transfer to mobile contact points on C; the curve C has additional mobile contact points with L, which come out of the intersection points of $\tilde{C}$ and $\tilde{L}$ in $\mathcal{Y}^\flat$.

We will provide an explicit geometric description of the degeneration of the curves X_γ in a neigbourhood of all these points in Subsection 3.4 below. Then, we will be able to track the limits of the δ nodes of the curves X_γ, and thus provide a geometric explanation for the formula relating δ' and δ.

3.2 – Simplified proof of Proposition (3.15)

We follow the presentation of [1], and first give a proof of Proposition (3.15) under some simplifying assumptions, in order to describe as clearly as possible what is actually going on in reality. We shall prove later on that these simplifying assumptions do indeed hold, in Subsection 3.3.

We have made six claims in Subsection 3.1 above, with numbers (3.9)–(3.14). Our simplifying assumptions consist in taking the three Claims (3.9), (3.11), and (3.12) for granted, for the moment.

(3.16) Proof of Proposition (3.15) assuming Claims (3.9), (3.11), and (3.12). Claim (3.12) ensures that the surface $\mathcal{Y}^\flat$ obtained from $\mathcal{Y}$ by contracting Z has at worst rational double points as singularities, and is still a nodal reduction of the family $\mathcal{X} \to \Gamma$. From now on we work on this new family, and denote any curve in $\mathcal{Y}^\flat$ image of a curve in $\mathcal{Y}$ by the same symbol (with the exception of the central fibre of $\mathcal{Y}^\flat$, which we will denote by $Y_0^\flat$).

Since Y_0 and $Y_0^\flat$ are semistable limits of the curves X_γ, which are smooth of genus g, it holds that

$$p_a(Y_0) = p_a(Y_0^\flat) = g.$$

Claim (3.11) ensures that $\tilde{L} \simeq L$ is rational. On the other hand, $Y_0^\flat = \tilde{L} \cup \tilde{C}$ is nodal, and the total number of points in $\tilde{C} \cap \tilde{L}$ is $|\beta^{C\cap L}|$. Thus one finds

$$g = p_a(Y_0^\flat) = p_a(\tilde{L} \cup \tilde{C}) = p_a(\tilde{C}) + |\beta^{C\cap L}| - 1. \tag{3.16.1}$$

We shall now apply Corollary (2.3), which gives a dimension bound on families of genus g curves and a dimensional characterization of generalized Severi varieties. Let V be the irreducible component of $V^{d,\delta}(\alpha,\beta) \cap p^\perp$ of which $X_0 = C \cup L$ is a general member. The dimension bound tells us that

$$\dim(V) \leqslant 2(d-1) + g(\tilde{C}) - 1 + |\beta^C + \beta^{C\cap L}|, \tag{3.16.2}$$

since the number of unassigned contact points of C with L is at most $|\beta^C + \beta^{C\cap L}|$, and C has degree at most $d-1$ (indeed, the unassigned contact points of C with L are the points in the support of $\pi_*\tilde{C} \cdot L$ off Ω, one has $\pi_*\tilde{C} \cdot L = \pi_*(C \cdot \pi^* L)$ by the projection formula, and the intersection points of $\tilde{C}$ with the sections Q_{ij} are mapped to Ω; besides, Claim (3.9) ensures that the number $|\beta^C + \beta^{C\cap L}|$ does not count any point in $\tilde{C} \cap \tilde{L} \subseteq Y_0^\flat$ several times).

But $p^\perp$ is a hyperplane which does not contain any irreducible component of $V^{d,\delta}(\alpha,\beta)(\Omega)$, by Proposition (2.2), and since $p \notin \Omega$. Hence

$$\dim(V) = \dim\big(V^{d,\delta}(\alpha,\beta)\big) - 1 = 2d + g - 2 + |\beta|. \tag{3.16.3}$$

In the upshot, we have the following sequence of equalities and inequalities:

$$\begin{aligned}
2d + g + |\beta| - 2 &\leqslant 2(d-1) + g(\tilde{C}) - 1 + |\beta^C + \beta^{C\cap L}| && \text{by (3.16.2) and (3.16.3)}\\
&\leqslant 2(d-1) + p_a(\tilde{C}) - 1 + |\beta^C + \beta^{C\cap L}| && \text{since } g(\tilde{C}) \leqslant p_a(\tilde{C})\\
&= 2(d-1) + g - |\beta^{C\cap L}| + |\beta^C + \beta^{C\cap L}| && \text{by (3.16.1)}\\
&= 2(d-1) + g + |\beta^C|\\
&\leqslant 2(d-1) + g + |\beta|.
\end{aligned} \tag{3.16.4}$$

We conclude that equality holds throughout. In particular, (3.16.2) is an equality and therefore, by Corollary (2.3), V is an irreducible component of the Severi variety $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$ where α', Ω', and β' are as stated in Proposition (3.15), and δ' is determined by the geometric genus of C (note in particular that the orders of contacts of C with L are given by the multiplicities in $\pi^* L \cdot \tilde{C}$, and therefore they are indeed α' and β'). Yet a couple of words are in order as to why the condition $\mathrm{Card}(C \cap L) = \mathrm{Card}(\pi^{-1}(C \cap L))$ of Corollary (2.3) is indeed verified: on the one hand, it follows from Claim (3.9) that the sections $Q_{i,j}$ are still mutually disjoint in $\mathcal{Y}^\flat$, so that

there is a unique point in $\tilde{C}$ over each point of Ω'; on the other hand, the equality in (3.16.2) implies (by Corollary (2.3) again) that C indeed intersects L at $|\beta^C + \beta^{C\cap L}|$ pairwise distinct points, so that eventually all points in $\pi^{-1}(C \cap L)$ have distinct images in L, as required.

To finish, let us explicitly derive the formula for δ': it follows from the series of equalities in (3.16.4) that the geometric genus of C is

$$g(C) = g(\tilde{C}) = p_a(\tilde{C}) = g - |\beta^{C\cap L}| + 1,$$

so that

$$\begin{aligned}\delta' = \binom{d-2}{2} - g(C) &= \binom{d-2}{2} - g + |\beta^{C\cap L}| - 1 \\ &= \binom{d-2}{2} - \binom{d-1}{2} + \delta + |\beta^{C\cap L}| - 1 = \delta - (d-1) + |\beta^{C\cap L}|,\end{aligned}$$

which is the value asserted in Proposition (3.15), as $\beta^{C\cap L} = \beta' - \beta$, since $\beta^C = \beta$ by the series of equalities in (3.16.4). □

(3.17) Proof of Claims (3.10), (3.13), and (3.14) assuming Claims (3.9), (3.11), and (3.12). As we have already observed, it follows from the series of equalities in (3.16.4) that $\beta = \beta^C$, which proves Claim (3.10).

On the other hand, the fact that equality holds in (3.16.2) implies by Corollary (2.3) that C is reduced. Thus, Claim (3.13) holds. Eventually, we have already observed that the series of equalities in (3.16.4) implies the equality $g(\tilde{C}) = p_a(\tilde{C})$, which proves that $\tilde{C}$ is smooth, i.e., Claim (3.14) holds. □

3.3 – Complete proof of Proposition (3.15)

(3.18) Labelling of the remaining sections. In order to prove Claims (3.9), (3.11), and (3.12), we need to consider the situation as complicated as it could a priori be, and therefore to introduce some more notation with respect to that already given in Subsection 3.1.

Thus, for all i we define (in the surface $\mathcal{Y}$) $\beta_i^{C,Z}$ as the number of the sections $(R_{i,j})_{1\leqslant j\leqslant \beta_i}$ passing through a connected component of Z that intersects $\tilde{C}$, and package these numbers in the sequence $\beta^{C,Z} = (\beta_i^{C,Z})$.

On the other hand we redefine $|\beta^{C\cap L}|$ as the sum of the number of points of $\tilde{C}$ meeting $\tilde{L}$ plus the number of connected components of Z meeting both $\tilde{C}$ and $\tilde{L}$. Thus it is the number of intersection points between $\tilde{C}$ and $\tilde{L}$ after we contract Z. Note that $|\beta^{C\cap L}|$ no longer corresponds to an actual sequence $\beta^{C\cap L} = (\beta_i^{C\cap L})$; the point is, without Claim (3.12), there could be a connected component of Z meeting $\tilde{C}$ at several distinct points with different multiplicities in $(\pi^* L)^{\mathrm{vert}}$.

(3.19) Bounding the number of mobile contact points of C and L. To this end, we note that every connected component of Z mapping to a point of L by π necessarily intersects either $\tilde{L}$, or a section among $Q_{i,j}$ and $R_{i,j}$. Indeed, otherwise there would exist a connected component of $\pi^{-1}(L)$ entirely contained in Z, and thus after contracting Z we would obtain a surface with an isolated point in the preimage of L, which is impossible.

Therefore the number of mobile contact points of C and L (in $\mathbf{P}^2$) is bounded from above as follows:

$$\mathrm{Card}\big(C \cap (L - \Omega)\big) \leqslant |\beta^C + \beta^{C,Z}| + |\beta^{C\cap L}|. \tag{3.19.1}$$

(3.20) Bounding the genus of C. As in (3.16), Y_0 is a semistable limit of the curves X_γ, and therefore $p_a(Y_0) = g$. Since the number of intersection points between $\tilde{C}$ and $\tilde{L}$ after we contract Z is $|\beta^{C\cap L}|$, we find that

$$g = p_a(Y_0) \geqslant p_a(\tilde{L}) + p_a(\tilde{C}) + |\beta^{C\cap L}| - 1. \tag{3.20.1}$$

Note that this inequality is strict if some connected component of Z has positive arithmetic genus, or intersects $\tilde{C}$ or $\tilde{L}$ at more than one point.

(3.21) Proof of Claim (3.11). Let e be the multiplicity of L in $X_0 = C \cup L$. Then C has degree $d-e$. On the other hand, $\tilde{L}$ is mapped $e:1$ to L by π, so by Riemann–Hurwitz we have the inequality $p_a(\tilde{L}) \geqslant -e+1$. Together with (3.20.1), this gives the bound:

$$p_a(\tilde{C}) \leqslant g - |\beta^{C\cap L}| + e. \tag{3.21.1}$$

Now, we argue exactly as in (3.16): by Corollary (2.3), we find that the irreducible component V of $V^{d,\delta}(\alpha,\beta)\cap p^\perp$ that C moves in has dimension at most

$$\begin{aligned} 2(d-e) + p_a(\tilde{C}) - 1 + |\beta^C + \beta^{C,Z}| + |\beta^{C\cap L}| &\leqslant 2(d-e) + g + |\beta^C + \beta^{C,Z}| + e - 1 \\ &= 2d + g + |\beta^C + \beta^{C,Z}| - e - 1. \end{aligned} \tag{3.21.2}$$

But since

$$\dim(V) = \dim\bigl(V^{d,\delta}(\alpha,\beta)\bigr) - 1 = 2d + g + |\beta| - 2,$$

and $|\beta| \leqslant |\beta^C + \beta^{C,Z}|$, one necessarily has $e = 1$. □

(3.22) Proof of Claim (3.12). By the fact that (3.21.2) is in fact an equality, we see that also (3.16.1) must be an equality. Thus, as already observed in (3.20), every connected component of Z has arithmetic genus 0, i.e., it is a tree of rational curves, and intersects each of $\tilde{L}$ and $\tilde{C}$ at most once.

In order to prove that such a connected component is a chain of rational curves connecting $\tilde{L}$ and $\tilde{C}$, it is therefore sufficient to prove that any "end component" of a connected component of Z meets either $\tilde{C}$ or $\tilde{L}$. By an end component, we mean an irreducible component Z_1 that meets at most one other irreducible component of Z (one if the corresponding connected component of Z has several irreducible components, zero if Z_1 is itself a connected component of Z).

The key argument is that if an end component Z_1 of Z intersects neither $\tilde{C}$ nor $\tilde{L}$, then by the minimality of $\mathcal{Y}$ there must be at least two of the sections $(Q_{i,j})_{1\leqslant j\leqslant\alpha_i}$ and $(R_{i,j})_{1\leqslant j\leqslant\beta_i}$ meeting Z_1, for otherwise we could blow down Z_1 in $\mathcal{Y}$ and still satisfy all the conditions imposed on $\mathcal{Y}$. (Note that Z_1, being a rational curve intersecting $Y_0 - Z_1$ in exactly one point since Y_0 is connected, is a (-1)-curve).

These two sections cannot be both of type α, for otherwise they would be contracted by π to the same point of L, which is excluded by the definition of generalized Severi varieties. To rule out the other possibilities, we use the fact that since (3.21.2) is an equality, then also (3.19.1) must be an equality. But if two sections either both of type β, or of types α and β respectively, meet the same connected component of Z, then the two corresponding points of C in $\mathbf{P}^2$ are the same, hence (3.19.1) is a strict inequality. This ends the proof of Claim (3.12). □

(3.23) Proof of Claim (3.9). Now we know that Claim (3.12) holds, i.e., the connected components of Z are chains of rational curves connecting $\tilde{C}$ and $\tilde{L}$. Therefore we can indeed consider the points $(r_{i,j}^{C\cap L})_{1\leqslant j\leqslant\beta_i^{C\cap L}}$ introduced in (3.8), and their respective images by π, the points $(s_{i,j}^{C\cap L})_{1\leqslant j\leqslant\beta_i^{C\cap L}}$ (see also (3.18)).

Our main argument here is that (3.19.1) is an equality, as we have already observed in (3.22) above. If a section of type α intersects a connected component of Z, then the image in $\mathbf{P}^2$ of the point $r_{i,j}^{C\cap L}$ corresponding to this component must lie in Ω, which implies that (3.19.1) is a strict inequality, a contradiction.

Similarly, if a section of type β intersects a connected component of Z, then the point $r_{i,j}^{C,Z}$ corresponding to this section and the point $r_{i,j}^{C\cap L}$ corresponding to the component of Z have the same image in $\mathbf{P}^2$, hence (3.19.1) is a strict inequality, a contradiction. □

We have now proved all three Claims (3.9), (3.11), and (3.12). In Subsection 3.2 we had proved Proposition (3.15) assuming these three claims. Thus, we now have a complete proof of Proposition (3.15), and the proof of Theorem (3.1) is over.

3.4 – Local picture of the degeneration of curves

We consider the situation set up in (3.4) and (3.5), when a general member X_γ of $V^{d,\delta}(\alpha,\beta)$ degenerates to a curve $X_0 = L\cup C$. We shall now give a complete description of this degeneration, in its concrete incarnation as a degeneration of plane curves. This will be useful in the next section, when we prove the theorem going the other direction than Theorem (3.1) above.

We carry on with the notation introduced above for Proposition (3.15). Something non-trivial happens only at the points where C and L intersect, which come in the following four types: (3.24)–(3.27).

(3.24) Vanishing assigned contact points. These are the points of $\Omega \setminus \Omega'$, i.e., the points $p_{i,j}^L$. Since C is general in the component V of $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$, by Proposition (2.2) it does not pass through any point $p_{i,j}^L$. Thus the curve X_0 is smooth at those points, i.e., it has only one local branch there, which is an open subset of the line L itself.

Therefore the degeneration happens like this: locally at a point $p_{i,j}^L$, the curves X_γ, $\gamma \neq 0$, have one smooth local branch tangent to the order i with L, which degenerates in X_0 to an open neighbourhood of $p_{i,j}^L$ in L.

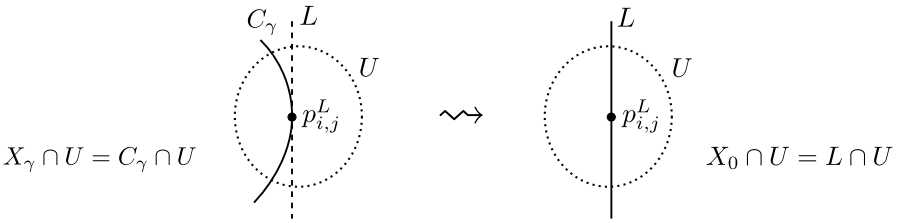


Figure 2: Degeneration in a neighbourhood U of a vanishing assigned contact point

(3.25) Non-vanishing assigned contact points. These are the points of Ω', i.e., the points $p_{i,j}^C$. Such a point is by definition the image of the point $q_{i,j}^C \in \tilde{C}$, and C is smooth at $p_{i,j}^C$ by Proposition (2.2). On the other hand $p_{i,j}^C$ also has a preimage in $\tilde{L}$, and exactly one as $\tilde{L} \to L$ is an isomorphism: let's call it $(q_{i,j}^C)'$. It follows that X_0 has two smooth local branches at $p_{i,j}^C$, tangent to i-th order, so that $p_{i,j}^C$ is an i-tacnode of X_0.

As we have just seen, the map $Y_0 \to X_0$ factors through the normalization of $p_{i,j}^C$ in X_0 (recall that Y_0 is the central fibre of the semistable reduction introduced in (3.5)), and thus the family $\mathcal{Y}$ is smooth in a neighbourhood of the two preimages of $p_{i,j}^C$ in Y_0. This implies that the curves X_γ, as they approach X_0, have two local branches analytically locally around $p_{i,j}^C \in \mathbf{P}^2$.

The difference in arithmetic genus between Y_b and X_γ over a neighbourhood of $p^C_{i,j}$ in $\mathbf{P}^2$ is constant as b moves (with $\gamma = \nu(\eta(b)$, see diagram (3.5.1)), because both $\mathcal{Y}$ and $\mathcal{X}$ are flat families. Moreover, $\mathcal{X}$ is equigeneric over $\Gamma \setminus \{0\}$, and $\mathcal{Y}$ is a nodal reduction of $\mathcal{X}$, so for $b \neq 0$, the difference in arithmetic genus between Y_b and X_γ is just the sum of the δ-invariants of the singularities of X_γ; since the curves X_γ are nodal for $\gamma \neq 0$, this is therefore the number of nodes of X_γ tending to the point $p^C_{i,j}$ as γ tends to 0. On the other hand, over a neighbourhood of $p^C_{i,j}$, $Y_0 \to X_0$ is the normalization of an i-tacnode, so the difference in arithmetic genus between Y_0 and X_0 over a neighbourhood of $p^C_{i,j}$ equals i.

The upshot is that locally at $p^C_{i,j}$, the curves X_γ, $\gamma \neq 0$, have two local branches intersecting transversely in i points tending to the i-tacnode of X_0 at $p^C_{i,j}$; one of the two branches of X_γ is tangent to order i with L at the fixed point $p^C_{i,j} \in \Omega' \subseteq \Omega$ (this is the branch corresponding to the branch of Y_b intersecting the section $Q^C_{i,j}$, namely that tending to the branch of Y_0 contained in $\tilde{C}$).

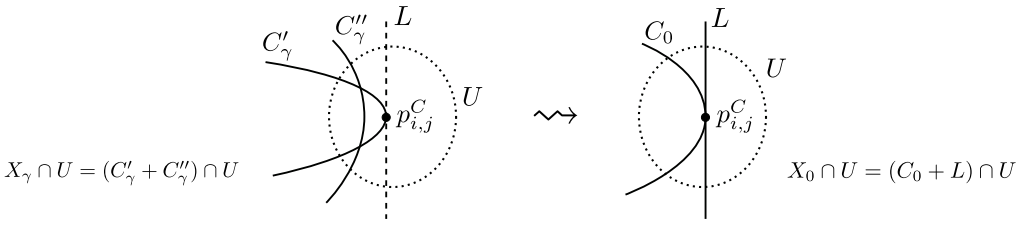


Figure 3: Degeneration in a neighbourhood U of a non-vanishing assigned contact point

(3.26) Limits of unassigned contact points. These are the points $s^C_{i,j}$. At those points the situation is almost exactly the same as that at the points $p^C_{i,j}$ described above, the key point being that the points $s^C_{i,j} \in L$ have two points in their preimage in Y_0, namely $r^C_{i,j} \in \tilde{C}$ and an additional point on $\tilde{L}$, which we may call $(r^C_{i,j})'$ so that, as in (3.25), the map $Y_0 \to X_0$ factors through the normalization of the i-tacnode of X_0 at $s^C_{i,j}$. The only difference is that here the branch of X_γ which is tangent to order i with L touches L at a mobile point, different from $s^C_{i,j}$ by Proposition (2.2) (this is the branch corresponding to the branch of Y_b intersecting the section $R^C_{i,j}$, which tends to a local branch of $\tilde{C}$).

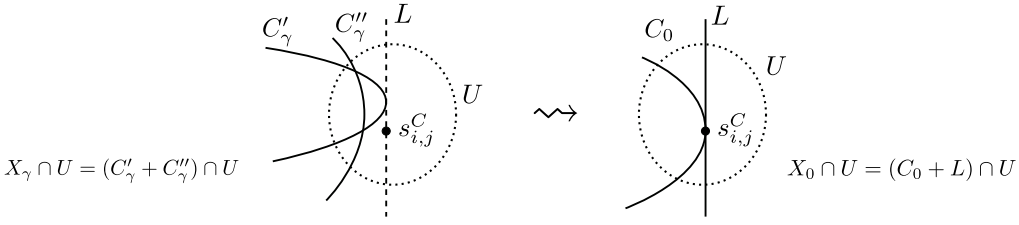


Figure 4: Degeneration in a neighbourhood U of a limit of unassigned contact points

(3.27) New unassigned contact point. These are the points $s^{C\cap L}_{i,j}$. Such a point is the image of a point $r^{C\cap L}_{i,j}$ which, in the surface $\mathcal{Y}^\flat$ obtained from $\mathcal{Y}$ by contracting Z, lies at the intersection of $\tilde{C}$ and $\tilde{L}$; it is thus an ordinary node of the central fibre $Y^\flat_0$ (in fact, seen in $\mathcal{Y}$ it is a node as well, although there it may lie on $\tilde{C} \cap Z$ rather than on $\tilde{C} \cap \tilde{L}$), and the point $s^{C\cap L}_{i,j}$ therefore has only one point in its preimage in Y_0. Since the nearby fibres Y_b are smooth (because $\mathcal{X}$ is equigeneric over $\Gamma \setminus \{0\}$, and $\mathcal{Y}$ is a nodal reduction of $\mathcal{X}$), they have only one

branch in an analytic neighbourhood of the preimage of $s_{i,j}^{C\cap L}$ in $\mathcal{Y}$, and therefore the curves X_γ, $\gamma \neq 0$, have only one local branch in a neighbourhood of $s_{i,j}^{C\cap L}$ in $\mathbf{P}^2$.

The points $s_{i,j}^{C\cap L}$ are again i-tacnodes of the curve X_0, this time because both L and C are smooth at $s_{i,j}^{C\cap L}$ by Proposition (2.2), and intersect there with multiplicity i because by definition $s_{i,j}^{C\cap L}$ lies on a component of multiplicity i of $(\pi^* L)^{\mathrm{vert}}$. In particular the difference in arithmetic genus between Y_0 and X_0 over $s_{i,j}^{C\cap L}$ is $i-1$. Since this difference for the families $\mathcal{Y}$ and $\mathcal{X}$ is constant over a neighbourhood of $s_{i,j}^{C\cap L}$, and the curves X_γ are nodal for $\gamma \neq 0$, we see that the latter will have $i-1$ nodes tending to $s_{i,j}^{C\cap L}$.

To sum up, analytically locally around $s_{i,j}^{C\cap L}$, the curves X_γ, $\gamma \neq 0$, have one irreducible, $(i-1)$-nodal local branch, transverse to L and not passing by $s_{i,j}^{C\cap L}$, degenerating to the i-tacnode at $s_{i,j}^{C\cap L}$ formed by C and L.

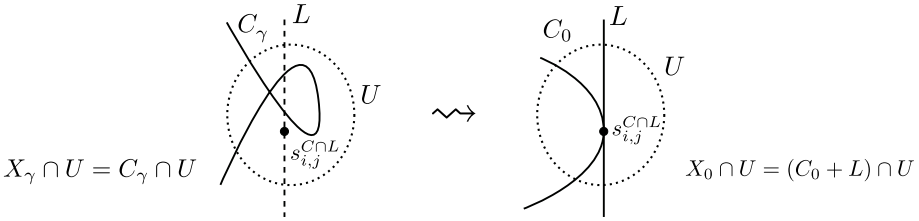


Figure 5: Degeneration in a neighbourhood U of a new unassigned contact point

We emphasize that this phenomenon is arguably the central topic of this whole volume; its local model has been studied in detail in the previous chapters [IV] and [A].

Whereas the above description of the degeneration will be needed in the next section, the remainder of this subsection is not necessary for the proof of the main theorem. I believe it is nevertheless helpful.

First of all, having the above descriptions at hand, we may now give the promessed geometric interpretation for the relation between δ and δ' in Theorem (3.1) and Proposition (3.15).

(3.28) Book-keeping of the nodes in the degeneration. As a result of (3.24)–(3.27) above, we see that as X_γ degenerates to $X_0 = C \cup L$ there are i nodes of X_γ tending to each point of types α_i^C and β_i^C (respectively the preserved tangencies at assigned points, and the limits of tangencies at unassigned points, which are all preserved), and $i-1$ nodes of X_γ tending to each point of type $\beta_i^{C\cap L}$ (the new tangency conditions, all at unassigned points); the degeneration is otherwise equisingular. Therefore, we have:

$$\delta - \delta' = I\alpha^C + I\beta^C + \big(I\beta^{C\cap L} - |\beta^{C\cap L}|\big).$$

On the other hand we know that, in the notation of Proposition (3.15), $\alpha' = \alpha^C$ and $\beta' = \beta^C + \beta^{C\cap L}$, so that

$$I\alpha^C + I\beta^C + I\beta^{C\cap L} = I\alpha' + I\beta' = C \cdot L = d-1;$$

since moreover $\beta = \beta^C$, we also have $|\beta^{C\cap L}| = |\beta' - \beta|$. Finally, we find

$$\delta - \delta' = (d-1) - |\beta' - \beta|$$

as required.

Lastly, let us analyze the multiplicities of the irreducible components of Y_0 in the divisor $\pi^* L$. This will be helpful in understanding the multiplicities of the components of the type $V^{d-1,\delta'}(\alpha', \beta')$ in the intersection of $V^{d,\delta}(\alpha, \beta)$ with the hyperplane $p^\perp$, cf. Theorem (4.1).

(3.29) Multiplicities of the divisor π^*L in $\mathcal{Y}$. Recall that the central fibre Y_0 of $\mathcal{Y}$ is reduced and consists of (i) $\tilde{L}$, an irreducible curve mapped isomorphically to L, (ii) $\tilde{C}$, the normalizations of the various irreducible components of C, and (iii) Z, the union of disjoint chains of rational curves joining $\tilde{L}$ to $\tilde{C}$.

Let m be the multiplicity of $\tilde{L}$ in the divisor π^*L on $\mathcal{Y}$. Let us remind that the integer m records the ramification of $\pi : \mathcal{Y} \to \mathbf{P}^2$ along $\tilde{L}$ in the direction normal to $\tilde{L}$, hence $m > 1$ does not contradict $\pi|_{\tilde{L}}$ being an isomorphism onto L; instead, if $\pi|_{\tilde{L}} : \tilde{L} \to L$ had degree k, this would translate into $\pi_*\tilde{L} = kL$.

The main output of this paragraph will be the divisibility of m by $\mathrm{lcm}(\beta^{C\cap L}) = \mathrm{lcm}(\beta' - \beta)$. To see this, consider a point $r_{i,j}^{C\cap L} \in \tilde{C}$. By definition, see Paragraph (3.8), $r_{i,j}^{C\cap L}$ sits both on $\tilde{C}$ and on a multiplicity i component of $(\pi^*L)^{\mathrm{vert}}$, and we have seen that $(\pi^*L)^{\mathrm{vert}}$ is supported on $\tilde{L} + Z$. Thus, if $i < m$, then $r_{i,j}^{C\cap L}$ sits on a smooth irreducible rational curve $Z_1 \subseteq Z$ which has multiplicity i in π^*L. Let Z_2 be the next curve in the chain of rational curves joining $\tilde{C}$ to $\tilde{L}$ and containing Z_1 (here, 'next' is intended relative to the direction pointing to $\tilde{L}$; if Z_1 directly joins $\tilde{C}$ to $\tilde{L}$, then we take $Z_2 = \tilde{L}$), and let i_2 be its multiplicity in π^*L. Since π contracts Z_1, we have $\pi^*L \cdot Z_1 = 0$, hence

$$(iZ_1 + i_2Z_2) \cdot Z_1 = 0.$$

Since Z_1 is a (-2)-curve[2] and $Z_1 \cdot Z_2 = 1$, we find that $i_2 = 2i$. We continue by induction to compute the successive multiplicities: the connected component of Z containing Z_1 is a chain of (-2)-curves; let us denote them by $Z_1, \ldots, Z_l$, in that order, and let $Z_{l+1} = \tilde{L}$. Let $i_1, \ldots, i_{l+1}$ be their multiplicities in π^*L. Once we know, for some $k \leqslant l$, that $i_s = si$ for all $s = 1, \ldots, k$, we have

$$\pi^*L \cdot Z_k = (i_{k-1}Z_1 + i_kZ_k + i_{k+1}Z_{k+1}) \cdot Z_k = 0,$$

hence $i_{k+1} = 2i_k - i_{k-1} = (k+1)i$, as required. Finally, the multiplicity m of $\tilde{L} = Z_{l+1}$ in π^*L equals $(l+1)i$; thus, m is divisible by i, and the length of the chain of (-2)-curves is $l = m/i - 1$.

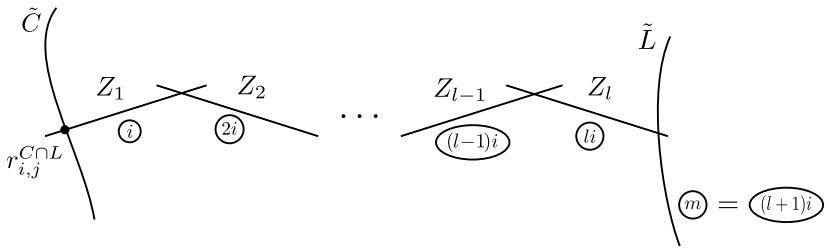


Figure 6: Multiplicities of the components of $(\pi^*L)^{\mathrm{vert}}$

The same argument shows that there are no points $r_{i,j}^{C\cap L}$ with $i > m$, and all points $r_{m,j}^{C\cap L}$ appear directly on $\tilde{L}$.

In conclusion, for all i such that $\beta_i^{C\cap L} = (\beta' - \beta)_i > 0$, the multiplicity m is divisible by i; equivalently, m is divisible by $\mathrm{lcm}(\beta' - \beta)$.

4 – Deformations of generalized Severi varieties

In this section we prove the result going the other direction than Theorem (3.1). It asserts that all the generalized Severi varieties which, according to Theorem (3.1), may appear in the hyperplane section $V^{d,\delta}(\alpha, \beta) \cap p^\perp$, do indeed appear.

[2]this is well-known and may be seen as follows: all fibres of $\mathcal{Y}$ are linearly equivalent, and have intersection number 0 with Z_1; therefore, $Y_0 \cdot Z_1 = (\tilde{C} + Z_1 + Z_2) \cdot Z_1 = 0$, hence $Z_1^2 = -(\tilde{C} + \tilde{L}) \cdot Z_1 = -2$.

(4.1) Theorem ([1, Theorem 1.3]). *Let p be a general point of L.*

a) Let V' be an irreducible component of $V^{d,\delta}(\alpha+e_k,\beta-e_k)(\Omega\cup\{p\})$ as in part a) of Theorem (3.1). Then V' is a component of the intersection $V^{d,\delta}(\alpha,\beta)(\Omega)\cap p^\perp$, and at a general point of V', the variety $V^{d,\delta}(\alpha,\beta)$ is smooth and has intersection multiplicity k with $p^\perp$ along V'.

b) Let V' be an irreducible component of $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$ as in part b) of Theorem (3.1). Then V' is a component of the intersection $V^{d,\delta}(\alpha,\beta)(\Omega)\cap p^\perp$ and, at a general point of V', the variety $V^{d,\delta}(\alpha,\beta)$ has $\binom{\beta'}{\beta}I^{\beta'-\beta}/\operatorname{lcm}(\beta'-\beta)$ local sheets, each of which has intersection multiplicity $\operatorname{lcm}(\beta'-\beta)$ with $p^\perp$ along V'. Moreover, each of these sheets has the generic point of V' as a point of multiplicity $\operatorname{lcm}(\beta'-\beta)/\max(\beta'-\beta)$,

In both cases a) and b) the general strategy will be to show that the general member of V' may be deformed to general members of $V^{d,\delta}(\alpha,\beta)(\Omega)$. We treat the two cases separately, in Subsections 4.1 and 4.2 respectively.

The multiplicity in case a) displays a phenomenon reminiscent of the fundamental principle of projective duality. The deformation argument for case b) is much more demanding, and eventually relies on the local material on deformations of tacnodes gathered in Chapters [IV] and [V], which is arguably the keystone of the present volume.

4.1 – Fixing an unassigned contact point

This section is devoted to the proof of part a) of Theorem (4.1), which is the case when V' parametrizes curves that do not contain the line L, and for which the condition that they lie on the hyperplane $p^\perp$ is accounted for by the fact that p is one of the assigned contact points.

It is fairly clear that $V^{d,\delta}(\alpha+e_k,\beta-e_k)(\Omega\cup\{p\})$ is contained in the hyperplane section of $V^{d,\delta}(\alpha,\beta)(\Omega)$ by $p^\perp$, so the point here is to establish the assertion on the smoothness and intersection multiplicity.

The multiplicity is as one would expect it to be from the point of view of projective duality. Indeed, the latter tells us the following. Consider a variety $X\subseteq\mathbf{P}^N$, and its dual $X^\vee\subseteq\check{\mathbf{P}}^N$ in the dual projective space, which parametrizes hyperplanes in $\mathbf{P}^N$ tangent to X. Then for a general point $p\in X$, the hyperplane $p^\perp\subseteq\check{\mathbf{P}}^N$ (consisting of those hyperplanes of $\mathbf{P}^N$ that contain the point p) is tangent to $X^\vee$ at the points $T^\perp\in\check{\mathbf{P}}^N$ corresponding to hyperplanes that are tangent to X at p. More generally, one may consider "osc-dual" varieties $X^{\vee_\beta}\subseteq\check{\mathbf{P}}^N$ of X, parametrizing hyperplanes of $\mathbf{P}^N$ osculating X to the order β, and then $p^\perp$ osculates $X^{\vee_\beta}$ to the order β at points $O^\perp$ corresponding to osculating hyperplanes to X at p. Some of these are discussed in Chapter [XII].

In the situation under consideration, this has to be applied to the case when X is the degree $d-I\alpha$ rational normal curve image of L by the linear system cut out by degree d plane curves with the contact conditions at assigned points determined by α and Ω. In fact this is exactly what we do in practice, following the approach of [1, §4.3], and establishing along the way the appropriate biduality principle for osculating hyperplanes to rational normal curves, which is the following.

(4.2) Proposition ([1, Lemma 4.7]). *Consider the line $L\subseteq\mathbf{P}^2$, and a general point p of L. We assume the data of d,α,β, and $\Omega=(p_{i,j})_{1\leqslant j\leqslant\alpha_i}$ is given as in Theorem (4.1); in particular, $I\alpha+I\beta=d$. We define the following three loci in $|\mathcal{O}_L(d)|$:*

- $p^\perp=\{D:p\in D\}$;
- $\Phi_{\alpha,\beta}(\Omega)=\big\{D=\sum_{1\leqslant j\leqslant\alpha_i}i\cdot p_{i,j}+\sum_{1\leqslant j\leqslant\beta_i}i\cdot s_{i,j}\quad\text{for some } s_{i,j}\text{'s in } L\big\}$;
- $\Psi_{\alpha,\beta}(\Omega)=\big\{D=k\cdot p+\sum_{1\leqslant j\leqslant\alpha_i}i\cdot p_{i,j}+\sum_{1\leqslant j\leqslant(\beta-e_k)_i}i\cdot s_{i,j}\quad\text{for some } s_{i,j}\text{'s in } L\big\}$.

In a neighbourhood of a general point $[D_0] \in \Psi_{\alpha,\beta}(\Omega)$*, the variety* $\Phi_{\alpha,\beta}(\Omega)$ *is smooth and has intersection multiplicity* k *with the hyperplane* $p^\perp$ *along* $\Psi_{\alpha,\beta}(\Omega)$*.*

Proof. We will proceed by an explicit local computation. Let x be an affine coordinate on L centered at the point p. Let the points $p_{i,j}$ have respective coordinates $\lambda_{i,j}$, and let the points $s^0_{i,j}$ such that

$$D_0 = k \cdot p + \sum_{1 \leqslant j \leqslant \alpha_i} i \cdot p_{i,j} + \sum_{1 \leqslant j \leqslant (\beta - e_k)_i} i \cdot s^0_{i,j}$$

have respective coordinates $\mu_{i,j}$. By generality of D_0, we may assume that these coordinates are altogether mutually distinct.

Then we can parametrize a neighbourhood of $[D_0]$ in $\Phi_{\alpha,\beta}(\Omega)$ by

$$(\varepsilon, \varepsilon_{i,j}) \mapsto [f(x)] = \Big[(x-\varepsilon)^k \cdot \prod_{1 \leqslant j \leqslant \alpha_i} (x - \lambda_{i,j})^i \cdot \prod_{1 \leqslant j \leqslant (\beta - e_k)_i} (x - \mu_{i,j} - \varepsilon_{i,j})^i\Big] \in \mathbf{C}_d[x],$$

from which we see first of all that $\Phi_{\alpha,\beta}(\Omega)$ is smooth at $[D_0]$. On the other hand, writing

$$f(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0 \in \mathbf{C}_d[x],$$

we get a system of affine coordinates on some chart of $|\mathcal{O}_L(d)|$, in which the hyperplane $p^\perp$ is defined by the linear equation $b_0 = 0$. Then, in the local coordinates $(\varepsilon, \varepsilon_{i,j})$ on $\Phi_{\alpha,\beta}(\Omega)$ around $[D_0]$, the intersection with $p^\perp$ is defined by the equation

$$\prod \lambda_{i,j}^i \cdot \varepsilon^k \cdot \prod (\varepsilon_{i,j} + \mu_{i,j})^i = 0 \iff \varepsilon^k \cdot \prod (\varepsilon_{i,j} + \mu_{i,j})^i = 0,$$

hence, locally around $[D_0]$, it is k times the divisor defined by the equation $\varepsilon = 0$, and the latter divisor is exactly $\Psi_{\alpha,\beta}(\Omega)$. □

In order to relate the local geometry of generalized Severi varieties to the model situation of Proposition (4.2), we consider the restriction map $\pi : |\mathcal{O}_{\mathbf{P}^2}(d)| \dashrightarrow |\mathcal{O}_L(d)|$. Note that this is merely the projection from the codimension $d+1$ linear subspace $L^\perp$ of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ parametrizing curves wich contain L. Finally, observe that the hyperplane $p^\perp$ of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ is the preimage by π of the hyperplane $p^\perp$ of $|\mathcal{O}_L(d)|$.

Let W be the plain Severi variety $V^{d,\delta}$, and denote by π_W the restriction of π to W. By definition, using the notation of Proposition (4.2), we have

$$V^{d,\delta}(\alpha,\beta)(\Omega) = \pi_W^{-1}\big(\Phi_{\alpha,\beta}(\Omega)\big) \qquad \text{and} \qquad V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p\}) = \pi_W^{-1}\big(\Psi_{\alpha,\beta}(\Omega)\big).$$

Therefore, part a) of Theorem (4.1) follows directly from the following proposition together with Proposition (4.2) above.

(4.3) Proposition. *Let* X_0 *be a general member of (any irreducible component of)* $V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p\})$*. The differential of the map* $\pi_W : W \dashrightarrow |\mathcal{O}_L(d)|$ *is surjective at the point* $[X_0]$*.*

Proof. First note that, being general, X_0 does not contain L and therefore π is well defined at $[X_0]$. Since the map π is a linear projection, it suffices to show that the projective tangent space $\mathbf{T}_{[X_0]}W$ to $W = V^{d,\delta}$ at the point $[X_0]$ intersects the center of the projection transversely, i.e., that the intersection $\mathbf{T}_{[X_0]}W \cap L^\perp$ has codimension $d+1$ in $\mathbf{T}_{[X_0]}W$.

Now, the projective tangent space $\mathbf{T}_{[X_0]}W$ identifies with the linear series of degree d curves passing through the nodes of X_0, and thus its intersection with $L^\perp$ identifies with the linear series of degree $d-1$ curves passing through the nodes of X_0. By Lemma (4.4), these two linear series have dimensions $\frac{d(d+3)}{2} - \delta$ and $\frac{(d-1)(d+2)}{2} - \delta$ respectively, so the codimension of $\mathbf{T}_{[X_0]}W \cap L^\perp$ in $\mathbf{T}_{[X_0]}W$ equals that of $|\mathcal{O}_{\mathbf{P}^2}(d-1)|$ in $|\mathcal{O}_{\mathbf{P}^2}(d)|$, which is $d+1$, as we wanted. □

The following lemma is the classical fact that, on a regular surface, the adjoint series cut out the complete canonical series. We include the proof for completeness.

(4.4) Lemma. *Let $C \subseteq \mathbf{P}^2$ be a δ-nodal curve of degree d. For all $e \geqslant d-3$, the nodes of C impose independent conditions on curves of degree e.*

Proof. It is sufficient to prove that the nodes of C impose independent conditions on curves of degree $d-3$. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbf{P}^2}$ be the ideal sheaf of the nodes of C as a subscheme of $\mathbf{P}^2$, and let $\mathcal{A} \subseteq \mathcal{O}_C$ be the conductor ideal sheaf of C (with respect to its normalization $\bar{C}$). The ideal $\mathcal{A}$ is the restriction of $\mathcal{I}$ to C (see, e.g., [II, Lemma (2.5)]), and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbf{P}^2}(d-3) \otimes \mathcal{I} \longrightarrow \mathcal{O}_C(d-3) \otimes \mathcal{A} \longrightarrow 0.$$

It follows that there is an isomorphism $H^0(\mathcal{O}_{\mathbf{P}^2}(d-3) \otimes \mathcal{I}) \cong H^0(\mathcal{O}_C(d-3) \otimes \mathcal{A})$. Now, by [II, Corollary (2.4)], $H^0(\mathcal{O}_C(d-3) \otimes \mathcal{A})$ is isomorphic to $H^0(\bar{C}, \omega_{\bar{C}})$, where $\bar{C}$ still denotes the normalization of C, hence it has dimension g, the geometric genus of C, so that finally

$$\begin{aligned} h^0\big(\mathcal{O}_{\mathbf{P}^2}(d-3) \otimes \mathcal{I}\big) &= g = p_a(d) - \delta \\ &= h^0\big(\mathcal{O}_{\mathbf{P}^2}(d-3)\big) - \delta, \end{aligned}$$

as we wanted to prove. □

4.2 – Merging a line

This section is devoted to the proof of part b) of Theorem (4.1), which is the case when V' parametrizes curves containing the line L. The proof will be divided in two parts (Sections 4.2.1 and 4.2.2) which we shall describe shortly, but we first need to introduce some notation. We advise the reader that he should have Sections 3.1 and 3.4 clear in mind in order to understand what is going on.

(4.5) The situation. We consider a curve $X_0 = C \cup L$, where $[C]$ is a general member of the component V' of $V^{d-1,\delta'}(\alpha', \beta')(\Omega')$ we have started with.

We have $\alpha' \leqslant \alpha$, and correspondingly Ω' is a subset of Ω; by this we mean that $\Omega' = (p'_{i,j})_{1 \leqslant j \leqslant \alpha'_i}$ such that for all i, $(p'_{i,j})_{1 \leqslant j \leqslant \alpha'_i}$ is a subsequence of $(p_{i,j})_{1 \leqslant j \leqslant \alpha_i}$.

On the other hand we have $\beta' \geqslant \beta$, and we choose a subset $\Lambda = (s_{i,j})_{1 \leqslant j \leqslant \beta_i}$ of cardinality β of the sequence $(s'_{i,j})_{1 \leqslant j \leqslant \beta'_i}$ of unassigned contact points of C with L.

(4.6) General strategy. The first part of the proof will consist in deforming the curve X_0 to degree d curves locally equigenerically around the points of Ω' and Λ, while maintaining the contact conditions encoded in α, β, and Ω. At a point $p'_{i,j}$ or $s_{i,j}$, the curve $C \cup L$ has an order i tacnode; the requirement that the deformation be locally equigeneric around such a point amounts in pratice to the requirement that the i-th order tacnode deforms into i nodes.

In the second part of the proof we will take care of the $\beta' - \beta$ remaining mobile contact points, i.e., those points $s'_{i,j}$ that are not in Λ. At these points, the curve $C \cup L$ has order i tacnodes as well, but these ones will be deformed into $i-1$ nodes only. This will ensure in particular that the line L be merged with C along the deformation, so that for instance if C is irreducible, then the curves deformed from $C \cup L$ will be irreducible as well.

The reader should consult Section 3.4 to see that these are indeed the appropriate deformations. The reason for this two steps deformation process is that we want to isolate the deformations of i-tacnodes to $i-1$ nodes from the rest, in order to apply the results from the two previous lectures [IV] and [V].

4.2.1 The relaxed Severi variety

Here we perform the first step of the proof, as described in (4.6) above. The deformation procedure described there will amount to defining a relaxed Severi variety, seeing $X_0 = C \cup L$ as one of its members, and showing that our relaxed Severi variety is smooth at $[X_0]$.

We use the notation introduced in (4.5) above, and in particular we fix the choice of a subsequence Λ. Moreover we label the points in $\Omega \setminus \Omega'$ as $(p^L_{i,j})_{1 \leqslant j \leqslant \alpha^L_i}$; then, letting $\alpha^L = (\alpha^L_i)_{i \geqslant 1}$, we have $\alpha = \alpha' + \alpha^L$. Note that around a point $p^L_{i,j}$, the curve X_0 has only one local branch, which is an open subset of the line L.

On the other hand, we label the mobile contact points of C with L that have not been included in Λ as $(s^{C \cap L}_{i,j})_{1 \leqslant j \leqslant \beta^{C \cap L}_i}$; then, letting $\beta^{C \cap L} = (\beta^{C \cap L}_i)_{i \geqslant 1}$, we have $\beta' = \beta + \beta^{C \cap L}$.

(4.7) The relaxed Severi variety. In an analytic neighbourhood of the point $[X_0]$ in $|\mathcal{O}_{\mathbf{P}^2}(d)|$, we define the *relaxed Severi variety* W_Λ to be the closure of the locus of reduced, degree d curves X_t satisfying the following six conditions:

(i) X_t preserves the δ' nodes of C, i.e., for every node of X_0 off L, X_t will have a node nearby;
(ii) at each point $p^L_{i,j} \in \Omega \setminus \Omega'$, X_t has contact of order i with L;
(iii) in a neighbourhood of each point $p'_{i,j} \in \Omega'$, X_t has singularities with total cogenus i, i.e., if we let $\bar{X}_t$ be the normalization of exactly these singularities, then $p_a(\bar{X}_t) = p_a(X_t) - i$;
(iv) in a neighbourhood of each point $s_{i,j} \in \Lambda$, X_t has singularities with total cogenus i;
(v) in a neighbourhood of each point $p'_{i,j} \in \Omega'$, it follows from condition (iii) that X_t has two local branches; we require that the one branch that is a deformation of an open subset of C has contact of order i with L at $p'_{i,j}$;
(vi) in a neighbourhood of each point $s_{i,j} \in \Lambda$, the curve X_t has two local branches; we require that the one branch that is a deformation of an open subset of C has contact of order i with L at a point near $s_{i,j}$.

Just to be sure, we recall that the curve $X_0 = C \cup L$ has an i-th order tacnode at each point $p'_{i,j} \in \Omega'$ or $s_{i,j} \in \Lambda$, with two local branches that are open subsets of C and L respectively. Conditions (iii) and (iv) imply that in a neighbourhood of such a point, the curves X_t have two local branches as well, which are deformations of open subsets of C and L respectively.

With this definition for the relaxed Severi variety, it is clear that $[X_0] \in W_\Lambda$. We could have defined W_Λ alternatively, by requiring in (iii) and (iv) that in a neighbourhood of each point $p'_{i,j}$ or $s_{i,j}$, the curve X_t has i nodes. The reason why W_Λ will be smooth at $[X_0]$ is because the only special feature of X_0 with respect to a general member of W_Λ is that it has several groups of i nodes coming together as i-th order tacnodes (for various i's), and this is perfectly harmless from the point of view of the deformation theory of nodal curves: somehow, this is the classical view that an i-th order tacnode is just i infinitely near ordinary nodes. This will be made precise when we compute the Zariski tangent space to W_Λ at the point $[X_0]$.

In order to show the smoothness of W_Λ at $[X_0]$ we will compare the dimension of W_Λ with that of its tangent space at $[X_0]$. We proceed to compute the dimension of W_Λ in the next paragraph.

(4.8) The relaxed Severi variety as a local sheet of a Severi variety. The basic observation in this paragraph is that the six conditions defining the relaxed Severi variety W_Λ in fact essentially define a generalized Severi variety in the plain sense. Namely, we are requiring that the curves X_t, (i) have a certain genus (or, equivalently, cogenus), defined by the δ' nodes deformations of those of C and the various groups of i nodes deformations of the points $p'_{i,j} \in \Omega'$ and $s_{i,j} \in \Lambda$, and (ii) satisfy certain contact conditions with L. Specifically, the cogenus, i.e.,

the difference in arithmetic genus between the curves X_t and their normalizations, is required to be:

$$\begin{aligned}\delta'' := \delta' + I\alpha' + I\beta &= \delta' + I\alpha' + I\beta' - I(\beta' - \beta) \\ &= \delta' + (d-1) - I(\beta' - \beta) = \delta - \big(I(\beta' - \beta) - |\beta' - \beta|\big)\end{aligned} \tag{4.8.1}$$

(for the last equality, recall the relation between δ and δ', given for instance in Theorem (3.1)). On the other hand the contact conditions with L are exactly those prescribed by α, β and Ω. Altogether, these are the conditions defining the (generalized) Severi variety $V^{d,\delta''}(\alpha,\beta)(\Omega)$.

The additional requirements in the definition of W_Λ have the effect of selecting an open subset of some local sheets of $V^{d,\delta''}(\alpha,\beta)(\Omega)$ at $[X_0]$. We are first of all requiring that the i nodes corresponding to a point $p'_{i,j}$ or $s_{i,j}$ be in a neighbourhood of this same point; in other words we prescribe among the tacnodes of X_0 which deform to i nodes, and which are ignored. This already selects local sheets of $V^{d,\delta''}(\alpha,\beta)(\Omega)$ at $[X_0]$, determined by Λ: indeed, if we consider W_Λ as a space of maps from curves of cogenus δ'' to $\mathbf{P}^2$, then the member of W_Λ corresponding to X_0 is the partial normalization of $X_0 = C \cup L$ at the points of Ω' and Λ; if one considers another partial normalization of X_0 at the points of Ω' and some other Λ', one gets a curve member of the space of maps corresponding to $V^{d,\delta''}(\alpha,\beta)(\Omega)$, which however does not belong to W_Λ.

We are moreover requiring that, locally around a point $p'_{i,j}$ or $s_{i,j}$, the order i contact with L be supported on the one local branch that is a deformation of the local branch of C. This again, and in the same way as above, has the effect of operating a selection among the local sheets of $V^{d,\delta''}(\alpha,\beta)(\Omega)$ at $[X_0]$.

The upshot is that the relaxed Severi variety W_Λ is an open subset in some local sheets of the generalized Severi variety $V^{d,\delta''}(\alpha,\beta)(\Omega)$. Since the latter is equidimensional, we have

$$\dim(W_\Lambda) = \dim\big(V^{d,\delta''}(\alpha,\beta)\big) = 2d + g'' + |\beta| - 1,$$

with $g'' = p_a(d) - \delta'' = \binom{d-1}{2} - \delta''$.

(4.9) The ideal defining the tangent space to the relaxed Severi variety. We shall prove, in Proposition (4.10) below, that the tangent space at $[X_0]$ of the relaxed Severi variety W_Λ identifies with the subspace $H^0(X_0, \mathcal{I}(d))$ of $H^0(X_0, \mathcal{O}(d))$ defined by an ideal sheal $\mathcal{I}$ of $\mathcal{O}_{X_0}$ which we shall now describe, based on the description of the tangent space of generalized Severi varieties given in [III].

If X is a δ-nodal curve member of the ordinary Severi variety $V^{d,\delta}$, then the tangent space $T_{[X]}V^{d,\delta}$ is $H^0(X, \mathcal{A}(d))$, with $\mathcal{A}$ the ideal sheaf of the points supporting the nodes of X. When i of the δ nodes become infinitely near so as to form an i-th order tacnode, the ideal sheaf $\mathcal{A}$ has to be modified in the most straightforward way, i.e., it is merely the ideal sheaf of the possibly infinitely near points supporting the nodes of X. In other words, it is cut out locally at an order i tacnode by curves osculating to the order i the two local branches of X at the tacnode.

On the other hand we have seen in a previous Lecture, see [III, Lemma (2.11)], that an order i contact condition with L at an assigned (resp. unassigned) point translates into the vanishing to the order i (resp. $i-1$) of the sections in $H^0(X, N_X)$ corresponding to vectors tangent to such deformations.

These considerations motivate the definition of the ideal sheaf $\mathcal{I}$.

(4.9.1) Definition. We consider the curve $X_0 = C \cup L$ introduced in (4.5) above. For all i, j with $1 \leqslant j \leqslant \alpha'_i$, we let ${}^\alpha\mathcal{I}_{i,j}$ be the subsheaf of $\mathcal{O}_{X_0}$ of those regular functions f, such that the restrictions $f|_C$ and $f|_L$ vanish at $p'_{i,j}$ to the orders i and $2i$ respectively.

Similarly, for all i, j with $1 \leqslant j \leqslant \beta_i$, we let ${}^{\beta}\mathcal{I}_{i,j}$ be the sheaf of regular functions f on X_0, such that $f|_C$ and $f|_L$ vanish at $s_{i,j}$ to the orders i and $2i-1$ respectively.

For all points $x \in X_0$, we denote by $\mathfrak{m}_x$ the ideal sheaf defining x in X_0. Lastly, we call $u_1, \ldots, u_{\delta'}$ the δ' nodes of C. Finally, we set:

$$\mathcal{I} = \prod_{1 \leqslant i \leqslant \delta'} \mathfrak{m}_{u_i} \cdot \prod_{1 \leqslant j \leqslant \alpha_i^L} \mathfrak{m}_{p_{i,j}^L}^i \cdot \prod_{1 \leqslant j \leqslant \alpha_i'} {}^{\alpha}\mathcal{I}_{i,j} \cdot \prod_{1 \leqslant j \leqslant \beta_i} {}^{\beta}\mathcal{I}_{i,j}.$$

(4.10) Proposition. *The relaxed Severi variety W_Λ is smooth at $[X_0]$, with tangent space*

$$T_{[X_0]}W_\Lambda = H^0(X_0, \mathcal{I}(d)) \ \subseteq \ T_{[X_0]}\big(|\mathcal{O}_{\mathbf{P}^2}(d)|\big) = H^0(X_0, \mathcal{O}_{X_0}(d)),$$

where $\mathcal{I}$ is the sheaf of ideals defined in (4.9.1) above.

Proof. We will compare the deformations of X_0 along W_Λ with those of the map given by a certain partial normalization of X_0. Let $\mu : \tilde{X}_0 \to X_0$ be the normalization of $X_0 = C \cup L$ at the nodes u_i of C and at the tacnodes $p'_{i,j} \in \Omega'$ and $s_{i,j} \in \Lambda$ (at which C and L touch), but not at the tacnodes $s'_{i,j}$, because the latter are off Λ. We will consider the deformations of the map $\phi : \tilde{X}_0 \to \mathbf{P}^2$ obtained as the composition of μ with the inclusion $X_0 \subseteq \mathbf{P}^2$.

The map ϕ is unramified, hence its normal sheaf N_ϕ is locally free, equal to $\omega_{\tilde{X}_0} \otimes \phi^* \omega_{\mathbf{P}^2}^{-1}$, see [II, Lemma (8.3)]. By Lemma (4.11) below, $H^1(\tilde{X}_0, N_\phi)$ vanishes, hence the deformation space of ϕ is smooth of dimension

$$h^0(\tilde{X}_0, N_\phi) = h^0(\tilde{X}_0, \omega_{\tilde{X}_0} \otimes \phi^* \omega_{\mathbf{P}^2}^{-1}).$$

The smoothness of this space ensures in particular the existence of a map F from a neighbourhood of $[\phi]$ to $|\mathcal{O}_{\mathbf{P}^2}(d)|$, mapping a deformation of ϕ to its image in $\mathbf{P}^2$, as in [III, (2.6)].

The image of F is what we will call the *superrelaxed Severi variety* $\tilde{W}_\Lambda$, which we define as W_Λ but only with the three requirements (i), (iii), and (iv) out of the six defining W_Λ; simply put, we just drop the contact conditions with L. Arguing as in (4.8) above, we see that $\tilde{W}_\Lambda$ is a local sheet of the plain Severi variety $V^{d,\delta''}$ at $[X_0]$, which implies that it has dimension

$$\dim(\tilde{W}_\Lambda) = \dim(V^{d,\delta''}) = 3d + g'' - 1.$$

That F surjects onto a neighbourhood of $[X_0]$ in $\tilde{W}_\Lambda$ is a consequence of Teissier's Résolution Simultanée Theorem, as in [III, (2.6)].

In order to analyze the tangent space to $\tilde{W}_\Lambda$ at $[X_0]$, we let $\mathcal{A} = (\mathcal{O}_{X_0} : \mu_* \mathcal{O}_{\tilde{X}_0})$ be the conductor of $\tilde{X}_0$ in X_0, see [II, §2]. It is a sheaf of ideals of $\mathcal{O}_{X_0}$, trivial on the open subset onto which μ is an isomorphism, equal to $\mathfrak{m}_{u_i}$ locally at the nodes u_i of C, while locally at the points $p'_{i,j} \in \Omega'$ and $s_{i,j} \in \Lambda$ it is the sheaf of regular functions vanishing to the order i on both C and L; this can be seen for instance with [II, Lemma (2.5)].

By [II, Corollary (8.4)], one has $\mu_* N_\phi = \mathcal{A} \otimes N_{X_0/\mathbf{P}^2}$. Therefore, there is a canonical isomorphism

$$H^0(\tilde{X}_0, N_\phi) \cong H^0(X_0, \mathcal{A} \otimes N_{X_0/\mathbf{P}^2}).$$

This isomorphism identifies with the differential of F at $[\phi]$, and thus we see that F is an isomorphism onto a neighbourhood of $[X_0]$ in $\tilde{W}_\Lambda$; in particular, the superrelaxed Severi variety $\tilde{W}_\Lambda$ is smooth at $[X_0]$, with tangent space $H^0(X_0, \mathcal{A} \otimes N_{X_0/\mathbf{P}^2})$.

Now, it follows from [III, (2.11)] that the Zariski tangent space at $[\phi]$ to the image by F^{-1} of the relaxed Severi variety $W_\Lambda \subseteq \tilde{W}_\Lambda$ is contained in

$$H^0\Big(\tilde{X}_0, N_\phi\Big(-\sum_{1 \leqslant j \leqslant \alpha_i^L} i \cdot q_{i,j}^L - \sum_{1 \leqslant j \leqslant \alpha_i'} i \cdot q'_{i,j} - \sum_{1 \leqslant j \leqslant \beta_i} (i-1) \cdot r_{i,j}\Big)\Big),$$

where (i) $q^L_{i,j}$ is the unique point of $\tilde{X}_0$ over $p^L_{i,j}$, and (ii) $q'_{i,j}$ and $r_{i,j}$ are the points on the proper transform of C in $\tilde{X}_0$ lying over $p'_{i,j}$ and $s_{i,j}$ respectively. Moreover, it follows from the description of $\mathcal{A}$ given above that

$$\mathcal{A} \otimes \phi_* \mathcal{O}_{\tilde{X}_0}\Big(- \sum_{1 \leqslant j \leqslant \alpha^L_i} i \cdot q^L_{i,j} - \sum_{1 \leqslant j \leqslant \alpha'_i} i \cdot q'_{i,j} - \sum_{1 \leqslant j \leqslant \beta_i} (i-1) \cdot r_{i,j}\Big) = \mathcal{I},$$

hence

$$\phi_* N_\phi\Big(- \sum_{1 \leqslant j \leqslant \alpha^L_i} i \cdot q^L_{i,j} - \sum_{1 \leqslant j \leqslant \alpha'_i} i \cdot q'_{i,j} - \sum_{1 \leqslant j \leqslant \beta_i} (i-1) \cdot r_{i,j}\Big) = \mathcal{I} \otimes N_{X_0/\mathbf{P}^2}.$$

Therefore, the Zariski tangent space to the relaxed Severi variety W_Λ at $[X_0]$ is contained in $H^0(X_0, \mathcal{I} \otimes N_{X_0/\mathbf{P}^2})$. Then, the proposition follows from the equality

$$h^0(X_0, \mathcal{I} \otimes N_{X_0/\mathbf{P}^2}) = \dim(W_\Lambda),$$

which is given by (4.12.2) below. □

The rest of the present Section 4.2.1 is devoted to some elementary considerations leading to the non-speciality of the invertible sheaves on $\tilde{X}_0$ involved in the above proof, and the computation of the dimensions of their spaces of global sections with the Riemann–Roch Formula.

(4.11) Lemma. *Let p be any point of the partial normalization $\tilde{X}_0$, and*

$$D = \sum_{1 \leqslant j \leqslant \alpha^L_i} i \cdot q^L_{i,j} + \sum_{1 \leqslant j \leqslant \alpha'_i} i \cdot q'_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} (i-1) \cdot r_{i,j},$$

with the notation as in the proof of Proposition (4.10) above. The two line bundles $N_\phi(-D)$ and $N_\phi(-D-p)$ on $\tilde{X}_0$ are non-special.

Proof. Let $\tilde{C}$ and $\tilde{L}$ be the proper transforms in $\tilde{X}_0$ of C and L respectively. Since $N_\phi = \omega_{\tilde{X}_0} \otimes \phi^* \omega^{-1}_{\mathbf{P}^2}$ as we have seen above, one has

$$\omega_{\tilde{X}_0} \otimes \big(N_\phi(-D)\big)^{-1} = \phi^* \mathcal{O}_{\mathbf{P}^2}(-3) \otimes \mathcal{O}_{\tilde{X}_0}(D).$$

The degree of the restriction of this line bundle to $\tilde{C}$ has degree

$$\begin{aligned} -3(d-1) + I\alpha' + \big(I\beta - |\beta|\big) &\leqslant -3(d-1) + I\alpha' + I\beta' \\ &= -2(d-1) < 0, \end{aligned}$$

hence every global section of $\omega_{\tilde{X}_0} \otimes N_\phi(-D)^{-1}$ vanishes identically on $\tilde{C}$. One proves similarly that the same holds for $\omega_{\tilde{X}_0} \otimes N_\phi(-D-p)^{-1}$, noting that it restricts on $\tilde{C}$ to a line bundle of degree $-2(d-1)+\varepsilon$, where ε equals 0 or 1 depending on whether p sits on $\tilde{C}$ or not.

Thus, every global section of $\omega_{\tilde{X}_0} \otimes N_\phi(-D)^{-1}$ restricts on $\tilde{L}$ to a global section of the line bundle

$$\phi^* \mathcal{O}_{\mathbf{P}^2}(-3) \otimes \mathcal{O}_{\tilde{L}}(D|_{\tilde{L}}) \otimes \mathcal{O}_{\tilde{L}}(-\tilde{C}|_{\tilde{L}}),$$

which has degree

$$\begin{aligned} -3 + I\alpha^L - I\beta^{C \cap L} &= -3 + I(\alpha - \alpha') - I(\beta' - \beta) \\ &= -3 + (I\alpha + I\beta) - (I\alpha' + I\beta') \\ &= -3 + d - (d-1) = -2. \end{aligned}$$

Therefore every global section of $\omega_{\tilde{X}_0} \otimes N_\phi(-D)^{-1}$ vanishes identically on $\tilde{L}$ as well, hence

$$h^0(\tilde{X}_0, \omega_{\tilde{X}_0} \otimes N_\phi(-D)^{-1}) = h^1(\tilde{X}_0, N_\phi(-D)) = 0.$$

One concludes in the same way that every global section of $\omega_{\tilde{X}_0} \otimes N_\phi(-D-p)^{-1}$ vanishes identically, hence $N_\phi(-D-p)$ is non-special. □

(4.12) Lemma. *The following equalities hold:*

$$h^0\big(\tilde{X}_0, N_\phi\big) = 3d + g'' - 1; \tag{4.12.1}$$

$$h^0\big(\tilde{X}_0, N_\phi(-D)\big) = 2d + g'' - 1 + |\beta|. \tag{4.12.2}$$

Moreover, the line bundle $N_\phi(-D)$ has no base point.

Proof. Both equalities follow from the application of the Riemann–Roch Formula on the curve $\tilde{X}_0$, which has arithmetic genus g'', see (4.8). The line bundle N_ϕ equals $\omega_{\tilde{X}_0} \otimes \phi^*\omega_{\mathbf{P}^2}^{-1}$ which is obviously non-special, hence

$$\begin{aligned} h^0(N_\phi) &= 1 - g'' + \deg\big(\omega_{\tilde{X}_0} \otimes \phi^*\omega_{\mathbf{P}^2}^{-1}\big) \\ &= 1 - g'' + (2g'' - 2) + 3d \\ &= 3d + g'' - 1. \end{aligned}$$

The line bundle $N_\phi(-D)$ is non-special by Lemma (4.11) above, hence

$$\begin{aligned} h^0\big(N_\phi(-D)\big) &= 1 - g'' + \deg(N_\phi) - \deg(D) \\ &= 3d + g'' - 1 - \big(I\alpha^L + I\alpha' + I\beta - |\beta|\big) \\ &= 2d + g'' - 1 + |\beta|, \end{aligned}$$

because

$$I\alpha^L + I\alpha' + I\beta = I\alpha + I\beta = d.$$

Eventually, since for all $p \in \tilde{X}_0$ the line bundle $N_\phi(-D-p)$ is non-special as well, we have

$$h^0\big(\tilde{X}_0, N_\phi(-D-p)\big) = h^0\big(\tilde{X}_0, N_\phi(-D)\big) - 1,$$

hence not all global sections of $N_\phi(-D)$ vanish at p. □

4.2.2 Deformation of order i tacnodes to $i-1$ nodes

Here we perform the second step of the proof of part b) of Theorem (4.1), as described in (4.6) above. We assume that $\beta' > \beta$, for otherwise there is nothing left to do in the second step (see Remark (4.21) below for what happens in this case).

We consider the map

$$F_\Lambda : W_\Lambda \to \Delta,$$

from the relaxed Severi variety W_Λ, introduced and studied in the previous Section 4.2.1, to the product Δ of the versal deformation spaces of the tacnodes $s_{i,j}^{C\cap L}$ of $X_0 = C \cup L$, i.e., those tacnodes of X_0 not included in Λ.

We refer to [IV] and [V] for background and the necessary material on deformation spaces of tacnodes and their product. The result we shall need here is [V, (1.1)], which we now recall in a form adapted to our context.

(4.13) The tacnodes $s_{i,j}^{C\cap L}$ are indexed by $\beta'-\beta$, and thus

$$\Delta = \prod_{i\in\mathbf{N}^*} \prod_{1\leqslant j\leqslant \beta'_i-\beta_i} \Delta_{i,j},$$

where each $\Delta_{i,j}$ is the versal deformation space of the i-th order tacnode $s_{i,j}^{C\cap L}$. For all i and j, we let $\Delta_{i,j;i}$ and $\Delta_{i,j;i-1}$ be the subvarieties of $\Delta_{i,j}$ of those deformations of $s_{i,j}^{C\cap L}$ into i and $i-1$ nodes respectively. We let

$$\mathbf{m} = (m_l)_{1\leqslant l\leqslant|\beta'-\beta|} = \big(\underbrace{1,\dots,1}_{\beta'_1-\beta_1},\quad \dots,\quad \underbrace{n,\dots,n}_{\beta'_n-\beta_n}\big)$$

with $n=\max(\beta'-\beta)$, and consider

$$\Delta_{\mathbf{m}} = \prod_{i\in\mathbf{N}^*} \prod_{1\leqslant j\leqslant \beta'_i-\beta_i} \Delta_{i,j;i}, \quad \text{and} \quad \Delta_{\mathbf{m-1}} = \prod_{i\in\mathbf{N}^*} \prod_{1\leqslant j\leqslant \beta'_i-\beta_i} \Delta_{i,j;i-1}.$$

Each $\Delta_{i,j}$ is a $(2i-1)$-dimensional affine space, with coordinates $a_0,\dots,a_{i-2},b_0,\dots,b_{i-1}$, such that the corresponding versal deformation is the hypersurface

$$y^2+y(x^i+a_{i-2}x^{i-2}+\cdots+a_0)+b_{i-1}x^{i-1}+\cdots+b_0 \tag{4.13.1}$$

in $\mathbf{A}^2_{x,y}\times\mathbf{A}^{2i-1}_{\mathbf{a},\mathbf{b}}$. In these terms, $\Delta_{i,j;i}$ is the dimension $i-1$ affine subspace defined by the equations $b_0=\cdots=b_{i-1}=0$. We consider in addition the hyperplane $\mathcal{H}_{i,j}\subseteq\Delta_{i,j}$ defined by the equation $b_0=0$ in this system of coordinates. Observe that $\mathcal{H}_{i,j}$ parametrizes those deformations of the tacnode defined by $y(y+x^m)=0$ that still pass through the origin in $\mathbf{A}^2_{x,y}$; in other words, these are the deformations maintaining one assigned contact point of order 1 with the line $y=0$. The general member of $\mathcal{H}_{i,j}$ is an irreducible curve. Eventually we let

$$\mathcal{H} = \bigcup_{i,j} \mathcal{H}_{i,j} \ \subseteq\ \Delta.$$

Now, the result is the following. Let $W\subseteq\Delta$ be a smooth subvariety of dimension $\dim(\Delta_{\mathbf{m-1}})+1$, containing $\Delta_{\mathbf{m-1}}$, and such that the tangent space at the origin T_0W is not contained in $\mathcal{H}$. Then, in an étale neighbourhood of the origin $0\in\Delta$,

$$W\cap\Delta_{\mathbf{m-1}} = \Delta_{\mathbf{m}}\cup\Gamma_1\cup\cdots\cup\Gamma_\kappa,$$

where

$$\kappa = \frac{\prod\limits_{1\leqslant l\leqslant|\beta'-\beta|} m_l}{\operatorname{lcm}\limits_{1\leqslant l\leqslant|\beta'-\beta|} m_l} = \frac{I^{\beta'-\beta}}{\operatorname{lcm}(\beta'-\beta)},$$

and $\Gamma_1,\dots,\Gamma_\kappa$ are distinct reduced unibranched curves, each of which has the origin as a point of multiplicity

$$\frac{\operatorname{lcm}\limits_{1\leqslant l\leqslant|\beta'-\beta|} m_l}{\max\limits_{1\leqslant l\leqslant|\beta'-\beta|} m_l} = \frac{\operatorname{lcm}(\beta'-\beta)}{\max(\beta'-\beta)},$$

and has intersection multiplicity $\operatorname{lcm}_{1\leqslant l\leqslant|\beta'-\beta|} m_l = \operatorname{lcm}(\beta'-\beta)$ with $\Delta_{\mathbf{m}}$ at the origin.

We now proceed to prove a series of claims in order to see that the hypotheses of the above statement are verified in our situation, taking $W=F_\Lambda(W_\Lambda)$.

(4.14) Claim. The inverse image $W_{\Lambda,\mathbf{m}} = F_\Lambda^{-1}(\Delta_\mathbf{m})$ is the locus of those members of W_Λ that contain L. It is also the intersection of W_Λ with the hyperplane $p^\perp \subseteq |\mathcal{O}_{\mathbf{P}^2}(d)|$, for general $p \in L$.

Proof. Let us first prove the first assertion. The inverse image $F_\Lambda^{-1}(\Delta_\mathbf{m})$ is the locus in W_Λ along which all the tacnodes $s_{i,j}^{C\cap L}$ of X_0 deform equigenerically. Taking in addition into account the requirements (i), (iii), and (iv) in the definition of the relaxed Severi variety W_Λ, we see that the curves in $F_\Lambda^{-1}(\Delta_\mathbf{m})$ are equigeneric deformations of X_0. Therefore they all come from deformations of the map $\bar\phi : \bar X_0 \to \mathbf{P}^2$ obtained from the total normalization $\nu : \bar X_0 \to X_0$, by the same argument that has been used in the proof of Proposition (4.10), which is essentially Teissier's Résolution Simultanée. Now since the normalization $\bar X_0$ is disconnected, the domain of any deformation of $\bar\phi$ is disconnected as well by the Principle of Connectedness (see, e.g., [4, Exercise III.11.4]). The upshot is that all curves in $F_\Lambda^{-1}(\Delta_\mathbf{m})$ consist of a deformation of C plus a deformation of L.

On the other hand we are assuming here that $\beta' > \beta$, for otherwise the second step of the proof of part b) of Theorem (4.1) is pointless. Since

$$I\alpha + I\beta = d \quad \text{and} \quad I\alpha' + I\beta' = d - 1,$$

it follows that $I(\alpha - \alpha') \geqslant 2$, i.e., $I\alpha^L \geqslant 2$. Now by requirement (ii) in the definition of the relaxed Severi variety, for each member of $F^{-1}(\Delta_\mathbf{m})$, the component that is a deformation of L must satisfy the contact conditions with L encoded in α^L. Therefore it intersects L in at least two points (possibly infinitely near), hence it is L itself, and our first assertion about $F^{-1}(\Delta_\mathbf{m})$ is proved.

As for the second assertion, the first assertion gives us of course the inclusion $F_\Lambda^{-1}(\Delta_\mathbf{m}) \subseteq W_\Lambda \cap p^\perp$ for all $p \in L$. To prove the other inclusion, we note that for all $p \in L$ neither in Ω nor in Λ, any curve in $W_\Lambda \cap p^\perp$ in a neighbourhood of $[X_0]$ has a total of

$$I\alpha + I\beta + 1 = d + 1$$

imposed points of intersection with L whereas it has degree d, so that it must contain L. □

(4.15) Claim. The image of F_Λ contains the locus $\Delta_\mathbf{m}$.

Proof. The projection of $F_\Lambda(W_\Lambda)$ to the factor $\Delta_{i,j}$ of Δ contains $\Delta_{i,j;i}$ if and only if we can find a deformation X_t of X_0 along W_Λ in which a neighbourhood of the tacnode $s_{i,j}^{C\cap L} \in X_0$ deforms to a reducible open curve $X_t^{\circ\prime} + X_t^{\circ\prime\prime}$, such that $X_t^{\circ\prime}$ and $X_t^{\circ\prime\prime}$ intersect transversely in i points that can be chosen arbitrarily on $X_t^{\circ\prime}$.

To see that this holds simultaneously for all tacnodes $s_{i,j}^{C\cap L} \in X_0$, we note that $F_\Lambda^{-1}(\Delta_\mathbf{m})$ identifies as in (4.8) with a local sheet at $[X_0]$ of the Severi variety

$$V^{d-1,\delta'}\big(\alpha', \beta + I(\beta' - \beta)e_1\big)(\Omega') :$$

we have seen that $F_\Lambda^{-1}(\Delta_\mathbf{m})$ consists of those curve in W_Λ having L as an irreducible component, and these curves correspond to equigeneric deformations of C maintaining the contact conditions with L in the points $p'_{i,j} \in \Omega'$ and $s_{i,j} \in \Lambda$, but dropping those in the points encoded in the points $s_{i,j} \notin \Lambda$; the latter dropping so to speak amounts to turn $\beta' - \beta$ into $I(\beta' - \beta)e_1$.

Then the condition we need to verify essentially follows from Proposition (2.2) applied to $V^{d-1,\delta'}\big(\alpha', \beta + I(\beta' - \beta)e_1\big)(\Omega')$. More explicitly, the latter Severi variety contains locally around $[X_0]$ the Severi varieties

$$V^{d-1,\delta'}\big(\alpha' + I(\beta' - \beta)e_1, \beta\big)(\Omega' \cup \Xi_t),$$

where Ξ_t is an arbitrary choice of i points on L around each of the points $s_{i,j} \notin \Lambda$, which is exactly the condition we needed to verify. □

(4.16) Claim. The inverse image $W_{\Lambda,\mathbf{m}} = F_\Lambda^{-1}(\Delta_\mathbf{m})$ has codimension one in W_Λ.

Proof. This follows from the identification given above of $F_\Lambda^{-1}(\Delta_\mathbf{m})$ with a local sheet at $[X_0]$ of the Severi variety $V^{d-1,\delta'}\big(\alpha', \beta + I(\beta'-\beta)e_1\big)$. Explicitly, this identifications tells us the dimension of $F_\Lambda^{-1}(\Delta_\mathbf{m})$ as in (4.8), then

$$\begin{aligned}\dim(W_\Lambda) - \dim\big(F_\Lambda^{-1}(\Delta_\mathbf{m})\big) &= \Big[2d + \tbinom{d-1}{2} - \delta'' - 1 + |\beta|\Big] - \Big[2(d-1) + \tbinom{d-2}{2} - \delta' - 1 + |\beta| + I(\beta'-\beta)\Big] \\ &= 2 + (d-2) - (\delta'' - \delta') - I(\beta'-\beta),\end{aligned}$$

and this equals one by (4.8.1). □

(4.17) Remark. It is instructive, although not strictly necessary for our purposes, to identify the fibre of F_Λ over a general point of $\Delta_\mathbf{m}$. We shall obtain such an identification from the considerations in the proof of Claim (4.15) above, after we observe that a point of $\Delta_{i,j;i} \subseteq \Delta_{i,j}$ corresponds to the choice of i points on one branch of the tacnode at $s_{i,j}$, but only up to reparametrization. This amounts to the fact that, in (4.13.1), the polynomial

$$x^i + a_{i-2}x^{i-2} + \cdots + a_0$$

defining the unions of these i points has no x^{i-1} term; any degree i polynomial in x can be put in this form by a change of variable, which amounts to translating the roots so that they sum up to zero.

Then the fibre of F_Λ over a general point of $\Delta_\mathbf{m}$ identifies with an open subset of the Severi variety

$$V^{d-1,\delta'}\big(\alpha' + \big[I(\beta'-\beta) - |\beta'-\beta|\big]e_1,\ \beta + |\beta'-\beta|e_1\big)(\Omega' \cup \Xi_t^\circ), \tag{4.17.1}$$

with Ξ_t° an arbitrary choice of $i-1$ points on L around each of the points $s_{i,j} \notin \Lambda$. The fibre of F_Λ over the origin of Δ on the other hand identifies with an open subset of

$$V^{d-1,\delta'}(\alpha', \beta')(\Omega'), \tag{4.17.2}$$

which was indeed our starting point. It may be useful for the conclusion in (4.20) below to recall that it has dimension one less than $V^{d,\delta}(\alpha, \beta)$.

The fibre over the general point of $\Delta_\mathbf{m}$ and the fibre over the origin have the same dimension: indeed the Severi varieties (4.17.1) and (4.17.2) have the same dimension

$$2(d-1) + g' - 1 + |\beta| + |\beta'-\beta| \ = \ 2(d-1) + g' - 1 + |\beta'|,$$

with $g' = p_a(d-1) - \delta' = \binom{d-2}{2} - \delta'$.

We leave it to the reader to verify that the latter common dimension of the fibre over the general point of $\Delta_\mathbf{m}$ and the fibre over the origin also equals

$$\dim(W_\Lambda) - \dim(\Delta_\mathbf{m}) - 1;$$

this is yet another straightforward computation after one notes that

$$\dim(\Delta_\mathbf{m}) = \sum\nolimits_{1 \leqslant l \leqslant |\beta'-\beta|} (m_l - 1) = I(\beta'-\beta) - |\beta'-\beta|.$$

(4.18) Claim. Let $dF_\Lambda : T_{[X_0]}W_\Lambda \to T_0\Delta$ be the differential map of F_Λ at the point $[X_0]$. One has

$$\dim\big(dF_\Lambda^{-1}(T_0\Delta_\mathbf{m})\big) = \dim\big(F_\Lambda^{-1}(\Delta_\mathbf{m})\big) = \dim(W_\Lambda) - 1.$$

Proof. The second equality is Claim (4.16) above. For the first one, the key observation is that, under the isomorphism

$$T_{[X_0]}W_\Lambda \cong H^0\big(X_0, \mathcal{I}(d)\big),$$

the inverse image of $T_0\Delta_{\mathbf{m}} \subseteq T_0\Delta$ by the differential dF_Λ at $[X_0]$ is the subspace of sections vanishing identically along $L \subseteq X_0$.

The restriction $\mathcal{I}(d)|_L$ has degree

$$d - I\alpha - I\beta = 0,$$

and is therefore trivial. Now, the restriction map on global sections,

$$H^0\big(X_0, \mathcal{I}(d)\big) \to H^0\big(L, \mathcal{I}(d)|_L\big) \cong \mathbf{C},$$

is non-zero by Lemma (4.12), hence

$$\dim\big(dF_\Lambda^{-1}(T_0\Delta_{\mathbf{m}})\big) = h^0\big(X_0, \mathcal{I}(d)\big) - 1 = \dim(W_\Lambda) - 1,$$

where the last equality comes from the smoothness of W_Λ at $[X_0]$. □

(4.19) Claim. The image of the differential map $dF_\Lambda : T_{[X_0]}W_\Lambda \to T_0\Delta$ is not contained in $\mathcal{H}$.

Proof. From the description of $\mathcal{H}$ in (4.13), the inverse image $(dF_\Lambda)^{-1}(\mathcal{H})$ is the union of the subspaces in $H^0\big(X_0, \mathcal{I}(d)\big)$ of sections vanishing at one point $s_{i,j}^{C\cap L}$. By Lemma (4.12) these are all proper subspaces (in fact, hyperplanes) of $H^0\big(X_0, \mathcal{I}(d)\big)$, and the claim is proved. □

(4.20) Conclusion. We may now finish the proof of part b) of Theorem (4.1). It follows from the description given in Section 3.4 that, locally around the point $[X_0]$, the sum of the local sheets of the Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ corresponding to the choice of Λ is given as the closure in the relaxed Severi variety W_Λ of the inverse image

$$F_\Lambda^{-1}(\Delta_{\mathbf{m-1}} \setminus \Delta_{\mathbf{m}}).$$

Now, it follows from Claim (4.18) that the differential map

$$dF_\Lambda : T_{[X_0]}W_\Lambda \to T_0\Delta$$

has rank $\dim(\Delta_{\mathbf{m}}) + 1$, equal to the dimension of $W = F_\Lambda(W_\Lambda)$. Thus W is smooth at $[X_0]$, of dimension $\dim(\Delta_{\mathbf{m}}) + 1$, and the map F_Λ is a submersion over W, locally around the origin in Δ.

The subvariety $W \subseteq \Delta$ is smooth and, by Claim (4.19), its tangent space is not contained in $\mathcal{H}$. We may thus apply the main result of [V], to the effect that

$$W \cap \Delta_{\mathbf{m-1}} = \Delta_{\mathbf{m}} \cup \Gamma_1 \cup \cdots \cup \Gamma_\kappa,$$

with $\Gamma_1, \ldots, \Gamma_\kappa$ curves as in (4.13) above. The upshot is thus that the local sheets of $V^{d,\delta}(\alpha,\beta)(\Omega)$ corresponding to the choice of Λ are in the number $\kappa = I^{\beta'-\beta}/\mathrm{lcm}(\beta'-\beta)$, each a fibration over a curve Γ_l, $l = 1, \ldots, \kappa$, with the fibre $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$ over the origin. The local sheet corresponding to the curve Γ_l has multiplicity $\mathrm{lcm}(\beta'-\beta)/\max(\beta'-\beta)$ at the generic point of the fibre over the origin, i.e., the generic point of $V^{d-1,\delta'}(\alpha',\beta')(\Omega')$, and intersection multiplicity $\mathrm{lcm}(\beta'-\beta)$ with the inverse image $F_\Lambda^{-1}(\Delta_{\mathbf{m}})$, i.e., with the hyperplane $p^\perp \subseteq |\mathcal{O}_{\mathbf{P}^2}(d)|$ for general $p \in L$ by Claim (4.14), along the fibre over the origin.

Finally, note that $\binom{\beta'}{\beta}$ is the number of possible choices for the subsequence Λ of the sequence of unassigned contact points of C with L. □

The proof of Theorem (4.1) is now over. We end this section with a remark about what happens when $\beta' = \beta$.

As we have already observed, in this case the first step performed in Subsubsection 4.2.1 above is sufficient to prove Theorem (4.1). Indeed, if $\beta = \beta'$, then the relaxed Severi variety W_Λ itself is the unique local sheet of the Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ at $[X_0]$, we already know that it is smooth at $[X_0]$, and there is nothing else to prove.

(4.21) Remark. If $\beta' = \beta$, then all the irreducible components of $V^{d,\delta}(\alpha,\beta)$ having an irreducible component of $V^{d-1,\delta'}(\alpha',\beta')$ in their intersection with the hyperplane $p^\perp \subseteq |\mathcal{O}_{\mathbf{P}^2}(d)|$ parametrize reducible curves.

Indeed, the members of such a component W in a neighourhood of $[X_0] = [C \cup L]$ with $[C] \in V^{d-1,\delta'}(\alpha',\beta')$ correspond to deformations of the total normalization $\bar{X}_0 \to X_0 \subseteq \mathbf{P}^2$, and therefore they are reducible curves because $\bar{X}_0$ is disconnected. This is the same argument we have used in the proof of Claim (4.14) above.

Conversely, if $\beta' > \beta$, and if V is an irreducible component of $V^{d-1,\delta'}(\alpha',\beta')$ that parametrizes irreducible degree $d-1$ curves, then all the irreducible components of $V^{d,\delta}(\alpha,\beta)$ having V in their intersection with $p^\perp$ parametrize irreducible curves.

Indeed, if $\beta' > \beta$, then there is at least one order i tacnode of $X_0 = C \cup L$ that is deformed to $i-1$ nodes only, so that locally around such a point the two local branches of X_0, corresponding respectively to an irreducible component C_1 of C and to L, deform to only one, irreducible, local branch, and thus C_1 and L merge into the same irreducible component of the curves X_t neighbouring X_0.

5 – The formula for irreducible curves

In this section we give and explain the version of Caporaso and Harris' recursion formula enabling the counting of irreducible curves. The formula is given in Theorem (5.4) below. The key observation to pass from the standard formula to the formula for irreducible curves is Remark (4.21) at the end of the previous section. Let us start with an example.

(5.1) Example. We consider the enumeration of quartics with three nodes and two assigned simple contact points with the line L, i.e., curves parametrized by $V^{4,3}(2,2)(\Omega)$. By the recursion formula, we have

$$N^{4,3}(2,2) = N^{4,3}(3,1) + 3N^{3,1}(0,3) + 2N^{3,0}(1,2). \tag{5.1.1}$$

The irreducible components of $V^{4,3}(2,2)(\Omega)$ parametrize either irreducible quartics, or quartics that are the sum of a cubic and a line. We let $N^{4,3}(2,2)_{\mathrm{irr}}$ and $N^{4,3}(2,2)_{3+1}$ be the respective contributions to $N^{4,3}(2,2)$ of irreducible and decomposable quartics. Thus,

$$N^{4,3}(2,2) = N^{4,3}(2,2)_{\mathrm{irr}} + N^{4,3}(2,2)_{3+1}.$$

Formula (5.1.1) splits accordingly into

$$\begin{aligned} N^{4,3}_{\mathrm{irr}}(2,2) &= N^{4,3}_{\mathrm{irr}}(3,1) + 3N^{3,1}_{\mathrm{irr}}(0,3) + 2N^{3,0}_{\mathrm{irr}}(1,2) \qquad \text{and} \\ N^{4,3}_{3+1}(2,2) &= N^{4,3}_{3+1}(3,1) + 3N^{3,1}_{3+1}(0,3) + 2N^{3,0}_{3+1}(1,2). \end{aligned} \tag{5.1.2}$$

Note that all components of $V^{3,1}(0,3)$ and $V^{3,0}(1,2)$ parametrize irreducible cubics, hence the distribution of $N^{3,1}(0,3)$ into $N^{3,1}_{\mathrm{irr}}(0,3)$ and $N^{3,1}_{3+1}(0,3)$ (and similarly that of $N^{3,0}(1,2)$) depends on whether the merging of these irreducible cubics with the line L produces an irreducible or a decomposable quartic. The upshot of Remark (4.21) is that

$$N^{3,0}(1,2) = N^{3,0}(1,2)_{3+1} \qquad \text{and} \qquad N^{3,1}(0,3) = N^{3,1}(0,3)_{\mathrm{irr}},$$

because $\beta' = \beta$ for the former and $\beta' > \beta$ for the latter, in the notation of Remark (4.21). Therefore, (5.1.2) reduces to

$$N^{4,3}_{\mathrm{irr}}(2,2) = N^{4,3}_{\mathrm{irr}}(3,1) + 3N^{3,1}(0,3) \qquad \text{and}$$
$$N^{4,3}_{3+1}(2,2) = N^{4,3}_{3+1}(3,1) + 2N^{3,0}(1,2).$$

Now let us show these formulas in action. First note that

$$N^{3,1}(0,3) = 12 \qquad \text{and} \qquad N^{3,0}(1,2) = 1$$

(the former is the plain number of rational cubics, while the latter is merely the number of cubics passing through one assigned point on L, and passing through eight general crossing points in both cases).

Let us compute the contributions $N^{4,3}(2,2)_{3+1}$ and $N^{4,3}(3,1)_{3+1}$. Since $V^{4,3}(2,2)$ has dimension 9, $N^{4,3}(2,2)_{3+1}$ is the number of quartics decomposed as a cubic plus a line passing through two assigned points p_1, p_2 on L, and through nine general points $a_1, \ldots, a_9$ in the plane. There are only the two following possibilities (recall that in the definition of Severi varieties it is requested that its members should not contain L itself):

(i) the line passes through one of p_1 and p_2 and through one of the a_i's, and the cubic passes through the remaining nine points among p_1, p_2, and $a_1, \ldots, a_9$;
(ii) the line passes through two of the a_i's, and the cubic passes through p_1, p_2 and the seven points remaining among $a_1, \ldots, a_9$.

We thus find

$$N^{4,3}(2,2)_{3+1} = 2 \times 9 + \tbinom{9}{2} = 54.$$

Similarly, $N^{4,3}(3,1)_{3+1}$ counts decomposed quartics passing through three assigned points p_1, p_2, p_3 on L and eight general points $a_1, \ldots, a_8$ on the plane. We have the same possibilities as in the previous case, and thus

$$N^{4,3}(3,1)_{3+1} = 3 \times 8 + \tbinom{8}{2} = 52.$$

We can now observe that, indeed,

$$N^{4,3}(2,2)_{3+1} = 54 = N^{4,3}(3,1)_{3+1} + 2N^{3,0}(1,2) = 52 + 2 \cdot 1.$$

Besides, one has $N^{4,3}(2,2)_{\mathrm{irr}} = N^{4,3}(0,4)_{\mathrm{irr}}$ because putting two crossing points on L doesn't put the crossing points in special position, and therefore $N^{4,3}(2,2)_{\mathrm{irr}} = 620$, see [I, Section 7.2.2], or [1, p. 349]. As a sanity check, note that

$$N^{4,3}(2,2) = 674 = N^{4,3}(2,2)_{\mathrm{irr}} + N^{4,3}(2,2)_{3+1} = 620 + 54$$

(see [1, p. 349] for $N^{4,3}(2,2) = 674$). On the other hand, $N^{4,3}(3,1) = 636$ by [1, p. 349], hence $N^{4,3}(3,1)_{\mathrm{irr}} = 636 - 52 = 584$. We can now observe that, indeed,

$$N^{4,3}_{\mathrm{irr}}(2,2) = 620 = N^{4,3}_{\mathrm{irr}}(3,1) + 3N^{3,1}(0,3) = 584 + 3 \cdot 12.$$

The main point in passing from the standard formula to the formula for irreducible curves is to reconsider Remark (4.21) carefully in order to make it cover all possible cases. The upshot is the following.

(5.2) Let the notation be as in Section 4.2 above. In particular, we assume that $V^{d-1,\delta'}(\alpha',\beta')$ appears in the intersection of $V^{d,\delta}(\alpha,\beta)$ with the hyperplane $p^\perp$, and let V' be an irreducible component of $V^{d-1,\delta'}(\alpha',\beta')$. Let C be a general member of V', and assume that it decomposes as a union of irreducible curves $C_1,\dots,C_k$.

Then, a local sheet of $V^{d,\delta}(\alpha,\beta)$ at the point $[C\cup L]$ parametrizes irreducible curves if and only if the corresponding choice of Λ as in Paragraph (4.5) is such that, for all $l=1,\dots,k$, there is at least one unassigned contact point of C with L on the component C_l that is off Λ.

Indeed, Λ is the subset of cardinality β of the set of unassigned contact points of C with L around which two local branches are maintained in the deformation of $C\cup L$. Moreover, two local branches are maintained at all assigned contact points of C with L. Therefore, the contact points of C with L at which the two local branches deform to a unique irreducible branch are exactly the points $s_{i,j}^{C\cap L}$ in the notation introduced at the beginning of Section 4.2.1, which deform to $i-1$ nodes only; those form the complement of Λ in the set of unassigned contact points of C with L.

The other ingredient of the proof of the recursive formula for irreducible curves is the following basic enumerative theoretic computation.

(5.3) Let V be the sum of irreducible components of $V^{d,\delta}(\alpha,\beta)$ that parametrizes all the curves that decompose as unions of irreducible curves $C_1,\dots,C_k$, such that for all $l=1,\dots,k$, the curve C_l is a general member of an irreducible component of $V^{d_l,\delta_l}(\alpha^l,\beta^l)$. For all l, let $V_l=V_{\mathrm{irr}}^{d_l,\delta_l}(\alpha^l,\beta^l)$ be the sum of irreducible components of $V^{d_l,\delta_l}(\alpha^l,\beta^l)$ that parametrizes all irreducible members of $V^{d_l,\delta_l}(\alpha^l,\beta^l)$. Then, V is the image of the product $\prod_{l=1}^k V_l$ by the Segre map

$$\Sigma:\prod_{l=1}^k|\mathcal{O}_{\mathbf{P}^2}(d_l)|\longrightarrow|\mathcal{O}_{\mathbf{P}^2}(d)|.$$

The degree of V is the intersection number $V\cdot H^{\dim(V)}$, where H denotes the hyperplane class in the Chow ring of $|\mathcal{O}_{\mathbf{P}^2}(d)|$. It may be computed by pulling-back by Σ. One has $\Sigma^*H=H_1+\cdots+H_k$, where H_l is the hyperplane class of $|\mathcal{O}_{\mathbf{P}^2}(d_l)|$ for all l, and then

$$\begin{aligned}\Big(\prod_{l=1}^k V_l\Big)\cdot(\Sigma^*H)^{\dim(V)}&=\Big(\prod_{l=1}^k V_l\Big)\cdot(H_1+\cdots+H_k)^{\dim(V)}\\&=\Big(\prod_{l=1}^k V_l\Big)\cdot\sum_{\nu_1+\cdots+\nu_k=\dim(V)}\binom{\dim V}{\nu_1,\dots,\nu_k}H_1^{\nu_1}\cdots H_k^{\nu_k}\\&=\binom{\dim V}{\dim V_1,\dots,\dim V_k}\prod_{l=1}^k V_l\cdot H_l^{\dim(V_l)},\end{aligned}$$

by computing in the Chow ring of $\prod_{l=1}^k|\mathcal{O}_{\mathbf{P}^2}(d_l)|$, as explained in [2, Section 2.1.4].

The above number equals $V\cdot H^{\dim(V)}$ if the restriction of Σ to $\prod_{l=1}^k V_l$ is birational, which happens if and only if the V_l's are pairwise distinct; otherwise the restriction of Σ has finite degree, equal to the order of the group of permutations of the factors that leaves the product unchanged. The upshot is that, denoting by σ the order of this group (we shall in fact use a refined version of this in the recursion formula),

$$\deg(V)=\frac{1}{\sigma}\binom{\dim V}{\dim V_1,\dots,\dim V_k}\prod_{l=1}^k\deg(V_l).$$

For the application below, recall that the dimensions of generalized Severi varieties is given in Proposition (2.2). Now, fasten your seatbelts, the formula is the following.

(5.4) Theorem. *For all d, δ, α, β as in Corollary (1.5), let $N^{d,\delta}_{\mathrm{irr}}(\alpha,\beta)$ be the number of irreducible members of the generalized Severi variety $V^{d,\delta}(\alpha,\beta)(\Omega)$ passing through $2d+g-1+|\beta|$ general points of the plane ($g = p_a(d) - \delta$), for any general Ω. Then,*

$$N^{d,\delta}_{\mathrm{irr}}(\alpha,\beta) = \sum_{k\geqslant 1:\, \beta_k>0} k \cdot N^{d,\delta}_{\mathrm{irr}}(\alpha+e_k, \beta-e_k)$$
$$+ \sum \left[\frac{1}{\sigma} \binom{2d+g-1+|\beta|}{2d_1+g_1-1+|\beta^1|,\ \ldots,\ 2d_k+g_k-1+|\beta^k|} \cdot \binom{\alpha}{\alpha^1,\ldots,\alpha^k,\alpha-\alpha'} \right.$$
$$\left. \cdot \prod_{l=1}^{k} \binom{\beta^l}{\beta^l-\gamma^l} \cdot \prod_{l=1}^{k} I^{\gamma^l} \cdot \prod_{l=1}^{k} N^{d_l,\delta_l}_{\mathrm{irr}}(\alpha^l,\beta^l) \right],$$

where the second sum is taken over all integers $k > 0$, and over all collections of integers $d_1,\ldots,d_k$, and $\delta_1,\ldots,\delta_k$, and all collections of sequences $\alpha^1,\ldots,\alpha^k$, $\beta^1,\ldots,\beta^k$, and $\gamma^1,\ldots,\gamma^k$, subject to the relations

$$\begin{aligned} & \alpha' = \alpha^1+\cdots+\alpha^k \leqslant \alpha \\ & \beta' = \beta^1+\cdots+\beta^k = \beta+\gamma^1+\cdots+\gamma^k \\ \forall l=1,\ldots,k: \quad & \gamma^l \neq 0 \\ & d_1+\cdots+d_k = d-1 \\ & \delta_1+\cdots+\delta_k = \delta+|\beta'-\beta|-d+1-\sum_{h<l} d_h d_l, \end{aligned}$$

and σ is defined as follows: define an equivalence relation on $\{1,\ldots,k\}$ by declaring $h \sim l$ if

$$d_h = d_l, \quad \delta_h = \delta_l, \quad \alpha^h = \alpha^l, \quad \beta^h = \beta^l, \quad \text{and} \quad \gamma^h = \gamma^l;$$

then σ is the product of the factorials of the cardinalities of the equivalence classes.

In the recursion formula, we have used multinomials for elements of $\underline{\mathbf{N}}$; we define them as

$$\binom{\alpha}{\alpha^1,\ldots,\alpha^k,\alpha^0} = \prod_{i\geqslant 1} \binom{\alpha_i}{\alpha_i^1,\ldots,\alpha_i^k,\alpha_i^0}.$$

The interpretation of the first sum in the formula should be fairly clear; let us decipher the second sum. Each summand corresponds to the contribution of

$$\prod_{l=1}^{k} V^{d_l,\delta_l}_{\mathrm{irr}}(\alpha^l,\beta^l) \ \subseteq\ V^{d_1+\cdots+d_k,\ \delta_1+\cdots+\delta_k+\sum_{h<l} d_h d_l}\left(\textstyle\sum_l \alpha^l, \sum_l \beta^l\right)$$

to $V^{d,\delta}_{\mathrm{irr}}(\alpha,\beta) \cap p^{\perp}$ determined by the sequences $\gamma^1,\ldots,\gamma^k \in \underline{\mathbf{N}}$ in the following way: for all $l = 1,\ldots,k$, $\gamma^l \in \underline{\mathbf{N}}$ is the number of unassigned contact points of the general member C_l of $V^{d_l,\delta_l}_{\mathrm{irr}}(\alpha^l,\beta^l)$ at which an open neighbourhood of $C_l \cup L$ deforms to only one irreducible branch; the deformation of $C_1 \cup \cdots \cup C_k \cup L$ is irreducible if and only if there is at least one such point for all l (see Paragraph (5.2) above), which corresponds to the requirement that $\gamma^l \neq 0$ for all l. Note that the number of nodes of $C_1 \cup \cdots \cup C_k$ is indeed $\delta_1+\cdots+\delta_k+\sum_{h<l} d_h d_l$.

This contribution in fact means the sum of the contributions of all $\prod_{l=1}^{k} V^{d_l,\delta_l}_{\mathrm{irr}}(\alpha^l,\beta^l)(\Omega^l)$, for all sets $\Omega^1,\ldots,\Omega^k$ of cardinalities $\alpha^1,\ldots,\alpha^k \in \underline{\mathbf{N}}$ such that $\Omega = \bigcup_{l=1}^{k} \Omega^l$; the multinomial $\binom{\alpha}{\alpha^1,\ldots,\alpha^k,\alpha-\alpha'}$ corresponds to the number of possible $\Omega^1,\ldots,\Omega^k$. On the other hand, the factor

$\frac{1}{\sigma}$ times the first multinomial is the degree of $\prod_{l=1}^{k} V_{\mathrm{irr}}^{d_l,\delta_l}(\alpha^l, \beta^l, \gamma^l)$ (see Paragraph (5.3)); note that we have added the decoration γ^l (with the meaning explained above), and the "stabilizer term" σ has been changed accordingly in the theorem with respect to Paragraph (5.3).

We leave to the reader the excellent exercise of filling out the details of the proof of Theorem (5.4).

References

Chapters in this Volume

[I] C. Ciliberto and Th. Dedieu, *Limits of nodal curves: generalities and examples.*

[II] Th. Dedieu and E. Sernesi, *Deformations of curves on surfaces.*

[III] Th. Dedieu, *Geometry of logarithmic Severi varieties at a general point.*

[IV] M. Lelli-Chiesa, *Deformations of an m-tacnode into $m-1$ nodes.*

[V] F. Bastianelli, *Products of deformation spaces of tacnodes.*

[A] Th. Dedieu, *Geometry of the deformation space of an m-tacnode into $m-1$ nodes.*

[VII] Th. Dedieu, *The Caporaso–Harris recursion within a degeneration of the plane.*

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces.*

External References

[1] L. Caporaso, J. Harris: *Counting plane curves of any genus*, Invent. Math. **131** (1998), no.2, 345–392.

[2] D. Eisenbud and J. Harris, *3264 and all that*, Cambridge: Cambridge University Press, 2016.

[3] J. Harris, *On the Severi problem*, Invent. math. **84**, 445–461 (1986).

[4] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[5] R. Vakil, *Counting curves on rational surfaces*, Manuscripta Math. **102**, 53–84 (2000).

[6] A. Zahariuc, *The irreducibility of the generalized Severi varieties*, Proc. London Math. Soc. (3) **119**, 1431–1463 (2019).

Lecture VII
The Caporaso–Harris recursion within a degeneration of the plane

by Thomas Dedieu

1 The degeneration procedure 141
2 Two-nodal quartics 142
3 An ancillary enumeration of cubics 146
4 Alternative degeneration procedure 149
References 150

In this text we show on examples how the Caporaso–Harris recursion procedure [1], explained in details in [VI], can be formulated in terms of a degeneration of the projective plane to the union of a projective plane and a rational ruled surface $\mathbf{F}_1$ intersecting transversely along a line.

We formulate the degeneration procedure in a general framework in Section 1, and work it out explicitly on two examples in Sections 2 and 3, namely for the enumerations of 2-nodal plane quartics, and of 1-nodal cubics tangent to a line, respectively. In Section 4 we follow a different degeneration procedure, by way of comparison, in order to enumerate 2-nodal plane quartics.

We assume the reader is familiar with the two chapters [I] and [VI], and will freely use terminology and notation from them.

1 – The degeneration procedure

(1.1) Consider the projective plane $\mathbf{P}^2$ and a line R in it. Let d, δ be integers and $\alpha, \beta \in \underline{\mathbf{N}}$ be sequences such that $d > 0$, $\delta \geqslant 0$, and $I\alpha + I\beta = d$. Let Ω be a (sequence of) set(s) of α points on the line R. Let Z be the union of $2d + p_a(d) - \delta - 1 + |\beta|$ general points in the plane, where $p_a(d) = \frac{1}{2}(d-1)(d-2)$. We want to count the (finitely many) plane curves of degree d with δ nodes, intersecting R at the points of Ω with the multiplicities prescribed by α and at further unassigned points with the multiplicities prescribed by β, and passing through all points of Z. The result is the number $N^{d,\delta}(\alpha, \beta)$ in the notation of [VI] and [1].

(1.2) Let $\mathcal{S}$ be the blow-up of $\mathbf{P}^2 \times \mathbf{A}^1$ along the line $R \times \{0\}$, and let $\pi : \mathcal{S} \to \mathbf{A}^1$ be the composition of the blow-up map with the second projection. For $t \neq 0$, the fibre S_t of π over t is a projective plane, whereas the central fibre S_0 is the union of a projective plane and a rational ruled surface $\mathbf{F}_1$, which we shall call $\mathbf{P}$ and $\mathbf{F}$ respectively, intersecting transversely along a curve E which is a line in $\mathbf{P}$, and the section of the ruling with self-intersection -1 in $\mathbf{F}$; we shall write $S_0 = \mathbf{P} \cup_E \mathbf{F}$.

Let $\mathcal{L}$ be the line bundle on $\mathcal{S}$ obtained by pulling-back $\mathcal{O}_{\mathbf{P}^2}(1)$ successively by the first projection $\mathbf{P}^2 \times \mathbf{A}^1 \to \mathbf{P}^2$ and by the blow-up map. The line bundle $\mathcal{L}$ restricts to $\mathcal{O}_{\mathbf{P}}(H)$ and $\mathcal{O}_{\mathbf{F}}(F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively, where H is the line class and F is the class of the ruling.

Let $\mathcal{R}$ be the proper transform of $R \times \mathbf{A}^1$ in $\mathcal{S}$. For $t \neq 0$, the fibre R_t of $\mathcal{R}$ over $t \in \mathbf{A}^1$ is the line R in $\mathbf{P}^2$, whereas the central fibre R_0 is a section of the ruling of $\mathbf{F}$ that has self-intersection $+1$. Let $\mathcal{W}$ be the proper transform of $\Omega \times \mathbf{A}^1$, and denote by Ω_0 its central fibre: this is merely Ω seen on the section $R_0 \cong \mathbf{P}^1$ of $\mathbf{F}$.

In order to perform the Caporaso–Harris degeneration procedure, we let the set of points Z degenerate as follows. Let p be one point of Z, and consider a general section σ_p of $\mathbf{P}^2 \times \mathbf{A}^1 \to \mathbf{A}^1$ which intersects $R \times \{0\}$, and whose fibre over $1 \in \mathbf{A}^1$ coincides with p. Then we consider the proper transform $\mathcal{Z}$ in $\mathcal{S}$ of $(Z - p) \times \mathbf{A}^1 + \sigma_p$; its fibre over $1 \in \mathbf{A}^1$ is $Z \subseteq \mathbf{P}^2$, whereas its central fibre is the sum of $Z - p \subseteq \mathbf{P}$ and a general point p_0 of $\mathbf{F}$.

(1.3) We consider the family of surfaces $\pi : \mathcal{S} \to \mathbf{A}^1$, equipped with the sheaf $\mathcal{L}^{\otimes d}$. It provides a degeneration to $S_0 = \mathbf{P} \cup_E \mathbf{F}$ of the linear system $|dH|$ on $\mathbf{P}^2$. Moreover, $\mathcal{R}$ provides a degeneration of the line $R \subseteq \mathbf{P}^2$, and $\mathcal{W}$ and $\mathcal{Z}$ provide degenerations of Ω and Z respectively. We want to solve the enumerative problem stated in (1.1) by considering its limit within this degeneration.

Let $\mathcal{I}_{\mathcal{Z}/\mathcal{S}}$ be the ideal sheaf defining $\mathcal{Z}$ in $\mathcal{S}$, and let $\mathcal{I}_{\mathcal{W}}$ be the ideal sheaf encoding the contact conditions with $\mathcal{R}$ at $\mathcal{W}$ corresponding to $\alpha \in \underline{\mathbf{N}}$. It follows from the results in [VI], which are taken from [1], that the pair $(\mathcal{S}/\mathbf{A}^1, \mathcal{L}^{\otimes d} \otimes \mathcal{I}_{\mathcal{Z}/\mathcal{S}} \otimes \mathcal{I}_{\mathcal{W}})$ is δ-well behaved, in the terminology of [I, Section 5]. Thus, our enumerative problem stated in (1.1) can be solved by computing the regular part of the limit Severi variety

$$\mathfrak{V}_\delta(\mathcal{S}/\mathbf{A}^1, \mathcal{L}^{\otimes d} \otimes \mathcal{I}_{\mathcal{Z}/\mathcal{S}} \otimes \mathcal{I}_{\mathcal{W}});$$

in other words, the number we are looking for equals the number of curves in the linear systems on $\mathbf{P} \cup \mathbf{F}$ corresponding to all possible twists of $\mathcal{L}^{\otimes d}$, passing through $Z - p$ on $\mathbf{P}$ and through p_0 on $\mathbf{F}$, with contact conditions with R_0 determined by $\alpha \in \underline{\mathbf{N}}$ at the points Ω_0 and by $\beta \in \underline{\mathbf{N}}$ at unassigned points, and with $\delta_{\mathbf{P}}$ nodes on $\mathbf{P}$, $\delta_{\mathbf{F}}$ nodes on $\mathbf{F}$, and τ tacnodes along $E = \mathbf{P} \cap \mathbf{F}$ ($\tau \in \underline{\mathbf{N}}$), such that

$$\delta = \delta_{\mathbf{P}} + \delta_{\mathbf{F}} + \nu(\tau) \tag{1.3.1}$$

with $\nu(\tau) = \sum_{m \geqslant 2} \tau_m \, (m-1)$; it is understood that these curves are counted with the multiplicity $\mu(\tau) = \prod_{m \geqslant 2} m^{\tau_m}$.[1]

The framework of [I] does not allow to formally take in consideration the contact conditions at unprescribed points; however this appendix is intended to be illustrative, so I will remain informal. Moreover, the framework of [I] would require to have an invertible sheaf instead of $\mathcal{L}^{\otimes d} \otimes \mathcal{I}_{\mathcal{Z}/\mathcal{S}} \otimes \mathcal{I}_{\mathcal{W}}$; this can be easily remedied by considering a suitable blow-up of $\mathcal{S}$, but I prefer not to do it here.

2 – Two-nodal quartics

In this section we enumerate 2-nodal plane quartics by repeatedly applying the procedure described in Section 1.

[1] Beware the difference in notation between $\alpha = (\alpha_m)_{m \geqslant 1}$ and $\beta = (\beta_m)_{m \geqslant 1}$ on the one hand, and $\tau = (\tau_m)_{m \geqslant 2}$ on the other hand: while the former encode all contact points with R_0, including the transverse ones, the latter only encodes m-tacnodes with $m > 1$. One can extend τ to $\tilde{\tau}$ to put it in α/β style, by setting $\tau_1 = \deg(\mathcal{L}^{\otimes d}(-W)\big|_E) - \sum_{m \geqslant 2} m \, \tau_m$ and $\tilde{\tau} = (\tau_1, \tau)$, where $\mathcal{L}^{\otimes d}(-W)$ is the twist of $\mathcal{L}^{\otimes d}$ under consideration; then, $\nu(\tau) = I\tilde{\tau} - |\tilde{\tau}|$, and $\mu(\tau) = I^{\tilde{\tau}}$.

(2.1) Our initial problem is to count 2-nodal plane quartics passing through 12 general points. In the notation of (1.1), we consider $d = 4$, $\delta = 2$, $\beta = 4$, and $\alpha = 0$. The result is the number $N^{4,2} = N^{4,2}(0, 4)$.

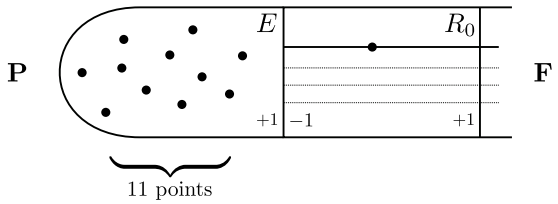
(2.2) We will let our 12 points degenerate one by one to general points on $\mathbf{F}$, and we will need to consider the restriction to $\mathbf{P} \cup \mathbf{F}$ of all possible twists of $\mathcal{L}^{\otimes 4}$. The line bundle $\mathcal{L}^{\otimes 4}$ itself restricts to $\mathcal{O}_{\mathbf{P}}(4H)$ and $\mathcal{O}_{\mathbf{F}}(4F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively. Thus, members of the corresponding linear system on S_0 consist of a plane quartic on $\mathbf{P}$ and four fibres of the ruling of $\mathbf{F}$, which have to match along their intersections with $E = \mathbf{P} \cap \mathbf{F}$, which consist of four points.

The only other relevant twist will be $\mathcal{L}^{\otimes 4}(-\mathbf{F})$, which restricts to $\mathcal{O}_{\mathbf{P}}(3H)$ and $\mathcal{O}_{\mathbf{F}}(E + 4F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively. The linear systems $|3H|_{\mathbf{P}}$ and $|E + 4F|_{\mathbf{F}}$ have dimensions 9 and 8 respectively. Members of the linear system on S_0 corresponding to $\mathcal{L}^{\otimes 4}(-\mathbf{F})$ consist of a member of $|3H|_{\mathbf{P}}$ and a member of $|E+4F|_{\mathbf{F}}$ which match along their intersections with E, which consist of three points; the dimension of the linear system on S_0 is therefore $9+8-3 = 14$ (both $|3H|_{\mathbf{P}}$ and $|E + 4F|_{\mathbf{F}}$ restrict to the complete linear system $|3H|_E$ on $E \simeq \mathbf{P}^1$), equal to that of the linear system of all plane quartics.

To rule out the other twists, we will use (i) that $\mathcal{L}^{\otimes 4}(-a\mathbf{F})$ with $a > 1$ restricts to $\mathcal{O}_{\mathbf{P}^2}(4-a)$ on $\mathbf{P}^2$, hence the corresponding curves on $\mathbf{P}$ have degree at most 2, and (ii) that $\mathcal{L}^{\otimes 4}(a\mathbf{F})$ with $a > 0$ restricts to $\mathcal{O}_{\mathbf{F}_1}(-aE + 4F)$ on $\mathbf{F}$, which is not effective, so that it is already clear that the latter twists will never give any contribution to our enumerations.

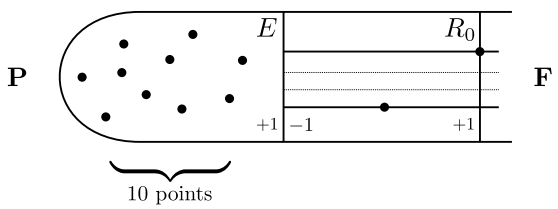
Now, let us start the enumeration. The relations we get at each step are all spelled out in Paragraph (2.8).

(2.3) $N^{4,2}(0, 4)$**.** Let us apply the procedure described in Section 1 to our initial enumerative problem: we end up with 11 general points on $\mathbf{P}$, and 1 general point on $\mathbf{F}$, as indicated on the figure below.



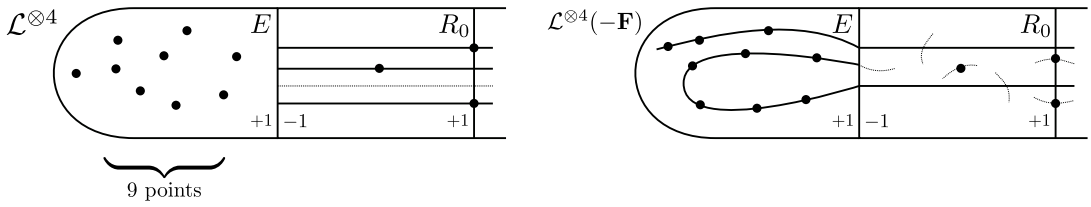
Only the trivial twist is relevant, because there is no curve of degree $d' < 4$ on $\mathbf{P}$ passing through the 11 base points. Thus we are only considering curves consisting of a plane quartic $C_{\mathbf{P}}$ on $\mathbf{P}$ and of four rulings of $\mathbf{F}$. One of these rulings must pass through the base point on $\mathbf{F}$, and has a fixed intersection point with E, which imposes a fixed passing point to $C_{\mathbf{P}}$ on E. On the other hand, the only way to fulfil Relation (1.3.1) with $\delta = 2$ with the curves under consideration is to have $\delta_{\mathbf{P}} = 2$ (and $\delta_{\mathbf{F}} = 0$ and $\tau = 0$). The upshot is that $N^{4,2}(0, 4) = N^{4,2}(1, 3)$.

(2.4) $N^{4,2}(1, 3)$**.** We now apply the procedure of Section 1 to the enumerative problem corresponding to the number $N^{4,2}(1, 3)$. We end up with 10 general points of $\mathbf{P}$, 1 general point of R_0, and 1 general point of $\mathbf{F}$, as pictured below.



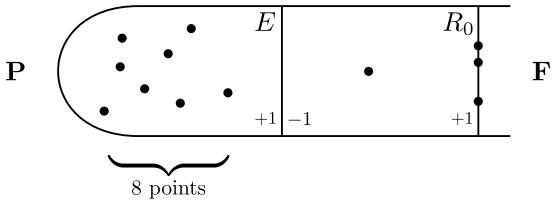
Again, only the trivial twist is relevant, and one must have $\delta_{\mathbf{P}} = 2$. This time, two of the rulings on $\mathbf{F}$ must pass through the two base points on $\mathbf{F}$, and they each impose one fixed passing point on E. The upshot is that $N^{4,2}(1,3) = N^{4,2}(2,2)$.

(2.5) $N^{4,2}(2,2)$**.** We repeat the procedure to compute this number. We end up with 9 general points of $\mathbf{P}$, 2 general point of R_0, and 1 general point of $\mathbf{F}$, as pictured below.



This time the two twists $\mathcal{L}^{\otimes 4}$ and $\mathcal{L}^{\otimes 4}(-\mathbf{F})$ will contribute. The trivial twist gives a contribution of $N^{4,2}(3,1)$ as in the previous cases. The contribution of the other twist comes from curves made of a cubic $C_{\mathbf{P}}$ on $\mathbf{P}$ and a curve $C_{\mathbf{F}}$ on $\mathbf{F}$ linearly equivalent to $E + 4F$. The cubic $C_{\mathbf{P}}$ is uniquely determined by the 9 base points on $\mathbf{P}$ (if one will, $N^{3,0}(0,3) = 1$), it is smooth and fixes 3 pairwise distinct passing points on E for $C_{\mathbf{F}}$. Therefore, the curve $C_{\mathbf{F}}$ must be 2-nodal. Since $C_{\mathbf{F}} \cdot F = 1$, this implies that it must be of the form $F_1 + F_2 + C'$ with F_1 and F_2 two rulings, and $C' \sim E + 2F$. The two rulings must take up two of the fixed points on E, which gives $\binom{3}{2}$ possibilities, and then C' is uniquely determined by the third point on E and the base points on $\mathbf{F}$, as the linear system $|E + 2F|$ has dimension 4.

(2.6) $N^{4,2}(3,1)$**.** We repeat the procedure, and end up with 8 general points of $\mathbf{P}$, 3 general point of R_0, and 1 general point of $\mathbf{F}$, as pictured below.



The trivial twist $\mathcal{L}^{\otimes 4}$ contributes by $N^{4,2}(4,0)$ as in the previous cases, and this time the twist $\mathcal{L}^{\otimes 4}(-\mathbf{F})$ contributes in several different ways: on the $\mathbf{P}$ side we have cubics through the 8 base points, which thus move in a pencil, and cut out a g^1_3 on E, while on the $\mathbf{F}$ side we have curves in $|E + 4F|$ through the $3 + 1$ base points, which impose independent linear conditions, so that we end up with a 4-dimensional linear system. Therefore we have the following three possibilities to fulfil (1.3.1) with $\delta = 2$.

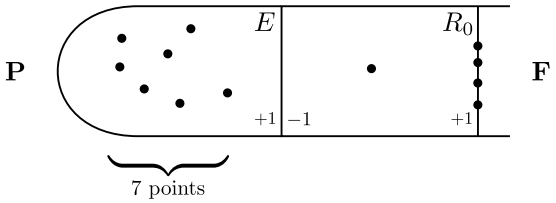
(a) $\delta_{\mathbf{P}} = \delta_{\mathbf{F}} = 1$. The number of 1-nodal cubics through the 8 points on $\mathbf{P}$ is $N^{3,1}(0,3)$, and each of these fixes 3 points on E. In turn, the curve on $\mathbf{F}$ has to split as $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E + 3F$ in order to be 1-nodal; moreover, in order to match with the curve on $\mathbf{P}$, the ruling F_1 must pass through one of the 3 fixed points on E. Since $|E + 3F|$ has dimension 6, the curve C_1 is then uniquely determined by the two other fixed points on E and the $3 + 1$ base points on $\mathbf{F}$. The number of curves of this kind is thus $\binom{3}{1} N^{3,1}(0,3)$.

(b) $\tau_2 = 1$ (i.e., one 2-tacnode) and $\delta_{\mathbf{F}} = 1$. The number of cubics through the 8 points on $\mathbf{P}$ that are tangent to E is $N^{3,0}(0,[1,1])$ (this number is the degree of the dual surface of a twisted cubic, which is 4). The curve on $\mathbf{F}$ has to split as $F_1 + C'$ as in the previous case, and this time the ruling F_1 can only pass through the simple point of the divisor fixed on E by the curve on $\mathbf{P}$. Then, the curve C_1 is uniquely determined by its passing through the double point

of the divisor on E and the $3+1$ base points on $\mathbf{F}$. The number of curves of this kind is thus $N^{3,0}(0,[1,1])$, to be counted with multiplicity 2 because of the tacnodal contribution in (1.3.1).

(c) $\delta_{\mathbf{F}}=2$. In this case the curve on $\mathbf{F}$ has to split as F_1+F_2+C' with $F_1,F_2\sim F$ and $C'\sim E+2F$. One of F_1 and F_2, say F_1, must pass through one of the three base points on R_0, and C' must pass through the other two: this gives $\binom{3}{1}$ choices. Then F_1 fixes a point on E, which determines a unique cubic through the 8 base points on $\mathbf{P}$ (which we can formulate as $N^{3,0}(1,2)=1$), which fixes two other points on E. The ruling F_2 must pass through one of them, which gives $\binom{2}{1}$ possibilities, after what the curve C' is uniquely determined by the third point on E, the two remaining points on R_0, and the general point on $\mathbf{F}$. The number of curves of this kind is thus $\binom{3}{1}\binom{2}{1}N^{3,0}(1,2)$.

(2.7) $N^{4,2}(4,0)$**.** We repeat the procedure, and end up with 7 general points of $\mathbf{P}$, 4 general point of R_0, and 1 general point of $\mathbf{F}$, as pictured below.



The trivial twist $\mathcal{L}^{\otimes 4}$ no longer contributes, because there is no curve in $\mathbf{F}$ linearly equivalent to $4F$ and passing through the $4+1$ base points on F. This is thus the point in the recursion where "contributions of degree 4 curves have been entirely exhausted.". For the contribution of the twist $\mathcal{L}^{\otimes 4}(-\mathbf{F})$, we have the following possibilities.

(a) $\delta_{\mathbf{P}}=2$. The number of 2-nodal cubics through the 7 points on $\mathbf{P}$ is $N^{3,2}(0,3)$ (which equals $\binom{7}{2}$ because all such cubics are decomposed as a conic plus a line). For each such cubic, there is a unique matching curve in $|E+4F|$ on $\mathbf{F}$ passing through the $4+1$ base points.

(b) $\delta_{\mathbf{P}}=\delta_{\mathbf{F}}=1$. The curve on $\mathbf{F}$ must split as F_1+C', with $F_1\sim F$ and $C'\sim E+3F$. The ruling F_1 must pass through one of the base points on R_0 (which gives 4 possibilities), and therefore fixes a point on E. Then, there is a finite number $N^{3,1}(1,2)$ of 1-nodal cubics on $\mathbf{P}$ passing through this point and the 7 base points on $\mathbf{P}$ (the number $N^{3,1}(1,2)$ is easily seen to equal $N^{3,1}(0,3)$, which is 12). In turn, each of these fixes two new points on E, and the curve C' is uniquely determined by these two points and the remaining $3+1$ base points on $\mathbf{F}$. Curves of this kind are thus in the number $\binom{4}{1}N^{3,1}(1,2)$.

(c) $\delta_{\mathbf{F}}=2$. The curve on $\mathbf{F}$ must split as F_1+F_2+C', with $F_1,F_2\sim F$ and $C'\sim E+2F$, and the two rulings F_1 and F_2 each must pass through one of the 4 base points on R_0, which gives $\binom{4}{2}$ possibilities. These two rulings then fix a unique cubic on $\mathbf{P}$ through the 7 points (which we can phrase as $N^{3,0}(2,1)=1$), which in turn fixes a third point on E. The curve C' is then uniquely determined by the latter point and the remaining $2+1$ base points on $\mathbf{F}$. We thus find $\binom{4}{2}N^{3,0}(2,1)$ curves.

(d) $\delta_{\mathbf{P}}=1$ and $\tau_2=1$ (i.e., one 2-tacnode). There is a finite number of possible curves on $\mathbf{P}$, in the number $N^{3,1}(0,[1,1])$, and each of these determines a unique matching curve in $|E+4F|$ on $\mathbf{F}$. These curves count with multiplicity 2.

(e) $\delta_{\mathbf{F}}=1$ and $\tau_2=1$. The curve on $\mathbf{F}$ must split as F_1+C', with $F_1\sim F$ and $C'\sim E+3F$. The ruling F_1 must pass through one of the base points on R_0 (which gives 4 possibilities), and therefore fixes a point on E. Then, there is a pencil of cubics on $\mathbf{P}$ passing through this point and the 7 base points on $\mathbf{P}$; it cuts out (residually) a g^1_2 on E, which has two members made of a double point, meaning that there are two cubics in the pencil that are tangent to E: in other words, $N^{3,0}(1,[0,1])=2$. Finally, for each such cubic there is a unique matching curve C'

which also pass through the $3+1$ remaining base points on $\mathbf{F}$. Thus we find $\binom{4}{1}N^{3,0}(1,[0,1])$ curves; they count with multiplicity 2.

(f) $\tau_3 = 1$. There is a finite number of possible curves on $\mathbf{P}$, which is $N^{3,0}(0,[0,0,1])$. This number equals the number of divisors of the form $3q$ in the g^2_3 cut out on E by the 2-dimensional linear system of cubics through the 7 base points on $\mathbf{P}$, which is 3 (this is the number of flexes of a 1-nodal cubic, see [XII, Section 1.3]). For each such cubic there is a unique matching curve in $|E+4F|$ on $\mathbf{F}$ which also passes through the $4+1$ base points. We thus find $N^{3,0}(0,[0,0,1])$ curves, and they count with multiplicity 3.

(2.8) Summary and conclusion. We have applied the degeneration procedure of Section 1 five times, and after that we are reduced to enumerations of curves of degrees $d' < 4$. The relations we have obtained are the following:

$$\begin{aligned}
N^{4,2}(0,4) &= N^{4,2}(1,3)\\
&= N^{4,2}(2,2)\\
&= N^{4,2}(3,1) + \tbinom{3}{2}\underbrace{N^{3,0}(0,3)}_{=1}\\
N^{4,2}(3,1) &= N^{4,2}(4,0) + \tbinom{3}{1}\underbrace{N^{3,1}(0,3)}_{=12} + 2\cdot\underbrace{N^{3,0}(0,[1,1])}_{=4} + \tbinom{3}{1}\tbinom{2}{1}\underbrace{N^{3,0}(1,2)}_{=1}\\
N^{4,2}(4,0) &= \underbrace{N^{3,2}(0,3)}_{=21} + \tbinom{4}{1}\underbrace{N^{3,1}(1,2)}_{=12} + \tbinom{4}{2}\underbrace{N^{3,0}(2,1)}_{=1} + 2\cdot\underbrace{N^{3,1}(0,[1,1])}_{=36}\\
&\qquad + 2\cdot\tbinom{4}{1}\underbrace{N^{3,0}(1,[0,1])}_{=2} + 3\cdot\underbrace{N^{3,0}(0,[0,0,1])}_{=3}.
\end{aligned}$$

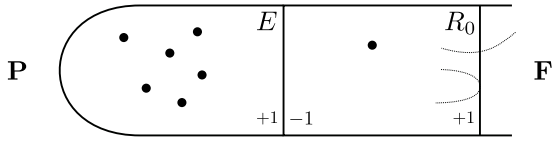
With the (possible) exception of $N^{3,1}(0,[1,1])$, which is computed in Section 3 below, the numbers of cubic curves appearing in the relations above can all be more or less elementarily computed; hints for these computations have been given in the course of the recursive application of the degeneration procedure. In the upshot, we get

$$\begin{aligned}
N^{4,2}(4,0) &= 21+48+6+72+16+9 = 172\\
N^{4,2}(3,1) &= 172+36+8+6 = 222\\
N^{4,2}(0,4) = N^{4,2}(1,3) &= N^{4,2}(2,2) = 222+3 = 225.
\end{aligned}$$

3 – An ancillary enumeration of cubics

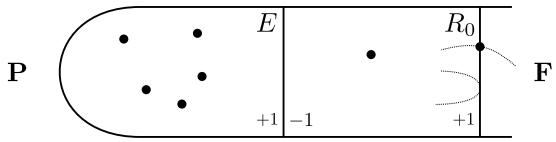
In this section we compute the number of 1-nodal plane cubics tangent to a line; the result is the number $N^{3,1}(0,[1,1])$. This is necessary for the enumeration of 2-nodal quartics in the previous section, and will serve as an additional illustration of the recursive degeneration procedure. We will need to consider the line bundles $\mathcal{L}^{\otimes 3}$, which restricts to $\mathcal{O}_{\mathbf{P}}(3H)$ and $\mathcal{O}_{\mathbf{F}}(3F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively, and $\mathcal{L}^{\otimes 3}(-\mathbf{F})$, which restricts to $\mathcal{O}_{\mathbf{P}}(2H)$ and $\mathcal{O}_{\mathbf{F}}(E+3F)$ on $\mathbf{P}$ and $\mathbf{F}$ respectively. The generalized Severi variety $V^{3,1}(0,[1,1])$ has dimension 7, so we start with 7 base points.

(3.1) $N^{3,1}(0,[1,1])$**.** We apply the degeneration procedure of Section 1, which gives 6 general base points on $\mathbf{P}$, and one general base point on $\mathbf{F}$.



Only the trivial twist contributes, because there is no conic on $\mathbf{P}$ through the 6 base points, and we must have $\delta_{\mathbf{P}} = 1$ in order to fulfil (1.3.1) with $\delta = 1$. To have the correct intersection pattern with R_0, the curve on $\mathbf{F}$ must be of the form $F_1 + 2F_2$ with $F_1, F_2 \sim F$. Then, either F_1 passes through the base point on $\mathbf{F}$, after what there are $N^{3,1}(1, [0, 1])$ matching 1-nodal cubics through the 6 base points on $\mathbf{P}$, or F_2 passes through the base point on $\mathbf{F}$, and then there are $N^{3,1}([0, 1], 1)$ matching 1-nodal cubics through the 6 base points on $\mathbf{P}$; it follows from the analysis in [1], reported on in [VI], that the latter curves count with multiplicity 2.

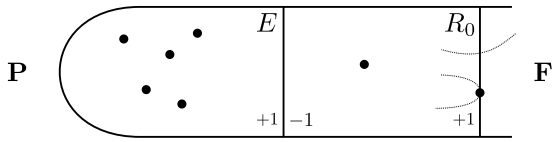
(3.2) $N^{3,1}(1, [0, 1])$**.** Applying the degeneration procedure, we get 5 general base points on $\mathbf{P}$, one general base point on R_0, and one general point on $\mathbf{F}$.



A priori, the two twists $\mathcal{L}^{\otimes 3}$ and $\mathcal{L}^{\otimes 3}(-\mathbf{F})$ contribute in this case. In the contribution of the trivial twist $\mathcal{L}^{\otimes 3}$, the curve on $\mathbf{F}$ must once again be of the form $F_1 + 2F_2$ with $F_1, F_2 \sim F$ in order to meet the contact condition with R_0. Then, F_1 must past through the base point on R_0, and F_2 through the general base point on $\mathbf{F}$. There are then $N^{3,1}([1, 1], 0)$ matching curves on $\mathbf{P}$, and the curves we find in this way count with multiplicity 2 because of the assignement of the tangency point with R_0.

In the contribution of the twist $\mathcal{L}^{\otimes 3}(-\mathbf{F})$, we find a unique conic through the five base points on $\mathbf{P}$. It is smooth and fixes two distinct points on E. The curve on $\mathbf{F}$ must be decomposed as $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E + 2F$ in order to be 1-nodal. The ruling F_1 must pass through one of the two points on E, and therefore does not pass through the base point on R_0. It follows that it is impossible to choose C' so that the contact conditions with R_0 are fulfilled (it is possible to choose C' tangent to R_0 at the base point, but this does not give an admissible curve, see the comment on the definition of logarithmic Severi varieties in [III, (1.6)]; this curve will contribute in (3.3) however).

(3.3) $N^{3,1}([0, 1], 1)$**.** We apply the degeneration procedure, and end up with 5 general base points on $\mathbf{P}$, one general base point on $\mathbf{F}$, and a fixed tangency point with R_0.

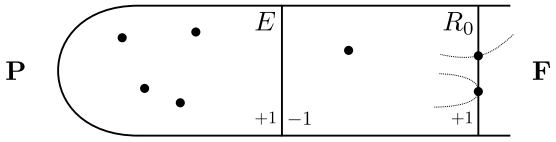


We will have contributions from the two twists $\mathcal{L}^{\otimes 3}$ and $\mathcal{L}^{\otimes 3}(-\mathbf{F})$. For the trivial twist $\mathcal{L}^{\otimes 3}$, the curve on $\mathbf{F}$ must be of the form $F_1 + 2F_2$ with F_1 and F_2 two rulings, with F_2 passing through the fixed point on R_0 and F_1 through the general base point on $\mathbf{F}$. This curve fixes a tangency point and a simple passing point on E, and the possible cubic curves on $\mathbf{P}$ are in the number $N^{3,1}([1, 1], 0)$.

In the situation defined by the twist $\mathcal{L}^{\otimes 3}(-\mathbf{F})$ the curve on $\mathbf{P}$ is a conic, and there is a unique one through the five base points (i.e., $N^{2,0}(0, 2) = 1$), which fixes two points on E. In order to

be 1-nodal, the curve on $\mathbf{F}$ must be decomposed as $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E + 2F$. The curve F_1 must pass through one of two points on E, which gives two possibilities, and then C' must pass through the second point on E, through the base point which is general on $\mathbf{F}$, and be tangent to R_0 at the prescribed point: these are 4 independent linear conditions, hence the curve C' is uniquely determined. The contribution of $\mathcal{L}^{\otimes 3}(-\mathbf{F})$ is thus by $\binom{2}{1} N^{2,0}(0,2)$.

(3.4) $N^{3,1}([1,1],0)$**.** We apply the degeneration procedure and get 4 general base points on $\mathbf{P}$, one general base point on $\mathbf{F}$, and a fixed tangency plus a fixed simple intersection point with R_0.



This time only the twist $\mathcal{L}^{\otimes 3}(-\mathbf{F})$ may contribute, because it is impossible for a curve in $|3F|$ to fulfil all the conditions on $\mathbf{F}$. This is therefore the point in the recursion procedure where there only remain numbers $N^{d',\delta'}(\alpha',\beta')$ with $d' < 3$. Thus, we have a conic on $\mathbf{P}$, which moves in a pencil determined by the four base points. We have the following possibilities to fulfil (1.3.1) with $\delta = 1$.

(a) $\delta_{\mathbf{P}} = 1$. There are $N^{2,1}(0,2)$ 1-nodal conics through the 4 base points on $\mathbf{P}$, and each fixes two points on E ($N^{2,1}(0,2) = 3$ because all such conics are decomposed in two lines). Then the curve on $\mathbf{F}$ must satisfy 6 independent linear conditions, and is thus uniquely determined.

(b) $\delta_{\mathbf{F}} = 1$. Then the curve on $\mathbf{F}$ must be of the form $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E + 2F$. The ruling F_1 must pass through the fixed simple intersection point with R_0. This fixes a point on E, which determines a unique conic on $\mathbf{P}$ through the 4 base points ($N^{2,0}(1,1) = 1$), which in turn fixes a second point on E. Then the curve C' must pass through this point, through the general base point on $\mathbf{F}$, and be tangent with R_0 at the prescribed point: it is thus uniquely determined.

(c) $\tau_2 = 1$. The curve on $\mathbf{P}$ must be a conic tangent to E: these are in the number $N^{2,0}(0,[0,1])$ (which equals 2, the number of double points in the g^1_2 cut out on E). Then the curve on $\mathbf{F}$ has an imposed tangency with E at a prescribed point, hence it must in total satisfy 6 independent linear conditions, and thus it is uniquely determined. These curve counts with the multiplicity 2.

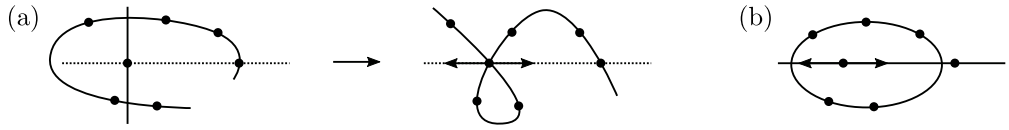
(3.5) Summary and conclusion. In the four above applications of the degeneration procedure, we have obtained the following relations:

$$
\begin{aligned}
N^{3,1}(0,[1,1]) &= N^{3,1}(1,[0,1]) + 2 \cdot N^{3,1}([0,1],1) \\
N^{3,1}(1,[0,1]) &= 2 \cdot N^{3,1}([1,1],0) \\
N^{3,1}([0,1],1) &= N^{3,1}([1,1],0) + \tbinom{2}{1} \underbrace{N^{2,0}(0,2)}_{=1} \\
N^{3,1}([1,1],0) &= \underbrace{N^{2,1}(0,2)}_{=3} + \underbrace{N^{2,0}(1,1)}_{=1} + 2 \cdot \underbrace{N^{2,0}(0,[0,1])}_{=2}
\end{aligned}
\tag{3.5.1}
$$

Putting these identities all together, we obtain $N^{3,1}([1,1],0) = 8$, $N^{3,1}([0,1],1) = 10$, $N^{3,1}(1,[0,1]) = 16$ and, finally, $N^{3,1}(0,[1,1]) = 36$.

(3.6) Remark. The numbers $N^{3,1}([1,1],0)$ and $N^{3,1}([0,1],1)$ may also be computed by appropriately enumerating singular cubics in a pencil, following [XI, Section 2.1], and doing so sheds some light on our degeneration procedure.

For the first number, we consider the pencil of plane cubics defined by 8 base points as follows: 3 of base points on a line R, 2 of which are infinitely near, and 5 general points of $\mathbf{P}^2$. There are 12 singular members (counted with multiplicities) in this pencil, as in any pencil of cubics, and the curves counted by $N^{3,1}([1,1],0)$ are all among them. Some singular members of the pencil must be excluded however, namely (a) the unique member of the pencil with a double point at the point where there are two infinitely near base points, and (b) the member of the pencil made of the line R itself plus the unique conic through the 5 general base points.



Both these curves have two nodes after we blow-up the base points of the pencil, so they contribute each by 2 to the number 12 of singular members in the pencil, and we thus find $N^{3,1}([1,1],0) = 12 - 2 - 2 = 8$.

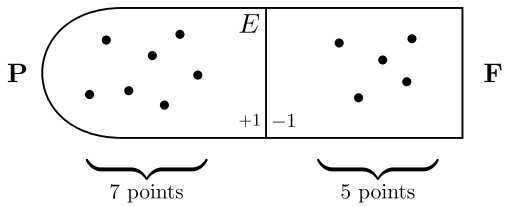
For the second number, we put the 8 base points as follows: 2 infinitely near base points on the line R, and 6 general points of $\mathbf{P}^2$. This time there is no member of the pencil made of the line R and a conic, and only the member as in case (a) above must be excluded. We thus find $N^{3,1}([1,1],0) = 12 - 2 = 10$.

The difference between these two enumerations explains Relation (3.5.1) above.

4 – Alternative degeneration procedure

In this section I present, by way of comparison, an alternative way of enumerating 2-nodal plane quartics by degeneration, which somehow packages all five degenerations in Paragraphs (2.3)–(2.7) in just one degeneration. This is the same degeneration procedure as in [I, Section 7.2.2], where 3-nodal quartics are enumerated, so I will be brief.

We consider always the same degeneration of the projective plane to the transverse union $\mathbf{P} \cup_E \mathbf{F}$, and this time we let 5 of the 12 base points on $\mathbf{P}^2$ degenerate to general points of $\mathbf{F}$.



This prevents the trivial twist $\mathcal{L}^{\otimes 4}$ to give any contribution, and therefore the total number $N^{4,2}$ will be accounted for by the contributions from the twist $\mathcal{L}^{\otimes 4}(-\mathbf{F})$. In this twist, we have a net of plane cubics on $\mathbf{P}$ defined by the 7 general base points on $\mathbf{P}$, which cuts out a g^2_3 on E; on the other hand, the 5 general base points on $\mathbf{F}$ define a 3-dimensional linear subsystem of $|E + 4F|$, which cuts out a g^3_3 on E. We have the following possibilities to fulfil (1.3.1) with $\delta = 2$.

(a) $\delta_{\mathbf{P}} = 2$. There are $N^{3,2}(0,3) = \binom{7}{2}$ two-nodal cubics through the 7 base points on $\mathbf{P}$, and for each of those there is a unique matching member of $|E + 4F|$ through the 5 base points on $\mathbf{F}$.

(b) $\delta_{\mathbf{P}} = \delta_{\mathbf{F}} = 1$. The 1-nodal curve on $\mathbf{F}$ necessarily decompose as $F_1 + C'$ with $F_1 \sim F$ and $C' \sim E + 3F$. There are the two following combinatorial possibilities.

(i) The ruling F_1 passes through 1 base point and C' through the other 4 ($\binom{5}{1} = 5$ possibilities). There are then $N^{3,1}(1,2) = 12$ 1-nodal cubics on $\mathbf{P}$ matching with F_1 along E, and for each of those there is a unique matching curve C'.
(ii) The ruling F_1 moves freely in the pencil $|F|$, while C' passes through the 5 base points and thus moves in a pencil. The divisors cut out on E by curves of this kind form a quadric surface in the g^3_3 cut out by the complete system $|E + 4F|$. It follows that the number of curves made of a curve of this kind and a nodal cubic on $\mathbf{P}$ through the 7 base points is $2 \times N^{3,1}(0,3) = 24$.

(c) $\delta_{\mathbf{F}} = 2$. The 2-nodal curve on $\mathbf{F}$ must decompose as $F_1 + F_2 + C'$ with $F_1, F_2 \sim F$ and $C' \sim E + 2F$, and there are the two following combinatorial possibilities.
(i) The two rulings F_1 and F_2 pass through 1 base point each, and C' moves in the pencil defined by the 3 remaining base points ($\binom{5}{2} = 10$ possibilities). There is a cubic through the 7 base points and matching with F_1 and F_2 ($N^{3,0}(2,1) = 1$), and it cuts out a third point on E. Then, there is a unique curve C' passing through this third point and the remaining 3 base points on $\mathbf{F}$.
(ii) The curves F_1 and C' pass through 1 and 4 base points respectively ($\binom{5}{1}$ possibilities), thus both are fixed, and F_2 moves freely in $|F|$. As in case (i), there is a unique cubic on $\mathbf{P}$ matching with $F_1 + C'$, and in turn a unique ruling F_2 through the third intersection point of this cubic with E.

(d) $\delta_{\mathbf{P}} = 1$ and $\tau_2 = 1$. There are $N^{3,1}(0,[1,1]) = 36$ one-nodal cubics tangent to E, and for each of those there is a unique matching curve on $\mathbf{F}$. These curves count with multiplicity 2.

(e) $\delta_{\mathbf{F}} = 1$ and $\tau_2 = 1$ (with multiplicity 2). The curve on $\mathbf{F}$ decomposes as $F_1 + C'$ with $F_1 \sim F$, and there are the two following possibilities.
(i) The ruling F_1 passes through 1 base point ($\binom{5}{1}$ possibilities). There are then $N^{3,0}(1,[0,1]) = 2$ cubics on $\mathbf{P}$ tangent to E and matching with F_1. Each of those determines a unique curve C'.
(ii) The ruling F_1 moves freely in $|F|$. There are finitely many curves C' in $|E + 3F|$ passing through the 5 base points and tangent to E (in the number 2, which is the number of double points in a g^1_2 on E). For each of those there is a unique matching cubic on $\mathbf{P}$ ($N^{3,0}([0,1],1) = 1$), which cuts out a third point on E and thus fixes F_1.

(f) $\tau_3 = 1$ (with multiplicity 3). There are $N^{3,0}(0,[0,0,1]) = 3$ cubics on $\mathbf{P}$ triply tangent with E at some point, and each of those determines a unique matching curve on $\mathbf{F}$.

These eventually add up as :

$$\begin{array}{ll}
N^{3,2}(0,3) & = 21 \\
+ \binom{5}{1} N^{3,1}(1,2) + \binom{5}{0} \times 2 \times N^{3,1}(0,3) & + 5 \cdot 12 + 2 \cdot 12 \\
+ \binom{5}{2} N^{3,0}(2,1) + \binom{5}{1} N^{3,0}(2,1) & + 10 + 5 \\
+ 2 \cdot N^{3,1}(0,[1,1]) & + 2 \cdot 36 \\
+ 2 \cdot \binom{5}{1} N^{3,0}(1,[0,1]) + 2 \cdot \binom{5}{5} \times 2 \times N^{3,0}([0,1],1) & + 2 \cdot 5 \cdot 2 + 2 \cdot 2 \cdot 1 \\
+ 3 \cdot N^{3,0}(0,[0,0,1]) & + 3 \cdot 3 \\
\hline
N^{4,2}(0,4) & = 225.
\end{array}$$

References

Chapters in this Volume

[I] C. Ciliberto and Th. Dedieu, *Limits of nodal curves: generalities and examples.*

[III] Th. Dedieu, *Geometry of logarithmic Severi varieties at a general point.*

[VI] Th. Dedieu and C. Galati, *Degenerations and deformations of generalized Severi varieties.*

[XI] Th. Dedieu, *Enumerative geometry of $K3$ surfaces.*

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces.*

External References

[1] L. Caporaso, J. Harris: *Counting plane curves of any genus*, Invent. Math. **131** (1998), no.2, 345–392.

Lecture VIII
Proving the existence of singular curves on smooth surfaces by smoothing tacnodes on reducible surfaces

by Concettina Galati

Abstract. Let $\mathcal{X} \to \mathbf{D}$ be any flat projective family of complex surfaces parametrized by the disc $\mathbf{D}$, with smooth total space $\mathcal{X}$, smooth general fiber $\mathcal{X}_t$, and reducible special fiber $\mathcal{X}_0 = \cup X_i$ with normal crossing singularities. Let $C = \cup_i C_i \subseteq \mathcal{X}_0$ be a reducible Cartier divisor with only nodes on the smooth locus of $\mathcal{X}_0$ and tacnodes and nodes on the singular locus E of $\mathcal{X}_0$. We will show how to study deformations of C to nodal curves $C_t \subseteq \mathcal{X}_t$ by using Caporaso and Harris' local analysis of the versal deformation space of an m-tacnode in [1] and [2]. We will also study in less detail deformations of C in $\mathcal{X}$ with A_k-singularities. Results of this paper are known. We especially refer to [3] and [11]. But they are usually proved in literature with some special assumptions on the surfaces $\mathcal{X}_t$. Here we work in full generality. This paper is in particular a generalization of [11, Section 3].

1 Introduction 153
2 Our deformation problem 154
2.1 Some local geometry: the versal deformation space of the singularities of C 156
2.2 Equisingular deformations of C 157
2.3 Image of the versal map 160
3 Global deformations of C 163
References 168

1 – Introduction

Let X be a smooth complex projective surface and let D be a smooth Cartier divisor of genus $p_a(D)$. Describing the singular elements in the linear system $|D|$ defined by D is not in general an easy problem, even if we restrict our interest to the existence of nodal divisors. There are various deformation arguments which may be helpful case by case. For example, if $X = \mathbf{P}^2$, we know that, for any $d \geqslant 3$, there exist degree d irreducible nodal curves of any genus $0 \leqslant g \leqslant \frac{(d-1)(d-2)}{2}$, cf. [17] and [13]. These curves are deformations of a rational nodal plane curve of degree d, obtained as a general projection of a rational normal curve $C_d \subseteq \mathbf{P}^d$ of degree d. Notice that $\frac{(d-1)(d-2)}{2}$ is the number of nodes of a rational nodal plane curve and so it is the maximal number of nodes of an irreducible plane curve of degree d. If X is any smooth complex projective surface and D is any smooth Cartier divisor, it is not in general obvious how to produce irreducible or even reducible curves in $|D|$ with "the maximum allowed number of nodes". One very classical

strategy is to degenerate the surface $X \simeq \mathcal{X}_t$ to a reducible normal crossing surface $\mathcal{X}_0$ and to prove the existence of divisors on $X = \mathcal{X}_t$ with certain singularities as deformations of suitable Cartier divisors on $\mathcal{X}_0$. This method was used in particular by Chen in [3] to prove the existence of irreducible rational nodal curves on a very general K3 surface, in the primitive linear system and in all its multiples. The key idea of Chen's theorem is to construct, on a suitable reducible K3 surface $\mathcal{X}_0 = A \cup B$, a curve with tacnodes at points of the singular locus of $\mathcal{X}_0$ and to study its deformations on a very general K3 surface $\mathcal{X}_t$, by using the analysis of the locus of $(m-1)$-nodal curves in the versal deformation space of the m-tacnode, carried out in [1] and [2], see [IV]. The local analysis in [1] and [2] of versal deformations of tacnodes has later been extended in [11], where the authors provide a method to construct curves with A_k singularities on smooth surfaces, for every $k \geqslant 1$. In particular [11] contains a new proof of the aforementioned Chen's theorem. Results in [11] have later been applied in [4] and [5].

In [11] the authors work in a family of regular surfaces, i.e. in a family of surfaces X satisfying the condition $h^1(\mathcal{O}_X) = 0$. An output of the present paper is that this regularity assumption is unnecessary. Here we provide a generalization of [11, Section 3] using a simpler notation. We consider a flat projective family of complex surfaces $\mathcal{X} \to \mathbf{D}$, parametrized by the disc $\mathbf{D}$, with smooth total space $\mathcal{X}$, smooth general fiber $\mathcal{X}_t$, and reducible special fiber $\mathcal{X}_0 = \cup_{i=1}^r X_i$ with normal crossing singularities. Given a reduced curve $C = \cup_i C_i \subseteq \mathcal{X}_0 = \cup_{i=1}^r X_i$ with only nodes and tacnodes as singularities, we study in details nodal deformations of C in $\mathcal{X}$, see Corollary (2.10) and Theorem (3.1). More generally, in Remark (2.11) and Remark (3.6), we study deformations of C in $\mathcal{X}$ with A_k-singularities, extending all local results in [11, Section 3] to this more general context.

We hope it will be clear to the reader how all results of this paper can be generalized to the study of deformations of reduced curves in a smooth total space with complete intersection singularities, see for example [10].

This paper is divided in four sections. In Section 2 we describe our degeneration argument and we state our deformation problem (Problem (2.2)). We moreover collect several local results from which, in Section 3, we will deduce our main result, Theorem (3.1). The latter proves in particular [I, Proposition (5.5)]; besides, [I] contains many applications of Theorem (3.1).

The degeneration argument which we are going to explain in the next section is useful to the study of many problems. For example, in [6], it has been applied to the study of the number of moduli of Severi varieties of nodal curves on K3 surfaces. Here we will restrict to the existence problem of singular curves on smooth surfaces.

(1.1) Terminology and notation. We will work over $\mathbf{C}$. A curve will be a reduced separated scheme of finite type and dimension 1.

(1.2) Acknowledgments. I would like to thank Thomas Dedieu for reading the preliminar version of this paper.

2 – Our deformation problem

(2.1) Let $\mathcal{X} \to \mathbf{D}$ be a flat projective family of complex surfaces parametrized by the disc $\mathbf{D}$, with smooth total space $\mathcal{X}$, smooth general fiber $\mathcal{X}_t$, and reducible special fiber $\mathcal{X}_0$ with normal crossing singularities, i.e., locally at every point in the analytic topology, $\mathcal{X}_0$ is isomorphic to the closed subset of an open polydisk of $\mathbf{C}^3$ defined by $\{(x_1, x_2, x_3) | \Pi_{i=1}^k x_i = 0\}$, where k may be $1, 2$, or 3, cf. [8]. Thus every irreducible component X_i of $\mathcal{X}_0 = \cup_{i=1}^r X_i$ is smooth, and the singular locus of $\mathcal{X}_0$ consists of the intersection locus of its irreducible components, where $\mathcal{X}_0$ has singularities of multiplicity 2 or 3. We denote by $E = \cup_{1 \leqslant i < j \leqslant r} E_{ij}$ the singular locus of $\mathcal{X}_0$,

where $E_{ij} = X_i \cap X_j$ (possibly empty), and by $E^o \subseteq E \subseteq \mathcal{X}_0$ the set of points of multiplicity exactly 2.

Let $p \in E^o$ be a double point of $\mathcal{X}_0$. Working locally at p, we may assume that $\mathcal{X}$ is isomorphic to a closed subset of an open polydisk of $\mathbf{C}^3 \times \mathbf{D}$, with coordinates (x, y, z, t), in such a way the local analytic equation of $\mathcal{X}$ at p is given by

$$xy = t,$$

and $xy = t = 0$ are the analytic equations of $\mathcal{X}_0$ at p.

Now we consider a reduced Cartier divisor

$$C = \cup_i C_i \subseteq \mathcal{X}_0 = \cup_{i=1}^r X_i,$$

where every $C_i = C \cap X_i \subseteq X_i$ is a reduced δ_i-nodal curve, that is smooth at every intersection point with E and does not contain any triple point of $\mathcal{X}_0$. The local equations of C at every intersection point $p \in E_{ij} \cap C = C_i \cap C_j$ are

$$\text{(2.1.1)} \qquad \begin{cases} y + x - z^m = 0 \\ xy = t \\ t = 0, \end{cases}$$

for some $m \geqslant 1$ (depending on p). Thus the singularity of C at $p \in C \cap E_{ij}$ is analytically isomorphic to the planar curve singularity of local equation

$$\text{(2.1.2)} \qquad f(y, z) = (y - z^m)y = 0,$$

which is called a *tacnode of order* m or an *m-tacnode*; observe that a 1-tacnode is a node.

Thus the curve

$$C = \cup_i C_i \subseteq \mathcal{X}_0 = \cup_{i=1}^r X_i,$$

has only planar singularities that are nodes and tacnodes.

(2.2) Problem. *We want to find sufficient conditions for $C \subseteq \mathcal{X}_0$ to be deformed to a nodal curve $C_t \subseteq \mathcal{X}_t$. More precisely we want to understand which is the maximal number of nodes of a curve $C_t \subseteq \mathcal{X}_t$ that is a deformation of $C \subseteq \mathcal{X}_0$.*

(2.3) Remark. Every tacnode of C at a moving point $p \in E_{ij} = X_i \cap X_j \subseteq E$ imposes at most $m - 1$ independent conditions to the linear system $|\mathcal{O}_{\mathcal{X}_0}(C)|$, with $m - 1$ being the expected number of conditions imposed to the linear system $|\mathcal{O}_{X_i}(C)|$ by a tangency of order m with $E_{ij} \subseteq E$ at an unprescribed point. Thus the naive expectation is that one may deform C to a curve $C_t \subseteq \mathcal{X}_t$ in such a way every m-tacnode of C on E deforms to $m - 1$ nodes of C_t and all nodes of C on $\mathcal{X}_0 \setminus E$ are preserved. In this case we will say that *an m-tacnode of C is smoothed to $m - 1$ nodes of C_t*.

(2.4) Remark. Let us recall that an m-tacnode is an A_{2m-1} planar curve singularity. An A_k planar curve singularity has analytic equation $y^2 - x^{k+1}$. Every plane curve singularity of multiplicity 2 is an A_k-singularity, for some k and two plane curve singularities of multiplicity two are topologically equivalent if and only if they are analytically equivalent. We refer to [7] and [II] for the notion of equisingularity from the topological and the analytic point of view for a family of reduced curves on a smooth surface. In particular, for an A_k-singularity the equisingular ideal coincides with the Jacobian ideal, given by $I = J = (y, x^k)$ [7, Proposition 5.6]. This implies that an A_k-singularity at a moving point of a smooth surface imposes as most $k = \dim(\mathbf{C}[x, y]/J)$ independent conditions to a linear system. Now, an A_k-singularity locally

deforms to an $A_{k'}$-singularity for every $k' \leqslant k$. Thus, in the same vein as in Problem (2.2), one could ask about deformations $C_t \subseteq \mathcal{X}_t$ of $C \subseteq \mathcal{X}_0$ with A_k-singularities. As in Remark (2.3), one expects that the curve singularity (2.1.1) deforms on $\mathcal{X}_t$ to d_i singularity of type A_i, for every multi-index $(d_1, ..., d_r)$ such that $\sum_i id_i = m - 1$.

Deformations of C in $\mathcal{X}$ are parametrized by suitable subschemes of the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ of $\mathcal{X} \to \mathbf{D}$ [16]. We denote by $\mathcal{H} \subseteq \mathcal{H}^{\mathcal{X}|\mathbf{D}}$ the union of irreducible components containing the point $[C]$ corresponding to the curve C, and by $\mathcal{H}_t$ the fibre of $\mathcal{H}$ over $t \in \mathbf{D}$. We say that *a Cartier divisor $C_t \subseteq \mathcal{X}_t$ is a deformation of $C \subseteq \mathcal{X}_0$ if the point $[C_t] \in \mathcal{H}_t$ belongs to a curve $\gamma \subseteq \mathcal{H}$, which is a multi-section of order $r \geqslant 1$ of $\mathcal{H} \to \mathbf{D}$, totally ramified at the point $[C]$, i.e., intersecting $\mathcal{H}_0$ at $r[C]$.*

2.1 – Some local geometry: the versal deformation space of the singularities of C

Every deformations of C in $\mathcal{X}$ induces a deformation of each singularity of C. Isolated singularities, i.e., germs of isolated singularities in the analytic topology, have a moduli space with the property of versality, cf. [7, Section 3], [16, Chapter 2], [12, Chapter 2] and [II].

For a plane curve singularity (D, q), a versal deformation family is given by

$$\mathcal{C}_p = \{g(x,y) + t_1 g_1(x,y) + ... + t_m g_m(x,y) = 0\} \hookrightarrow \mathbf{A}^2 \times \Delta_{D,q} \tag{2.4.1}$$

$$\Delta_{D,q} = \mathrm{Spec}(\mathbf{C}[t_1, ..., t_m])$$

where, denoting by $J_p = (g, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z})$ the Jacobian ideal of the local equation $g(y,z)$ of D at p, the polynomials $g_1(x,y), ..., g_m(x,y)$ are such that their images in $\mathbf{C}(y,z)/J_q$ form a basis of $\mathbf{C}(y,z)/J_q$, as a $\mathbf{C}$-vector space. The parameter space

$$\Delta_{D,q} = \mathrm{Spec}(\mathbf{C}[t_1, ..., t_m]),$$

is a (mini)-versal deformation space of (D, q). Using the terminology of [7], we will refer to (2.4.1) as *the* étale versal deformation family of the plane curve singularity (D, q), and to $\Delta_{D,q}$ as *the* étale (mini)-versal deformation space of (D, q).

We now provide an explicit description of the étale versal deformation family of every singularity of our curve $C \subseteq \mathcal{X}$. At every node $p \in \mathcal{X}_0 \setminus E$, the curve C is analytically equivalent to the plane curve $f(y,z) = yz = 0$, thus

$$\mathbf{C}[y,z]/J_p \simeq \mathbf{C}[y,z]/(y,z) \simeq \mathbf{C},$$

and a versal family for (C, p) is $\mathcal{C}_p \to \Delta_{(C,p)} = \mathrm{Spec}(\mathbf{C}[\mathrm{t}])$, with

$$\mathcal{C}_p = \big\{xy + t = 0\big\} \subseteq \mathbf{A}^2 \times \mathrm{Spec}(\mathbf{C}[\mathrm{t}]).$$

The only singular fibre is that over $t = 0$, implying that a deformation of a nodal curve is either a nodal curve or a smooth curve.

At every tacnodal singularity $p \in E_{ij}$, a local equation f of C at p is given by (2.1.2), so that $J_f = (2y - z^m, mz^{m-1}y)$ and

$$\mathbf{C}[y,z]/J_f \simeq \mathbf{C}^{2m-1}.$$

In particular, at every tacnodal singularity $p \in E_{ij}$ of C, choosing

$$\{1,\, z,\, z^2,\, \ldots,\, z^{m-1},\, y,\, yz,\, yz^2,\, \ldots,\, yz^{m-2}\}$$

as a basis for $\mathbf{C}[y,z]/J_f$, a versal deformation family of (C,p) is given by $\mathcal{C}_p \to \Delta_{(C,p)}$, where

$$\Delta_{(C,p)} = \mathrm{Spec}(\mathbf{C}[\alpha_0, ..., \alpha_{m-2}, \beta_0, ..., \beta_{m-1}])$$

and $\mathcal{C}_p \subseteq \mathbf{A}^2 \times \Delta_{(C,p)}$ has equation

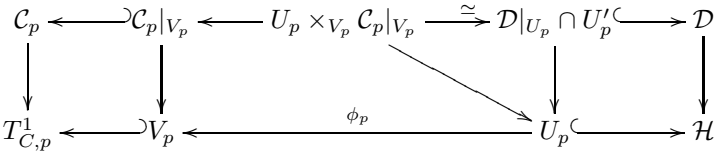
$$\mathcal{C}_p : F(y,z;\underline{\alpha},\underline{\beta}) = y^2 + \Big(\sum_{i=0}^{m-2} \alpha_i z^i + z^m\Big)y + \sum_{i=0}^{m-1} \beta_i z^i = 0, \tag{2.4.2}$$

where we are using the same notation as in [1] and [7], see [IV, V].

We now want to introduce the versal property of the versal family $\mathcal{C}_p \to \Delta_{(C,p)}$. Keeping in mind [14, Appendix B], we will do this in the analytic topology instead of the étale topology as in [7, Section 3], to simplify notation.

Let us denote by $\mathcal{D} \to \mathcal{H}$ the universal family parametrized by the Hilbert scheme $\mathcal{H}$. By versality, for every singular point p of C, there exist analytic neighborhoods U_p of $[C]$ in $\mathcal{H}$, U'_p of p in $\mathcal{D}$, and V_p of $\underline{0}$ in $\Delta_{(C,p)}$, and a map $\phi_p : U_p \to V_p$, such that the family $\mathcal{D}|_{U_p} \cap U'_p$ is isomorphic to the pull-back of $\mathcal{C}_p|_{V_p}$ by the map ϕ_p, making the following diagram

(2.4.3)



commutative. Thus, answering Problem (2.2) is the same as describing the image of the versal map

$$\phi = \prod_{p\in \mathrm{Sing}(C)} \phi_p : \bigcap_{p\in \mathrm{Sing}(C)} U_p \;\to\; \prod_{p\in \mathrm{Sing}(C)} \Delta_{(C,p)}, \tag{2.4.4}$$

where $\mathrm{Sing}(C)$ is the singular locus of C.

2.2 – Equisingular deformations of C

In order to understand the image of the versal map ϕ of (2.4.4), we describe its differential $d\phi_{[C]}$ at the point $[C] \in \bigcap_p U_p \subseteq \mathcal{H}$, which splits in the direct sum $d\phi_{[C]} = \bigoplus_{p\in \mathrm{Sing}(C)} d\phi_{p,[C]}$, where $d\phi_{p,[C]}$ is the differential of ϕ_p at the point $[C]$.

The differential map $d\phi_{[C]}$ can be identified with the map

$$H^0(\beta) = d\phi_{[C]} = \bigoplus_{p\in \mathrm{Sing}(C)} d\phi_{p,[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}}) \;\to\; H^0(C, T^1_C) = \bigoplus_{p\in \mathrm{Sing}(C)} T^1_{C,p} \tag{2.4.5}$$

induced by the exact sequence

$$0 \longrightarrow \Theta_C \longrightarrow \Theta_{\mathcal{X}}|_C \xrightarrow{\alpha} \mathcal{N}_{C|\mathcal{X}} \xrightarrow{\beta} T^1_C \longrightarrow 0, \tag{2.4.6}$$

where Θ_C and $\Theta_{\mathcal{X}}$ are the tangent sheaves of C and $\mathcal{X}$, respectively, $\mathcal{N}_{C|\mathcal{X}}$ is the normal bundle of C in $\mathcal{X}$, and T^1_C is the first cotangent sheaf of C, cf. [16, Prop. 1.1.9]. We are in particular using

standard identifications of the tangent space $T_{[C]}\mathcal{H}$ at $[C]$ with $H^0(C,\mathcal{N}_{C|\mathcal{X}})$ and of $T_0\Delta_{(C,p)}$ with $T^1_{C,p}$.

The kernel of the sheaf map β in (2.4.6) is usually denoted by $\mathcal{N}'_{C|\mathcal{X}}$, and is called the *equisingular normal sheaf of C in $\mathcal{X}$*. This terminology is due to the fact that the origin $\underline{0}\in\Delta_{(C,p)}$ is the only point in the mini-versal deformation space corresponding to a singularity analytically equivalent to (C,p). Thus, by the short exact sequence

$$0\longrightarrow\mathcal{N}'_{C|\mathcal{X}}\longrightarrow\mathcal{N}_{C|\mathcal{X}}\xrightarrow{\beta}T^1_C\longrightarrow 0, \tag{2.4.7}$$

we have that $H^0(C,\mathcal{N}'_{C|\mathcal{X}})$ can be identified with the kernel of the differential $d\phi_{[C]}$ in (2.4.5), and it parametrizes the tangent space to the analytically equisingular deformation locus of C in $\mathcal{X}$. One says that $H^0(C,\mathcal{N}'_{C|\mathcal{X}})$ parametrizes first order infinitesimal deformations of C preserving all singularities of C from the analytic point of view. Similarly, the kernel

$$\ker(d\phi_{p,[C]})$$

of the differential $d\phi_{p,[C]}: H^0(C,\mathcal{N}_{C|\mathcal{X}})\to T^1_{C,p}$ in (2.4.5) parametrizes first order infinitesimal deformations of C in $\mathcal{X}$ which are analytically equisingular at p, and there is an obvious inclusion $H^0(C,\mathcal{N}'_{C|\mathcal{X}})\subseteq\ker(d\phi_{p,[C]})$.

(2.5) Definition. *We denote by*

$$\mathcal{ES}(C)=\phi^{-1}(\underline{0},...,\underline{0})$$

the (scheme theoretic) inverse image of the origin with respect to the versal map ϕ in (2.4.4). *We will call it the* equisingular deformation locus of C.

(2.6) Remark. As already observed, the inverse image $\phi^{-1}(\underline{0},...,\underline{0})$ of $(\underline{0},...,\underline{0})$ by the versal map (2.4.4) is, in general, the locus of analytically equisingular deformations of C in $\mathcal{X}$. Since we assume C with only nodes and tacnodes, analytically equisingular deformations of C in $\mathcal{X}$ coincide with topologically equisingular deformations of C in $\mathcal{X}$, cf. Remark (2.4). For this reason we call $\mathcal{ES}(C)=\phi^{-1}(\underline{0},...,\underline{0})$ the equisingular deformation locus of C, with no further specification. Similarly, we refer to its tangent space $H^0(C,\mathcal{N}'_{C|\mathcal{X}})\simeq T_{[C]}(\mathcal{ES}(C))$ at $[C]$ as the space of equisingular first order infinitesimal deformations of C in $\mathcal{X}$.

(2.7) Remark. The versal map ϕ in (2.4.4), as well as the description of its differential $d\phi_{[C]}$ in (2.4.5), and the exact sequences (2.4.6) and (2.4.7), are defined for any reduced Cartier divisor C on $\mathcal{X}_0$, whose singularities (C,p) are all reduced complete intersection singularities in $\mathbf{C}^3$. Moreover, the description of the étale versal deformation family of (C,p) provided in (2.4.1) under the assumption that (C,p) is a planar curve singularity, extends to complete intersection curve singularities in $\mathbf{C}^n$ (see [12], or [10, Section 3] for an example of explicit computation when (C,p) is a complete intersection curve singularity in $\mathbf{C}^3$).

(2.8) Lemma. *Let $\mathcal{X}\to\mathbf{D}$ be a flat projective family of complex surfaces as in (2.1).*

(a) For any *reduced Cartier divisor $C\subseteq\mathcal{X}_0$ with non-empty intersection with E, one has*

$$H^0(C,\mathcal{N}'_{C|\mathcal{X}})=H^0(C,\mathcal{N}'_{C|\mathcal{X}_0})\subseteq H^0(C,\mathcal{N}_{C|\mathcal{X}_0}). \tag{2.8.1}$$

More generally, for every intersection point $p\in C\cap E$, the kernel of the differential $d\phi_{p,[C]}: H^0(C,\mathcal{N}_{C|\mathcal{X}})\to T^1_{C,p}$ in (2.4.5) *is such that*

$$H^0(C,\mathcal{N}'_{C|\mathcal{X}})=H^0(C,\mathcal{N}'_{C|\mathcal{X}_0})\subseteq\ker(d\phi_{p,[C]})\subseteq H^0(C,\mathcal{N}_{C|\mathcal{X}_0}). \tag{2.8.2}$$

(b) If $C \subseteq \mathcal{X}_0$ is a reduced Cartier divisor as above with only $\delta = \sum_i \delta_i$ nodes on $\mathcal{X}_0 \setminus E$ and tacnodes on E, then $H^0(C, \mathcal{N}'_{C|\mathcal{X}}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}_0})$ is contained in the linear subspace of sections of $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ vanishing at every node p of C on $\mathcal{X}_0 \setminus E$ and vanishing at every m-tacnode $p \in E$ of C with multiplicity $m-1$.

Proof. We first prove part (a) of the lemma and, in particular, equality (2.8.1) for any reduced Cartier divisor C on $\mathcal{X}_0$ with non-empty intersection with the singular locus E of $\mathcal{X}_0$. Fix any intersection point $p \in E \cap C$. Keeping in mind Remark (2.7), we want to write equisingular infinitesimal first order deformations of C in $\mathcal{X}$ locally at the point p. We consider the localized exact sequence

$$0 \longrightarrow \mathcal{N}'_{C|\mathcal{X},p} \longrightarrow \mathcal{N}_{C|\mathcal{X},p} \longrightarrow T^1_{C,p} \longrightarrow 0. \tag{2.8.3}$$

Fix local analytic coordinates x, y, z, t at p in such a way that C has equations

$$t = 0, \quad f_1(x,y,z) = 0, \quad \text{and} \quad f_2(x,y,z) = xy = 0$$

at p. Then we may identify:

- the local ring $\mathcal{O}_{C,p} = \mathcal{O}_{\mathcal{X},p}/\mathcal{I}_{C|\mathcal{X},p}$ of C at p with the localization at the origin of $\mathbf{C}[x,y,z]/(f_1,f_2)$,
- the $\mathcal{O}_{C,p}$-module $\mathcal{N}_{C|\mathcal{X},p}$ with the free $\mathcal{O}_{\mathcal{X},p}$-module $\mathfrak{hom}_{\mathcal{O}_{\mathcal{X},p}}(\mathcal{I}_{C|\mathcal{X},p}, \mathcal{O}_{C,p})$ generated by the morphisms f_1^* and f_2^*, defined by

$$f_i^*\big(s_1(x,y,z)f_1(x,y,z) + s_2(x,y,z)f_2(x,y,z)\big) = s_i(x,y,z), \quad \text{for } i = 1, 2,$$

and, finally,

- the $\mathcal{O}_{C,p}$-module

$$\begin{aligned}(\Theta_{\mathcal{X}}|_C)_p &\simeq \Theta_{\mathcal{X},p}/(I_{C,p} \otimes \Theta_{\mathcal{X},p}) \\ &\simeq \langle \partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/\partial t\rangle_{\mathcal{O}_{C,p}} \Big/ \langle \partial/\partial t - x\partial/\partial y - y\partial/\partial x\rangle\end{aligned}$$

  with the free $\mathcal{O}_{\mathcal{X},p}$-module generated by the derivatives $\partial/\partial x, \partial/\partial y, \partial/\partial z$.

With these identifications, the localization $\alpha_p : (\Theta_{\mathcal{X}}|_C)_p \to \mathcal{N}_{C|\mathcal{X},p}$ of the sheaf map α from (2.4.6) is defined by

$$\begin{aligned}\alpha_p(\partial/\partial x) &= \Big(s = s_1f_1 + s_2f_2 \longmapsto \partial s/\partial x =_{\mathcal{O}_{C,p}} s_1\partial f_1/\partial x + s_2\partial f_2/\partial x\Big) \\ &= \partial f_1/\partial x f_1^* + y f_2^*, \\ \alpha_p(\partial/\partial y) &= \partial f_1/\partial y f_1^* + x f_2^*, \text{ and} \\ \alpha_p(\partial/\partial z) &= \partial f_1/\partial z f_1^*.\end{aligned}$$

By definition of $\mathcal{N}'_{C|\mathcal{X}}$, a germ $s \in \mathcal{N}_{C|\mathcal{X},p}$ is equisingular at p, i.e., $s \in \mathcal{N}'_{C|\mathcal{X},p}$, if and only if there exists a germ

$$u = u_x(x,y,z)\partial/\partial x + u_y(x,y,z)\partial/\partial y + u_z(x,y,z)\partial/\partial z \in (\Theta_{\mathcal{X}}|_C)_p$$

such that $s = \alpha_p(u)$. Hence, locally at p, an equisingular first order infinitesimal deformation of C in $\mathcal{X}$ has equation

$$\begin{cases} f_1(x,y,z) + \varepsilon(u_x\partial f_1/\partial x + u_y\partial f_1/\partial z + u_z\partial f_1/\partial z) = 0 \\ xy + \varepsilon(yu_x + xu_y) = 0. \end{cases} \tag{2.8.4}$$

We claim that the second equation of (2.8.4) is the local equation at p of an equisingular deformation of $\mathcal{X}_0$ in $\mathcal{X}$. Indeed, by the exact sequence

$$0 \longrightarrow \Theta_{\mathcal{X}_0} \longrightarrow \Theta_{\mathcal{X}}|_{\mathcal{X}_0} \longrightarrow \mathcal{N}_{\mathcal{X}_0|\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}_0} \longrightarrow T^1_{\mathcal{X}_0} \cong \mathcal{O}_E \longrightarrow 0,$$

one finds that $\mathcal{N}'_{\mathcal{X}_0|\mathcal{X}} \simeq \mathcal{I}_{E|\mathcal{X}_0}$.

Now if $s = \alpha_p(u)$ is the localization of a global section of $\mathcal{N}'_{C|\mathcal{X}}$, by the equality $H^0(\mathcal{X}_0, \mathcal{N}'_{\mathcal{X}_0|\mathcal{X}}) = H^0(\mathcal{X}_0, \mathcal{I}_{E|\mathcal{X}_0}) = 0$, we find that the analytic function

$$yu_x(x,y,z) + xu_y(x,y,z)$$

must be identically zero in the second equation of (2.8.4). This proves inclusion (2.8.1) and (2.8.2).

We now prove part (b) of the lemma. Assume that C is a Cartier divisor with only $\delta = \sum_i \delta_i$ nodes on $\mathcal{X}_0 \setminus E$ and tacnodes on E, as in (2.1), and consider $p \in E \cap C$. In this case we may assume

$$f_1(x,y,z) = x + y + z^m$$

so that the equations in (2.8.4) become

$$\left\{ \begin{array}{l} x + y + z^m + \varepsilon(u_x + u_y + mz^m u_z) = 0 \\ xy + \varepsilon(yu_x + xu_y) = 0. \end{array} \right. \tag{2.8.5}$$

If these equations are the equations of the localization of a global section in $H^0(C, \mathcal{N}_{C|\mathcal{X}})$, then, as above, we find that $yu_x + xu_y$ is identically zero. We deduce in particular that

- $u_x(0,0,0) = u_y(0,0,0) = 0$, and
- for every $n \geqslant 1$, no z^n-terms appear in $u_x(x,y,z)$ and $u_y(x,y,z)$, no y^n-terms and $y^n z^m$-terms appear in $u_x(x,y,z)$ and, finally, no x^n-terms and $x^n z^m$-terms appear in $u_y(x,y,z)$.

In particular, if X_i, with local equations $x = t = 0$, and X_j, with local equations $y = t = 0$, are the two irreducible components of $\mathcal{X}_0$ containing p, then the local equations at p on X_i of equisingular infinitesimal deformations of C are given by

$$\left\{ \begin{array}{l} y + z^m + \varepsilon(mz^{m-1}u_z(x,y,z) + yq(y,z)) = 0 \\ x = 0, \end{array} \right. \tag{2.8.6}$$

where $q(y,z)$ is an analytic function in the variables y and z, and similarly on X_j. This proves that all sections in $H^0(C, \mathcal{N}'_{C|\mathcal{X}})$ vanish with multiplicity $m-1$ at every m-tacnode of C on E.

The fact that all sections of $H^0(C, \mathcal{N}'_{C|\mathcal{X}_0})$ vanish at every node p of C on $\mathcal{X}_0 \setminus E$ is a very classical result, which can be proved here with the same approach as above, by using that at a node on $\mathcal{X}_0 \setminus E$ the local equations of C are $x - t = t = 0 = yz$ (see also [II]).

Thus the lemma is proved. □

2.3 – Image of the versal map

We now consider:

- $\mathcal{X} \to \mathbf{D}$ be a flat projective family of complex surfaces with smooth total space $\mathcal{X}$, smooth general fibre $\mathcal{X}_t$ and central fibre $\mathcal{X}_0$ with normal crossing singularities,
- $C \subseteq \mathcal{X}_0 = \cup X_i$ a reduced Cartier divisor with only tacnodes at the intersection points with the singular locus $E = \cup_{ij} E_{ij}$ of $\mathcal{X}_0$ and nodes on $\mathcal{X}_0 \setminus E$,

as in (2.1).

(2.9) Lemma. *In the above setting, let $p \in C \cap E_{ij} \subseteq \mathcal{X}_0 \subseteq \mathcal{X}$ be an m-tacnode, with $m \geqslant 1$. Then, using the parametrization* (2.4.2) *for the mini-versal family of (C,p), we have that the image of the differential map*

$$d\phi_{p,[C]} : H^0(C,\mathcal{N}_{C|\mathcal{X}}) \to T^1_{C,p} = T_{\underline{0}}\Delta_{(C,p)}$$

is always contained in the linear subspace $H_p : \beta_1 = \cdots = \beta_{m-1} = 0$, while the image of the differential map

$$d\phi_{p,[C]} : H^0(C,\mathcal{N}_{C|\mathcal{X}_0}) \to T^1_{C,p} = T_{\underline{0}}\Delta_{(C,p)}$$

is always contained in the tangent space $\Gamma_p = T_{\underline{0}}EG_{(C,p)} : \beta_0 = \cdots = \beta_{m-1} = 0$ at $\underline{0}$ to the equigeneric locus $EG_{(C,p)}$ in $\Delta_{(C,p)}$ (cf. [7] and [II]). In particular, the subspace

$$\ker(d\phi_{p,[C]}) \subseteq H^0(C,\mathcal{N}'_{C|\mathcal{X}_0}) \tag{2.9.1}$$

has dimension

$$\dim(\ker(d\phi_{p,[C]})) \geqslant h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) - m + 1 \geqslant h^0(C,\mathcal{N}_{C|\mathcal{X}}) - m.$$

The differential map $d\phi_{p,[C]}$ is surjective on H_p if and only if the subspace $\ker(d\phi_{p,[C]})$ *of infinitesimal first order deformations of C in $\mathcal{X}_0$ that are equisingular at p has codimension exactly $m-1$ in $H^0(C,\mathcal{N}_{C|\mathcal{X}_0})$.*

Proof. We first observe that H_p contains in codimension 1 the tangent space

$$\Gamma_p = T_{\underline{0}}EG_{(C,p)} : \beta_0 = \cdots = \beta_{m-2} = 0$$

at $\underline{0}$ to the equigeneric locus $EG_{(C,p)} \subseteq \Delta_{(C,p)}$.

The image of $U_p \cap \mathcal{H}_0$ with respect to ϕ_p (recall the notation from Diagram (2.4.3)) is contained in the equigeneric locus $EG_{(C,p)} \subseteq \Delta_{(C,p)}$. Indeed, no matter how we deform C in $\mathcal{X}_0$, the smooth tangency of the irreducible components C_i and C_j with E_{ij} at p deforms to smooth tangencies with E_{ij} at $r \leqslant m$ points p_l, with $1 \leqslant l \leqslant r$, of multiplicity m_l with $\sum_l m_l = m$. This corresponds locally to a deformation of the m-tacnode of C at p into r tacnodes of orders $m_1,\ldots,m_r$, each decreasing the geometric genus by m_l with respect to the arithmetic genus. Thus, $d\phi_{p,[C]}(H^0(C,\mathcal{N}_{C|\mathcal{X}_0})) \subseteq \Gamma_p = T_{\underline{0}}EG_{(C,p)}$.

Now, by the exact sequence

$$0 \to \mathcal{N}_{C|\mathcal{X}_0} \to \mathcal{N}_{C|\mathcal{X}} \to \mathcal{N}_{\mathcal{X}_0|\mathcal{X}} = \mathcal{O}_{\mathcal{X}_0} \to 0,$$

we have that $H^0(C,\mathcal{N}_{C|\mathcal{X}_0})$ has codimension less or equal to 1 in $H^0(C,\mathcal{N}_{C|\mathcal{X}})$. If $h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}})$, the lemma is true because $T_{\underline{0}}EG_{(C,p)} \subseteq H_p$. If $h^0(C,\mathcal{N}_{C|\mathcal{X}}) = h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) + 1$, it is enough to find a section

$$\sigma \in H^0(C,\mathcal{N}_{C|\mathcal{X}}) \setminus H^0(C,\mathcal{N}_{C|\mathcal{X}_0})$$

such that $d\phi_{p,[C]}(\sigma) \in H_p \setminus T_{\underline{0}}EG_{(C,p)}$. In this case, one has

$$H^0(C,\mathcal{N}_{C|\mathcal{X}}) \simeq H^0(C,\mathcal{N}_{C|\mathcal{X}_0}) \oplus H^0(C,\mathcal{O}_{\mathcal{X}_0}) \simeq H^0(C,\mathcal{N}_{C|\mathcal{X}_0}) \oplus \mathbf{C}.$$

The section $\sigma \in H^0(C,\mathcal{N}_{C|\mathcal{X}}) \setminus H^0(C,\mathcal{N}_{C|\mathcal{X}_0})$ corresponding to the pair $(\underline{0},1)$ via the above isomorphism has local equations

$$\begin{cases} x + y + z^m = 0 \\ xy = \varepsilon. \end{cases} \tag{2.9.2}$$

It follows that the image of σ via $d\phi_{p,[C]}$ is the point corresponding to the curve $y(y+z^m) = \beta_0$. This proves the lemma. □

(2.10) Corollary. *In the same setting as above and with the same notation and hypotheses as in Lemma (2.9) and Diagram* (2.4.3)*, assume that:*

a) $h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}}) - 1$*;*

b) the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ *is smooth at* $[C]$ *of dimension* $h^0(C,\mathcal{N}_{C|\mathcal{X}})$*;*

c) $\dim(\ker(d\phi_{p,[C]})) = h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) - m + 1$.

Then the image of the versal map

$$\phi_p : \mathcal{H} \cap U_p \to \Delta_{(C,p)}$$

is a smooth variety of dimension m *in* $\Delta_{(C,p)}$*, with tangent space at* $\underline{0}$ *given by the linear subspace* $H_p : \beta_1 = ... = \beta_{m-1} = 0$*, while the image of the versal map*

$$\phi_p : \mathcal{H}_0 \cap U_p \to \Delta_{(C,p)}$$

coincides with the equigeneric deformation locus $EG_{(C,p)} \subseteq \Delta_{(C,p)}$*, which is smooth of dimension* m *at* $\underline{0}$*, with tangent space given by* $\Gamma_p = T_{\underline{0}}EG_{(C,p)} : \beta_0 = ... = \beta_{m-1} = 0$*, and whose general element corresponding to an* m*-nodal curve.*

Moreover, $\phi_p(\mathcal{H} \cap U_p)$ *intersects the locus of* $(m-1)$*-nodal curves in* $\Delta_{(C,p)}$ *along* $\Gamma_p \cup \gamma$*, where* γ *is a curve that is smooth at* $\underline{0}$ *and intersects* Γ_p *with multiplicity* m*. In particular the curve* C *may be deformed in* $\mathcal{H}$ *in such a way the tacnode at* p *is deformed into* $m-1$ *nodes.*

Proof. First observe that, by hypothesis b), there is only one irreducible component $\mathcal{H}$ of $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ containing $[C]$, which is smooth at $[C]$ of dimension $h^0(C,\mathcal{N}_{C|\mathcal{X}})$. Moreover, by hypothesis a), the central fibre of $\mathcal{H}_0$ of $\mathcal{H}$ is smooth at $[C]$ of dimension $h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}}) - 1$.

In particular, in Diagram (2.4.3), we have that the analytic varieties $\mathcal{H} \cap U_p$ and $\mathcal{H}_0 \cap U_p$ are irreducible. It follows that the image of $\phi_p : \mathcal{H} \cap U_p \to \Delta_{(C,p)}$ is an irreducible variety of dimension $\leqslant m$, since $\phi_p(\mathcal{H}_0 \cap U_p)$ is contained in the equigeneric locus, which has dimension $m-1$ and $\phi_p(\mathcal{H} \cap U_p)$ has dimension $\leqslant \dim(\phi_p(\mathcal{H}_0 \cap U_p)) + 1$. In particular, ϕ_p has general fibre of dimension at least $\dim_{[C]}(\mathcal{H}) - m$. Under the hypotheses of this corollary, the differential $d\phi_{p,[C]}$ has rank m. Hence, under hypotheses a), b) and c), the fibre $\phi_p^{-1}(\underline{0})$ has dimension exactly $\dim_{[C]}(\mathcal{H}) - m = h^0(C,\mathcal{N}_{C|\mathcal{X}}) - m$, and $\phi_p(\mathcal{H} \cap U_p)$ is irreducible of dimension m and smooth at $\underline{0}$. The fact that $\phi_p(\mathcal{H} \cap U_p)$ intersects the locus of $(m-1)$-nodal curves in $\Delta_{(C,p)}$ along $\Gamma_p \cup \gamma$, where γ is a curve that is smooth at $\underline{0}$ and intersects Γ_p with multiplicity m follows from [1, Lemma 4.1], cf. [IV, Theorem (1.2)]. □

(2.11) Remark. Under the hypotheses of Lemma (2.10), by [11, Proposition 3.7 and Corollary 3.12], one has more generally that there exist deformations $C_t \subseteq \mathcal{X}_t$ of $C \subseteq \mathcal{X}_0$ such that the m-tacnode of C at p deforms to d_i singularity of type A_i, for every multi-index $(d_1, ..., d_r)$ such that $\sum_i id_i = m - 1$.

The following lemma provides a standard way to verify when the hypothesis c) in Corollary (2.10) is satisfied.

Let $W^{|C|}_{p,m-1} \subseteq |\mathcal{O}_{\mathcal{X}_0}(C)|$ be the linear system of Cartier divisors in $|\mathcal{O}_{\mathcal{X}_0}(C)|$ intersecting E at p with multiplicity $m-1$. We have that

$$\dim(W^{|C|}_{p,m-1}) \geqslant \dim(|\mathcal{O}_{\mathcal{X}_0}(C)|) - m + 1.$$

(2.12) Lemma. *In the same setting as above, and with the same notation and hypotheses as in Lemma (2.9), assume that:*

$$\dim(W^{|C|}_{p,m-1}) = \dim(|\mathcal{O}_{\mathcal{X}_0}(C)|) - m + 1. \tag{2.12.1}$$

Then we have

$$\dim(\ker(d\phi_{p,[C]})) = h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) - m + 1.$$

Proof. By the hypothesis that $\dim(W^{|C|}_{p,m-1}) = \dim(|\mathcal{O}_{\mathcal{X}_0}(C)|) - m + 1$, we have

$$\dim(W^{|C|}_{p,r}) = \dim(|\mathcal{O}_{\mathcal{X}_0}(C)|) - r + 1,$$

for every $r \leqslant m-1$. Via the natural inclusion

$$|\mathcal{O}_{X_0}(C)| = H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(C))/H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}) \subseteq H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$$

induced by the standard exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}_0} \to \mathcal{O}_{\mathcal{X}_0}(C) \to \mathcal{O}_C(C) \to 0,$$

we find a chain a codimension 1 linear subsystems

$$W^{|C|}_{p,m-1} \subsetneq W^{|C|}_{p,m-2} \subsetneq \ldots \subsetneq W^{|C|}_{p,1} \subsetneq |\mathcal{O}_{X_0}(C)| \subseteq H^0(C, \mathcal{N}_{C|\mathcal{X}_0}).$$

By Lemma (2.8), we know that $W^{|C|}_{p,r} \cap \ker(d\phi_{p,[C]}) \subseteq W^{|C|}_{p,m-1}$, for every $r \leqslant m-2$. Since $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ is mapped by $d\phi_{p,[C]}$ to the $(m-1)$-dimensional linear subspace $T_{\underline{0}}EG \subseteq T_{\underline{0}}\Delta_{(C,p)}$, we find that $d\phi_{p,[C]} : H^0(C, \mathcal{N}_{C|\mathcal{X}_0}) \to T_{\underline{0}}EG$ is surjective by standard linear algebra, and thus $\dim(\ker(d\phi_{p,[C]})) = h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) - m + 1$. □

3 – Global deformations of C

Let $\mathcal{X} \to \mathbf{D}$ and $C = \cup_i C_i \subseteq \mathcal{X}_0 = \cup_i X_i$ be respectively the flat projective family of surfaces and the Cartier divisor C on $\mathcal{X}_0$ as in (2.1). We recall that $\mathcal{X}$ has smooth general fibre $\mathcal{X}_t$ and reducible central fibre $\mathcal{X}_0 = \cup_i X_i$ with normal crossing singularities. We denoted by X_i the irreducible components of $\mathcal{X}_0$, and by $E = \cup_{i,j} E_{ij}$ the singular locus of $\mathcal{X}_0$, where $E_{ij} = X_i \cap X_j$. Moreover, we assume that $C \cap X_i = C_i$ is a reduced δ_i-nodal curve intersecting E at smooth points, in such a way that the only singularities of C are nodes on $\mathcal{X}_0 \setminus E$ and tacnodes on E. For every tacnode p of C, we denote by m_p its order and by

$$\lambda = \mathrm{lcm}\{m_p | \, p \text{ tacnode of } C\} \tag{3.0.1}$$

the least common multiple of the m_p. We furthermore set

$$\mu = \prod_{p \text{ tacnode of } C} m_p \quad \text{and} \quad k = \frac{\mu}{\lambda}. \tag{3.0.2}$$

Finally, we assume that C has

$$\delta = \sum_i \delta_i \quad \text{nodes on } \mathcal{X}_0 \setminus E,$$

and we denote by

$$\tau_m \quad \text{the number of } \ m - \text{tacnodes of } C \text{ on } E,$$

for every $m \geqslant 1$.

Here we will provide sufficient conditions for $C \subseteq \mathcal{X}_0$ to be deformed to a nodal curve $C_t \subseteq \mathcal{X}_t$.

We recall that we denoted by $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ the relative Hilbert scheme of the family $\mathcal{X} \to \mathbf{D}$, by $\mathcal{H}$ the union of the irreducible components of $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ containing $[C]$, and by $\mathcal{H}_t$ the fibre of $\mathcal{H}$ over $t \in \mathbf{D}$. Let now $\mathbf{D}^o = \mathbf{D} \setminus \underline{0}$, $\mathcal{X}^o = \mathcal{X} \setminus \mathcal{X}_0$, and $\mathcal{H}^o = \mathcal{H} \setminus \mathcal{H}_0$. For all $\sigma \in \mathbf{N}$, we denote by $\mathcal{U}_\sigma^{\mathcal{X}^o|\mathbf{D}^o} \subseteq \mathcal{H}^o$ the relative Severi variety of σ-nodal curves, the fibre of which over $t \in \mathbf{D}^o$ is the locally closed (possibly empty) subscheme $\mathcal{U}_\sigma^{\mathcal{X}_t} \subseteq \mathcal{H}_t$ (endowed with its reduced structure) parametrizing reduced σ-nodal Cartier divisors on $\mathcal{X}_t$. We moreover denote by $\mathcal{V}_\sigma^{\mathcal{X}|\mathbf{D}} \subseteq \mathcal{H}$ the Zariski closure in $\mathcal{H}$ of $\mathcal{U}_\sigma^{\mathcal{X}^o|\mathbf{D}^o}$. Observe that, for every $t \neq 0$, the fiber $\mathcal{V}_\sigma^{\mathcal{X}_t}$ of $\mathcal{V}_\sigma^{\mathcal{X}|\mathbf{D}}$ over t is the Zariski closure of $\mathcal{U}_\sigma^{\mathcal{X}_t}$, while $\mathcal{V}_\sigma^{\mathcal{X}|\mathbf{D}} \cap \mathcal{H}_0$ will have several irreducible components, whose general point corresponds to a curve on $\mathcal{X}_0$ with singularities possibly different than nodes.

(3.1) Theorem. *In the above setting, let*

$$W^{|C|}_{[C],es} \subseteq |\mathcal{O}_{\mathcal{X}_0}(C)|$$

be the linear system of Cartier divisors in $|\mathcal{O}_{\mathcal{X}_0}(C)|$ passing through every node of C on $\mathcal{X}_0 \setminus E$, and intersecting E with multiplicity $m-1$ at every m-tacnode of C. Assume that:

1. *$h^0(C, \mathcal{N}_{C|\mathcal{X}_0}) = h^0(C, \mathcal{N}_{C|\mathcal{X}}) - 1$;*
2. *the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbf{D}}$ is smooth at $[C]$ of dimension $h^0(C, \mathcal{N}_{C|\mathcal{X}})$. In particular, there is only one irreducible component $\mathcal{H} \subseteq \mathcal{H}^{\mathcal{X}|\mathbf{D}}$ containing $[C]$, which is smooth at $[C]$;*
3. *$\dim(W^{|C|}_{[C],es}) = \dim\big(|\mathcal{O}_{\mathcal{X}_0}(C)|\big) - \delta - \sum_m \tau_m(m-1)$.*

Then $C \subseteq \mathcal{X}_0$ may be deformed to $(\delta + \sum_m \tau_m(m-1))$-nodal curves $C_t \subseteq \mathcal{X}_t$, in such a way every node of C on $\mathcal{X}_0 \setminus E$ is preserved, and every m-tacnode of C on E is smoothed to $m-1$ nodes. All nodal curves C_t obtained in this way are such that

$$h^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}_t}) = h^0(C_t, \mathcal{N}_{C_t|\mathcal{X}_t}) - \delta - \sum_m \tau_m(m-1), \tag{3.1.1}$$

thus, the corresponding point $[C_t] \in \mathcal{H}_t$ is a smooth point of the Severi variety $\mathcal{V}^{\mathcal{X}_t}_{\delta+\sum_m \tau_m(m-1)}$. In particular the nodes of C_t may be smoothed independently , i.e. the versal map (2.4.4) for C_t is surjective. This implies that

$$[C] \in \mathcal{V}_\sigma^{\mathcal{X}|\mathbf{D}}, \text{ for every } \sigma \leqslant \delta + \sum_m \tau_m(m-1).$$

Moreover, the equisingular deformation locus $\mathcal{ES}(C) \subseteq \mathcal{X}_0$ of $[C]$ in $\mathcal{X}_0$ (cf. Definition (2.5)) is an analytic open set of the smooth locus of an irreducible component V_0 of the central fibre

$$\mathcal{V}_0 = \mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)} \cap \mathcal{H}_0 = \mu V_0 + \cdots$$

of $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)} \subseteq \mathcal{H}$, of scheme theoretic multiplicity

$$\mu = \prod_{p \text{ tacnode of } C} m_p.$$

More precisely, at an analytic neighborhood of $[C]$*, the universal Severi variety* $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)}$ *will be the union of* k *branches, each intersecting* $\mathcal{H}_0$ *at* $[C]$ *with multiplicity* λ*, where* k *and* λ *are defined by* (3.0.2)*. These* k *local analytic branches of* $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)}$ *will all be smooth at* $[C]$ *if* $\lambda = m_p$*, for some tacnode* p *of* C*; otherwise they will all be singular at* $[C]$*, with multiplicity* $\frac{\lambda}{\max\{m_p | \, p \text{ tacnode of } C\}}$.

(3.2) Remark. Hypothesis 3 in Theorem (3.1) can be weakened by asking

$$h^0(C,\mathcal{N}'_{C|\mathcal{X}}) = h^0(C,\mathcal{N}'_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) - \delta - \sum_m \tau_m(m-1). \tag{3.2.1}$$

As we will observe in the proof of the theorem, under hypotheses 1 and 2, we have that hypothesis 3 is a sufficient condition for the equality (3.2.1). The reason we preferred this statement of Theorem (3.1) is that hypothesis 3 in the theorem is usually easier to be verified than (3.2.1).

We finally observe that, if

$$H^0(C,\mathcal{N}_{C|\mathcal{X}_0}) = H^0(C,\mathcal{O}_C(C)), \tag{3.2.2}$$

which happens in particular if the family $\mathcal{X} \to \mathbf{D}$ is a family of regular surfaces (i.e., $h^1(\mathcal{X}_t,\mathcal{O}_{\mathcal{X}_t}) = 0$ for all t), then, under hypotheses 1, 2, and 3 of Theorem (3.1), we have that $H^0(C,\mathcal{N}'_{C|\mathcal{X}}) = W^{|C|}_{[C],es}$. Indeed, if (3.2.2) holds, then $H^0(C,\mathcal{N}'_{C|\mathcal{X}}) \subseteq W^{|C|}_{[C],es}$ by Lemma (2.8), and this inclusion is in fact an equality for dimensional reason.

(3.3) Remark. In Theorem (3.1), the nodes of C on E cannot be preserved by deforming C to a curve in $\mathcal{X}_t$, with $t \neq 0$. This is a very general result that holds for any reduced Cartier divisor on X_0 with a node on E under our hypotheses on $\mathcal{X} \to \mathbf{D}$, i.e., just assuming $\mathcal{X} \to \mathbf{D}$ a flat projective family of surfaces with smooth total space, smooth general fibre and central fibre $\mathcal{X}_0$ with normal crossing singularities (or more generally, with normal crossing singularity at a neighborhood of the node). A proof of this can be found in [9, Section 2]. This also follows from the hypotheses 1, 2 of Theorem (3.1), by Lemma (2.8). In order to see this, we fix a node p of C on E, and we consider the diagram (2.4.3). What we want to prove is that

$$\phi_p^{-1}(\underline{0}) = U_p \cap \mathcal{H}_0. \tag{3.3.1}$$

By Lemma (2.8), we know that

$$T_{\underline{0}}(\phi_p^{-1}(\underline{0})) = \ker(d\phi_{p,[C]}) \subseteq H^0(C,\mathcal{N}_{C|\mathcal{X}_0}),$$

and, by the hypothesis 1 of Theorem (3.1), we get that $\dim(T_{\underline{0}}(\phi_p^{-1}(\underline{0}))) = h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}}) - 1$. Now observe that $U_p \cap \mathcal{H}_0 \subseteq \phi_p^{-1}(\underline{0})$ because every deformation of C in $\mathcal{X}_0$ preserves the node of C at p. Finally, by the hypotheses 1 and 2 in Theorem (3.1), we have that $\mathcal{H}_0$ is smooth at $[C]$ of dimension $h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}}) - 1$. This proves (3.3.1), which in particular says that Theorem (3.1) completely answers Problem (2.2) for a curve C verifying the hypothesis (3.2.1).

Proof of Theorem (3.1). Let $\mathrm{Sing}(C)$ be the singular locus of C. For every singular point $p \in \mathrm{Sing}(C)$, we consider the mini-versal deformation space $\Delta_{(C,p)}$ described in Section 2.1, the versal map (2.4.4)

$$\phi = \prod_{p\in\mathrm{Sing}(C)} \phi_p : \bigcap_{p\in\mathrm{Sing}(C)} U_p \to \prod_{p\in\mathrm{Sing}(C)} \Delta_{(C,p)},$$

and its differential

$$d\phi_{[C]} = \bigoplus_{p\in \mathrm{Sing}(C)} d\phi_{p,[C]} : H^0(C,\mathcal{N}_{C|\mathcal{X}}) \ \to \ H^0(C,T^1_C) = \bigoplus_{p\in \mathrm{Sing}(C)} T^1_{C,p}$$

at $[C]\in\mathcal{H}$, described in Sections 2.2 and 2.3.

Arguing exactly as in the proof of Lemma (2.12), we see that

$$W^{|C|}_{[C],es} \subseteq |\mathcal{O}_{\mathcal{X}_0}(C)| \subseteq H^0(C,\mathcal{N}_{C|\mathcal{X}_0}),$$

and the hypothesis 3 of the theorem implies that the image of the differential $d\phi_{[C]}$ has dimension $\delta+\sum_m \tau_m(m-1)$, hence

$$\dim\big(\ker(d\phi_{[C]})\big) = h^0(C,\mathcal{N}'_{C|\mathcal{X}}) = h^0(C,\mathcal{N}'_{C|\mathcal{X}_0}) = h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) - \delta - \sum_m \tau_m(m-1). \tag{3.3.2}$$

This in turn implies that

$$\dim\big(\ker(d\phi_{p,[C]})\big) = h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) - m + 1, \quad \text{for every } m\text{-tacnode } p\in E\cap C \text{ of } C$$

and

$$\dim\big(\ker(d\phi_{p,[C]})\big) = h^0(C,\mathcal{N}_{C|\mathcal{X}_0}) - 1, \quad \text{for every node of } C \text{ in } \mathcal{X}_0\setminus E.$$

In particular, Corollary (2.10) applies at every m-tacnode p of C on E.

Now, by Remark (3.3), we know that every 1-tacnode (i.e., node) of C on E is necessarily smoothed as we deform $C\subseteq\mathcal{X}_0$ outside the central fibre of $\mathcal{X}\to\mathbf{D}$. Thus, denoting by

$$\mathrm{Sing}^\circ(C) = \mathrm{Sing}(C) \ \setminus \ \{\text{nodes on } E\},$$

we may restrict our attention to the versal map

$$\varphi = \prod_{p\in\mathrm{Sing}^\circ(C)} \phi_p : \bigcap_{p\in\mathrm{Sing}^\circ(C)} U_p \ \to \ \Big(\prod_{m\geqslant 2}\prod_{\substack{p\in E\cap C\\ m\text{-tacnode}}} \Delta_{(C,p)})\Big) \times \prod_{\substack{p\in C\setminus E\\ \text{node}}} \Delta_{(C,p)} \tag{3.3.3}$$

and its differential at $[C]$,

$$d\varphi_{[C]} : H^0(C,\mathcal{N}_{C|\mathcal{X}}) \ \to \ \Big(\bigoplus_{m\geqslant 2}\bigoplus_{\substack{p\in E\cap C\\ m\text{-tacnode}}} T^1_{C,p}\Big) \oplus \bigoplus_{\substack{p\in C\setminus E\\ \text{node}}} T^1_{C,p},$$

obtained by composing $d\phi_{[C]}$ with the natural projection map. By Lemma (2.9) and Corollary (2.10), and using the notation therein, we know that

$$d\varphi_{[C]}\big(H^0(C,\mathcal{N}_{C|\mathcal{X}})\big) \subseteq \Big(\bigoplus_{m\geqslant 2}\bigoplus_{\substack{p\in E\cap C\\ m\text{-tacnode}}} H_p\Big) \oplus \bigoplus_{\substack{p\in C\setminus E\\ \text{node}}} T^1_{C,p}$$

and

$$d\varphi_{[C]}(H^0(C,\mathcal{N}_{C|\mathcal{X}_0})) \subseteq \Big(\bigoplus_{m\geqslant 2}\bigoplus_{\substack{p\in E\cap C\\ m\text{-tacnode}}} \Gamma_p\Big) \oplus \bigoplus_{\substack{p\in C\setminus E\\ \text{node}}} T^1_{C,p}.$$

More precisely, observing that

$$\ker(d\varphi_{[C]}) = H^0(C,\mathcal{N}'_{C|\mathcal{X}_0}) \subseteq H^0(C,\mathcal{N}_{C|\mathcal{X}_0})$$

because every node on E is trivially preserved if we deform C in $\mathcal{X}_0$, by (3.3.2) we get that $d\varphi_{[C]}(H^0(C,\mathcal{N}_{C|\mathcal{X}_0}))$ has dimension

$$\sum_m \tau_m(m-1)+\delta = \dim\left[\left(\bigoplus_{m\geqslant 2}\bigoplus_{\substack{p\in E\cap C\\ m\text{-tacnode}}}\Gamma_p\right)\oplus\bigoplus_{\substack{p\in C\setminus E\\ \text{node}}}T^1_{C,p}\right].$$

Thus we obtain that

$$d\varphi_{[C]}\big(H^0(C,\mathcal{N}_{C|\mathcal{X}_0})\big) = \left(\bigoplus_{m\geqslant 2}\bigoplus_{\substack{p\in E\cap C\\ m\text{-tacnode}}}\Gamma_p\right)\oplus\bigoplus_{\substack{p\in C\setminus E\\ \text{node}}}T^1_{C,p}, \tag{3.3.4}$$

and

$$d\varphi_{[C]}(H^0(C,\mathcal{N}_{C|\mathcal{X}})) = \Omega\oplus\bigoplus_{\substack{p\in C\setminus E\\ \text{node}}}T^1_{C,p}, \tag{3.3.5}$$

for some linear subspace $\Omega\subseteq\bigoplus_{m\geqslant 2}\bigoplus_p H_p$ containing $\bigoplus_{m\geqslant 2}\bigoplus_p\Gamma_p$ in codimension 1. Arguing as in the proof of Corollary (2.10), we get that:

- the versal map φ defined in (3.3.3) maps $\mathcal{H}_0$ surjectively onto the product
$$\Big(\prod_{m\geqslant 2}\prod_{\substack{p\in E\cap C\\ m\text{-tacnode}}}\Gamma_p\Big)\times\underline{0}\times\ldots\times\underline{0},$$
- the image of φ is an analytic variety, smooth of dimension $\sum_m\tau_m(m-1)+\delta+1$ at $\underline{0}$, which is a product
$$\mathcal{W}\times\prod_{p\in C\setminus E\text{ node}}\Delta_{(C,p)},$$
where $\mathcal{W}\subseteq\prod_{m\geqslant 2}\big(\prod_{p\in E\cap C\ m\text{-tacnode}}\Delta_{(C,p)}\big)$ may be identified with the image of the versal map
$$\psi=\prod_{p\in\mathrm{Sing}^\circ(C)\cap E}\phi_p:\bigcap_{p\in\mathrm{Sing}^\circ(C)\cap E}U_p\ \to\ \prod_{m\geqslant 2}\prod_{\substack{p\in E\cap C\\ m\text{-tacnode}}}\Delta_{(C,p)}. \tag{3.3.6}$$

In particular, $\mathcal{W}$ is a smooth analytic variety of dimension $\sum_m\tau_m(m-1)+1$, with tangent space $T_0\mathcal{W}=\Omega$ at $\underline{0}$, where Ω is as in (3.3.5). Keeping in mind the affine equations of the subspaces $\Gamma_p\subseteq H_p\subseteq\Delta_{(C,p)}$ in Lemma (2.9), the theorem follows by [1, Lemma 4.4] (see [V, Lemma (1.3)]), as we shall now explain.

To help the reader comparing our notation with that in [1], we observe that, for every m-tacnode (C,p), our subspace $\Gamma_p\subseteq\Delta_{(C,p)}$ is denoted in [1, Lemma 4.4] by Δ_{i,m_i}. Moreover, the product of the subspaces Γ_p is denoted in [1, Lemma 4.4] by Δ_m. Finally, the curves denoted by $\Gamma_1,\ldots,\Gamma_k$ in the statement of [1, Lemma 4.4], will be denoted here by $\gamma_1,\ldots,\gamma_k$.

By [1, Lemma 4.4], we have that the intersection of $\mathcal{W}$ with the locus of $\sum_m\tau_m(m-1)$-nodal curves in $\prod_{m\geqslant 2}\prod_{p\in E\cap C\ m\text{-tacnode}}\Delta_{(C,p)}$ is given by

$$\big(\prod_{m\geqslant 2}\prod_{p\in E\cap C\ m\text{-tacnode}}\Gamma_p\big)\cup\gamma_1\cup\cdots\cup\gamma_k,$$

where $\gamma_1,\ldots,\gamma_k\subseteq\mathcal{W}$ are distinct, reduced, unibranched curves having intersection multiplicity exactly λ with $\prod_{m\geqslant 2}\prod_{p\in E\cap C\ m\text{-tacnode}}\Gamma_p$ at the origin, with λ and k as defined in (3.0.1) and

(3.0.2). The curves $\gamma_1, ..., \gamma_k$ are all smooth at $[C]$ if $\lambda = m_p$, for some tacnode p of C; otherwise they are all singular at $[C]$, with multiplicity $\lambda/\max\{m_p | \, p \text{ tacnode of } C\}$.

The proof of the theorem is now complete if we observe that the versal map φ defined by (3.3.3) has differential of maximal rank at $[C]$, thus

$$\varphi^{-1}(\gamma_1 \cup ... \cup \gamma_k \times \underline{0} \times ... \times \underline{0})$$

is locally a fibration over $\gamma_1 \cup ... \cup \gamma_k \times \underline{0} \times ... \times \underline{0}$ with smooth fibre of dimension $h^0(C, \mathcal{N}_{C|\mathcal{X}}) - \delta - \sum_m \tau_m(m-1)$, and it actually consists of k analytic branches of $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)}$ at $[C]$. The fact that, if $[C_t]$ is general in one of these analytic branches of $\mathcal{V}^{\mathcal{X}|\mathbf{D}}_{\delta+\sum_m \tau_m(m-1)}$, then (3.1.1) holds, follows by semicontinuity. □

(3.4) Remark. Theorem (3.1) proves in particular [I, Proposition (5.5)]; besides, [I] contains many applications of Theorem (3.1). In order to help the reader compare Theorem (3.1) with [I, Proposition (5.5)], we observe that [I, Proposition (5.5)] is stated under the assumption that Hilb($\mathcal{L}$) (notation from loc. cit.) is an irreducible component of the relative Hilbert scheme of curves in the family of surfaces $S \to \mathbf{D}$, see [I, Section 4].

(3.5) Remark. The failure of hypothesis 3 in Theorem (3.1) is not always an obstruction to study deformations of C outside $\mathcal{X}_0$. It may happen for example that the singularities of C do not impose the expected number of conditions because several nodes of $\mathcal{X}_0 \setminus E$ are "disconnecting nodes", i.e., normalizing C at these nodes produces a reducible curve, and for some geometric reason it is not possible to deform C on $\mathcal{X}_t$ to a reducible curve. In this case the disconnecting nodes cannot be preserved by deforming C outside the central fibre, and the singularities of C do not impose the expected number of conditions. This happens for example for curves constructed in [11, Section 4].

When hypothesis 3 in (3.1) Theorem fails, one has to restrict the attention on the subset of singularities imposing the right number of conditions (namely, 1 condition for a node on $\mathcal{X}_0 \setminus E$, and $m-1$ conditions for an m-tacnode on E) to the linear system $|\mathcal{O}_{\mathcal{X}_0}(C)|$, and study the versal map only for this subset of singularities.

(3.6) Remark. Under the hypotheses of Theorem (3.1), by [11, Proposition 3.7 and Corollary 3.12], one has more generally that there exist deformations $C_t \subseteq \mathcal{X}_t$ of $C \subseteq \mathcal{X}_0$ so that every m-tacnode p of C is deformed to $d_{i,p}$ singularity of type A_i, for every multi-index $(d_{1,p}, ..., d_{r,p})$ such that $\sum_i i d_{i,p} = m-1$, and every node of $C \subseteq \mathcal{X}_0$ is preserved. For these curves C_t the equality (3.1.1) still holds, thus the corresponding generalized universal Severi variety is smooth at $[C_t]$, but describing its local geometry at $[C_0]$ is much more complicated.

References

Chapters in this Volume

[I] C. Ciliberto, *Limits of nodal curves: generalities and examples.*

[II] Th. Dedieu and E. Sernesi, *Deformations of curves on surfaces.*

[IV] M. Lelli-Chiesa, *Deformations of an m-tacnode into m − 1 nodes.*

[V] F. Bastianelli, *Products of deformation spaces of tacnodes.*

External References

[1] L. Caporaso, J. Harris: *Counting plane curves of any genus*, Invent. Math. **131** (1998), no.2, 345–392.

[2] L. Caporaso, J. Harris: *Parameter spaces for curves on surfaces and enumeration of rational curves*, Compositio Math. **113** (1998), no.2, 155–208.

[3] X. Chen: *Rational curves on K3 surfaces*, J. Alg. Geom. **8** (1999), no.2, 245–278.

[4] C. Ciliberto, T. Dedieu, C. Galati, A. L. Knutsen: *Nonemptiness of Severi varieties on Enriques surfaces*, Forum Math. - Sigma **11** Paper No. e52, 32 pp. (2023).

[5] C. Ciliberto, Th. Dedieu, C. Galati, A. Knutsen: *Severi varieties on blow-ups of the symmetric square of an elliptic curve*, Math. Nacthr. **296** (2), (2023), 574–587.

[6] C. Ciliberto, F. Flamini, C. Galati, A. L. Knutsen: *Moduli of nodal curves on K3 surfaces*, Adv. Math. **309** (2017), 624–654.

[7] S. Diaz, J. Harris: *Ideals associated to deformations of singular plane curves*, Trans. Amer. Math. Soc. **309** (1988), 433–468.

[8] R. Friedman: *Global smoothings of varieties with normal crossings*, Ann. of Math. **118** (1983), 75–114.

[9] C. Galati: *Degenerating curves and surfaces: first results*, Rend. Circ. Mat. Palermo **58** (2009), no. 2, 211–243.

[10] C. Galati: *On the existence of curves with a triple point on a K3 surface*, Rend. Lincei Mat. Appl. **23** (2012), no. 3, 295–317.

[11] C. Galati, A. L. Knutsen: *On the existence of curves with A_k–singularities on $K3$ surfaces*, Math. Res. Lett. **21** (2014), no.5, 1069–1109.

[12] G. M. Greuel, C. Lossen, E. Shustin: *Introduction to Singularities and Deformations*, Springer Monographs in Mathematics, Springer (2007).

[13] J. Harris: *On the Severi problem* Inventiones Math. **84** (1986), 445–461.

[14] R. Hartshorne: *Algebraic Geometry*, Graduate Texts in Mathematic **52**, Springer (1997).

[15] Z. Ran: *Families of plane curves and their limits: Enriques' conjecture and beyond*, Annals of Math. **130** (1), (1989), 121–157.

[16] E. Sernesi: *Deformations of Algebraic Schemes*, A Series of Comprehensive Studies in Mathematics **334**, Springer–Verlag, Berlin, 2006.

[17] F. Severi: *Vorlesungen über Algebraische Geometrie*, Teubner, Leipzig (1921).

Dipartimento di Matematica e Informatica, Università degli Studi della Calabria, Via Pietro Bucci, cubo 31B, 87036 Arcavacata di Rende (CS), Italy
galati@mat.unical.it

Appendix B
Flag semi-stable reduction of tacnodal curves at the limit of nodal curves

by Ciro Ciliberto, Thomas Dedieu, and Concettina Galati

1 Synthetic descriptions of nodal curves degenerating to tacnode 172
2 The local situation 175
3 Abstract reduction of nodes degenerating to a tacnode, via normalization of the nodes 177
4 Abstract reduction of nodes degenerating to a tacnode, maintaining the nodes 179
4.1 The construction 179
4.2 Ready for use toroidal computations 181
5 The flag reduction 183
References 187

0 – Introduction

The objective of this note is to provide down-to-earth explanations of the following phenomenon, which is a central theme of the present collection of articles. It is studied in detail in [IV], following Caporaso and Harris [2, 3], and in [VIII] in the same context that we are considering here; it also appears in various guises in many other places.

Let $\mathcal{S} \to \mathbf{A}^1$ be a family of surfaces such that the general member S_t, $t \neq 0$, is a smooth projective surface, and the central member S_0 is the union of two smooth projective surfaces intersecting transversely along a smooth curve B. Let $\mathcal{L}$ be a line bundle on $\mathcal{S}$, and consider the family of Severi varieties

$$\mathring{\mathcal{V}} = \coprod_{t \neq 0} V^1(S_t, L_t) \longrightarrow \mathring{\mathbf{A}}^1 = \mathbf{A}^1 - \{0\},$$

where $V^1(S_t, L_t)$ is the family of 1-nodal curves in the linear system $|L_t|$ on the surface S_t, with $L_t = \mathcal{L}|_{S_t}$. Let V_0 be the family of curves in the linear system $|L_0|$ on S_0 having a *tacnode* along B, i.e., V_0 is the family of curves $C_0 \in |L_0|$, such that there exists a point $p \in B \cap C_0$, at which C_0 locally consists of two smooth branches $C_0' \subseteq S_0'$ and $C_0'' \subseteq S_0''$, both of which are tangent to B at p. Then, at least in principle, V_0 appears in the central fibre of the closure $\overline{\mathcal{V}}$ of $\mathring{\mathcal{V}}$, with multiplicity 2.

We carry out a detailed local study of the nodes degenerating to a tacnode in the above situation, with stable reduction as a central concept. We investigate in particular the following questions, with various points of view: (i) in which way is such a tacnodal curve of the appropriate genus? (ii) why does it count with multiplicity 2? (iii) how should one modify the family of surfaces in order to see a tacnodal limit curve in S_0 as a nodal curve in an alternative semi-stable limit of the S_t's?

The multiplicity 2 is fairly straightforward to understand, cf. (2.5). Seeing the tacnodal curve as a curve of the appropriate genus may be done in various ways, according to the chosen

point of view, and is not always straightforward. We do provide a solution to problem (iii), but the conclusion is that although it is possible to see one tacnodal limit curve as a nodal curve in some semi-stable limit of the S_t's, it is impossible to do it simultaneously for all curves of the family V_0. Therefore, tacnodal curves in the degeneration cannot be avoided, and have to be considered as definitely acceptable limits of nodal curves in their own right: in [I, Section 5], a degeneration of surfaces is said to be well-behaved if the limits of Severi varieties can all be accounted for, up to some reduction, by families of nodal and tacnodal curves: the upshot of our discussion here is that it is hopeless to ask that the limits of Severi varieties be accounted for only by families of nodal curves.

The text is organized as follows. Section 1 is meant as an overview of this note, and hopefully clarifies in what sense we study the questions (i-ii-iii) mentioned above: in this section, we outline various ways of viewing a tacnodal curve as an appropriate limit of 1-nodal curves, in order to illustrate all the local computations performed in the subsequent sections. In Section 2, we give the equations of the local situation we want to study, namely 1-nodal curves in smooth surfaces degenerating to a tacnodal curve in the transverse union of two smooth surfaces. In Section 3, we perform the stable reduction of a family of 1-nodal curves degenerating to a tacnodal curve, forgetting about the ambient degeneration of surfaces. The stable reduction process in this section is rather simple (we first normalize the total space, which normalizes the general fibres, then do some modifications, and finally return to 1-nodal general fibres), but it is unfortunately not suitable to answer question (iii) above. Therefore, in Section 4, we compute the same stable reduction without normalizing the total space. Finally, we are able to answer question (iii) in Section 5, providing what we call a *flag semi-stable reduction* of the nodes degenerating to a tacnode inside surface sheets degenerating to two smooth transverse sheets. The name refers to the fact that we perform semi-stable reduction simultaneously for the family of curves and for the ambient family of surfaces.

1 – Synthetic descriptions of nodal curves degenerating to tacnode

In this section we outline various ways to see a tacnodal curve as the (equigeneric) limit of nodal curves, in the context described in the introduction above. We do not enter in too many details for the moment; a fully rigorous treatment will be given in the next sections.

(1.1) An example of ambient deformation space. We will follow throughout this section the guiding example of smooth quartics degenerating to two quadrics, and their hyperplane sections.

Let $\mathcal{S} \to \mathbf{A}^1$ be a pencil of quartic hypersurfaces in $\mathbf{P}^3$, generated by a general quartic S_∞, and the sum S_0 of two smooth quadrics S_0' and S_0'' intersecting transversely along a quartic elliptic normal curve B. The total space $\mathcal{S}$ has ordinary double points at the finitely many points of $S_\infty \cap B$, which one usually prefers to resolve in one way or another; in our context we can safely ignore this, as our study is local, away from the incriminated points. This example is described in more detail in [I, 7.1].

(1.2) Degeneration of smooth curves. A general hyperplane section C_t of S_t, $t \neq 0$, is a smooth plane quartic, and thus has genus 3. On the other hand, a general hyperplane section C_0 of S_0 is the union of two non-degenerate conics meeting transversely at 4 points: it has arithmetic genus 3, and it is a stable curve of genus 3.

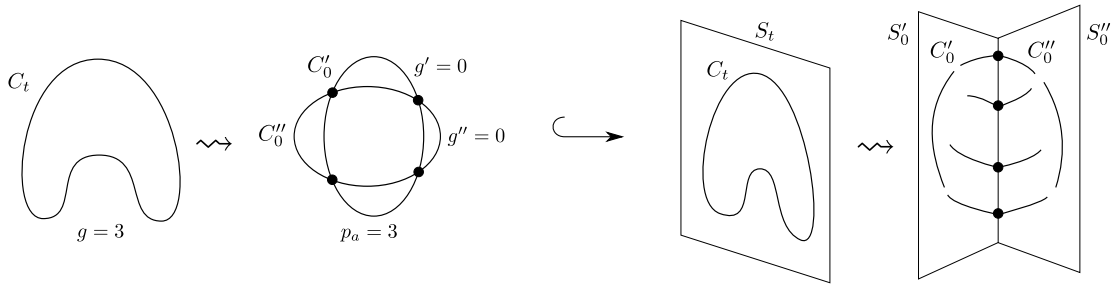


Figure 1: Degeneration of smooth curve sections

On Figure 1 above, the curves C'_0 and C''_0 are pictured in $S_0 = S'_0 \cup S''_0$ following a tropically inspired design, which is well suited for our drawings, see [X].

(1.3) Degeneration of 1-nodal curves. Continuing with our example of a pencil of quartic surfaces, we now consider for the rest of this section a family of 1-nodal hyperplane sections C_t of S_t, $t \neq 0$, degenerating to a tacnodal hyperplane section $C'_0 \cup C''_0$ of S_0, i.e., C'_0 and C''_0 are non-degenerate conics on S'_0 and S''_0 respectively, tangent to B at some point q, and meeting transversely at two other points $p_1, p_2 \in B$ (recall that B is the curve $S'_0 \cap S''_0$).
(1.3.1) An abuse of presentation. In fact, as we shall see in Section 2, such a family must be 2-valued, i.e., the curve C_t must consist of two 1-nodal hyperplane sections for $t \neq 0$, and the central fibre C_0 must be the double of $C'_0 \cup C''_0$ on S_0. This is an important point but we shall ignore it for the moment, while we are outlining the situation, as it is not the aspect we want to focus on in this note. Thus, for the sake of keeping this outline simple, we shall pretend for the rest of this section that the curves C_t are irreducible 1-nodal curves, and C_0 is a reduced tacnodal curve, without any further mention. In particular, on Figure 2 and the other figures in this section, we only picture one 1-nodal curve tending to $C'_0 \cup C''_0$, whereas in fact there are two. From Section 2 on, our treatment will be fully rigorous without any abuse of presentation.

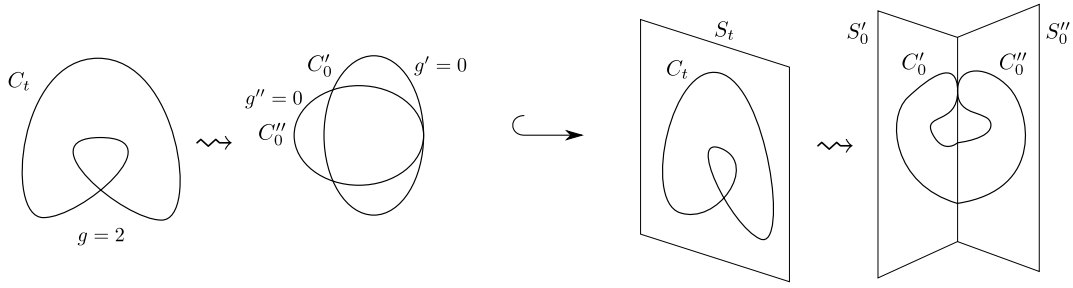


Figure 2: Degeneration of 1-nodal curves to a tacnodal curve

The curves C_t, $t \neq 0$, have geometric genus 2. In the next paragraphs we give various ways of appropriately seeing C_0 as a genus 2 limit of the C_t's.

(1.4) Degeneration of the normalizations. We may incarnate the 1-nodal curves C_t, $t \neq 0$, as the stable maps $\bar{C}_t \to S_t$ given by the normalizations $\bar{C}_t \to C_t$; these are maps from smooth, genus 2, sources. From this perspective, the relevant incarnation of the curve C_0 is as the image of a stable map as pictured on Figure 3 below, with source a stable, genus 2, curve consisting of two smooth rational components meeting transversely at three points (this source is a partial normalization of the tacnode of C_0).

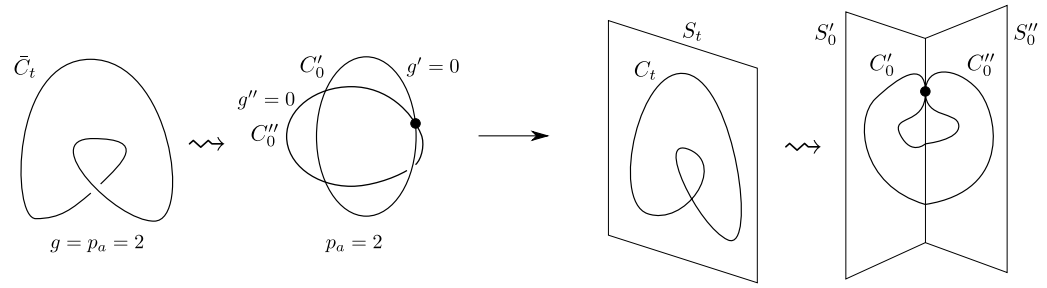


Figure 3: Degeneration of the normalizations of 1-nodal curve sections

To see this, one considers the family $\mathcal{C}$ of the curves C_t, $t \in \mathbf{A}^1$, and normalize it. It follows from Teissier's simultaneous resolution theorem [6] that the fiber of the normalization $\bar{\mathcal{C}}$ over $t \neq 0$ is the normalization $\bar{C}_t$. The fibre over 0 is computed in (3.2) below; it is singular over the tacnode, because there is a jump in δ-invariant when the node of C_t degenerates to the tacnode of C_0. This is compensated for by the jump in the number of irreducible components, which allows for the genus to be constant.

The latter point of view is arguably the more transparent. It is yet also desirable to understand the situation by seeing the curves C_t as singular curves in the surfaces S_t.

(1.5) Degeneration of the stable maps. The 1-nodal curves C_t are stable curves of arithmetic genus 3. As such, their limit C_0 can be modified into a stable curve of genus 3. In other words, the closed immersions $C_t \hookrightarrow S_t$ are stable maps of genus 3, and have a limit as such. These limits are pictured on Figure 4 below.

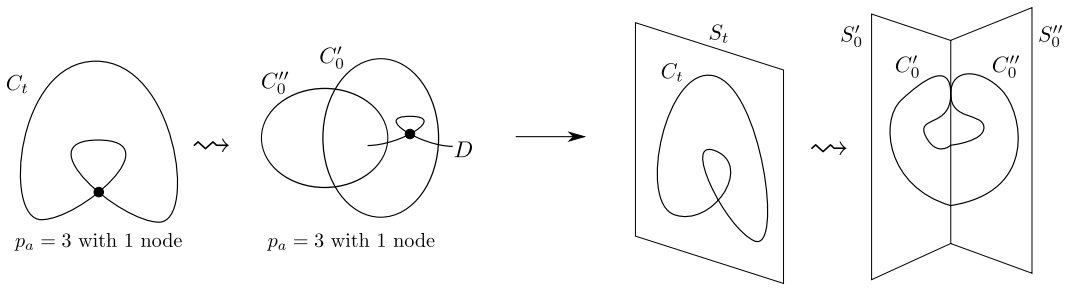


Figure 4: Degeneration as stable maps of the immersions of 1-nodal curve sections

The component D on Figure 4 is a 1-nodal rational curve ($p_a = 1$ and $p_g = 0$); it is contracted to the tacnode of C_0 by the stable map with target S_0 which is the limit of the maps $C_t \hookrightarrow S_t$. The computations are carried out in Section 3.

One nice feature of this point of view is that we see our limit of curves of arithmetic genus 3 with 1 node as a curve of arithmetic genus 3 with 1 node as well; the node in question is that of the component D (of course, the source of the limit stable map has other nodes; what we are saying is that the node of D is the limit of the node of C_t as t tends to 0). The price one pays is that this stable limit of maps has a contracted component.

(1.5.1) Observation. The above model may be used to recover that described in (1.4). To do so, one simultaneously normalizes the nodes of all the curves C_t: on the central fibre, this normalizes the node of D, which then becomes a destabilizing smooth rational component; the stable model of the central fibre is obtained by contracting this component, which gives the central fibre of the family of curves described in (1.4).

(1.6) Degeneration of the stable maps, II. It frequently happens in practice that one has to modify the family of surfaces $\mathcal{S} \to \mathbf{A}^1$ by adding one or more ruled surfaces over B between S_0' and S_0. Typically, one performs a base change $t \in \mathbf{A}^1 \mapsto t^m \in \mathbf{A}^1$, and then resolves the singularities of the obtained family. In particular, the description below is very useful to understand [IX].

Let us assume here that there is one ruled surface Σ_B over B between our two quadrics S_0' and S_0'', and that our family of surfaces has reduced central fibre. In this model of $\mathcal{S}$, our tacnodal curve C_0 is modified as indicated on Figure 5, by the adding of (i) reduced fibres of Σ_B to connect the points on C_0' and C_0'' that previously were transverse intersection points of C_0' and C_0'', and (ii) one double fibre to connect the two points that previously were the tacnode.

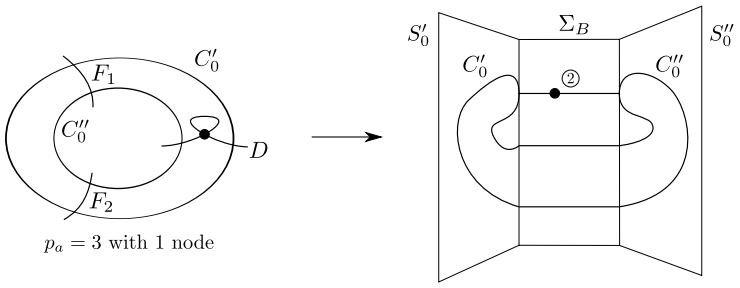


Figure 5: Degeneration as stable maps of the immersions of 1-nodal curve sections, II

The stable map limit of the immersions $C_t \hookrightarrow S_t$ in this context is easily deduced from the limit in paragraph (1.5). The two intersection points between C_0' and C_0'' are replaced by smooth rational curves F_1 and F_2, mapped isomorphically to the corresponding fibres of Σ_B. The nodal curve D on the other hand is mapped $2:1$ to the double fibre "at the tacnode", with ramification divisor the sum of the two points $D \cap C_0'$ and $D \cap C_0''$; the node of D is mapped to the limit of the nodes of the curves C_t as t approaches 0, which lies on the double fibre of Σ_D; the pre-image of this point in D consists only of the node of D.

Alternatively, this limit may be computed using the reduction which we discuss next, and which is constructed in Section 5.

(1.7) Flag semi-stable reduction. Ultimately, one may desire to see the stable limit of the 1-nodal curves C_t, depicted in Figure 4, not mapped to S_0 with some contracted components, but rather immersed in some semi-stable limit of the surfaces S_t. It is of course asserted by the general stable reduction theorems that this is possible, provided one allows the replacement of the stable limit of the C_t's by another semi-stable model. We carry this out explicitly in Section 5, and the result is depicted in Figure 13 there. We dub this a *flag semi-stable reduction* of the pairs (C_t, S_t), $t \neq 0$.

2 – The local situation

In this section, we set up an explicit incarnation of the situation that we study throughout this text, of a family of 1-nodal curves degenerating to a tacnodal curve inside a degeneration of smooth surfaces to the transverse union of two smooth surfaces. This is the same setup as in [IV].

(2.1) Notation. The notation $\mathbf{A}^4_{xyzt}$ denotes the affine 4-space equipped with affine coordinates (x, y, z, t); we will also use obvious variations, e.g., $\mathbf{A}^1_t$ denotes the affine line with affine

coordinate t. Similarly, $\mathbf{P}^1_{x:y}$ denotes the projective line with homogeneous coordinates $(x : y)$, and obvious variations will be used.

The arrow $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$ without further indication denotes the affine projection map $(x, y, z, t) \in \mathbf{A}^4 \to t \in \mathbf{A}^1$.

(2.2) We consider the degeneration of surfaces $\mathcal{S} \to \mathbf{A}^1_t$, where the total space $\mathcal{S}$ is the hypersurface of $\mathbf{A}^4_{xyzt}$ defined by the equation

$$\mathcal{S} : xy = t,$$

and the map $\mathcal{S} \to \mathbf{A}^1_t$ is the restriction of the affine projection $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$.

For all $t \neq 0$, the fibre S_t of $\mathcal{S}$ over t is smooth and irreducible, whereas the central fibre S_0 is the union of two planes S'_0 and S''_0, defined by the equations $t = y = 0$ and $t = x = 0$ respectively, which intersect transversely along the line defined by the equations $x = y = t = 0$.

(2.3) We want to study degenerations of nodal curves to a tacnodal curve inside this degeneration of surfaces, i.e., families of curves $\mathcal{C} \to \mathbf{A}^1$ fitting in a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \subseteq & \mathcal{S} \\ & \searrow & \downarrow \\ & & \mathbf{A}^1 \end{array}$$

such that the fibres C_t with $t \neq 0$ are 1-nodal curves, and the central fibre C_0 is a curve with a tacnode along the double curve of S_0, by which we mean that C_0 consists of two smooth branches C'_0 and C''_0 contained in S'_0 and S''_0 respectively, which intersect in only one point where they are both tangent to the curve $S'_0 \cap S''_0$.

(2.4) As a matter of fact, families of curves as in (2.3) cannot exist, because if they would, then the singularities of the curves would form a section of $\mathcal{S} \to \mathbf{A}^1$ meeting the central fibre along its double curve, which is impossible since the total space $\mathcal{S}$ is smooth.

As we shall now see, there exist instead families of curves with central fibre the double of the curve C_0 as in (2.3) above, and fibre over $t \neq 0$ the sum of two 1-nodal curves. We let, indeed, $\sharp\mathcal{C} \to \mathbf{A}^1_t$ be defined in $\mathcal{S}$ by the equation

$$\sharp\mathcal{C} : \big(x + y - z^2\big)^2 = 4t.$$

Its central fibre is indeed the double of a tacnodal curve C_0 as described in (2.3): it is defined by $t = 0$, and thus it is the double of the curve with the following equations in $\mathbf{A}^3_{xyz}$ (the latter being the central fibre of $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$):

$$\begin{cases} xy = 0 \\ x + y - z^2 = 0 \end{cases} \iff \begin{cases} y(y - z)^2 = 0 \\ -x = y - z^2. \end{cases}$$

On the other hand, for all $t_0 \neq 0$, the fibre $\sharp C_{t_0}$ consists of two irreducible 1-nodal curves $C^{\pm}_{t_0}$, with the following equations in $\mathbf{A}^3_{xyz}$, the fibre of $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$ over t_0:

$$\begin{cases} -xy + t_0 = 0 \\ x + y - z^2 \mp 2\sqrt{t_0} = 0 \end{cases} \iff \begin{cases} (y \mp \sqrt{t_0})^2 - yz^2 = 0 \\ -x = y - z^2 \mp 2\sqrt{t_0}. \end{cases} \tag{2.4.1}$$

(2.5) The family $\sharp\mathcal{C}$ inside $\mathcal{S}$ over $\mathbf{A}^1$ is the universal local model for 1-nodal curves in smooth surfaces degenerating to a tacnodal curve along the transverse intersection of two smooth surface sheets.

The necessity of having two 1-nodal curves in each fibre S_t, $t \neq 0$, tending to the tacnodal curve C_0 explains why a tacnodal curve in the limit amounts for (at least) two 1-nodal curves.

(2.6) Next, we shall perform a degree 2 base change in order to separate $\sharp\mathcal{C}$ in two distinct families $\mathcal{C}^+$ and $\mathcal{C}^-$ with the same central fibre C_0.

Thus, let $\tilde{\mathcal{S}} \to \mathbf{A}^1_t$ be the family of surfaces with total space defined by

$$\tilde{\mathcal{S}} : xy = t^2$$

in $\mathbf{A}^4_{xyzt}$, and with the map $\tilde{\mathcal{S}} \to \mathbf{A}^1_t$ given by the affine projection $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$. Note that $\tilde{\mathcal{S}}$ is singular along the double curve of its central fibre S_0. There are two families of 1-nodal curves tending to a curve with a tacnode along the double curve of S_0, namely $\mathcal{C}^{\pm}$ defined by

$$\mathcal{C}^{\pm} : x + y - z^2 = \pm 2t$$

inside $\tilde{\mathcal{S}}$. We will usually only look at one of these two families, namely $\mathcal{C}^-$, which we will simply denote by $\mathcal{C}$, in spite of the conflict of notation with (2.3).

3 – Abstract reduction of nodes degenerating to a tacnode, via normalization of the nodes

In this section, we compute the stable reduction of the family $\mathcal{C} \to \mathbf{A}^1$ of 1-nodal curves degenerating to a tacnodal curve described in the previous section. We start by normalizing the total space $\mathcal{C}$, which has the effect of normalizing the fibres C_t with $t \neq 0$. Then, we make some modifications in order for the pre-images of the nodes to form two disjoint sections. The last operation is to glue these two sections, so that we come back to a family with general member a 1-nodal curve.

(3.1) Setup. We consider the family $\mathcal{C}$ defined as $\mathcal{C}^-$ in (2.6) (recall also Notation (2.1)). It may be identified with its affine projection in $\mathbf{A}^3_{yzt}$, which is defined by the single equation

$$\mathcal{C} : \quad y(y - z^2 + 2t) + t^2 = 0 \iff (y + t)^2 - yz^2 = 0, \tag{3.1.1}$$

obtained by eliminating x from the two equations $xy = t^2$ and $x + y - z^2 + 2t = 0$ defining $\mathcal{C}^-$.

The total space $\mathcal{C}$ is a surface with a double curve along

$$R : \ y + t = z = 0,$$

and a pinch point at the origin (which is the intersection point of R with the central fibre), and is otherwise smooth. For all $t \neq 0$, C_t is a 1-nodal curve with its node at $C_t \cap R$, and C_0 is a 1-tacnodal curve with its tacnode at the pinch point $C_0 \cap R$. We let C_0', C_0'' be the two irreducible components of C_0, defined respectively by $y = t = 0$ and $y - z^2 = t = 0$.

(3.2) Normalization of the total space. The first move in the present approach is to normalize the total space $\mathcal{C}$. We do this by blowing-up the double curve R in $\mathcal{C}$. Let $\varepsilon_1 : \mathcal{C}_1 \to \mathcal{C}$ be the corresponding morphism.

The surface $\mathcal{C}_1$ lives in the subscheme of the product $\mathbf{A}^3_{yzt} \times \mathbf{P}^1_{t_1:z_1}$ defined by the equation

$$z_1(y+t) = t_1 z, \tag{3.2.1}$$

in which it is defined by the equation

$$t_1^2 - yz_1^2 = 0. \tag{3.2.2}$$

First note that the surface $\mathcal{C}_1$ is smooth. The exceptional locus of ε_1 (i.e., the total transform via ε_1 of R) is the curve R_1 defined by

$$y + t = z = 0$$

in $\mathcal{C}_1$; it is a bisection of $\mathcal{C}_1 \to \mathbf{A}^1_t$ (which is the composed map $\mathcal{C}_1 \to \mathcal{C} \to \mathbf{A}^1_t$), mapped $2:1$ onto R by ε_1 with ramification over the tacnode. The central fibre of $\mathcal{C}_1$ is defined by $t = 0$, which gives $z_1 y = t_1 z$ by (3.2.1), hence (3.2.2) becomes

$$t_1(t_1 - zz_1) = 0;$$

thus the central fibre of $\mathcal{C}_1$ consists of two branches intersecting transversely, which we shall still denote by C_0' and C_0''. The situation is summed up in Figure 6.

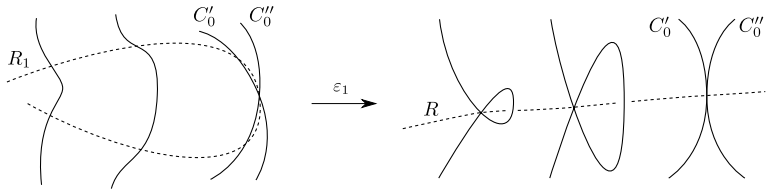


Figure 6: Normalization of the total space, $\varepsilon_1 : \mathcal{C}_1 \to \mathcal{C}$

(3.3) Base change. The next step is to perform the base change $t \in \mathbf{A}^1 \mapsto t^2 \in \mathbf{A}^1$ in order to separate the pre-images of the nodes of the C_t's in two sections. Let $\tilde{\mathcal{C}}_1 \to \mathcal{C}_1$ be the corresponding double cover. In practice, this amounts to replacing t by t^2 everywhere in the equations defining $\mathcal{C}_1$. The notable fact is that this yields an ordinary double point on the surface $\tilde{\mathcal{C}}_1$ at the point where C_0' and C_0'' intersect. We call R_1' and R_1'' the two irreducible components of the pre-image of R_1.

(3.4) Resolution. The next thing we do is to resolve the ordinary double point of $\tilde{\mathcal{C}}_1$. We call $\bar{\mathcal{C}}_1 \to \tilde{\mathcal{C}}_1$ the corresponding map. It is a birational morphism, with exceptional locus a (-2)-curve which we call E. The resulting family of curves is depicted on the right-hand-side of Figure 7.

(3.5) Reconstruction of the nodes. Finally, since what we are looking for is the stable reduction of the 1-nodal curves C_t, $t \neq 0$, we reconstruct the nodes that have been normalized in the first step by identifying the two disjoint sections R_1' and R_1'' of the family $\bar{\mathcal{C}}_1 \to \mathbf{A}^1$. We call $\mathcal{C}_2 \to \mathbf{A}^1$ the obtained family of curves, see the left-hand-side of Figure 7. Its central fibre is reduced, with a 1-nodal rational curve D (image of E) connecting C_0' and C_0''; the node of D is the limit position of the nodes of the curves C_t, $t \neq 0$.

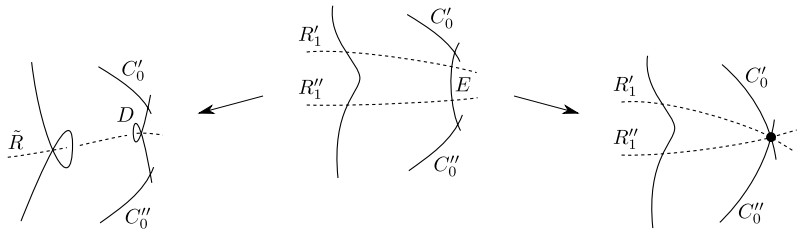


Figure 7: After the base change, $\mathcal{C}_2 \leftarrow \bar{\mathcal{C}}_1 \rightarrow \tilde{\mathcal{C}}_1$

(3.6) Why we're not satisfied with this. We want to perform the stable reduction of the nodal curves C_t (or possibly a semi-stable variant of it), while simultaneously performing the semi-stable reduction of the degeneration of surfaces $\tilde{\mathcal{S}} \to \mathbf{A}^1$, with notation as in Section 2. The main problem with the above described stable reduction process is that the normalization of the family of curves and later the reconstruction of the nodes necessitate unwanted modifications of the surfaces S_t, $t \neq 0$.

Indeed, the normalization of $\mathcal{C}$ inside $\tilde{\mathcal{S}}$ requires to blow-up every surface S_t, $t \neq 0$ at the point where the curve C_t has a node, which introduces a (-1)-curve. Finally, once the remaining steps of the reduction of the C_t's are performed, we have to recontract all these (-1)-curves, and this messes everything up, in particular this destabilizes the central fibre. We leave it to the reader to check for himself what indeed happens.

Our solution is to follow another path to the stable reduction of $\mathcal{C}$, which doesn't involve the normalization of the nodal C_t's. This is what we do in the next section.

4 – Abstract reduction of nodes degenerating to a tacnode, maintaining the nodes

4.1 – The construction

In this section we present our construction of the stable reduction of the family of 1-nodal curves C_t without normalizing the nodes. The setup is the same as that formulated in (3.1), and we don't repeat it here. We use Notation (2.1), as always.

(4.1) Blow-up of the tacnode. Let $\varepsilon_1 : \mathcal{C}_1 \to \mathcal{C}$ be the blow-up of $\mathcal{C}$ at the origin of $\mathbf{A}^3_{yzt}$ (here we use the same notation as in paragraph (3.2) for a different blow-up, but we believe this will not cause any confusion). We will see that the reduced exceptional locus of ε_1 is a smooth $\mathbf{P}^1$, which we shall call E_1, and the central fibre of $\mathcal{C}_1$ consists of the proper transform of C_0, which is the transverse union of two smooth curves, namely the proper transforms of C_0' and C_0'', which we continue to denote by the same symbols, plus E_1 with multiplicity 2; the total space $\mathcal{C}_1$ has a double curve along the proper transform of R, still with a pinch point at its intersection with the central fibre; besides this double curve, $\mathcal{C}_1$ has an ordinary double point at $C_0' \cap C_0''$, which lies on E_1, and no other singularities. See Figure 8.

We realize $\mathcal{C}_1$ as the proper transform of $\mathcal{C}$ in the subvariety of $\mathbf{A}^3_{yzt} \times \mathbf{P}^2_{y_1:z_1:t_1}$ defined by the equations

$$\operatorname{rk}\begin{pmatrix} y & z & t \\ y_1 & z_1 & t_1 \end{pmatrix} < 2.$$

In the affine chart $t_1 = 1$, $\mathcal{C}_1$ has the equation

$$\mathcal{C}_1 \text{ in } [t_1 = 1]: \quad (y_1 + 1)^2 - y_1 z_1^2 t = 0. \tag{4.1.1}$$

We thus see that $\mathcal{C}_1$ has a double curve along the proper transform R_1 of R, which has equations $y_1 + 1 = z_1 = 0$, with a pinch point at the point $(-1 : 0 : 1)$ on the exceptional divisor (the latter is the fibre of $\mathcal{C}_1$ over the origin in the projection $\mathbf{A}^3_{yzt} \times \mathbf{P}^2_{y_1:z_1:t_1} \to \mathbf{A}^3_{yzt}$). In this chart, the proper transform of C_0 is at infinity, and the central fibre $t = 0$ is the exceptional divisor, namely the double line $(y_1 + t_1)^2$ in $\mathbf{P}^2$.

In the affine chart $z_1 = 1$, we get the equation

$$\mathcal{C}_1 \text{ in } [z_1 = 1] : \quad (y_1 + t_1)^2 - y_1 z = 0. \tag{4.1.2}$$

We thus see that $\mathcal{C}_1$ has an ordinary double point at the point $(0 : 1 : 0)$ on the exceptional divisor. The central fibre is defined by the equation $t = 0$, which gives $t_1 z = 0$: $t_1 = 0$ is the proper transform of C_0, and has equation $y_1(y_1 - z) = 0$, while $z = 0$ defines the exceptional divisor: it is the divisor defined by $(y_1 + t_1)^2 = 0$, i.e., E_1 with multiplicity 2.

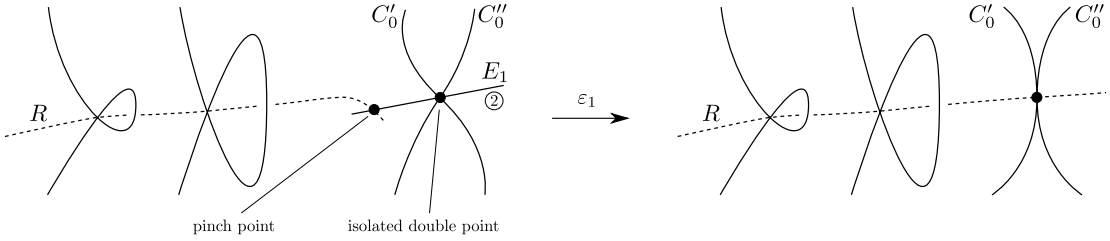


Figure 8: Blow-up of the tacnode, $\varepsilon_1 : \mathcal{C}_1 \to \mathcal{C}$

(4.2) Blow-up of the ordinary double point. Let $\varepsilon_2 : \mathcal{C}_2 \to \mathcal{C}_1$ be the blow-up at the point $C_0' \cap C_0'' \cap E_1$. The three curves C_0', C_0'', E_1 in $\mathcal{C}_1$ intersect transversely at this point, hence they are separated by this blow-up. The reduced exceptional locus is a smooth $\mathbf{P}^1$, which we shall call E_2; it appears with multiplicity 2 in the central fibre. See the right-hand-side of Figure 9 for a picture of the situation.

Note that the central fibre of $\mathcal{C}_1$ is $C_0' + C_0'' + 2E_1$, hence it has multiplicity 4 at the centre of the blow-up ε_2. However, as we shall now see, E_2 indeed appears only with multiplicity 2 in the central fibre of $\mathcal{C}_1$, because the centre of ε_2 is a singular point (of multiplicity 2) of $\mathcal{C}_1$.

This is fairly elementary, so we'll only sketch the computations. It is enough to consider the part of $\mathcal{C}_1$ in the affine chart $z_1 = 1$, which has equation (4.1.2). We add the new set of homogeneous variables $(y_2 : z_2 : t_2)$ with the new equations

$$\operatorname{rk} \begin{pmatrix} y_1 & z & y_1 + t_1 \\ y_2 & z_2 & t_2 \end{pmatrix} < 2.$$

Then, $\mathcal{C}_2$ has equation $t_2^2 - y_2 z_2 = 0$. Its central fibre has equation $t = 0$, which, in the affine chart $y_2 = 1$, since $t = t_1 z$, $t_1 = y_1(t_2 - 1)$, and $z = y_1 z_2$, reads

$$(t_2 - 1) z_2 y_1^2 = 0.$$

Now, $t_2 - 1 = 0$ defines one of the two branches C_0' and C_0'' (the other is invisible in this affine chart), $z_2 = 0$ is the equation of the proper transform of E_1, and $y_1^2 = 0$ that of the exceptional curve E_2 with multiplicity two.

(4.3) Base change and normalization. Let $\tilde{\mathcal{C}}_2 \to \mathcal{C}_2$ be the double covering of $\mathcal{C}_2$ ramified over the central fibre, and $\bar{\mathcal{C}}_2 \to \tilde{\mathcal{C}}_2$ the partial normalization obtained by leaving only the

singularity along the proper transform of R. As we shall see, the family $\bar{\mathcal{C}}_2 \to \mathbf{A}^1$ is a semi-stable reduction of $\mathcal{C} \to \mathbf{A}^1$.

Algebraically, $\tilde{\mathcal{C}}_2$ is defined by adding a square root of t, i.e., a new variable s with the new equation $s^2 = t$, to the equations of $\mathcal{C}_2$. The normalization process may be chopped in several local computation units, which we gather in Section 4.2 below. The upshot is as follows. Denote by abuse of notation $E_1, E_2 \subseteq \tilde{\mathcal{C}}_2$ the reduced pre-images of $E_1, E_2 \subseteq \mathcal{C}_2$. The normalization may be obtained by successively blowing-up the Weil divisors E_1 and E_2.

First, the blow-up of E_1 produces two smooth rational curves E_1' and E_1'', disposed as pictured in Figure 9, and both mapped isomorphically onto E_1 (to see this, use (4.7) at the pinch point, and (4.6.1) at $E_1 \cap E_2$). Then, the blow-up of E_2 produces a double covering $\bar{E}_2 \to E_2$, with $\bar{E}_2$ a smooth rational curve disposed as pictured in Figure 9; this double covering branches at the two points $C_0' \cap E_2$ and $C_0'' \cap E_2$, and the associated involution exchanges the two points $E_1' \cap \bar{E}_2$ and $E_1'' \cap \bar{E}_2$ (to see this use (4.5) at the two points $C_0' \cap E_2$ and $C_0'' \cap \bar{E}_2$, and (4.6.2) at the point $(E_1' + E_1'') \cap E_2$).

(4.3.1) A useful variant. Although arguably less natural, blowing-up first E_2 and then E_1 produces the same result. The blow-up of E_2 produces a double cover $\tilde{E}_2 \to E_2$, ramified at the two points $C_0' \cap \tilde{E}_2$ and $C_0'' \cap \tilde{E}_2$, and with a node over the point $E_1 \cap E_2$, as depicted in Figure 9. Then, the blow-up along E_1 produces two curves E_1' and E_1'' over E_1, and resolves the node of $\tilde{E}_2$.

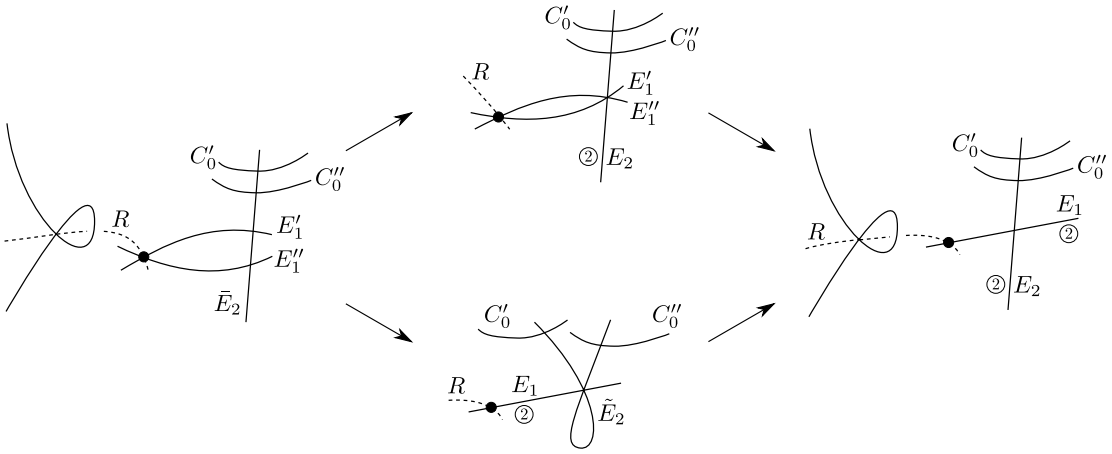


Figure 9: Removing the multiplicities, $\bar{\mathcal{C}}_2 \to \tilde{\mathcal{C}}_2$

(4.4) Conclusion. It follows from the previous considerations that the family $\bar{\mathcal{C}}_2 \to \mathbf{A}^1$ is indeed semi-stable. One may check that the final family has an ordinary double point all along the proper transform of R, i.e., the pinch point has disappeared. To build a stable model out of this semi-stable model, one has to contract the two rational curves E_1' and E_1'', which turns the curve $\bar{E}_2$ into a 1-nodal rational curve, with its node at the limit of the nodes of the curves C_t, $t \neq 0$.

4.2 – Ready for use toroidal computations

In the following paragraphs we gather some local computation, useful to normalize families of curves gotten by base change from families with non-reduced but normal crossing central fibre; the normalization will be expressed as a succession of blow-ups. We only include the

computations needed for (4.3) above, but the reader will undoubtly figure out how to generalize them to similar situations. [5, p. 125] is a useful reference on this subject.

(4.5) Local situation $x^2y = t^2$. We consider the surface $\mathcal{C}$ defined by this equation in $\mathbf{A}^3_{xyt}$, together with the fibration over $\mathbf{A}^1_t$ induced by the projection $\mathbf{A}^3_{xyt} \to \mathbf{A}^1_t$. This is the family gotten by a degree two base change totally ramified at the origin from a central fibre with one double component $2A$ and one reduced component B meeting transversely, defined by the equations $x^2 = 0$ and $y = 0$ respectively.

The normalization of $\mathcal{C}$ amounts to the blow-up $\bar{\mathcal{C}} \to \mathcal{C}$ of the Weil divisor A, defined by the two equations $x = t = 0$ (note that $\mathcal{C}$ has an ordinary double point at the generic point of A, and a pinch point at the origin). To write down this blow-up, we add the set of homogeneous coordinates $(x_1 : t_1)$ with the new equation $x_1 t = t_1 x$.

The surface $\bar{\mathcal{C}}$ is defined by the homogeneous equation

$$x_1^2 y = t_1^2. \tag{4.5.1}$$

One thus sees that $\bar{\mathcal{C}}$ is smooth. We claim that its central fibre has the form $\bar{A} + \bar{B}$, with two maps $\bar{A} \to A$ and $\bar{B} \to B$ induced by $\bar{\mathcal{C}} \to \mathcal{C}$, respectively a double covering of A ramified at the point $\bar{A} \cap \bar{B}$, and an isomorphism.

Indeed, the central fibre is defined by $t = 0$, which gives $t_1 x = 0$. The equations $x = t = 0$ define the exceptional divisor in $\bar{\mathcal{C}}$; using (4.5.1), one sees that the exceptional divisor is a double cover of the line $x = t = 0$ in $\mathbf{A}^3_{xyt}$, which is the divisor A. The equation $t_1 = 0$, on the other hand, implies $y = 0$ by (4.5.1), and one sees that it defines a divisor projecting isomorphically to B.

(4.6) Local situation $x^2y^2 = t^2$. We consider the surface $\mathcal{C}$ defined by this equation in $\mathbf{A}^3_{xyt}$, together with the fibration $\mathcal{C} \to \mathbf{A}^1_t$ defined by affine projection. The central fibre of $\mathcal{C}$ consists of the two Weil divisors A ($y = t = 0$) and B ($x = t = 0$), each with multiplicity 2.

(4.6.1) Blow-up of the Weil divisor A. We add the homogenous coordinates $(y_1 : t_1)$, with the equation $y_1 t = t_1 y$; the blow-up $\mathcal{C}_1$ of $\mathcal{C}$ is defined by the equation $x^2 y_1^2 = t_1^2$. Let us restrict to the chart $y_1 = 1$, to fix ideas; then $\mathcal{C}_1$ is defined by the equation $x^2 = t_1^2$. The central fibre is defined by $t = 0$, which implies $t_1 y = 0$, hence it consists of the proper transform B_1 of B, still with multiplicity 2 (defined by $t = t_1 = 0$, hence $x^2 = 0$), plus the exceptional divisor, defined by $t = y = 0$ and $(x - t_1)(x + t_1) = 0$, which is thus a sum $A' + A''$ of two divisors mapping isomorphically to A.

(4.6.2) Successive blow-up of the second Weil divisor B. Now, we consider the blow-up $\mathcal{C}_2$ of $\mathcal{C}_1$ along the Weil divisor B_1, which is the proper transform of B in $\mathcal{C}_1$. Thus we blow-up along the subscheme defined by $t_1 = x = 0$, and take the proper transform $\mathcal{C}_2$ of $\mathcal{C}_1$: we add the homogenous coordinates $(x_2 : t_2)$, with the equation $x_2 t_1 = t_2 x$; the blow-up $\mathcal{C}_2$ of $\mathcal{C}_1$ is defined by the equation $x_2^2 = t_2^2$ which gives $(x_2 : t_2) = (1 : \pm 1)$. The central fibre is defined by $t = 0$, which implies $t_1 y = 0$. If $t_1 = 0$, we get $x = 0$ and we obtain two reduced curves since $(x_2 : t_2) = (1 : \pm 1)$. These are the curves $B' + B''$ because $x = 0$. If $y = 0$, we get the equations $x = \pm t_1$, which give the two proper transforms of A' and A'' respectively.

Of course the situation in the present paragraph is more or less trivial, as the surface $\mathcal{C}$ has two smooth irreducible components, defined by $xy = t$ and $xy = -t$ respectively, hence its normalization is the disjoint union of these two components. However, we find it useful to describe the normalization as an explicit composition of blow-ups.

(4.7) Local situation $x^2 = y^2t^2$. We consider the surface $\mathcal{C}$ defined by this equation in $\mathbf{A}^3_{xyt}$, together with the fibration over $\mathbf{A}^1_t$ defined by the projection $\mathbf{A}^3_{xyt} \to \mathbf{A}^1_t$. The family $\mathcal{C} \to \mathbf{A}^1$ is gotten by a degree 2 base change from the family defined by $x^2 = y^2t$, which has a double curve dominating $\mathbf{A}^1$, and a pinch point on the central fibre; the central fibre is a smooth curve with multiplicity 2.

We blow-up the Weil divisor $x = t = 0$: this introduces homogeneous coordinates $(x_1 : t_1) \in \mathbf{P}^1$, with the equation $x_1t = t_1x$, and the blown-up surface is defined by the homogeneous equation $x_1^2 = y^2t_1^2$. The exceptional divisor is given by the latter equation together with $x = t = 0$, hence it is the union of two lines meeting at the point $(0, 0, 0, (0 : 1))$. The central fibre is defined by $t = 0$, hence also $t_1x = 0$, and one sees that it equals the exceptional divisor.

The total space is singular along the curve $y = x_1 = 0$, where it uniformly has an ordinary double point. In particular, the pinch point has been resolved by the blow-up.

5 – The flag reduction

We now use the construction of Section 4 in order to perform the semi-stable reduction of both families $\mathcal{C} \to \mathbf{A}^1$ and $\tilde{\mathcal{S}} \to \mathbf{A}^1$ at the same time (in the notation of (2.6)).

(5.1) Setup. We briefly recapitulate the situation. We consider the threefold $\tilde{\mathcal{S}}$ defined by the equation

$$\tilde{\mathcal{S}} : \quad xy = t^2 \tag{5.1.1}$$

in $\mathbf{A}^4_{xyzt}$, and the map $\tilde{\mathcal{S}} \to \mathbf{A}^1$ induced by the projection $\mathbf{A}^4_{xyzt} \to \mathbf{A}^1_t$. Thus $\tilde{\mathcal{S}} \to \mathbf{A}^1$ is a family of surfaces. We call S_0' and S_0'' the two components of its central fibre, defined by $y = t = 0$ and $x = t = 0$ respectively. The family of curves $\mathcal{C}$ studied in Sections 3 and 4 is a Cartier divisor of $\tilde{\mathcal{S}}$, defined in $\mathbf{A}^4_{xyzt}$ by the equations

$$\mathcal{C} : \quad \begin{cases} y + x - z^2 + 2t = 0 \\ xy = t^2. \end{cases} \tag{5.1.2}$$

(5.2) Blow-up of the tacnode. Let $\varepsilon_1 : \mathcal{S}_1 \to \tilde{\mathcal{S}}$ be the blow-up of $\tilde{\mathcal{S}}$ at the point $\mathbf{z} = (0, 0, 0, 0) \in \mathbf{A}^4$, and let $\mathcal{C}_1$ be the proper transform of $\mathcal{C}$ in $\mathcal{S}_1$; the restriction $\varepsilon_1|_{\mathcal{C}_1}$ is the same map $\mathcal{C}_1 \to \mathcal{C}$ as in (4.1). We call Σ_1 the exceptional divisor of ε_1.

The blow-up amounts to adding the new homogeneous variables $(x_1 : y_1 : z_1 : t_1) \in \mathbf{P}^3$, with the equations

$$\operatorname{rk} \begin{pmatrix} x & y & z & t \\ x_1 & y_1 & z_1 & t_1 \end{pmatrix} < 2.$$

The total space $\mathcal{S}_1$ is defined by these equations plus the homogeneous equation

$$x_1y_1 = t_1^2 \tag{5.2.1}$$

in $\mathbf{A}^4_{xyzt} \times \mathbf{P}^3_{x_1:y_1:z_1:t_1}$. The exceptional divisor Σ_1 is the corank 1 quadric defined by the equation (5.2.1) in $\mathbf{P}^3_{x_1:y_1:z_1:t_1}$ (and it sits over the origin of $\mathbf{A}^4_{xyzt}$). The central fibre of $\mathcal{S}_1$ is defined by the equation $t = 0$, which implies $t_1x = t_1y = t_1z = 0$, hence it is the reduced sum of divisors $S_0' + S_0'' + \Sigma_1$, where we denote by abuse of notation $S_0', S_0'' \subseteq \mathcal{S}_1$ the respective proper transforms of $S_0', S_0'' \subseteq \tilde{\mathcal{S}}$ (the former are the respective blow-ups of the latter at their smooth point $\mathbf{z}$).

The surface $\mathcal{C}_1$ on the other hand is defined inside $\mathcal{S}_1$ by the same equations as in (4.1). We leave it to the reader to check that the central fibres of $\mathcal{S}_1$ and $\mathcal{C}_1$ are arranged as indicated on Figure 10 (compare also with Figure 8, right-hand-side).

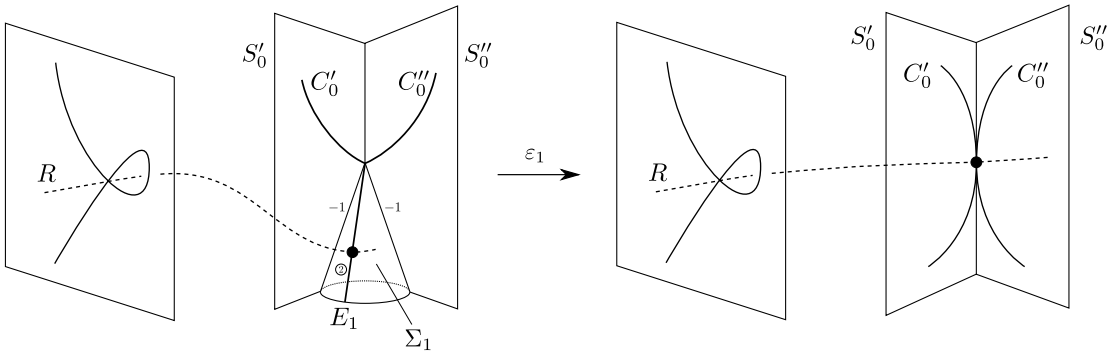


Figure 10: Blow-up of the tacnode, $\varepsilon_1 : \mathcal{S}_1 \to \mathcal{S}$

(5.3) Blow-up of the vertex of Σ_1. Let $\varepsilon_2 : \mathcal{S}_2 \to \mathcal{S}_1$ be the blow-up at the point $\mathbf{z}_1 = (0,0,0,0,(0:0:1:0)) \in \Sigma_1$, $\mathcal{C}_2$ be the proper transform of $\mathcal{C}_1$, and Σ_2 be the exceptional divisor. As usual, we call Σ_1 its own proper transform in $\mathcal{S}_2$.

To fix ideas, we consider the affine chart $z_1 = 1$ with respect to the homogeneous coordinates introduced in (5.2) for the previous blow-up; there, $\mathcal{S}_1$ is defined by the affine equation $x_1 y_1 = t_1^2$ (compare (5.2.1)) in $\mathbf{A}^4$ with coordinates (x_1, y_1, z, t_1). We add the new homogeneous variables $(x_2 : y_2 : z_2 : t_2) \in \mathbf{P}^3$, with the new equations

$$\operatorname{rk} \begin{pmatrix} x_1 & y_1 & z & t_1 \\ x_2 & y_2 & z_2 & t_2 \end{pmatrix} < 2.$$

Then $\mathcal{S}_2$ is defined by the homogeneous equation

$$x_2 y_2 = t_2^2. \tag{5.3.1}$$

Exactly as in (5.2), the exceptional divisor Σ_2 is a corank 1 quadric surface. The central fibre of $\mathcal{S}_2$ is defined by $t = 0$, or equivalently (in the affine chart $z_1 = 1$) by $t_1 z = 0$. Let us work in the chart $x_2 = 1$. Then we have $z = x_1 z_2, t_1 = x_1 t_2$ and therefore $t_1 z = x_1^2 z_2 t_2$. Hence the central fibre of $\mathcal{S}_2$ is the sum of the three divisors respectively defined in $\mathcal{S}_2$ by the three equations $x_1^2 = 0, z_2 = 0, t_2 = 0$. The first equation gives a double component; the second a simple component; the third gives $x_2 y_2 = 0$, hence two reduced components. The double component is necessarily the exceptional divisor over the vertex of the cone. The upshot is that the central fibre is the non-reduced sum $S'_0 + S''_0 + \Sigma_1 + 2\Sigma_2$, with our usual abuse of notation: $S'_0, S''_0 \subseteq \mathcal{S}_2$ are the blow-ups of $S'_0, S''_0 \subseteq \mathcal{S}_1$ at their smooth point $\mathbf{z}_1$, and $\Sigma_1 \subseteq \mathcal{S}_2$ is the blow-up of $\Sigma_1 \subseteq \mathcal{S}_1$ at its vertex (in particular, it is isomorphic to an $\mathbf{F}_2$ minimal rational ruled surface).

Again, the surface $\mathcal{C}_2$ is defined inside $\mathcal{S}_2$ by the same equations as in (4.2), and we leave it to the reader to check that the central fibres of $\mathcal{S}_2$ and $\mathcal{C}_2$ are arranged as indicated on Figure 11, right-hand-side (compare also with Figure 9, right-hand-side). Note in particular that the curve E_2 is contained in Σ_2 and tangent to the curve $\Sigma_2 \cap \Sigma_1$.

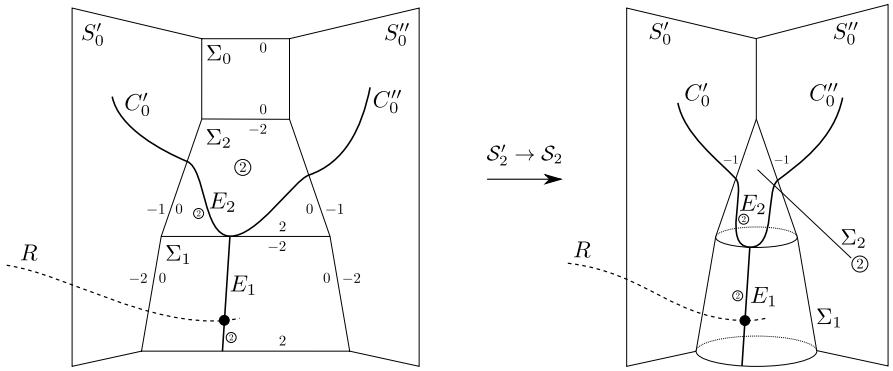


Figure 11: Blow-up of the vertex of Σ_1, $\varepsilon_2 : \mathcal{S}_2 \to \mathcal{S}_1$, and its resolution $\mathcal{S}_2' \to \mathcal{S}_2$

(5.4) Resolution of singularities of $\mathcal{S}_2$. The threefold $\mathcal{S}_2$ is singular along the curve $S_0' \cap S_0''$, at the generic point of which it has an A_1 singularity. Let $\mathcal{S}_2' \to \mathcal{S}_2$ be the blow-up along this curve, with exceptional divisor Σ_0 a $\mathbf{P}^1$-bundle over $S_0' \cap S_0''$. Then, $\mathcal{S}_2'$ is non-singular, with central fibre

$$S_0' + S_0'' + \Sigma_1 + 2\Sigma_2 + \Sigma_0; \tag{5.4.1}$$

all summands are isomorphic to their counterparts in $\mathcal{S}_2$ except $\Sigma_2 \subseteq \mathcal{S}_2'$, which is the minimal resolution of the quadric cone $\Sigma_2 \subseteq \mathcal{S}_2$. On Figure 11, left-hand-side, we picture the central fibre of $\mathcal{S}_2'$ with the self-intersections of the intersection curves of any two of its irreducible components. One may check that they concord with the triple-point-formula, which we recall as Lemma (5.5) below.

One may prefer to skip step (5.4); this induces only minor changes.

(5.5) Lemma (Triple Point Formula, see [1, 4])**.** *Let $f : \mathcal{X} \to \mathbf{A}^1$ be a family of surfaces with smooth total space, such that the central fibre X_0 has simple normal crossing support but may be non-reduced. Let Q, Q' be irreducible components of X_0, with respective multiplicities m and m' in X_0, intersecting along the* double curve *B. Then*

$$m' B_Q^2 + m B_{Q'}^2 + \sum_{Q'' \neq Q, Q'} \mathrm{mult}_{X_0}(Q'') \cdot \mathrm{Card}(B \cap Q'') = 0,$$

where Q'' ranges through all irreducible components of X_0 besides Q' and Q''.

In the above statement, $\mathrm{mult}_{X_0}(Q'')$ denotes the multiplicity of Q'' in the central fibre X_0, and $B_Q^2 = \deg(N_{B/Q})$ (resp. $B_{Q'}^2 = \deg(N_{B/Q'})$) is the self-intersection of B as a curve in the surface Q (resp. Q').

(5.6) Base change and resolution. Let $\tilde{\mathcal{S}}_2 \to \mathcal{S}_2'$ be the double covering ramified over the central fibre. It is defined by adding a square root of t, i.e., we introduce a new variable s with the new equation $s^2 = t$, and consider the family of surfaces $\tilde{\mathcal{S}}_2 \to \mathbf{A}^1_s$ defined by the projection map. The proper transform of $\mathcal{C}_2$ in $\tilde{\mathcal{S}}_2$ is isomorphic to the family $\tilde{\mathcal{C}}_2$ of (4.3).

The central fibre of $\tilde{\mathcal{S}}_2$ is, of course, the same as that of $\mathcal{S}_2'$. We let $\tilde{\mathcal{S}}_2' \to \tilde{\mathcal{S}}_2$ be the blow-up of the Weil divisor Σ_2. This has the effect of replacing Σ_2 by its double cover $\tilde{\Sigma}_2$ branched over the anticanonical divisor $(S_0' + S_0'' + \Sigma_1 + \Sigma_0) \cap \Sigma_2$, and doesn't change anything else.

Let $\tilde{\mathcal{C}}_2'$ be the proper transform of $\tilde{\mathcal{C}}_2$ in $\tilde{\mathcal{S}}_2'$. It is obtained by replacing the curve E_2 with its double cover $\tilde{E}_2$ branched along the divisor $(C_0' + C_0'' + 2E_1) \cap E_2$: this is a rational curve with

an ordinary double point at the intersection with E_1 (see (4.3.1)). The central fibre is depicted on Figure 12, right-hand-side.

One checks with local computations in the style of those in subsection 4.2[1] that $\tilde{\mathcal{S}}'_2$ is singular only along the four curves at the intersection of one of S'_0, S''_0 with one of Σ_0, Σ_1, where it has lines of A_1 double points. Correspondingly, the surface $\bar{\Sigma}_2$ has four ordinary double points over the double points of the anticanonical divisor of Σ_2.

It follows that the blow-up of $\tilde{\mathcal{S}}'_2$ along the four aforementioned curves is non-singular. We let $\bar{\mathcal{S}}_2 \to \tilde{\mathcal{S}}'_2$ be the corresponding blow-up morphism. The central fibre of $\bar{\mathcal{S}}_2 \to \mathbf{A}^1_s$ is pictured on the left-hand-side of Figure 12 below (see also Figure 9 central-bottom). The self-intersections of the double curves of the central fibre are indicated as well.

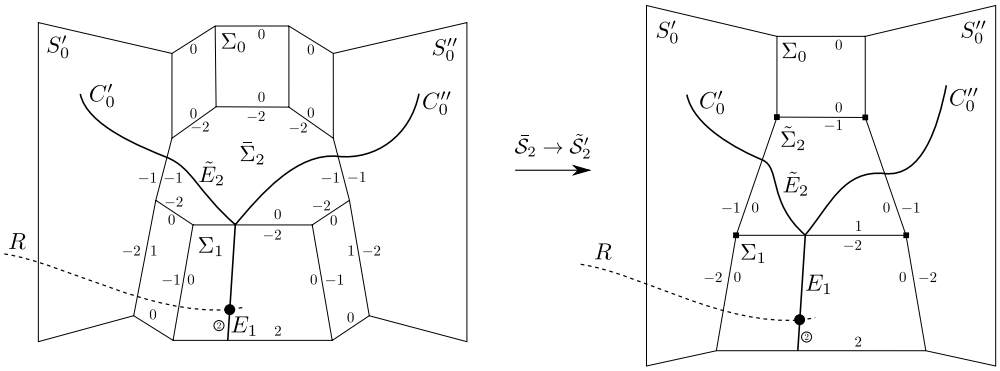


Figure 12: Base change and resolution

(5.7) Conclusion. Eventually, let $\mathcal{S}_3 \to \bar{\mathcal{S}}_2$ be the blow-up along the curve E_1, and Σ^1_1 its exceptional divisor. The curve E_1 is contained in $\tilde{\mathcal{C}}'_2$, hence the proper transform of the latter family of curves in $\mathcal{S}_3$ is isomorphic to $\bar{\mathcal{C}}_2$ in (4.3), see (4.3.1). The result is indicated on Figure 13 below (see also Figure 9, left-hand-side).

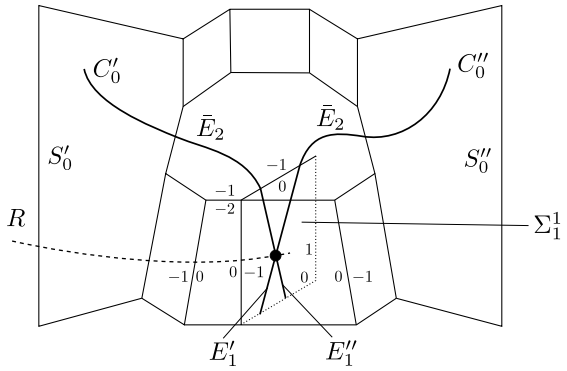


Figure 13: Flag semi-stable reduction, $\mathcal{S}_3$

This is the flag semi-stable reduction we were looking for: the family of surfaces $\mathcal{S}_3$ is semi-stable, and so is the family of curves $\bar{\mathcal{C}}_2$ it contains; thus, we have realized a tacnodal limit curve

[1] for instance, at the generic point of $S'_0 \cap \Sigma_2$, $\tilde{\mathcal{S}}_2$ has an equation equivalent to $a^2 b = u^2$, and $\tilde{\mathcal{S}}'_2 \to \tilde{\mathcal{S}}_2$ is locally the blow-up of $a = u = 0$.

of 1-nodal curves as a curve with one distinguished node, in a semi-stable limit of the ambient surfaces.

One may in turn modify this flag semi-stable reduction in order to obtain whatever variant we fancy. For instance, it is possible to understand in this way the model described in (1.6).

References

Chapters in this Volume

[I] C. Ciliberto, *Limits of nodal curves: generalities and examples*.

[IV] M. Lelli-Chiesa, *Deformations of an m-tacnode into $m-1$ nodes*.

[VIII] C. Galati, *Proving the existence of curves on smooth surfaces by smoothing tacnodes and other singularities on reducible surfaces*.

[IX] Th. Dedieu, *Enumeration of curves by floor diagrams, via degenerations of the projective plane*.

[X] Th. Dedieu, *Enumeration of curves in toric surfaces, tropical geometry, and floor diagrams*.

External References

[1] A. Calabri, C. Ciliberto, F. Flamini and R. Miranda: *On the K^2 of degenerations of surfaces and the multiple point formula*, Ann. of Math., **165** (2007), 335–395.

[2] L. Caporaso, J. Harris, *Parameter spaces for curves on surfaces and enumeration of rational curves*, Compositio Math. **113** (1998), no. 2, 155–208.

[3] L. Caporaso, J. Harris: *Counting plane curves of any genus*, Invent. Math. **131** (1998), no.2, 345–392.

[4] R. Friedman: *Global smoothings of varieties with normal crossings*, Ann. of Math. **118** (1983), 75–114.

[5] J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998.

[6] B. Teissier, *Résolution simultanée I, II*, in *Séminaire sur les singularités des surfaces, Cent. Math. Éc. Polytech., Palaiseau 1976–77*, Lecture Notes in Mathematics, vol. 777, Springer-Verlag, Berlin, 1980, 71–146.

Lecture IX
Enumeration of curves by floor diagrams, via degenerations of the projective plane

by Thomas Dedieu

1 Definitions and statement of the result 190
1.1 Floor diagrams of plane curves relative to a line 190
1.2 The enumerative problem 193
1.3 The Correspondence Theorem 194
2 Log-GW invariants of $\mathbf{P}^2$ and Hirzebruch surfaces 195
3 Degeneration of $\mathbf{P}^2$ to a chain of ruled surfaces of maximal length 198
4 Combinatorial translation 204
5 Projective geometric interpretation 206
References 208

The goal of this text is to present a combinatorial translation of the same enumerative problem considered in [VI], namely that of counting plane curves of given degree and genus, with a prescribed intersection pattern with a fixed line, both at assigned and unassigned point, and passing through the appropriate number of fixed general points in the plane. The result is the Correspondence Theorem (1.9), which says that it is equivalent to count curves as above, and to count certain combinatorial gadgets called marked floor diagrams. Although, as is well-known, being a problem combinatorial doesn't make it easy, it is possible to teach a computer to count floor diagrams, whereas teaching it to directly solve the initial algebro-geometric problem is arguably hopeless.

The Correspondence Theorem has first been obtained by Brugallé and Mikhalkin [3], using tropical geometry. Here we present a proof which is purely algebro-geometric, due to Brugallé, see [2]. This proof is by degeneration of the projective plane. The idea in Brugallé and Mikhalkin's first proof, see [X], was to strech the general base points of the enumerative problem along a preferred direction in the plane; in Brugallé's proof presented here, this idea takes the following form. Let r be the number of fixed general points in the enumerative problem. One lets the plane degenerate to a chain of r rational ruled surfaces $\mathbf{F}_1$, in such a way that the fixed points of the problem end up one on each of the surfaces of the chain. This has the effect that the curves one has to count in the limit in order to solve the enumerative problem are so to speak maximally degenerate, in the sense that all their irreducible components are rational curves, which are moreover either sections or fibres of rational normal scrolls. One is thus left with objects from which all irrationality has vanished, so that it is possible to encode them in combinatorial terms.

The recursive formula obtained by Caporaso and Harris about the same enumerative problem, presented in the previous lecture [VI], may be interpreted in terms of a degeneration of the projective plane to a transverse union of two rational surfaces, see [VII]. The base cases of this recursion are the enumerative problems for degree one plane curves, i.e., lines. In Brugallé's

proof, one somehow lets all the degenerations corresponding to the various iterations of the Caporaso–Harris' recursive procedure, all the way to the base cases, happen at once.

In order to directly follow Brugallé's proof, it is necessary to consider spaces of maps instead of the genuine Severi varieties, which parametrize curves embedded in surfaces. Indeed, multiple curves (or rather non-birational maps) inevitably show up. It follows that the results on degenerations of nodal curves along degenerations of surfaces usually considered in this volume, see [I] say, are not directly applicable. Following Brugallé, we use instead Jun Li's theory of relative stable maps [7, 8]. This theory will regrettably not be presented in this volume, and here I shall directly state its output without trying to give any explanation. Based on my personal experience, I believe that they should nevertheless look reasonable enough to anyone familiar with the results presented in due form in this volume. Finally, note that it should be possible (albeit certainly less convenient) to recast this proof using the theory expound in [I]; indications to that effect are given in Section 5.

In Section 1 we define floor diagrams and their markings, and state the Correspondence Theorem. In Section 2 we introduce the moduli spaces of maps that are needed for the proof of the Correspondence Theorem. Section 3 is the heart of the text, it contains the description of the degeneration (see Proposition (3.3)) that will finally give the proof of the Correspondence Theorem in Section 4, when we interpret it in combinatorial terms. Lastly, Section 5 is devoted to the interpretation of the considerations in this text in terms of the theory expound in [I].

Thanks. Many thanks to Erwan Brugallé for explaining all this to me and for answering unflustered my endless questions. Thanks also for your observations on this text. I know the secret meaning of the "French adage" you quote in [2].

(0.1) Notation. We use freely the notation introduced in [III, (1.2)]. In particular, $\underline{\mathbf{N}}$ denotes the set of sequences $\alpha = [\alpha_i]_{i\geqslant 1}$ of non-negative integers with finitely many α_i non-zero, and for all $\alpha \in \underline{\mathbf{N}}$, $|\alpha| = \sum_{i\geqslant 1} \alpha_i$ and $I\alpha = \sum_{i\geqslant 1} i\alpha_i$.

For any two integers $a \leqslant b$, we let $[\![a, b]\!] = [a, b] \cap \mathbf{Z}$.

In regard to the projective plane $\mathbf{P}^2$, we let H denote the line class. In regard to the Hirzebruch ruled surface $\mathbf{F}_1$ (equivalently, $\mathbf{P}^2$ blown-up at one point), we let E be the class of the (-1)-curve, F be the class of the fibers of the ruling $\mathbf{F}_1 \to \mathbf{P}^1$, and $H = E + F$.

1 – Definitions and statement of the result

1.1 – Floor diagrams of plane curves relative to a line

In this section we define the floor diagrams corresponding to plane algebraic curves relative to a line. They are a particular instance of the floor diagrams considered in [X], and they are the only floor diagrams that will be considered in this text. Thus we simply call them floor diagrams, with no additional epithet.

(1.1) A special class of weighted oriented graphs. We shall consider connected, acyclic, oriented graphs $\mathcal{D}$, with finitely many vertices and edges. We denote by $\mathrm{Vert}(\mathcal{D})$ and $\mathrm{Edges}(\mathcal{D})$ the sets of vertices and edges of $\mathcal{D}$, respectively (or simply Vert and Edges if no confusion is likely). We see Edges as a subset of Vert $\times$ Vert, with the first (resp. second) factor corresponding to the origin (resp. target) of an edge. We let $\mathcal{D}_{\mathrm{top}}$ be the topological incarnation of $\mathcal{D}$ as a 1-dimensional CW-complex.

We let the *vertices at infinity* of $\mathcal{D}$ be those 1-valent vertices of $\mathcal{D}$ for which the unique adjacent edge is outgoing. The other vertices of $\mathcal{D}$ will be called *floors* (this name will be explained in [X]). We denote by $\mathrm{Vert}^{-\infty}$ and $\mathrm{Vert}^{\mathrm{fin}}$ the sets of vertices at infinity and floors,

respectively. By definition, there cannot be any edge joining two vertices at infinity. Thus, there is a partition $\mathrm{Edges} = \mathrm{Edges}^{\infty} \coprod \mathrm{Edges}^{\mathrm{fin}}$, where Edges^{∞} is the set of edges originating from a vertex at infinity, and $\mathrm{Edges}^{\mathrm{fin}}$ is the set of those edges joining two floors. The former edges are called *infinite*, and the latter *finite*.

We will in addition assume that our graphs are weighted, which means that they come equipped with a function $w : \mathrm{Edges} \to \mathbf{N}^*$, which we call *weight function* One then defines the *divergence* $\mathrm{div}(v)$ of a vertex v (either at infinity or a floor) as

$$\mathrm{div}(v) = \sum_{(s,v)\in \mathrm{Edges}} w(s,v) - \sum_{(v,t)\in \mathrm{Edges}} w(v,t)$$

("sum of weights of incoming edges minus sum of weights of outgoing edges").

We adopt the following graphical convention: floors will be represented by ellipses, and vertices at infinity will not be represented; edges will be represented by vertical line segments, oriented from the bottom to the top. The weight of an edge is indicated only if it is greater than 1, as a circled integer.

(1.2) Definition (floor diagram)**.** *Let d, g be nonnegative integers. A (plane) floor diagram of degree d and genus g is a connected, acyclic, weighted oriented graph of the class described in (1.1), such that the following conditions hold:*

(i) the first Betti number $b_1(\mathcal{D}_{\mathrm{top}})$ equals g;
(ii) every floor has divergence 1;
(iii) $\sum_{e\in \mathrm{Edges}^{\infty}} w(e) = -d$.

(1.3) Example. All possible floor diagrams of degree 3 and genus 1 are pictured in Figure 1 below. When computing the genus, i.e., $b_1(\mathcal{D}_{\mathrm{top}})$, one should keep in mind that ellipses represent floors, which are vertices, hence the real ellipse does not contribute to b_1.[1] In the three diagrams below, the only cycle contributing to b_1 is that formed by the two lower floors and the two edges between them (it consists of two edges joining the two same vertices).

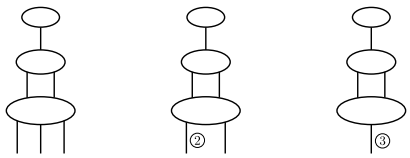


Figure 1: Floor diagrams of degree 3 and genus 1

(1.4) We let D be the abstract set $\mathrm{Vert}^{\mathrm{fin}} \coprod \mathrm{Edges}$, whose elements are all floors and edges. Since $\mathcal{D}$ is acylic, the set D inherits a partial ordering from the orientation.

Before we state the next definition, it is worth noting the following relations.

(1.5) Lemma. *Let $\mathcal{D}$ be a floor diagram of degree d and genus g. The two following equalities hold:*

(i) $\mathrm{Card}(\mathrm{Vert}^{\mathrm{fin}}) = d$;
(ii) $\mathrm{Card}(\mathsf{D}) = 2d + g - 1 + \mathrm{Card}(\mathrm{Vert}^{-\infty})$.

Proof. One has

$$\begin{aligned} 0 = \sum_{v\in \mathrm{Vert}} \mathrm{div}(v) &= \sum_{v\in \mathrm{Vert}^{\mathrm{fin}}} \mathrm{div}(v) + \sum_{v\in \mathrm{Vert}^{-\infty}} \mathrm{div}(v) \\ &= \mathrm{Card}(\mathrm{Vert}^{\mathrm{fin}}) - d, \end{aligned}$$

[1] To make this a rational choice (pun intended), think of the ellipses as rational curves (after all ellipses are conics!), hence as having genus 0.

which proves (i). The Euler characteristic of $\mathcal{D}_{\mathrm{top}}$ is

$$1 - g = \mathrm{Card}(\mathrm{Vert}) - \mathrm{Card}(\mathrm{Edges}),$$

so

$$\begin{aligned}\mathrm{Card}(\mathsf{D}) &= \mathrm{Card}(\mathrm{Vert}^{\mathrm{fin}}) + \big(g - 1 + \mathrm{Card}(\mathrm{Vert})\big) \\ &= 2\,\mathrm{Card}(\mathrm{Vert}^{\mathrm{fin}}) + g - 1 + \mathrm{Card}(\mathrm{Vert}^{-\infty}),\end{aligned}$$

which together with (i) proves (ii). □

(1.6) Definition (marking)**.** *Let $\mathcal{D}$ be a floor diagram of degree d and genus g, and $\alpha, \beta \in \underline{\mathbf{N}}^2$, such that $I\alpha + I\beta = d$. A* marking of type (α, β) *of $\mathcal{D}$ is a strictly increasing function*

$$m : \big\{-|\alpha| + 1, \dots, 1, \dots, 2d + g - 1 + |\beta|\big\} \to \mathsf{D}$$

such that:

(i) $\forall k \in \mathbf{N}^*$, $\forall i \in \big[\!\big[-|\alpha| + (\sum_{j \leqslant k-1} \alpha_j) + 1,\ -|\alpha| + \sum_{j \leqslant k} \alpha_j \big]\!\big]$, $m(i) \in \mathrm{Edges}^{\infty}$ *and* $w(m(i)) = k$*;*

(ii) $\forall k \in \mathbf{N}^*$, *there are exactly β_k infinite edges of weight k in* $m\big(\big[\!\big[1, 2d + g - 1 + |\beta|\big]\!\big]\big)$.

A floor diagram $\mathcal{D}$ enhanced with a marking m of type (α, β) is called a *marked floor diagram* $(\mathcal{D}, m)$ of type (α, β).

Two marked floor diagrams of the same type are *equivalent* if there exists an isomorphism of oriented graphs $\phi : \mathcal{D} \cong \mathcal{D}'$ such that $w = w' \circ \phi$ and $m = \phi \circ m'$.

Markings will be indicated on graphic representations by plain numbers. Beware not to confuse them with weights, which are circled integers.

(1.7) Examples. Up to equivalence, there is a unique marked floor diagram of degree 3, genus 1, and type $(0, 3)$.

(1.7.1) Marked floor diagrams of degree 3 *and genus* 0 *of type* $(0, 3)$. Below I give all possibilities for such diagrams. For the leftmost diagram there is a unique marking (up to equivalence). For the next one, there are 5: the label i may be chosen arbitrarily in $[\![1, 5]\!]$, and this choice determines uniquely the four labels not indicated on the picture. For the rightmost diagram, there are 3: the index i may be chosen arbitrarily in $[\![6, 8]\!]$, and then the two remaining labels are uniquely determined.

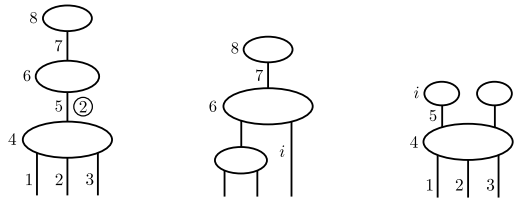


Figure 2: Marked floor diagrams of degree 3 and genus 0 of type $(0, 3)$.

(1.7.2) Marked floor diagrams of degree 3 *and genus* 0 *of type* $(0,[1,1])$. All possible marked diagrams of this type are displayed below. The leftmost diagram has two possible markings, determined by the choice of $i \in [\![1,2]\!]$. The central diagram has four possible markings, determined by the choice of $i \in [\![1,4]\!]$. The rightmost diagram has six possible markings, determined first by the choice of $i \in [\![1,2]\!]$, which fixes i', and then by the choice of $j \in [\![5,7]\!]$ which fixes the rest.

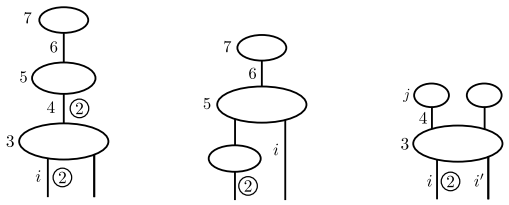


Figure 3: Marked floor diagrams of degree 3 and genus 0 of type $(0,[1,1])$.

(1.7.3) Marked floor diagrams of degree 3 *and genus* 0 *of type* $([0,1],1)$. All possible marked diagrams of this type are displayed below. It is instructive to make comparisons with the type $(0,[1,1])$ considered in the previous paragraph (1.7.2). The leftmost diagram has a unique possible marking. The central diagram has three possible markings, determined by the choice of $i \in [\![1,3]\!]$. The rightmost diagram has three possible markings, determined by the choice of $j \in [\![4,6]\!]$.

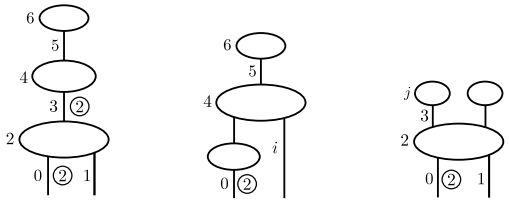


Figure 4: Marked floor diagrams of degree 3 and genus 0 of type $([0,1],1)$.

1.2 – The enumerative problem

We shall now state precisely the enumerative problem we are interested in. The definitions below are specializations of those given in [III].

Assume a line $\Lambda \subseteq \mathbf{P}^2$ is fixed. Let $d > 0$ and $g \geqslant 0$ be integers, and $\alpha, \beta \in \underline{\mathbf{N}}$, such that $I\alpha + I\beta = d$. Let Ω be a general set of α points on Λ. We let $V_g^{d,\mathrm{irr}}(\alpha,\beta)(\Omega)$ be the Zariski closure in the projective space $|\mathcal{O}_{\mathbf{P}^2}(d)|$ of the localy closed set $\mathring{V}_g^{d,\mathrm{irr}}(\alpha,\beta)(\Omega)$ parametrizing irreducible curves $C \subseteq \mathbf{P}^2$ of degree d, genus g, and having intersection pattern with Λ prescribed by α, β and Ω as follows: write $\Omega = \left(\{p_{i,j}\}_{1 \leqslant j \leqslant \alpha_i}\right)_{i \geqslant 1}$, and let $\nu : \bar{C} \to C$ be the normalization of C; we ask that there exist α points $q_{i,j} \in C$, $1 \leqslant j \leqslant \alpha_i$, and β points $r_{i,j} \in C$, $1 \leqslant j \leqslant \beta_i$, such that

$$\forall\, 1 \leqslant j \leqslant \alpha_i : \quad \phi(q_{i,j}) = p_{i,j} \qquad \text{and} \qquad \phi^*\Lambda = \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} i\, r_{i,j}.$$

The variety $V_g^{d,\mathrm{irr}}(\alpha,\beta)(\Omega)$ has dimension $2d+g-1+|\beta|$. We call it a logarithmic Severi variety of irreducible[2] plane curves, relative to the line Λ.

[2] in the definition given in [III], the curve C was not required to be irreducible; the superscript 'irr' indicates this additional requirement.

We let $N_g^{d,\mathrm{irr}}(\alpha,\beta)$ be the degree of $V_g^{d,\mathrm{irr}}(\alpha,\beta)(\Omega)$ as a projective subvariety of $|\mathcal{O}_{\mathbf{P}^2}(d)|$; it is independent of the choice of Λ and Ω. This is the number of plane curves of degree d, genus g, and having contact of the required type with Λ, that pass through $2d+g-1+|\beta|$ general points on $\mathbf{P}^2$.

1.3 – The Correspondence Theorem

The Correspondence Theorem (1.9) is the central result of this text. It will be proved in the subsequent sections. It states that one can compute the numbers $N_g^{d,\mathrm{irr}}(\alpha,\beta)$ by enumerating floor diagrams. To do so, one needs to count the latter with the appropriate multiplicity.

(1.8) Definition. *Let $(\mathcal{D},m)$ be a marked floor diagram of type (α,β). Its multiplicity is*

$$\mu^{\mathbf{C}}(\mathcal{D},m)=I^{\beta}\prod_{e\in\mathrm{Edges}^{\mathrm{fin}}}w(e)^2.$$

Thus, the multiplicity of a marked floor diagram is the product of the multiplicities of its edges where:
(a) a finite edge of weight w has multiplicity w^2;
(b) an infinite edge of weight w corresponding to a mobile boundary condition[3] (i.e., such that its label in the marking is positive) has multiplicity w;
(c) an infinite edge of weight w corresponding to a fixed boundary condition (i.e., such that its label in the marking is non-positive) has multiplicity 1.

One may simply write $\mu^{\mathbf{C}}(\mathcal{D})$ in place of $\mu^{\mathbf{C}}(\mathcal{D},m)$ since the multiplicity only depends on the type and not on the marking. We leave the original superscript '$\mathbf{C}$' as a reminder that this is the correct multiplicity only in order to count *complex* plane curves.

(1.9) Theorem (Brugallé–Mikhalkin [3], see also [1])**.** *Let $d>0$ and $g\geqslant 0$ be integers, and $\alpha,\beta\in\underline{\mathbf{N}}$, such that $I\alpha+I\beta=d$. One has*

$$N_g^{d,\mathrm{irr}}(\alpha,\beta)=\sum_{(\mathcal{D},m)}\mu^{\mathbf{C}}(\mathcal{D},m),$$

where the sum on the right-hand-side is taken over all equivalence classes of marked floor diagrams of degree d, genus g, and type (α,β).

(1.10) Example. Let us compute the numbers $N_0^{3,\mathrm{irr}}(\alpha,\beta)$ in the cases considered in Example (1.7), by enumerating floor diagrams.

For $(\alpha,\beta)=(0,3)$, the diagram of the leftmost shape has multiplicity 4, and all the other have multiplicity 1. Thus

$$N_0^{3,\mathrm{irr}}(0,3)=1\times 4+5+3=12$$

(the first number of each summand indicates the number of possible markings for a diagram of the given shape, and the second one indicates the multiplicity; it is omitted if it is one).

For $(\alpha,\beta)=(0,[1,1])$, the diagrams of the leftmostshape have multiplicity 8, and all the other have multiplicity 2. Thus

$$N_0^{3,\mathrm{irr}}(0,[1,1])=2\times 8+4\times 2+6\times 2=36.$$

For $(\alpha,\beta)=([0,1],1)$, the diagram of the leftmost shape has multiplicity 4, and all the other have multiplicity 1. Thus

$$N_0^{3,\mathrm{irr}}(0,[1,1])=1\times 4+3+3=10.$$

[3] the term 'boundary' comes from the definition (2.2), see Paragraph (4.1).

2 – Log-GW invariants of $\mathbf{P}^2$ and Hirzebruch surfaces

Despite the definition of logarithmic Severi varieties having been given in terms of embedded curves in Section 1.2, it will be necessary in the context of these notes to consider their counterparts as spaces of stable maps. Indeed we will need to consider stable maps that are not birational. Moreover, we will degenerate $\mathbf{P}^2$ to the unions of $\mathbf{P}^2$ and various copies of $\mathbf{F}_1$, and therefore we need to define these spaces not only for the projective plane. The results stated in this section initially appeared in [4] and [10].

(2.1) Definition. *Let X be a smooth variety. A* stable map *to X is a map $f : C \to X$ with source C a nodal curve, such that C has only finitely many automorphisms commuting with f.*

The finiteness of automorphisms condition boils down to the following requirement: for all irreducible components C_i of C such that $p_a(C_i) \leqslant 1$ and C_i is contracted by f, if $p_a(C_i) = 0$ (resp. $p_a(C_i) = 1$), then C_i has at least three (resp. one) intersection points with the other components of C.

(2.2) Stable maps relative to a curve. Let S be a smooth projective surface, and $R \subseteq S$ be a fixed smooth curve (not necessarily connected). The curve R is often called the *boundary* of the pair (S, R). Let $g \in \mathbf{Z}_{\geqslant 0}$, $\xi \in \mathrm{NS}(S)$, and $\alpha, \beta \in \underline{\mathbf{N}}$, such that $I\alpha + I\beta = \xi \cdot R$. Let $n \in \mathbf{N}$. Let $\Omega = \left(\{p_{i,j}\}_{1 \leqslant j \leqslant \alpha_i}\right)_{i \geqslant 1}$ be a set of α general points on R. We define $M_g^{\xi,\mathrm{con}}(\alpha, \beta)(\Omega)$ as the moduli space of stable maps $f : C \to S$ such that:

(i) C is a connected projective curve of arithmetic genus g;
(ii) $f_*[C] = \xi \in \mathrm{H}^2(S, \mathbf{Z})$;
(iii) $f(C)$ and R have no irreducible component in common;
(iv) there exist α points $q_{i,j} \in C$, $1 \leqslant j \leqslant \alpha_i$, and β points $r_{i,j} \in C$, $1 \leqslant j \leqslant \beta_i$, such that

$$f(q_{i,j}) = p_{i,j} \quad \forall\, 1 \leqslant j \leqslant \alpha_i, \qquad \text{and} \qquad f^*R = \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} i\, r_{i,j}.$$

If $x = \{x_1, \ldots, x_n\}$ is a set of n general points on S, we define $M_g^{\xi,\mathrm{con}}(\alpha, \beta)(\Omega, x)$ as the moduli subspace of $M_g^{\xi,\mathrm{con}}(\alpha, \beta)(\Omega)$ determined by the further requirement:

(v) $f(C) \supset x$ as sets.

Similarly, for all components M of $M_g^{\xi,\mathrm{con}}(\alpha, \beta)(\Omega)$, $M(x)$ is the subspace of M determined by condition (v) above.

The superscript 'con' stands for 'connected'; note that C is required to be connected but may be reducible. We do not require that f be birational.

An isomorphism between two stable maps $f_1 : C_1 \to X$ and $f_2 : C_2 \to X$ is an isomorphism $\phi : C_1 \to C_2$ such that $f_1 = f_2 \circ \phi$. The existence of moduli spaces of stable maps is a highly non-trivial result. We refer to [5] for an exposition in the case of plain stable maps; the result for relative stable maps (i.e., stable maps with additional features as in (2.2)) is due to Jun Li, see [7, 9].

(2.3) We shall consider instances of the space $M_g^{\xi,\mathrm{irr}}(\alpha, \beta)(\Omega)$ in the two following situations.

(2.3.1) $S = \mathbf{P}^2$, $\xi = dH$ with for some integer $d > 0$, and $R = \Lambda$ a line; we may then write $M_g^{d,\mathrm{con}}(\alpha, \beta)(\Omega)$ or even $M_g^{d,\mathrm{con}}(\alpha, \beta)$ in place of $M_g^{\xi,\mathrm{con}}(\alpha, \beta)(\Omega)$.

(2.3.2) $S = \mathbf{F}_1$, $\xi = \lambda H + kF$ for some nonnegative integers λ, k, and $R = \Lambda_E \sqcup \Lambda_H$, the disjoint union of two curves $\Lambda_E \in |E|$ and $\Lambda_H \in |H|$ (recall Notation (0.1)). In this case it is better to impose directly contact conditions with R that are decomposed in two disjoint set of contact conditions $(\alpha_E, \beta_E, \Omega_E)$ and $(\alpha_H, \beta_H, \Omega_H)$ with Λ_E and Λ_H respectively. One thus

obtains a set $M_g^{\xi,\mathrm{con}}(\alpha_E,\alpha_H,\beta_E,\beta_H)(\Omega_E,\Omega_H)$ of (isomorphism classes of) stable maps; we may abuse the notation of (2.2) and still denote this space by $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega)$, with $\alpha=(\alpha_E,\alpha_H)$, $I\alpha = I\alpha_E + I\alpha_H$ and $|\alpha| = |\alpha_E| + |\alpha_H|$, and similarly for $\beta=(\beta_E,\beta_H)$. One has

$$I\alpha_E + I\beta_E = \xi\cdot E = k, \qquad I\alpha_H + I\beta_H = \xi\cdot H = \lambda + k,$$

and $-(K_S+R) = 2F$.

We shall be interested only in those irreducible components M of $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega)$ that are *enumeratively relevant*, i.e., such that for a set x of $n=\dim(M)$ general points on S, the subspace $M(x)$ of those stable maps in M whose image contains x is not empty. The other irreducible components of M are said to be *enumeratively irrelevant*; their enumerative contribution is zero, and therefore they will always be ignored. Note that if the general member of M is birational onto its image, then M is enumeratively relevant.

(2.4) Theorem. *Assume we are either in situation (2.3.1) above, or in situation (2.3.2) and $\lambda>0$. Let M be an irreducible component of $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega)$. The family of those curves in S that are the image of a stable map in M has dimension at most*

$$-(K_S+R)\cdot\xi + g - 1 + |\beta| = \begin{cases} 2d+g-1+|\beta| & \text{in situation (2.3.1)} \\ 2\lambda+g-1+|\beta_E|+|\beta_H| & \text{in situation (2.3.2).}\end{cases}$$

The component M is enumeratively relevant if and only if equality holds, and in this case its general member $[f:C\to S]$ enjoys the following properties:

(i) the curve C is smooth (hence irreducible);
(ii) f is an immersion, birational onto its image;
(iii) $f(C)$ intersects R at smooth points;
(iv) for all finite subset $y\subseteq S$, if f is general with respect to y, then $f(C)$ does not intersect y.

In situation (2.3.1), the statement on the dimension and birationality follow from [III, Prop. (4.3)], and the rest from [III, Thm. (1.7)]. In situation (2.3.2) the proof is similar; the first part is proven in [10, Prop. 3.4], and the rest again follows from [III, Thm. (1.7)].

(2.5) In the cases not covered by (2.4), i.e., in situation (2.3.2) when $\xi=kF$ ($k>0$), let M be an irreducible component of $M_g^{kF,\mathrm{con}}(\alpha,\beta)(\Omega)$. The family of those curves in $\mathbf{F}_1$ that are the image of a stable map in M has dimension 0 or 1, always less or equal than

$$-(K_S+R)\cdot\xi + g - 1 + |\beta| = g - 1 + |\beta_E| + |\beta_H|,$$

and M is enumeratively relevant if and only if equality holds. The space $M_g^{kF,\mathrm{con}}(\alpha,\beta)(\Omega)$ has an enumeratively relevant irreducible component if and only if $g=0$, and either

$$\alpha_E=\beta_H=[0,\dots,0,1] \quad\text{and}\quad \beta_E=\alpha_H=0 \quad\text{or viceversa,}$$

or

$$\beta_E=\beta_H=[0,\dots,0,1] \quad\text{and}\quad \alpha_E=\alpha_H=0.$$

In those cases there is a unique enumeratively relevant component, which parametrizes degree k maps from $\mathbf{P}^1$ to a fibre F of $\mathbf{F}_1$, with total ramification over the two points $F\cap\Lambda_E$ and $F\cap\Lambda_H$.

This is a consequence of the basic observation that a connected member of the linear system $|kF|$ necessarily consists of one single fibre F with multiplicity k, hence is uniquely determined

by its passing through one single point $x_1 \in \mathbf{F}_1$. When $g > 0$, irreducible components of $M_g^{kF,\mathrm{con}}(\alpha,\beta)(\Omega)$ even have dimension strictly larger than the expected $g-1+|\beta|$, which itself is already strictly larger than the dimension of the corresponding families of curves in $\mathbf{F}_1$.

Note that condition (iv) of Thm. (2.4) holds for an enumeratively relevant component of $M_0^{kF,\mathrm{con}}(0,\beta)(\Omega)$.

(2.6) Log-GW invariants. In favourable circumstances (see, e.g., [10, Subsec. 2.3]), and in particular in the situations we are concerned with in this text, one may define the *logarithmic Gromov–Witten invariant* (log-GW invariant for short) of the pair (S,R) in genus g and class ξ, with boundary conditions (α,β) as the length, for a general choice of a set Ω of α points on R and of a set x of

$$n = -(K_S + R)\cdot\xi + g - 1 + |\beta|$$

points on S, of the 0-dimensional space $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega,x)$. The moduli space $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega)$ is endowed with the structure of a stack, and each point $[f : C \to S] \in M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega,x)$ should be counted with the multiplicity

$$\mu(f,x) = \frac{1}{\mathrm{Card}\,(\mathrm{Aut}\, f)} \prod_{1\leqslant i\leqslant r} \mathrm{Card}\, f^{-1}(x_i).$$

(2.7) In situations (2.3.1) and (2.3.2), it follows from (2.4) and (2.5) that the space $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega,x)$ is finite, and only those enumeratively relevant components of $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega)$ contribute to the corresponding log-GW invariant. It holds also true that the cardinality of $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega,x)$ is independent of the choice of Ω and x (as long as they are general). Moreover, conditions (ii) and (iii) in (2.4) together imply that $\mu(f,x) = 1$ for all $[f : C \to S] \in M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega,x)$. Correspondingly, in situation (2.3.1) say, there is a $1:1$ correspondence between $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega,x)$ and $V_g^{d,\mathrm{irr}}(\alpha,\beta)(\Omega) \cap \bigcap_{1\leqslant i\leqslant r} x_i^{\perp}$, where $x_i^{\perp}$ denotes the hyperplane of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ consisting of those divisors that contain x_i, so that

$$\sum_{f\in M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega,x)} \mu(f,x) = N_g^{d,\mathrm{irr}}(\alpha,\beta).$$

A similar result holds in situation (2.3.2) with $\lambda > 0$, see [10].

(2.8) Enumeratively relevant components of dimension at most 1. It will be useful in the proof of Proposition (3.3) to have a list of the enumeratively relevant components of dimension at most 1 of $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega)$ in situation (2.3.2), i.e., for the ruled surface $\mathbf{F}_1$. For such a component, we must have $\lambda \leqslant 1$. If $\lambda = 0$, we are in the situation of (2.5): enumerative relevance imposes $g = 0$, and all enumeratively relevant components have dimension at most 1. The corresponding log-GW invariants are computed in (2.9) below. If $\lambda = 1$, we must have $g = 0$ and $\beta = 0$. Then the space $M_0^{H+kF,\mathrm{irr}}(\alpha,0)(\Omega)$ is isomorphic to the sublinear system of $|H + kF|$ defined by the contact conditions imposed at Ω: indeed the arithmetic genus of $H + kF$ is 0, and the contact conditions at fixed points are linear. Therefore for a general point $x \in \mathbf{F}_1$, $M_0^{H+kF,\mathrm{irr}}(\alpha,0)(\Omega,x)$ has exactly one member, which has multiplicity 1.

Note moreover that in situation (2.3.1), i.e., for the projective plane, there is no enumeratively relevant components of of dimension 0 of $M_g^{\xi,\mathrm{con}}(\alpha,\beta)(\Omega)$, as follows from similar considerations.

(2.9) Multiplicity for multiple fibres. Now let us compute the log-GW invariants in the cases considered in (2.5) above, i.e., we consider the spaces $M_0^{kF,\mathrm{irr}}(\alpha,\beta)$ for the pair $(\mathbf{F}_1, \Lambda_E +$

Λ_H), and either (a) $\alpha_E = \beta_H = [0, \ldots, 0, 1]$ and $\beta_E = \alpha_H = 0$ (or viceversa), or (b) $\beta_E = \beta_H = [0, \ldots, 0, 1]$ and $\alpha_E = \alpha_H = 0$.

In case (a) Ω consists of one point, and $M_0^{kF,\mathrm{irr}}(\alpha, \beta)(\Omega)$ has a unique enumeratively meaningful component, of dimension 0, and in case (b) Ω is empty, the unique enumeratively meaningful component of $M_0^{kF,\mathrm{irr}}(\alpha, \beta)$ has dimension 1, hence one should impose the passing through 1 point x. In either case there is a unique stable map to consider, i.e., $M_0^{kF,\mathrm{irr}}(\alpha, \beta)(\Omega)$ in case (a) and $M_0^{kF,\mathrm{irr}}(\alpha, \beta)(x)$ in case (b) are supported on exactly one point, and this map is a degree k cover of the form $z \in \mathbf{P}^1 \mapsto z^k \in \mathbf{P}^1$ where the target $\mathbf{P}^1$ identifies with the relevant fibre of $\mathbf{F}_1$, and $f(0)$ and $f(\infty)$ are on Λ_E and Λ_H respectively. The automorphism group of this map is thus isomorphic to $\mathbf{Z}/k\mathbf{Z}$.

On the other hand, there is no point x in case (a), and the unique point x in case (b) has k different points in its preimage by f. Therefore the log-GW invariants are $\frac{1}{k}$ in case (a), and 1 in case (b).

3 – Degeneration of $\mathbf{P}^2$ to a chain of ruled surfaces of maximal length

This section contains the heart of Brugallé's proof of the Correspondence Theorem. We introduce the necessary degeneration of the projective plane, and provide (in Proposition (3.3)) the description of the limit enumerative problem that will be encoded in terms of floor diagrams in the next section.

(3.1) Degeneration of the plane. We shall construct a sequence $\pi_r : \mathcal{P}_r \to \mathbf{D}$, indexed by $r \in \mathbf{N}$, of degenerations over the unit disk $\mathbf{D}$ of the projective plane together with a distinguished line, with central fibre $\pi_r^{-1}(0)$ a reduced chain of rational surfaces

$$S_{r+1} \cup S_r \cup \cdots \cup S_1 \cup S_0$$

with $S_{r+1} = \mathbf{P}^2$ and $S_i = \mathbf{F}_1$ for $0 \leqslant i \leqslant r$: for all $i = 1, \ldots, r$, the surface S_i intersects only S_{i+1} and S_{i-1}, along two smooth rational curves, respectively, while S_{r+1} intersects only S_r and S_0 intersects only S_1, along one smooth rational curve. We set $\Lambda_i = S_i \cap S_{i-1}$ for all $i = 1, \ldots, r+1$, and require that it is linearly equivalent to H on S_i and to E on S_{i-1}. We require moreover that the limit as $t \in \mathbf{D}^*$ tends to 0 of the distinguished lines in the planes $\pi_r^{-1}(t)$, which we will denote by $R(t)$, be a smooth rational curve Λ_0 in S_0, linearly equivalent to H.

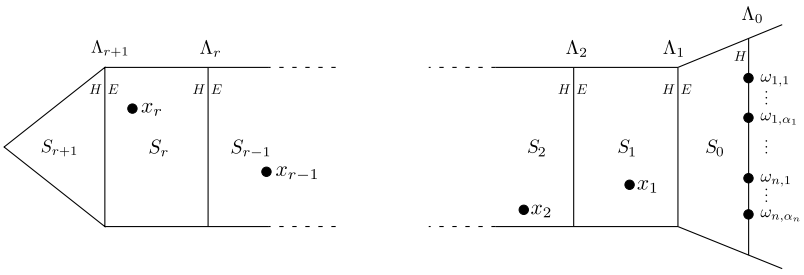


Figure 5: The degeneration $\pi_r : \mathcal{P}_r \to \mathbf{D}$, together with limit base points and tangency points along the boundary

The construction is by induction on $r \geqslant 0$. To construct $\mathcal{P}_0$, consider the trivial family $\mathbf{P}^2 \times \mathbf{D}$ together with a trivial family of lines $\Lambda \times \mathbf{D}$, and let $\varepsilon_0 : \mathcal{P}_0 \to \mathbf{P}^2 \times \mathbf{D}$ be the blow-up

of $\Lambda \times \{0\}$, the distinguished line in the central fibre, and let $\pi_0 := \mathrm{pr}_2 \circ \varepsilon_0$; then $S_1 \cong \mathbf{P}^2$ is the proper transform of the initial central plane $\mathbf{P}^2 \times \{0\}$, $S_0 \cong \mathbf{F}_1$ is the exceptional divisor of ε_0, and Λ_0 is the intersection with S_0 of the proper transform of the initial family of lines $\Lambda \times \mathbf{D}$.

The inductive construction is pretty much identical. Assuming $\pi_r : \mathcal{P}_r \to \mathbf{D}$ is constructed, let $\varepsilon_{r+1} : \mathcal{P}_{r+1} \to \mathcal{P}_r$ be the blow-up of $\Lambda_0 \subseteq \mathcal{P}_r$, and $\pi_{r+1} := \pi_r \circ \varepsilon_{r+1}$. In $\mathcal{P}_{r+1}$ the surface S_0 is the exceptional divisor of ε_{r+1}, and the curve Λ_0 is the central fibre of the proper transform of the family of lines in $\mathcal{P}_r$ (or equivalently of the proper transform by $\varepsilon_0 \circ \cdots \varepsilon_{r+1}$ of the initial family of lines $\Lambda \times \mathbf{D}$). For $i = 1, \ldots, r+2$, the surface S_i in $\mathcal{P}_{r+1}$ is the proper transform of S_{i-1} in $\mathcal{P}_r$.

(3.2) Degenerations of stable maps. We now fix $d, g \in \mathbf{Z}_{\geqslant 0}$ and $\alpha, \beta \in \underline{\mathbf{N}}$ such that $I\alpha + I\beta = d$, and a set Ω of α points on Λ. We want to look at degenerations of the space $M_g^{d,\mathrm{irr}}(\alpha, \beta)(\Omega, x)$ for the pair $(\mathbf{P}^2, \Lambda)$ as the projective plane degenerates as in (3.1) above, for some specific degeneration of a set x of $r = 2d + g - 1 + |\beta|$ points, as follows.

We consider a set $x = \{x_1, \ldots, x_r\}$ of sections of $\pi_r : \mathcal{P}_r \to \mathbf{D}$ such that for all $t \in \mathbf{D}^*$, the set $x(t)$ consists of r general points in the plane $\pi_r^{-1}(t)$, and for $t = 0$, $x(0) = \{x_1(0), \ldots, x_r(0)\}$ with $x_i(0)$ a general point of $S_i \subseteq \pi_r^{-1}(0)$ for all $i \in [\![1, r]\!]$. We choose in addition a set Ω of α sections of $\pi_r : \mathcal{P}_r \to \mathbf{D}$, such that for all $t \in \mathbf{D}$, $\Omega(t)$ is a set of α general points on the line $R(t)$.

The points $x_1(0), \ldots, x_r(0)$ are indicated on Figure 5 (where they are simply indicated as $x_1, \ldots, x_r$ to lighten the picture), as well as the points of $\Omega(0)$, indicated on the picture as $\omega_{1,1}, \ldots, \omega_{1,\alpha_1}, \ldots, \omega_{n,1}, \ldots, \omega_{n,\alpha_n}$. Note that S_{r+1}, the proper transform of the central fibre of $\mathbf{P}^2 \times \mathbf{D}$, does not support anything, S_i supports the base point $x_i(0)$ for all $i = 1, \ldots, r$, and S_0 supports all the fixed contact points on the boundary Λ_0.

We then let $M_g^{d,\lim}(\alpha, \beta)(\Omega, x)$ be the limit as $t \in \mathbf{D}^*$ tends to 0 of the moduli spaces of stable maps $M_g^{d,\mathrm{irr}}(\alpha, \beta)(\Omega(t), x(t))$ for the pairs $(\pi_r^{-1}(t), R(t))$. It is a moduli space of stable maps $f : C \to \pi_r^{-1}(0)$ satisfying the following conditions:

(i) C is a connected nodal curve of arithmetic genus g;
(ii) $f(C) \supset x(0)$;
(iii) $f(C)$ does not contain Λ_0, and there exist α points $q_{i,j} \in C$, and β points $r_{i,j} \in C$ such that
$$\big\{\phi(q_{i,j})\big\} = \Omega(0) \qquad \text{and} \qquad \phi^*\Lambda_0 = \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} i\, r_{i,j};$$
(iv) if $p \in f^{-1}(S_{i+1} \cap S_i)$ does not lie on a component of C which is entirely mapped to $S_{i+1} \cap S_i$, then it is a node of C, the intersection of two components C' and C'' mapped to S_{i+1} and S_i respectively, and appears with the same multiplicity μ_p in $(f|_{C'})^*\Lambda_{i+1}$ and $(f|_{C''})^*\Lambda_{i+1}$.

That such a limit of moduli spaces of stable maps indeed exists is a highly non-trivial fact which is due to Jun Li, see [8, 9].

The following result is the core of the proof by degeneration of the Correspondence Theorem (1.9).

(3.3) Proposition (Brugallé)**.** *The moduli space $M_g^{d,\lim}(\alpha, \beta)(\Omega, x)$ (defined in (3.2) above) is finite, and depends on (Ω, x) only by way of $(\Omega(0), x(0))$.*

Let $\big[f : C \to \pi_r^{-1}(0)\big] \in M_g^{d,\lim}(\alpha, \beta)(\Omega, x)$. Then all the irreducible components $C_1, \ldots, C_n$ of C are smooth rational curves, none of them is entirely mapped to one of the curves Λ_i, $0 \leqslant i \leqslant r+1$, and the following conditions hold after suitable relabelling:

(i) $C_1, \ldots, C_d$ are pairwise disjoint smooth rational curves, and there exist $1 \leqslant a_1 < \cdots < a_d \leqslant r$ such that for each $i \in [\![1, d]\!]$ the curve $f(C_i)$ is a member of the linear system

$|H + k_i F|$ on S_{a_i} (for some nonnegative integer k_i) and passes through the point $x_{a_i}(0) \in S_{a_i}$;

(ii) the remaining curves form $r - d$ pairwise disjoint chains of smooth rational curves

$$\bar{C}_j = C_{d+\ell_1+\cdots+\ell_{j-1}+1} \cup \cdots \cup C_{d+\ell_1+\cdots+\ell_{j-1}+\ell_j}$$

(i.e., for all $1 \leqslant s < s' \leqslant \ell_j$, $C_{d+\ell_1+\cdots+\ell_{j-1}+s}$ and $C_{d+\ell_1+\cdots+\ell_{j-1}+s'}$ intersect if and only if $s' = s+1$, and in that case transversally in exactly one point, and $\bar{C}_j \cap \bar{C}_{j'}$ is empty if $j \neq j'$); there are integers $b_1, \ldots, b_{r-d} \in [\![0, r]\!]$ and $\mu_1, \ldots, \mu_{r-d} \geqslant 1$ such that the divisor $f_*(C_{d+\ell_1+\cdots+\ell_{j-1}+s})$, $1 \leqslant j \leqslant r-d$ and $1 \leqslant s \leqslant \ell_j$, belongs to the linear system $|\mu_j F|$ on the ruled surface S_{b_j+s-1}; for all $j = 1, \ldots, r-d$, the chain $\bar{C}_j$ intersects exactly 1 or 2 curves among $C_1, \ldots, C_d$, necessarily at points mapped to $\Lambda_r \cup \cdots \cup \Lambda_1$ by f, and $f(\bar{C}_j)$ passes through exactly one of the points in $(\Omega(0), x(0))$.

(3.4) Example. Depicted in Fig. 6a is an example of a source curve C of a member $\big[f : C \to \pi_7^{-1}(0)\big]$ of $M_0^{3,\lim}(0, [1,1])$, together with its image by f in Fig. 6b. The circled numbers on Fig. 6b indicate that the restrictions of f to C_4, C_5, C_6 have degree 2; the restrictions to all other irreducible components of C are birational.

The picture on S_3 means that $f(C_1)$ is the only curve in S_3: it is irreducible, linearly equivalent to $H + 2F$.[4] On the other hand, there are two irreducible components of $f(C)$ in S_6, linearly equivalent to H and F, respectively. Note that f normalizes the one node indicated by a square dot on S_6.

To help the reader get their way around, I indicated on Fig. 6a one point in the pre-image of each element of $x(0)$; there is only one point in $f^{-1}(x_i(0))$ if $i \neq 1$, but there are two points in $f^{-1}(x_1(0))$, both in C_5. In this example one has

$$(a_1, a_2, a_3) = (3, 6, 7) \qquad \bar{C}_1 = C_4 \cup C_5 \cup C_6 \qquad \bar{C}_3 = C_{10} \cup C_{11}$$
$$(b_1, b_2, b_3, b_4) = (0, 0, 4, 4), \qquad \bar{C}_2 = C_7 \cup C_8 \cup C_9 \qquad \bar{C}_4 = C_{12} \cup C_{13} \cup C_{14}.$$

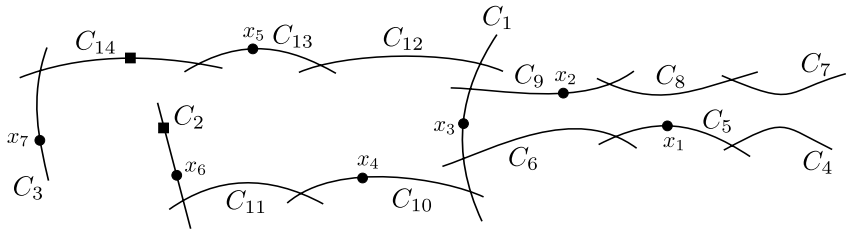


Figure 6a: Source curve of a member of $M_0^{3,\mathrm{irr}}(0, [1,1])(x(0))$

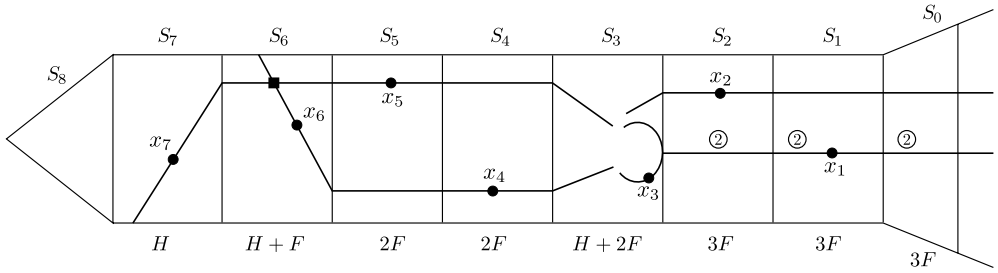


Figure 6b: Image in $\pi_7^{-1}(0)$

[4] the picture is slightly incorrect, as $f(C_1)$ should be unisecant to all fibres of the ruling of S_3; we will only be able to fix this when we use tropical representations of curves, see [X, Section 9]. There is the same issue with the two curves linearly equivalent to H on S_6 and S_7, respectively.

(3.5) Proof of Proposition (3.3). Let $\big[f : C \to \pi_r^{-1}(0)\big] \in M_g^{d,\lim}(\alpha,\beta)(\Omega, x)$. The fact that no component of C is entirely mapped to one of the curves Λ_i follows from [6, Example 11.4 and Lemma 14.6], and we will take it for granted. The other assertions essentially follow from a series of dimension counts, which are now going to expound.

Let $C_1, \ldots, C_n$ be the irreducible components of C. For all $j = 1, \ldots, n$, we let $f_j : C_j \to S_{s(j)}$ be the restriction of f to C_j, g_j the arithmetic genus of C_j, ξ_j the linear equivalence class of $f_*(C_j)$ on the surface $S_{s(j)}$ (note that a priori ξ_j may vanish, in case C_j is contracted), $\zeta_j \in \{0,1\}$ the number of points in $x(0)$ contained in $f_j(C_j)$, and $\beta_j^E, \beta_j^H \in \underline{\mathbf{N}}$ the types of contact of $f_j(C_j)$ with $\Lambda_{s(j)+1}$ and $\Lambda_{s(j)}$ respectively, pretending that all contacts are at unassigned points. Thus

$$\big[f_j\big] \in M_{g_j}^{\xi_j,\mathrm{irr}}(0,0,\beta_j^E,\beta_j^H)$$

as a moduli space of stable maps to $S_{s(j)}$ with boundary $R = \Lambda_{s(j)} \coprod \Lambda_{s(j)+1}$, where the set Ω is empty, and where we choose to forget the datum of the possible point of $x(0)$ contained in $f(C_j)$. (This all comes with a harmless abuse of notation in case $s(j) = r+1$; in fact we shall see that this case never occurs).

Then we consider $M_j \subseteq M_{g_j}^{\xi_j,\mathrm{irr}}(0,0,\beta_j^E,\beta_j^H)$ the smallest irreducible subvariety in which $[f_j]$ moves as we let x vary, maintaining the requirements of (3.2). We let CM_j be the corresponding family of curves in $S_{s(j)}$. By generality of the point $x_{s(j)}(0)$ in $S_{s(j)}$, we must have $\dim(CM_j) \geqslant \zeta_j$; let $\varepsilon_j = \dim(CM_j) - \zeta_j$. By Theorem (2.4) and Paragraph (2.5) we also have:

$$\zeta_j \leqslant \dim(CM_j) = \zeta_j + \varepsilon_j \leqslant 2F \cdot \xi_j + g_j - 1 + |\beta_j^E| + |\beta_j^F|. \tag{3.5.1}$$

On the other hand, the various $f(C_j)$'s are subject to matching conditions at the boundary of the various $S_{s(j)}$, in order for condition (iv) of (3.2) to hold. (A priori there may be other matching conditions if the C_j's intersect at points mapped off $\Lambda_{r+1} \cup \cdots \cup \Lambda_1$ by f, but we don't need to consider them for the moment; we shall see that in fact this does not happen). These matching conditions are in the number of

$$a = \mathrm{Card}\big(f^{-1}(\Lambda_{r+1} \cup \cdots \cup \Lambda_1)\big),$$

and they are independent by Theorem (2.4), (2.5), and [III]: consider some curve $f(C_{j_0})$, and assume all the other components $f(C_j)$ are fixed; then the matching conditions imposed to $f(C_{j_0})$ amount to turn some of the contact conditions β_j^E, β_j^H at unassigned points to contact conditions at assigned points, and it follows from [III, (1.7.0)] that each assigned contact point makes the dimension drop by 1. Note that the assigned contact points may not be in general position; fortunately this does not cause any trouble, as we have observed in [III].

Lastly, the assigned contact points $\Omega(0)$ on Λ_0 impose $|\alpha|$ further conditions, independent from all the rest for the same reasons as before. Therefore, if we let CM be the smallest irreducible variety in which $f(C)$ moves as we let x vary, then CM has codimension at least $a + |\alpha|$ in $\prod_j CM_j$.

In the upshot, we must have:

$$\textstyle\sum_j \dim(CM_j) \geqslant \big(\sum_j \zeta_j\big) + a + |\alpha| = r + a + |\alpha|. \tag{3.5.2}$$

On the other hand, summing the inequalities (3.5.1) for all $j = 1, \ldots, n$, we get

$$\begin{aligned}\sum_j \dim(CM_j) &\leqslant \sum_j \Big(2F \cdot \xi_j + g_j - 1 + |\beta_j^E| + |\beta_j^F|\Big) \\ &\overset{\star}{=} 2d + g - 1 + |\beta| + |\alpha| + a - a' = r + |\alpha| + a - a',\end{aligned} \tag{3.5.3}$$

where a' is the number of intersection points of the C_j's mapped off $\Lambda_{r+1}\cup\cdots\cup\Lambda_1$ by f; equality $(\star)$ is given by the three relations of Lemma (3.6) below. Comparing with (3.5.2), we get that

$$a' = 0 \quad \text{and} \quad \textstyle\sum_j \dim(CM_j) = r + |\alpha| + a,$$

which in particular implies that the family of curves corresponding to $M_g^{d,\lim}(\alpha,\beta)(\Omega,x)$ is finite and depends only on $(\Omega(0),x(0))$.

The inequalities (3.5.2) and (3.5.3) also imply that for all $j=1,\dots,n$ one has

$$\dim(CM_j) = \zeta_j + \varepsilon_j = 2F\cdot\xi_j + g_j - 1 + |\beta_j^E| + |\beta_j^F|, \tag{3.5.4}$$

and moreover $\sum_j \varepsilon_j = |\alpha| + a$. By Theorem (2.4) and (2.5), Equation (3.5.4) implies that each M_j is an enumeratively relevant irreducible component of $M_{g_j}^{\xi_j,\mathrm{irr}}(0,0,\beta_j^E,\beta_j^H)$. In particular, the finiteness of the family of curves in $\bigcup_j S_j$ corresponding to $M_g^{d,\lim}(\alpha,\beta)(\Omega,x)$ implies the finiteness of $M_g^{d,\lim}(\alpha,\beta)(\Omega,x)$ itself.

These finiteness results imply that for all j, if we fix the intersection points of $f(C_j)$ with $\Lambda_{s(j)+1}$ and $\Lambda_{s(j)}$ (again with a slight abuse of notation if $s(j)=r+1$), then $[f_j]$ moves in dimension at most ζ_j. By (2.8), this implies that for all j one has $g_j=0$, $s(j)<r+1$, and

$$\xi_j = \begin{cases} k_jF & \text{if } \zeta_j = 0 \\ k_jF \text{ or } H+k_jF & \text{if } \zeta_j = 1. \end{cases}$$

By relation (3.6.3) below, there are exactly d indices j such that ξ_j is of the form $H+k_jF$ and, relabelling if necessary, we will assume that they are $j=1,\dots,d$. Then, the indices $s(1),\dots,s(d)$ are pairwise distinct and in the interval $[\![1,r]\!]$, because there is only one point of $x(0)$ on each of $S_1,\dots,S_r$.

Condition (iv) of (3.2) imposes that the curves $C_{d+1},\dots,C_n$ (i.e., those C_j such that $\xi_j = k_jF$) organize in chains of fibres of the same multiplicity as stated in Proposition (3.3). The vanishing of a' imposes that each chain meets exactly one or two among $C_1,\dots,C_d$, at points mapped to $\Lambda_{r+1}\cup\cdots\cup\Lambda_1$. The finiteness results above imply that each chain must pass through exactly one point of $(\Omega(0),x(0))$.

It remains to show that two curves among $C_1,\dots,C_d$ cannot intersect each other. Assume by contradiction that $f(C_1)\subseteq S_1$, $f(C_2)\subseteq S_2$, and C_1 and C_2 intersect at a point mapped to $y\in S_1\cap S_2=\Lambda_2$. Then we would find a 1-dimensional family of stable maps in $M_g^{d,\lim}(\alpha,\beta)(\Omega,x)$ by letting y move along Λ_2, a contradiction. This ends the proof of Proposition (3.3). □

(3.6) Lemma. *In the above notation, we have the following relations:*

$$\textstyle\sum_j\big(|\beta_j^E| + |\beta_j^H|\big) = 2a + |\alpha| + |\beta|; \tag{3.6.1}$$

$$g = \big(\textstyle\sum_j g_j\big) + 1 - n + a + a'. \tag{3.6.2}$$

$$\textstyle\sum_j (\xi_j\cdot F)_{S_{s(j)}} = d; \tag{3.6.3}$$

Proof. The points corresponding to those β_j^E such that $0\leqslant s(j)\leqslant r$, and equally those corresponding to those β_j^H such that $1\leqslant s(j)\leqslant r+1$, are exactly the points in the singular locus of $\pi_r^{-1}(0)$ at which the matching conditions referred to in the proof of Proposition (3.3) above happen, which are in the number a. Each such point is counted twice, once in some number $\beta_{s(j)}^E$, and once in the corresponding number $\beta_{s(j)+1}^H$. On the other hand, those β_j^H such that $s(j)=0$ sum up as the number of points in $\Omega(0)$, which is $|\alpha|+|\beta|$. This proves (3.6.1).

Relation (3.6.2) is the expression of the arithmetic genus of the nodal curve C, which has n irreducible components $C_1\dots,C_n$ of respective genera $g_1,\dots,g_n$, and $a+a'$ nodes, where a' is the number of intersection points among the C_j's that are mapped off $\Lambda_{r+1}\cup\cdots\cup\Lambda_1$ by f.

Relation (3.6.3) comes with an abuse of notation, namely that for those j such that $s(j) = r+1$, F should be replaced by H. Then the curve class on the limit surface $\pi_r^{-1}(0)$ consisting of the class F on all S_i, $i = 0, \dots, r+1$ (which thus has to be taken as F on all S_i, $i = 0, \dots, r$, and H on S_{r+1}), is a limit of the class H on the projective planes $\pi_r^{-1}(t)$ as $t \in \mathbf{D}^*$ tends to $0 \in \mathbf{D}$. Relation (3.6.3) is then a consequence of the fact that $\left[f : C \to \pi_r^{-1}(0)\right] \in M_g^{d,\lim}(\alpha,\beta)(\Omega, x)$ is a limit of stable maps realizing the curve class dH on the projective planes $\pi_r^{-1}(t)$, $t \in \mathbf{D}^*$. See also (5.1) for another point of view on this argument. □

Proposition (3.3) has the following corollary, which we will translate in Section 4 below as saying that the number of plane curves $N_g^{d,\mathrm{irr}}(\alpha,\beta)$ may be computed by enumerating floor diagrams.

(3.7) Corollary. *In the notation of (3.1) to (3.3), one has*

$$N_g^{d,\mathrm{irr}}(\alpha,\beta) = \sum_{[f:C\to\pi_r^{-1}(0)]\in M_g^{d,\lim}(\alpha,\beta)(\Omega,x)} \mu(f)$$

where $\mu(f)$ is defined as

$$\mu(f) = \left(\prod_{p\in f^{-1}(\Lambda_{r+1}\cup\cdots\cup\Lambda_1)} \mu_p\right)\left(\prod_{p\in x(0)} \mathrm{Card}\, f^{-1}(p)\right)\left(\prod_{1\leqslant j\leqslant n} \mathrm{Card}(\mathrm{Aut}(f|_{C_j}))\right)^{-1}$$

(recall in particular the definition of μ_p in Condition (iv) *of (3.2)).*

Proof. By Proposition (3.3) and Jun Li's Formula, see [8, 9], for all $t \in \mathbf{D}^*$, one has

$$\sum_{f\in M_g^{\xi,\mathrm{irr}}(\alpha,\beta)(\Omega(t),x(t))} \mu(f,x(t)) = \sum_{[f:C\to\pi_r^{-1}(0)]\in M_g^{d,\lim}(\alpha,\beta)(\Omega,x)} \mu(f).$$

The result follows, since the left-hand-side of the equation above equals $N_g^{d,\mathrm{irr}}(\alpha,\beta)$, as we have seen in (2.7). □

(3.8) It will be useful to have a ready-to-use expression for the multiplicity $\mu(f)$ of a stable map $[f] \in M_g^{d,\lim}(\alpha,\beta)(\Omega,x)$ as in Proposition (3.3). Only the chains of multiple fibers $\bar{C}_1, \dots, \bar{C}_{r-d}$ may contribute non-trivially to the multiplicity $\mu(f)$, and their respective contributions may be computed in a similar way as in (2.9).

Let $\bar{C}_j$ be such a chain, with multiplicity μ_j and length ℓ_j (the number of its irreducible components). There are the following three cases:

(a) no irreducible components of $\bar{C}_j$ is mapped to S_0: then $f(\bar{C}_j)$ passes through exactly one point p of $x(0)$, for which $f^{-1}(p)$ has cardinality μ_j, there are $\ell_j + 1$ points $p \in f^{-1}(\Lambda_{r+1} \cup \cdots \cup \Lambda_1)$ lying on $\bar{C}_j$, all with multiplicity $\mu_p = \mu_j$, and for each irreducible component of $\bar{C}_j$, there is an automorphism group of cardinality μ_j. Thus the contribution of the chain $\bar{C}_j$ to the multiplicity $\mu(f)$ is
$$\mu_j^{\ell_j+1} \cdot \mu_j \cdot \mu_j^{-\ell_j} = \mu_j^2;$$

(b) there is an irreducible components of $\bar{C}_j$ mapped to S_0, and $f(\bar{C}_j)$ passes through exactly one point p of $x(0)$ and no point of $\Omega(0)$: then there are only ℓ_j points $p \in f^{-1}(\Lambda_{r+1}\cup\cdots\cup\Lambda_1)$ lying on $\bar{C}_j$, but otherwise the situation is the same as in case (a). The contribution of the chain $\bar{C}_j$ to the multiplicity $\mu(f)$ is thus
$$\mu_j^{\ell_j} \cdot \mu_j \cdot \mu_j^{-\ell_j} = \mu_j;$$

(c) there is an irreducible components of $\bar{C}_j$ mapped to S_0, and $f(\bar{C}_j)$ passes through exactly one point p of $\Omega(0)$ and no point of $x(0)$: then there are ℓ_j points $p \in f^{-1}(\Lambda_{r+1} \cup \cdots \cup \Lambda_1)$ lying on $\bar{C}_j$ as in case (b), but now there is no point $p \in x(0)$ contributing to the multiplicity. The contribution of the chain $\bar{C}_j$ to the multiplicity $\mu(f)$ is thus

$$\mu_j^{\ell_j} \cdot 1 \cdot \mu_j^{-\ell_j} = 1.$$

4 – Combinatorial translation

We are now ready to give the proof of [2] of the Correspondence Theorem (1.9). In fact, most of the work has already been done in Section 3, and it only remains to interpret Proposition (3.3) and Corollary (3.7) in terms of floor diagrams.

(4.1) Proof of the Correspondence Theorem (1.9). We associate each stable map $[f : C \to \pi_r^{-1}(0)] \in M_g^{d,\lim}(\alpha,\beta)(\Omega, x)$ as in Proposition (3.3) with a marked floor diagram of genus g, degree d, and type (α,β) as follows.

We first identify the vertices. To each irreducible component C_i of C such that $f_*C_i = H + kF$ on some $S_{s(i)}$ we associate a floor, and to each point in $f(C) \cap \Lambda_0$ we associate a vertex at infinity. We provide the vertices with an ordering, which in turn will determine the orientation of the edges: each vertex v corresponds to an object lying on a unique ruled surface $S_{s(v)}$, and for all vertices v and v', we declare that v precedes v' if $s(v) \leqslant s(v')$.

Now let us define the edges: they are in bijective correspondence with the chains $\bar{C}_j$ described in (ii) of Proposition (3.3), and the weight of the edge corresponding to $\bar{C}_j$ is the positive integer μ_j. The edge corresponding to $\bar{C}_j$ is adjacent to the floor corresponding to C_i (resp. the vertex at infinity corresponding to $p \in \Lambda_0$) if $\bar{C}_j$ and C_i intersect at a point (resp. if $f(\bar{C}_j)$ passes through p).

It follows from Proposition (3.3) that the object constructed so far is a floor diagram $\mathcal{D}$ of genus g and degree d, and the corresponding set $\mathsf{D} = \mathrm{Vert}^{\mathrm{fin}} \coprod \mathrm{Edges}$ is in bijection with $\Omega(0) \coprod x(0)$. The set $x(0)$ naturally comes with a bijection with $[\![1, r]\!]$ (remember that each $x_i(0)$ lies on the surface S_i). By appropriately labelling the points in $\Omega(0)$ with indices in $[\![-|\alpha| + 1, 0]\!]$, we obtain a marking of $\mathcal{D}$ of type (α,β). There are several possible ways of labelling the points in $\Omega(0)$, but they are equivalent up to automorphisms of the marked floor diagram.

Conversely, from any marked floor diagram of type (α,β) one can construct a stable map as in Proposition (3.3), and this is a point in $[f : C \to \pi_r^{-1}(0)] \in M_g^{d,\lim}(\alpha,\beta)(\Omega, x)$. It only remains to observe that the multiplicity of the map $[f : C \to \pi_r^{-1}(0)] \in M_g^{d,\lim}(\alpha,\beta)(\Omega, x)$ in the formula of Corollary (3.7) equals that of the corresponding floor diagram, which follows from (3.8) and Definition (1.8). □

(4.2) Example. I shall now illustrate the correspondence of (4.1) between floor diagrams and limit curves on various examples pertaining to plane cubic curves. I will only consider curves with trivial boundary conditions, i.e., $(\alpha,\beta) = (0, 3)$, so nothing relevant happens on S_0 (and on S_{r+1} either, as always), and thus I will only represent the S_i's for $i = 1, \ldots, r$.

Under each ruled surface S_i, I have indicated the total curve class supported on this component, i.e., the sum of the $\xi_j = f_*C_j$ for all j such that C_j is mapped to S_i. The limit base points $x_1(0), \ldots, x_r(0)$ on $S_1, \ldots, S_r$ respectively are indicated by black disks.

Note that (obviously, I hope) to obtain a proper graphic correspondence between our representations of floor diagrams and limit curves, the diagrams should be rotated by $+\frac{\pi}{2}$.

(4.2.1) Limit of a smooth cubic. Below is displayed the limit curve of a smooth cubic, corresponding to the unique possible marked floor diagram of degree 3 and genus 1 (and type $(0,3)$). The two curves in S_7 and S_4 are irreducible. Later on (in Lecture [X]) we will have a better way of representing these curves, coming from tropical geometry.

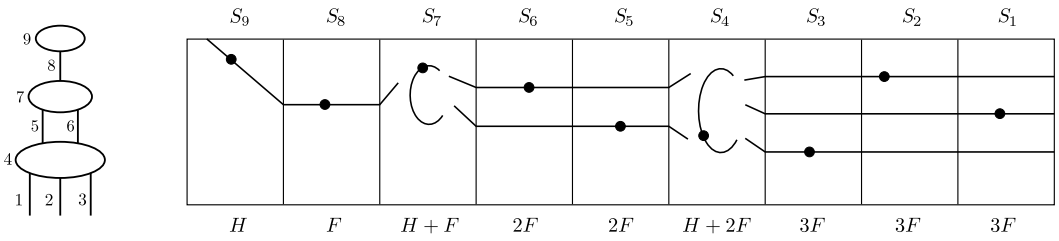


Figure 7: Limit of a smooth cubic

(4.2.2) Various limits of rational cubics. Below I display limit curves corresponding to the various diagrams of degree 3 and genus 0 (and type $(0,3)$), see Example (1.7.1). For the diagram shape at the center of Figure 2, I consider two different markings (Figures 9 and 10 below), so that one can better see what is encoded in the marking.

On Figure 8, the two curves in S_6 and S_4 are irreducible, and tangent to $S_6 \cap S_5$ and $S_5 \cap S_4$ respectively; the curve on S_5 is a fibre with multiplicity 2.

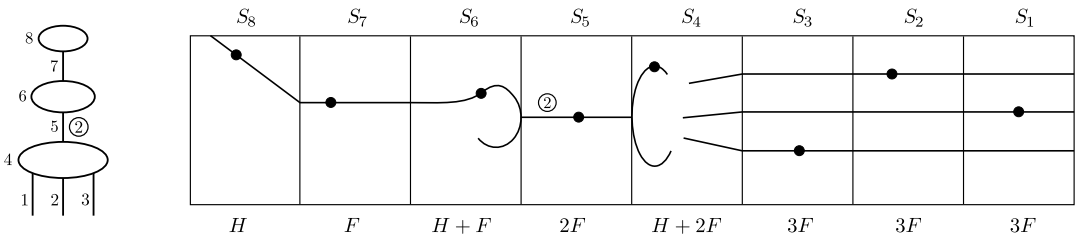


Figure 8: Limit of rational cubics (a)

On Figure 9, the curve in S_6 is irreducible, and that in S_4 has two components of classes $H+F$ and F respectively, which intersect at one point represented by a full black square. On Figure 10 the situation is similar.

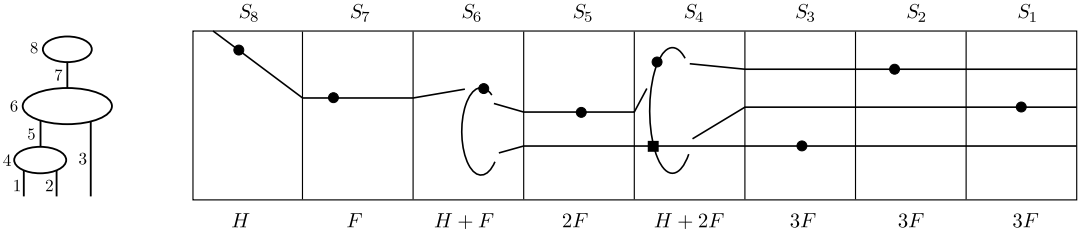


Figure 9: Limit of rational cubics (b)

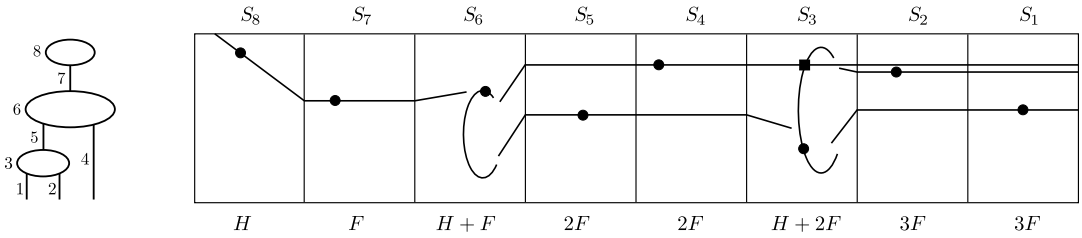


Figure 10: Limit of rational cubics (b')

On Figure 11, the curve in S_4 is irreducible, and that in S_7 has two components of classes H and F respectively, which intersect at one point represented by a black square.

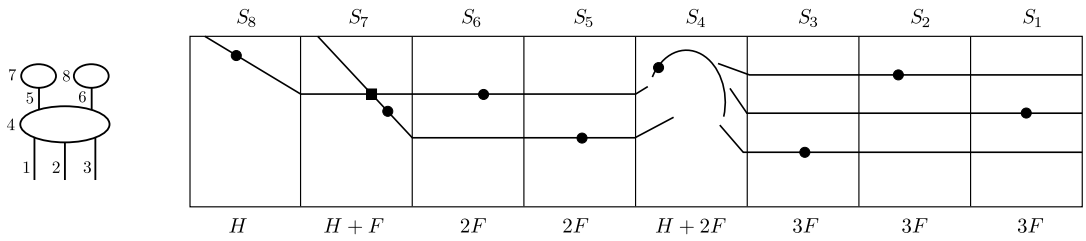


Figure 11: Limit of rational cubics (c)

I have tried to maintain the black dots representing the points of $x(0)$ in the series of Figures 8–11 at a fixed position. I have only cheated a little bit on Figure 11, on which I have moved the base point on S_7, for otherwise the picture was outrageously unclear.

5 – Projective geometric interpretation

In this section, I propose a formulation of the degenerations of stable maps considered in this lecture in terms of degenerations of nodal curves, in the framework of the theory expound in [I].

(5.1) Twists. Let us consider the degeneration $\pi_r : \mathcal{P}_r \to \mathbf{D}$ of the projective plane introduced in (3.1). It comes with a map $\bar{\varepsilon}_r : \mathcal{P}_r \to \mathcal{P} := \mathbf{P}^2 \times \mathbf{D}$ which is the composition of blow-ups $\bar{\varepsilon}_r = \varepsilon_r \cdots \varepsilon_0$. Consider the line bundle

$$\mathcal{L}_{r,d} = \bar{\varepsilon}_r^* \operatorname{pr}_1^*\big(\mathcal{O}_{\mathbf{P}^2}(d)\big);$$

it restricts to $\mathcal{O}_{\mathbf{P}^2}(d)$ on the fibres $\pi_r^{-1}(t)$, $t \in \mathbf{D}^*$, and thus its restriction to the central fibre $\pi_r^{-1}(0)$ defines a possible linear system for the linear system of plane degree d curves to degenerate to within the degeneration $\pi_r : \mathcal{P}_r \to \mathbf{D}$. In this paragraph I will describe the various *twists* $\mathcal{L}_{r,d}(-W)$ that have the same property: the requirements are that W is an effective divisor of $\mathcal{P}_r$ supported on the central fibre, such that the restriction $\mathcal{L}_{r,d}(-W)|_{\pi_r^{-1}(0)}$ defines a linear system on the central fibre that has the same dimension as $|\mathcal{O}_{\mathbf{P}^2}(d)|$.

It is elementary to prove that these twists all have the form

$$\mathcal{L}_{r,d}\big(-\textstyle\sum_{i=0}^r m_i \bar{S}_i\big),$$

where $\bar{S}_i = \sum_{i' \leqslant i} S_{i'}$, and the m_i's are non-negative integers subject to the inequalities

$$d \geqslant m_r \geqslant \cdots \geqslant m_0.$$

As a sideremark, note that this gives an alternative way of proving the relation (3.6.3). Since $\mathcal{L}_{r,d}$ restricts to dH on S_{r+1} and to dF on the other S_i's, the line bundle $\mathcal{L}_{r,d}\big(-\sum_{i=0}^{r} m_i \bar{S}_i\big)$ restricts to the following linear equivalence classes on the various irreducible components of the central fibre of π_r:

— on S_{r+1}: $(d-m_r)H$;
— on S_i, $0<i\leqslant r$: $dF+m_iE-m_{i-1}H=(m_i-m_{i-1})H+(d-m_i)F$;
— on S_0: $dF+m_0=m_0H+(d-m_0)F$.

Now, the choice of the limiting positions of the base points that has been made in (3.2), i.e., the choice that $x_i(0)$ lies on S_i for all $i=1,\dots,r$, implies that the twists that are relevant for the enumeration of limit curves are subject to the further restrictions:

$$m_r=d;\quad m_0=0;\quad m_i-m_{i-1}\leqslant 1\quad \text{for all } i=r,\dots,1.$$

This corresponds to the fact that the classes ξ_j, in the notation of Proposition (3.3), are necessarily of the form $H+k_jF$ or k_jF.

(5.1.1) Example. The twist giving the limiting linear system in which the curve represented on Figure 6b lives is the following:

$$\mathcal{L}_{7,3}\big(-3\bar{S}_7-2\bar{S}_6-\bar{S}_5-\bar{S}_4-\bar{S}_5\big).$$

(5.2) Projective models of the degeneration. I now describe the projective model of the degeneration $\pi_r:\mathcal{P}_r\to\mathbf{D}$ given by a twist which is relevant for the enumeration of limit curves with the limit base points as in (3.2).

By what we have seen above, such a twist gives the classes $H, H+F,\dots,H+(d-1)F$ respectively on the ruled surfaces $S_{r_0},S_{r_1},\dots,S_{r_{d-1}}$, with $r=r_0>r_1>\cdots>r_{d-1}$, and multiples of the class F on the other ruled surfaces (specifically, it gives the class kF on the surfaces S_i for $r_{k-1}>i>r_k$, for all $k=1,\dots,d$, setting $r_{-1}=r+2$ and $r_d=-1$ for convenience, and with the abuse of considering S_{r+1} as a ruled surface).

It follows that the restriction of such a twist to the central fibre maps $S_{r_0},S_{r_1},\dots,S_{r_{d-1}}$ to rational normal scrolls of types $\Sigma(0,1),\Sigma(1,2),\dots,\Sigma(d-1,d)$ respectively, and contracts all the other irreducible components to rational normal curves (with the notation that a rational normal scroll of type $\Sigma(k_1,k_2)$ is built on two rational normal curves of degrees k_1 and k_2 which span two supplementary linear spaces). Two successive scrolls $\Sigma(k,k+1)$ and $\Sigma(k+1,k+2)$ in the previous list are glued along the rational normal degree $k+1$ curves Λ_{r_k} and $\Lambda_{r_{k+1}-1}$.

The upshot is that the restriction to the fibres of $\pi_r:\mathcal{P}_r\to\mathbf{D}$ of the twist of $\mathcal{L}_{r,d}$ under consideration gives a degeneration of the d-th Veronese image of $\mathbf{P}^2$ to a chain of rational normal scrolls. Correspondingly, we see limits of plane degree d curves as hyperplane sections of this chain of rational normal scrolls.

As a sanity check, let us verify that the degrees and embedding dimension of such a projective model are correct. The total degree is

$$\sum_{k=0}^{d-1}\deg\big(\Sigma(k,k+1)\big)=\sum_{k=0}^{d-1}(2k+1)=d^2,$$

and the dimension of the projective linear space spanned by our chain of scrolls is

$$\sum_{k=0}^{d-1}h^0(\mathcal{O}_{\mathbf{P}^1}(k+1))=\sum_{k=0}^{d-1}(k+2)=2d+\sum_{k=1}^{d-1}k=\sum_{k=1}^{d+1}k-1,$$

which indeed coincide respectively to the degree and embedding dimension of d-th Veronese image of $\mathbf{P}^2$.

We will be able to give a nice graphic representation of this degeneration of the d-th Veronese to a chain of rational normal scrolls when we know some toric and tropical geometry, see [X, (9.4)].

(5.3) Limits of nodes. It is a central theme of this set of notes (in particular, see [I, 5]) that the limit of a node may be of the two following types.

(a) It may be a node on one of the irreducible components of the degeneration. In our situation the irreducible components of the central fibre are rational normal scrolls, so the only way a hyperplane section has a node is that it splits off a fibre.

This is indeed what we have observed in the previous examples. See in particular (4.2.2): nodes as above are indicated by a black square. There is such a node on Figures 9, 10, and 11.

(b) It may be a tacnode along the transverse intersection curve of two irreducible components of the central fibre (or, more generally, $\mu - 1$ nodes may degenerate together to a μ-tacnode). In our situation in this lecture, this phenomenon occurs when two curves of respective classes $H + kF$ and $H + (k + 1)F$ are linked by a chain of fibres of multiplicity μ.

This happens for instance on Figure 8. In the projective model given by the appropriate twist for the curve displayed on Figure 8, S_5 is contracted to a conic, and the two scrolls images of S_6 and S_4 intersect along this conic; the two curves on S_6 and S_4 are then cut out by a hyperplane which is tangent to this conic. Thus, there is an ordinary tacnode, and one can find the correct multiplicity 4 for this limit curve as follows.

A multiplicity μ chain of fibres (of type (a) in the notation of (3.8), say) contributes enumeratively with multiplicity μ^2, whereas a μ-tacnode contributes with multiplicity μ in the count of $(\mu - 1)$-nodal curves. The principle to obtain the multiplicity μ^2 in the tacnodal context is the following: the points at which one can have a μ-tacnode should be searched within a $g^{\mu-1}_{\mu}$ on a smooth rational curve, and there are μ divisors of the form $\mu.p$ in such a linear series (for instance, this may be seen by applying the de Jonquières formula, see [XII, Section B.5]). Then, each of those μ possibilities counts with multiplicity μ, thus giving the total contribution μ^2 in the end, as required.

It is nevertheless not obvious how to understand in full generality the limits of the nodes along the degeneration considered in the present lecture in terms of the theory presented in [I]. For instance, one ought to deal with situations as in Figure 12 below, where the multiple chain of fibres is not entirely contracted in the appropriate projective model. The curve on Figure 12 is a limit of rational plane quartics, hence carries the limits of three nodes. One node is amounted for by the tacnode, and the two others by the intersection point of the double fibre with the curve of class H on S_7. In order to understand this situation within the theory of [I], it is necessary to consider an alternative model of the degeneration, in such a way that one gets rid of the multiple curves.

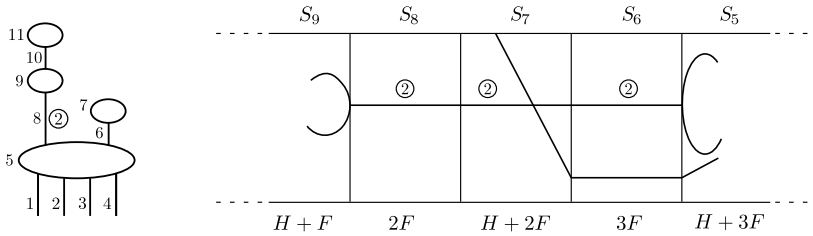


Figure 12: Limit of a rational plane quartic with multiple fibres (detail)

References

Chapters in this Volume

[I] C. Ciliberto and Th. Dedieu, *Limits of nodal curves: generalities and examples*.

[III] Th. Dedieu, *Geometry of logarithmic Severi varieties at a general point*.

[VI] Th. Dedieu and C. Galati, *Degenerations and deformations of generalized Severi varieties*.

[VII] Th. Dedieu, *The Caporaso–Harris recursion within a degeneration of the plane*.

[X] Th. Dedieu, *Enumeration of curves in toric surfaces, tropical geometry, and floor diagrams*.

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces*.

External References

[1] A. Arroyo, E. Brugallé and L. López de Medrano, *Recursive formulas for Welschinger invariants of the projective plane*, Int. Math. Res. Not. IMRN (2011), no. 5, 1107–1134.

[2] E. Brugallé, *Floor diagrams relative to a conic, and GW–W invariants of Del Pezzo surfaces*, Adv. Math. **279** (2015), 438–500.

[3] E. Brugallé and G. Mikhalkin, *Floor decompositions of tropical curves: the planar case*, in *Proceedings of Gökova Geometry-Topology Conference 2008*, Gökova Geometry/Topology Conference (GGT), Gökova, 2009, 64–90.

[4] L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392.

[5] S. Druel, *Courbes sur les variétés et leurs espaces de modules – Quelques applications*, lecture notes available at `http://druel.perso.math.cnrs.fr/textes/ecole2010.pdf`.

[6] E.-N. Ionel and T. H. Parker, *The symplectic sum formula for Gromov-Witten invariants*, Ann. of Math. (2) **159** (2004), no. 3, 935–1025.

[7] Jun Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57** (2001), no. 3, 509–578.

[8] ———, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293.

[9] ———, *Lecture notes on relative GW-invariants*, in *Intersection theory and moduli*, ICTP Lect. Notes, vol. XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, 41–96.

[10] R. Vakil, *Counting curves on rational surfaces*, Manuscripta Math. **102** (2000), no. 1, 53–84.

Lecture X
Enumeration of curves in toric surfaces, tropical geometry, and floor diagrams

by Thomas Dedieu

1 Introduction 211
1.1 Introduction and overview 211
1.2 About this text 213
1.3 Further developments 214
1.4 General notation 215
2 Toric projective surfaces 216
2.1 The prototypical examples 216
2.2 The general construction 217
2.3 Geometric properties and numerical invariants 218
3 The toric enumerative problem 220
3.1 Toric Cayley–Bacharach 220
3.2 The enumerative problem 221
3.3 Examples from degenerations of $K3$ surfaces 222
4 From the torus to the tropics 227
5 Parabolic polarity, b.k.a. Legendre–Fenchel duality 229
6 Tropical curves 231
6.1 Plane tropical curves 231
6.2 Abstract tropical curves and tropical maps 234
7 The tropical enumerative problem and the Correspondence Theorem . . . 239
7.1 General theory 239
7.2 Regeneration of tropical curves 242
7.3 Examples 247
8 Floor diagrams and enumeration 250
9 Tropical representation of algebraic curves 261
References 263

1 – Introduction

1.1 – Introduction and overview

The theme of this text is the application of tropical geometry to an enumerative problem for curves on toric surfaces, analogous to the one studied in previous chapters for the projective plane. We will thus discuss an equivalent enumerative problem for plane tropical curves, as well as the translation of the latter into a purely combinatorial enumerative problem for objects called floor diagrams. The text is intended for readers with absolutely no prior knowledge either

of toric varieties or of tropical geometry (e.g., the author himself), and includes the necessary introductory material on these two topics.

Toric surfaces are compact surfaces containing a torus $(\mathbf{C}^*)^2$, the complement of which is an anticanonical divisor made of a cycle of rational curves. The enumerative problem we are interested in consists in counting curves of given genus g and Newton polygon Δ, with prescribed contact pattern with the anticanonical cycle, both at assigned and unassigned points, and passing through the appropriate number of general base points; the shape of the polygon Δ determines which toric surface the curves live in; the lengths of its sides, in turn, determine their linear equivalence class. Problems of this kind naturally arise when one studies an enumerative problem by degeneration as discussed in [I], much in the same way that logarithmic Severi varieties of the pair $(\mathbf{P}^2, \text{line})$ show up when one studies plain Severi varieties of the projective plane, as explained in [VI].

Plane tropical[1] curves are piecewise linear objects in the real plane $\mathbf{R}^2$ (equipped with the integral structure determined by $\mathbf{Z}^2$), which appear as a sort of projected shadows of algebraic curves in $(\mathbf{C}^*)^2$, following seminal ideas of Kapranov [19] and others. Of course much information is lost in the projection, but tropical curves are nevertheless excellent at representing algebraic curves. In particular, it has been shown by Mikhalkin [36], following a suggestion by Kontsevich (see [32]), that the aforementioned enumerative problem for curves in toric surfaces is equivalent to its tropical shadow: this is the main theme of this text.

An advantage of tropical curves over algebraic curves is that they are much more hands-on: it is possible to play around with them using a ruler and a sheet of gridded paper without losing any piece of information. The tropical enumerative problem is therefore more tractable than the original toric problem, and in principle it can be solved with elementary considerations only: we shall see several instances of this in the text. Yet, of course, when the size of the problem gets too large, it is no longer possible to solve it with bare hands.

It has been realized by Brugallé and Mikhalkin [14] that stretching the base points of the tropical enumerative problem along a chosen direction in $\mathbf{R}^2$ helps structuring its solution. In particular, under rather mild favourable technical conditions, if the base points are far enough one from another along this direction, then for all the tropical curves one wants to count, the base points are distributed one by one on certain "components"; these "components" are of two types, floors and elevators, and their incidence relations may be neatly encoded in a combinatorial device called a floor diagram. Then, the tropical enumerative problem translates into a problem of enumeration of floor diagrams enhanced with a marking, the latter encoding the distribution of the base points on the various components. The tropical enumerative problem may also be solved using the so-called lattice path algorithm, which was already given by Mikhalkin in [36], but we will not discuss it here.

The text starts by a quick introduction to toric surfaces (Section 2), based on my personal down-to-earth understanding of it (but supported by the reference [17]) and guaranteed free of fans. Next we state the enumerative problem for curves on toric surfaces in Section 3.2. Section 3.3 gives several examples, showing how the toric enumerative problem in question arises when one studies, say, the enumerative geometry of $K3$ surfaces by degeneration to unions of planes. Plane tropical curves are introduced in Section 6 in the way we need them in order to state and study the relevant tropical enumerative problem. Before that, we briefly explain in Section 4 the ideas that led to the definition of these objects. Section 5 is a parenthesis about the Legendre–Fenchel transformation (often only called Legendre transformation), which is a useful tool in the study of corner loci and tropical hypersurfaces. It is possible to skip Sections 4 and 5, as Section

[1]The word 'tropical' has no obvious mathematical connotations: it was given in honor of Imre Simon, who resided in Brazil. The name "tropical geometry" was coined by Mikhalkin and Sturmfels, as the meeting between algebraic geometry and tropical algebra. See, e.g., [34, p. 283].

6 should be self-contained, and sufficient to understand the remainder of the text. In Section 7 we state the tropical enumerative problem (using the definitions of Section 6) and explain the correspondence with the toric enumerative problem. Finally, Section 8 is dedicated to the translation of the tropical enumerative problem in terms of floor diagrams. The last Section 9 is a sort of appendix, in which we develop the idea that tropical curves are great for the graphic representation of algebraic curves; we will see in particular that the graphic representation of the degeneration procedure of [IX] (which is an alternative path to floor diagrams) makes for striking revelations.

1.2 – About this text

There already exist several texts written by experts of tropical geometry which the reader can consult: [13] is written for a broad audience of mathematicians, and does not discuss enumerative geometry; [25] does discuss enumerative geometry, and targets an audience of graduate students; [11] and [29] are intended for geometers. I also recommend the somehow historical [45], as well as [44].

The modest originality of the present text is that it is written for algebraic geometers by an algebraic geometer. As such, it proposes answers to questions which come naturally for someone who has lived by studying Severi varieties of algebraic curves on surfaces, but may seem outlandish to enumerative or tropical geometers. Moreover, we usually adopt an intuitive and elementary point of view; while this is, I believe, best in order to understand things, it is arguably ill-suited to developing a rigorous and elegant theory. Therefore, readers already familiar with the abstract theory may find this presentation somewhat awkward. I hope, however, that newcomers will find it a helpful gateway to the fascinating subject of tropical enumerative geometry.

Another salient feature of this text is its emphasis on the connections of tropical enumerative geometry to other degeneration techniques presented in this volume, in particular in [I] and [VI]; Section 7.2 is particularly relevant with this respect. Conversely, Section 9 is important in order to understand various drawings in other chapters, most notably [I]. In addition, many examples are taken from the study of degenerations of $K3$ surfaces to unions of planes (a situation not unlike that described in Section 7.2), see in particular Section 3.3.

For this presentation I have drawn from various sources, to which I owe a great intellectual debt. The general presentation of tropical geometry follows rather closely a mini course given by Ilia Itenberg, *Géométrie tropicale énumérative*, at the 2010 GAGC meeting at CIRM. The presentation of plane and abstract tropical curves, and of the tropical enumerative problem, exists only thanks to personal lectures offered by Erwan Brugallé to the author in Jussieu, 2016. Section 7.2 has been written after conversations with Thomas Blomme. The presentation of floor diagrams and their enumeration follows Brugallé and Mikhalkin's article [14]. Finally, I wish to mention a wonderful talk by Kontsevich [31], which in principle was targeted for master students, but contains illuminating insights on tropical geometry and its interplay with algebraic geometry.

Thanks. I am deeply grateful to Erwan Brugallé, without whom this text would have never existed. I have also benefited from discussions with Thomas Blomme, which were fundamental to this work despite taking place only at a late stage. Thanks to Ilia Itenberg for his comments on a preliminary version of this text.

1.3 – Further developments

This text is by no means a general introduction to tropical enumerative geometry, and instead focuses on the most basic enumerative applications of tropical geometry. Yet it seems fair to conclude this introduction by mentioning some more elaborate aspects and developments (which will not be discussed any further in the text), in order to give a more up-to-date picture of this very active area.

Let us start by a few remarkable applications of floor diagrams themselves. By the way, note that the correspondence between algebraic curves and floor diagrams can be established without resorting to tropical geometry, by degeneration, as has been showed by Brugallé [10] (this is explained in [IX]). The formulation as an enumerative problem about floor diagrams has enabled Fomin and Mikhalkin [20] to prove various nice results about complex algebraic plane curves. In particular, they have obtained a proof of the Di Francesco–Itzykson conjecture, to the effect that for all $\delta \geqslant 0$, there exists a degree 2δ polynomial $P_\delta \in \mathbf{Q}[d]$ such that for all large enough d, $P_\delta(d)$ is the number of degree d plane curves with δ nodes passing through the appropriate number of general base points. This is a particular case of the Göttsche conjecture, and it is related to the results discussed in [XIII] and [XIV]. Floor diagrams have been used by Brugallé and Mikhalkin [12] to give an equivalent formulation of the problem of enumerating rational curves in a complex projective space of arbitrary dimension. Also, adapted versions of floor diagrams have been applied to the study of Gromov–Witten invariants with (stationary) descendant insertions of Hirzebruch surfaces, see [6] and [15] (the descendant insertions mean that we are imposing more elaborate conditions than the passing through base points in order to arrive at a finite number of curves).

Quite remarkably, tropical geometry and floor diagrams also enable the study of enumerative problems in real algebraic geometry. In particular they give a way to compute Welschinger invariants (this is aleady in [36]), which count real rational curves with a sign depending on the real nature of their singularities (see the nice survey [11]). This has been recently generalized by Jaramillo Puentes and Pauli [39], see also [38] with Markwig and Röhrle, to the enumeration of curves over a field $\mathbf{k}$, counted with the Levine–Welschinger quadratic weight in the Grothendieck–Witt ring of $\mathbf{k}$ (this also involves techniques from $\mathbf{A}^1$-homotopy theory). In the real setting, Gathmann, Markwig, and Schroeter [22] have given a tropical proof of invariance and an effective combinatorial way to compute Welschinger invariants in the toric del Pezzo case. As another sample application of tropical geometry to real algebraic geometry, see [18] by Cueto and Markwig about the real bitangents of real plane quartics.

On the other hand, tropical geometry makes it possible to define new enumerative invariants. The emblematic instance is the count of plane tropical curves with multiplicities that are rational fractions in the (so-called quantum) indeterminate q, which was introduced by Block and Göttsche [7], and related by them to the refined Severi degrees introduced by Göttsche and Shende [24]; see also [7], in which marked floor diagrams are used in an essential way. The invariance of the number of curves counted with the Block–Göttsche multiplicity under the choice of a generic tropical configuration of points has been shown by Itenberg and Mikhalkin [28]. The Block–Göttsche invariants are polynomials in q interpolating between Welschinger (given by $q = -1$) and Gromov–Witten (given by $q = 1$) invariants; the (algebraic geometric) enumerative interpretation of their coefficients is still unclear, in spite of substantial progress. Mikhalkin [37] has refined the enumeration of real rational curves in toric surfaces (satisfying certain conditions) in a way consistent with the Block-Göttsche invariants, by defining a half-integer multiplicity, called the quantum index, which is the area encompassed by the logarithmic image of such curves divided by π^2; this has been generalized to curves of genus one or two by Itenberg and Shustin [30]. Blomme [8] has generalized Mikhalkin's work in another direction, by giving a way to compute the quantum index of any oriented real rational curve, relating

the corresponding classical refined invariants to Block-Göttsche invariants. On the other hand, Bousseau [9] has shown that, under the change of variables $q = e^{iu}$, Block-Göttsche invariants are generating series of higher genus log Gromov–Witten invariants with insertion of a lambda class, thus giving a geometric interpretation of the Block-Göttsche invariants (connected to invariants of some Calabi–Yau threefolds), and making their deformation invariance manifest. In [3], Argüz and Bousseau generalize the Block–Göttsche invariants to arbitrary surfaces, showing that they arise as special cases of BPS polynomials; they show that their evaluation at $q = -1$ yields Welschinger invariants for $\mathbf{P}^2$ blown-up at up to six general points.

Last but not least, tropical geometry is instrumental in Abramovich-Chen-Gross-Siebert's degeneration formula [1]: the latter somehow fulfils many wishes formulated in [I], in the context of logarithmic[2] Gromov–Witten invariants. While logarithmic geometry gives a natural framework for degenerations, tropical geometry records the combinatorics of logarithmic objects: given a one-parameter log smooth family $X \to B$ with singular fibre over $b_0 \in B$, one has a family $\mathcal{M}(X/B, \beta) \to B$ of moduli stacks of stable logarithmic maps, and the degeneration formula gives a virtual decomposition of the fibre of this family over b_0 in terms of rigid tropical maps to the tropicalization of X/B. Closely related is the identification by Ranganathan [41] of the Berkovich skeleton of the moduli space of logarithmic stable maps with the moduli space of tropical stable maps. Mandel and Ruddat [33] have taken advantage of this technology to push further the toric degeneration approach of Nishinou and Siebert [40], in order to prove the correspondence of tropical curve counts and log Gromov-Witten invariants with general incidence and psi-class conditions in toric varieties for genus zero curves (as well as the correspondence for the non-superabundant part of the invariant in higher genus). The logarithmic framework has also been used by Argüz and Bousseau [2] to give a tropical description of the counting of real log curves in toric degenerations of toric varieties; in dimension two, they study the counting with Welschinger signs and obtain a new proof of Mikhalkin's tropical correspondence theorem for Welschinger invariants.

1.4 – General notation

We use the notation of [III, (1.2)] and [VI, (1.2)]. In particular $\underline{\mathbf{N}}$ denotes the set of sequences $\alpha = [\alpha_1, \alpha_2, \ldots]$ of non-negative integers with only finitely many α_i non-zero, and for all $\alpha \in \underline{\mathbf{N}}$,

$$|\alpha| = \sum_i \alpha_i, \qquad I\alpha = \sum_i i\,\alpha_i, \qquad \text{and} \qquad I^\alpha = \prod_i i^{\alpha_i}.$$

For any vector u in the real plane $\mathbf{R}^2$, we denote by $u^\perp$ the vector obtained by rotating u by $+\pi/2$. A vector $u = (a, b) \in \mathbf{Z}^2$ is *primitive* if a and b are relatively prime.

On the projective plane $\mathbf{P}^2$, we let H denote the line class. On Hirzebruch ruled surfaces $\mathbf{F}_e$, we let E be the exceptional section, F the class of fibres of the ruling, and $H = E + eF$.

[2] beware that the meaning of 'logarithmic' here is different from that in [III], say; for 'logarithmic' in the context of [III], enumerative geometers use the word 'relative'.

2 – Toric projective surfaces

2.1 – The prototypical examples

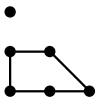
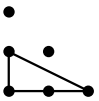
Let S be a surface equipped with a system of (pluri) homogeneous coordinates[3], e.g., $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$, and fix a (multi) degree d. The projectivization of the space V_d of all homogeneous polynomials of degree d is a complete linear system of curves on S. The notion of *Newton polygon* enables one to define linear subsystems of $\mathbf{P}V_d$ in a remarkably meaningful way, as follows.

Degree d monomials form a basis of V_d, and may be represented as lattice points $\mathsf{p}_1, \ldots, \mathsf{p}_{N_d}$ in some real affine space (by *lattice points*, or *integral points*, we mean points with integral coordinates). Consider a convex polygon Δ, the vertices of which belong to $\{\mathsf{p}_1, \ldots, \mathsf{p}_{N_d}\}$. One lets V_Δ be the subspace of V_d generated by those monomials whose corresponding lattice point p lies in Δ (thus, we see Δ as a 2-dimensional convex domain). We will see that the subsystem $\mathbf{P}V_\Delta$ of $\mathbf{P}V_d$ may alternatively be defined in a natural way by base points, possibly infinitely near. This is best understood by looking at a couple of examples.

(2.1) Example ($\mathbf{P}^2$)**.** It comes with a system of homogeneous coordinates $(x : y : z)$. Fix a degree $d \in \mathbf{N}$; to each monomial $x^a y^b z^c$ of degree d, we associate the lattice point $(a, b) \in \mathbf{R}^2$ (of course this determines $c = d - a - b$). Degree d monomials then correspond bijectively to points with integral coordinates in the triangle

$$T_d \stackrel{\text{def.}}{=} \{(a, b) \in \mathbf{R}^2 : a, b \geqslant 0 \text{ and } a + b \leqslant d\}.$$

We may for instance consider the two Newton polygons pictured below.

(a)  (b) 

In case (a), the elements of V_Δ are the quadratic polynomials of the form

$$f_2(x, z) + y f_1(x, z),$$

with f_2 and f_1 homogeneous polynomials of degrees 2 and 1, respectively. Thus, $\mathbf{P}V_\Delta$ is the linear system of conics passing through the point $(0{:}1{:}0)$.

In case (b), V_Δ consists of the polynomials of the form

$$f_2(x, z) + \alpha \cdot yz,$$

with f_2 a degree 2 homogeneous polynomial, and $\alpha \in \mathbf{C}$. Thus, $\mathbf{P}V_\Delta$ is the linear system of conics passing by the point $(0{:}1{:}0)$, and tangent to the line $z = 0$ there (if you set $z = 0$ in the above expression, only the x^2 term does not vanish).

(2.2) Example ($\mathbf{P}^1 \times \mathbf{P}^1$)**.** It comes with a system of bihomogeneous coordinates $((x_0 : x_1), (y_0 : y_1))$. Fix a bidegree $d = (d_1, d_2) \in \mathbf{N}^2$; to each monomial $x_0^{a_0} x_1^{a_1} y_0^{b_0} y_1^{b_1}$ of bidegree d, we associate the lattice point $(a_1, b_1) \in \mathbf{R}^2$. Bidegree d monomials then correspond bijectively to points with integral coordinates in the rectangle

$$R_{d_1,d_2} \stackrel{\text{def.}}{=} \{(a, b) \in \mathbf{R}^2 : a, b \geqslant 0,\ a \leqslant d_1, \text{ and } b \leqslant d_2\}.$$

[3] In order to keep this introduction elementary, we will not give a general definition of a system of homogeneous coordinates, but you can look up [17, Chapter 5]. It is fine to just think of S as one of $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$; other examples are weighted projective spaces, and Hirzebruch surfaces $\mathbf{F}_e$ (equipped with homogeneous coordinates s, t, u, v of weights $(1, 0), (1, 0), (0, 1), (-e, 1)$, respectively).

On the picture below, the polygon (a) (resp. (b)) corresponds to the linear system of hyperplane sections (resp. of sections by quadrics) of $\mathbf{P}^1 \times \mathbf{P}^1$ embedded as a smooth quadric surface in $\mathbf{P}^3$.

(a) (b)

Observe that the polygon (a) may also be realized inside the triangle T_2 of Example (2.1). In this incarnation, it corresponds to the linear system of plane conics with the two base points $(1:0:0)$ and $(0:1:0)$. The map associated to this linear system is birational from $\mathbf{P}^2$ to a smooth quadric surface. It follows that the two incarnations of the polygon (a) give the same linear system of curves.

2.2 – The general construction

(2.3) Let $\Delta \subseteq \mathbf{R}^2$ be a convex polygon with vertices at integral points (this is called a *lattice polygon*), and let $\mathsf{p}_i = (a_i, b_i)$, $i = 1, \ldots, N_\Delta$, be the integral points of $\mathbf{R}^2$ lying inside Δ. We associate the following map to Δ:

$$\begin{aligned} \phi_\Delta : \quad & (\mathbf{C}^*)^2 \to \mathbf{P}^{N_\Delta - 1} \\ & (t_1, t_2) \mapsto \left(t_1^{a_1} t_2^{b_1} : \ldots : t_1^{a_{N_\Delta}} t_2^{b_{N_\Delta}}\right). \end{aligned}$$

Then, we define $X_\Delta \subseteq \mathbf{P}^{N_\Delta - 1}$ to be the Zariski closure in $\mathbf{P}^{N_\Delta - 1}$ of the image of ϕ_Δ; it is a projectively normal surface, see [17, Thm. 2.4.1, Def. 2.3.14, and Cor. 2.2.13].

Thus, the object associated to Δ is a normal surface in a given projective embedding; equivalently, it is a surface together with a linear system of Cartier divisors. It is possible to build a combinatorial object from Δ that forgets about the embedding, which is called a *fan*; it is usually the preferred object in toric geometry. In this text however, our primary interest resides in linear systems (hence in polytopes), and we shall not discuss fans.

Quite remarkably, the surface X_Δ thus constructed may always be equipped with a system of homogeneous coordinates, see [17, Chapter 5], so that we can play the game of Section 2.1 with it, and associate the complete linear system $|\mathcal{O}_{X_\Delta}(1)|$ with a polygon. The latter polygon is the very Δ we started with.

(2.4) Remark. One may rephrase the above definition in the terms of Section 2.1. For instance, any Δ as above may be realized inside a triangle T_d as in (2.1), for sufficiently large d (note that the definition of ϕ_Δ is invariant under translation of Δ). Then, Δ determines a linear system of degree d plane curves of dimension $N_\Delta - 1$, and ϕ_Δ equals the restriction to

$$(\mathbf{C}^*)^2 \simeq \left\{(t_0 : t_1 : t_2) : t_0 t_1 t_2 \neq 0\right\} \subseteq \mathbf{P}^2$$

of the rational map $\mathbf{P}^2 \dashrightarrow \mathbf{P}^{N_\Delta - 1}$ associated with this linear system.

Similarly, Δ may always be realized inside a rectangle R_{d_1,d_2} as in (2.2), and ϕ_Δ identified with the restriction to

$$(\mathbf{C}^*)^2 \simeq \left\{(s_1 : t_1), (s_2 : t_2)) : s_1 t_1 s_2 t_2 \neq 0\right\} \subseteq \mathbf{P}^1 \times \mathbf{P}^1$$

of the rational map $\mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow \mathbf{P}^{N_\Delta - 1}$ associated with the corresponding linear system.

We thus see that the construction in Section 2.1 does not depend on the choice of the surface S at the beginning, but only on the polygon Δ.

(2.5) Equations. One may compute the equations of X_Δ in $\mathbf{P}^{N_\Delta-1}$ by elimination from a parametrization of the affine cone $\widehat{X}_\Delta$ over X_Δ. To do so, beware that the map

$$\widehat{\phi}_\Delta : (t_1, t_2) \in (\mathbf{C}^*)^2 \mapsto \left(t_1^{a_1} t_2^{b_1} : \ldots : t_1^{a_{N_\Delta}} t_2^{b_{N_\Delta}}\right) \in (\mathbf{C}^*)^{N_\Delta}$$

of course does not parametrize (an open subset of) the affine cone $\widehat{X}_\Delta$. To obtain such a parametrization, one may apply the trick described in [17, top of p. 58], and consider instead the map

$$(t_0, t_1, t_2) \in (\mathbf{C}^*)^3 \mapsto \left(t_0 t_1^{a_1} t_2^{b_1} : \ldots : t_0 t_1^{a_{N_\Delta}} t_2^{b_{N_\Delta}}\right) \in (\mathbf{C}^*)^{N_\Delta}.$$

Then, the equations of X_Δ in $\mathbf{P}^{N_\Delta-1}$ may be obtained by eliminating (t_0, t_1, t_2) from the ideal

$$\left(x_1 - t_0 t_1^{a_1} t_2^{b_1}, \ldots, x_{N_\Delta} - t_0 t_1^{a_{N_\Delta}} t_2^{b_{N_\Delta}}\right) \subseteq \mathbf{C}[t_0, t_1, t_2] \otimes \mathbf{C}[x_1, \ldots, x_{N_\Delta}].$$

Alternatively, one may consider the map

$$(t_0, t_1, t_2) \in (\mathbf{C}^*)^3 \mapsto \left(t_0^{d-a_1-b_1} t_1^{a_1} t_2^{b_1} : \ldots : t_0^{d-a_{N_\Delta}-b_{N_\Delta}} t_1^{a_{N_\Delta}} t_2^{b_{N_\Delta}}\right) \in (\mathbf{C}^*)^{N_\Delta}, \tag{2.5.1}$$

corresponding to the extension of ϕ_Δ to a rational map $\mathbf{P}^2 \dashrightarrow \mathbf{P}^{N_\Delta-1}$ discussed in Remark (2.4). It does parametrize an open subset of the affine cone $\widehat{X}_\Delta$ by [17, Prop. 2.1.4], since the linear functional on monomials $t_0^c t_1^a t_2^b \mapsto a+b+c$ takes the value d on all monomials appearing in the map (2.5.1). Then, one may obtain the equations of X_Δ in $\mathbf{P}^{N_\Delta-1}$ by eliminating (t_0, t_1, t_2) from the ideal

$$\left(x_1 - t_0^{d-a_1-b_1} t_1^{a_1} t_2^{b_1}, \ldots, x_{N_\Delta} - t_0^{d-a_{N_\Delta}-b_{N_\Delta}} t_1^{a_{N_\Delta}} t_2^{b_{N_\Delta}}\right)$$

as above.

Similarly, one may consider the map

$$(s_1, s_2, t_1, t_2) \in (\mathbf{C}^*)^4 \mapsto \left(s_1^{d_1-a_1} s_2^{d_2-b_1} t_1^{a_1} t_2^{b_1} : \ldots : s_1^{d_1-a_{N_\Delta}} s_2^{d_2-b_{N_\Delta}} t_1^{a_{N_\Delta}} t_2^{b_{N_\Delta}}\right) \in (\mathbf{C}^*)^{N_\Delta} \tag{2.5.2}$$

corresponding to the extension of ϕ_Δ to a rational map $\mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow \mathbf{P}^{N_\Delta-1}$ discussed in Remark (2.4), which parametrizes an open subset of the affine cone $\widehat{X}_\Delta$ by [17, Prop. 2.1.4], since the two linear functionals $s_1^{a'} s_2^{b'} t_1^a t_2^b \mapsto a + a'$ and $s_1^{a'} s_2^{b'} t_1^a t_2^b \mapsto b + b'$ take the values d_1 and d_2, respectively, on all monomials appearing in the map (2.5.2).

(2.6) Example. The trapezium

$$\mathrm{Tz}_{a,b}^e = \left\{(x, y) \in \mathbf{Z}^2 : 0 \leqslant y \leqslant a \text{ and } 0 \leqslant x \leqslant b + (e - y)a\right\}$$

corresponds to the complete linear system $|aH + bF|$ on the Hirzebruch surface $\mathbf{F}_e$, where F is the class of a fibre, and $H = E + eF$ with E the class of the section with self-intersection $-e$.

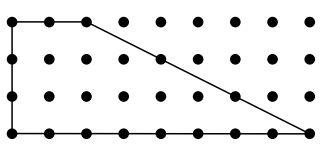


Figure 1: The trapezium $\mathrm{Tz}_{3,2}^2$: $|3H + 2F|$ on $\mathbf{F}_2$

2.3 – Geometric properties and numerical invariants

Consider a surface X_Δ constructed from a convex integral polygon $\Delta \subseteq \mathbf{R}^2$ as in (2.3).

(2.7) Anticanonical cycle. The complement $X_\Delta \setminus \phi_\Delta((\mathbf{C}^*)^2)$ is a cycle of rational curves, and it is a Weil divisor belonging to the anticanonical linear equivalence class $-K_{X_\Delta}$, see [17, Theorem 8.2.3]; we shall call it the *anticanonical cycle* of X_Δ, and denote it by $\curlyvee_\Delta$. We will also often call it the *toric boundary* of X_Δ.

Being a cycle of rational curves means that $\curlyvee_\Delta$ is a union of rational curves $C_1, \dots, C_n$, where each C_i intersects C_{i-1} and C_{i+1} transversely[4] at one point (with the convention that $C_0 = C_n$ and $C_{n+1} = C_1$) and no other curve.

The curves $C_1, \dots, C_n$ correspond to the edges $\Sigma_1, \dots, \Sigma_n$ (labelled in a cyclic way) of the polygon Δ, and the intersection points $C_i \cap C_{i+1}$ to its vertices. In the projective embedding given by Δ, the curve C_i is a rational normal curve of degree $\ell_{\mathbf{Z}}(\Sigma_i)$, which we define as the *integral length* of the segment Σ_i, i.e., the number of integral points lying on Σ_i minus one.

(2.8) Degree and number of sections. We let H_Δ be the class $c_1(\mathcal{O}_{X_\Delta}(1))$, corresponding to the embedding defined by Δ. The dimension $h^0(\mathcal{O}_{X_\Delta}(1))$ is the number N_Δ of integral points lying in Δ minus one. One may think of N_Δ as an integral measure of the area of Δ, in analogy with the integral length $\ell_{\mathbf{Z}}$, but beware that the integral area of Δ is usually defined as $2\mathscr{A}(\Delta)$, in the notation introduced in (2.9) below.

The anticanonical degree of H_Δ is

$$-K_{X_\Delta} \cdot H_\Delta = \curlyvee_\Delta \cdot H_\Delta = \sum_{i=1}^n \ell_{\mathbf{Z}}(\Sigma_i) = \mathrm{Card}(\partial\Delta_{\mathbf{Z}}),$$

where $\partial\Delta_{\mathbf{Z}}$ stands for the set of integral points in the boundary of Δ, see [17, Proposition 10.5.6].

(2.9) Arithmetic genus. The arithmetic genus of the divisors belonging to the linear system $|\mathcal{O}_{X_\Delta}(1)|$ is

$$p_a(\Delta) \overset{\text{def.}}{=} p_a(H_\Delta) = \mathrm{Card}(\mathring{\Delta}_{\mathbf{Z}}),$$

where $\mathring{\Delta}_{\mathbf{Z}}$ stands for the set of integral points in the interior of Δ, see [17, Proposition 10.5.8].

Equivalently, one has

$$(H_\Delta)^2 = 2\mathscr{A}(\Delta),$$

where $\mathscr{A}(\Delta)$ is the *Euclidean area* of Δ, taking the area of the quadrilateral defined by any lattice basis of $\mathbf{R}^2$ as the unit, [17, Proposition 10.5.6].

With the two above identities, the adjunction formula

$$2p_a(H_\Delta) - 2 = (K_{X_\Delta} + H_\Delta) \cdot H_\Delta$$

amounts to Pick's formula, see [17, Example 9.4.4], which expresses the area of a lattice polygon in terms of its lattice points.

(2.10) Singularities. The surface X_Δ is smooth except (possibly) at the points corresponding to the vertices of the anticanonical cycle $\curlyvee_\Delta$, where it has cyclic quotient singularities, the types of which are determined as follows (see [17, Section 10.1]).

Consider a vertex v of Δ, and let u'_v and u''_v be the two primitive integral vectors giving the directions of the two edges of Δ intersecting at v and pointing away from v, in such a way that the angle (u'_v, u''_v) points inside Δ. Then the two vectors $(u'_v)^\perp$ and $-(u''_v)^\perp$ are the normal vectors of the two edges at v, pointing towards the interior of Δ (see Figure 2 for an example).

[4] two consecutive curves C_i and C_{i+1} of $\curlyvee_\Delta$ certainly intersect transversely in $\mathbf{P}^{N_\Delta - 1}$, because they span two linear subspaces $\mathbf{P}^{\ell_i}$ and $\mathbf{P}^{\ell_{i+1}}$ which intersect at one point only. However, the surface X_Δ may be singular at the point $C_i \cap C_{i+1}$, and the intersection number $C_i \cdot C_{i+1}$ differs from 1 in general.

It is always possible to find an integral basis (e_1, e_2) of $\mathbf{Z}^2$ such that the two vectors $(u'_v)^\perp$ and $-(u''_v)^\perp$ write

$$e_2 \quad \text{and} \quad de_1 - ke_2,$$

where d and k are relatively prime integers such that $d > 0$ and $0 \leqslant k < d$. Then the singularity of the surface X_Δ at the point corresponding to the vertex v is of type $\frac{1}{d}(1, k)$, i.e., it is analytically locally isomorphic to the quotient of $\mathbf{C}^2$ by the group of d-th roots of unity acting as $\zeta.(x, y) = (\zeta x, \zeta^k y)$.

In particular, the Euclidean area of the parallelogram defined by u'_v and u''_v, namely

$$\mathscr{A}\big(\langle u'_v, u''_v\rangle\big) = \det(u'_v, u''_v)$$

computes the order d of the cyclic group, the action of which defines the singularity at the point of X_Δ corresponding to v. The surface is smooth at this point if (u'_v, u''_v) forms a basis of $\mathbf{Z}^2$, and it has an ordinary double point if $\det(u'_v, u''_v) = 2$.

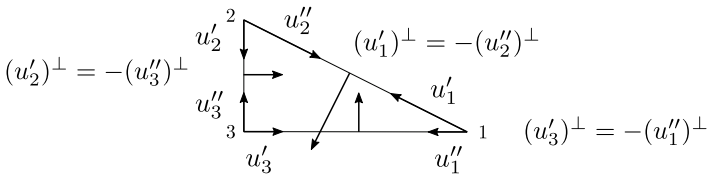


Figure 2: Determination of the types of singularities at the vertices: the triangle $\mathrm{Tz}^2_{3,0}$, $|3H|$ on $\mathbf{F}_2$

(2.11) Example. Consider v the lower-right corner of the trapezium $\mathrm{Tz}^r_{a,b}$, see (2.6) and Figure 2 above. Then $u'_v = (-r, 1)$ and $u''_v = (-1, 0)$, hence $(u'_v)^\perp = (-1, -r)$ and $-(u''_v)^\perp = (0, 1)$, and we find that the corresponding point in the image of $\mathbf{F}_r$ by $|aH + bF|$ is a smooth point.

Consider on the other hand v the upper vertex of the triangle $\mathrm{Tz}^r_{a,0}$. Then $u'_v = (0, -1)$ and $u''_v = (r, -1)$, hence $(u'_v)^\perp = (1, 0)$ and $-(u''_v)^\perp = (-1, -r)$. We may take $e_1 = (0, -1)$ and $e_2 = (1, 0)$, to the effect that

$$(u'_v)^\perp = e_2 \quad \text{and} \quad -(u''_v)^\perp = re_1 - e_2.$$

Thus the singularity at the point corresponding to v is of type $\frac{1}{r}(1, 1)$, which analytically locally is the vertex of the cone over a rational normal curve of degree r. And indeed the image of $\mathbf{F}_r$ by $|aH|$ is a generalized cone over a degree r rational normal curve, and the point under consideration is its vertex.

3 – The toric enumerative problem

3.1 – Toric Cayley–Bacharach

The following may be considered as a toric Cayley–Bacharach property, and needs to be taken in consideration in order to properly state the toric enumerative problem. Later on, we will also see a tropical incarnation of this phenomenon, see Figure 12 and Figure 27. We refer to [26, V, Corollary 4.5 and Exercise 4.6] for the original Cayley–Bacharach property.

(3.1) Lemma. *Let Z be a subscheme of $\mathcal{X}_\Delta$ cut out by a (properly intersecting) hyperplane in $\mathbf{P}^{N_\Delta - 1}$. It imposes only* $\deg Z - 1$ *independent conditions to the linear system $|\mathcal{O}_{X_\Delta}(1)|$, i.e., the linear subsystem of divisors containing Z has codimension* $\deg Z - 1$ *in $|\mathcal{O}_{X_\Delta}(1)|$.*

This follows from a basic cohomological argument, valid regardless of our toric setup, see also [III, Example (5.8.1)]. In the toric setup however, there is a nice elementary way of understanding it.

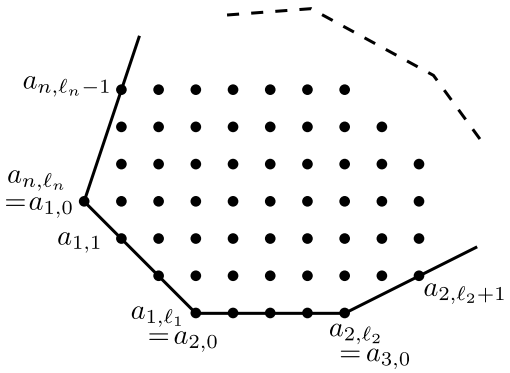
Proof. The lemma follows from the long exact sequence in cohomology induced by the restriction exact sequence,

$$0 \to \mathcal{O}_{X_\Delta}(K_{X_\Delta} + H) \to \mathcal{O}_{X_\Delta}(H) \to \mathcal{O}_{\curlyvee_\Delta}(H) \to 0.$$

Indeed, $h^1(O_{X_\Delta}(K_{X_\Delta} + H)) = 0$ by Kawamata-Viehweg vanishing, and $h^0(\mathcal{O}_{\curlyvee_\Delta}(H)) = \deg Z$ by Riemann–Roch, since $p_a(\curlyvee_\Delta) = 1$. □

Let me now give an elementary argument showing why Z cannot impose independent conditions to hyperplane sections of X_Δ. Let $C_1, \dots, C_n$ be the smooth rational curves forming the cycle $\curlyvee_\Delta$. We shall exhibit an explicit relation between the roots of the polynomials $P|_{C_1}, \dots, P|_{C_n}$, for all polynomials P with Newton polygon Δ, which implies that if P vanishes at all the points of Z but one, then it also vanishes at the last point.

Let P be a polynomial with Newton polygon Δ, and name its coefficients regarding the monomials on the boundary of Δ as indicated on the picture below.



Thus, for all $i = 1, \dots, n$, we have $P|_{C_i} = a_{i,0} + \cdots + a_{i,\ell_i} X^{\ell_i}$, hence the product of the roots of $P|_{C_i}$ equals $(-1)^{\ell_i} a_{i,0}/a_{i,\ell_i}$. Therefore, the product of the roots of $P|_{C_1}, \dots, P|_{C_n}$ altogether equals

$$(-1)^{\ell_1+\cdots+\ell_n} \frac{a_{1,0}}{a_{1,\ell_1}} \cdots \frac{a_{n,0}}{a_{n,\ell_n}} = (-1)^{\ell_1+\cdots+\ell_n},$$

which is the announced relation.

(3.2) Remark. The above relation between the roots of the polynomials $P|_{C_i}$ is Carnot's generalization of the Menelaüs theorem (or rather, its adaptation to an arbitrary toric variety). It is also interesting to note that Carnot's theorem may be obtained from the Weil reciprocity law. See [37] for more details.

3.2 – The enumerative problem

(3.3) Let $\Delta \subseteq \mathbf{R}^2$ be a convex integral polygon as in Section 2. We call $\Sigma_1, \dots, \Sigma_n$ its edges, and correspondingly $C_1, \dots, C_n$ the irreducible components of the anticanonical cycle $\curlyvee_\Delta$.

Let g be an integer such that $0 \leqslant g \leqslant p_a(\Delta) = \mathrm{Card}(\mathring{\Delta}_{\mathbf{Z}})$, and $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \underline{\mathbf{N}}^n$. We assume that for $s = 1, \dots, n$,

$$I\alpha_s + I\beta_s = (C_s \cdot H_\Delta) = \ell_{\mathbf{Z}}(\Sigma_s),$$

and

$$I\alpha \stackrel{\text{def.}}{=} \sum_{s=1}^{n} I\alpha_s < (\curlywedge_\Delta \cdot H_\Delta) = \mathrm{Card}(\partial\Delta_{\mathbf{Z}}).$$

The problem is to find the number $N_g^\Delta(\alpha,\beta)$ of genus g curves in the linear system $|H_\Delta|$ that (i) are the Zariski closure of a curve in $\phi_\Delta((\mathbf{C}^*)^2) \cong (\mathbf{C}^*)^2 \subseteq X_\Delta$, (ii) verify the contact conditions with $\curlywedge_\Delta$ at assigned and unassigned points fixed by α and β, respectively, and (iii) pass through the appropriate number of fixed general points in X_Δ.

The precise setup is as follows. We fix $\underline{x} = \{x_1, \ldots, x_{g-1+|\beta|}\}$ a set of

$$g - 1 + |\beta| = \mathrm{Card}(\partial\Delta_{\mathbf{Z}}) + g - 1 - I\alpha - (I\beta - |\beta|) \tag{3.3.1}$$

general points in X_Δ, and $\Omega = (\{p_{s,i,j}\}_{1\leqslant j\leqslant\alpha_{s,i}})_{1\leqslant s\leqslant n, 1\leqslant i}$ a set of α general points on $\curlywedge_\Delta$, i.e., for $s = 1, \ldots, n$, $\Omega_s = (\{p_{s,i,j}\}_{1\leqslant j\leqslant\alpha_{s,i}})_{1\leqslant i}$ is a set of $\alpha_s \in \underline{\mathbf{N}}$ general points on the curve C_s. We want the number of birational maps $\phi : C \to X_\Delta$, where C is a reduced, nodal curve of arithmetic genus g, such that (i) $\phi(C)$ equals the Zariski closure of $\phi(C) \cap (X_\Delta \setminus \curlywedge_\Delta)$, (ii) there exist α points $q_{s,i,j}$ and β points $r_{s,i,j}$ on C such that $\phi(q_{s,i,j}) = p_{s,i,j}$ and

$$\phi^*\curlywedge_\Delta = \sum_{1\leqslant s\leqslant n} \sum_{1\leqslant i} \Bigl(\sum_{1\leqslant j\leqslant\alpha_i} i\, q_{i,j} + \sum_{1\leqslant j\leqslant\beta_i} i\, r_{i,j} \Bigr),$$

and (iii) $\phi(C)$ contains all $g - 1 + |\beta|$ points of $\underline{x}$.

(3.4) Remarks. This is exactly the same problem that we considered in [III] and subsequent chapters. In particular, [III, Theorem (1.7)] applies to our present situation.

The condition that $I\alpha < (\curlywedge_\Delta \cdot H_\Delta)$ is necessary because of Lemma (3.1): if we impose all contact points with $\curlywedge_\Delta$ to be assigned at general points, then there is no curve at all.

The condition (i) that the curves are the Zariski closures of their intersections with $X_\Delta \setminus \curlywedge_\Delta$ really means that we count curves that do not contain any irreducible component of the anticanonical cycle.

3.3 – Examples from degenerations of $K3$ surfaces

(3.5) Introduction. An emblematic situation where enumerative problems of the kind described in Section 3.2 occur is when one undertakes the enumeration of curves on $K3$ surfaces by considering type III degenerations, e.g., degenerations of $K3$ surfaces $X_t \subseteq \mathbf{P}^N$, $t \neq 0$, to unions of planes $X_0 \subseteq \mathbf{P}^N$ realizing a triangulation of the sphere $\mathbf{S}^2$ (i.e., the dual complex of X_0 is the dual complex of a triangulation of $\mathbf{S}^2$). A basic example is quartic surfaces $X_t \subseteq \mathbf{P}^3$ degenerating to a *tetrahedron*, i.e., the union of four general planes in $\mathbf{P}^3$. Typically, as we shall illustrate below, the irreducible components of the central fibre X_0 will be toric surfaces, with the anticanonical cycle cut out by the other components of the central fibre.

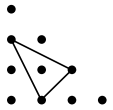
In general, such a degeneration comes with various alternative degenerations of the same surfaces $X_t \subseteq \mathbf{P}^N$ to other limits $X_0' \subseteq \mathbf{P}^N$, which zoom in on a particular aspect (typically an infinitesimal one) of the initial degeneration. For instance, one may obtain an alternative degeneration by blowing up the total space of the initial degeneration at a smooth point p of X_0; subtracting the exceptional divisor E to the pull-back of the initial relative polarization then gives a degeneration to $X_0' = \pi_p(X_0) \cup E \subseteq \mathbf{P}^N$, where $\pi_p(X_0)$ is the projection of X_0 from p to a hyperplane $H \subseteq \mathbf{P}^N$, and E is embedded as a plane in $\mathbf{P}^N$ intersecting H (and $\pi_p(X_0)$) along a line. This is explained in [I].

The examples in (3.6), (3.7) and (3.8) below show up when one considers a degeneration of quartic hypersurfaces $X_t \subseteq \mathbf{P}^3$ to a *tetrahedron* $X_0 \subseteq \mathbf{P}^3$, i.e., the union of four general planes in $\mathbf{P}^3$ (equivalently, there exist homogeneous coordinates $(x\,{:}\,y\,{:}\,z\,{:}\,w)$ in which X_0 is defined by

the equation $xyzw = 0$). A *face*, an *edge*, and a *vertex* of the tetrahedron X_0 are, respectively, the intersections of one, two, and three distinct irreducible components of X_0. We refer to [16] for a detailed study, and to [I, Section 8.2] for a brief discussion.

The example in (3.9) shows up when one considers a degeneration of octic $K3$ surfaces $X_t \subseteq \mathbf{P}^5$ (i.e., smooth complete intersections of three quadrics) to an *octahedron*, i.e., the union of eight 2-planes $X_0 \subseteq \mathbf{P}^5$, the dual complex of which is a plain, Platonic, cube in $\mathbf{R}^3$ (equivalently, there exist homogeneous coordinates $(x : y : z : w : u : v)$ in which X_0 is defined by the equations $xy = zw = uv = 0$).

(3.6) A cubic surface with an anticanonical triangle.
(3.6.1) Description of the surface. The surface $X_\Delta \subseteq \mathbf{P}^3$ corresponding to the Newton polygon Δ below



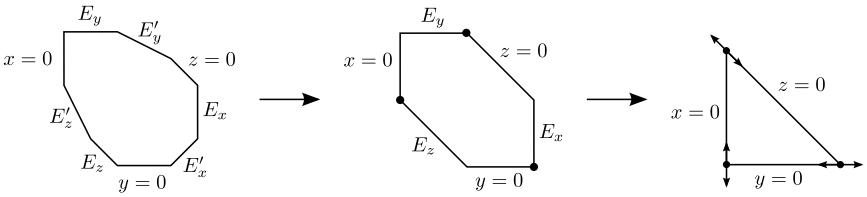
is the cubic hypersurface $xyz = w^3$ in homogeneous coordinates $(x : y : z : w)$. Its anticanonical cycle $\mathfrak{H}_\Delta$ is the triangle $xyz = w = 0$. The three sides of this triangle are the only lines contained in X_Δ; the surface X_Δ has an A_2 double point at each vertex of $\mathfrak{H}_\Delta$, and is smooth elsewhere.

Let us verify the type of singularity at the vertex at the bottom, following the rule of (2.10). One has $u'_v = (1, 1)$ and $u''_v = (-1, 2)$, hence $(u'_v)^\perp = (-1, 1)$ and $-(u''_v)^\perp = (2, 1)$. We may take $e_1 = (0, 1)$ and $e_2 = (-1, 1)$, to the effect that

$$(u'_v)^\perp = e_2 \quad \text{and} \quad -(u''_v)^\perp = 3e_1 - 2e_2.$$

The singularity corresponding to this vertex is of type $\frac{1}{3}(1, 2)$, i.e., an A_2 double point.

The linear system of hyperplane sections $|\mathcal{O}_{X_\Delta}(1)|$ may be realized, much in the same way as in Example (2.1), (b), above, as the linear system of plane cubics with six base points in the configuration indicated on the picture below. On the picture, two successive blow-ups of $\mathbf{P}^2$ at three points are drawn, with exceptional divisors E_x, E_y, E_z and E'_x, E'_y, E'_z respectively; base points are represented by black dots when they are proper, and arrows when they are infinitely near.



The three sides of $\mathfrak{H}_\Delta$ are the three (-1)-curves: $C_1 = E'_y$, $C_2 = E'_z$, and $C_3 = E'_x$. The cubic X_Δ is the main character of [I, Section 8.1].

Alternatively, the surface X_Δ may be realized as a cyclic triple cover of $\mathbf{P}^2$ branched over a general triangle.

(3.6.2) Occurrence in a degeneration to a tetrahedron. In the degeneration context discussed in (3.5), the surface X_Δ shows up in the following situation. Consider a degeneration of quartic surfaces $X_t \subseteq \mathbf{P}^3$ to a tetrahedron $X_0 \subseteq \mathbf{P}^3$, and assume we want to study limits of hyperplane sections $H_t \cap X_t$ such that the planes H_t tend to a plane H_0 passing through a vertex v of X_0 (recall the terminology introduced in (3.5)). To do so, it is meaningful to substitute the

degeneration to X_0 by the alternative degeneration of the same surfaces $X_t \subseteq \mathbf{P}^3$ ($t \neq 0$) to the quartic surface $X_0' = X_\Delta \cup \Pi_v \subseteq \mathbf{P}^3$, where Π_v is the face of X_0 opposite to v, identified with the plane spanned by λ_Δ: indeed, the limit of the curves $H_t \cap X_t$ may be realized as a hyperplane section $H_0' \cap X_0'$ (provided H_0 is general among the planes passing through v); note that H_0' depends not only on H_0, but also on how the planes H_t tend to H_0. We refer to [I] for more details. The alternative degeneration is obtained by blowing-up the total space of the initial degeneration at the vertex v of X_0, exactly as in [I, 8.1].

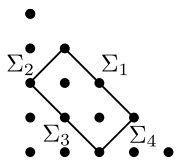
For instance, let $x_t, x_t' \in \mathbf{P}^3$ be two families of points tending to two general points $x_0, x_0' \in \mathbf{P}^3$, and assume we want to enumerate the limits at $t = 0$ of 1-nodal curves $H_t \cap X_t$ cut out by planes H_t passing through x_t and x_t'. There is a unique plane H_0 through x_0 and x_0', and we want to determine to which extent $H_0 \cap X_0$ is a limit of 1-nodal curves $H_t \cap X_t$; thus H_0 is fixed, and we want to find the possible families H_t tending to H_0 such that $H_t \cap X_t$ is a 1-nodal curve for all $t \neq 0$. To do so, it is relevant to consider the degeneration to X_0'. The plane H_0 fixes the line $H_0 \cap \Pi_v$ and, as it turns out, what we need to do is find the 1-nodal hyperplane sections of X_Δ that match with this line in X_0'. The line $H_0 \cap \Pi_v$ cuts out three points on the triangle $\lambda_\Delta = X_\Delta \cap \Pi_v$, and in the end we need to enumerate 1-nodal hyperplane sections of X_Δ passing through these three points.

(3.6.3) The toric enumerative problem. In the situation described above, we want to assign three points to hyperplane sections of $X_\Delta \subseteq \mathbf{P}^3$, one on each side of the triangle λ_Δ; these three points lie on a line, hence together they impose only two independent linear conditions on $|\mathcal{O}_{X_\Delta}(1)|$, in line with Lemma (3.1). To reduce to the exact terms of the enumerative problem (3.3), one may impose the passing through a fixed point on only two sides of λ_Δ, and let the intersection point with the third side move freely. For instance, one may set $\alpha = (1, 1, 0)$ and $\beta = (0, 0, 1)$. The upshot of Lemma (3.1) is that this third point, although free to move, will turn out to remain fixed. About this trick, see also [III, Paragraph (5.8)].

One may see that $N_0^\Delta(\alpha, \beta) = 3$ (i.e., there are three 1-nodal hyperplane sections of X_Δ through our three aligned points on λ_Δ) with the topological formula giving the number of singular members in a pencil of curves, see [XI, Section 2.1], as follows. The two fixed points on λ_Δ (or the three aligned fixed points) define a pencil in $|\mathcal{O}_{X_\Delta}(1)|$; this is a pencil of plane cubics, hence it has twelve singular members, counted with multiplicities. There is a special member in this pencil that corresponds to λ_Δ itself, incarnated as its pull-back to the minimal desingularization of X_Δ, namely the 9-gon in the picture above; it counts with multiplicity 9 among the singular members of the pencil. It follows that there are $12-9 = 3$ irreducible 1-nodal curves in this pencil. This is the same argument as in [I, 8.1]; we will confirm this number by solving the tropical counterpart of this enumerative problem in (7.18); see also Example (7.14.2) for another confirmation.

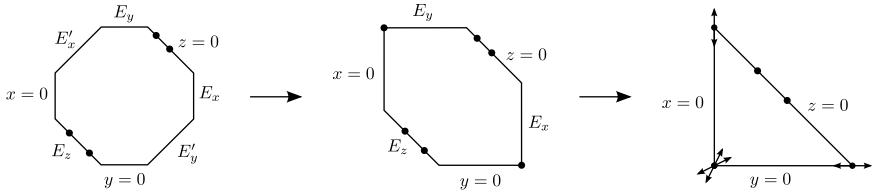
The other examples are similar in spirit, and we will deal with them more briefly.

(3.7) Consider the surface $X_\Delta \subseteq \mathbf{P}^7$ corresponding to the Newton polygon depicted below.



Its anticanonical cycle λ_Δ consists of two conics C_1, C_3 corresponding to the sides Σ_1, Σ_3, and two lines C_2, C_4 corresponding to the sides Σ_2, Σ_4.

Along a degeneration of quartic $K3$ surfaces $X_t \subseteq \mathbf{P}^3$ to a tetrahedron $X_0 \subseteq \mathbf{P}^3$, the limits of hyperplane sections $H_t \cap X_t$ such that the planes H_t tend to a plane H_0 containing an edge W of X_0 may be realized as the hyperplane sections of X_Δ passing through two base points on C_1 and two base points on C_3. The four base points on C_1 and C_3 are determined by how the X_t tend to X_0; for instance, if the X_t move in a pencil of quartic surfaces, then they correspond to the intersection $X_t \cap W$ for any $t \neq 0$, as in [I, Examples (3.3) and (3.4)]. They impose four independent conditions to the linear system corresponding to Δ, and therefore determine a 3-dimensional linear subsystem of $|\mathcal{O}_{X_\Delta}(1)|$ (and, dually, a projection of $X_\Delta \subseteq \mathbf{P}^7$ to $\mathbf{P}^3$). The latter can be realized as the linear system of plane quartics with base points as indicated on the picture below (with the same graphic convention as in (3.6): there are four simple and one double proper base points, and four infinitely near base points, all simple).



The projection map $\phi : X_\Delta \to \mathbf{P}^3$ from the four base points contracts the two conics C_1 and C_3 to two triple points, and maps the two lines C_2 and C_4 to the same image; the image $\phi(X_\Delta)$ is a rational quartic surface $X_W \subseteq \mathbf{P}^3$, with a double line (the image of both C_2 and C_4), two triple points on the latter (the images of C_1 and C_3, respectively), and no further singularity. The rational quartic surface $X_W \subseteq \mathbf{P}^3$ is one of the relevant alternative degenerations X'_0 of the degeneration to the tetrahedron X_0; in this perspective, the double line of X_W is reminiscent of the edge of X_0 opposite to W.

In relation with the enumeration of limits of 2-nodal hyperplane sections $H_t \cap X_t$ such that the planes H_t contain a general point $x_t \in \mathbf{P}^3$ and tend to a plane H_0 containing the edge W, one wants to enumerate genus 0 hyperplane sections of $X_W \subseteq \mathbf{P}^3$ passing through a general point of the double line $\phi(C_2) = \phi(C_4)$. The latter point corresponds to the intersection of H_0 and the edge of X_0 opposite to W, where H_0 is the unique plane containing the line W and the point x_0 limit of the x_t.

At the level of $X_\Delta \subseteq \mathbf{P}^7$, this amounts to adding two more base points to our initial four: one on C_2 and one on C_4, having the same image by the map ϕ to $\mathbf{P}^3$. Equivalently, these two base points together with the four on C_1 and C_3 impose only five independent linear conditions to $|\mathcal{O}_{X_\Delta}(1)|$, again in line with Lemma (3.1). To set up our enumerative problem in the terms of (3.3), we fix the intersection with C_2, and let that with C_4 move freely; in other words, we set $\alpha = (2, 1, 2, 0)$ and $\beta = (0, 0, 0, 1)$.

By carefully analysing the geometry, one finds with the Plücker formulae that $N_1^\Delta(\alpha, \beta) = 10$ and $N_0^\Delta(\alpha, \beta) = 16$, see [16, Section 5.5]. We will come back to this with a tropical point of view in Example (8.19). For related floor diagram considerations, see Examples (8.5), (8.8), (8.11), and (8.18.2).

(3.8) Consider the 4-Veronese surface $X_\Delta = v_4(\mathbf{P}^2) \subseteq \mathbf{P}^{14}$, corresponding to the triangle $\Delta = T_4$ as in (2.1). The anticanonical cycle $\mathfrak{K}_\Delta$ is the image by v_4 of the triangle $xyz = 0$ in $\mathbf{P}^2$.

Along a degeneration of quartic $K3$ surfaces $X_t \subseteq \mathbf{P}^3$ to a tetrahedron $X_0 \subseteq \mathbf{P}^3$, to the limits of hyperplane sections $H_t \cap X_t$ such that the planes H_t tend to a face F of X_0, there correspond hyperplane sections of X_Δ passing through twelve base points on $\mathfrak{K}_\Delta$. These base points are determined by how the X_t tend to X_0, as in [I, Examples (3.3) and (3.4)], and together they

form the image by v_4 of the divisor cut out on the triangle $xyz = 0$ in $\mathbf{P}^2$ by a general quartic curve. As in [III, Example (5.8.1)], they impose only eleven independent linear conditions to $|\mathcal{O}_{X_\Delta}(1)|$. The projection of $X_\Delta \subseteq \mathbf{P}^{14}$ from these twelve points is a quartic surface $X_F \subseteq \mathbf{P}^3$, with a triple point as a result of the contraction of $\lambda\!\!\!^{-}_\Delta$. The surface $X_F \subseteq \mathbf{P}^3$ is one of the relevant alternative degenerations of the degeneration to X_0.

Correspondingly, one wants to enumerate hyperplane sections of this quartic surface $X_F \subseteq \mathbf{P}^3$ of genera $2, 1, 0$, respectively, and passing through $2, 1, 0$ general points of X_F, respectively. Again, to set this up as in (3.3), we impose the passing through only eleven out of the twelve base points. For instance, we set $\alpha = (4, 4, 3)$ and $\beta = (0, 0, 1)$.

By degeneration, one finds

$$N_2^\Delta(\alpha, \beta) = 21; \quad N_1^\Delta(\alpha, \beta) = 132; \quad N_0^\Delta(\alpha, \beta) = 304,$$

cf. [16, Theorem 3]. Mikhalkin (private communication) has been able to find these numbers with floor diagrams, but we won't do it in these notes.

(3.9) Consider the surface $X_\Delta \subseteq \mathbf{P}^4$ corresponding to the polygon Δ represented below.

It is the quartic surface $xy - t^2 = zw - t^2 = 0$ in homogeneous coordinates $(x : y : z : w : t)$. It has four ordinary double points and no further singularity. Its anticanonical cycle $\lambda\!\!\!^{-}_\Delta$ is a quadrilateral made of four lines, the vertices of which coincide with the ordinary double points of the surface.

The linear system $|\mathcal{O}_{X_\Delta}(1)|$ may be thought of either as a linear system of plane cubics with five base points — as indicated by the picture on the left —, or as a linear system of curves of bidegree $(2, 2)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ with four base points — as indicated by the picture on the right. Alternatively, X_Δ may be realized as the double cover of $\mathbf{P}^1 \times \mathbf{P}^1$ branched over an anticanonical quadrilateral.

The quartic surface $X_\Delta \subseteq \mathbf{P}^4$ and the complete linear system of its hyperplane sections is the key to understand the limits, in terms of the theory expound in [I], of hyperplane sections $H_t \cap X_t$ of smooth surfaces $X_t \subseteq \mathbf{P}^N$ which degenerate to some surface $X_0 \subseteq \mathbf{P}^N$, such that the limit of the H_t is a hyperplane H_0 passing through a point $v \in X_0$, at which the tangent cone of X_0 is the cone in $\mathbf{P}^4$ over the quadrilateral $(xy = zw = 0) \subseteq \mathbf{P}^3$; e.g., v is a vertex of an octahedron $X_0 \subseteq \mathbf{P}^5$.[5] This is analogous to the cubic $X_\Delta \subseteq \mathbf{P}^3$ of (3.6) corresponding to limits of sections $H_t \cap X_t$ when the limit of the H_t is a plane passing through a triple point of X_0, at which X_0 looks like the union of three general planes in $\mathbf{P}^3$, see [I, Section 8.1].

In this context, the enumerative problem one needs to consider is that of counting the hyperplane sections of X_Δ of genus 0 intersecting $\lambda\!\!\!^{-}_\Delta$ along a prescribed member of $|\mathcal{O}_{\lambda\!\!\!^{-}_\Delta}(1)|$; such a divisor consists of four coplanar points, one on each side of $\lambda\!\!\!^{-}_\Delta$. Correspondingly, one sets (for example) $\alpha = (1, 1, 1, 0)$ and $\beta = (0, 0, 0, 1)$. One finds $N_0^\Delta(\alpha, \beta) = 4$ in the same way as in (3.6). We will compute this number from the tropical perspective in (7.17), see also (7.19), and using floor diagrams in (8.18.1).

[5] In this case, the relevant alternative degeneration is $X_0' = X_\Delta \cup \bigcup_{i=1}^4 \Pi_i \subseteq \mathbf{P}^5$, where $\Pi_1, \ldots, \Pi_4$ are the four faces of the octahedron which do not pass through v; X_Δ and $\bigcup_{i=1}^4 \Pi_i$ each span a hyperplane in $\mathbf{P}^5$, and their intersection is the quadrilateral $\lambda\!\!\!^{-}_\Delta$.

4 – From the torus to the tropics

In this section we present amœbæ and tropical hypersurfaces. The amœba of a hypersurface in the torus $(\mathbf{C}^*)^n$ has first been considered by Gelfand–Kapranov–Zelevinsky in [23]. Non-archimedean amœbæ have been introduced by Kapranov ([19], see footnote 6), who showed that they coincide with tropical hypersurfaces.

(4.1) Amœba of a hypersurface in a torus. Let X be a hypersurface of the torus $(\mathbf{C}^*)^n$. The *amœba* of X is the image of X by the logarithmic moment map

$$\begin{aligned} \mathrm{Log}: \quad & (\mathbf{C}^*)^n \to \mathbf{R}^n \\ & (z_1, \dots, z_n) \mapsto \big(\log|z_1|, \dots, \log|z_n|\big). \end{aligned}$$

Pictured below is the amœba of a general line in $(\mathbf{C}^*)^2$. From this you should guess whence the name 'amœba' came from.

(4.2) Limits of amœbæ. One of the ideas at the origin of tropical geometry is to consider limits of amœbæ. To this effect, consider a hypersurface $X \subseteq (\mathbf{C}(t)^*)^n$, which may be seen as a rational family of hypersurfaces in $(\mathbf{C}^*)^n$. The $\mathbf{C}(t)$-points of X are n-tuples $z(t)$ of rational functions $(z_1(t), \dots, z_n(t))$. Renormalizing, it is possible to consider the limit of $\mathrm{Log}(z(t))$ at $t = 0$, namely

$$\lim_{t\to 0} \frac{1}{\log|t|}\big(\log|z_1(t)|, \dots, \log|z_n(t)|\big) = \big(v(z_1), \dots, v(z_n)\big),$$

where v is the t-adic valuation, i.e., for $f(t) \in \mathbf{C}(t)$, $v(f(t))$ is the integer a such that one may write $f(t) = t^a(\alpha + \sum_{n=1}^{+\infty} \alpha_n t^n)$ for some non-zero α.

The field $\mathbf{C}(t)$ however is not algebraically closed, so in general X doesn't have enough $\mathbf{C}(t)$-points, and the previous considerations don't give a fully satisfactory definition of the limit at $t = 0$. The straightforward workaround to this issue is to replace $\mathbf{C}(t)$ by its algebraic closure; another nice possibility is to work over the field of Puiseux series — the latter being the algebraic closure of the field of formal Laurent series $\mathbf{C}((t))$. Recall that a Puiseux series of the variable t and with coefficients in $\mathbf{C}$ is a formal power series of the form

$$f(t) = \sum_{n=n_0}^{+\infty} \alpha_n t^{n/N}$$

for some $n_0 \in \mathbf{Z}$ and $N \in \mathbf{N}^*$.

The idea of taking limits of amœbæ is one inspiration for the definitions in the following paragraph.

(4.3) Non-Archimedean amœbae. Let $\mathbf{K}$ be a valuated, non-Archimedean, algebraically closed field. Being non-Archimedean means that $\mathbf{K}$ has some infinitely large elements, i.e., elements that are larger than all integers: in the context of series in t observed at $t = 0$, these are the series with negative t-adic valuation.

Precisely, we say that $\mathbf{K}$ is *valuated non-Archimedean* if it comes with a function $v : \mathbf{K}^* \to \mathbf{R}$, the *valuation*, such that

$$\forall f, g \in \mathbf{K}^* : \quad v(fg) = v(f) + v(g), \quad \text{and} \quad v(f+g) \geqslant \min\{v(f), v(g)\} \text{ (if } f+g \neq 0).$$

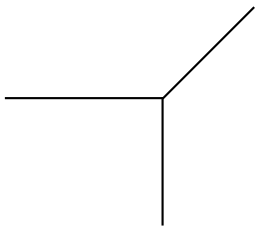
This determines a *non-Archimedean norm* $|\cdot|$, by setting

$$|f| = \begin{cases} e^{-v(f)} & \text{if } f \in \mathbf{K}^* \\ 0 & \text{if } f = 0. \end{cases}$$

Now, let $X \subseteq (\mathbf{K}^*)^n$ be a hypersurface. The (non-Archimedean) *amœba* of X is, by definition, its image by the valuative moment map

$$\begin{aligned} \mathrm{Val} : \qquad (\mathbf{K}^*)^n &\longrightarrow \mathbf{R}^n \\ (x_1, \dots, x_n) &\longmapsto \big(v(x_1), \dots, v(x_n)\big). \end{aligned}$$

Pictured below is the amœba of a general line in $(\mathbf{K}^*)^2$. One way to think about it is as the limit of the amœbæ of a rational family of lines in $(\mathbf{R}^*)^2$, i.e., the limit as t goes to 0 of the amœbæ of a family of lines defined by an equation $a(t)x + b(t)y + c(t) = 0$, where a, b, c are rational functions in the parameter t.



(4.4) Theorem (Kapranov[6], see [19])**.** *Let $\mathbf{K}$ be a valuated, non-Archimedean, algebraically closed field, and assume its valuation $v : \mathbf{K}^* \to \mathbf{R}$ is surjective.*

Consider a hypersurface $X \subseteq (\mathbf{K}^*)^n$ defined by a polynomial

$$P(x) = \sum_{I \in \mathrm{Supp}(P)} a_I x^I$$

($\mathrm{Supp}(P)$ *is the support of P, i.e., the set of those multi-indices I such that the coefficient of P in the monomial x^I is non-zero).*

The amœba of X is the corner locus of the piecewise affine function

$$p \in \mathbf{R}^n \longmapsto \max_{I \in \mathrm{Supp}(P)} \{I \cdot p - v(a_I)\}.$$

The corner locus of a piecewise affine function $f : \mathbf{R}^n \to \mathbf{R}$ is the set of points where f is not smooth. For a function f as in the theorem, the graph of f is a polyhedron in $\mathbf{R}^n \times \mathbf{R}$, and the corner locus of f is the projection to $\mathbf{R}^n$ of the "corners" of this polyhedron, i.e., of

[6] this theorem has apparently appeared in a preprint of Kparanov only, in 2000; this preprint is referenced in [34] for instance.

its faces of less than maximal dimensions. Equivalently, the maximal subsets of $\mathbf{R}^n$ on which the maximum in the definition of f is realized by a unique affine function $p \mapsto I \cdot p - v(a_I)$ are interiors of polyhedra, which together with their faces form a polyhedral complex, and the corner locus of f is the union of the polyhedra separating these maximal subsets, equivalently, the union of those cells of the polyhedral complex having less than maximal dimensions. We will come back to this in the next section.

The theorem also holds in case the valuation is not surjective, provided one takes the closure of the amœba. This typically happens when $\mathbf{K}$ is the field of Puiseux series.

(4.5) The tropical semi-field. Still applying the idea of taking the limit at $t = 0$, one defines a structure of semi-field (without an additive zero) on $\mathbf{R}$ as the limit of the usual field structure.

We consider the family indexed by $t \in \mathbf{R}_+^*$ of bijective maps

$$\frac{\log}{\log t} : \mathbf{R}_+^* \to \mathbf{R},$$

and use it to degenerate the field structure on $\mathbf{R}$: One defines the two operation laws "$+$" and "$\times$" as the limits of $+$ and $\times$ respectively, i.e., for all $x, y \in \mathbf{R}$:

$$x \text{ “}+\text{” } y \overset{\text{def.}}{=} \lim_{t\to 0^+} \frac{1}{\log t} \log\big((\log t)e^x + (\log t)e^y\big) = \max\{x, y\}$$

$$\text{and} \quad x \text{ “}\times\text{” } y \overset{\text{def.}}{=} \lim_{t\to 0^+} \frac{1}{\log t} \log\big((\log t)e^x \cdot (\log t)e^y\big) = x + y.$$

We call ($\mathbf{R}$, "$+$" , "$\times$") the *tropical semi-field*, and denote it by $\mathbf{T}$. Note that $\mathbf{T}$ does not have an additive zero, which would be $-\infty$. It is another convention to consider $\mathbf{T} \cup \{-\infty\}$ instead of $\mathbf{T}$ as the tropical semi-field; ours follows Mikhlakin's in [36]. See Remark (7.3) for indications on how the adjunction of $-\infty$ to $\mathbf{T}$ can be used to compactify the tropical plane $\mathbf{T}^2$.

A convenient notation for the operations "$+$" and "$\times$" is to simply surround algebraic expressions by quotation marks. Thus, "$x + y$" $= x$ "$+$" y and "xy" $= x$ "$\times$" y.

(4.6) Tropical polynomials and hypersurfaces. The definition of a tropical polynomial (function) is pretty straightforward, one just replace the ordinary algebraic expression by its counterpart in the tropical semifield. Thus, a *tropical polynomial* is a function $P : \mathbf{T}^n \to \mathbf{T}$ defined by an expression of the form

$$\begin{aligned} P(x_1, \dots, x_n) &= \text{“} \sum_I a_I x^I \text{”} \\ &= \max_I \{a_I + i_1 \cdot x_1 + \cdots i_n \cdot x_n\}, \end{aligned}$$

where the sum is taken on a finite set of multi-indices.

It does not make much sense to consider the zero locus of a function as above. Instead, a *tropical hypersurface* in $\mathbf{T}^n$ is by definition the corner locus of a tropical polynomial.

With these definitions, the content of Theorem (4.4) is that amœbæ of hypersurfaces in $(\mathbf{K}^*)^n$ and tropical hypersurfaces in $\mathbf{T}^n$ ($\mathbf{R}^n$, if one prefers) are the same objects.

5 – Parabolic polarity[7], b.k.a. Legendre–Fenchel duality

This section is about what is often called Legendre transformation. A nice reference is [27, Part E]. For the case of twice differentiable functions, a classical reference is [4, Section 14].

[7] about this name, imagined (but not used) by Fenchel, see [27, p. 247].

(5.1) Definition. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function. The* Legendre–Fenchel transform *of f is the function*

$$\begin{aligned} f^\vee : \quad & \check{\mathbf{R}}^n \to \mathbf{R} \\ & p \mapsto \sup_{x \in \mathbf{R}^n} \bigl\{ p \cdot x - f(x) \bigr\}, \end{aligned}$$

where $\check{\mathbf{R}}^n$ is the dual of $\mathbf{R}^n$.

The definition is intended for all functions defined on a subset U of $\mathbf{R}^n$, so in fact the supremum in the definition is to be taken on all $x \in U$ only.

In fact we don't allow $f^\vee$ to take the value $+\infty$, and thus $f^\vee$ is in general only defined on a subset of $\check{\mathbf{R}}^n$. To make sure that $f^\vee$ is defined on a non-empty subset of $\check{\mathbf{R}}^n$, one may assume that there exists an affine function m such that $m \leqslant f$. We make this assumption from now on and for the rest of the section; it is true in particular if f is convex.

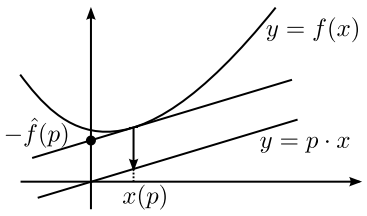
The Legendre–Fenchel transform is always a closed convex function, see [27, Example B.2.1.3]. The Legendre–Fenchel transformation is involutive on closed convex functions. More precisely, for all functions f, the "bi-conjugate" $(f^\vee)^\vee$ is the closed convex hull of f, see [27, Theorem E.1.3.5]. By definition, the closed convex hull of f is obtained by considering the epigraph $\operatorname{epi}(f)$ (i.e., the part of $\mathbf{R}^n \times \mathbf{R}$ lying above the graph of f), taking the closed convex hull $\overline{\operatorname{co}} \operatorname{epi}(f)$ of the latter, and finally taking the unique function of which this is the epigraph; in other words, it is the function

$$x \mapsto \inf \bigl\{ y : (x, y) \in \overline{\operatorname{co}} \operatorname{epi}(f) \bigr\}.$$

(5.2) Link with projective duality. Consider a strictly convex function $f : \mathbf{R}^n \to \mathbf{R}$ that is twice differentiable (thus, $f'' > 0$). Then, for all linear functional $p \in \check{\mathbf{R}}^n$, the supremum of all $p \cdot x - f(x)$ is attained at a unique point $x(p) \in \mathbf{R}^n$, which may be characterized as the unique $x \in \mathbf{R}^n$ such that the tangent space to the graph of f at the point $(x, f(x))$ has slope p; moreover, the equation of this very tangent space is

$$y = p \cdot x - f^\vee(p).$$

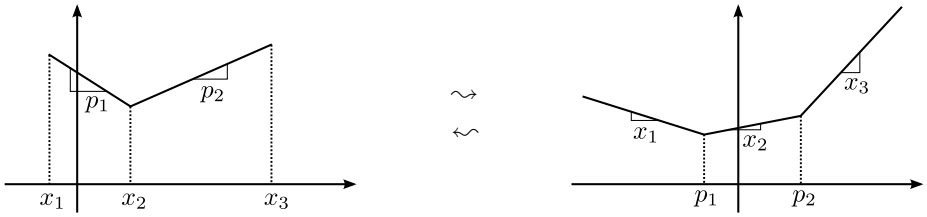
Therefore, the points of the graph of the Legendre–Fenchel transform $f^\vee$ naturally correspond to the tangent spaces of the graph of f: this is an instance of projective duality! (see [XII]).



The Legendre–Fenchel transformation is also linked with classical mechanics, hence the notation 'p' for linear functionals. Luckily enough, the letter 'p' can also stand for *pente* and *pendenza* (French and Italian for slope).

(5.3) For convex piecewise affine functions, there is an interpretation akin to that of (5.2). In this context, the Legendre–Fenchel transformation exchanges, loosely speaking, the roles of the corner loci and the slopes; this is perhaps best understood by looking at the figure below, which pictures the graphs of a piecewise affine linear function (defined on a finite interval) and its Legendre–Fenchel transform (defined on the whole $\mathbf{R}$). It is elementary to check that things indeed happen as on the picture, and that taking the Legendre–Fenchel transform of the function

on the right gives back the function on the left; I of course encourage the reader to try it for him or herself.



There is a similar story for functions f defined at finitely many points $x_1, \dots, x_n$ only. In this case, the Legendre–Fenchel transform is a convex piecewise affine function defined on the whole dual space, and the bitransform is a convex piecewise affine function defined on the convex hull of $x_1, \dots, x_n$, and characterized by the fact that its graph is the bottom of the convex hull of the graph of f. Consider for instance the restriction to $\{x_1, x_2, x_3\}$ of the function on the left in the above figure. Its transform is the function on the right of the picture, and its bi-transform is the function on the left itself. This is again elementary to check.

(5.4) Application to tropical geometry. In the context of Theorem (4.4), the function $p \in \mathbf{R}^n \mapsto \max_{I \in \mathrm{Supp}(P)}\{I \cdot p - v(a_I)\}$ is the Legendre–Fenchel transform of the function

$$v_P : I \in \mathrm{Supp}(P) \mapsto v(a_I). \tag{5.4.1}$$

Therefore the above discussion shows that, dual to the amœba of X, which may be seen as a subdivision of $\mathbf{R}^n$ into convex polytopes, there is a subdivision of the Newton polytope $\Delta(P)$ of P (i.e., the convex hull of $\mathrm{Supp}(P)$) into convex polytopes, given by the corner locus of the real-valued function on $\Delta(P)$ obtained by "extending by convexity" the function v_P, i.e., of the closed convex hull $\bar{v}_P$ of v_P. In other words, the tiles of this subdivision of $\Delta(P)$ are the maximal polytopes on which the function $\bar{v}_P$ is affine.

6 – Tropical curves

In this section we introduce two notions of tropical curves in $\mathbf{R}^2$, following Mikhalkin. First, *plane tropical curves* are rational piecewise linear objects in $\mathbf{R}^2$, which we will see coincide with the tropical hypersurfaces that have been presented in Section 4; second, *parametrized tropical curves* are certain maps from graphs to $\mathbf{R}^2$.

6.1 – Plane tropical curves

(6.1) Definition. *A* plane tropical curve *is the data of a finite union $\Gamma \subseteq \mathbf{R}^2$ of closed segments and semi-lines $\Sigma_1, \dots, \Sigma_n$, and of weights $w_1, \dots, w_n \in \mathbf{N}^*$ attached to $\Sigma_1, \dots, \Sigma_n$, respectively, such that:*

(i) for all $i \neq j$, Σ_i and Σ_j may intersect only at a point which is an end for both of them;

(ii) for all i, Σ_i has rational slope;

(iii) the following balancing condition *holds: for all $v \in \Gamma$, let $I(v) \subseteq \{1, \dots, n\}$ be the set of those indices i such that v is an end of Σ_i, and for all $i \in I(v)$, let $u_{v,i} \in \mathbf{Z}^2$ be the primitive (see Section 1.4) integral vector directing Σ_i and pointing from v outwards; then either $I(v)$ is empty, or $\mathrm{Card}(I(v)) \geqslant 3$, and*

$$\sum_{i \in I(v)} w_i . u_{v,i} = 0. \tag{6.1.1}$$

We will sometimes call just $\Gamma \subseteq \mathbf{R}^2$ a plane tropical curve, leaving the weights implicit.

Condition (i) says that the set of polyhedra containing $\Sigma_1, \dots, \Sigma_n$, their ends, and the empty set, is a polyhedral complex.[8]

It is sometimes convenient to compactify Γ by adding one end at infinity to each semi-line among $\Sigma_1, \dots, \Sigma_n$ (beware that these points at infinity are not intended in the usual sense of projective geometry; rather, see Remark (7.3)). One can then attach a graph to Γ: its vertices are the ends of $\Sigma_1, \dots, \Sigma_n$, both in $\mathbf{R}^2$ and at infinity, and its edges are in bijective correspondence with $\Sigma_1, \dots, \Sigma_n$ in the obvious way.

We will thus call *vertices* of Γ the endpoints in $\mathbf{R}^2$ of $\Sigma_1, \dots, \Sigma_n$, and *vertices at infinity* of Γ their endpoints at infinity. We call *edges* of Γ the one-dimensional polyhedra $\Sigma_1, \dots, \Sigma_n$. The *infinite edges* of Γ are those Σ_i which are unbounded in $\mathbf{R}^2$, i.e., the semi-lines. Note that the vertices at infinity of Γ are 1-valent vertices of the associated graph, while all other vertices are at least 3-valent.

For each vertex $v \in \mathbf{R}^2$ and edge Σ ending at v, we denote by $u_{v,\Sigma}$ the primitive integral vector directing Σ and pointing from v outwards, and set $u_{\Sigma,v} = -u_{v,\Sigma}$. Similarly, if Σ is a semi-line, we denote by $u_{\infty,\Sigma}$ and $u_{\Sigma,\infty}$ the primitive integral vectors directing Σ and pointing, respectively, from infinity outwards and towards infinity.

There is also the possibility of considering plane tropical curves that are the union of finitely many parallel lines. This is a degenerate case and we will not consider it in this text.

(6.2) Examples. Examples of plane tropical curves are pictured on the left in Figures 3 and 4. The weight of an edge is indicated as a circled integer, only if it is larger than one. The grids on the pictures are drawn so that one can easily read the rational slopes, but they do not mean that the vertices are integral points, which indeed is not requested in the definition of plane tropical curves.

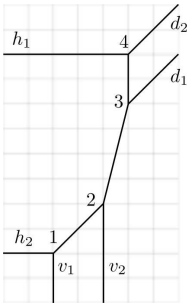
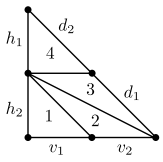
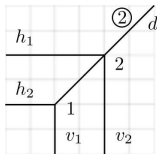


Figure 3





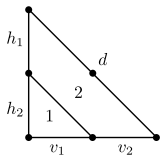


Figure 4

(6.3) Theorem (Mikhalkin [35, Section 2.1])**.** *Plane tropical curves as defined in (6.1) are exactly the hypersurfaces of the tropical plane* $\mathbf{T}^2$*, i.e., the corner loci of tropical polynomials, as defined in (4.6).*

The main content of this theorem is that the balancing condition (6.1.1) characterizes tropical hypersurfaces. To make this statement work, one needs to equip the corner locus of a tropical polynomial P with a weight function on the set of its edges. By definition, the *weight* of an

[8] Recall that a *polyhedral complex* $\mathcal{K}$ is a set of polyhedra such that: (i) for all polyhedra $P \in \mathcal{K}$, all faces of P belong to $\mathcal{K}$, and (ii) for all polyhedra $P, P' \in \mathcal{K}$, the intersection $P \cap P'$ belongs to $\mathcal{K}$. Note that the empty set is a face of every polyhedron.

edge Σ is the integral length (see (2.7)) of the dual segment in the decomposition of the Newton polygon $\Delta(P)$ discussed in (5.4). This boils down to the following: let Σ correspond to a corner between the graphs of two affine functions $(x,y) \mapsto a+ix+jy$ and $(x,y) \mapsto b+kx+ly$ with rational slopes; then the weight of Σ in the corner locus is the integral distance between the two integral points (i,j) and (k,l) in $\check{\mathbf{R}}^2$, i.e., the number of integral points on the segment $\big[(i,j),(k,l)\big]$ minus one, which we have denoted by $\ell_{\mathbf{Z}}\big[(i,j),(k,l)\big]$ in Section 2.

(6.4) Newton polygon of a plane tropical curve. Let $\Gamma \subseteq \mathbf{R}^2$ be a tropical curve. By Theorem (6.3) it is the corner locus of some tropical polynomial P. We define the *Newton polygon* of Γ, denoted by $\Delta(\Gamma)$, to be the Newton polygon of P, i.e., the convex hull of the multi-indices I such that the coefficient a_I of P is non-zero.

The Newton polygon of Γ may be found without resorting to the polynomial P, this is very useful in practice and goes as follows. Consider the list $(\Sigma_1,\dots,\Sigma_n)$ of the infinite edges of Γ as a list ordered up to cyclic permutation, following the orientation of $\mathbf{R}^2$.[9] Then, start at an arbitrary integral point of $\mathbf{R}^2$, and draw the vectors $w(\Sigma_1).u^{\perp}_{\Sigma_1,\infty},\dots,w(\Sigma_n).u^{\perp}_{\Sigma_n,\infty}$ consecutively, so as to obtain a chain in $\mathbf{R}^2$: when one vector is drawn, the next one starts where the previous one ended (recall that $u_{\Sigma,\infty}$ is the unitary direction vector of Σ oriented towards the end at infinity). It is a consequence of the balancing condition that this circuit of vectors is a closed path, and actually a convex integral polygon. It is, in fact, the Newton polygon of Γ.

(6.4.1) Example. For each example of plane tropical curve in (6.2), the corresponding Newton polygon is indicated next to it, on the right. The subdivision corresponds to the following construction.

(6.5) Subdivision of the Newton polygon. Going on with the same idea as in (6.4), one may enhance the Newton polygon of a plane tropical curve with a subdivision. By definition, this is the subdivision of $\Delta(P)$ introduced in (5.4) using the Legendre–Fenchel transformation, for any tropical polynomial P such that Γ is the tropical hypersurface defined by P. In particular, the subdivision of the Newton polygon of a plane tropical curve Γ is dual to Γ itself, seen as a subdivision of $\mathbf{R}^2$.

Again, this subdivision may be found without resorting to the tropical polynomial P. Let $\Gamma \subseteq \mathbf{R}^2$ be a tropical curve. For all vertices v of Γ, we consider the list $(\Sigma_1,\dots,\Sigma_n)$ of the edges of Γ ending at v, ordered up to cyclic permutation. Because of the balancing condition at v, drawing the vectors $w(\Sigma_1)\cdot u^{\perp}_{v,\Sigma_1},\dots,w(\Sigma_n)\cdot u^{\perp}_{v,\Sigma_n}$ consecutively produces an integral convex polygon Δ_v.

There is a canonical way of arranging all the Δ_v together as a polygonal decomposition of the Newton polygon Δ of Γ. Let Σ be an edge of Γ. If it is a segment, it has two ends v_1 and v_2 in $\mathbf{R}^2$, which are vertices of Γ, and one shall glue Δ_{v_1} and Δ_{v_2} along their faces supported along $w(\Sigma)u^{\perp}_{v_1,\Sigma}$ and $w(\Sigma)u^{\perp}_{v_2,\Sigma}$, respectively, so that the 2-dimensional polygons Δ_{v_1} and Δ_{v_2} are each on one side of this segment. If Σ is a semi-line, and it has only one end $v \in \mathbf{R}^2$ which is a vertex of Γ; then one glues Δ_v and $\Delta(\Gamma)$ along the two segments $w(\Sigma)u^{\perp}_{v,\Sigma}$ and $w(\Sigma)u^{\perp}_{\Sigma,\infty}$, so that Δ_v lies inside Δ (note that the segment $w(\Sigma)u^{\perp}_{\Sigma,\infty}$ is in general strictly contained in a face of $\Delta(\Gamma)$).

Once arranged in this way, the polygons Δ_v and their faces form a polyhedral complex, such that the union of all Δ_v equals $\Delta(\Gamma)$. This complex is the subdivision of $\Delta(\Gamma)$.

(6.5.1) Remark. To a subdivision of a lattice polygon Δ into a tiling of lattice polygons as above, one may associate a "reducible toric surface" with the same degree and embedding dimension

[9] To see this ordering, one may wish to draw a circle of sufficiently large radius in $\mathbf{R}^2$, so that it intersects Γ at exactly one point on each infinite edge and nowhere else; then the order on the set of infinite edges of Γ is that of their intersection points with the circle, according to the orientation of $\mathbf{R}^2$.

as X_Δ, and a (toric) degeneration of X_Δ to this reducible toric surface, see (7.9). One may wonder if and how plane tropical curves with the same subdivised polygon Δ are related to this degeneration: we shall provide answers in Section 7.2. See also Paragraph (9.4), which explores the relation of plane tropical curves with toric degenerations of the projective plane, following [10] and with evocative pictures.

(6.5.2) Example. For each plane tropical curve in (6.2), the Newton polygon is indicated with its subdivision pictured.

(6.6) Arithmetic genus. We define the *arithmetic genus* of a plane tropical curve as the number of integral points in the interior of its Newton polygon, as in (2.9).

6.2 – Abstract tropical curves and tropical maps

The definitions in this section are due to Mikhalkin [36] (beware however that the versions that we give here slightly differ from [36]).

(6.7) Definition. *An* abstract tropical curve *is the data of a finite graph $\bar\Gamma$ (which we see as a compact topological space), without isolated vertex, and equipped with a complete metric γ on the complement $\bar\Gamma \setminus \Gamma^\infty$, denoted by Γ, where Γ^∞ is the set of 1-valent vertices of $\bar\Gamma$.*

We will often write $\mathbf{\Gamma} = (\bar\Gamma, \gamma)$. The points of Γ^∞ are called the *vertices at infinity* of $\bar\Gamma$. The condition on the completeness of the metric means that every edge of Γ should be isometric either to a segment, a semi-line, or (rarely) a line, Edges with at least one end at infinity are called *unbounded edges* of Γ, and the others are called *bounded edges*.

An abstract tropical curve $(\bar\Gamma, \gamma)$ is *irreducible* if the space $\bar\Gamma$ is connected. The *arithmetic genus* $p_a(\bar\Gamma)$ of $(\bar\Gamma, g)$ is defined by the topological Euler–Poincaré characteristic of $\bar\Gamma$, as

$$p_a(\bar\Gamma) \stackrel{\text{def.}}{=} 1 - \chi_{\text{top}}(\bar\Gamma) = b_1(\bar\Gamma) - b_0(\bar\Gamma) + 1;$$

if the curve is irreducible, this equals the first Betti number $b_1(\bar\Gamma)$.

(6.8) Definition. *A* tropical morphism *from an abstract tropical curve $\mathbf{\Gamma} = (\bar\Gamma, \gamma)$ to the tropical plane is a continuous map $f : \Gamma \to \mathbf{R}^2$ such that: (i) the restriction of f to each edge is affine integral; and (ii) the balancing condition holds at every vertex at finite distance.*

We will interchangeably use the terms tropical morphism and tropical map.

Some more precision is in order. Let Σ be an edge of Γ. The restriction of f to Σ is *affine integral* if (i) the image $f(\Sigma) \subseteq \mathbf{R}^2$ is supported on a line directed by an integral unitary vector u_Σ, and (ii) for all $x \in \Sigma$, the differential of $f|_\Sigma$ at x is $\mathbf{Z}$-linear with respect to the integral structure on the tangent spaces defined by γ and u_Σ, respectively, and the $\mathbf{Z}$-linearity factor is the same for all x; its absolute value is called the *stretching factor*, and denoted by $w_{\Sigma,f}$.

Let v be a plurivalent vertex of Γ. The balancing condition at v is

$$\sum_{\Sigma \ni v} w_{\Sigma,f} \cdot u_{v,\Sigma} = 0,$$

where as before $u_{v,\Sigma}$ is the unitary direction vector for $f(\Sigma)$ directed from v to the other end.

A pair $(\mathbf{\Gamma}, f) = ((\bar\Gamma, \gamma), f)$ as in the above definition will be called a *parametrized tropical curve*. Such a pair determines, via the image $f(\Gamma)$, a plane tropical curve, which we will denote by $f(\mathbf{\Gamma})$ and call the image of $(\mathbf{\Gamma}, f)$. Note that the definition of $f(\mathbf{\Gamma})$ is not completely straightforward, as it may involve the adding of new vertices (at finite distance). This happens

for instance in Example (6.9) below: the image curve must have the vertex #6 since, by definition, two edges may only intersect at a vertex, see (6.1)); thus in case (b) the graph has to be modified.

If no confusion is likely, we may simply call tropical curve, or even just curve, a plane tropical curve, an abstract tropical curve, or a parametrized tropical curve.

(6.9) Example. Two abstract tropical curves are pictured on Figure 5, of genera 1 and 0, respectively. They both admit a tropical morphism to the plane tropical curve pictured on Figure 6.

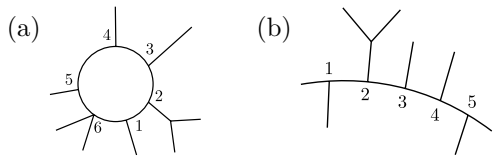


Figure 5

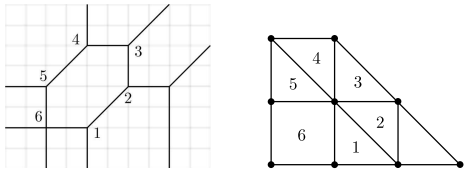


Figure 6

(6.10) If $\mathbf{\Gamma}$ is an abstract tropical curve with a tropical morphism f to the plane, one defines the *Newton polygon* of the map f, denoted by $\Delta(\mathbf{\Gamma}, f)$, as the Newton polygon of the plane tropical curve image of $\mathbf{\Gamma}$ by f; as such, it comes with a subdivision.

(6.11) Geometric genus and resolutions. So far, we have defined the arithmetic genus of a plane tropical curve (as well as that of an abstract tropical curve); one may next want to define the geometric genus of a plane tropical curve as the arithmetic genus of its normalization, as in algebraic geometry. However, it is not clear how to appropriately define the normalization in general; this works well for simple curves (see Definition (6.12)).

One may instead define the *geometric genus* of a plane tropical curve $\Gamma \subseteq \mathbf{R}^2$ as the smallest possible arithmetic genus of all abstract tropical curves $\mathbf{\Gamma'}$ such that there exists a tropical map $f : \mathbf{\Gamma'} \to \mathbf{R}^2$ mapping $\mathbf{\Gamma'}$ to Γ. In general, it is arguably best to proceed as in the theory of stable maps in algebraic geometry and, instead of trying to consider plane tropical curves of geometric genus g, to consider parametrized tropical curves of genus g, i.e., tropical maps $f : \mathbf{\Gamma'} \to \mathbf{R}^2$ such that the corresponding abstract tropical curve $\mathbf{\Gamma'}$ has arithmetic genus g.

(6.12) Definition. *A plane tropical curve* $\Gamma \subseteq \mathbf{R}^2$ *is* simple *if (i) it is reduced, i.e., it cannot be written as the sum*[10] *of two non-empty plane tropical curves* Γ_1 *and* Γ_2 *such that* Γ_2 *is contained in* Γ_1*, and (ii) the tiles of the subdivision of its Newton polygon* Δ *(i.e., the* 2*-dimensional cells of the subdivision) are all triangles and parallelograms.*

Condition (ii) means that all vertices of Γ are either trivalent, or locally look like the crossing of two lines (like vertex 6 on Figure 6).

If a plane tropical curve $\Gamma \subseteq \mathbf{R}^2$ is simple, then it is the image of a unique parametrized tropical curve of minimal genus, which is obtained by "resolving" the 4-valent vertices of Γ, as in case (b) of Example (6.9).

[10] the sum of two plane tropical curves Γ_1 and Γ_2 is defined by taking the union of Γ_1 and Γ_2, adding some vertices if necessary to maintain the condition that it is a ployhedral complex, and summing the weights of those edges appearing in both Γ_1 and Γ_2.

(6.13) Geometric genus and singularities. Alternatively, there is in principle the possibility of defining the geometric genus of a plane tropical curve Γ as its arithmetic genus minus some corrections determined by its singularities (the latter concept yet to be defined). It is however delicate to give a general definition along these lines, and we will therefore only consider simple curves. It will follow from Proposition (7.4) that this is sufficient in view of the tropical enumerative problem we will be interested in.

Let Γ be a simple plane tropical curve; its *δ-invariant* is

$$\delta(\Gamma) = \sum_{e \text{ segmentone of } \Gamma} \big(w(e)-1\big) + \sum_{\substack{v \text{ vertex of } \Gamma: \\ \Delta_v \text{ parallelogram}}} \mathscr{A}(\Delta_v) + \sum_{\substack{v \text{ vertex of } \Gamma: \\ \Delta_v \text{ triangle}}} p_a(\Delta_v)$$

where w is the weight function, $\mathscr{A}$ is the Euclidean area, and $p_a(\Delta_v)$ is the arithmetic genus of the lattice polygon Δ_v, see (2.9); a *segmentone*[11] is a maximal connected chain of parallel edges of Γ which is bounded; thus, it is a chain of parallel segments which cannot be continued by a parallel semi-line. Note that, since Γ is simple, a segmentone is a chain of segments which all have the same weight; the weight of the segmentone is this common weight.

We then define the *geometric genus* of the simple plane tropical curve Γ as its arithmetic genus $p_a(\Delta)$ minus its δ-invariant $\delta(\Gamma)$. This is coherent with our previous definition by Lemma (6.16) below.

(6.14) Examples. With the above definitions, the plane tropical curve in Example (6.9) has arithmetic genus 1 and geometric genus 0. There are tropical maps to this plane tropical curve from the two abstract tropical curves (a) and (b), having genus 1 and 0, respectively. Note that vertex #6, which gives the only non-trivial contribution to the δ-invariant of the plane curve, does not appear in the abstract curve (b).

The plane tropical curves on Figure 7 and 8 have arithmetic genus one and geometric genus zero. The only non-trivial contribution to the δ-invariant of the curve on Figure 7 (resp., Figure 8) is from its weight two edge (resp., from the vertex with associated polygon the triangle with an integral point in its interior).

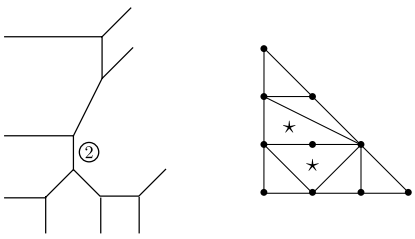


Figure 7

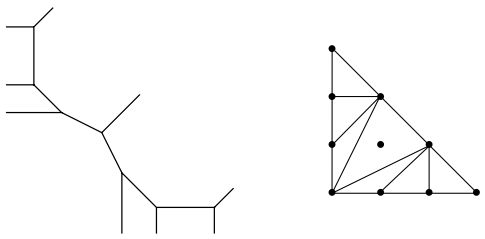


Figure 8

The plane tropical curve on Figure 9 has arithmetic genus three and geometric genus zero. The vertical weight two finite segment contributes by one to the δ-invariant, and its crossing with another edge contributes by two.

(6.15) Remark. Remembering that tropical curves are in essence limits of families of curves, one way to look at the above definition of the δ-invariant is in terms of the description of the limit of Severi varieties given in [I, Section 5]. We will see in Section 7.2 that a plane tropical

[11] pronounce seg-men-TOH-nay; this word is built by the addition of the Italian productive augmentative suffix -one to segment, in the same way a *pallone* (balloon) is a big *palla* (ball); French readers (of any nationality) might want to Frenchify it as *segmenton*. This is personal terminology of the author.

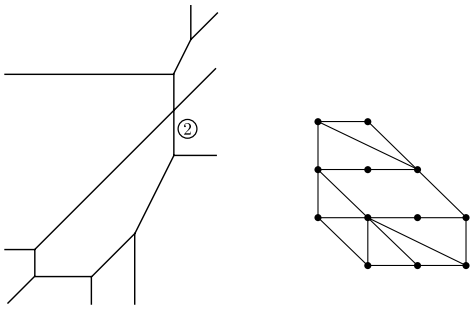


Figure 9

curve Γ with Newton polygon Δ encodes data about limits C_0 of algebraic curves C_t along a degeneration of the toric surface X_Δ to a union of the surfaces X_{Δ_v} over all vertices v of C. For each vertex v, the curve $C_0 \cap X_{\Delta_v}$ may have up to $p_a(\Delta_v)$ nodes off the toric boundary of X_{Δ_v} if Δ_v is a triangle, and up to $\mathcal{A}(\Delta_v)$ nodes if Δ_v is a parallelogram, and each of these nodes may be the limit of a node on the curves C_t. Moreover, for each finite segment e of weight w of Γ, the curve C_0 may have up to a w-tacnode along the intersection $X_{\Delta_v} \cap X_{\Delta_{v'}}$, where v and v' are the two ends of e, and each such w-tacnode may be the limit of $w-1$ nodes on the curves C_t. In sum, the curve C_0 may be the limit of curves C_t having up to $\delta(C)$ nodes. We are being colloquial here; see the end of Section 7.2 for a more precise discussion.

(6.16) Lemma. *Let Γ be a simple plane tropical curve. Then the two definitions of the geometric genus of a plane tropical curve in (6.11) and (6.13) agree.*

The following folklore proof has been communicated to me by Erwan Brugallé. We will need the following equivalent formulation of the δ-invariant of (6.13).

Let v be a vertex of Γ, and let Δ_v be the corresponding lattice polygon in the subdivision of Δ. An edge σ of Δ_v is said to *border* Δ if it is linked to an edge in the boundary of Δ by a (possibly empty) chain of parallelograms[12]. For example, for the subdivision on Figure 6, in addition to the edges directly lying on the boundary of Δ, the left edge of triangle 1 and the bottom edge of triangle 5 border Δ. Recall that $\ell_{\mathbf{Z}}(\sigma)$ denotes the integral length of σ, i.e., the number of integral points on σ minus one. We define

$$\delta(v) = \begin{cases} p_a(\Delta_v) + \frac{1}{2} \sum_{\substack{\sigma \text{ edge of } \Delta_v \\ \text{not bordering } \Delta}} \big(\ell_{\mathbf{Z}}(\sigma) - 1\big) & \text{if } \Delta_v \text{ triangle} \\ \mathcal{A}(\Delta_v) & \text{if } \Delta_v \text{ parallelogram.} \end{cases}$$

Then $\delta(\Gamma)$ equals the sum of $\delta(v)$ over all vertices of Γ.

Note that a segmentone of Γ corresponds to a maximal chain of parallelograms in the subdivision of Δ, empty if the segmentone is made of only one segment, such that none of the extremal edges in the corresponding sequence of parallel edges lies on the boundary of Δ.

Proof. For the sake of simplicity, let us assume that Γ is irreducible, i.e., it cannot be written as the sum of two non-empty plane tropical curves. Let Δ be the Newton polygon of Γ. As already noted, since Γ is simple, it is the image of a unique parametrized tropical curve $f : \mathbf{\Gamma}' \to \mathbf{R}^2$ of minimal genus, which is obtained by "resolving" the "nodes" of Γ, i.e., by seeing each crossing

[12] two segments σ and σ' are *linked by a chain of parallelograms* if they are parallel and there exist parallelograms $\Delta_{v_1}, \ldots, \Delta_{v_n}$ in the subdivision of Δ such that σ and σ' are edges of Δ_{v_1} and Δ_{v_n} respectively and, for all $i = 1, \ldots, n-1$, Δ_{v_i} and $\Delta_{v_{i+1}}$ are adjacent along an edge parallel to σ and σ'.

of two line segments as the image of two disjoint line segments. The graph Γ' (or rather $\overline{\Gamma}'$) underlying $\mathbf{\Gamma}'$ has only trivalent vertices (besides the univalent vertices at infinity), in the number t equal to the number of triangles in the subdivision of Δ. The finite edges of Γ' (i.e., those joining two vertices at finite distance) correspond to the segmentoni[13] of Γ, and their number is the number a of unordered pairs of parallel edges $\{\sigma, \sigma'\}$ in the subdivision of Δ such that σ and σ' belong to two distinct triangles Δ_v and $\Delta_{v'}$ of the subdivision and are linked by a (possibly empty) chain of parallelograms (the most common situation is that $\sigma = \sigma'$ is the intersection of two adjacent triangles); note that an edge of a triangle Δ_v appears in such a pair if and only if it does not border Δ, and in that case it appears in a unique pair. The topological Euler–Poincaré characteristic of Γ' is

$$\chi_{\text{top}}(\Gamma') = 1 - g = t - a,$$

where g is the genus of Γ', hence $g = 1 - t + a$.[14]

In order to relate this value with the genus of Γ computed as in (6.13), we shall use Pick's formula (see (2.8) and (2.9)); it says that for all lattice polygons Π,

$$p_a(\Pi) = \mathscr{A}(\Pi) - \frac{1}{2}\sum\nolimits_{\sigma \text{ edge of } \Pi} \ell_{\mathbf{Z}}(\sigma) + 1.$$

Thus, for all vertices v of Γ,

$$\delta(v) = \begin{cases} \mathscr{A}(\Delta_v) - \frac{1}{2}\sum_{\sigma \text{ edge of } \Delta_v} \ell_{\mathbf{Z}}(\sigma) + 1 + \frac{1}{2}\sum_{\substack{\sigma \text{ edge of } \Delta_v \\ \text{not bordering } \Delta}} \big(\ell_{\mathbf{Z}}(\sigma) - 1\big) & \text{if } \Delta_v \text{ triangle} \\ \mathscr{A}(\Delta_v) & \text{if } \Delta_v \text{ parallelogram,} \end{cases}$$

and therefore

$$\begin{aligned} \delta(\Gamma) &= \mathscr{A}(\Delta) - \frac{1}{2} \sum_{\substack{(\sigma,\Delta_v),\\ \sigma \text{ edge of } \Delta_v,\\ \Delta_v \text{ triangle}}} \ell_{\mathbf{Z}}(\sigma) + t + \frac{1}{2} \sum_{\substack{(\sigma,\Delta_v),\\ \sigma \text{ edge of } \Delta_v,\\ \Delta_v \text{ triangle},\\ \sigma \text{ not bordering } \Delta}} \ell_{\mathbf{Z}}(\sigma) - a \\ &= \mathscr{A}(\Delta) - \frac{1}{2} \sum_{\substack{(\sigma,\Delta_v),\\ \sigma \text{ edge of } \Delta_v,\\ \Delta_v \text{ triangle},\\ \sigma \text{ bordering } \Delta}} \ell_{\mathbf{Z}}(\sigma) + t - a. \end{aligned}$$

Now, we claim that the big sum in the last expression equals the sum of the lengths of the edges of Δ. Indeed, since Γ is irreducible it cannot contain a line between two vertices at infinity. Therefore, an edge in the boundary of Δ cannot be linked by a chain of parallelograms to another edge in the boundary of Δ, and thus every edge in the boundary of Δ is linked by a chain of parallelograms to an edge of a triangle of the subdivision of Δ.

Finally, by Pick's formula again, $\delta(\Gamma) = p_a(\Delta) - 1 + t - a$, hence $p_a(\Delta) - \delta(\Gamma) = 1 - t + a$, which coincides with the genus g computed in the first place. □

[13] plural of segmentone.

[14] Note how this computation of $\chi_{\text{top}}(\Gamma)$ uses the irreducibility of Γ: if Γ contained a line joining two vertices at infinity (corresponding to two edges in the boundary of Δ linked by a chain of parallelograms), then Γ would have a connected component not containing any vertex at finite distance, and the formula for $\chi_{\text{top}}(\Gamma)$ would need to be suitably adjusted.

7 – The tropical enumerative problem and the Correspondence Theorem

In this section, we first state the tropical enumerative problem and describe its correspondence with the toric enumerative problem. Then we give several detailed examples.

7.1 – General theory

(7.1) The tropical enumerative problem. Let $\Delta \subseteq \mathbf{R}^2$ be a convex integral polygon, and call $\Sigma_1, \dots, \Sigma_n$ its edges. We consider the data of an integer g such that $0 \leqslant g \leqslant \mathrm{Card}(\mathring{\Delta}_{\mathbf{Z}})$, and $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \underline{\mathbf{N}}^n$ such that

$$\forall s = 1, \dots, n, \qquad I\alpha_s + I\beta_s = \ell_{\mathbf{Z}}(\Sigma_s),$$

and

$$I\alpha < \mathrm{Card}(\partial\Delta_{\mathbf{Z}}).$$

In a nutshell, the problem is to find the number of plane tropical curves of genus g and Newton polygon Δ that (i) have assigned and unassigned infinite edges as prescribed by α and β, respectively, and (ii) pass through the appropriate number of fixed general points of $\mathbf{R}^2$. It is important to count these curves with the appropriate multiplicities, for their plain number may depend on the choice of the assigned infinite edges and the fixed general points in $\mathbf{R}^2$.

To set this up precisely, fix a set $\underline{x} = \{x_1, \dots, x_{g-1+|\beta|}\}$ of $g - 1 + |\beta|$ general points of $\mathbf{R}^2$, and a set $\Omega = (\{p_{s,i,j}\}_{1 \leqslant j \leqslant \alpha_{s,i}})_{1 \leqslant s \leqslant n, 1 \leqslant i}$ of α general real numbers. Let $u_1, \dots, u_n \in \mathbf{R}^2$ be the unitary vectors such that $u_1^\perp, \dots, u_n^\perp$ give the oriented directions of $\Sigma_1, \dots, \Sigma_n$, respectively ($\Sigma_1, \dots, \Sigma_n$ are ordered, up to cyclic permutation, and oriented according to the orientation of the $\mathbf{R}^2$ in which Δ dwells). For all $s = 1, \dots, n$ and $p \in \mathbf{R}$, we let $\mathsf{L}_{u_s,p}$ be the line in $\mathbf{R}^2$ that is directed by the vector u_s and contains the point with coordinates $p \cdot u_s^\perp$.

We then define $\mathcal{E}_g^\Delta(\alpha, \beta)(\underline{x}, \Omega)$ as the set of tropical maps f from an abstract tropical curve $\mathbf{\Gamma}$ of arithmetic genus g to the tropical plane, such that the Newton polygon of f equals Δ, and in addition: (i) for all $1 \leqslant s \leqslant n$ and $i \geqslant 1$, the plane tropical curve $f(\mathbf{\Gamma})$ has $\alpha_{s,i} + \beta_{s,i}$ infinite edges E of weights $w(E) \geqslant i$ such that $u_{E,\infty} = u_s$, with $\alpha_{s,i}$ of them supported on the lines $\mathsf{L}_{u_s,p_{s,i,1}}, \dots, \mathsf{L}_{u_s,p_{s,i,\alpha_{s,i}}}$, respectively, and (ii) $f(\mathbf{\Gamma})$ contains all the points $x_1, \dots, x_{g-1+|\beta|}$.

The problem is to count the elements of the set $\mathcal{E}_g^\Delta(\alpha, \beta)(\underline{x}, \Omega)$ with the multiplicities defined in (7.6). The set $\mathcal{E}_g^\Delta(\alpha, \beta)(\underline{x}, \Omega)$ is finite by Proposition (7.4).

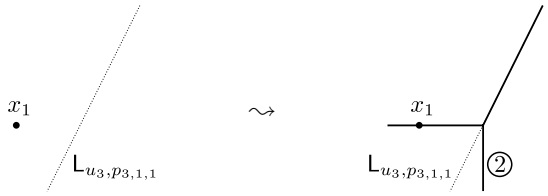
We will often refer to the conditions encoded in α, β, and Ω as the *boundary conditions* of our tropical enumerative problem. The lines $\mathsf{L}_{u_s,p_{s,i,1}}$ (and, by extensions, the elements of Ω) will be referred to as the *assigned supports at infinity* of the problem.

(7.2) Example. Consider the same Newton polygon Δ as in (2.1), (b), with the edges labelled as in the picture below.

Next to the diagram is depicted a typical plane tropical curve with Newton polygon Δ; it has arithmetic genus 0.

We consider the data $g = 0$, $\alpha = (0, 0, 1)$, $\beta = (1, [0, 1], 0)$, $\underline{x} = \{x_1\}$, and $\Omega = \{p_{3,1,1}\}$: this means that no condition is imposed on the infinite edge with direction u_1, and we require that (i) one edge with direction u_2 have weight $\geqslant 2$, with unassigned support, and (ii) one edge with direction u_3 be supported on the line $\mathsf{L}_{u_3,p_{3,1,1}}$ determined by the real number $p_{3,1,1}$; we require in addition that the curve contain the general point x_1.

One checks that there is a unique such tropical map, as indicated on the picture below (the fixed line $\mathsf{L}_{u_3,p_{3,1,1}}$ is represented as a dotted line).



(7.3) Remark. The condition on infinite edges being supported by the assigned lines L may be formulated in terms of the passing through assigned points at infinity, for a suitable notion of points at infinity. The latter are the points one needs to add to $\mathbf{R}^2$ in order to obtain the tropical toric surface associated to Δ, see [11, Section 7.3].

For instance the tropical projective plane, i.e., the tropical toric surface associated to a triangle T_d as in Example (2.1), looks like the triangle T_d itself, with the interior corresponding to $\mathbf{R}^2$ and the edges to the points at infinity; each edge may be identified with $\mathbf{R} \cup \{\pm\infty\}$. It is the quotient of $(\mathbf{R} \cup \{-\infty\})^3 \setminus \big\{(-\infty, -\infty, -\infty)\big\}$ by the equivalence relation $\sim$ defined by

$$(x_0, x_1, x_2) \sim (x_0', x_1', x_2') \quad \Longleftrightarrow \quad \exists \lambda \in \mathbf{R} : \ (x_0', x_1', x_2') = (x_0 + \lambda, x_1 + \lambda, x_2 + \lambda).$$

If $x_0 \neq -\infty$, then (x_0, x_1, x_2) is equivalent to a unique point of the form $(0, x, y)$ with $(x, y) \in (\mathbf{R} \cup \{-\infty\})^2$, namely $(0, x_1 - x_0, x_2 - x_0)$; if $(x_0, x_1, x_2) \in \{-\infty\} \times \mathbf{R}^2$, then (x_0, x_1, x_2) is equivalent to a unique point of the form $(-\infty, x, -x)$ with $x \in \mathbf{R}$, namely $(-\infty, \frac{x_1 - x_2}{2}, \frac{x_2 - x_1}{2})$.

The following result states that for a general choice of Ω and $\underline{x}$, the set $\mathcal{E}_g^{\Delta}(\alpha, \beta)(\underline{x}, \Omega)$ is indeed finite, and its members have the simplest possible shape. This is thus a tropical analogue of the main theorem in [III]. It is mainly due to Mikhalkin [36]; the article [36] does not comprise the possibility of boundary conditions, even if it contains all the necessary ingredients to handle them. Boundary conditions in this context have first appeared in [21].

(7.4) Proposition. *Let Δ, g, α, β be as in (7.1). For a general choice of $\underline{x}, \Omega$, the set $\mathcal{E}_g^{\Delta}(\alpha, \beta)(\underline{x}, \Omega)$ is finite. Moreover, for every $(\mathbf{\Gamma}, f) \in \mathcal{E}_g^{\Delta}(\alpha, \beta)(\underline{x}, \Omega)$, the following hold:*

(i) the graph $\Gamma = \bar{\Gamma} \setminus \Gamma^{\infty}$ underlying $\mathbf{\Gamma} = (\bar{\Gamma}, \gamma)$ is trivalent;
(ii) the map f is a topological immersion;
(iii) the preimage of any point $x \in \mathbf{R}^2$ has at most two elements;
(iv) the infinite edges of $f(\mathbf{\Gamma})$ have minimal weight;[15]
(v) the set of vertices of the plane curve $f(\mathbf{\Gamma})$ is disjoint from the set $\underline{x}$;
(vi) an edge of the plane curve $f(\mathbf{\Gamma})$ contains at most one point from the set $\underline{x}$.

Condition (ii) asserts in particular that no neighbourhood of a trivalent vertex of Γ is ever mapped to a segment. Conditions (i) and (iii) together imply that the curve $f(\mathbf{\Gamma})$ is simple.

One should beware that Proposition (7.4) does not say that the shapes of the plane tropical curves corresponding to tropical maps in the set $\mathcal{E}_g^{\Delta}(\alpha, \beta)(\underline{x}, \Omega)$ are independent of the choice of $\underline{x}$ and Ω, and this is indeed not true in general. For instance, their respective subdivisions of Δ may vary as $\underline{x}$ moves, cf. Example (8.15) and Figure 20.

[15] i.e., for all $s = 1, \ldots, n$ and $i \geqslant 1$, the plane tropical curve $f(\mathbf{\Gamma})$ has exactly $\alpha_{s,i} + \beta_{s,i}$ infinite edges of weight precisely i in the oriented direction u_i.

(7.5) Remark. It most often happens that some members of $\mathcal{E}_g^{\Delta}(\alpha,\beta)(\underline{x},\Omega)$ have their corresponding plane tropical curve that has finite edges of weight larger than one (see Example (8.15) for instance). This is analogous to the fact that when considering degenerations of nodal algebraic curves, one more or less inevitably ends up with some tacnodal curves, which is one of the central themes of this whole volume. See Section 7.2 for more details.

In order to correspond to the counting of algebraic curves, the counting of tropical curves must be made with appropriate multiplicities. Besides, the mere cardinality (i.e., without multiplicities) of $\mathcal{E}_g^{\Delta}(\alpha,\beta)(\underline{x},\Omega)$ does in general depend on the choice of $\underline{x}$ and Ω.

(7.6) Tropical multiplicity. Let f be a tropical morphism from an abstract tropical curve $\mathbf{\Gamma} = (\bar{\Gamma},\gamma)$ to the tropical plane. Assume that every vertex $v \in \Gamma$ is trivalent (e.g., the plane curve $f(\mathbf{\Gamma})$ is simple).

For each such vertex v, we define its contribution $\mu^{\mathbf{C}}(v)$ to the multiplicity of $(\mathbf{\Gamma},f)$ to be twice the Euclidean area of the triangle Δ_v dual to v in the subdivision of the Newton polygon of f, i.e.,

$$\mu^{\mathbf{C}}(v) = 2\mathscr{A}(\Delta_v) = \left|\det\big(w(\Sigma)\cdot u_{v,\Sigma},\, w(\Sigma')\cdot u_{v,\Sigma'}\big)\right|, \tag{7.6.1}$$

where in the last equality Σ,Σ' are any two edges ending at v (out of three, since v is assumed to be trivalent; the balancing condition implies that the multipicity is independent of the choice of those two edges).

The *complex tropical multiplicity* of the map $(\mathbf{\Gamma},f)$ is

$$\mu^{\mathbf{C}}(\mathbf{\Gamma},f) \overset{\text{def.}}{=} \prod_{v\in\Gamma} \mu^{\mathbf{C}}(v).$$

The enumeration of plane tropical curves may also be related with the enumeration of real algebraic plane curves, albeit with a different multiplicity: this is why we keep an index $\mathbf{C}$ in the notation.

We refer to Section 7.2 for a discussion of why $\mu^{\mathbf{C}}(\mathbf{\Gamma},f)$ is indeed the correct multiplicity with which $(\mathbf{\Gamma},f)$ should be counted in order for the toric and tropical enumerative problems to correspond. In particular, we will interpret the complex tropical multiplicity in terms of the theory of limit Severi varieties, see [I] and [VI].

(7.7) Example. The tropical map from the genus zero curve (b) of Example (6.9) has complex tropical multiplicity one. Note that the abstract tropical curve has the property that all its vertices at finite distance are trivalent, so that its complex tropical multiplicity is indeed well defined; vertex #6 of the image plane curve has two points in its preimage, and does not correspond to a vertex of the abstract curve.

The plane tropical curve in Figure 7 is the image of a unique tropical map from a genus zero curve, with only trivalent vertices at finite distance; this tropical map has multiplicity four. Indeed, the two triangles marked with a star in the decomposition of the Newton polygon correspond to vertices v of the source curve such that $\mu^{\mathbf{C}}(v) = 2$.

The following result, known as Mikhalkin's Correspondence Theorem, states the correspondence between the algebraic and tropical enumerative problems stated in (3.3) and (7.1), respectively. It is mainly due to Mikhalkin [36]; as for Proposition (7.4), the original result does not mention boundary conditions, which have first appeared in [21].

(7.8) Theorem. *Let Δ, g, α, β be as in (7.1), and choose $\underline{x},\Omega$ general enough. The following equality holds:*

$$\sum_{(\mathbf{\Gamma},f)\in\mathcal{E}_g^{\Delta}(\alpha,\beta)(\underline{x},\Omega)} \mu^{\mathbf{C}}(C,f) = I^{\alpha}\cdot N_g^{\Delta}(\alpha,\beta).$$

Of course (perhaps), the main insight behind this result is the idea presented in Section 4 that tropical hypersurfaces are limits of toric hypersurfaces. In fact, the strategy of the proof is very similar to that of the the Caporaso–Harris Formula, explained in details in [VI]. It has two parts, which we will call the degeneration and the regeneration (or deformation) parts. The degeneration part consists in showing that the limits (or rather, the tropicalizations) of the toric curves solution of the toric enumerative problem underlying the number $N_g^\Delta(\alpha,\beta)$ are parametrized tropical curves solution of the tropical enumerative problem encoded in $\mathcal{E}_g^\Delta(\alpha,\beta)(\underline{x},\Omega)$. The regeneration part goes the other direction, and consists in showing that every parametrized tropical curve $(\mathbf{\Gamma},f)\in\mathcal{E}_g^\Delta(\alpha,\beta)(\underline{x},\Omega)$ is indeed the tropicalization of precisely $\mu^{\mathbf{C}}(\mathbf{\Gamma},f)/I^\alpha$ toric curves solution of the problem corresponding to $N_g^\Delta(\alpha,\beta)$.

We give some indications about the regeneration part of the proof in Section 7.2 below, in order to show the relations with the theory of limit Severi varieties, presented in [I, VI].

7.2 – Regeneration of tropical curves

In this section we outline the deformation part of the proof of the Correspondence Theorem (7.8). The idea is to consider, for each solution $(\mathbf{\Gamma},f)\in\mathcal{E}_g^\Delta(\alpha,\beta)(\underline{x},\Omega)$ of the tropical enumerative problem, a degeneration $X_\mathcal{S}$ of the toric surface X_Δ obtained from the subdivision $\mathcal{S}$ of Δ associated to $(\mathbf{\Gamma},f)$. The data of $(\mathbf{\Gamma},f)$ determines finitely many curves in the reducible surface $X_\mathcal{S}$, which in turn may be deformed to solutions of the toric enumerative problem on X_Δ. The number of these deformed solutions is the complex tropical multiplicity $\mu^{\mathbf{C}}(\mathbf{\Gamma},f)$.

This is inspired by Viro's patchworking construction, see [45]. The proof is due to Mikhalkin [36], and has been revisited by Shustin [43] (see also [29, Chapter 2]) in terms closer to the presentation given here. The same idea (with additional techniques from log deformation theory) has been applied by Nishinou and Siebert [40] to the enumeration of rational curves in toric varieties of arbitrary dimensions.

Let us first describe the degeneration of a projectively embedded toric variety associated to a subdivision of the corresponding polytope.

(7.9) Toric degeneration associated to a subdivision. Let Δ be a lattice polytope, equipped with a subdivision $\mathcal{S}$ (suposed to be convex, see below). This data determines a degeneration of the toric variety $X_\Delta\subseteq\mathbf{P}^{N_\Delta-1}$ to a "reducible toric variety" $X_\mathcal{S}\subseteq\mathbf{P}^{N_\Delta-1}$. The variety $X_\mathcal{S}$ has the same degree as X_Δ, and generates $\mathbf{P}^{N_\Delta-1}$; its irreducible components are the various X_{Δ_v} corresponding to the maximal dimensional cells of the subdivision $\mathcal{S}$ of Δ, and they are glued along toric varieties of smaller dimensions, according to the dual graph of the subdivision.

Let us simply give an example instead of making a formal definition of the "reducible toric variety" $X_\mathcal{S}$ (you can also consult (9.4) for another example, with evocative pictures). Consider the subdivised polygon Δ from Figure 4. The polygon Δ is the triangle T_2 of (2.1), which defines the 2-Veronese surface $X_\Delta\subseteq\mathbf{P}^5$, of degree 4. There are two maximal dimensional cells in the subdivision. One is the trapezium with a label 2, which is a $T^1_{1,2}$ in the notation of (2.6), and thus defines a degree 3 rational normal scroll $X_2\subseteq\mathbf{P}^4$; this scroll is built on a line $X_{1,2}$, which corresponds to the length one segment at the intersection of the trapezium with label 2 and the triangle with label 1, and a conic which corresponds to the opposite edge of the trapezium, which has length two. These line and conic respectively span two supplementary linear spaces $\mathbf{P}^1$ and $\mathbf{P}^2$ in the $\mathbf{P}^4$ spanned by X_2. We may view this $\mathbf{P}^4$ as a linear subspace in $\mathbf{P}^5$. The other maximal dimensional cell in the subdivision is the triangle with label 1, which defines a plane $X_1=\mathbf{P}^2$. We may view this $\mathbf{P}^2$ as a linear subspace of $\mathbf{P}^5$, intersecting the $\mathbf{P}^4$ spanned by X_2 exactly along the $\mathbf{P}^1$ spanned by the line $X_{1,2}$; thus, the spans $\langle X_1\rangle$ and $\langle X_2\rangle$ generate $\mathbf{P}^5$. Then, the "reducible toric surface" $X_\mathcal{S}$ is the union $X_1\cup X_2\subseteq\mathbf{P}^5$, with $X_1\cap X_2=X_{1,2}$.

The total space of the degeneration associated to $\mathcal{S}$ is a toric variety $\mathcal{X}_\mathcal{S} \subseteq \mathbf{P}^{N_\Delta - 1} \times \mathbf{A}^1$, flat over $\mathbf{A}^1$, with fibres over all $t \neq 0$ isomorphic to X_Δ, and central fibre isomorphic to $X_\mathcal{S}$. We will now give a down-to-earth description of $\mathcal{X}_\mathcal{S}$; see, e.g., [40, Section 3] for a description in terms of fans.

By definition, the subdivision $\mathcal{S}$ being *convex* means that there exists a function

$$\nu : \Delta_\mathbf{Z} \to \mathbf{Z}$$

defined on the integral points of $\Delta \subseteq \mathbf{R}^n$, such that the subdivision is the corner locus of the double Legendre–Fenchel transform of ν (see (5.3)); in other words, the polyhedra forming the subdivision are the projections to $\mathbf{R}^n$ of the faces of the convex hull of the semi-lines

$$\big(I, \nu(I)\big) + \{0\}^n \times \mathbf{R}_+$$

for all $I \in \Delta_\mathbf{Z}$. Then, the toric variety $\mathcal{X}_\mathcal{S}$ is the Zariski closure of the image of the map

$$(\mathbf{x}, t) \in (\mathbf{C}^*)^n \times \mathbf{C}^* \longmapsto \big([t^{\nu(I)}\mathbf{x}^I]_{I \in \Delta_\mathbf{Z}},\, t\big) \in \mathbf{P}^{N_\Delta - 1} \times \mathbf{A}^1,$$

where $\mathbf{x}^I = x_1^{i_1} \cdots x_n^{i_n}$ for all $\mathbf{x} = (x_1, \ldots, x_n)$ and $I = (i_1, \ldots, i_n)$, as usual.

We may now discuss the deformation part of the proof of the Correspondence Theorem.

(7.10) Setup. Let us consider the setup of (7.1), and assume for simplicity that there are no boundary conditions, i.e., $\alpha_\Sigma = 0$ and $\beta_\Sigma = [l_\mathbf{Z}(\Sigma)]$ for all edges Σ of Δ, hence $\Omega = \varnothing$. Thus, we will simply write $\mathcal{E}_g^\Delta(\underline{x})$ for $\mathcal{E}_g^\Delta(\alpha, \beta)(\underline{x}, \Omega)$.

Let us fix an element $(\mathbf{\Gamma}, f)$ of $\mathcal{E}_g^\Delta(\underline{x})$ until the end of the discussion. Recall that it enjoys the nice properties listed in Proposition (7.4), in particular it is a simple curve. We let $\mathbf{\Gamma} = (\bar{\Gamma}, \gamma)$ and $\Gamma = \bar{\Gamma} \setminus \Gamma^\infty$, following our usual notation.

(7.11) The degenerate enumerative problem. We now describe an enumerative problem on a degenerate toric variety associated to the solution $(\mathbf{\Gamma}, f)$ of the tropical enumerative problem.

Let $\mathcal{S}$ be the subdivision of Δ associated to $(\mathbf{\Gamma}, f)$, and consider the corresponding variety $X_\mathcal{S} \subseteq \mathbf{P}^{N_\Delta - 1}$ as in (7.9) above. The problem is to find all curves C cut out by a hyperplane on $X_\mathcal{S}$, and satisfying conditions (i)–(iv) below. Recall that, by Proposition (7.4), the polygons in $\mathcal{S}$ are either triangles or parallelograms.

(i) For all triangles $T \in \mathcal{S}$, we require that the curve $C \cap X_T$ (denoted by C_T) on the component of $X_\mathcal{S}$ corresponding to T be (irreducible) rational (note that, being C a hyperplane section of $X_\mathcal{S}$, $C \cap X_T$ is a membre of the linear system $|H_T|$ corresponding to T); this curve will be nodal.

(ii) For all parallelograms $P \in \mathcal{S}$, the linear equivalence class H_P on X_P corresponding to P writes as $wF + w'F'$, where F and F' are two rulings by rational curves; w and w' are the integral lengths of the edges of P. We require that the curve $C \cap X_P$ be of the form $wC_P + w'C'_P$, with $C_P \sim F$ and $C'_P \sim F'$; thus, $C \cap X_P$ is the sum of a degree w' rational normal curve with multiplicity w, and a degree w rational normal curve with multiplicity w'.

(iii) To all segments $E \in \mathcal{S}$ lying at the intersection of two polygons Π and Π' of $\mathcal{S}$, there corresponds a rational normal curve $X_E \subseteq \mathbf{P}^{N_\Delta - 1}$ which is the intersection of the two components X_Π and $X_{\Pi'}$ of $X_\mathcal{S}$. For all such E, we require that the intersection of C with X_E be concentrated at only one point.

(iv) By Proposition (7.4), each edge of $f(\mathbf{\Gamma})$ contains at most one point of $\underline{x}$. Each point $x \in \underline{x}$ determines a point p_x on the corresponding curve $X_E \subseteq X_\mathcal{S}$, where $E \in \mathcal{S}$ is the segment

dual to the edge of $f(\mathbf{\Gamma})$ on which x lies. We require that the curve C passes through the points p_x for all $x \in \underline{x}$.[16]

(7.12) Remark. The curves C solution of the enumerative problem of (7.11) may be seen as the image of genus g stable maps $\phi : \tilde{C} \to X_{\mathcal{S}}$ such that all irreducible components of $\tilde{C}$ are rational curves. Thus, they are somehow maximally degenerate.

(7.13) Finiteness. Let us now explain why the enumerative problem of (7.11) has finitely many solutions. The idea is that the assigned passing points p_x, $x \in \underline{x}$, propagate among the various components of $X_{\mathcal{S}}$ so that, in the upshot: (i) for all triangles $T \in \mathcal{S}$, the curve C_T has two assigned passing points on the toric boundary of X_T, and (ii) for all parallelograms $P \in \mathcal{S}$, the curves C_P and C'_P each have one assigned passing point on the toric boundary of X_P.

This implies that there are finitely many possibilities for each irreducible component of C, let us briefly recall why. For a parallelogram P, in the notation of (7.11), the linear systems $|F|$ and $|F'|$ are one-dimensional, therefore the assignment of one passing point to each of C_P and C'_P determines these curves uniquely. For a triangle T, finiteness follows from the discussion in (3.3): in the notation of (3.3), the curve C_T must belong to the linear system $|H_T|$ on X_T, have geometric genus 0, and is subject to conditions on the toric boundary of X_T corresponding to

$$\alpha_{\Sigma_i} = [\underbrace{0, \ldots, 0}_{\ell_{\mathbf{Z}}(\Sigma_i)-1}, 1] \quad \text{and} \quad \beta_{\Sigma_i} = 0$$

on the two edges Σ_1 and Σ_2 on which the passing point is assigned, and to $\alpha_{\Sigma_{\text{out}}} = 0$ and $\beta_{\Sigma_{\text{out}}} = [0, \ldots, 0, 1]$ on the remaining edge Σ_{out}. Thus, $|\beta| = 1$, and the corresponding toric enumerative problem on X_T has finitely many solutions, see (3.3.1).

We will now indicate, first on an example, how the assigned points p_x propagate. Let Δ be the trapezium $\mathrm{Tz}^1_{2,1}$, in the notation of (2.6), and let $g = 0$; let then $\underline{x}$ consist of 7 general points in $\mathbf{R}^2$. Assume the plane tropical curve corresponding to our fixed solution $(\mathbf{\Gamma}, f) \in \mathcal{E}_0^{\Delta}(\underline{x})$ is as pictured in Figure 10 below; the associated subdivision $\mathcal{S}$ of Δ is the same as that in Example (6.9), Figure 6, and the vertices of $f(\mathbf{\Gamma})$ are labelled accordingly. The surface $X_{\mathcal{S}} \subseteq \mathbf{P}^8$ consists of six planes X_i, $i \neq 6$, and a smooth quadric X_6.

Let us see that there is a unique, in this case, solution C to the degenerate enumerative problem (7.11) associated to $(\mathbf{\Gamma}, f)$. The curve $C_4 = C \cap X_4$ must be a line passing through the two points p_1 and p_2 corresponding to x_1 and x_2; of course there is a unique such line. Similarly, the curves C_3 and C_7 on X_3 and X_7 are uniquely determined lines. Consider now the curve $C_5 = C \cap X_5$: it must be a line passing through the point p_4, but it must also pass through the point cut out by C_4 on the line $X_5 \cap X_4$, hence it is uniquely determined. Similarly, the curve C_2 is the unique line on X_2 passing through p_6 and intersecting C_3 along the line $X_3 \cap X_2$. Next, the curve C_6 must be the sum of two intersecting lines on the quadric X_6; these two lines are uniquely determined, respectively, by the point p_5, and by the intersection of C_5 with the line $X_5 \cap X_6$. Finally, the curve C_1 is a line, uniquely determined by the points cut out on the two lines $X_1 \cap X_6$ and $X_1 \cap X_2$ by the curves C_6 and C_2, respectively.

The arrows on Figure 10 indicate how the initial assigned points $p_1, \ldots, p_7$ (corresponding to $x_1, \ldots, x_7$) induce further assigned points, e.g., p_1 and p_2 fix the line C_4, hence induce an assigned point at the intersection of C_4 with the line $X_4 \cap X_5$, and so on.

The proof that things always work as in this example (except that there may be several possibilities for the curves C_i corresponding to trivalent vertices) is based on the fact, proved by

[16] actually, one had better modify the degeneration in order to replace the double curves Σ of $X_{\mathcal{S}}$ supporting a point p_x by surfaces X_Σ, in such a way that p_x corresponds to a general point of X_Σ; this can be done with the techniques introduced in [I]. Since we don't give a complete, rigorous proof of the regeneration part of the Correspondence Theorem, we have chosen to stick with the simpler model of the degeneration.

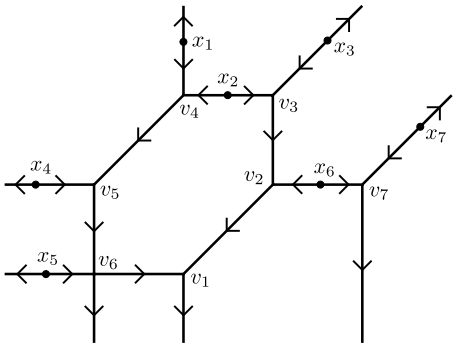


Figure 10

Mikhalkin, that if one removes all points x_i from the plane tropical curve $f(\mathbf{\Gamma})$, one obtains a forest, i.e., a disjoint union of trees. It implies that one may orient these trees by imposing that the orientation always points in the direction opposite to the points x_i (this corresponds to the arrows on Figure 10). This orientation induces a partial order on the vertices of $f(\mathbf{\Gamma})$, following which one may determine the components C_i of C one after another. Each vertex v of $f(\mathbf{\Gamma})$ has two incoming edges, meaning that the curve C_v on X_v must pass through two assigned points, either a point p_i or a point $C_{v'} \cap X_v$ for some vertex v' linked to v by an edge.

(7.14) Number of solutions. We claim that the number of solutions of the degenerate enumerative problem set up in (7.11) is

$$\prod_T \frac{\mu^{\mathbf{C}}(T)}{w_1 w_2}, \tag{7.14.1}$$

where the product is over all triangles $T \in \mathcal{S}$, $\mu^{\mathbf{C}}(T)$ is twice the Euclidean area of T, and w_1 and w_2 denote the integral lengths of the two edges Σ_1 and Σ_2 of T on which the passing point is assigned, as in (7.13) above (these two edges correspond to the two incoming edges at the vertex v dual to T of the plane tropical curve $f(\mathbf{\Gamma})$). Note that $\mu^{\mathbf{C}}(T)$ is the contribution to the multiplicity $\mu^{\mathbf{C}}(\mathbf{\Gamma}, f)$ of $(\mathbf{\Gamma}, f)$ of the vertex of $\mathbf{\Gamma}$ mapping to the vertex of $f(\mathbf{\Gamma})$ dual to T, cf. (7.6.1), while w_1 and w_2 are the weights of the two edges of $f(\mathbf{\Gamma})$ dual to Σ_1 and Σ_2.

Indeed, the degenerate enumerative problem boils down, as we have seen in (7.13), to a disjoint union of ordered trees of enumerative problems on the various irreducible components of $X_{\mathcal{S}}$, and the claim follows from the two facts that (i) for all parallelograms P, the enumerative problem on X_P has a unique solution, and (ii) for all triangles T, the enumerative problem on X_T has $\mu^{\mathbf{C}}(T)/w_1 w_2$ solutions.

Fact (i) has already been explained at the beginning of (7.13). Fact (ii) may be seen by realizing each X_T as a quotient of $\mathbf{P}^2$, as follows. For all triangles T, there exists a degree $\mu^{\mathbf{C}}(T)$ toric morphism $\nu : \mathbf{P}^2 \to X_T$, which may be obtained by "rectifying the triangle T", i.e., by producing an affine integral map from T to the standard triangle T_d of (2.1), for some d; this is fairly standard toric geometry, but still slightly more advanced than what has been introduced in this text, so let us simply accept it. For all divisors Σ of the toric boundary of $\mathbf{P}^2$, the restriction $\nu|_\Sigma$ is a degree $\mu^{\mathbf{C}}(T)/w'$ toric morphism to a divisor Σ' of the toric boundary of X_T, where $w' = \ell_{\mathbf{Z}}(\Sigma')$.

Now, consider the enumerative problem on X_T posed by a given solution of the preceding enumerative problems in the tree. It asks the number of curves in the linear system $|H_T|$ intersecting each of the three components $\Sigma_1, \Sigma_2, \Sigma_{\text{out}}$ at only one point, with maximal order of

contact, and passing through two assigned points q_1 and q_2 on Σ_1 and Σ_2, respectively. Each such curve is the image by ν of a line in $\mathbf{P}^2$ (in general, this line is not unique), and the problem boils down to counting the lines in $\mathbf{P}^2$ passing through one preimage of q_1 and one preimage of q_2. The points q_1 and q_2 have $\mu^{\mathbf{C}}(T)/w_1$ and $\mu^{\mathbf{C}}(T)/w_2$ preimages, respectively, so the lines we are looking for in $\mathbf{P}^2$ are $(\mu^{\mathbf{C}}(T))^2/w_1w_2$ in total. Since there are $\mu^{\mathbf{C}}(T)$ such lines over each of the curves in X_T that we are looking for, the final answer to the enumerative problem on X_T is $\mu^{\mathbf{C}}(T)/w_1w_2$ indeed.

(7.14.2) Example. Consider the triangle T of (3.6), and recall the description of the surface $X_T \subseteq \mathbf{P}^3$ given there. The toric boundary ∂X_T of X_T consists of three coplanar lines, and the enumerative problem on X_T is to count the hyperplane sections of X_T that are rational and pass through assigned points p_1 and p_2 on two sides Σ_1 and Σ_2 of the triangle ∂X_T. One has $\mu^{\mathbf{C}}(T) = 3$ and $w_1 = w_2 = 1$, and indeed the curves we are interested in are 3 in total, see (3.6.3).

The surface X_T is defined in $\mathbf{P}^3$ by the equation $xyz = t^3$, in homogeneous coordinates $(x:y:z:t)$. The degree 3 toric morphism $\nu : \mathbf{P}^2 \to X_T$ is given by the mapping

$$(u : v : w) \mapsto (u^3 : v^3 : w^3 : uvw).$$

(7.15) Deformations of each solution. We now briefly discuss the claim that each solution C of the degenerate enumerative problem of (7.11) associated to $(\mathbf{\Gamma}, f)$ is the limit of exactly

$$\prod_{\Sigma \text{ edge of } \Gamma} m(\Sigma)$$

genus g curves in the fibres of $\mathcal{X}_{\mathcal{S}} \to \mathbf{A}^1$ neighbouring the central fibre $X_{\mathcal{S}}$ (see below for the definition of $m(\Sigma)$). In doing so, we will provide an algebraic geometric interpretation of the δ-invariant of tropical curves introduced in (6.13).

Let us first note that this claim ends the deformation part of the proof of the Correspondence Theorem: For all edges Σ of Γ, $m(\Sigma)$ is defined by

$$m(\Sigma) = \begin{cases} w_{\Sigma,f} & \text{if } \Sigma \text{ joins two vertices of } \Gamma \text{ and } f(\Sigma) \text{ does not contain any point of } \underline{x}; \\ w_{\Sigma,f}^2 & \text{if } \Sigma \text{ joins two vertices of } \Gamma \text{ and } f(\Sigma) \text{ contains one point of } \underline{x}; \\ 1 & \text{if } \Sigma \text{ joins a vertex of } \Gamma \text{ to a vertex at infinity and } f(\Sigma) \text{ does not contain any point of } \underline{x}; \\ w_{\Sigma,f} & \text{if } \Sigma \text{ joins a vertex of } \Gamma \text{ to a vertex at infinity and } f(\Sigma) \text{ contains one point of } \underline{x}, \end{cases}$$

where $w_{\Sigma,f}$ is the stretching factor of f on Σ, equivalently the weight of all edges of $f(\mathbf{\Gamma})$ issued from Σ (recall also that $\bar{\Gamma} = \Gamma \coprod \Gamma^\infty$ is the graph underlying $\mathbf{\Gamma}$). It is an exercise to verify that

$$\prod_{\Sigma \text{ edge of } \Gamma} m(\Sigma) \cdot \prod_T \frac{1}{w_1 w_2} = 1,$$

where the second product is defined as in (7.14.1). Thus, one concludes from the claim that all solutions of the degenerate enumerative problem associated to $(\mathbf{\Gamma}, f)$ together are the limits of $\mu^{\mathbf{C}}(\mathbf{\Gamma}, f)$ genus g curves in the nearby fibres of $\mathcal{X}_{\mathcal{S}} \to \mathbf{A}^1$.

Now, we would like to explain the claim along the lines of the theory presented in [I, VI], by the fact that deformations of C to genus g curves in the fibres of $\mathcal{X}_{\mathcal{S}} \to \mathbf{A}^1$ are obtained by (i) preserving the nodes of the curves C_Π for all irreducible components X_Π of $X_{\mathcal{S}}$, and (ii) deforming all w-tacnodes of the curve C along the double curves of $X_{\mathcal{S}}$ to $w-1$ nodes, and that the number of such deformations is the product $\prod w$ over all tacnodes of C. To this effect,

note that if T and T' are two triangles of $\mathcal{S}$ intersecting along a length w segment, then the two components C_T and $C_{T'}$ of C intersect at exactly one point which is a w-tacnode of C along the double curve $X_T \cap X_{T'}$ of $X_{\mathcal{S}}$.

In general however, the situation is more complicated due to the presence of multiple components in the curve C, so that it is delicate to fully explain the claim along these lines. It is then arguably better to think of C in terms of a genus g stable map $\phi : \tilde{C} \to X_{\mathcal{S}}$. In these terms, a curve $wC_P + w'C'_P$ on X_P for some parallelogram P of $\mathcal{S}$, as in (7.11), (ii), is the image of a map $\tilde{C}_P \cup \tilde{C}'_P \to X_P$ as pictured on Figure 11 below, such that $\tilde{C}_P \to C_P$ and $\tilde{C}'_P \to C'_P$ have degrees w and w', respectively, and are totally ramified over the intersection points of C_P and C'_P with the toric boundary of X_P.

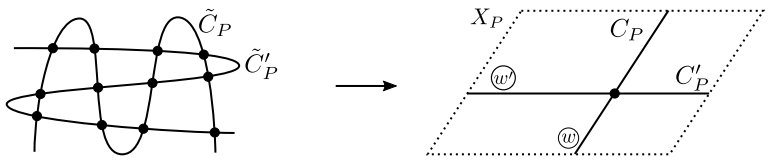


Figure 11

The curve $\tilde{C}_P \cup \tilde{C}'_P$ has $\mathscr{A}(P)$ nodes, which are preserved along the genus g deformations of C; this explains the contributions of the parallelograms of $\mathcal{S}$ to the δ-invariant of the plane tropical curve $f(\mathbf{\Gamma})$, see (6.13). The contributions of the triangles, on the other hand, correspond to the fact that for all triangles T, the curve C_T is rational and linearly equivalent to H_T, hence it has $p_a(T)$ nodes, which are preserved as well along genus g deformations of C. Finally, the contributions of the segmentoni (see (6.13)) correspond (at least morally) to each w-tacnode being deformed to $w-1$ nodes.

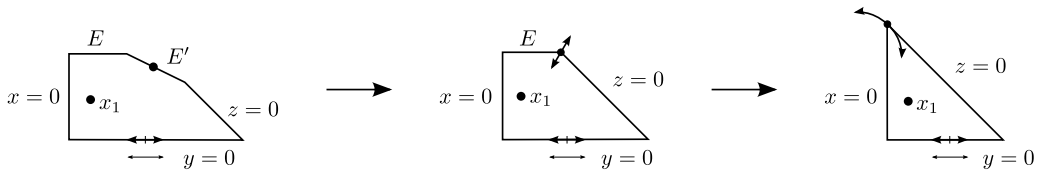
The contribution of the multiple components of C (all part of a curve $wC_P + w'C'_P$ for some parallelogram P, by Proposition (7.4), (iv), given our assumption that there are no conditions at infinity) to the multiplicity of C as a limit of genus g curve, on the other hand, may be computed as in [IX, (3.8)] and [IX, (2.9)] (see also [IX, (5.3)]). Roughly speaking, the fact that this contribution is multiplied by w if $f(\Sigma)$ contains a point x_i of $\underline{x}$ comes from the fact than one must choose a point in the preimage $\phi^{-1}(x_i)$.

7.3 – Examples

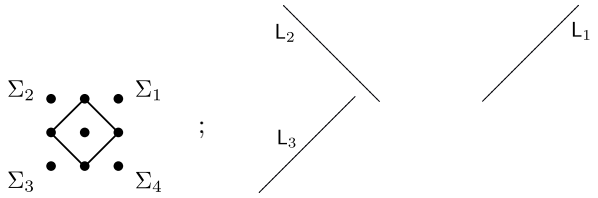
(7.16) In the situation of Example (7.2) above, there is exactly one element $(\mathbf{\Gamma}, f)$ in $\mathcal{E}_g^{\Delta}(\alpha, \beta)(\underline{x}, \Omega)$. The corresponding decomposition of Δ is the trivial one, consisting only of the triangle Δ itself, which has Euclidean area $\mathscr{A}(\Delta) = 1$; one therefore has $\mu^{\mathbf{C}}(\mathbf{\Gamma}, f) = 2$.

As the Correspondence Theorem tells us, this gives $N_g^{\Delta}(\alpha, \beta) = 2$, which indeed is the number of conics tangent to the line $z = 0$ at $(0 : 1 : 0)$, with prescribed general second order osculatory behaviour at this point, tangent to the line $y = 0$ at an unprescribed point, and passing through a general point x_1.

This is pictured below: two successive blow-ups of $\mathbf{P}^2$ are drawn, with exceptional divisors E and E'; base points are represented by black dots or arrows when they are infinitely near, and the unassigned tangency condition is represented by an arrow with a dash.



(7.17) Let us return to the enumerative problem (3.9): we let Δ be the Newton polygon indicated below, and fix $\alpha = (1,1,1,0)$, $\beta = (0,0,0,1)$. The set Ω consists of three real values, fixing three lines $\mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_3$ as indicated on the picture below.



The tropical curves defined by this data move in a 1-dimensional family, in the way pictured below (two curves are pictured, one in green and one in blue): the deformations of the curves amount to change the size of the central rectangle by sliding one of its corners along a line L_i, which lets all the rest follow. As the curves move in this family, one notices that the semi-line corresponding to the edge Σ_4 of the Newton polygon is always supported on the same line, which we shall call $\tilde{\mathsf{L}}_4$: this is the tropical incarnation of the toric Cayley–Bacharach property stated in Lemma (3.1).

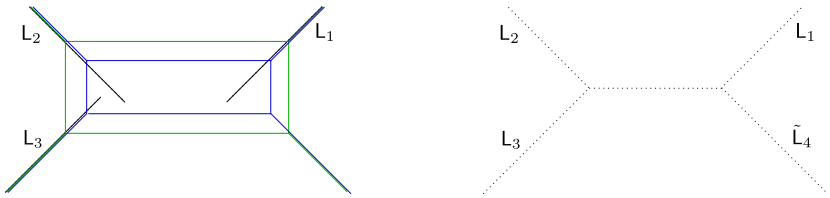
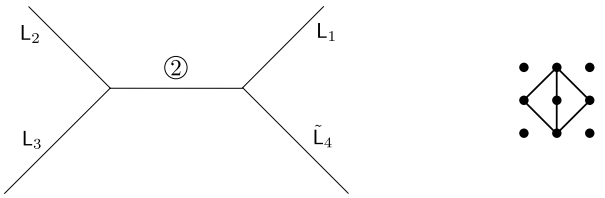


Figure 12: A pencil of tropical curves and the Cayley–Bacharach property

We want to count curves of genus $g = 0$ in this system. There is a finite number of such curves, thus the set $\underline{x}$ of base points is empty. There is a unique rational curve, depicted below together with the corresponding triangulation of the Newton polygon (note that a different choice for the lines $\mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_3$ may lead to another, similar, triangulation).

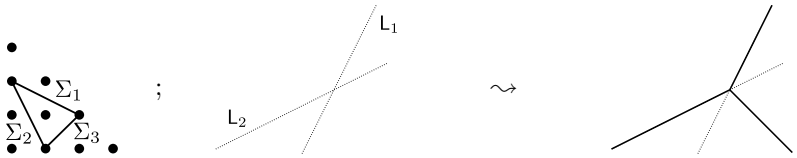


Both triangles in this triangulation yield the same multiplicity,

$$\mu(\Delta_v) = \left|\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right| = 2.$$

One thus finds $N_0^{\Delta}\big((1,1,1,0),(0,0,0,1)\big) = 4$, in tune with the number previously found in (3.9).

(7.18) We now consider the enumerative problem (3.6): we let Δ be the Newton polygon indicated below, and set $g = 0$, $\alpha = (1, 1, 0)$, $\beta = (0, 0, 1)$. The set Ω consists of two real values, fixing two lines $\mathsf{L}_1, \mathsf{L}_2$ as indicated on the picture below, and the set $\underline{x}$ of general base points is empty.



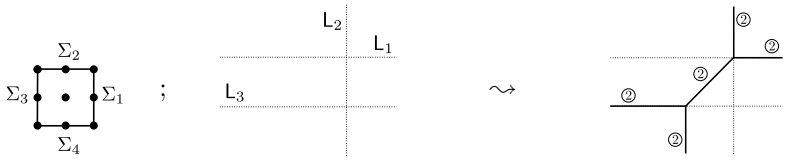
This enumerative problem is very similar to (3.9) and may be solved exactly as in (7.17) above. There is only one tropical map in $\mathcal{E}_0^{\Delta}\big((1,1,0),(0,0,1)\big)$; the corresponding subdivision of Δ is the trivial one, with only one triangle Δ_v giving the multiplicity

$$\mu(\Delta_v) = \left|\det \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}\right| = 3.$$

One thus finds $N_0^{\Delta}\big((1,1,0),(0,0,1)\big) = 3$, in tune with the number previously found in (3.6).

(7.19) Let us consider one last enumerative problem, which is linked to (3.9). The problem is to enumerate genus 0 curves of type $(2,2)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ tangent to all four points cut out on an anticanonical quadrilateral by a general curve of type $(1,1)$. As this imposes only seven independent linear conditions, this amounts to requiring the tangency at three fixed points and one mobile point.

Correspondingly, we consider the Newton polygon Δ indicated below, and set $g = 0$, $\alpha = (2,2,2,0)$ and $\beta = (0,0,0,2)$. The set Ω determines three lines $\mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_3$, and the set $\underline{x}$ is empty.



There is only one tropical map $(\mathbf{\Gamma}, f)$ in $\mathcal{E}_0^{\Delta}(\alpha, \beta)$; note that its shape depends on the relative vertical position of the two parallel lines L_1 and L_3. In any event, there are five edges of weight 2, and the subdivision of Δ consists of two triangles, each with Euclidean area 2. The Correspondence Theorem thus gives

$$N_0^{\Delta}(\alpha, \beta) = \frac{1}{I^{\alpha}} \times \mu^{\mathbf{C}}(C, f) = \frac{4 \times 4}{2^3} = 2.$$

The relation with (3.9) comes from seeing the quartic Del Pezzo surface with four ordinary double points of (3.9) as a double cover of $\mathbf{P}^1 \times \mathbf{P}^1$ branched over an anticanonical quadrilateral. The number 2 found above fits with the number 4 given in (3.9) and (7.17), the factor 2 between 4 and 2 being accounted for by the double cover.

For one more example, slightly more involved than the above ones, you can consult Example (8.19), in which the enumerative problem of (3.7) is considered.

8 – Floor diagrams and enumeration

This section is dedicated to the translation of the tropical enumerative problem (7.1) to a purely combinatorial enumerative problem, under certain technical assumptions. We first define floors and elevators of a plane tropical curve, which will enable its encoding as a floor diagram. Next we give an abstract definition of floor diagrams, and discuss the relevant technical conditions. When this is done we state Brugallé and Mikhalkin's Correspondence Theorem. Many examples are discussed throughout. We use the notation of Section 6.2.

The floor diagrams that I define here are a mixture of those defined in [14] and [5], with a presentation according to my personal taste; as Brugallé taught me, floor diagrams are more a philosophy than a specific object. Besides, the floor diagrams considered in [20] are themselves slightly different from those in [14], see [20, Remark 1.2]. Note that the floor diagrams introduced in [IX] are a particular instance of those presented here, corresponding to the Newton polygons of complete linear systems of plane curves (i.e., the triangles T_d of Example (2.1)).

(8.1) Definition. *Let* $\mathsf{d} = (a, b) \in \mathbf{Z}^2$ *be a primitive vector, and let* $(\mathbf{\Gamma}, f)$ *be a parametrized plane tropical curve, with underlying graph* $\Gamma = \Gamma \coprod \Gamma^\infty$. *An* elevator *of* $(\mathbf{\Gamma}, f)$ *is an edge* Σ *of* Γ *with oriented direction vector in the plane* $u_\Sigma = \pm\mathsf{d}$ *(see Definition (6.8)). A* floor *of* $(\mathbf{\Gamma}, f)$ *is a connected component of* $\Gamma \setminus \mathrm{Elev}$, *where* Elev *is the union of all elevators of* $(\mathbf{\Gamma}, f)$.

(8.2) Example. The respective decompositions of a genus 1 and a genus 0 plane tropical cubics into floors and elevators are given below, taking d to be the vertical direction. On the left, I picture the plane tropical curve; in the middle, I separate the floors and elevators (the former are chains of line segments and semi-lines with no vertical components, the latter are vertical segments or semi-lines); on the right, I provide a schematic picture of the incidence between floors and diagrams, on which the floors are represented by ellipses and the elevators by vertical segments.

On Figure 14, note that we see the curve as a genus 0 parametrized tropical curve, so that the two segments intersecting at the black square are separate in the source curve, as in Example (6.9), case (b). The floors and elevators are defined at the level of the source curve.

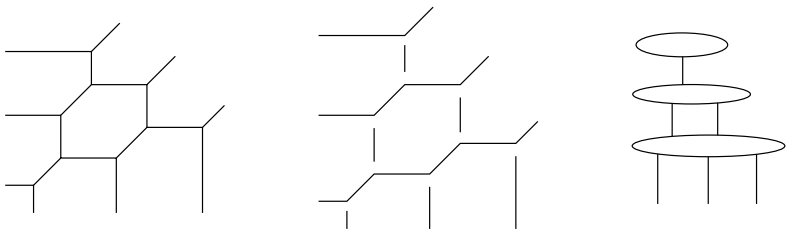


Figure 13: Floor decomposition of a genus 1 tropical cubic

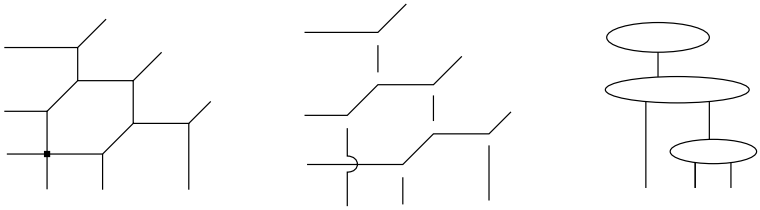


Figure 14: Floor decomposition of a genus 0 tropical cubic

(8.3) Remark. In general it is not true that floors are chains of line segments, or that an elevator intersects a given floor only once, see the examples in Figure 15. This will however be the case when we impose the passing through base points $\underline{x}$ sufficiently stretched out along direction d.

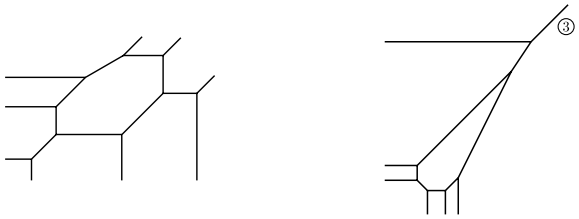


Figure 15: Tropical curves with messed up floor decomposition (see Remark (8.3))

It is also possible that the planar incarnations of two floors intersect, even under our general favourable framework when the points $\underline{x}$ are sufficiently stretched out along direction d. See (8.16.1) for an example.

The feature that will enable the translation of a tropical enumerative problem into an enumeration of floor diagrams, under appropriate conditions, is the following: if the the base points $\underline{x}$ are sufficiently streched in the direction d, then for all curves $(\mathbf{\Gamma}, f) \in \mathcal{E}_g^{\Delta}(\alpha, \beta)(\underline{x}, \Omega)$, each elevator (resp. floor) of $(\mathbf{\Gamma}, f)$ will either pass through exactly one point x_i of $\underline{x}$, or match exactly one assigned support at infinity L_p, $p \in \Omega$.

The above mentioned appropriate conditions for the translation into an enumeration of floor diagrams to work correctly are in the number two, given in (8.4) and (8.6) below. The first one is that the Newton polygon Δ should be "transverse to the direction $^{\mathsf{T}}\mathsf{d}$", and the second one is that non-trivial boundary conditions (i.e., conditions of types α and β) may only be imposed corresponding to specific components of the boundary of Δ, orthogonal to the direction d.

(8.4) Definition. *Set* $\mathsf{d} = (a, b) \in \mathbf{Z}^2$ *with* $\gcd(a, b) = 1$. *Let* Δ *be a Newton polygon. The* left and right boundaries *of* Δ *with respect to* d *are, respectively,*

$$\partial_l\Delta = \left\{p \in \Delta : \ \forall t > 0, \ p + t.\mathsf{d}^{\perp} \notin \Delta\right\} \quad \textit{and} \quad \partial_r\Delta = \left\{p \in \Delta : \ \forall t > 0, \ p - t.\mathsf{d}^{\perp} \notin \Delta\right\}.$$

The Newton polygon Δ *is* $\mathsf{d}^{\perp}$-transverse *if for all primitive vectors* $u \in \mathbf{Z}^2$ *parallel to an edge of* $\partial_l\Delta$ *or* $\partial_r\Delta$, *one has* $\det(u, \mathsf{d}^{\perp}) = \pm 1$.

The original definition [14, 2.2] corresponds to $\mathsf{d} = (0, 1)$, and is called h-transversality there, 'h' standing for 'horizontal'.

The geometric meaning of $\mathsf{d}^{\perp}$-transversality is that for all tropical curves Γ with Newton diagram Δ, any (weight one) line in $\mathbf{R}^2$ directed by d should intersect each infinite branch e of Γ with multiplicity equal to the weight $w(e)$ (except if the branch has the same direction d). This implies that the total intersection number of such a line with Γ is the number of floors of Γ. The definition of intersection multiplicity to make this work is the following: two lines directed by primitive vectors $u_1, u_2 \in \mathbf{Z}^2$ and with weights w_1 and w_2 intersect with multiplicity $w_1 \cdot w_2 \cdot \det(u_1, u_2)$.

(8.5) Examples. The triangle T_d, corresponding to the complete linear system $|dH|$ on $\mathbf{P}^2$ (see (2.1)), is $\mathsf{d}^{\perp}$-transverse for $\mathsf{d} = (0, 1)$.

— The trapezium $\mathrm{Tz}^r_{a,b}$, corresponding to the complete linear system $|aH + bF|$ on the Hirzebruch surface $\mathbf{F}_r$ (see (2.6)), is $\mathsf{d}^{\perp}$-transverse for $\mathsf{d} = (0, 1)$, but if $r > 1$ it is not for $\mathsf{d} = (1, 0)$.

— The two polygons given in (3.7) and (3.9) are $\mathsf{d}^\perp$-transverse for $\mathsf{d} = (0,1)$, but neither for $\mathsf{d} = (1,1)$, nor for $\mathsf{d} = (1,-1)$.
— The triangle in (3.6) is not $\mathsf{d}^\perp$-transverse either for $\mathsf{d} = (0,1)$, or $(1,0)$, or $(1,1)$, or $(1,-1)$.

(8.6) The second condition for a tropical enumerative problem to be translatable into a floor diagram problem is that one should impose non-trivial boundary conditions only along those edges of Δ that are orthogonal to d, or equivalently along the zero, one, or two edges of Δ off $\partial_l\Delta$ and $\partial_r\Delta$. In other words, in the notation of (7.1), for all $s = 1, \dots, n$, if Σ_s lies on the left or right boundary of Δ with respect to d, then $\alpha_s = 0$ and $\beta_s = [\ell_{\mathbf{Z}}(\Sigma_s)]$.

The edges of Δ that are orthogonal to d (if any), are call *bottom* and/or *top* edges of Δ, in the obvious terminology with respect to the direction d.

We need the following before we can define floor diagrams.

(8.7) Let Δ be a $\mathsf{d}^\perp$-transverse Newton polygon, for some primitive vector $\mathsf{d} \in \mathbf{Z}^2$. Let Σ be an edge of Δ lying on its left or right boundary, and view it as a vector going down with respect to the direction d: this vector writes $\ell_{\mathbf{Z}}(\Sigma).u$, with u a primitive integer vector such that $\det(\mathsf{d}^\perp, u) = +1$. The list of directions associated to Σ is the unordered list $[u^\perp, \dots, u^\perp]$ consisting of the primitive vector $u^\perp$ repeated $\ell_{\mathbf{Z}}(\Sigma)$ times. The *lists of left and right directions* of Δ are the unordered list $D_l(\Delta)$ and $D_r(\Delta)$) obtained by concatenating the lists of directions of all edges Σ lying on the left and righ) boundaries $\partial_l\Delta$ and $\partial_r\Delta$ of Δ, respectively.

Note that, for a plane tropical curve $\Gamma \subseteq \mathbf{R}^2$ with Newton polygon Δ, the list of left directions of Δ corresponds to the direction vectors $u_{\infty,\Sigma}$ (pointing inwards from infinity) of infinite branches Σ of Γ coming from the left with respect to the direction d, while the list of right directions corresponds to vectors $u_{\Sigma,\infty}$ (pointing outwards to infinity) of infinite branches Σ of Γ going to the right.

The $\mathsf{d}^\perp$-transversality condition ensures that $D_l(\Delta)$ and $D_r(\Delta)$ have the same number of elements[17], which we shall call the d-height of the polygon Δ, and denote by $d_{\mathsf{d}}(\Delta)$. Thus

$$\begin{aligned} d_{\mathsf{d}}(\Delta) &= \mathrm{length}(D_l(\Delta)) = \mathrm{Card}(\partial_l(\Delta) \cap \mathbf{Z}^2) - 1 \\ &= \mathrm{length}(D_r(\Delta)) = \mathrm{Card}(\partial_r(\Delta) \cap \mathbf{Z}^2) - 1. \end{aligned}$$

In addition, let $d_+(\Delta)$ and $d_-(\Delta)$ be the integral lengths of the top and bottom edges of Δ, see (8.6) (if there is no such edge, we let the corresponding value $d_\pm(\Delta)$ be 0). The following relation holds,

$$2d_{\mathsf{d}}(\Delta) + d_+(\Delta) + d_-(\Delta) = \mathrm{Card}(\partial\Delta_{\mathbf{Z}}). \tag{8.7.1}$$

(8.8) Examples. The following are the examples from (8.5) continued. taking $\mathsf{d} = (0,1)$ for all of them.
— For the triangle T_d, one has $d_{\mathsf{d}} = d_- = d$, $d_+ = 0$, $D_l = \big[(1,0), \dots, (1,0)\big]$ and $D_r = \big[(1,1), \dots, (1,1)\big]$.
— For the trapezium $\mathrm{Tz}^r_{a,b}$, one has $d_{\mathsf{d}} = a$, $d_- = b + ar$, $d_+ = b$, $D_l = \big[(1,0), \dots, (1,0)\big]$ and $D_r = \big[(1,r), \dots, (1,r)\big]$.
— For the polygon in (3.9), one has $d_{\mathsf{d}} = 2$, $d_- = d_+ = 0$, and $D_l = D_r = \big[(1,-1), (1,1)\big]$.
— For the polygon in (3.7), one has $d_{\mathsf{d}} = 3$, $d_- = d_+ = 0$, and $D_l = D_r = \big[(1,-1), (1,1), (1,1)\big]$.

[17] by a suitable change of coordinates in $\mathrm{SL}_2(\mathbf{Z})$ we may reduce to the case $\mathsf{d} = (0,1)$, and then the transversality condition is that any primitive vector $u \in \mathbf{Z}^2$ parallel to an edge on the left or right boundary of Δ is of the form $(\alpha, \pm 1)$, $\alpha \in \mathbf{Z}$.

Let us now give an abstract definition of floor diagrams. Beware not to confuse the graphs that we introduce below with those used in Sections 6 and 7 to define tropical curves; floor diagrams are purely combinatorial objects retaining only some information of plane tropical curves, whereas the latter are geometric objects.

(8.9) Weighted oriented graphs. Consider a connected, acyclic, oriented graph $\mathcal{D}$, with finitely many vertices and edges. We let $\mathrm{Vert}(\mathcal{D})$ and $\mathrm{Edges}(\mathcal{D})$ be the sets of vertices and edges of $\mathcal{D}$ respectively (or simply Vert and Edges if no confusion is likely).

We assume that there is a distinguished subset of 1-valent vertices of $\mathcal{D}$, which we call the *vertices at infinity*. We let $\mathrm{Vert}^{-\infty}$ (resp. $\mathrm{Vert}^{+\infty}$) be the set of vertices at infinity whose unique adjacent edge is outgoing (resp. ingoing). We let $\mathrm{Vert}^{\mathrm{fin}}$ be the complement in Vert of $\mathrm{Vert}^{-\infty} \coprod \mathrm{Vert}^{+\infty}$, and call *floors* those vertices in $\mathrm{Vert}^{\mathrm{fin}}$.

We assume that there is no edge joining two vertices at infinity. Thus we have a partition $\mathrm{Edges} = \mathrm{Edges}^{-\infty} \coprod \mathrm{Edges}^{\mathrm{fin}} \coprod \mathrm{Edges}^{+\infty}$, where $\mathrm{Edges}^{\pm\infty}$ is the set of those edges adjacent to a vertex in $\mathrm{Vert}^{\pm\infty}$. Thus $\mathrm{Edges}^{\mathrm{fin}}$ is the set of those edges joining two floors. The edges in $\mathrm{Edges}^{-\infty} \coprod \mathrm{Edges}^{+\infty}$ are called *infinite*, and the others *finite*. In order to match the terminology of Definition (8.1), one calls the edges of $\mathcal{D}$ (either finite or infinite) *elevators*.

We let $|\mathcal{D}|$ be the abstract set $\mathrm{Vert}^{\mathrm{fin}} \coprod \mathrm{Edges}$, consisting of all floors and edges (finite or infinite) of $\mathcal{D}$: thus $|\mathcal{D}|$ has one element for each floor of $\mathcal{D}$ and one element for each edge. Since $\mathcal{D}$ is acylic and oriented, the set $|\mathcal{D}|$ inherits a partial ordering from the orientation.

We will in addition assume that $\mathcal{D}$ is weighted, which means that it comes equipped with a *weight function* $w : \mathrm{Edges} \to \mathbf{N}^*$. One then defines the *divergence* $\mathrm{div}(v)$ of a vertex v (either a vertex at infinity or a floor) as

$$\mathrm{div}(v) = \sum_{\substack{e \in \mathrm{Edges}: \\ v \text{ target of } e}} w(e) - \sum_{\substack{e \in \mathrm{Edges}: \\ v \text{ source of } e}} w(e)$$

("sum of weights of incoming edges minus sum of weights of outgoing edges").

(8.10) Definition. *Let Δ be a $\mathsf{d}^\perp$-transverse lattice polygon, and g be a non-negative integer. A (plane)* floor diagram *$(\mathcal{D}, \theta)$ of genus g and Newton polygon Δ with respect to direction d is the data of an acyclic connected weighted oriented graph $\mathcal{D}$ as in (8.9), together with a map $\theta : \mathrm{Vert}(\mathcal{D})^{\mathrm{fin}} \to \mathbf{Z}^2$, such that the following conditions hold:*

(i) the first Betti number $b_1(\mathcal{D})$ equals g;
(ii) one has $\sum_{v \in \mathrm{Vert}^{-\infty}(\mathcal{D})} \mathrm{div}(v) = -d_-(\Delta)$ and $\sum_{v \in \mathrm{Vert}^{+\infty}(\mathcal{D})} \mathrm{div}(v) = d_+(\Delta)$;
(iii) the unordered list $\left[\theta(v), v \in \mathrm{Vert}(\mathcal{D})^{\mathrm{fin}}\right]$ equals $D_l(\Delta)$;
(iv) the unordered list $\left[\theta(v) + \mathrm{div}(v).\mathsf{d}, v \in \mathrm{Vert}(\mathcal{D})^{\mathrm{fin}}\right]$ equals $D_r(\Delta)$.

To understand condition (iv), note that it follows from the balancing condition for plane tropical curves that if a floor of some tropical curve has direction at infinity from the left u, and is intersected from below by k edges (counted with weights) with direction d and from above by l edges (counted with weights) with direction d, then its direction at infinity to the right is $u + (k-l).\mathsf{d}$. For conditions (iii) and (iv) to be compatible, one needs that $D_l(\Delta)$ and $D_r(\Delta)$ have the same length, which is granted by the $\mathsf{d}^\perp$-transversality condition, see (8.7); in this situation, the number $d_{\mathsf{d}}(\Delta)$ defined in (8.7) is the number of floors of a floor diagram with Newton polygon Δ.

In condition (ii), $\pm\sum_{v \in \mathrm{Vert}^{\pm\infty}(\mathcal{D})} \mathrm{div}(v)$ is the number of edges in $\mathrm{Edges}^{\pm\infty}$, counted with weights; we may also refer to it as the number of vertices in $\mathrm{Vert}^{\pm\infty}$ counted with weights (even though, strictly speaking, vertices at infinity don't have weights).

We adopt the following (standard) graphical convention: floors will be represented by ellipses, and vertices at infinity will be omitted; edges will be represented by vertical line segments, oriented from the bottom to the top. The weight of an edge is indicated only if it is greater than 1, as a circled integer. We indicate $\theta(v)$ inside the ellipse v, except if θ takes the same value on all floors, in which case we don't indicate it.

(8.11) Examples. Let us continue with the examples from (8.5) and (8.8). For each of these I picture below the unique floor diagram corresponding to curves with the maximal genus, with respect to the direction $\mathsf{d} = (0, 1)$.

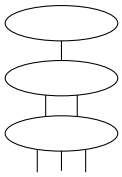


Figure 16: Floor diagram of Newton polygon T_3 and genus 1

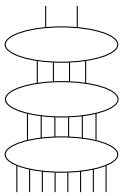


Figure 17: Floor diagram of Newton polygon $\mathrm{Tz}^2_{3,2}$ (see Figure 1) and genus 8

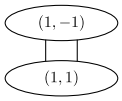


Figure 18: Floor diagram of genus 1 for the Newton polygon of (3.9)

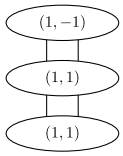


Figure 19: Floor diagram of genus 2 for the Newton polygon of (3.7)

In order to appropriately count floor diagrams, we need to enhance them with markings, which will encode the distribution of the base points $\underline{x}$ and of the boundary conditions Ω. To understand the next definition, keep in mind that the idea is that if the points $\underline{x}$ are sufficiently stretched in the direction d, then for all curves $(\mathbf{\Gamma}, f) \in \mathcal{E}_g^{\Delta}(\alpha, \beta)(\underline{x}, \Omega)$, there is exactly one point of $\underline{x}$ on each floor or elevator, except on those elevators that are already taken by a type α boundary condition, on which there is none. Moreover, we shall label the points of $\underline{x}$ according to their order along the direction d, which is where the monotonicity condition (ii) comes from.

(8.12) Setup. Let Δ be a $\mathsf{d}^{\perp}$-transverse lattice polygon, and consider a tropical enumerative problem as in (7.1), with data $\Delta, g, \alpha, \beta, \underline{x}, \Omega$, satisfying the condition stated in (8.6) that the non-trivial boundary conditions are all supported on the top and bottom edges $\Delta_{\pm}$ of Δ, with corresponding data $\alpha^{\pm}, \beta^{\pm} \in \underline{\mathbf{N}}$ such that

$$I\alpha^{\pm} + I\beta^{\pm} = d_{\pm}(\Delta).$$

Recall that $\mathrm{Card}(\partial\Delta_{\mathbf{Z}}) = 2d_{\mathsf{d}}(\Delta) + d_+(\Delta) + d_-(\Delta)$, see (8.7.1), and note that, by Condition (8.6), the number of points in $\underline{x}$ is

$$s = g - 1 + 2d_{\mathsf{d}}(\Delta) + |\beta^+| + |\beta^-|.$$

(8.13) Definition. *A* marking of type $(\alpha^+, \alpha^-, \beta^+, \beta^-)$ *of a floor diagram* $(\mathcal{D}, \theta)$ *of genus* g *and Newton polygon* Δ *is a map* $m : \{-|\alpha^-| + 1, \ldots, 1, \ldots s + |\alpha^+|\} \to |\mathcal{D}|$ *such that the following conditions hold:*

(i) the map m is injective;
(ii) if $m(i) > m(j)$ for the order on $|\mathcal{D}|$, then $i > j$;
(iii) for all $k \in \mathbf{N}^*$, if $-|\alpha^-| + 1 + \sum_{l=1}^{k-1} \alpha_l^- \leqslant i \leqslant -|\alpha^-| + \sum_{l=1}^{k} \alpha_l^-$ (resp. $s + 1 + \sum_{l=1}^{k-1} \alpha_l^+ \leqslant i \leqslant s + \sum_{l=1}^{k} \alpha_l^+$), then $m(i) \in \mathrm{Edges}^{-\infty}$ (resp. $m(i) \in \mathrm{Edges}^{+\infty}$), and the edge $m(i)$ has weight k;
(iv) for all $k \in \mathbf{N}^*$, there are exactly $\beta_k^{\pm}$ elements of $\mathrm{Edges}^{\pm\infty}$ of weight k in $m\big(\{1, \dots, s\}\big)$.

A floor diagram $\mathcal{D}$ enhanced with a marking m of type $(\alpha^+, \alpha^-, \beta^+, \beta^-)$ is called a *marked floor diagram* $(\mathcal{D}, m)$ of type $(\alpha^+, \alpha^-, \beta^+, \beta^-)$.

Two marked floor diagrams of the same type are *equivalent* if there exists a homeomorphism of oriented graphs $\phi : \mathcal{D} \cong \mathcal{D}'$ such that $w = w' \circ \phi$, $\theta = \theta' \circ \phi$, and $m = \phi \circ m'$.

Now, the fundamental result is the following. The enumerative application is given in Corollary (8.17) below.

(8.14) Theorem (Brugallé–Mikhalkin [14, Prop. 5.9])**.** *Consider a tropical enumerative problem as in Setup (8.12). If the points $\underline{x}$ are sufficiently strectched along direction d, then there is a bijection between $\mathcal{E}_g^{\Delta}(\alpha, \beta)(\underline{x}, \Omega)$ and the set of equivalence classes of marked floor diagrams of type $(\alpha^+, \alpha^-, \beta^+, \beta^-)$.*

I shall not prove this result and refer instead to [14, §5], but I will try to convey some ideas why it is true. First of all, Definition (8.1) shows how to associate a floor diagram to a tropical curve, with the same genus and Newton polygon. Moreover, each tangency condition of order k along the boundary imposed to the tropical curve translates into a condition that the floor diagram has an infinite edge of weight k.

These features of the curve being fixed, we need to let it move so as to meet the points $\underline{x}$ and supports at infinity Ω. One has to convince oneself that the degrees of freedom in order to do so are to let the floors move along the direction d (which correspondingly stretches the elevators), and let the elevators move along the direction $\mathsf{d}^{\perp}$. By the way, note that the number of floors and elevators (i.e., the cardinality of $|\mathcal{D}|$) indeed equals the number of elements in $\underline{x}$ and Ω together (i.e., $s + |\alpha^+| + |\alpha^-|$): the number of vertices of $\mathcal{D}$ is

$$\mathrm{Card}(\mathrm{Vert}(\mathcal{D})) = d_{\mathsf{d}}(\Delta) + |\alpha^+| + |\beta^+| + |\alpha^-| + |\beta^-|,$$

where $d_{\mathsf{d}}(\Delta)$ is the number of floors of $\mathcal{D}$, hence

$$\mathrm{Card}(|\mathcal{D}|) = \mathrm{Card}(\mathrm{Edges}(\mathcal{D})) + \mathrm{Card}(\mathrm{Vert}(\mathcal{D})) - \big(|\alpha^+| + |\beta^+| + |\alpha^-| + |\beta^-|\big),$$

and by the Euler formula,

$$\mathrm{Card}(|\mathcal{D}|) = g - 1 + 2d_{\mathsf{d}}(\Delta) + |\alpha^+| + |\beta^+| + |\alpha^-| + |\beta^-| = s + |\alpha^+| + |\alpha^-|,$$

as we claimed.

How the elements of $\underline{x}$ and Ω are taken up by the curve is encoded in the marking of the corresponding diagram. The supports at infinity of Ω must be taken up by pairwise distinct infinite elevators, which correspond to infinite edges of the floor diagram. Next, when the points $\underline{x}$ are sufficiently distant one from another in direction d, it happens that they are taken up one by one by the floors and remaining elevators. The condition that the marking m be increasing, i.e., (ii) in Definition (8.13), is satisfied when one orders the points $\underline{x}$ following direction d; note that the increasing condition is meaningful only in the range $\{1, \dots, s\}$ (which corresponds to points $\underline{x}$), as it is otherwise implied by the requirement that $m(i)$ is in $\mathrm{Edges}^{-\infty}$ if $i < 1$ and in $\mathrm{Edges}^{+\infty}$ if $i > s$.

Lastly one has to convince oneself that, given the incidence relations prescribed by a marked floor diagram, there is a unique corresponding tropical curve. The idea is this: consider a floor in the diagram; in the plane it corresponds to a chain of line segments, intersecting various elevators coming from below and from above (with respect to direction d), in a way prescribed by the diagram; each of these elevators has to pass through one base point prescribed by the marking, so that the line supporting each elevator in the plane is known. The slope of the leftmost (with respect to d) line segment in the chain is prescribed by the function θ attached to the diagram, and this in turn determines the slopes of the other segments via the balancing condition, see also the proof of Corollary (8.17) below. At this point, we know the slopes of all segments in the floor, and we know that the changes of slopes occur along the lines supporting the elevators; therefore the incarnation of the floor in the plane is determined up to translation along the direction d. Finally, there is a unique translate passing through the base point prescribed by the marking. It is arguably best to experiment this construction by one's self, playing around with examples; I provide some material to do so in the following example.

(8.15) Example. We display all marked floor diagrams of rational plane cubics with trivial boundary conditions in Figures 20, 21, and 22 below.

For the first diagram (Figure 20), we picture the corresponding tropical curve for two different distributions of the base points (always stretched out in the vertical direction). For both of them, I try to explain how to construct the curves in practice by showing the constructions lines that I have used to find the curves, following the procedure explained in the previous paragraph.

The five next marked diagrams (Figure 21) correspond to five different markings of the same diagram. I give the corresponding plane tropical curves for a given distribution of the base points (the same distribution for all five curves). I use black circles to picture the base points, and black squares to picture the "nodes" of the plane tropical curve (one on each curve in this case).

The three remaining marked diagrams are pictured on Figure 22, and I leave it as an exercise to construct the corresponding tropical curves for a given sufficiently stretched distribution of base points.

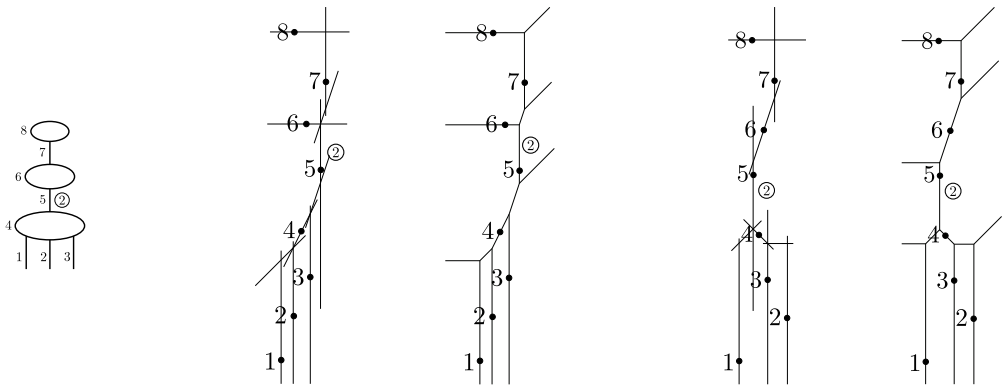


Figure 20: Tropical cubics with the same marked floor diagram for two different distributions of the base points (for each, on the left: construction lines)

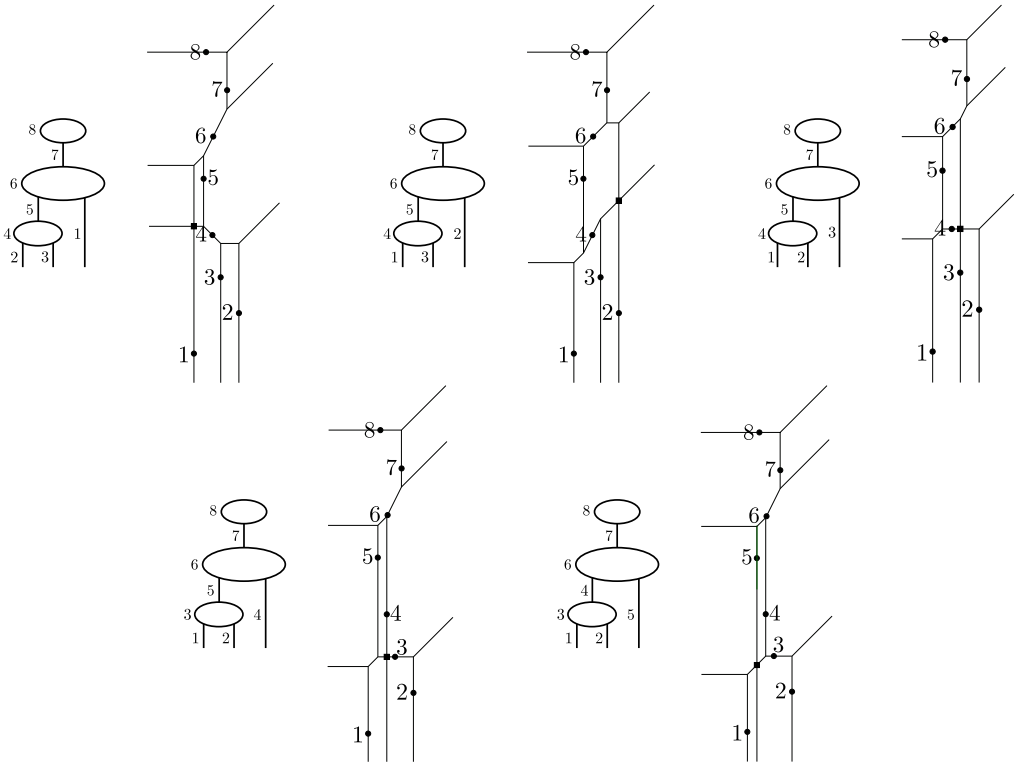


Figure 21: Tropical cubics with the same floor diagram, all possible markings

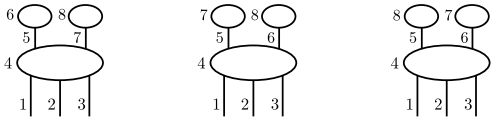
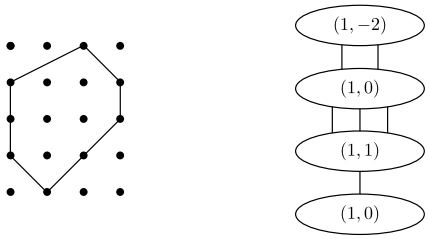


Figure 22: Remaining marked floor diagrams for rational plane cubics

(8.16) Warnings. Here are a few caveats about the correspondence between tropical curves and floor diagrams, which I feel are helpful to properly understand what is going on.

(8.16.1) There is no reason why the left infinite directions of floors (those encoded by θ) would be ordered following the order in which they appear on the Newton polygon: this is the reason why the lists $D_l(\Delta)$ and $D_r(\Delta)$ are unordered.

An example is given in the figure below (you should try and draw the tropical curve for yourself). We take $\mathsf{d} = (0, 1)$; the Newton polygon pictured on the left is indeed $\mathsf{d}^\perp$-transverse. Note that the floor diagram has genus 3, while the arithmetic genus of the Newton polygon is 5. The singularities of a curve with the indicated floor diagram are to be found as intersection points between the planar incarnation of two distinct floors.



(8.16.2) In general the datum of the floor diagram does not impose a particular subdivision of the Newton polygon. This can be observed on the examples of Figure 20 above.

Finally, let us give the application of Theorem (8.14) to the toric enumerative problem (3.3).

(8.17) Corollary. *Consider a tropical enumerative problem as in Setup (8.12). The number $N_g^\Delta(\alpha,\beta)$ equals the number of equivalence classes of marked floor diagrams of genus g, Newton polygon Δ, and type $(\alpha^+,\alpha^-,\beta^+,\beta^-)$, where each marked diagram $(\mathcal{D},m)$ is counted with the multiplicity*

$$\mu^{\mathbf{C}}(\mathcal{D},m) = I^\beta \prod_{e\in \mathrm{Edges}^{\mathrm{fin}}} w(e)^2.$$

Proof. By Mikhalkin's Correspondence Theorem (7.8), the number $N_g^\Delta(\alpha,\beta)$ equals $I^{-\alpha}$ times the number of parametrized plane tropical curves $(\mathbf{\Gamma},f)\in\mathcal{E}_g^\Delta(\alpha,\beta)(\underline{x},\Omega)$, for a general choice of points $\underline{x}$ and Ω, each tropical curve being counted with the multipicity $\mu^{\mathbf{C}}(\mathbf{\Gamma},f)$ defined in (7.6). In turn, this number counts marked floor diagrams if we take the points $\underline{x}$ as in Theorem (8.14). It thus only remains to show that the multiplicities of the tropical curves in $\mathcal{E}_g^\Delta(\alpha,\beta)(\underline{x},\Omega)$ relate to that of the corresponding marked floor diagrams in the appropriate way.

Let us consider a curve $(\mathbf{\Gamma},f)$ as above. In the corresponding subdivision of the Newton polygon Δ, each sub-polygon is a triangle corresponding to a vertex at which a floor intersects an elevator and changes slope. Let us thus consider a floor of $(\mathbf{\Gamma},f)$, and name $u_0,\dots,u_n$ the direction vectors of the various line segments composing this floor, oriented and numbered from left to right with respect to direction d as indicated on the figure below.

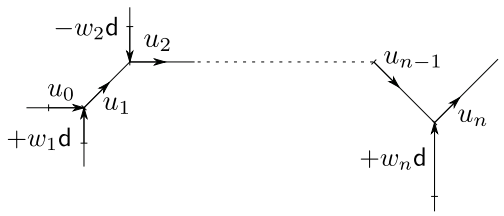


Figure 23: Slopes of line segments along a floor

Label the vertices on this floor from 1 to n, and for each let w_i be the weight of the corresponding elevator, and $\varepsilon_i=+1$ (resp. -1) if it arrives from the bottom (resp. from the top). The balancing condition says that, for all $i=0,\dots,n$,

$$u_i = u_{i-1} + \varepsilon_i w_i \mathsf{d}. \tag{8.17.1}$$

By $\mathsf{d}^\perp$-transversality, one has $\det(u_0,\mathsf{d}) = \pm 1$. Then it follows from the induction relation (8.17.1) that $\det(u_i,\mathsf{d})=\pm1$ for all $i=0,\dots,n$, hence the vertex $\#i$ contributes to the multiplicity $\mu^{\mathbf{C}}(\mathbf{\Gamma},f)$ by w_i, cf. (7.6.1).

Thus, the multiplicity $\mu^{\mathbf{C}}(\mathbf{\Gamma},f)$ may be computed as a product of contributions from the elevators, the contribution of an elevator of weight w being w (resp. w^2) if it is attached to one (resp. two) floor(s). Therefore

$$\mu^{\mathbf{C}}(\mathbf{\Gamma},f) = \prod_{e\in\mathrm{Edges}^{\pm\infty}} w(e)\cdot \prod_{e\in\mathrm{Edges}^{\mathrm{fin}}} w(e)^2,$$

where the products on the right-hand-side are indexed on the edges of the floor diagram associated to $(\mathbf{\Gamma},f)$. This coincides with the multiplicity $\mu^{\mathbf{C}}(\mathcal{D},m)$ after we multiply by the factor $I^{-\alpha}$ indicated by Mikhalkin's Correspondence Theorem. □

Applications of Corollary (8.17) to the enumeration of plane cubics with various tangency conditions along a fixed line have already been given in [IX, (1.7)]. Let us now give a few other examples, in continuation of Section 3.3.

(8.18) Examples. We continue with some of the examples from (8.5), (8.8), and (8.11).

(8.18.1) Class of the quartic surface of (3.9). This amounts to counting marked floor diagrams of type Δ and genus 0 (with trivial boundary conditions), where Δ is the square displayed in (3.9), with respect to the direction $\mathsf{d} = (0, 1)$. There is only one such diagram up to equivalence, which is the following.

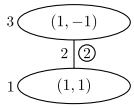


Figure 24: Floor diagram of genus 0 for the Newton polygon of (3.9)

Thus the number of irreducible rational curves in a pencil of hyperplane sections of the surface $(xy - t^2 = zw - t^2 = 0) \subseteq \mathbf{P}^4$ is 4, by Corollary (8.17).

(8.18.2) Invariants of the octic surface of (3.7). We want to count irreducible genus 1 (resp. genus 0) curves in a pencil (resp. a web) of hyperplane sections of the octic surface in $\mathbf{P}^7$ corresponding to the rectangle Δ displayed in (3.7). The corresponding marked floor diagrams, with respect to direction $\mathsf{d} = (0, 1)$, are the following.

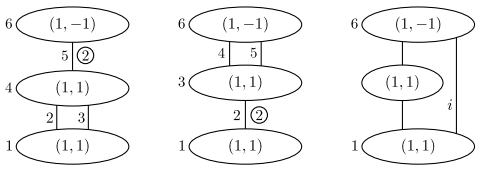


Figure 25: Genus 1 floor diagrams

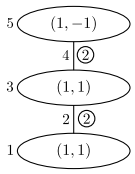
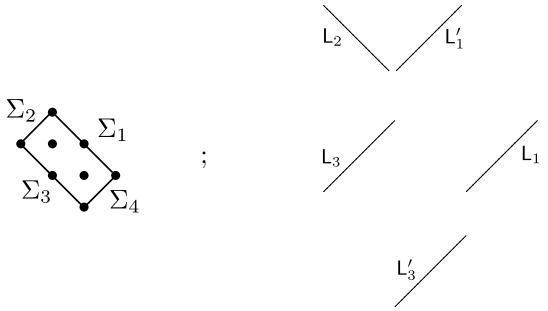


Figure 26: Genus 0 floor diagram

In the genus 1 case, the rightmost diagram has four possible markings, as i may take any value in $[\![1, 4]\!]$, which then determines the rest of the marking (i.e., the marking of the one floor and two elevators with nothing indicated on the figure). We thus find $N_1^\Delta = 12$, and $N_0^\Delta = 16$.

Note that the number in genus 1 does not coincide with the one given in (3.7), which is not a contradiction because this is not the same enumerative problem: in the computation above, no base points have been imposed along the toric anticanonical cycle, whereas in (3.7) all the points along the toric anticanonical cycle are fixed. The latter computation cannot be performed purely in terms of floor diagrams, because we want too many boundary conditions with respect to what is possible, according to the condition given in (8.6). We solve the tropical problem corresponding to the enumerative problem of (3.7) in the next example.

(8.19) Example. So, let us consider the enumerative problem of (3.7): we let Δ be the Newton polygon indicated below, and fix $\alpha = (2, 1, 2, 0)$, $\beta = (0, 0, 0, 1)$. Thus the set Ω of boundary conditions consists of five real values, fixing five lines $\mathsf{L}_1, \mathsf{L}_1', \mathsf{L}_2, \mathsf{L}_3, \mathsf{L}_3'$ as indicated on the picture below.



The tropical curves defined by this data move in a 2-dimensional family, in the way pictured below (three curves are pictured, in green, blue, and purple, respectively): the deformations of the curves amount to change the sizes of the two rectangles by letting their corners slide along the lines L_i and L'_i. As the curves move in this family, one notices that the semi-line corresponding to the edge Σ_4 of the Newton polygon is always supported on the same line, which we shall call $\tilde{\mathsf{L}}_4$: this is yet another tropical incarnation of the toric Cayley–Bacharach property stated in Lemma (3.1).

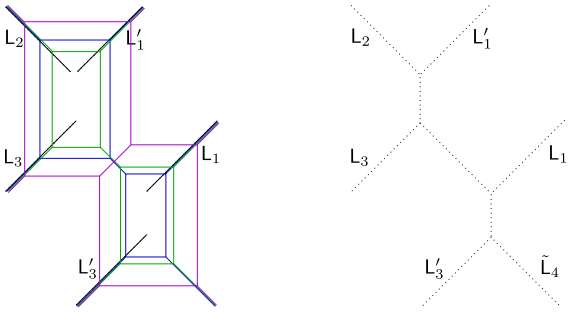


Figure 27: A pencil of tropical curves and the Cayley–Bacharach property

We want to count curves of genus $g = 1$ and $g = 0$ in this system. Let us focus on the case of genus $g = 1$. We need to fix a set $\underline{x}$ of base points consisting of one point only, which is represented by a black dot on Figure 28 below. With the configuration depicted on this figure, we find the four physical curves represented below: the first two count with multiplicity 1 each, and the last two with multiplicity 4 each. This confirms the number $N_1^\Delta = 10$ given in (3.7) and found in [16].

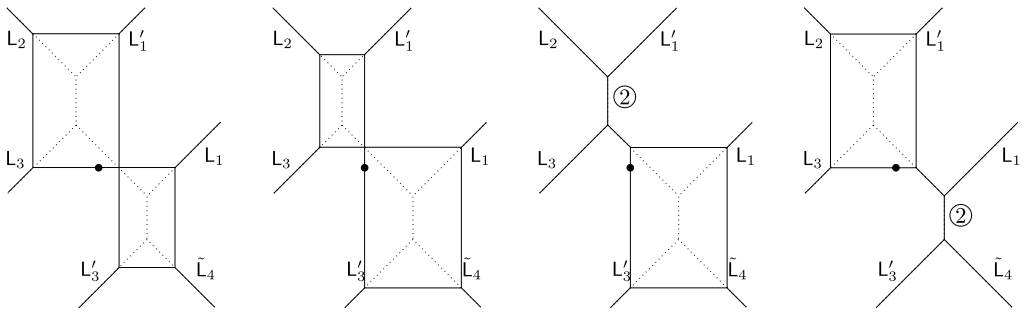


Figure 28: Genus 1 curves in the pencil defined by one base point

Note that these four curves all have a perfectly fine floor decomposition, with corresponding (unmarked) floor diagram one of the diagrams in Figure 25 above. It is however not true that

the base points and boundary conditions are taken up exactly one by one by the floors and elevators.

To complete the picture, note that there is a unique genus $g = 0$ curve in this tropical linear system, which you can see by filling out the spaces between the dots in the picture on the right of Figure 27. It counts with multiplicity 16, and has the (unmarked) floor diagram given in Figure 26.

9 – Tropical representation of algebraic curves

In this section we elaborate on the idea that tropical curves are great for the graphic representation of algebraic curves in toric surfaces, and illustrate this with meaningful examples. A natural attempt to represent complex algebraic plane curves is to draw the set of their real points, in $\mathbf{R}^2$ say, but as anyone who has tried knows, this does not reflect the reality of things very well. For instance, the set of real points may be disconnected, it may contain isolated points, and it does not respect the Bezout Theorem. These problems do not occur with tropical curves.

(9.1) Theorem (Bezout Theorem, see [42, Section 4])**.** *Let $C, D \subseteq \mathbf{R}^2$ be two plane tropical curves with Newton polygon Δ. If C and D intersect at finitely many points, then the number of their intersection points, counted with multiplicities, is $c_1(\mathcal{O}_{X_\Delta}(1))^2 = 2\mathscr{A}(\Delta)$, in the notation of (2.9).*

The intersection multiplicity of two line segments with primitive direction vectors u and v and multiplicities m and n is defined as $m \cdot n \cdot \big|\det(u, v)\big|$. Note that the theorem says not only that all intersection points are visible in the tropical shadow, but also that none of them has wandered at infinity to some component of the anticanonical cycle.

The next property is merely an observation, as it follows directly from the definition of the Newton polygon of a plane tropical curve. At a more conceptual level, plane tropical curves are by design projected shadows of curves in the torus $(\mathbf{C}^*)^2$; it is therefore expected that they behave well with respect to the completion of $(\mathbf{C}^*)^2$ in a toric surface by the addition of a toric boundary.

(9.2) Remark. Let $C \subseteq \mathbf{R}^2$ be a plane tropical curve with Newton polygon Δ. The number, counted with multiplicities, of infinite branches of C in the direction corresponding to a segment Σ of the boundary of Δ equals the intersection number $c_1(\mathcal{O}_{X_\Delta}(1)) \cdot C_\Sigma$, where C_Σ is the component of the anticanonical cycle $\curlywedge_\Delta$ corresponding to Σ.

Now, the principles of "my" tropically inspired way of graphically representing algebraic curves on (surfaces birational to) toric surfaces are the following.

(9.3) Graphic representation. Let C be a curve in a toric surface X_Δ. We represent the surface X_Δ by drawing its anticanonical cycle $\curlywedge_\Delta$ as a polygon (which looks much like the Newton polygon Δ); then the surface X_Δ is the inside of this polygon. To represent the curve C, we draw a (finite length portion of a) tropical curve C_{trop} with the same Newton polygon as C, inside the polygon representing X_Δ: the infinite branches of C_{trop} are represented by finite segments hitting the appropriate segment of $\curlywedge_\Delta$, and the finite segments of C_{trop} do not hit $\curlywedge_\Delta$. The concept of tropical toric varieties, see [11, Section 7.3], gives a mathematical meaning to this graphic process; see Remark (7.3) for a construction in the case $\Delta = T_d$, which gives $X_\Delta \simeq \mathbf{P}^2$.

Note that the Newton polygon of C may differ from Δ: for instance, we may want to represent a fibre of the ruling of a Hirzebruch surface $\mathbf{F}_e$, or a curve C in $\mathbf{F}_e$ such that the complete linear system $|C|$ represents $\mathbf{F}_e$ as a cone over a rational normal curve. In this case

the curve C is not ample in X_Δ, and some segments of $\nexists_\Delta$ will not be hit by any infinite branch of C_{trop}. We give first examples in Figure 29.

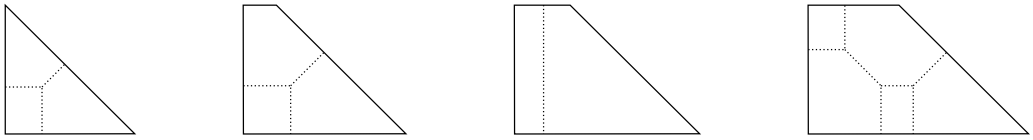


Figure 29: Curves linearly equivalent to H in $\mathbf{P}^2$, H in $\mathbf{F}_1$, F in $\mathbf{F}_1$, and $H+F$ in $\mathbf{F}_1$ (in the notation introduced in Section 1.4)

One more nice feature of this way of picturing curves is that it lets graphically see the genus, as it is the first Betti number of the real object that we are actually seeing. Thus, in Figure 30, we see that the curve on the left (resp., on the right) has genus one (resp., two).

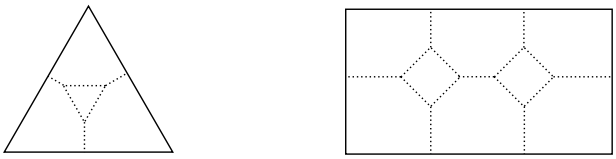


Figure 30: Hyperplane sections of the cubic of (3.6) and the quartic of (3.7)

Note that what we are considering is really representations of algebraic curves inspired by tropical geometry, and not actual tropical curves. In particular, we may forget about the integral structure underlying $\mathbf{R}^2$ (i.e., , $\mathbf{Z}^2$), the balancing condition, the subdivision of the Newton polygon of the curve, and so on. This allows for an essentialization of our drawings: we may for instance simply represent a Hirzebruch surface by a rectangle instead of a trapezium (or by whatever quadrilateral we fancy), and simplify the representations of curves as indicated on Figure 31.

Figure 31: Curves linearly equivalent to $3H$ in $\mathbf{P}^2$, and $H+F$ in $\mathbf{F}_1$

In the same vein it frequently happens, as we have seen in [I], that one needs to represent curves that actually lie on some blow-up of a toric surface at points lying on the toric boundary. In that case we adapt our representation in a rather flexible way without much trouble.

Our interest resides especially in representing curves lying on reducible surfaces, the irreducible components of which are toric surfaces (or blow-ups thereof). In that case we may simply tile our polygons so as to reflect the dual graph of the various irreducible components, as in (7.9). A nice example is given in [I, Figure 11]. In that example, the dual graph of our reducible surface is a tiling of the sphere $\mathbf{S}^2$, and we have tweaked some of the sides of our polygons to obtain a better looking drawing; incidentally, we have also tweaked some line segments of the representation of our curve. Note that the surface T^1 is actually outside of its polygon, as we have performed a kind of stereographic projection from a point inside this surface T^1, in order to represent the sphere $\mathbf{S}^2$ on the plane. You may observe that the representation of the curve

is topologically the same as the union of the edges of a tetrahedron, which is none other than the dual of the tetrahedron (in the sense of Section 3.3) we had started with.

(9.4) Application to Brugallé degenerations (see [IX])**.** I will simply give an example which, I believe, displays all I want to show. Let us follow a degeneration of a plane cubic along the degeneration of $\mathbf{P}^2$ to a chain of Hirzebruch $\mathbf{F}_1$ surfaces $S_0, \ldots, S_9$, with a (useless) projective plane S_{10} at one end of the chain. The plane cubic degenerates (in our example) to a curve which restricts to curves linearly equivalent to H on S_9, $H+F$ on S_7, $H+2F$ on S_4, and multiples of F on all other $\mathbf{F}_1$ surfaces (and 0 on the plane), in the notation introduced in Section 1.4; see [IX, Section 4] for more details. We may see our plane cubic as a hyperplane section of the Veronese surface $v_3(\mathbf{P}^2) \subseteq \mathbf{P}^9$, and its degeneration as a hyperplane section of a "reducible toric surface" as in (7.9), made of a projective plane, attached to a rational normal scroll of type $(1,2)$ along a line, and the latter attached to a scroll of type $(2,3)$ along a conic.

In Figure 32a, we have drawn "proper tropical curves" inside the various polygons (those on which the curve has class F are skipped). Now the wonder[18] is that when we put all the pieces together, what we obtain is a plane tropical cubic! In Figure 32b, we have drawn "essentialized tropical curves"; if you look at their patching together with the right kind of eyes, you should see only floors and elevators.

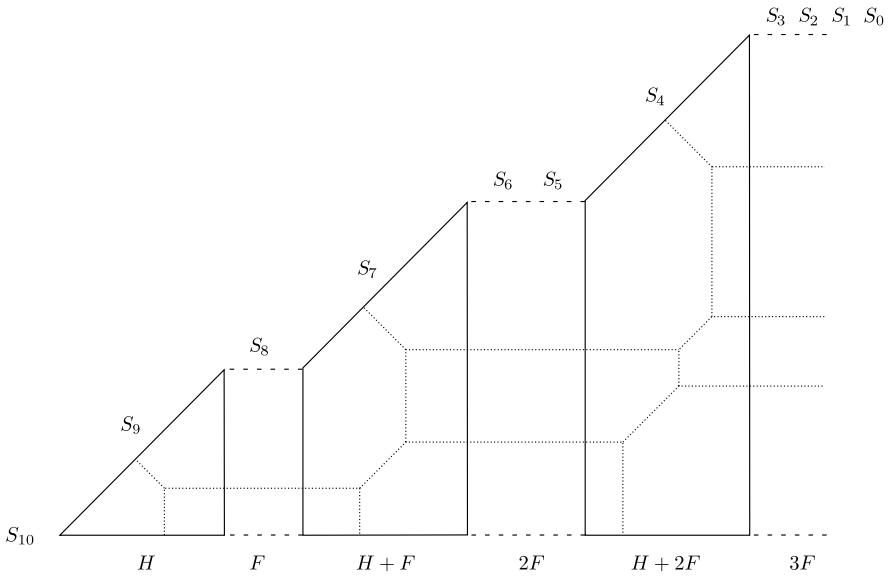


Figure 32a: Smooth plane cubic in a degeneration of $v_3(\mathbf{P}^2)$ to a union of scrolls (tropical version)

References

Chapters in this Volume

[I] C. Ciliberto and Th. Dedieu, *Limits of nodal curves: generalities and examples.*

[III] Th. Dedieu, *Geometry of logarithmic Severi varieties at a general point.*

[VI] Th. Dedieu and C. Galati, *Degenerations and deformations of generalized Severi varieties.*

[18] of course, once one has digested Sections 6 and 7, this is sadly obvious.

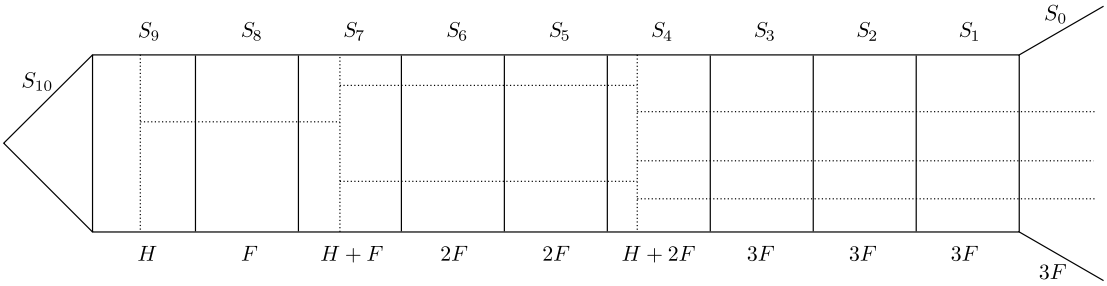


Figure 32b: Smooth plane cubic in a degeneration of $v_3(\mathbf{P}^2)$ to a union of scrolls (arts déco version)

[IX] Th. Dedieu, *Enumeration of curves by floor diagrams, via degenerations of the projective plane.*

[XI] Th. Dedieu, *Enumerative geometry of $K3$ surfaces.*

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces.*

[XIII] Th. Dedieu, *Node polynomials for curves on surfaces.*

[XIV] F. Bastianelli and Th. Dedieu, *The Göttsche conjecture and the Göttsche–Yau–Zaslow formula.*

External References

[1] D. Abramovich, Q. Chen, M. Gross and B. Siebert, *Decomposition of degenerate Gromov-Witten invariants*, Compos. Math. **156** (2020), no. 10, 2020–2075.

[2] H. Argüz and P. Bousseau, *Real log curves in toric varieties, tropical curves, and log Welschinger invariants*, Ann. Inst. Fourier (Grenoble) **72** (2022), no. 4, 1547–1620.

[3] ———, *BPS polynomials and Welschinger invariants*, 2025, preprint arXiv:2506.02770.

[4] V. I. Arnold, *Mathematical methods of classical mechanics*, second ed., Graduate Texts in Mathematics, vol. 60, Springer-Verlag, New York, 1989, Translated from the Russian by K. Vogtmann and A. Weinstein.

[5] A. Arroyo, E. Brugallé and L. López de Medrano, *Recursive formulas for Welschinger invariants of the projective plane*, Int. Math. Res. Not. IMRN (2011), no. 5, 1107–1134.

[6] F. Block, A. Gathmann and H. Markwig, *Psi-floor diagrams and a Caporaso-Harris type recursion*, Israel J. Math. **191** (2012), no. 1, 405–449.

[7] F. Block and L. Göttsche, *Refined curve counting with tropical geometry*, Compos. Math. **152** (2016), no. 1, 115–151.

[8] T. Blomme, *A tropical computation of refined toric invariants*, Geom. Topol. **29** (2025), no. 6, 3129–3185.

[9] P. Bousseau, *Tropical refined curve counting from higher genera and lambda classes*, Invent. Math. **215** (2019), no. 1, 1–79.

[10] E. Brugallé, *Floor diagrams relative to a conic, and GW–W invariants of Del Pezzo surfaces*, Adv. Math. **279** (2015), 438–500.

[11] E. Brugallé, I. Itenberg, G. Mikhalkin and K. Shaw, *Brief introduction to tropical geometry*, in *Proceedings of the Gökova Geometry-Topology Conference 2014*, Gökova Geometry/Topology Conference (GGT), Gökova, 2015, 1–75.

[12] E. Brugallé and G. Mikhalkin, *Floor decompositions of tropical curves in any dimension*, `http://erwan.brugalle.perso.math.cnrs.fr/miscellaneous.html`.

[13] E. Brugallé and K. Shaw, *A bit of tropical geometry*, Amer. Math. Monthly **121** (2014), no. 7, 563–589.

[14] ———, *Floor decompositions of tropical curves: the planar case*, in *Proceedings of Gökova Geometry-Topology Conference 2008*, Gökova Geometry/Topology Conference (GGT), Gökova, 2009, 64–90.

[15] R. Cavalieri, P. Johnson, H. Markwig and D. Ranganathan, *Counting curves on Hirzebruch surfaces: tropical geometry and the Fock space*, Math. Proc. Cambridge Philos. Soc. **171** (2021), no. 1, 165–205.

[16] C. Ciliberto and Th. Dedieu, *Limits of pluri-tangent planes to quartic surfaces*, in *Algebraic and complex geometry*, Springer Proc. Math. Stat., vol. 71, Springer, Cham, 2014, 123–199.

[17] D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[18] M. A. Cueto and H. Markwig, *Combinatorics and real lifts of bitangents to tropical quartic curves*, Discrete Comput. Geom. **69** (2023), no. 3, 597–658.

[19] M. Einsiedler, M. Kapranov and D. Lind, *Non-Archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157.

[20] S. Fomin and G. Mikhalkin, *Labeled floor diagrams for plane curves*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 6, 1453–1496.

[21] A. Gathmann and H. Markwig, *The Caporaso-Harris formula and plane relative Gromov-Witten invariants in tropical geometry*, Math. Ann. **338** (2007), no. 4, 845–868.

[22] A. Gathmann, H. Markwig and F. Schroeter, *Broccoli curves and the tropical invariance of Welschinger numbers*, Adv. Math. **240** (2013), 520–574.

[23] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008, Reprint of the 1994 edition.

[24] L. Göttsche and V. Shende, *Refined curve counting on complex surfaces*, Geom. Topol. **18** (2014), no. 4, 2245–2307.

[25] P. Harinck, A. Plagne and C. Sabbah eds, *Géométrie tropicale (X-UPS 2008)*, Éditions de l'École Polytechnique, Palaiseau, 2008. With contributions by E. Brugallé, I. Itenberg, B. Teissier.

[26] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[27] J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of convex analysis*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2001.

[28] I. Itenberg and G. Mikhalkin, *On Block-Göttsche multiplicities for planar tropical curves*, Int. Math. Res. Not. IMRN (2013), no. 23, 5289–5320.

[29] I. Itenberg, G. Mikhalkin and E. Shustin, *Tropical algebraic geometry*, second ed., Oberwolfach Seminars, vol. 35, Birkhäuser Verlag, Basel, 2009.

[30] I. Itenberg and E. Shustin, *Real enumerative invariants relative to the toric boundary and their refinement*, 2025, preprint arXiv:2401.06718.

[31] M. Kontsevich, *What is tropical mathematics?*, talk at the Fields Institute, Toronto, 2013, `https://video-archive.fields.utoronto.ca/list/event/170`.

[32] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, in *Symplectic geometry and mirror symmetry (Seoul, 2000)*, World Sci. Publ., River Edge, NJ, 2001, 203–263.

[33] T. Mandel and H. Ruddat, *Descendant log Gromov-Witten invariants for toric varieties and tropical curves*, Trans. Amer. Math. Soc. **373** (2020), no. 2, 1109–1152.

[34] G. Mikhalkin, *Amoebas of algebraic varieties and tropical geometry*, in *Different faces of geometry*, Int. Math. Ser. (N. Y.), vol. 3, Kluwer/Plenum, New York, 2004, 257–300.

[35] ———, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, Topology **43** (2004), no. 5, 1035–1065.

[36] ———, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), no. 2, 313–377.

[37] ———, *Quantum indices and refined enumeration of real plane curves*, Acta Math. **219** (2017), no. 1, 135–180.

[38] A. J. Puentes, H. Markwig, S. Pauli and F. Röhrle, *Quadratically enriched plane curve counting via tropical geometry*, 2026, preprint arXiv:2502.02569.

[39] A. J. Puentes and S. Pauli, *A quadratically enriched correspondence theorem*, 2024, preprint arXiv:2309.11706.

[40] T. Nishinou and B. Siebert, *Toric degenerations of toric varieties and tropical curves*, Duke Math. J. **135** (2006), no. 1, 1–51.

[41] D. Ranganathan, *Skeletons of stable maps I: rational curves in toric varieties*, J. Lond. Math. Soc. (2) **95** (2017), no. 3, 804–832.

[42] J. Richter-Gebert, B. Sturmfels and T. Theobald, *First steps in tropical geometry*, in *Idempotent mathematics and mathematical physics*, Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, 289–317.

[43] E. Shustin, *A tropical approach to enumerative geometry*, Algebra i Analiz **17** (2005), no. 2, 170–214, a.k.a. Patchworking singular algebraic curves, non-Archimedean amoebas and enumerative geometry.

[44] B. Sturmfels, *Polynomial equations and convex polytopes*, Amer. Math. Monthly **105** (1998), no. 10, 907–922.

[45] O. Viro, *Dequantization of real algebraic geometry on logarithmic paper*, in *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, Progr. Math., vol. 201, Birkhäuser, Basel, 2001, 135–146.

Lecture XI
Enumerative geometry of $K3$ surfaces

by Thomas Dedieu

1 Introduction 267
2 Rational curves in a primitive class 270
2.1 An elementary topological counting formula 271
2.2 Proof of the Beauville–Yau–Zaslow formula 273
2.3 Compactified Jacobian of an integral curve 274
2.4 Two fundamental interpretations of the multiplicity 276
3 Curves of any genus in a primitive class 277
3.1 Reduced Gromov–Witten theories for $K3$ surfaces 277
3.2 The Göttsche–Bryan–Leung formula 278
3.3 Proof of the Göttsche–Bryan–Leung formula 283
4 BPS state counts 289
4.1 Rational curves on the quintic threefold 289
4.2 Degree 8 rational curves on a sextic double plane 292
4.3 Elliptic curves in a 2-divisible class on a $K3$ surface 293
4.4 The Yau–Zaslow formula for non-primitive classes 294
5 Relations with threefold invariants 296
5.1 Two obstruction theories 296
5.2 The Katz–Klemm–Vafa formula 297
5.3 Further Gromov–Witten integrals 298
6 Noether–Lefschetz theory and applications 300
6.1 Lattice polarized $K3$ surfaces and Noether–Lefschetz theory 300
6.2 Invariants of families of lattice polarized $K3$ surfaces 301
6.3 Application to the Yau–Zaslow conjecture 303
References 306

1 – Introduction

The aim of these notes is to explain various enumerative results about $K3$ surfaces without assuming familiarity with Gromov–Witten theory; in fact, they represent an attempt on my part to understand the meaning of these results in classical terms. The enumerative results in question are due to Beauville, Bryan and Leung, Pandharipande, Maulik, Thomas, and others, and confirm conjectures made by Yau–Zaslow, Göttsche, and Katz–Klemm–Vafa. They are listed in Paragraph (1.1) below.

They fall in three categories: (i) results that don't really need Gromov–Witten theory at all, either to be formulated or to be proved; (ii) results that may be formulated without Gromov–Witten theory, but which necessitate techniques from this theory to be proven (as far as current knowledge goes); (iii) results which require an understanding of Gromov–Witten theory to be fully appreciated. The text provides the necessary background in Gromov–Witten theory in

order to understand results from all these three categories. In particular we shall give a brief idea of what Gromov–Witten invariants are in Paragraph (1.2).

(1.1) Contents description. In Section 2 we state a formula of Yau and Zaslow giving the number of rational curves in a primitive linear system $|L|$ on a $K3$ surface S, and give its proof by Beauville following the strategy suggested by Yau–Zaslow, using the universal compactified Jacobian; this falls in the above mentioned category (i). In this formula, each rational curve is counted with multiplicity equal to the topological Euler number of its compactified Jacobian, which depends only on the singularities of the curve. We also present results due to Fantechi, Göttsche, and van Straten which show that this is also the multiplicity of the origin in a semi-universal deformation space of the singularities of the curve.

The Yau–Zaslow formula is generalized in Section 3 to a formula giving the number of genus g curves in the system $|L|$ passing through g general points on S, for arbitrary g; this generalized formula had been conjectured by Göttsche. We give an outline of its proof by Bryan and Leung, by degeneration to an elliptic $K3$ surface, with as much details as the scope of these notes and the ability of the author permit. This proof requires the formulation of the result in terms of some special version of Gromov–Witten invariants specifically designed for algebraic $K3$ surfaces (see Section 3.1), and relies among other things on a multiple cover formula for nodal rational curves. This result thus falls in category (ii).

Essentially all remaining results fall in category (iii). The goal of Section 4 is to explain the extension of the Yau–Zaslow formula to a formula for non-primitive linear systems, which has been proven by Klemm–Maulik–Pandharipande–Scheidegger (this proof is streamlined in Section 6.3). This features the Aspinwall–Morrison multiple cover formula, and its application to define corrected Gromov–Witten invariants known as BPS states numbers. I also discuss other degenerate contributions to these invariants besides plain multiple curves, striving to sort out the relation between the number given by the formula and the actual number of integral rational curves.

In Section 5 we introduce various generalizations of the Gromov–Witten invariants on $K3$ surfaces considered so far, in particular the Hodge integrals and their BPS states version. The main theme in this section is the relation between between Gromov–Witten integrals on $K3$ surfaces and curve counting Gromov–Witten invariants on threefolds. For instance, we discuss the Katz–Klemm–Vafa formula, proved by Pandharipande and Thomas, which may be seen as computing the excess contribution of an algebraic $K3$ surface to the Gromov–Witten invariants of any fibered threefold in which it appears as a fibre. Some of the material presented in this section is necessary for the proof (discussed in Section 6) of the Yau–Zaslow formula for non primitive classes (stated in Section 4).

In Section 6 we introduce Noether–Lefschetz numbers for families of lattice-polarized $K3$ surfaces, and state a theorem due to Maulik and Pandharipande which shows, on a threefold fibered in lattice-polarized $K3$ surfaces, how these Noether–Lefschetz numbers give an explicit relation between Gromov–Witten invariants of the threefold and of the $K3$ fibres. Finally, we discuss the application of this formula to the proof of the Yau–Zaslow formula for non-primitive linear systems. It involves a mirror symmetry theorem that enables the computation of Gromov–Witten invariants of anticanonical sections of toric 4-manifolds, as well as modularity results for Noether–Lefschetz numbers following from the work of Borcherds and Kudla–Millson; the latter enable the computation of all Noether–Lefschetz numbers of the family of lattice-polarized $K3$ surfaces considered in the proof, viz. the so-called STU model.

(1.2) Gromov–Witten theory. Let X be a compact complex manifold. In Gromov–Witten theory, one views genus g curves on X as *stable maps*, i.e., morphisms $f : C \to X$ where C is a connected, nodal, curve of arithmetic genus g, such that there are only finitely many

automorphisms ϕ of C satisfying the identity $f\circ\phi = f$. The latter condition is called the *stability condition*, and amounts to the requirement that each irreducible component of arithmetic genus 0 (resp., 1) of C which is contracted by f carries at least 3 (resp., 1) *special points*, i.e., either intersection points with other irreducible components of C or, if relevant, marked points. An integral embedded curve $C \subseteq X$ is then encoded as the map $f : \bar{C} \to X$ obtained by composing the normalization of C with its embedding in X.

Stable maps have moduli spaces that are compact. Let g and k be non-negative integers, and β be a homology class in $H_2(X, \mathbf{Z})$. We denote by $\overline{M}_{g,k}(X,\beta)$ the moduli space of genus g stable maps $f : C \to X$, with k marked points on C, such that $[f_*(C)] = \beta$. It is compact, has expected dimension

$$\operatorname{expdim}\big(\overline{M}_{g,k}(X,\beta)\big) = (\dim(X)-3)(1-g) - K_X\cdot\beta + k, \tag{1.2.1}$$

[1]

and all its irreducible components have dimension larger than or equal to the expected dimension. The moduli space $\overline{M}_{g,k}(X,\beta)$ has in general many irreducible components, including some parametrizing stable maps that one would arguably rather discard if one is willing to enumerate genuine curves on X: for instance, note that the contraction of a curve C of genus $g \geqslant 2$ to a point of X is a stable map. It turns out nevertheless that considering stable maps instead of embedded curves provides much better behaved (e.g., deformation invariant) enumerative numbers. It is possible to consider different devices instead of stable maps in order to obtain well-behaved enumerative numbers, each such possibility coming with its own specific pros and cons; about this, see the enlightening survey [32].

Now, Gromov–Witten invariants are[2] intersection numbers

$$\int_{[\overline{M}_{g,k}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1)\cup\ldots\cup\mathrm{ev}_k^*(\gamma_k), \tag{1.2.2}$$

where: $\gamma_1,\ldots,\gamma_k$ are cohomology classes in $H^*(X,\mathbf{Z})$; for all $i = 1,\ldots,k$, ev_i is the *evaluation map* at the i-th marked point, i.e., the map $\overline{M}_{g,k}(X,\beta) \to X$ defined by $[f : C \to X, x_1,\ldots,x_k] \mapsto f(x_i)$; $[\overline{M}_{g,k}(X,\beta)]^{\mathrm{vir}}$ is a rational homology class in $H_{2\,\mathrm{vdim}}\big(\overline{M}_{g,k}(X,\beta),\mathbf{Q}\big)$, called the *virtual fundamental class*, where vdim is the expected (a.k.a. virtual) dimension of $\overline{M}_{g,k}(X,\beta)$. The virtual fundamental class is the usual fundamental class when the moduli space $\overline{M}_{g,k}(X,\beta)$ has the expected dimension, but in general it is given by an excess formula (it is the top Chern class of the obstruction bundle when $\overline{M}_{g,k}(X,\beta)$ is non-singular). Thus, the virtual fundamental class allows the computation of the intersection product $\mathrm{ev}_1^*(\gamma_1)\cup\ldots\cup\mathrm{ev}_k^*(\gamma_k)$ on $\overline{M}_{g,k}(X,\beta)$ "as if the latter were equidimensional of the expected dimension". Note that, by definition, the integral (1.2.2) is 0 if the degree of the integrand does not match the dimension of the virtual fundamental class.

Typically, the cohomology classes $\gamma_1,\ldots,\gamma_k$ are the Poincaré duals of algebraic cycles $\Gamma_1,\ldots,\Gamma_k$ on X; in this case, the condition that the degrees of $\mathrm{ev}_1^*\gamma_1,\ldots,\mathrm{ev}_k^*\gamma_k$ sum up to $2\,\mathrm{vdim}\,\overline{M}_{g,k}(X,\beta)$ is equivalent to the equality

$$\sum\nolimits_{i=1}^{k}\big(\mathrm{codim}_X(\Gamma_i)-1\big) = \mathrm{expdim}\,\overline{M}_{g,0}(X,\beta),$$

[1] at a point of $\overline{M}_{g,k}(X,\beta)$ corresponding to a map $f : C \to X$ such that C is smooth and f is unramified, this may be computed as $\chi(N_f)+k$, where N_f (the *normal sheaf* of f) is the locally free cokernel of the injective map of locally free sheaves $T_C \to f^*T_X$, see [II, Section 8], [38, § 3.4.2], or [32, § $1\frac{1}{2}$]; these references also explain how to compute the expected dimension in general.

[2] in fact there are other Gromov–Witten invariants, which are more general intersection numbers against $[\overline{M}_{g,k}(X,\beta)]^{\mathrm{vir}}$ than those in (1.2.2) (we will see some examples in Section 5.3), but our primary interest in these notes resides in the latter.

which means that the incidence conditions imposed by $\Gamma_1, \ldots, \Gamma_k$ to genus g curves in the class β are expected to define a finite number of curves. Then, under suitable transversality assumptions, and provided the moduli space $\overline{M}_{g,k}(X, \beta)$ (or equivalently $\overline{M}_{g,0}(X, \beta)$) has the expected dimension, the Gromov–Witten invariant (1.2.2) gives the number of genus g curves in the class β (interpreted as stable maps, and counted with multiplicities) which pass through the cycles $\Gamma_1, \ldots, \Gamma_k$. We will be mainly concerned in this text with the case when all Γ_i's are points, which is the only relevant case when X is a surface.

(1.3) Terminology and conventions. We always work over the field of complex numbers.

Let C be a curve. Its arithmetic genus, denoted by $p_a(C)$, is the integer $1 - \chi(\mathcal{O}_C)$. If C is reduced, its geometric genus is the arithmetic genus of its normalization, and is denoted by $p_g(C)$. When I write 'genus', this means 'geometric genus'.

A reduced curve C is *immersed* when the differential of its normalization map is everywhere non-degenerate. Concretely this means that C has no cuspidal points; it may have however points of any multiplicity, and non-ordinary singularities (e.g., a tacnode, i.e., a point at which there are two smooth local branches tangent one to another). A *node* is an ordinary double point.

A $K3$ surface S is a smooth surface with trivial canonical bundle and vanishing irregularity (i.e., $q(S) = h^1(S, \mathcal{O}_S) = 0$); we may occasionally qualify as $K3$ a surface with canonical singularities, the minimal smooth model of which is a smooth $K3$ surface. Let p be a positive integer. A $K3$ surface of genus p is a pair (S, L), where S is a $K3$ surface and L an effective line bundle on S, such that $L^2 = 2p - 2$ (in particular, the $K3$ surface S is algebraic). Under these assumptions, the complete linear system $|L|$ has dimension p, and its general member has arithmetic genus p (it is smooth if $|L|$ is free of fixed components). The pair (S, L) is *primitive* if the line bundle L is indivisible, i.e., there is no line bundle L' on S such that $L \cong (L')^{\otimes m}$ for some integer $m > 1$.

In the notation of (1.2), we write $\overline{M}_g(X, \beta)$ for $\overline{M}_{g,0}(X, \beta)$. If S is a surface equipped with an effective line bundle $L \to S$, we write $\overline{M}_{g,k}(X, L)$ for $\overline{M}_{g,k}(X, \beta)$ where β is the homology class of the members of $|L|$.

(1.4) Let (S, L) be a $K3$ surface of genus p. Members of $|L|$ with exactly δ nodes as singularities have geometric genus $p - \delta$, and are expected to fill up a locus of codimension δ in $|L|$. For this reason (and because $|L|$ has dimension p), the locus of genus g curves in $|L|$ has expected dimension g; note that this does not match with the virtual dimension (1.2.1) of $\overline{M}_g(S, L)$, see (3.1). One can actually prove that this is indeed the correct dimension, and that the locus of genus g curves is equidimensional (see [12, § 4.2]). This implies that for a general set of g points $x_1, \ldots, x_g \in S$, there is a finite number of genus g curves in $|L|$ passing through all points $x_1, \ldots, x_g$, by Kleiman and Piene's little lemma [XIII, Lemma (5.1)].

(1.5) Acknowledgements. I thank Jim Bryan and Rahul Pandharipande for patiently answering my naive questions, and Yifan Zhao for his observations and help on Section 2.4.

2 – Rational curves in a primitive class

In this section we discuss the following result, proved by Beauville [4] following a strategy proposed by Yau and Zaslow [44]. This strategy was inspired by physics; it is an elaboration of the elementary argumentation using Euler numbers presented in Section 2.1.

(2.1) Theorem (Yau–Zaslow, Beauville)**.** *Let (S, L) be a smooth primitive $K3$ surface of genus p_0, and assume that $\mathrm{Pic}(S) = \mathbf{Z}\,L$. Then there is a finite number N^{p_0} of rational curves in the complete linear system $|L|$, and it is determined by the formula*

$$1 + \sum_{p=1}^{+\infty} N^p q^p = \prod_{n=1}^{+\infty} \frac{1}{(1-q^n)^{24}} = 1 + 24q + 324q^2 + 3200q^3 + \cdots \tag{2.1.1}$$

As one may expect, the number N^{p_0} is the number of rational curves counted with appropriate multiplicities. As we shall see, the multiplicity of an integral rational curve C is the topological Euler number $e(\bar{J}C)$ of its compactified Jacobian, and it depends only on the equisingularity class of its singularities. It may be explicitly computed in many cases; in particular, it is 1 whenever the curve is immersed. For a very general (S, L), all rational curves in $|L|$ are nodal by [8], hence in this case all rational curves count with multiplicity one.

The assumption that $\mathrm{Pic}(S) = \mathbf{Z}\,L$ is made to ensure that all members of $|L|$ are integral curves. While it may be possible to adapt the arguments given by Beauville and Yau–Zaslow to the case when some members of $|L|$ are reducible but reduced, the presence of non-reduced members of $|L|$ poses problems of a different nature, see Section 4.

Theorem (2.1) is a particular case of the more general result discussed in Section 3.2. We will recall there the relevant facts from the theory of modular forms needed to explore the modular aspects of formula (2.1.1), and give more values of N^p for small p.

2.1 – An elementary topological counting formula

Let X be a complex compact manifold and $\mathfrak{f}$ a 1-dimensional family of divisors of X, such that the general member of $\mathfrak{f}$ is smooth and $\mathfrak{f}$ has no fixed part. It is possible to count the number of singular members of $\mathfrak{f}$ using the topological formula stated in Lemma (2.3) below.

(2.2) Euler number. Let X be a topological space. The *(topological) Euler number* of X is

$$e(X) = \sum_i (-1)^i \dim H^i_c(X, \mathbf{Z}),$$

where the subscript c means cohomology with compact support.

If X is compact and $F \subseteq X$ is a closed subset, then the following *additivity formula* holds:

$$e(X) = e(X - F) + e(F).$$

It follows from the long exact sequence

$$\cdots \to H^i_c(X - F, \mathbf{Z}) \to H^i(X, \mathbf{Z}) \to H^i(F, \mathbf{Z}) \to H^{i+1}_c(X - F, \mathbf{Z}) \to \cdots$$

see [18, Théorème 4.10.1].

(2.3) Lemma. *Let $f : X \to C$ be a surjective morphism from a complex compact manifold onto a smooth curve. One has*

$$e(X) = e(F_{\mathrm{gen}})\, e(C) + \sum_{y \in \mathrm{Disc}(f)} \big(e(F_y) - e(F_{\mathrm{gen}})\big), \tag{2.3.1}$$

where F_{gen} is the fibre of f over a general point of B, and for all $y \in B$, F_y is the fibre over y; $\mathrm{Disc}(f) \subseteq C$ is the set of points above which f is not smooth.

Proof. Set $U = X - \bigcup_{y\in \mathrm{Disc}(f)} F_y$. Then $f : U \to C - \mathrm{Disc}(f)$ is a topological fibre bundle, hence

$$e(U) = e(C - \mathrm{Disc}(f))\, e(F_{\mathrm{gen}}).$$

Now, by the additivity formula,

$$e(X) = e(U) + \sum_{y\in \mathrm{Disc}(f)} e(F_y) \qquad \text{and} \qquad e(C - \mathrm{Disc}(f)) = e(C) - \sum_{y\in \mathrm{Disc}(f)} 1,$$

and Formula (2.3.1) follows. □

(2.4) The upshot of Lemma (2.3) is that the number of singular fibres of $f : X \to C$ is $e(X) - e(F_{\mathrm{gen}})\, e(C)$, provided each singular fibre F_y is counted with multiplicity $e(F_y) - e(F_{\mathrm{gen}})$. When X is a surface and the schematic fibre over y is reduced, the difference $e(F_y) - e(F_{\mathrm{gen}})$ is determined by the singularities of F_y, as we shall now explain.

Let D be a reduced projective curve, Σ its singular locus, $\nu : \bar{D} \to D$ its normalization, and $\bar{\Sigma} = \nu^{-1}(\Sigma)$. Since $\bar{D} - \bar{\Sigma}$ and $D - \Sigma$ are isomorphic one has, by the additivity formula,

$$\begin{aligned} e(D) &= e(D - \Sigma) + e(\Sigma) \\ &= e(\bar{D} - \bar{\Sigma}) + e(\Sigma) \\ &= e(\bar{D} - \bar{\Sigma}) + e(\bar{\Sigma}) + e(\Sigma) - e(\bar{\Sigma}) = e(\bar{D}) - \big(\mathrm{Card}(\bar{\Sigma}) - \mathrm{Card}(\Sigma)\big). \end{aligned} \tag{2.4.1}$$

Now, let $f : S \to C$ be a surjective morphism from a smooth projective surface to a smooth curve, and consider a point $y \in C$ such that the schematic fibre F_y is reduced. Then the curve F_y has the same arithmetic genus as the general fibre F_{gen}, and $\bar{F}_y$ is a smooth curve of genus $p_a(F_{\mathrm{gen}}) - \delta$, where $\delta = p_a(F_y) - p_g(F_y)$ is the sum of the δ-invariants of all singularities of F_y. Therefore,

$$e(\bar{F}_y) = 2 - 2(p_a(F_{\mathrm{gen}}) - \delta) = e(F_{\mathrm{gen}}) + 2\delta,$$

hence

$$e(F_y) - e(F_{\mathrm{gen}}) = 2\delta - \big(\mathrm{Card}(\bar{\Sigma}_y) - \mathrm{Card}(\Sigma_y)\big)$$

by (2.4.1). This proves the following.

(2.5) Lemma. *In the above notation, the multiplicity with which the fibre F_y is counted in Formula* (2.3.1) *is a sum of local multiplicities computed at the singular points of F_y, namely*

$$e(F_y) - e(F_{\mathrm{gen}}) = \sum_{x\in \mathrm{Sing}\, F_y} \left(\delta(F_y, x) - \#\binom{\textit{local branches}}{\textit{of } F_y \textit{ at } x} + 1\right).$$

This gives for example the local multiplicity 1 for a node, 2 for an ordinary cusp, 3 for a tacnode, and 4 for an ordinary triple point.

(2.6) Application to $K3$ surfaces of genus one. Let (S, L) be a primitive $K3$ surface of genus one. If $\mathrm{Pic}(S) = \mathbf{Z}\, L$, then $|L|$ is a base-point-free pencil of elliptic curves, and all its members are reduced since L is not divisible. All $K3$ surfaces S have $e(S) = 24$, and smooth elliptic curves have Euler number 0. Thus, it follows from formula (2.3.1) that $|L|$ has 24 singular members, counted with the multiplicity given in Lemma (2.5). This agrees with Theorem (2.1).

(2.7) Finally, if $\mathfrak{f}$ is a 1-dimensional family of divisors on a compact complex manifold X, free of fixed component and such that the general member of $\mathfrak{f}$ is smooth, in order to find the number of singular members of $\mathfrak{f}$ one may consider the minimal smooth blow-up $\tilde{X} \to X$ such that the proper transform of $\mathfrak{f}$ is base-point-free; then there is a regular map $\tilde{X} \to \mathfrak{f}$, the fibres of which are exactly the members of $\mathfrak{f}$, and one may apply Lemma (2.3) to this fibration.

2.2 – Proof of the Beauville–Yau–Zaslow formula

In this section, we outline the proof of Theorem (2.1). Let p be a positive integer, (S, L) a smooth primitive $K3$ surface of genus p, and $\mathfrak{L}$ the complete linear system $|L|$. We assume that all members of $\mathfrak{L}$ are integral.

We let $\mathcal{C}$ be the universal curve over $\mathfrak{L}$, and consider the component

$$\pi : \bar{\mathcal{J}}^p\mathcal{C} \to \mathfrak{L}$$

of the compactified Picard scheme of the family $\mathcal{C} \to \mathfrak{L}$ which parametrizes pairs (C, M) made of a member C of $\mathfrak{L}$ and a rank 1, torsion-free, coherent sheaf M of degree p on C. For all $[C] \in \mathfrak{L}$, the fibre $\pi^{-1}([C])$ is the compactified Jacobian $\bar{J}^pC$ of C. The total space $\bar{\mathcal{J}}^p\mathcal{C}$ is a projective variety of dimension $2p$, which we will call the universal compactified Jacobian.

(2.8) Claim. The Euler number $e(\bar{\mathcal{J}}^p\mathcal{C})$ is the number of rational curves in $\mathfrak{L}$, where each rational curve C is counted with multiplicity $e(\bar{J}^pC)$.

Proof. The claim relies on the fact that for all $[C] \in \mathfrak{L}$, the fibre $\pi^{-1}([C]) = \bar{J}^pC$ has non-vanishing Euler number if and only if C is rational, see Propositions (2.10) and (2.13) in Section 2.3 below.

The rest is essentially an elaboration of Lemma (2.3) and its proof. There exists a stratification $\mathfrak{L} = \coprod_\alpha \Sigma_\alpha$ by locally closed subsets such that π is topologically locally trivial over each stratum Σ_α [42]. Then,

$$e(\bar{\mathcal{J}}^p\mathcal{C}) = \sum_\alpha e\big(\pi^{-1}(\Sigma_\alpha)\big) \tag{2.8.1}$$

by the additivity formula, and for each α one has

$$e\big(\pi^{-1}(\Sigma_\alpha)\big) = e(\Sigma_\alpha) \times e(\bar{J}_\alpha)$$

where $\bar{J}_\alpha$ stands for the fibre of π over any point in the stratum Σ_α. Since $e(\bar{J}^pC) = 0$ if the curve C is not rational, the sum in (2.8.1) reduces to the sum over all α such that the curves parametrized by the stratum Σ_α are rational. Now, the union of these strata is a finite set (see, e.g., [12, Prop. (4.7)]), which we shall denote by $\mathfrak{L}_{\mathrm{rat}}$, and we finally find

$$e(\bar{\mathcal{J}}^p\mathcal{C}) = \sum_{[C] \in \mathfrak{L}_{\mathrm{rat}}} e(\bar{J}^pC), \tag{2.8.2}$$

which proves the claim. □

Let $S^{[p]}$ be the component of the Hilbert scheme of S parametrizing 0-dimensional subschemes of length p. It is a smooth Hyperkähler variety of dimension $2p$, see [3].

(2.9) Claim. The universal compactified Jacobian $\bar{\mathcal{J}}^p\mathcal{C}$ and the Hilbert scheme $S^{[p]}$ have the same Euler numbers, given by Formula (2.1.1).

Proof. First, the universal compactified Jacobian $\bar{\mathcal{J}}^p\mathcal{C}$ is smooth and Hyperkähler, as noted in [30, Example 0.5], because it is a connected component of the moduli space of simple sheaves on the $K3$ surface S. It is sometimes called the Beauville–Mukai system.

Next, let us prove that $\bar{\mathcal{J}}^p\mathcal{C}$ and $S^{[p]}$ are birational. This will imply, by a theorem of Huybrechts [22, p. 65], that $\bar{\mathcal{J}}^p$ and $S^{[p]}$ are deformation equivalent, hence share the same Betti numbers, so that $e(\bar{\mathcal{J}}^p) = e(S^{[p]})$.

There is an open subset $U \subseteq \bar{\mathcal{J}}^p\mathcal{C}$ whose points are pairs (C, M) such that C a smooth (genus p) member of $\mathfrak{L}$ and M is a non-special, degree p, line bundle on C. For such a pair one has $h^0(C, M) = 1$, and the unique divisor D in the complete linear system $|M|$ may be seen as a point of $S^{[p]}$: this gives a regular map $U \to S^{[p]}$. On the other hand, the fact that $h^1(C, \mathcal{O}_C(D)) = 0$ implies by geometric Riemann–Roch that D spans the whole space in the canonical embedding of C, hence C is the only member of $\mathfrak{L}$ that contains D. Therefore, the map $(C, M) \in U \mapsto D$ is injective, and this ends the proof of the birationality of $\bar{\mathcal{J}}^p\mathcal{C}$ and $S^{[p]}$.

Finally, the fact that $e(S^{[p]})$ is the coefficient of q^p in the Fourier expansion of $\prod_{n=1}^{\infty}(1-q^n)^{-24}$ as in (2.1.1) follows from the computation by Göttsche of the Betti and Euler numbers of $S^{[p]}$ for any smooth, complex, projective surface [20]. This computation is based on the idea of using the Weil conjectures (proved by Deligne) to translate the question into that of counting the points of $S^{[p]}$ over finite fields. □

Claims (2.8) and (2.9) together imply Theorem (2.1).

2.3 – Compactified Jacobian of an integral curve

Let C be an integral curve. We now turn to the study of the Euler number of the compactified Jacobian $\bar{J}^dC$. We will in particular prove some results used in the proof of Claim (2.8) in Section 2.2. As is well-known, the choice of an invertible sheaf of degree d on C induces an isomorphism between $\bar{J}^dC$ and $\bar{J}C = \bar{J}^0C$, so we will actually consider the latter variety.

(2.10) Proposition. *If C is an integral curve of positive geometric genus, then $e(\bar{J}C) = 0$.*

Proof. There is an exact sequence

$$0 \to H \to JC \to J\tilde{C} \to 0$$

where $\tilde{C}$ is the normalization of C,[3] J denotes the Jacobian Pic^0, and H is a product of copies of $(\mathbf{C}, +)$ and $(\mathbf{C}^*, \times)$ (see, e.g., [25, Thm. 7.5.19]). Since H is divisible, the sequence splits as an exact sequence of Abelian groups, hence $JC = J\tilde{C} \times H$. Since $J\tilde{C}$ is a positive dimensional torus, for all positive integers n, there exists an order n group G_n which is a subgroup of both JC and $J\tilde{C}$.

Then, [4, Lemma 2.1] tells us that for all $M \in G_n$ and $\mathcal{F} \in \bar{J}C$, the two sheaves $\mathcal{F}$ and $\mathcal{F} \otimes M$ are not isomorphic. Thus G_n acts freely on $\bar{J}C$, hence n divides $e(\bar{J}C)$. Since this is true for all n, we conclude that $e(\bar{J}C) = 0$. □

We need the following local construction (cf. [4, §3.6] and the references therein) in order to make $e(\bar{J}C)$ explicit for a rational curve C.

(2.11) Let (C, x) be a germ of curve which we assume to be unibranch (see (2.12) below), and let $\tilde{C}$ be the normalization of C; there is only one point in the preimage of x, which we call x as well. Set $\delta_x = \dim \mathcal{O}_{\tilde{C},x}/\mathcal{O}_{C,x}$ (this is the number by which a singularity equivalent to (C, x) makes the geometric genus drop), and let $\mathbf{c}'_x$ be the ideal $\mathcal{O}_{\tilde{C}}(-2\delta \cdot x)$.[4] We shall consider the two finite-dimensional algebras $A_x := \mathcal{O}_{C,x}/\mathbf{c}'$ and $\tilde{A}_x := \mathcal{O}_{\tilde{C},x}/\mathbf{c}'$.

Let $\mathbf{G}_x$ be the closed subvariety of the Grassmannian $\mathbf{G}(\delta_x, \tilde{A}_x)$ parametrizing those codimension δ_x subspaces of $\tilde{A}_x$ that are also sub-A_x-modules of $\tilde{A}_x$; $\mathbf{G}_x$ may also be seen as the variety parametrizing codimension δ_x sub-$\mathcal{O}_{C,x}$-modules of $\mathcal{O}_{\tilde{C},x}$. It only depends on the completion $\hat{\mathcal{O}}_{C,x}$, hence only on the analytic type of the singularity (C, x).

[3] exceptionally, I do not use the notation $\bar{C}$ in order to avoid unpleasant confusions between $\bar{J}C$ and $J\bar{C}$

[4] we reserve the notation $\mathbf{c}_x$ for the conductor ideal (see [II]), which contains $\mathbf{c}'_x$

(2.12) A curve is *unibranch* if its normalization is a homeomorphism, or equivalently if it is everywhere analytically locally irreducible. All curves C have a *unibranchization* $\breve{\nu} : \breve{C} \to C$: this is the unique partial normalization of C such that $\breve{C}$ is unibranch, and such that all others partial normalizations $\nu' : C' \to C$ with C' unibranch factor through $\breve{\nu}$.

If C is a unibranch curve with singular locus $\Sigma \subseteq C$, the product $\prod_{x\in\Sigma} \mathbf{G}_x$ parametrizes sub-$\mathcal{O}_C$-modules $\mathcal{F} \subseteq \mathcal{O}_{\tilde{C}}$ such that $\dim \mathcal{O}_{\tilde{C},x}/\mathcal{F}_x = \delta_x$ for all x. Such an $\mathcal{F}$ enjoys the property that $\chi(\mathcal{F}) = \chi(\mathcal{O}_C)$, which implies $\mathcal{F} \in \bar{J}C$. This defines a morphism

$$\varepsilon : \prod_{x\in\Sigma} \mathbf{G}_x \to \bar{J}C.$$

Part (ii) in the following statement is proved by showing that ε is a homeomorphism if C is rational, though in general not an isomorphism.

(2.13) Proposition. (i) [4, Prop. 3.3] *If C is an integral curve, then $e(\bar{J}C) = e(\bar{J}\breve{C})$.*
(ii) [4, Prop. 3.8] *If C is a unibranch rational curve with singular locus Σ, then $e(\bar{J}C) = \prod_{x\in\Sigma} e(\mathbf{G}_x)$.*

By (i), one has $e(\bar{J}C) = 1$ for an immersed rational curve C (C is immersed if all its analytic local branches are smooth, equivalently if the unibranchization $\breve{C}$ is smooth). If C is not immersed, then $e(\bar{J}C) = e(\bar{J}\breve{C})$ and the latter number is computed by (ii); it follows from Theorem (2.17) in the next subsection that this number is positive if C is a rational curve lying on a smooth surface.

Part (ii) also shows that for all rational curves C (unibranch or not, thanks to (i)), $e(\bar{J}C)$ is entirely determined by the singularities of C. In fact it only depends on their topological types if C has planar singularities, see Corollary (2.18) below.

(2.14) Example ([4, § 4])**.** If (C, x) is the germ of curve given by the equation $u^p + v^q = 0$ at the origin in the affine plane, with p and q relatively prime, then

$$e(\mathbf{G}_x) = \frac{1}{p+q}\binom{p+q}{p}. \tag{2.14.1}$$

Formula (2.14.1) also gives $e(\mathbf{G}_{x'})$ for all singularities (C', x') equisingular to (C, x) by the aforementioned Corollary (2.18).

In particular, $e(\mathbf{G}_x) = \ell + 1$ for the cuspidal singularity defined by the equation $u^2 + v^{2\ell+1} = 0$, i.e., for an $A_{2\ell}$-double point; on the other hand $A_{2\ell+1}$-double points are immersed, hence $e(\mathbf{G}_x) = 1$ for them. Using Proposition (2.13), (i), to the effect that the local contribution $e(\mathbf{G}_x)$ of a germ (C, x) is the product of the local contributions of all analytic local irreducible branches of (C, x), Formula (2.14.1) is enough to determine the local contribution $e(\mathbf{G}_x)$ of any simple curve singularity, see [4, Prop. 4.5].

(2.15) Remark. The fact that all immersed rational curves count with multiplicity one seems to disagree with the results of Section 2.1, see in particular Lemma (2.5). However, if $|F|$ is a complete pencil of elliptic curves, the assumption that all curves in $|F|$ are integral implies that all rational curves in $|F|$ are curves of arithmetic genus 1 with either a node or an ordinary cusp as their unique singular point, and in this case the two multiplicities predicted by Proposition (2.13) and Lemma (2.5) do agree.

(2.16) Remark. Returning to the case of a p-dimensional linear system $\mathfrak{L}$ with all members integral, Beauville deems "rather surprising" the fact that "some highly singular [immersed] curves count with multiplicity one", and considers the case $p = 2$ to provide a confirming example.

(2.16.1) In hindsight, a good conceptual explanation comes from the fact that the numbers N^p should be seen as counting stable maps rather than embedded curves, since stable maps don't make any difference between nodal and arbitrary immersed curves: if p is a singular point of a curve C and p' is a point of the normalization $\tilde{C}$ mapping to p, analytically locally at p' one does not see how the local branch of C at p corresponding to p' intersects the other local branches (one does see however if this branch of C is singular, by looking at the differential of the map $\tilde{C} \to C$). This heuristic argument is confirmed by Theorem (2.20) in the next subsection, see also Lemma (3.4).

(2.16.2) Another comforting observation is that the equigeneric deformation space of an immersed singularity is smooth, as we shall see in (2.19) in the next subsection. Indeed, let C be a rational member of $\mathfrak{L}$. There exists an analytic neighbourhood of $[C]$ in $\mathfrak{L}$ and a map ϕ from U to the semi-universal deformation space B of the singularities of C, which encodes the singularities of the curves neighbouring C in $\mathfrak{L}$. Then, the locus of rational curves in U is the inverse image by ϕ of the equigeneric locus $\mathrm{EG}(C)$ in B, hence it is reasonable to imagine that the multiplicity of C in the count of rational curves is the multiplicity of the 0-cycle $\phi^*\mathrm{EG}(C)$ at the point $[C] \in U$. The fact that rational curves in $\mathfrak{L}$ are finitely many implies as in (1.4) that the intersection $\phi(U) \cap \mathrm{EG}(C)$ has the expected dimension. If we assume furthermore that ϕ is an immersion and $\phi(U)$ intersects $\mathrm{EG}(C)$ transversely, then the multiplicity of $\phi^*\mathrm{EG}(C)$ at $[C]$ equals the multiplicity of the origin of B as a point of $\mathrm{EG}(C)$.

In the upshot, it is plausible that a rational curve counts with multiplicity equal to the multiplicity of the origin in the equigeneric deformation space of its singularities. Since the latter multiplicity is one if all singularities of C are immersed, it is plausible that any immersed rational curve counts with multiplicity one. In fact, we shall see in Section 2.4 that things happen exactly as described above.

2.4 – Two fundamental interpretations of the multiplicity

We now conclude Section 2 by stating two enlightening geometric interpretations of the local multiplicities $e(\mathbf{G}_x)$ defined in the previous subsection, due to Fantechi, Göttsche and van Straten [14].

In the present subsection we only consider *planar* curve singularities (C, x), i.e., singularities that may be realized by the germ of a plane curve. All singularities occuring on a curve that lies on a smooth surface are of this kind.

(2.17) Theorem ([14, Thm. 1])**.** *Let (C, x) be a reduced, planar, curve singularity, and let $\mathbf{G}_x$ be as in (2.11). Then the topological Euler number $e(\mathbf{G}_x)$ equals the multiplicity at the point $[(C, x)]$ of the equigeneric locus $\mathrm{EG}(C, x)$ in the semi-universal deformation space of the singularity (C, x).*

Recall that the *equigeneric locus* $\mathrm{EG}(C, x)$ is defined as the reduced subscheme of the semi-universal deformation space of the singularity (C, x) supported on those points corresponding to singularities with the same δ-invariant as (C, x). In practice this is the closure of the locus along which the singularity deforms to δ nodes, where δ is the number by which the singularity makes the geometric genus drop. See, e.g., [II] for more details.

Together with Proposition (2.13), Theorem (2.17) implies that for a rational, integral, curve C with only planar singularities, the Euler number $e(\bar{J}C)$ of the compactified Jacobian of C equals the multiplicity at the point $[C]$ of the equigeneric stratum $\mathrm{EG}(C)$ in a semi-universal deformation space of the singularities of C. See (2.16.2) above for a discussion of why this provides a reasonable notion of multiplicity for C in a count of rational curves.

Theorem (2.17) has been generalized by Shende [40], to the effect that the multiplicity at $[C]$ of all δ-constant strata in a semi-universal deformation space of the singularities of C are

determined by the Euler numbers of the Hilbert schemes of points on C (for a curve C of arbitrary genus).

On the other hand, Maulik [27] has proved that the Euler numbers of the Hilbert schemes of points $(C,x)^{[n]}$ of a planar curve singularity only depend on the topological type of the singularity. This implies the following.

(2.18) Corollary (see [27, § 6.5])**.** *Let (C,x) be a reduced, planar, curve singularity. The multiplicities at the point $[(C,x)]$ of the δ-constant (a.k.a. constant genus) strata of the semi-universal deformation space of the singularity (C,x) only depend on the topological type (or equisingularity class) of (C,x).*

(2.19) As a corollary of Theorem (2.17) and Proposition (2.13), one obtains that if (C,x) is an immersed planar curve singularity, then the equigeneric locus $\mathrm{EG}(C,x)$ in the space of semi-universal deformations is smooth at the point $[(C,x)]$. Indeed, note that for all reduced plane curve singularities (C,x), there exists a rational curve $\tilde{C}$ with (C,x) as its only singularity (this follows for instance from [29]), and then one has $e(\mathbf{G}_x) = e(\bar{J}\tilde{C})$.

In fact the converse holds as well, i.e., for a reduced planar curve singularity (C,x), the equigeneric locus $\mathrm{EG}(C,x)$ is smooth at the point $[(C,x)]$ if and only if (C,x) is immersed, see [13, Proposition (4.17), (2)].

Now let us turn to the second interpretation of the multiplicity $e(\mathbf{G}_x)$, or rather of $e(\bar{J}C) = \prod_{x\in\Sigma} e(\mathbf{G}_x)$. It tells us that what the Yau–Zaslow formula (2.1.1) really computes is the numbers of genus 0 stable maps realizing primitive classes on $K3$ surfaces. Recall that if C is an integral curve of geometric genus g which is a subvariety of a complex variety X, then $\overline{M}_g(X,[C])$ is the space of genus g stable maps with target X realizing the homology class $[C] \in H_2(C,\mathbf{Z})$.

(2.20) Theorem ([14, Thm. 2] and [45])**.** *Let C be an integral rational curve contained in a smooth $K3$ surface S. Then the topological Euler number $e(\bar{J}C)$ equals the length of the space of stable maps $\overline{M}_0(S,[C])$ at the closed point corresponding to the normalization of C.*

Yifan Zhao has found a problem in the proof of [14, Thm. 2], and provides a corrected proof in the preprint [45], to which I refer for the full details.

Fantechi–Göttsche–van Straten first consider the space $\overline{M}_g(C,[C])$ of stable maps to C itself which realize the fundamental class $[C]$. It is a 0-dimensional scheme, which contains a single closed point, corresponding to the normalization $\nu : \bar{C} \to C$. If C is an isolated, genus g curve in a smooth manifold X, then $\overline{M}_g(C,[C])$ is a subscheme of $\overline{M}(X,[C])$ and, at the normalization of C, the length of the former scheme is a lower bound for the length of the latter.

Now, in [14, Thm. 2] the authors claim that for an integral projective curve C of genus g with planar singularities (not necessarily lying on a $K3$ surface), the length of $\overline{M}_g(C,[C])$ equals the multiplicity at $[C]$ of the semi-universal deformation space of the singularities of C. However, [31, §7.2.2] provides a counterexample, namely a rational curve C with planar singularities such that the Euler number $e(\bar{J}C)$ is larger than the length of $\overline{M}_0(C,[C])$ (as we have seen, since C is rational, $e(\bar{J}C)$ equals the multiplicity at $[C]$ of the semi-universal deformation space).

3 – Curves of any genus in a primitive class

3.1 – Reduced Gromov–Witten theories for $K3$ surfaces

(3.1) A vanishing phenomenon. It happens that all Gromov–Witten invariants of $K3$ surfaces are trivial. The fundamental reason for this is that Gromov–Witten invariants are defor-

mation invariant (this is indeed a desirable feature of any well-behaved counting invariants, and the very reason why they are called invariants and not merely numbers, or integrals), and all algebraic $K3$ surfaces deform to non-algebraic ones, which in general do not contain any curve at all.

Somewhat more concretely, the explanation is that the virtual and actual dimensions of the moduli spaces of stable maps on $K3$ surfaces do not match, as we already pointed out in (1.4). Let S be a $K3$ surface, and $C \subseteq S$ an integral curve of geometric genus g. Consider the stable map $f : \bar{C} \to S$ obtained by composing the normalization $\bar{C} \to C$ with the inclusion $C \subseteq S$. The normal sheaf N_f of f is isomorphic to the canonical bundle $\omega_{\bar{C}}$, and therefore $h^0(N_f) = g$ and $h^1(N_f) = 1$. It follows that the virtual dimension of $\overline{M}_g(S, [C])$ is $g - 1$, whereas the curve C actually moves in a g-dimensional family of curves of genus g on S (see [12, § 4.2] for more details). This implies that the Gromov–Witten invariant counting genus g curves on S passing through the appropriate number of points (namely g) vanishes for mere degree reasons.

The following two paths have been successfully followed to circumvent this issue, and define modified invariants for algebraic $K3$ surfaces which capture the relevant enumerative information.

(3.2) Invariants of families of symplectic structures [6, § 2–3]**.** This was chronologically the first workaround to be proposed, and it enabled the counting of curves of any genus in a primitive class on a $K3$ surface reported on in Section 3.2 below.

Let S be a polarized $K3$ surface. The idea here is really to take into account the existence of non-algebraic deformations of S. To this effect, instead of counting curves directly on S, one counts curves in a family of Kähler surfaces defined over the 2-sphere $\mathbf{S}^2$, canonically attached to S, and in which roughly speaking S is the only one to be algebraic, so that all the curves we count are actually concentrated on S.

This family of Kähler surfaces is the twistor family of S (cf. [37, p. 124]): the polarization on S determines a Kähler class α, and Yau's celebrated theorem asserts that there is a unique Kählerian metric g in α with vanishing Ricci curvature. Then the holonomy defines an action of $\mathbf{H}$ (the field of quaternion numbers) on the holomorphic tangent bundle TS by parallel endomorphisms. The quaternions of square -1 define those complex structures on the differentiable manifold S for which the metric g remains Kählerian. There is a 2-sphere worth of such quaternions, and it parametrizes the family we are interested in.

(3.3) Reduced Gromov–Witten theory ([28, 2.2], see also [26])**.** In this second approach, the idea is to plug in the fact that, for a stable map f as in (3.1), the space $H^1(\bar{C}, N_f)$ although non-trivial does not contain any actual obstruction to deform f as a map with target S, following for instance Ran's results on deformation theory and the semiregularity map. To this end, Maulik and Pandharipande define a suitable perfect obstruction theory which they dub *reduced*, and which provides, following the construction pioneered by Behrend and Fantechi, a reduced virtual fundamental class $[\overline{M}_g(S, \beta)]^{\mathrm{red}}$, for all integers $g \geqslant 0$ and algebraic classes $\beta \in H_2(S, \mathbf{Z})$, which has the appropriate (real) dimension $2g$.

This in turn gives reduced Gromov–Witten integrals, by replacing the virtual fundamental class by its reduced version in the integral (1.2.2). They are invariant under *algebraic* deformations of $K3$ surfaces.

3.2 – The Göttsche–Bryan–Leung formula

In this subsection, I discuss a result of Bryan and Leung [6] giving the number of curves in a primitive linear system on a $K3$ surface that have a given genus and pass through the appropriate

number of base points. The formula was conjectured by Göttsche [21] as a particular case of a more general framework, see [XIV].

Let

$$N_g^p := \int_{[\overline{M}_{g,g}(S,L)]^{\mathrm{red}}} \mathrm{ev}_1^*(\mathrm{pt}) \cup \cdots \cup \mathrm{ev}_g^*(\mathrm{pt})$$

be the reduced Gromov–Witten invariant counting curves of genus g in the linear system $|L|$ on a primitive $K3$ surface (S, L) of genus p ($\mathrm{pt} \in H^4(S, \mathbf{Z})$ is the point class, and the $\mathrm{ev}_1, \ldots, \mathrm{ev}_g$ are the evaluation maps $\overline{M}_{g,g}(S, L) \to S$, as in (1.2)). The following result tells us that these invariants do indeed count curves.

(3.4) Lemma. *The invariants N_g^p are strongly enumerative, in the following sense: let (S, L) be a very general primitive $K3$ surface of genus p; then N_g^p is the actual number of genus g curves in $|L|$ passing through a general set of g points, all counted with multiplicity* 1.

Proof. The key fact is that genus g curves on a $K3$ surface move in g-dimensional families, see e.g., [12, Prop. (4.7)]; for $g > 0$, this implies by a deformation argument that the general member of such a family is an immersed curve [12, Prop. (4.8)]. The same holds for $g = 0$ by the more difficult result of Chen [8], which asserts that all rational curves in $|L|$ are nodal (this requires the very generality assumption).

Let $\mathbf{x} = (x_1, \ldots, x_g)$ be a general (ordered) set of g points on S. The invariant N_g^p may be computed by integration against a virtual class on the so-called cut-down moduli space $\overline{M}(S, \mathbf{x}) \subseteq \overline{M}_{g,g}(S, L)$, consisting of those genus g stable maps sending the i-th marked point to $x_i \in S$ for $i = 1, \ldots, g$, see [6, Appendix A].

Let $[f : C \to S] \in \overline{M}(S, \mathbf{x})$. Thanks to the generality assumption on (S, L), we may and will assume that L generates the Picard group of S. The condition $f_*C \in |L|$ thus imposes that f_*C is an integral cycle, hence that f contracts all irreducible components of C but one, and restricts to a birational map on the latter component, which we will call C_1. Now the points $x_1, \ldots, x_g$, being general, impose g general independent linear conditions on $|L|$, which implies that the curve $f(C) = f(C_1)$ must have geometric genus at least g by the key fact mentioned at the beginning of the proof. Therefore C_1 as well must have geometric genus at least g, and because of the inequality of arithmetic genera

$$p_a(C_1) \leqslant p_a(C) = g,$$

this implies that C_1 is smooth of genus g; moreover, the stability conditions then imply that $C = C_1$, hence f is the normalization of the integral genus g curve $f(C_1)$.

This already tells us that the space $\overline{M}(S, \mathbf{x})$ is 0-dimensional, and isomorphic as a set to the space of genus g curves in $|L|$ passing through $x_1, \ldots, x_g$. Since $\overline{M}(S, \mathbf{x})$ has the expected dimension, the virtual class on it simply encodes its schematic structure. It follows that the number N_g^p is weakly enumerative, i.e., it gives the number of genus g curves passing through $x_1, \ldots, x_g$ counted with multiplicities.

The fact that these multiplicities all equal 1 follows from the fact that all the curves we consider are immersed, as follows. Let $[f : C \to S] \in \overline{M}(S, \mathbf{x})$ as above. Since f is an immersion, it has normal bundle

$$N_f = f^*\omega_S \otimes \omega_C = \omega_C$$

by [12, (2.3)], hence $\mathrm{h}^0(N_f) = g$ and the full moduli space $\overline{M}_{g,0}(S, L)$ is smooth of dimension g at the point $[f]$. By [12, Lemma (2.5)], there is a surjective map e from a neighbourhood of $[f]$ in $\overline{M}_{g,0}(S, L)$ onto a neighbourhood of $f(C)$ in the locally closed subset of $|L|$ parametrizing genus g curves. By generality of the points $x_1, \ldots, x_g$, the latter space is smooth of dimension g at the point $[f(C)]$. Therefore e is a local isomorphism at $[f]$, and this implies that the scheme $\overline{M}(S, \mathbf{x})$ is reduced at $[f]$, which shows that the stable map f counts with multiplicity 1. □

For all integers $g \geqslant 0$, set

$$F_g(q) := \sum_{p=g}^{+\infty} N_g^p \, q^p \tag{3.5.1}$$

as a formal power series in the variable q, where we set N_0^0 by convention so that F_0 equals the power series of (2.1.1). Beware the shift in degree between the definition (3.5.1) of the series F_g and that given in [6]. Note that $N_g^p = 0$ whenever $p < g$.

(3.5) Theorem (Bryan–Leung)**.** *The power series F_g is the Fourier expansion of*

$$\left(\sum_{n=1}^{+\infty} n\sigma_1(n)\, q^n\right)^g \prod_{m=1}^{+\infty} \frac{1}{(1-q^m)^{24}}, \tag{3.5.2}$$

where $\sigma_1(n) := \sum_{d|n} d$ is the sum of all positive integer divisors of n.

This formula gives the possibility to explicitly compute as many numbers N_g^p as we want. Table 1 (p. 281) gives sample values for small p and g. Note that since columns are indexed by $\delta := p - g$, the Fourier coefficients of a given F_g are read along a diagonal; this gives for instance F_0 as in (2.1.1),

$$F_1(q) = q + 30q^2 + 480q^3 + 5460q^4 + \cdots$$

and so on.

We will discuss the proof of Theorem (3.5) in Section 3.3.

(3.6) Modularity. Formula (3.5.2) implies that $qF_g(q)$ is a quasi-modular form for all g. In the present paragraph I explain what this means; this is also an occasion to introduce notation for further use (I follow [39]). I will however not try to give a meaningful interpretation of the (quasi-)modularity of $qF_g(q)$.

(3.6.1) For every integer $k > 1$, define the k-th *Eisenstein series* to be

$$G_k(z) := \sum_{\substack{(m,n)\in\mathbf{Z}^2:\\ (m,n)\neq(0,0)}} \frac{1}{(m+nz)^{2k}};$$

it is a modular form of weight $2k$ [39, Prop. VII.4], which means that it is holomorphic and $G_k(z)dz^k$ is invariant under the action of $\mathrm{PSL}_2(\mathbf{Z})$. Its Fourier expansion at infinity is

$$G_k(z) = \frac{2^{2k}}{(2k)!} B_k \pi^{2k} + 2\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{+\infty} \sigma_{2k-1}(n) q^n,$$

where $q = e^{2\pi i z}$, $\sigma_k(n) = \sum_{d|n} d^k$, and B_k is the k-th Bernoulli number, defined by the formula

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{+\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}$$

[39, Prop. VII.8]. We set

$$\begin{aligned} E_k(z) := \frac{(2k)!}{2^{2k} B_k \pi^{2k}} G_k(z) &= 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{+\infty} \sigma_{2k-1}(n) q^n \\ &= 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{+\infty} n^{2k-1} \frac{q^n}{1-q^n}. \end{aligned}$$

p \ δ	1	2	3	4	5	6	7	8	9
1	24								
2	30	324							
3	36	480	3200						
4	42	672	5460	25650					
5	48	900	8728	49440	176256				
6	54	1164	13220	88830	378420	1073720			
7	60	1464	19152	150300	754992	2540160	5930496		
8	66	1800	26740	241626	1412676	5573456	15326880	30178575	
9	72	2172	36200	371880	2499648	11436560	36693360	84602400	143184000
10	78	2580	47748	551430	4213332	22116456	81993600	219548277	432841110
11		3024	61600	791940	6808176	40588544	172237344	531065070	1210781880
12			77972	1106370	10603428	71127680	342358560	1205336715	3154067950
13				1508976	15990912	119665872	647773200	2582847180	7698660544
14					23442804	194196632	1172896512	5255204625	17710394230
15						305225984	2041899840	10205262330	38607114200
16							3431986848	19002853575	80149394030
17								34070137272	159184435520
18									303705014550

Table 1: First values of N_g^p ($\delta = p - g$)

(3.6.2) Define

$$\Delta(z) := \big(60G_2(z)\big)^3 - 27\big(140G_3(z)\big)^2,$$

the discriminant of the cubic polynomial $4X^3 - 60G_2X - 140G_3$ divided by 16. It is a modular form of weight 12 vanishing at infinity, and it is a theorem of Jacobi [39, Thm. VII.6] that

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{+\infty} (1-q^n)^{24}.$$

(3.6.3) In the $k = 1$ case, we set

$$G_1(z) := \sum_{n\in\mathbf{Z}} \sum_{\substack{m\in\mathbf{Z}:\\(m,n)\neq(0,0)}} \frac{1}{m+nz^2}$$

(note that the order of summation is significant). It has Fourier expansion at infinity

$$G_1(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{+\infty} \sigma_1(n) q^n$$

[39, § VII.4.4], and we set

$$E_1(z) := \frac{3}{\pi^2} G_1(z) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) q^n.$$

One has the identity [39, § VII.4.4]

$$\frac{d\Delta}{\Delta} = 2\pi i E_1(z) dz.$$

(3.6.4) The function G_1 is *not* a modular form, but still it does satisfy a functional equation close to that equivalent to the invariance of $G_1(z)dz$ under the action of $\mathrm{PSL}_2(\mathbf{Z})$ [5, Prop. 6 p. 19]. For this reason, it is called a *quasi-modular* form.

One may then define the ring of quasi-modular forms as the $\mathbf{C}$-algebra generated by G_1 and the algebra of modular forms (see [5] for a more intrinsic definition). Since the ring of modular forms is generated by G_2 and G_3 [39, Cor. VII.2], the ring of quasi-modular forms may be concretely described as $\mathbf{C}[G_1, G_2, G_3]$. The ring of quasi-modular forms is closed under differentiation by the operator

$$D := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$$

[5, Prop. 15 p. 49].

(3.6.5) Quasi-modularity of $qF_g(q)$. Taking into account the various stunning formulæ above, (3.5.2) may be rewritten as

$$qF_g(q) = \left(-\frac{1}{24} q \frac{dE_1}{dq}\right)^g \left(\frac{\Delta}{(2\pi)^{12}}\right)^{-1},$$

from which it follows that $qF_g(q)$ is a quasi-modular form, with a simple pole at infinity (i.e., at $q = 0$) if $g = 0$.

3.3 – Proof of the Göttsche–Bryan–Leung formula

This subsection is dedicated to the proof of Theorem (3.5). As the latter really is about counting actual curves (i.e., subvarieties of the surface), as Lemma (3.4) attests, one may prefer in the first place to avoid the complications of Gromov–Witten theory to prove it. Bryan and Leung's proof however goes the orthogonal direction: it fundamentally relies on the agile possibilities featured in Gromov–Witten theory, precisely as a reward to the aforementioned complications.

(3.7) Degeneration to an elliptic $K3$. Let $g \geqslant 0$, $p > 0$ be integers. By deformation invariance, we are free to compute the number N_g^p on our favourite primitively polarized $K3$ surface (S, L) of genus p. We let S be an elliptic $K3$ surface with a section E, and denote by F the class of the ellitic fibres; the intersection form (or its restriction to the subspace $\langle E, F\rangle$) is given in the basis (E, F) by the matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

We set $L := \mathcal{O}_S(E + pF)$. Then $L^2 = 2p - 2$, and L is a primitive polarization of genus p. We shall compute the numbers N_g^p on the pair (S, L).

(3.8) The linear system $|E + pF|$ has dimension p and consists solely of reducible curves $E + F_1 + \cdots + F_p$, where the F_i's are in the class F. From this it readily follows that if we fix a general set of g points $\mathbf{y} = (y_1, \ldots, y_g)$ on S, then the moduli space

$$\overline{M}_{g,\mathbf{y}}(S, E + pF) := \overline{M}_{g,g}(S, E + pF) \cap \mathrm{ev}_1^*(y_1) \cap \ldots \cap \mathrm{ev}_g^*(y_g)$$

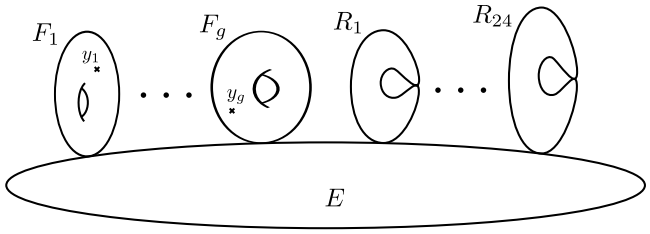
of genus g stable maps passing through the points $y_1, \ldots, y_g$ decomposes as the disjoint union

$$\coprod_{\mathbf{a},\mathbf{b}} \overline{M}_{\mathbf{a},\mathbf{b}},$$

where $\mathbf{a} = (a_1, \ldots, a_{24})$ and $\mathbf{b} = (b_1, \ldots, b_g)$ range through $\mathbf{Z}_{\geqslant 0}^{24}$ and $\mathbf{Z}_{>0}^{g}$ respectively, subject to the condition that $\sum_i a_i + \sum_j b_j = p$, and $\overline{M}_{\mathbf{a},\mathbf{b}}$ is the moduli space of genus g stable maps $(f : C \to S, x_1, \ldots, x_g)$ such that $f(x_j) = y_j$ for $j = 1, \ldots, g$ and

$$f_*C = E + \sum_{i=1}^{24} a_i R_i + \sum_{j=1}^{g} b_j F_j,$$

where $R_1, \ldots, R_{24}$ are the 24 rational members of the pencil $|F|$, and F_j being the unique member of $|F|$ containing the point y_j for all $j = 1, \ldots, g$ (see figure below); note that we may and do assume that all members of $|F|$ are irreducible, and the R_i's are 1-nodal.



(3.9) Partition function. For all positive integers n, we let $p(n)$ be the *number of partitions* of n, i.e., the number of ways to write $n = \lambda_1 + \cdots + \lambda_k$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 1$ (k is not fixed). The numbers $p(n)$ may be computed using the generating series

$$\begin{aligned} 1 + \sum_{n=1}^{+\infty} p(n)\, t^n &= (1 + t^1 + t^{1+1} + t^{1+1+1} + \cdots) \times (1 + t^2 + t^{2+2} + t^{2+2+2} + \cdots) \\ &\qquad \times (1 + t^3 + t^{3+3} + t^{3+3+3} + \cdots) \times \cdots \\ &= \prod_{n=1}^{+\infty} \frac{1}{1 - t^n}. \end{aligned} \tag{3.9.1}$$

See [15, Chap. 4] for much more about this.

The following proposition is the key to Formula (3.5.2) for an elliptic $K3$ surface (S, L) as above. It yields Formula (3.5.2) after a series of manipulations of generating functions, involving Identity (3.9.1). I don't reproduce them here, and refer to [6, p. 383] for details. The rest of this subsection is dedicated to the proof of the proposition.

(3.10) Proposition. *The contribution of $\overline{M}_{\mathbf{a},\mathbf{b}}$ to N_g^p is*

$$\Big(\prod\nolimits_{i=1}^{24} p(a_i)\Big)\Big(\prod\nolimits_{j=1}^{g} b_j \sigma_1(b_j)\Big). \tag{3.10.1}$$

(3.11) Enumeration of elliptic multiple covers. We first explain the factors $b_j\sigma_1(b_j)$ in (3.10.1); they are of a combinatorial nature.

Let $f : C \to S$ be a member of $\overline{M}_{\mathbf{a},\mathbf{b}}$. It is necessarily shaped as follows: the curve C consists of (i) a smooth rational curve mapped isomorphically to the section $E \subseteq S$, which we will abusively call E as well, (ii) g smooth elliptic curves $G_1, \ldots, G_g$, pairwise disjoint and each attached at one point to E, and (iii) 24 trees of smooth rational curves $T_1, \ldots, T_{24}$, pairwise disjoint, each disjoint from the G_j's and attached to E at one point. For all $j = 1, \ldots, g$, f maps G_j to the elliptic fibre F_j with degree b_j, and there is a marked point $x_j \in G_j$ mapped to y_j; for all $i = 1, \ldots, 24$, one has $f_*T_i = a_i R_i$.

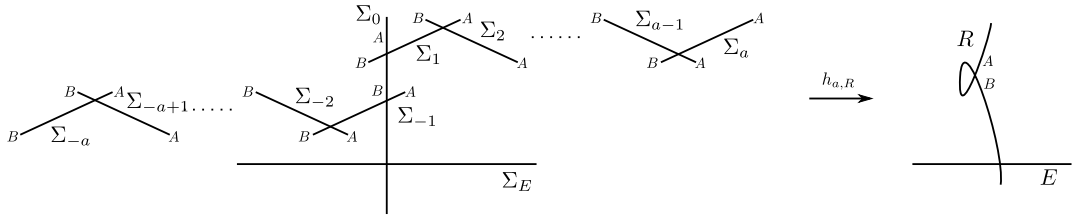
For all j, we may fix the intersection point with E as the origin of G_j and F_j respectively, which makes $f|_{G_j} : G_j \to F_j$ a degree b_j homomorphism of elliptic curves. Such homomorphisms are in $1:1$ correspondence with index b_j sublattices of the lattice defining F_j as a complex torus, and the number of such sublattices is $\sigma_1(b_j)$ [39, § VII.5.2]. In addition, there are b_j possibilities to choose the marked point x_j in the preimage of y_j in G_j.

Once the data of the homomorphisms $G_j \to F_j$ and the marked points $x_j \in G_f$ is fixed, the corresponding sub-moduli space of $\overline{M}_{\mathbf{a},\mathbf{b}}$ decomposes as a product

$$\prod_{i=1}^{24} \overline{M}_{a_i\mathbf{e}_i,0},$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_{24} = (0, \ldots, 0, 1) \in \mathbf{Z}^{24}$. The moduli space $\overline{M}_{\mathbf{a},\mathbf{b}}$ thus consists of $\prod_{j=1}^{g} b_j\sigma_1(b_j)$ disjoint copies of the space $\prod_{i=1}^{24} \overline{M}_{a_i\mathbf{e}_i,0}$, and the rest of the proof of Proposition (3.10) consists in showing that each of those contributes by $\prod_{i=1}^{24} p(a_i)$ to N_g^p.

(3.12) Description of $\overline{M}_{a_i\mathbf{e}_i,0}$. Let us start by defining a model stable map $h_{a,R} : \mathbf{\Sigma}_a \to S$ for all positive integers a and 1-nodal rational curves $R \in \{R_1, \ldots, R_{24}\}$. The curve $\mathbf{\Sigma}_a$ is a tree of $2a + 2$ smooth rational curves $\Sigma_E, \Sigma_{-a}, \ldots, \Sigma_0, \ldots \Sigma_{+a}$ as depicted on the figure below.



The map $h_{a,R}$ is chosen so that it restricts to an isomorphism $\Sigma_E \cong E$ (hence from now on we will denote Σ_E by E) and to $2a+1$ copies $\Sigma_i \to R$ of the normalization of the 1-nodal rational curve R, in such a way that it is everywhere a local isomorphism between $\mathbf{\Sigma}_a$ and $E \cup R$. Concretely, the latter requirement is that locally at every node $\Sigma_i \cap \Sigma_{i+1}$, the map $h_{a,R}$ should send Σ_i to one of the two local branches of R at its node and Σ_{i+1} to the other.

There are basically two possible (indifferent) choices. We indicate one of them on the above figure by decorating each local branch at a node of $\Sigma_{-a} \cup \ldots \cup \Sigma_{+a}$ with a letter A or B, with the convention that A and B label the two local branches of R at its node.

(3.13) Lemma. *For every $(f : C \to S) \in \overline{M}_{a\mathbf{e}_i,0}$, there is a unique lift of f to a stable map $\tilde{f} : C \to \mathbf{\Sigma}_a$, meaning that $f = h_{a,R_i} \circ \tilde{f}$.*

Proof. The curve C is necessarily made of a smooth rational component mapped isomorphically to E, which we denote by E as well, and a tree of smooth rational curves T attached at one point to E, as in (3.11). For each irreducible component C_s of T, one has $f_*C_s = k_sR_i$ for some non-negative integer k_s. We determine the lift $\tilde{f}$ by exploring the dual graph of T along all possible paths from its root to one of its leaves, as follows.

There is a unique irreducible component C_0 of T intersecting E; we call the corresponding vertex of the dual graph of T the *root* of the dual graph. The *leaves* are those vertices corresponding to irreducible components of T intersecting only one other irreducible component. Now the lift $\tilde{f}$, should it exist, necessarily maps C_0 to Σ_0, and there is a unique suitable map $C_0 \to \Sigma_0$ (possibly contracting C_0 to the point $\Sigma_0 \cap E$) by the universal property of normalization.

Suppose a putative lift $\tilde{f}$ is determined on an irreducible component C_s of T, and consider an arbitrary component C_{s+1} of T intersecting C_s at one point z_{s+1}. We claim that the behaviour of $\tilde{f}$ on C_{s+1} is uniquely determined by the already constructed piece $\tilde{f}\big|_{C_s}$. If C_{s+1} is contracted by f, this is clear; otherwise it is enough, by the universal property of normalization, to determine which component of $\mathbf{\Sigma}_a$ the lift $\tilde{f}$ should map C_{s+1} to. If $\tilde{f}(z_{s+1})$ is a smooth point of $\mathbf{\Sigma}_a$, then $\tilde{f}$ has to map C_{s+1} to the same component it maps C_s to; if not, $\tilde{f}(z_{s+1})$ is a node $\Sigma_{t_s} \cap \Sigma_{t_s+\varepsilon}$, $\varepsilon \in \{\pm 1\}$, and $\tilde{f}$ has to map C_{s+1} to Σ_{t_s} or $\Sigma_{t_s+\varepsilon}$, depending on which of the local branches A and B of R_i at its node the local branch of C_{s+1} at z_{s+1} is mapped to by f.

This discussion shows that one may algorithmically construct an $\tilde{f}$ such that $h_{a,R_i} \circ \tilde{f} = f$, and that there is a unique such lift. (Note that the chain of rational curves $\Sigma_{-a} \cup \ldots \cup \Sigma_a$ is long enough for the construction to go through without any trouble: since $f_*C = E + aR_i$, if at some point during the algorithm we hit an irreducible component C_s of C that has to be mapped to Σ_{-a+1} or Σ_{a-1}, then the push-forward by f of the sum of all components already visited by the algorithm fills out the class aR_i, hence all components of C not yet touched by the algorithm are contracted by f, and we don't have to go beyond Σ_{-a+1} or Σ_{a-1} in $\mathbf{\Sigma}_a$). □

Using Lemma (3.13), one can associate to every stable map $(f : C \to S) \in \overline{M}_{a\mathbf{e}_i,0}$ a combinatorial datum called an *admissible sequence* of weight a: this is a sequence of $2a+1$ non-negative integers $\mathbf{k} = (k_l)_{-a \leqslant l \leqslant a}$ with the property that there exist $m, n \geqslant 0$ such that $k_l = 0$ for all $l < -m$ and $l > n$, and $k_l > 0$ for all $l = -m, \ldots, n$. The association goes as

follows: let $h_{a,R_i} \circ \tilde{f}$ be the factorization of f, and write

$$\tilde{f}_* C = E + \sum_{s=-a}^{a} k_s \Sigma_s.$$

It follows from the construction of $\tilde{f}$ in the proof of Lemma (3.13) that $(k_{-a}, \ldots, k_a)$ is an admissible sequence of weight a. This gives the following.

(3.14) Lemma. *The moduli space $\overline{M}_{a\mathbf{e}_i,0}$ decomposes as the disjoint union*

$$\overline{M}_{a\mathbf{e}_i,0} = \coprod_{\mathbf{k}} \overline{M}_{\mathbf{k}},$$

where $\mathbf{k}$ ranges over all weight a admissible sequences, and $\overline{M}_{\mathbf{k}}$ is the sub-moduli space of $\overline{M}_{a\mathbf{e}_i,0}$ parametrizing those f with associated admissible sequence $\mathbf{k}$.

(3.15) Identification of the virtual class. It follows from the above that each moduli space $\overline{M}_{\mathbf{a},\mathbf{b}}$ decomposes as a disjoint union of various products $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$. A key point in the proof of Bryan and Leung is the explicit identification of the restriction to the product $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$ of the (reduced) virtual class giving rise to the invariant N_g^p [6, § 5.2]. This is by far the most demanding part of their article, and I will not attempt to give any idea of the proof.

The upshot is that (i) the virtual class on $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$ is a product of virtual classes on the various factors, and (ii) the virtual class on the factor $\overline{M}_{\mathbf{k}_i}$ is computed by means of a *virtual tangent bundle* on the target curve $\mathbf{\Sigma}_{a_i}$. This virtual tangent bundle T is the vector bundle on $\mathbf{\Sigma}_{a_i}$ defined by the conditions that it is isomorphic to $h_{a_i,R_i}^* T_S$ on $\Sigma_{-a} \cup \ldots \cup \Sigma_a$ and to $T_E \oplus \mathcal{O}_E(-1)$ on E.

Note that $h_{a_i,R_i}^* T_S$ restricts to $T_E \oplus \mathcal{O}_E(-2)$ on E; the correction made to define T corresponds to the fact that we want to kill the obstruction space $H^1(C, N_f)$ as we know the actual obstruction space is trivial although $H^1(C, N_f)$ is not, see (3.3).

(3.16) Planar model. Let a be a non-negative integer. Thanks to the result discussed in (3.15), it is possible to construct a model for $\mathbf{\Sigma}_a$ embedded in a familiar surface P, such that the actual deformation theory of $\mathbf{\Sigma}_a$ in P is isomorphic to the reduced virtual theory leading to the invariants N_g^p. This will eventually let us compute the contribution of the $\overline{M}_{\mathbf{k}_i}$'s (hence also of the $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$'s) to the invariant N_g^p.

Let p, p_{-1}, p_1 be three distinct points lying on a line in the projective plane $\mathbf{P}^2$. Let $P_1 \to \mathbf{P}^2$ be the blow-up at these three points, and call E the exceptional divisor over p, Σ_0 the proper transform of the line through p, p_{-1}, p_1, and Σ_{-1}, Σ_1 the exceptional divisors over p_{-1}, p_1 respectively. Next, recursively for all $s = 1, \ldots, a$, we perform the blow-up $P_{s+1} \to P_s$ at two general points of Σ_{-s} and Σ_s respectively, and let Σ_{-s-1} and Σ_{s+1} be the two corresponding exceptional divisors; we call $E, \Sigma_{-s}, \ldots, \Sigma_s$ the proper transforms in P_{s+1} of the curves with the same name in P_s.

The curve $\Sigma_{-a} \cup \ldots \cup \Sigma_0 \cup \ldots \cup \Sigma_a$ in P_{a+1} (note that this excludes the last two exceptional curves Σ_{-a-1} and Σ_{a+1}) is isomorphic as an abstract curve to $\mathbf{\Sigma}_a$. Moreover, the tangent bundle of P_{a+1} restricts to $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ on E, and to $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$ on $\Sigma_{-a}, \ldots, \Sigma_a$; it is therefore isomorphic to the virtual tangent bundle T introduced in (3.15). Then, Bryan–Leung prove the following.

(3.17) Lemma. *For all admissible sequences $\mathbf{k} = (k_{-a}, \ldots, k_0, \ldots, k_a)$, the "local" contribution $\int_{[\overline{M}_{\mathbf{k}}]^{\mathrm{vir}}} 1$ of $\overline{M}_{\mathbf{k}}$ to the invariant N_g^p equals the ordinary genus 0 Gromov–Witten integral $\int_{[\overline{M}_0(P_{a+1},\beta)]^{\mathrm{vir}}} 1$ for the class $\beta = E + \sum_{-a}^{a} k_s \Sigma_s$.*

The lemma follows from (3.15) and the isomorphism between the restriction of $T_{P_{a+1}}$ and T to $\mathbf{\Sigma}_a$, provided the two moduli spaces $\overline{M}_{\mathbf{k}}$ and the ordinary $\overline{M}_0(P_{a+1}, E + \sum_{-a}^{a} k_s \Sigma_s)$ are isomorphic as sets. Bryan and Leung are able to prove this by elementary arguments using a more elaborate set-up: they start with a linear $\mathbf{C}^*$ action (not $(\mathbf{C}^*)^2$) on $\mathbf{P}^2$ leaving the line $\langle p, p_{-1}, p_1 \rangle$ and a point q fixed, and this $\mathbf{C}^*$ action survives in P_{a+1}. See [6, Lemma 5.7] for the proof.

Finally, by deformation invariance of Gromov–Witten invariants, we may transport the computation on the projective plane blown-up at $2a+3$ general points. This gives the following.

(3.18) Lemma. *Let $\tilde{\mathbf{P}}^2$ be the blow-up of $\mathbf{P}^2$ at a general set of $2a+3$ points, with exceptional divisors $E, E_{-a-1}, \ldots, E_{-1}, E_1, \ldots, E_{a+1}$ (all (-1)-curves of course). We call H the pull-back of the line class. Then for all admissible sequences $\mathbf{k} = (k_{-a}, \ldots, k_0, \ldots, k_a)$, the "local" contribution $\int_{[\overline{M}_{\mathbf{k}}]^{\mathrm{vir}}} 1$ of $\overline{M}_{\mathbf{k}}$ to the invariant N_g^p equals the ordinary genus 0 Gromov–Witten integral $\int_{[\overline{M}_0(\tilde{\mathbf{P}}^2, \beta_{\mathbf{k}})]^{\mathrm{vir}}} 1$ for the class*

$$\begin{aligned}\beta_{\mathbf{k}} &= E + \sum_{s=1}^{a} k_{-s}(E_{-s} - E_{-s-1}) + k_0(H - E - E_1 - E_{-1}) + \sum_{s=1}^{a} k_s(E_s - E_{s+1}) \\ &= k_0 H - (k_0 - 1)E - \sum_{s=1}^{a}(k_{s-1} - k_s)E_s - k_a E_{a+1} - \sum_{s=1}^{a}(k_{-s+1} - k_{-s})E_{-s} - k_{-a}E_{-a-1}.\end{aligned}$$

Note that

$$(k_0 - 1) + \sum_{s=1}^{a}(k_{s-1} - k_s) + k_a + \sum_{s=1}^{a}(k_{-s+1} - k_{-s}) + k_{-a} = 3k_0 - 1,$$

so the virtual class $[\overline{M}_0(\tilde{\mathbf{P}}^2, \beta_{\mathbf{k}})]^{\mathrm{vir}}$ has dimension 0, see (3.19) below.

(3.19) The computation on the blown-up plane. The Gromov–Witten invariants gotten in Lemma (3.18) are computable in practice thanks to the analysis of genus 0 Gromov–Witten invariants of blow-ups of $\mathbf{P}^2$ carried out by Göttsche and Pandharipande [19].

Let n be a non-negative integer, $d, \alpha_1, \ldots, \alpha_n$ integers. We call $N(d; \alpha_1, \ldots, \alpha_n)$ the genus 0 Gromov–Witten invariant for $\tilde{\mathbf{P}}^2$ and the class $dH - \sum_i \alpha_i E_i$ (mind the minus sign, introduced for obvious geometric reasons), where $\tilde{\mathbf{P}}^2$ is the projective plane blown-up at a general set of n points, H the pull-back of the line class, and $E_1, \ldots, E_n$ the exceptional (-1)-curves. The corresponding moduli space of stable maps has virtual dimension $3d - 1 - \sum_i \alpha_i$; if this is positive, then we impose in addition the appropriate number of point constraints, and if this is negative, then we set the invariant to 0.

These invariants enjoy the following properties (see [19]):
(i) $N(1) = 1$;
(ii) $N(d; \alpha_1, \ldots, \alpha_{n-1}, 1) = N(d; \alpha_1, \ldots, \alpha_{n-1}, 0) = N(d; \alpha_1, \ldots, \alpha_{n-1})$;
(iii) $N(d; \alpha_1, \ldots, \alpha_n) = N(d; \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$ for any permutation $\sigma \in \mathfrak{S}_n$;
(iv) $N(d; \alpha_1, \ldots, \alpha_n) = 0$ if there is an index i for which $\alpha_i < 0$, unless $dH - \sum_i \alpha_i E_i = E_{i_0}$ for some i_0 in which case the invariant is 1;
(v) The invariant $N(d; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n)$ is invariant under the isomorphism given by the quadratic Cremona transformation corresponding to the linear system $|2H - E_1 - E_2 - E_3|$, i.e.,

$$\begin{aligned}&N(d; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n) = \\ &\qquad N(2d - \alpha_1 - \alpha_2 - \alpha_3, d - \alpha_2 - \alpha_3, d - \alpha_1 - \alpha_3, d - \alpha_1 - \alpha_2, \alpha_4, \ldots, \alpha_n).\end{aligned}$$

We need the following definition to state the result. An admissible sequence $(k_{-a},\dots,k_0,\dots,k_a)$ is 1-*pyramidal*[5] if

$$k_s - 1 \leqslant k_{s+1} \leqslant k_s \qquad \text{and} \qquad k_{-s} - 1 \leqslant k_{-s-1} \leqslant k_{-s}$$

for all $s = 0,\dots,a-1$.

(3.20) Lemma. *Let* $\mathbf{k} = (k_{-a},\dots,k_0,\dots,k_a)$ *be an admissible sequence of weight* a*. Then the genus* 0 *Gromov–Witten invariant*

$$\begin{aligned} N(\mathbf{k}) &:= \int_{[\overline{M}_0(\tilde{\mathbf{P}}^2,\beta_{\mathbf{k}})]^{\mathrm{vir}}} 1 \\ &= N(k_0;\, k_0-1,\, k_{-a},\, k_{-a+1}-k_{-a},\, \dots,\, k_0-k_{-1},\, k_0-k_1,\, \dots,\, k_{a-1}-k_a,\, k_a) \end{aligned}$$

equals 1 *if* $\mathbf{k}$ *is* 1*-pyramidal, and* 0 *otherwise.*

Proof. We first show that $N(\mathbf{k}) = 0$ if $\mathbf{k}$ is not 1-pyramidal. Since $k_0 > 0$ by definition of an admissible sequence, it follows from Property (iv) above that $N(\mathbf{k}) \neq 0$ implies

$$k_{-s} \leqslant k_{-s+1} \qquad \text{and} \qquad k_{s-1} \geqslant k_s \tag{3.20.1}$$

for all $s = 1,\dots,a$. Next, we apply for all $s = 2,\dots,a-1$ the Cremona transformation defined by $|2H - E - E_1 - E_{s+1}|$ and get

$$N(\mathbf{k}) = N(1 + k_1 + k_{s+1} - k_s;\, \underbrace{k_1 - k_s + k_{s+1}}_{E},\, \dots,\, \underbrace{1 - k_s + k_{s+1}}_{E_1},\, \dots,\, \underbrace{1 - k_0 + k_1}_{E_{s+1}},\, \dots)$$

by Property (v) above. If $N(\mathbf{k}) \neq 0$, we have $k_s \leqslant k_1$ by (3.20.1), hence $1 + k_1 + k_{s+1} - k_s > 0$, and Property (iv) then implies that $k_1 \geqslant k_0 - 1$ and $k_{s+1} \geqslant k_s - 1$. An analogous move shows that $k_{-s-1} \geqslant k_{-s} - 1$ for all $s = 0,\dots,a-1$ if $N(\mathbf{k}) \neq 0$. Thus, the non-vanishing of $N(\mathbf{k})$ implies that $\mathbf{k}$ is 1-pyramidal.

Conversely, assume that $\mathbf{k}$ is 1-pyramidal of weight a. Then $k_a = k_{-a} = 0$ (otherwise the weight exceeds a; we have somehow already made this observation in the course of the proof of Lemma (3.13)), and all coefficients $k_s - k_{s-1}$ and $k_{-s} - k_{-s+1}$ for $s = 1,\dots,a$, equal 0 or 1. It thus follows from Property (ii) that $N(\mathbf{k}) = N(k_0; k_0 - 1)$, which is readily seen to equal 1: the moduli space of genus 0 stable maps $\overline{M}_0\big(\tilde{\mathbf{P}}^2, k_0H - (k_0-1)E\big)$ has only one enumeratively meaningful irreducible component, isomorphic to the family of degree k_0 plane curves with multiplicity $k_0 - 1$ at a fixed point $x_E \in \mathbf{P}^2$, and this is a linear system. □

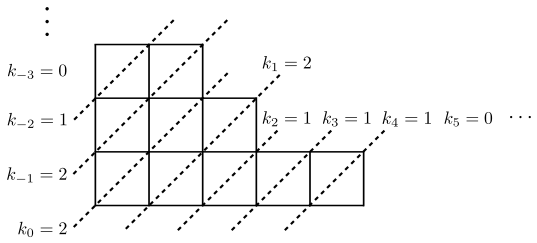
The proof of Theorem (3.5) will be complete once we show the following combinatorial result.

(3.21) Lemma. *The number of* 1*-pyramidal admissible sequences* $(k_{-a},\dots,k_0,\dots,k_a)$ *of weight* a *equals the partition number* $p(a)$ *(cf. (3.9)).*

Indeed, together with Lemmata (3.14), (3.18) and (3.20), this shows that the "local" contribution of $\overline{M}_{a_i\mathbf{e}_i,0}$ equals $p(a_i)$, hence the contribution of each copy of the product $\prod_1^{24}\overline{M}_{a_i\mathbf{e}_i,0}$ equals $\prod_1^{24} p(a_i)$; this proves Proposition (3.10), thanks to the enumeration of elliptic multiple covers performed in (3.11). Finally, as already mentioned, Proposition (3.10) implies Theorem (3.5).

[5] this is called 1-*admissible* in [6].

Proof of Lemma (3.21). Partitions of an integer a are in bijective correspondence with Young diagrams of size a [15, Chap. 4]. We shall exhibit a bijective correspondence between Young diagrams of size a and 1-pyramidal admissible sequences $(k_{-a}, \dots, k_0, \dots, k_a)$ of weight a. We see Young diagrams as embedded in the upper-right quadrant of a Cartesian plane, leaning on both the x and y axes, and with blocks squares of size 1. Given such a Young diagram, we let k_s be the number of blocks on the line $y - x = s$ for all $s = -a, \dots, 0, \dots, a$: see the figure below for an example of this procedure.



We leave it to the reader to check that $(k_{-a}, \dots, k_a)$ is indeed admissible, 1-pyramidal, of weight a, and that this correspondence establishes the announced bijection. □

4 – BPS state counts

In this Section, I discuss why and how curve counting in non-primitive classes imply the use of multiple covers formulæ. This features the generalization of the Yau–Zaslow formula of Theorem (2.1) to non-primitive classes.

4.1 – Rational curves on the quintic threefold

To describe the picture in its simplest form, let us first discuss a question that lies slightly outside the scope of these notes, namely that of counting rational curves on a general quintic hypersurface V of $\mathbf{P}^4$.

(4.1) The Clemens conjecture (see (4.5) below) predicts that there are finitely many such curves of any given degree d, in keeping with the virtual dimension

$$\operatorname{vdim} \overline{M}_g(V, \beta) = (\dim V - 3)(g - 1) - K_V \cdot \beta \tag{4.1.1}$$

being 0 for any homology class on a Calabi–Yau threefold V. This suggests that the number

$$N_d^V := \int_{[\overline{M}_0(V, d\ell)]^{\mathrm{vir}}} 1$$

(where ℓ denotes the homology class of a line) may give the actual number of rational curves of degree d in V. This would be particularly appealing, since the numbers N_d^V may in theory be rigorously computed using the predictions of mirror symmetry; this has been done in practice for small values of d, see [10] for a thorough discussion.

A first objection against such an ideal correspondence is that there may be rational curves with non-trivial infinitesimal deformations, but the Clemens conjecture also predicts that this does not happen.

(4.2) Multiple cover formula. An actual issue preventing N_d^V from directly reflecting the number of degree d rational curves on V is the existence of components of $\overline{M}_0(V, d\ell)$ having dimension larger than expected (i.e., positive dimension). Suppose there is a smooth degree $d_0 = d/k$ rational curve $\Gamma \subseteq V$ for some integer $k > 1$, and let $f_0 : \mathbf{P}^1 \to V$ be the stable map induced by the normalization of Γ. Then, for all degree k covers $\phi_k : \mathbf{P}^1 \to \mathbf{P}^1$, the composed map $f_0 \circ \phi_k$ is a member of $\overline{M}_0(V, d\ell)$, and the locus in $\overline{M}_0(V, d\ell)$ corresponding to such composed maps over Γ is an irreducible variety $M_k(\Gamma)$ of dimension $2k - 2$.

The Aspinwall–Morrison formula asserts that the corresponding irreducible component $\overline{M}_k(\Gamma)$ of $\overline{M}_0(V, d\ell)$ contributes by $\frac{1}{k^3}$ to the integral N_d^V. This has been mathematically proved by Kontsevich and Manin, and Voisin, see [43, § 5.6] and [10, Thm. 7.4.4].

To make this plausible it is convenient to consider, instead of the integrals N_d^V, their close cousins

$$\langle I^V_{0,3,d\ell}\rangle(\omega_1, \omega_2, \omega_3) := \int_{[\overline{M}_{0,3}(V,d\ell))]^{\mathrm{vir}}} \mathrm{ev}_1^*(\omega_1) \wedge \mathrm{ev}_2^*(\omega_2) \wedge \mathrm{ev}_3^*(\omega_3),$$

where $\overline{M}_{0,3}(V, d\ell)$ is the space of genus 0 stable maps with 3 marked points, and the ω_i are Kähler forms on V. The moduli space $\overline{M}_{0,3}(V, d\ell)$ has the advantage over $\overline{M}_0(V, d\ell)$ that it identifies with the Hilbert scheme $\mathrm{Hom}(\mathbf{P}^1, V)$, locally at those stable maps having source $\mathbf{P}^1$. Now, it follows from the divisorial axiom of Gromov–Witten theory that

$$\langle I^V_{0,3,d\ell}\rangle(\omega_1, \omega_2, \omega_3) = \int_{d\ell} \omega_1 \times \int_{d\ell} \omega_2 \times \int_{d\ell} \omega_3 \times N_d^V.$$

On the other hand, each physical degree d rational curve $C \subseteq V$ contributes via its normalization $f : \mathbf{P}^1 \to V$ by

$$\int_{\mathbf{P}^1} f^*\omega_1 \times \int_{\mathbf{P}^1} f^*\omega_2 \times \int_{\mathbf{P}^1} f^*\omega_3$$

to the integral $\langle I^V_{0,3,d\ell}\rangle(\omega_1, \omega_2, \omega_3)$, hence by

$$\frac{\left(\int_{\mathbf{P}^1} f^*\omega_1\right)\left(\int_{\mathbf{P}^1} f^*\omega_2\right)\left(\int_{\mathbf{P}^1} f^*\omega_3\right)}{\left(\int_{d\ell} \omega_1\right)\left(\int_{d\ell} \omega_2\right)\left(\int_{d\ell} \omega_3\right)}$$

to the invariant N_d^V. The contribution of Γ to $N_{d_0}^V$ may be expressed in the same way.

Now, suppose we knew what the compactification $\overline{M}_k(\Gamma)$ of $M_k(\Gamma)$ in $\overline{M}_{0,3}(V, d\ell)$ looks like, and we had a vector bundle E over it having fibre over $(f_0 \circ \phi_k) \in M_k(\Gamma)$ the obstruction space $H^1(\mathbf{P}^1, (f_0 \circ \phi_k)^* T_V)$. Then we could compute the contribution of $\overline{M}_k(\Gamma)$ to $\langle I^V_{0,3,d\ell}\rangle(\omega_1, \omega_2, \omega_3)$ by the excess formula

$$\int_{\overline{M}_k(\Gamma)} c_{2k-2}(E) \wedge \mathrm{ev}_1^*(\omega_1) \wedge \mathrm{ev}_2^*(\omega_2) \wedge \mathrm{ev}_3^*(\omega_3). \tag{4.2.1}$$

The computation of [43, p. 115–116] shows that a heuristic model for $\overline{M}_k(\Gamma)$ leads to the following expression for this excess contribution:

$$\int_{\mathbf{P}^1} f_0^*\omega_1 \times \int_{\mathbf{P}^1} f_0^*\omega_2 \times \int_{\mathbf{P}^1} f_0^*\omega_3.$$

The upshot is that the direct contribution of Γ to $N_{d_0}^V$ and its contribution via $\overline{M}_k(\Gamma)$ to N_d^V (in this heuristic model) are, respectively,

$$\frac{\left(\int_{\mathbf{P}^1} f_0^*\omega_1\right)\left(\int_{\mathbf{P}^1} f_0^*\omega_2\right)\left(\int_{\mathbf{P}^1} f_0^*\omega_3\right)}{\left(\int_{d_0\ell} \omega_1\right)\left(\int_{d_0\ell} \omega_2\right)\left(\int_{d_0\ell} \omega_3\right)} \quad \text{and} \quad \frac{\left(\int_{\mathbf{P}^1} f_0^*\omega_1\right)\left(\int_{\mathbf{P}^1} f_0^*\omega_2\right)\left(\int_{\mathbf{P}^1} f_0^*\omega_3\right)}{\left(\int_{d\ell} \omega_1\right)\left(\int_{d\ell} \omega_2\right)\left(\int_{d\ell} \omega_3\right)}.$$

The latter is $\frac{1}{k^3}$ times the former, since $\int_{d\ell} \omega_i = k \int_{d_0\ell} \omega_i$ for all $i = 1, 2, 3$.

(4.3) Instanton numbers. The idea is to define new invariants n_d^V from the Gromov–Witten integrals N_d^V that better reflect the enumerative geometry of the Calabi–Yau threefold V, taking into account the multiple cover phenomenon described in paragraph (4.2) above. The relations that these numbers should satisfy are

$$N_d^V = \sum_{k|d} \tfrac{1}{k^3}\, n_{d/k}^V \tag{4.3.1}$$

for all positive integers d, where the sum runs over all positive integral divisors of d. They form an invertible triangular set of relations, and each number n_d^V is therefore uniquely determined by the invariants $N_{d'}^V$ for all positive divisors d' of d. These numbers n_d^V are traditionally called *instanton numbers*. This name obviously bears some physical meaning, but let me not try to discuss it.

(4.4) Reducible covers of singular curves. There is yet another phenomenon, first observed by Pandharipande, that prevents the instanton numbers n_d^V defined in (4.3) above from being the actual numbers of integral degree d rational curves on V. It comes from the existence of singular integral rational curves.

To describe the simplest instance of this phenomenon, assume there is a degree d integral rational curve $C \subseteq V$ with an ordinary double point x. Let $f : \mathbf{P}^1 \to V$ be the normalization of C, and denote by x_1 and x_2 the two points in the preimage of x in the normalization, Now, consider the map

$$f_x : \mathbf{P}^1 \cup_{x_1,x_2} \mathbf{P}^1 \to V,$$

obtained by glueing tranversaly two copies of the normalization of C, in such a way that x_1 in the first copy is identified with x_2 in the second copy. It is a stable map of genus 0 realizing the homology class $2[C]$ on V, hence it contributes to the Gromov–Witten invariant N_{2d}^V. This contribution is by 1 if C is infinitesimally rigid, cf. [10, § 9.2.3]. It follows that a δ-nodal rational curve of degree d (i.e., a curve with exactly δ ordinary double points as singularities) contributes in the above described fashion by δ to the Gromov–Witten integral N_{2d}^V.

The Clemens conjecture below predicts that the complications don't go beyond this in the case of a general quintic threefold. Note in particular that part (iii) of the conjecture implies that the numbers N_d^V (or n_d^V) count *irreducible* physical rational curves $C \subseteq V$, since by definition the source of a stable map is connected.

(4.5) Conjecture (Clemens)**.** *Let $V \subseteq \mathbf{P}^4$ be a general quintic hypersurface.*
(i) For each integer $d \geqslant 1$, there are only finitely many irreducible rational curves $C \subseteq V$ of degree d.
(ii) For every integral rational curve $C \subseteq V$ with normalisation $f : \mathbf{P}^1 \to V$, the normal bundle N_f of the map f is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ (i.e., C is infinitesimally rigid).
(iii) All *the integral rational curves on V (of any degree) are pairwise disjoint.*

It is completely proved in degree $\leqslant 11$ (cf. [9, 11] for the latest steps), and part (i) is known in degree 12 [1].

We end this prologue about quintic threefolds by an explicit example displaying all these phenomena together.

(4.6) Rational curves of degree 10 on a quintic threefold. (cf. [10, § 9.2.3] for a thorough analysis). First note that, by definition of the instanton numbers, the Gromov–Witten integral N_{10}^V decomposes as

$$N_{10} = \frac{1}{10^3} n_1 + \frac{1}{5^3} n_2 + \frac{1}{2^3} n_5 + n_{10}$$

(we have now dropped the superscript V to lighten notations), and one has

$$\begin{aligned} n_1 &= 2,875 \\ n_2 &= 609,250 \\ n_5 &= 229,305,888,887,625 \\ n_{10} &= 704,288,164,978,454,686,113,488,249,750, \end{aligned}$$

cf. [7]. While n_1 and n_2 are simply the numbers of lines and conics on a general quintic threefold, n_5 counts two kinds of rational curves of degree 5. Indeed, the planes in $\mathbf{P}^4$ are parametrized by a 6-dimensional Grassmannian, and for a general quintic $V \subseteq \mathbf{P}^4$, finitely many of them are 6-tangent to V; the corresponding plane sections of V are 6-nodal plane quintic curves, and in particular they are rational curves. Vainsencher [41] has been able to compute their number

$$n_5' = 17,601,000,$$

see [XIII, Paragraph (3.6)]. Each such curve contributes to n_5 (or N_5) by 1 [10, Lemma 9.2.4], and

$$n_5'' := n_5 - n_5'$$

is indeed the number of *smooth* rational curves of degree 5 on V.

Whereas n_5 still is the number of rational curves of degree 5 on V, this is no longer true for n_{10} as the discussion in (4.4) indicates. The actual number of degree 10 integral rational curves on V is

$$n_{10}^\circ := n_{10} - 6n_5'$$

[10, Thm. 9.2.6][6].

4.2 – Degree 8 rational curves on a sextic double plane

In this subsection we give the enumerative interpretation due to Gathmann [17] of the reduced Gromov–Witten invariant

$$N_{0,2}^2 := \int_{[\overline{M}_0(S,2L)]^{\mathrm{red}}} 1$$

of a general primitively polarized $K3$ surface (S, L) of genus 2 (i.e., S is a double covering of the plane $\pi : S \to \mathbf{P}^2$ branched over a general sextic curve B, and L is the pull-back of the line class). Note that the polarization $2L$ has genus 5, as $(2L)^2 = 8$.

(4.7) The analysis carried out in Section 4.1 indicates that the integral $N_{0,2}^2$ is a sum of contributions corresponding to the following types of curves.
(i) 5-nodal integral curves in $|2L|$; these are the preimages of the conics in $\mathbf{P}^2$ tangent to the branch sextic B at 5 distinct points. There are 70,956 of those, as Gathmann was able to compute using his theory of relative Gromov–Witten invariants for hypersurfaces.
(ii) Reducible rational curves made of two distinct rational curves in $|L|$: let $C_1, C_2 \in |L|$ be two rational curves (each of these is the pull-back of a line bitangent to B); they intersect transversely in two points x, x' where both of them are smooth, if (S, L) is general; then, the partial normalization of $C_1 + C_2$ at the four nodes of C_1 and C_2 and one node among x and x' gives a stable map with source a chain of two smooth rational curves, mapping birationally onto the physical curve $C_1 + C_2 \in |2L|$.

These two stable maps contribute each by 1 to $N_{0,2}^2$, so in the upshot each of the $\binom{324}{2}$ pairs of distinct rational curves in $|L|$ contributes by 2 in this way, thus giving a total contribution of

[6] there is a misprint there: $6\frac{1}{8}$ should be replaced by $6 + \frac{1}{8}$.

$104,652$ to $N^2_{0,2}$ (recall that there are 324 bitangent lines to B, as can be classicaly computed, or extracted from Theorem (2.1)).
(iii) Reducible double coverings of one rational curve in $|L|$, as in (4.4). Since all rational curves in $|L|$ are 2-nodal, all 324 of them give 2 stable maps with reducible source, contributing by 1 each to $N^2_{0,2}$, for a total contribution of 648.
(iv) Irreducible double covers of one rational curve in $|L|$, as in (4.2). Gathmann [17, Lemma 4.1] proved that each of the 324 corresponding positive-dimensional irreducible components of $\overline{M}_0(S, 2L)$ gives a contribution by $\frac{1}{8}$ to $N^2_{0,2}$, similar to the prediction of the Aspinwall–Morrison formula, although we are in a significantly different situation (we are not on a Calabi–Yau threefold, and the rational curves we are considering double covers of are singular, for instance).

(4.8) Remark. An important difference with the case of the quintic threefold, although rather innocent-looking, is that in the present situation integral rational curves do intersect each other, contrary to the prediction of part (iii) of Clemens' Conjecture. In the above discussion (4.7), this amounts to case (ii) needing to be added with respect to the discussion for quintic threefolds, which is still manageable. When looking at linear systems $|mL|$ with $m \geqslant 3$ however, this soon gets much more complicated, see (4.15).

(4.9) From the enumeration of (4.7), one may deduce the value of $N^2_{0,2}$, which was not known before. But the striking observation of [17] is that the sum of the contributions (i)–(iii) above, which would be the instanton number $n^2_{0,2}$ in the language of (4.3), actually equals

$$N^5_{0,1} := N^5_0 = 176,256,$$

the number of degree 8 rational curves in a primitively polarized $K3$ surface of genus 5, computed by formula (2.1.1). This suggests the amazing possibility that the number of rational curves in a linear system $|L|$ on a $K3$ surface is independent of the divisibility of the line bundle L. We will come back to this in detail in Section 4.4 below.

4.3 – Elliptic curves in a 2-divisible class on a $K3$ surface

Here, I report on the computation by Lee and Leung [24] of the reduced Gromov–Witten invariant

$$N^p_{1,2} := \int_{[\overline{M}_{1,1}(S,2L)]^{\mathrm{red}}} \mathrm{ev}_1^*(\mathrm{pt})$$

of a general primitively polarized $K3$ surface (S, L) of genus p, which "counts" genus 1 curves in $|2L|$ passing through 1 general point on S. Using a suitable version of topological recursion, they prove the following formula.

(4.10) Theorem. [24] *One has*

$$N^p_{1,2} = N^{4p-3}_{1,1} + 2N^p_{1,1}.$$

The numbers $N^q_{1,1} := N^q_1$ are those giving the number of elliptic curves through a general point in the primitive class of a $K3$ surface of genus q, as in formula (3.5). Note that $p' := 4p - 3$ is the genus of the polarization $2L$, i.e., $(2L)^2 = 2p' - 2$.

(4.11) Lee and Leung propose the following interpretation of their formula (4.10), to put it in tune with (4.9), and more generally with the results of Section 4.4 below.

Given a smooth elliptic curve E, there are $\sigma_1(2) = 1 + 2 = 3$ morphisms of elliptic curves $E' \to E$ of degree 2 (note that we require that the origin is respected), as we have already seen

in (3.11). This implies that each of the $N^p_{1,1}$ elliptic curves C in the primitive linear system $|L|$ passing through a general point $x_1 \in S$ gives via double covers of its normalization three genus 1 stable maps realizing the homology class $2[C]$, each contributing by 1 to the number $N^p_{1,2}$.

Lee and Leung deduce from this that N_1^{4p-3} is the actual number of physical elliptic curves in $|2L|$, meaning that it counts each integral elliptic curve $C \in |2L|$ for 1 and each $2C$ where $C \in |L|$ is an integral elliptic curve for 1 as well. This is indeed a striking interpretation, although arguably debatable.

There is at any rate a phenomenon that prevents this interpretation to be anything more than philosophical, namely that reducible curves $C_0 + C_1$, where C_0 (resp. C_1) is a rational (resp. elliptic) integral curve in $|L|$, also contribute to the invariant $N^p_{1,2}$. The two curves C_0 and C_1 intersect (transversely, say) at $2p-2$ points $y_1, \dots, y_{2p-2}$, and this gives $2p-2$ genus 1 stable maps

$$\mathbf{P}^1 \cup_{y_i} \bar{C}_1 \to S$$

realizing the class $2L$ and passing through the appropriate fixed point whenever C_1 does (the source is the transverse union of the normalization $\mathbf{P}^1$ and $\bar{C}_1$ of C_0 and C_1, attached at one point lying in the preimages of one of the points y_i).

4.4 – The Yau–Zaslow formula for non-primitive classes

We now come to the general statement confirming (4.9) and recently proved by Klemm, Maulik, Pandharipande, and Scheidegger.

(4.12) Let S be a $K3$ surface, and $L \in \mathrm{Pic}(S)$. It follows from deformation invariance of reduced Gromov–Witten integrals and the global Torelli theorem for $K3$ surfaces that the integral $\int_{[\overline{M}_0(S,L)]^{\mathrm{red}}} 1$ only depends on the self-intersection L^2 and the divisibility index of L in $\mathrm{Pic}(S)$ (the latter is the largest integer m for which there exists $L' \in \mathrm{Pic}(S)$ such that $L = mL'$). We may thus make the following definition.

For all integers $p > 0$ and $m \geqslant 1$, we let

$$N^p_{0,m} := \int_{[\overline{M}_0(S,mL)]^{\mathrm{red}}} 1$$

where (S, L) is any primitively polarized $K3$ surface of genus p.

(4.13) BPS states. Similar to what has been done in (4.3), and following the insight of (4.9), we now define new invariants from the $N^p_{0,m}$ of (4.12) above by applying the corrections indicated by the Aspinwall–Morrison formula. Let us formulate this in terms of generating series as follows. Given a positive integer p, set

$$F^p(v) := \sum_{m \geqslant 1} N^p_{0,m}\, v^m \tag{4.13.1}$$

as a formal power series in the variable v. Then the new set of invariants $n^p_{0,m}$ is uniquely determined by the rewriting of the generating series as

$$F^p(v) := \sum_{m \geqslant 1} n^p_{0,m} \left(\sum_{d>0} \frac{1}{d^3}\, v^{dm} \right) \tag{4.13.2}$$

(this is exactly the same modification as that of (4.3.1)).

Note that this is not mere makeshift reformulation. The invariants $n^p_{0,m}$ are believed to count objects named BPS states by the physicists, after Bogomol'nyi, Prasad, and Sommerfield, and have been introduced by Gopakumar and Vafa. The mathematical nature of these BPS states is however not clear yet. In particular, it should be possible to define the $n^p_{0,m}$ intrinsically, without relying on the $N^p_{0,m}$; the relation (4.13.2) would then tie together these two sets of independently defined invariants. See the enlightening survey [32, § $2\frac{1}{2}$] for more about this. There is moreover a physical meaning to the introduction of generating series, but I will not discuss it.

(4.14) Theorem ([23])**.** *The genus* 0 *invariants* $n^p_{0,m}$ *do not depend upon the divisibility index, i.e., for all integers* $m, p \geqslant 1$*, one has*

$$n^p_{0,m} = n^{m^2p-m^2+1}_{0,1} = N_0^{m^2p-m^2+1} \tag{4.14.1}$$

(the integer $p' = m^2p - m^2 + 1$ *is designed such that* $(mL)^2 = 2p' - 2$ *if* $L^2 = 2p - 2$*).*

Recall that N^p_0 was defined in section 2; the second equality in (4.14.1) is by definition of $n^p_{0,1}$ and N^p_0. This statement was part of the Yau–Zaslow conjecture [44]. Together with Theorem (2.1), which was also part of the Yau–Zaslow conjecture, it implies that all the $n^p_{0,m}$'s may be computed by means of formula (2.1.1). The set of relations (4.13.1) being triangular invertible, this also gives all genus 0 reduced Gromov–Witten invariants of $K3$ surfaces. Section 6 below contains an overview of the proof given by Klemm, Maulik, Pandharipande, and Scheidegger [23] of Theorem (4.14).

The truly remarkable feature of the invariants $n^p_{0,m}$ displayed by this statement is that the number of rational curves of prescribed degree in an algebraic $K3$ surface does not depend on the algebraic geometry of the surface.

(4.15) In spite of formula (4.13.2) taking into account the Aspinwall-Morrison multiple cover correction, the invariants $n^p_{0,m}$ do not in general count the actual number of rational curves in $|mL|$.

One reason for this is the existence of more non-reduced curves with rational support than those taken in consideration in the correction (4.13.2), namely curves with reducible support. For instance let $m = 3$ and consider two integral rational curves C_1, C_2. Then $2C_1 + C_2 \in |3L|$, and there are correspondingly finitely many positive-dimensional components of $\overline{M}_0(S, 3L)$, the general points of which correspond to stable maps

$$\mathbf{P}^1 \cup_x \mathbf{P}^1 \to S$$

with source a transverse union of two $\mathbf{P}^1$'s, consisting of a double cover of C_1 on the first component and the normalization of C_2 on the other. These certainly give an excess contribution to the invariant $N^p_{0,3}$, which is not taken into account in the definition of $n^p_{0,3}$.

Such problematic phenomena do not occur for $m \leqslant 2$, so that $n^p_{0,1}$ is directly enumerative as was already noted in Section 2, and $n^p_{0,2}$ counts reduced rational curves in the way described in Section 4.2. It would be very interesting to relate $n^p_{0,m}$ to the number of integral rational curves in $|mL|$ for $m \geqslant 3$.

There were, at least conjecturally, no such phenomena at work in the case of the quintic threefold discussed in subsection 4.1, as part (iii) of the Clemens conjecture (4.5) asserts that two integral rational curves in a general quintic threefold never intersect. In a surface, there is of course not enough space for two curves to avoid each other, so we inevitably have to deal with the aforementoined degenerate contributions.

The philosophy, as R. Pandharipande communicated to me, is that what the BPS numbers for $K3$ surfaces are virtually counting, are rational curves in some perturbation of the twistor

family of the $K3$ surface (a threefold, cf. (3.2)). We shall consider in more detail the close interplay between counting invariants for $K3$ surfaces and Calabi–Yau threefolds in the next Section 5.

5 – Relations with threefold invariants

It is already visible in the very foundation of the theory of Gromov–Witten invariants for algebraic $K3$ surfaces developed by Bryan–Leung [6], see (3.2), that these invariants are fundamentally attached to a threefold (even though the approach by Maulik–Pandharipande [28], see (3.3), bypasses this). Another revealing evidence of the 3-dimensional nature of these invariants is the meaningful role played by the Aspinwall–Morrison in the Yau–Zaslow statement discussed in subsection 4.4 above, a tool specifically designed for Calabi–Yau threefolds. In this section we will try to describe this relation in a more conceptual way. It it wise to keep in mind the symplectic nature of Gromov–Witten invariants throughout.

5.1 – Two obstruction theories

(5.1) A threefold degenerate contribution. Let V be a Calabi–Yau threefold. It follows from formula (4.1.1) that the virtual dimension of all spaces of stable maps is 0 regardless of the genus. This makes the following phenomenon happen.

Let $C_0 \subseteq V$ be a rational curve (smooth and infinitesimally rigid, say). Its normalization $f : \mathbf{P}^1 \to V$ contributes regularly by 1 to the integral $\int_{[\overline{M}_0(V,[C_0])]^{\mathrm{vir}}} 1$. But for any stable curve C' of genus $g \geqslant 1$ we may obtain a genus g stable map realizing the class $[C_0]$ by attaching C' to the normalization of C_0 over a smooth point x, and letting

$$f_{C',x} : \mathbf{P}^1 \cup_x C_0 \to V$$

equal to f along $\mathbf{P}^1$ and collapsing C' to x. This produces a positive dimensional moduli space of genus g stable maps which all have the same image $C_0 \subseteq V$; its contribution to the Gromov–Witten invariant $\int_{[\overline{M}_g(V,[C_0])]^{\mathrm{vir}}} 1$ must be computed via Hodge integrals over the moduli space of stable curves of genus g, see below. This has been studied by Faber and Pandharipande, see [32, § $1\frac{1}{2}$] and the references therein.

(5.2) Curves in the twistor space of a $K3$. Let S be a $K3$ surface, together with an algebraic class $\beta \in H_2(S, \mathbf{Z})$. We consider its twistor space $T \to \mathbf{S}^2$ described in (3.2) above (we emphasize that this is a real 6-dimensional variety), and denote by $\iota : S \hookrightarrow T$ the canonical inclusion of S. Since curves in T can only appear in the fibre S, we have the equality of moduli spaces of stable maps

$$\overline{M}_g(T, \iota_*\beta) = \overline{M}_g(S, \beta),$$

a priori only as sets but in fact as Deligne–Mumford stacks. They come however with two different obstructions theories, hence also with two different virtual classes. Gromov–Witten invariants on T are related to those on S (within the reduced theory for $K3$ surfaces, cf. (3.3)) by the formula

$$\int_{[\overline{M}_g(T,\iota_*\beta)]^{\mathrm{vir}}} 1 \;=\; \int_{[\overline{M}_g(S,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g, \tag{5.2.1}$$

where λ_g stands for the top Chern class $c_g(\mathbf{E}_g)$ of the Hodge bundle $\mathbf{E}_g \to \overline{M}_g(S, \beta)$, whose fibre over the stable map $f : C \to S$ is $H^0(C, \omega_C)$.

(5.3) Hodge integrals. It follows from the invariance of reduced Gromov–Witten invariants under algebraic deformation and the global Torelli theorem for $K3$ surfaces that the right-hand side of (5.2.1) depends only on the self-intersection β^2 and the divisibility index of β as an algebraic class. We may thus formulate the following definition.

For integers $g \geqslant 0$ and $p, m \geqslant 1$, let

$$R^p_{g,m} := \int_{[\overline{M}_g(S,mL)]^{\mathrm{red}}} (-1)^g \lambda_g \tag{5.3.1}$$

where (S, L) is any primitively polarized $K3$ surface of genus p, and λ_g is the top Chern class of the Hodge bundle as in (5.2) above.

This extends the definition of the invariants $N^p_{0,m}$ in (4.12) above, in the sense that $N^p_{0,m} = R^p_{0,m}$ (note however that the invariants $N^p_{1,2}$ used in subsection 4.3 do not coincide with the $R^p_{1,2}$). For $g > 0$, these invariants are certainly not counting curves on S; rather, formula (5.2.1) tells us that they virtually give the excess contribution of S to the vertical Gromov–Witten theory of any $K3$-fibred threefold in which it appears as a fibre. This philosophy is put into concrete form by Theorem (6.9) below.

5.2 – The Katz–Klemm–Vafa formula

This is an extension of the Yau–Zaslow conjecture discussed in Section 4.4 above to the invariants $R^p_{g,m}$. It has been proved by Pandharipande and Thomas [36], see also [35].

(5.4) BPS invariants. It is admittedly better to organize the invariants $R^p_{g,m}$ in BPS form as in (4.13). We have now a clear justification for this, as we have seen in Section 5.1 above that these invariants really count objects on threefolds.

We first let

$$F^p(u,v) := \sum_{g\geqslant 0} \sum_{m>0} R^p_{g,m}\, u^{2g-2} v^m$$

as formal power series in the two variables u, v for all positive integers p. One then defines new invariants $r^p_{g,m}$ for all integers $p, m > 0$, $g \geqslant 0$, by setting

$$\begin{aligned}
F^p(u,v) &= \sum_{m>0} \sum_{g\geqslant 0} r^p_{g,m} u^{2g-2} \left(\sum_{d>0} \frac{1}{d} \left(\frac{\sin d\frac{u}{2}}{\frac{u}{2}} \right)^{2g-2} v^{dm} \right) \\
&= \sum_{m>0} \Bigg(r^p_{0,m} u^{-2} \sum_{d>0} \Big(\frac{1}{d^3} + \frac{1}{12d} u^2 + \frac{d}{240} u^4 + \frac{d^3}{6048} u^6 + \frac{d^5}{172800} u^8 + \cdots \Big) v^{dm} \\
&\qquad + r^p_{1,m} \sum_{d>0} \frac{1}{d}\, v^{dm} \\
&\qquad + r^p_{2,m} u^2 \sum_{d>0} \Big(d - \frac{d^3}{12} u^2 + \frac{d^5}{360} u^4 - \frac{d^7}{20160} u^6 + \frac{d^9}{1814400} u^8 + \cdots \Big) v^{dm} \\
&\qquad + r^p_{3,m} u^4 \sum_{d>0} \Big(d^3 - \frac{d^5}{6} u^2 + \frac{d^7}{80} u^4 - \frac{17 d^9}{30240} u^6 + \frac{31 d^{11}}{1814400} u^8 + \cdots \Big) v^{dm} \\
&\qquad + \cdots \Bigg).
\end{aligned}$$

The modifications for genus $g > 0$ objects did not appear earlier in this text. Note that every object counted by $r^p_{g,m}$ contributes to the invariants $R^p_{g',m}$ for all $g' \geqslant g$ (except when $g = 1$),

with alternated sign if $g \geqslant 2$. This is in accord with what the phenomenon described in (5.1) suggests.

(5.5) Theorem (Katz–Klemm–Vafa formula, [36])**.** *The invariants $r^p_{g,m}$ do not depend on the divisibility index, meaning that one has*

$$r^p_{g,m} = r^{m^2p-m^2+1}_{g,1} = R^{m^2p-m^2+1}_g$$

for all integers $p, m > 0$ and $g \geqslant 0$.

They are all determined by the formula

$$\sum_{p\geqslant 0}\sum_{g\geqslant 0}(-1)^g\, r^p_g \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^{2g} q^p = \prod_{n\geqslant 1} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}, \tag{5.5.1}$$

where we set $r^p_g := r^p_{g,1}$ (and for convenience, $r^0_0 = 1$, and $r^0_g = 0$ for all $g > 0$).

Setting $y = 1$ in the formula restricts to the invariants r^p_0, and recovers the Yau–Zaslow formula of Theorem (2.1). As a first corollary, one gets that $r^p_g = 0$ if $g > p$, and $r^p_p = (-1)^p(p+1)$. The first values of r^p_g are tabulated below.

g \ p	0	1	2	3	4
0	1	24	324	3200	25650
1		−2	−54	−800	−8550
2			3	88	1401
3				−4	−126
4					5

Table 2: First values of r^p_g

5.3 – Further Gromov–Witten integrals

We close this section with a short discussion of further results about Gromov–Witten integrals on $K3$ surfaces. They all come from [26].

(5.6) Hodge integrals with point insertions. As a direct generalization of the invariants $R^p_{g,m}$ from (5.3.1), one may consider the integrals

$$R^p_{g,k,m} := \int_{[\overline{M}_{g,k}(S,mL)]^{\mathrm{red}}} (-1)^{g-k}\lambda_{g-k} \cup \mathrm{ev}_1^*(\mathrm{pt}) \cup \cdots \cup \mathrm{ev}_k^*(\mathrm{pt}),$$

where (S, L) is a primitive $K3$ surface of genus p, $\overline{M}_{g,k}(S, mL)$ is the moduli space of genus g stable maps with k marked points, and λ_i is the i-th Chern class of the Hodge bundle $\mathbf{E}_{g,k} \to \overline{M}_{g,k}(S, mL)$ as in (5.2).

For *primitive* classes on $K3$ surfaces, the following formula was proved by Maulik, Pandharipande and Thomas [26, Thm. 3]:

$$\sum_{g=0}^{+\infty}\sum_{p=0}^{+\infty} R^p_{g,k,1}\, u^{2g-2}q^p \;= \tag{5.6.1}$$

$$q\,\frac{(2\pi)^{12}}{u^2\Delta(q)} \cdot \exp\left(\sum_{g=1}^{+\infty} u^{2g}\frac{B_{2g}}{g(2g)!}E_g(q)\right) \cdot \left(\sum_{m=1}^{+\infty} q^m \sum_{d|m}\frac{m}{d}\left(2\sin d\tfrac{u}{2}\right)^2\right)^k$$

(notation is as in (3.6)).

Note that in the $k = 0$ case, this contains nothing new with respect to the formula given in Theorem (5.5), as the expression (5.6.1) may be deduced from (5.5.1) using known identities, see [26, § 5.4].

(5.7) Descendent Gromov–Witten invariants. So far, we have been essentially concerned with the reduced Gromov–Witten invariants (see (3.3))

$$N_g(S,\beta) = \int_{[\overline{M}_{g,g}(S,\beta)]^{\mathrm{red}}} \mathrm{ev}_1^*(\mathrm{pt}) \cup \cdots \cup \mathrm{ev}_g^*(\mathrm{pt})$$

counting curves in the algebraic class $\beta \in H_2(S,\mathbf{Z})$ on the $K3$ surface S and passing through g general fixed points. It is of course possible to pull-back more general cohomology classes $\gamma_i \in H^*(S,\mathbf{Z})$ by the evaluation maps, thus encoding more general incidence conditions than the passing through a given point (although this is not really relevant for surfaces due to the divisor axiom of Gromov–Witten theory). Beware that when doing so one gets integrals that do depend on the class β itself, and not only on its self-intersection and divisibility index, as classes in $H^2(S,\mathbf{Z})$ are not monodromy invariant.

A more sensible generalization is to integrate *descendent classes*. Let $\overline{M}_{g,k}(S,\beta)$ be the moduli space of genus g stable maps with k marked points realizing the class β, and $\mathrm{ev}_1, \ldots, \mathrm{ev}_k$ the corresponding evaluation maps $\overline{M}_{g,k}(S,\beta) \to S$. For all $i = 1, \ldots, k$, define the *i-th cotangent line bundle* L_i to be the line bundle over $\overline{M}_{g,k}(S,\beta)$ the fibre of which over the point $(f : C \to S, p_1, \ldots, p_k)$ is the $\mathbf{C}$-line Ω^1_{C,p_i}. The descendent classes on $\overline{M}_{g,k}(S,\beta)$ are those gotten from the Chern classes of these line bundles.

Let $\psi_i := c_1(L_i) \in H^2\big(\overline{M}_{g,k}(S,\beta),\mathbf{Q}\big)$. For all cohomology classes $\gamma_1, \ldots, \gamma_k \in H^*(S,\mathbf{Z})$ and non-negative integers $n_1, \ldots, n_k$, we set

$$\big\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \big\rangle_g^{S,\beta} := \int_{[\overline{M}_{g,k}(S,\beta)]^{\mathrm{red}}} \psi_1^{k_1} \cup \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \psi_k^{k_k} \cup \mathrm{ev}_k^*(\gamma_k) \tag{5.7.1}$$

whenever the degree of the integrand equals the (real) dimension $2g + 2k$ of the reduced virtual class, and $\big\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \big\rangle_g^{S,\beta} := 0$ otherwise. These integrals are called *reduced descendent Gromov–Witten invariants*.

The geometric interpretation of the insertion of the classes ψ_i is not straightforward; I refer to [34] and [16] for some discussions about this. See however [16, Thm. 2.2.6], where descendent classes are used to define Gromov–Witten invariants of a projective manifold X *relative* to a smooth very ample hypersurface Y, i.e., invariants virtually counting curves in X with prescribed tangency conditions along Y.

(5.8) Quasi-modularity. The integrals (5.7.1) for fixed integrand and fixed g and divisibility index of β are expected to fit together as the Fourier coefficients of a quasi-modular form, as in Theorem (3.5). Due to their dependency on the class β and not only on its numerical characters, this is formulated as follows.

Let S be an arbitrarily fixed $K3$ surface possessing an elliptic fibration $\pi : S \to \mathbf{P}^1$ and a section E of π. Call $\mathsf{e}, \mathsf{f} \in H_2(S,\mathbf{Z})$ the classes of E and the fibres of π respectively. It follows from deformation invariance and the same standard degeneration argument as in the proof of Theorem (3.5) that any integral of the form (5.7.1) on an algebraic $K3$ surface equals an integral of the same kind on S with $\beta = a\mathsf{e} + b\mathsf{f}$ for some non-negative integers a, b. For all integers $g \geqslant 0$ and $m > 0$, we set

$$F^S_{g,m}\big(\tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k)\big) := \sum_{n \geqslant 0} \big\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \big\rangle_g^{S, m\mathsf{e}+n\mathsf{f}} q^{m(n-m)}$$

as a formal power series in the variable q.

(5.9) Conjecture (Maulik and Pandharipande, [33, Conj. 3] and [26, § 7.5])**.** *The power series* $F^S_{g,m}\big(\tau_{n_1}(\gamma_1)\cdots\tau_{n_k}(\gamma_k)\big)$ *is the Fourier expansion in* q *of a quasi-modular form of level* m^2 *with pole at* $q=0$ *of order at most* m^2*.*

(A quasi-modular form of level N with possible pole at $q=0$ is by definition an element of the $\mathbf{C}$-algebra generated by the Eisenstein series G_1 (see (3.6)) and modular forms of level N; recall in addition that a modular form of level N is a form satisfying the modular equation for transformations in the congruence subgroup $\Gamma_0(N)$ consisting of elements of $\mathrm{PSL}_2(\mathbf{Z})$ congruent to the identity matrix modulo N).

For $m=1$, i.e., for primitive classes, this was proved by Maulik, Pandharipande and Thomas [26, Thm. 4]. Note however that, even in the primitive case, there is no general explicit formula for the modular form in question, as far as I know. Theorem (3.5) provides particular instances of such a formula. At any rate, modularity strongly constrains the invariants and in favorable cases enables one to compute them all (see (6.11) for an example in a different context).

We will only say the following about the proofs of the results presented in this section. A fundamental ingredient for them is the use of other counting invariants than those coming from Gromov–Witten theory, together with correspondence theorems between the two. They are more algebraic in nature than Gromov–Witten invariants, and more agile to study the problems we have been discussing. These invariants virtually count *stable pairs*; they were defined by Pandharipande and Thomas, specifically for threefolds up to now. See [32] for a presentation.

6 – Noether–Lefschetz theory and applications

6.1 – Lattice polarized $K3$ surfaces and Noether–Lefschetz theory

In this subsection we define Noether–Lefschetz divisors in the moduli spaces of lattice polarized $K3$ surfaces. While the version we will use is the refined one of (6.4), the elementary version of (6.3) is needed to give a proper definition.

Let $\mathbf{L}_{K3} := U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}$ be the $K3$ lattice (see, e.g., [2]). Throughout this subsection, we consider a fixed lattice Λ of rank r and signature $(1,r-1)$, together with a primitive embedding $\iota:\Lambda\hookrightarrow\mathbf{L}_{K3}$ (the embedding is *primitive* if the corresponding quotient $\mathbf{L}_{K3}/i(\Lambda)$ is torsion-free).

(6.1) Definition. *Let* S *be a* $K3$ *surface. A* Λ*-polarization on* S *is a primitive embedding* $j:\Lambda\hookrightarrow\mathrm{Pic}(S)$ *such that*

(i) there is a nef and big class in $j(\Lambda)\subseteq\mathrm{Pic}(S)$*;*

(ii) there exists an isometry $\phi: H^2(S,\mathbf{Z})\to\mathbf{L}_{K3}$ *such that* $\phi\circ j=\iota$*.*

A Λ*-polarized* $K3$ *surface is a pair* (S,j) *where* S *is a* $K3$ *surface and* j *is a* Λ*-polarization on* S*.*

There exists a moduli space $\mathcal{K}_\Lambda$ of Λ-polarized $K3$ surfaces, which may be constructed relying on the global Torelli theorem by adapting the method of [37, Exp. XIII, §3].

(6.2) Define the discriminant of a rank s lattice L to be the signed determinant

$$\mathrm{Disc}(L) := (-1)^{s-1}\det\big(\langle v_i,v_j\rangle\big)_{1\leqslant i,j\leqslant s}$$

where $(v_1,\dots,v_s)$ is an integral basis of L (the sign has been added to the usual definition so that $\mathrm{Disc}(\Lambda)>0$); this does not depend on the choice of the basis.

Let $\mathbf{L}$ be a rank $r+1$ lattice with an even symmetric bilinear form, together with a primitive embedding $i : \Lambda \hookrightarrow \mathbf{L}$. There is an invariant of the pair $(\mathbf{L}, i)$ called the coset, which is defined as follows. Consider any vector $v \in \mathbf{L}$ such that $\mathbf{L} = i(\Lambda) \oplus v$; the pairing with v determines an element $\ell_v \in \Lambda^\vee$ in the lattice dual to Λ. On the other hand, let $G_\Lambda := \Lambda^\vee/\Lambda$ be the quotient of the injection $\Lambda \hookrightarrow \Lambda^\vee$ defined by the pairing on Λ; it is an abelian group of order $\mathrm{Disc}(\Lambda)$. The *coset* δ of $(\mathbf{L}, i)$ is the class of ℓ_v in $G_\Lambda/\pm$; it does not depend on the choice of v.

Two pairs $(\mathbf{L}, i)$ and $(\mathbf{L}', i')$ as above are isomorphic (i.e., there exists an isometry $\phi : \mathbf{L} \to \mathbf{L}'$ such that $\phi \circ i = i'$) if and only if the following two conditions hold: (i) $\mathrm{Disc}(\mathbf{L}) = \mathrm{Disc}(\mathbf{L}')$, and (ii) $\delta(\mathbf{L}, i) = \delta(\mathbf{L}', i')$.

(6.3) Elementary Noether–Lefschetz divisors. The *Noether–Lefschetz divisor* $P^\Lambda_{\Delta,\delta} \subseteq \mathcal{K}_\Lambda$ is the closure of the locus of Λ-polarized $K3$ surfaces (S, j) such that $\mathrm{Pic}(S)$ has rank $r+1$ and discriminant Δ, and the coset $\delta(\mathrm{Pic}(S), j)$ equals δ.

It follows from the Hodge index theorem that the divisor $P^\Lambda_{\Delta,\delta}$ is empty when $\Delta \leqslant 0$.

(6.4) Refined Noether–Lefschetz divisors. Let us now fix an integral basis $\mathbf{v}_\Lambda = (v_1, \ldots, v_r)$ for Λ, and let $m \in \mathbf{Z}_{>0}$, and $(p, \mathbf{d}) = (p, d_1, \ldots, d_r) \in \mathbf{Z}^{r+1}$. We want to define a Noether–Lefschetz divisor $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} \subseteq \mathcal{K}_\Lambda$ corresponding to Λ-polarized $K3$ surfaces (S, j) with an extra class $\beta \in \mathrm{Pic}(S)$ of divisibility index m such that $\langle \beta, \beta \rangle = 2p-2$, and $\langle \beta, v_i \rangle = d_i$ for all $i = 1, \ldots, r$. The definition depends on the following cases: let

$$\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} := (-1)^r \begin{vmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle & d_r \\ d_1 & \cdots & d_r & 2p-2 \end{vmatrix};$$

— if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} > 0$, set

$$D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} := \sum_{\Delta,\delta} \mu^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}(\Delta, \delta) \cdot P^\Lambda_{\Delta,\delta}$$

where the sum runs over all Δ and δ such that there exists a pair $(\mathbf{L}, i)$ as in (6.2) with $\mathrm{Disc}(\mathbf{L}) = \Delta$ and $\delta(\mathbf{L}, i) = \delta$ (the pair $(\mathbf{L}, i)$ is then unique up to isomorphism), and $\mu^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}(\Delta, \delta)$ is the number of elements $\beta \in \mathbf{L}$ having divisibility index m and satisfying $\langle \beta, \beta \rangle = 2p-2$ and $\langle \beta, v_i \rangle = d_i$ for all $i = 1, \ldots, r$. Note that $\mu^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}(\Delta, \delta)$ may be 0; its non-vanishing implies that Δ divides $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}}$, so the above sum has only finitely many terms. The condition $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} > 0$ implies that any β such that $\langle \beta, \beta \rangle = 2p-2$ and $\langle \beta, v_i \rangle = d_i$ for all i does not belong to $i(\Lambda)$;
— if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} < 0$, set $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} := 0$;
— if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} = 0$ and $m = \gcd(d_1, \ldots, g_r)$, let $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}$ be the divisor associated to the dual of the Hodge line bundle $\mathcal{E} \to \mathcal{K}_\Lambda$ (the fibre of $\mathcal{E}$ over the point (S, i) is $H^{2,0}(S)$);
— if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} = 0$ and $m \neq \gcd(d_1, \ldots, g_r)$, set $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} := 0$.

6.2 – Invariants of families of lattice polarized $K3$ surfaces

(6.5) Families of lattice polarized $K3$ surfaces. Let $\iota : \Lambda \hookrightarrow \mathbf{L}_{K3}$ be a primitive embedding of a lattice Λ of rank r and signature $(1, r-1)$. A *1-parameter family of Λ-polarized $K3$ surfaces* is a smooth family $\pi : X \to C$ of $K3$ surfaces equipped with line bundles $L_1, \ldots, L_r$ on X such that:

(i) X is a compact 3-dimensional complex manifold (not necessarily algebraic), C is a complete smooth complex curve, and π is a holomorphic submersion;

(ii) for all $t \in C$, the fibre X_t of π over t is a (smooth) $K3$ surface;
(iii) there exists a linear combination L^π of the holomorphic line bundles L_i on X, the restriction of which to every fibre of π is nef and big;
(iv) there exists an integral basis $(v_1, \dots, v_r)$ of Λ such that for all $t \in C$, the map $j_t : \Lambda \to \mathrm{Pic} X_t$ defined by $v_i \mapsto L_{i,t}$ (the restriction of L_i to X_t) is a Λ-polarization of X_t.

For the remainder of this subsection, we consider a 1-parameter family of Λ-polarized $K3$ surfaces $(\pi : X \to C, L_1, \dots, L_r)$ as in (6.5) above.

(6.6) Noether–Lefschetz numbers. Let $m \in \mathbf{Z}_{>0}$ and $(p, \mathbf{d}) = (p, d_1, \dots, d_r) \in \mathbf{Z}^{r+1}$. The *Noether–Lefschetz number* $\mathrm{NL}^\pi_{m,p,\mathbf{d}}$ is defined as

$$\mathrm{NL}^\pi_{m,p,\mathbf{d}} := \int_C f_\pi^* \big(D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}\big),$$

where $f : C \to \mathcal{K}_\Lambda$ is the morphism corresponding to $(\pi : X \to C, L_1, \dots, L_r)$ by the universal property of $\mathcal{K}_\Lambda$, and $\mathbf{v}_\Lambda$ is the integral basis of Λ defined by $(L_1, \dots, L_r)$ via point (iv) of (6.5).

Note that this is a classical intersection product (i.e., there is no need to define a virtual class), although it may be given by an excess formula in case the image $f_\pi(C)$ is fully contained in the divisor $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}$.

(6.7) Gromov–Witten invariants for vertical curve classes. Although it may not be a projective variety, the total space X carries a $(1,1)$-form ω_π which is Kähler on the fibres of π; this is sufficient to define a Gromov–Witten theory for non-zero *vertical* classes $\gamma \in H_2(X, \mathbf{Z})^\pi$, i.e., classes $\gamma \in H_2(X, \mathbf{Z})$ such that $\pi_*(\gamma) = 0$ (see [28, §2.1] for details).

In this theory, the moduli spaces of genus g stable maps $\overline{M}_g(X, \gamma)$ all have virtual dimension 0, and we have a set of invariants

$$N^X_{g,\gamma} := \int_{[\overline{M}_g(X,\gamma)]^{\mathrm{vir}}} 1$$

for non-zero vertical classes γ. We consider the invariants $n^X_{g,\gamma}$ obtained from the $N^X_{g,\gamma}$ by applying the BPS corrections packaged in the formula of (5.4): we let

$$F^X(u, v) := \sum_{g \geqslant 0} \sum_{0 \neq \gamma \in H_2(Z, \mathbf{Z})^\pi} N^X_{g,\gamma}\, u^{2g-2} v^\gamma$$

as a formal power series in the variables u, v, where the powers of v are indexed by $H_2(Z, \mathbf{Z})^\pi$, and set

$$F^X(u, v) := \sum_{g \geqslant 0} \sum_{0 \neq \gamma \in H_2(Z, \mathbf{Z})^\pi} n^X_{g,\gamma}\, u^{2g-2} \left(\sum_{d>0} \frac{1}{d} \left(\frac{\sin d\frac{u}{2}}{\frac{u}{2}} \right)^{2g-2} v^{d\gamma} \right).$$

Finally, for a non-zero multidegree $\mathbf{d} = (d_1, \dots, d_r) \in \mathbf{Z}^r$, we let $n^X_{g,\mathbf{d}}$ be the invariant counting genus g stable maps in vertical classes of degree $d_1, \dots, d_r$ with respect to $L_1, \dots, L_r$ respectively, i.e.,

$$n^X_{g,\mathbf{d}} := \sum_{\gamma \in H_2(X, \mathbf{Z})^\pi :\ \int_\gamma L_i = d_i} n^X_{g,\gamma}. \tag{6.7.1}$$

(6.8) Reduced Gromov–Witten invariants of $K3$ fibres. We also consider the invariants $r^p_{g,m}$ for $K3$ surfaces which have been defined in (5.4); recall that they are the reduced Hodge integrals (5.3.1) put in BPS form. Note that we need to maintain the dependency on the divisibility index m, because Theorem (6.9) below is needed for the proof of the independence on m conjectured by Yau–Zaslow.

A multidegree $\mathbf{d} = (d_1, \dots, d_r) \in \mathbf{Z}^r$ is positive with respect to L^π if for any line bundle M on some fibre X_t of π, $(M, L_{i,t}) = d_i$ for all i implies $(M, L^\pi) > 0$; since L^π is a linear combination of the L_i, this is an elementary linear algebraic condition.

(6.9) Theorem ([28])**.** *Let $\mathbf{d} = (d_1, \dots, d_r) \in \mathbf{Z}^r$ be a multidegree which is positive with respect to L^π. Then*

$$n^X_{g,\mathbf{d}} = \sum_{p=0}^{+\infty} \sum_{m=1}^{+\infty} r^p_{g,m} \cdot \mathrm{NL}^\pi_{m,p,\mathbf{d}}. \tag{6.9.1}$$

(This is stated in [28] in the $r = 1$ case (*i.e.*, $\mathbf{d} \in \mathbf{Z}$), but as noted in [23] the same proof goes through in general).

The philosophy behind this relation is rather natural, and ought to be compared to the discussion of Section 5.1 above. Consider the genus $g = 0$ case for simplicity; then the invariant $n^X_{0,\mathbf{d}}$ counts vertical rational curves in X of prescribed degrees with respect to $L_1, \dots, L_r$, and these are virtually in finite number. There are on the other hand finitely many members of the family π with algebraic divisor classes of the prescribed degrees with respect to $L_1, \dots, L_r$, each of which provides a finite number of rational curves. The theorem morally says that the number of rational curves in X is the sum of these isolated contributions from the fibres.

Of course the actual story is more complicated than this, if only because of the existence of 1-dimensional families of rational curves on X, coming from finitely many rational curves in *all* $K3$ members of the family (which all have algebraic divisor classes, as they are Λ-polarized), in spite of the virtual dimension being 0. In other words, a Calabi-Yau threefold X as in Theorem (6.9) above is far from satisfying the same properties than the perturbations of the twistor families of algebraic $K3$ surfaces on which BPS numbers are supposed to count curves.

6.3 – Application to the Yau–Zaslow conjecture

In this subsection we give an outline of the proof by Klemm, Maulik, Pandharipande, and Scheidegger of the Yau–Zaslow conjecture (Theorem (4.14) above). Recall that the invariants $r^p_{g,m}$ being invariant under algebraic deformations of the $K3$ surface, it is enough to prove the result for our favourite $K3$ surface. These invariants for certain elliptic $K3$ surfaces are expressed by means of the relation (6.9.1) for a particular family.

(6.10) The *STU* model. The central character of the proof is a smooth projective Calabi–Yau 3-fold X, known as the *STU* model and coming from physics (quoting [23]: the letter S stands for the dilaton and T and U label the torus moduli in the heterotic string). It is contructed as an anticanonical section of a smooth projective toric 4-fold Y defined by an explicit fan in $\mathbf{Z}^4$.

The variety X has the structure of a fibration $\pi : X \to \mathbf{P}^1$, the general fibre of which is a smooth $K3$ surface, itself with an elliptic fibration. It comes with two line bundles $L_1, L_2 \to X$, defining a Λ-polarization on the family $\pi : X \to \mathbf{P}^1$ (leaving aside the fact that there are inevitably singular members), where Λ is the lattice with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The family π has the shape of a Lefschetz pencil, in particular each of its singular members has a unique ordinary double point as its only singularity. One may thus build an actual family $\tilde{\pi} : \tilde{X} \to C$ of Λ-polarized $K3$ surfaces from π as follows. One first performs a base change by $t \mapsto t^2$ around each singular member; to do so, one considers the $2:1$ covering $\varepsilon : C \to \mathbf{P}^1$ with branch divisor $\mathrm{Disc}(\pi)$, the set of points above which π fails to be smooth, and let $\pi^\flat : X^\flat \to C$ be the family obtained from π by applying the base change $\varepsilon : C \to \mathbf{P}^1$. The new total space $X^\flat$ is singular, precisely it has an ordinary 3-fold double point at each singular point of a fibre (analytically locally around such a point, X is defined by the equation $x^2 + y^2 + z^2 = t$ in a 4-dimensional complex ball, hence $X^\flat$ is defined by $x^2 + y^2 + z^2 = t^2$). One then chooses for $\tilde{X}$ any small resolution of all these singularities: this may be understood as first blowing-up once all singular points, and then contracting one ruling of each exceptional divisor (they are all smooth quadric surfaces). This has the effect of replacing each fibre of $X^\flat$ by its minimal model.

One may determine the number of singular members of π by the same topological Euler characteristic computation as in Section 2.1. The Euler number $e(X)$ is found to be -480 by toric intersection computations in the 4-fold Y, hence the number of singular fibres equals

$$e(K3) \cdot e(\mathbf{P}^1) - e(X) = 528$$

($e(K3)$ denotes the common Euler number of all $K3$ surfaces, i.e., 24).

(6.11) Modularity for Noether–Lefschetz numbers. It is a stunning application of a theory developed by Borcherds and Kudla–Millson (see [28, 23] and the references therein) that the Noether–Lefschetz numbers of a family of Λ-polarized $K3$ surfaces fit into a vector valued modular form.

In order to state this precisely, let us return to the setup of (6.6) for a moment (see [28, § 4] for a complete treatment). One may define divisors $D^{\mathbf{v}_\Lambda}_{p,\mathbf{d}}$ and numbers $\mathrm{NL}^\pi_{p,\mathbf{d}}$ by dropping the requirement on the divisibility index m in (6.4). It is an elementary result [23, Lemma 1] that the full set of the numbers $\mathrm{NL}^\pi_{p,\mathbf{d}}$ determine the refined Noether–Lefschetz numbers $\mathrm{NL}^\pi_{m,p,\mathbf{d}}$. Let $\mathrm{Mp}_2(\mathbf{Z})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbf{Z})$. There is a canonical representation

$$\rho^*_\Lambda : \mathrm{Mp}_2(\mathbf{Z}) \to \mathrm{End}\big(\mathbf{C}[G_\Lambda]\big)$$

associated to Λ (recall that $G_\Lambda = \Lambda^\vee/\Lambda$).

(6.12) Theorem (Borcherds, Kudla–Millson, Maulik–Pandharipande)**.** *There exists a vector-valued modular form*

$$\Phi^\pi(q) = \sum_{\gamma \in G_\Lambda} \Phi^\pi_\gamma(q)\, u^\gamma \in \mathbf{C}[[q^{\frac{1}{2\mathrm{Disc}\Lambda}}]] \otimes \mathbf{C}[G_\Lambda]$$

of weight $\frac{22-r}{2}$ *and type* ρ^*_Λ*, such that the Noether–Lefschetz number* $\mathrm{NL}^\pi_{p,\mathbf{d}}$ *is the coefficient of* Φ^π_γ *in* q *to the power* $\frac{\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}}}{2\mathrm{Disc}\Lambda}$*, where* $\gamma \in G_\Lambda$ *is any of the two liftings of the coset* $\delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} \in G_\Lambda/\pm$ *represented by the linear functional* $v_i \mapsto d_i$*.*

Taking advantage of the strong structure results for modular forms, Maulik and Pandharipande are able to use this theorem to derive explicitly the Noether–Lefchetz numbers of classical families of $K3$ surfaces of genus $2 \leqslant p \leqslant 5$ (i.e., double planes and complete intersection $K3$'s). A similar calculation is carried out in [23] for the *STU* family, as one of the key steps in the proof of the Yau–Zaslow conjecture.

(6.13) Let us now return to the setup of (6.10). Theorem (6.12) tells that the Noether–Lefschetz numbers of the family $\tilde{\pi} : \tilde{X} \to C$ are the Fourier coefficients of a scalar modular form of weight 10. The vector space of such forms has dimension 1 and is generated by the Eisenstein series

$$E_5(q) = E_2(q)E_3(q) = 1 - 264\sum_{n=1}^{+\infty}\sigma_9(n)q^n$$

[39, § VII.3.2] (notation as in (3.6)). It follows that it is enough to know one Noether–Lefschetz number to determine the full modular form, and since we do know one of them, namely the number 528 of singular members of the STU family, one obtains that the number $\mathrm{NL}^{\tilde{\pi}}_{p,d_1,d_2}$ is the coefficient in q to the power $\frac{1}{2}\Delta(p,d_1,d_2)$ of the modular form $-4E_2(q)E_3(q)$, where

$$\Delta(p,d_1,d_2) = \begin{vmatrix} 0 & 1 & d_1 \\ 1 & 0 & d_2 \\ d_1 & d_2 & 2p-2 \end{vmatrix}.$$

(6.14) Mirror symmetry. The STU model X being an anticanonical section of a smooth semi-positive toric variety, its genus 0 Gromov–Witten invariants are known by mathematically proven mirror symmetry results. This gives the corresponding invariants of $\tilde{X}$, the latter being twice those of X [28, § 5.2].

Precisely, Givental has proven the relation of the genus 0 Gromov–Witten invariants of X by mirror transformation to hypergeometric solutions of the Picard–Fuchs equations of the Batyrev–Borisov mirror, see [28, 23] and the references therein. This gives the following formula of Klemm–Mayr–Lerche [23, Prop. 5]

$$\sum_{(d_1,d_2)\in\mathcal{P}} (d_2)^3 N^X_{0,(d_1,d_2)}\, q_1^{d_1}q_2^{d_2} = -2 + 2\,\frac{E_2(q_1)E_3(q_1)}{(2\pi)^{-12}\Delta(q_1)}\,\frac{E_2(q_2)}{j(q_1)-j(q_2)}, \tag{6.14.1}$$

where

$$j(q) := 1728\,\frac{\big(60G_2(q)\big)^3}{\Delta(q)} = (2\pi)^{12}\frac{E_2(q)^3}{\Delta(q)} = \frac{1}{q} + 744 + 196884q + \cdots$$

(notation as in (3.6)) is the normalized j function, $\mathcal{P} = \big\{(d_1,d_2) \neq (0,0) : d_1 \geqslant 0, d_1 \geqslant -d_2\big\}$, and $N^X_{0,(d_1,d_2)}$ is defined by formula (6.7.1) from the various $N^X_{0,\gamma}$, $\gamma \in H_2(X,\mathbf{Z})^\pi$.

(6.15) Conclusion: the Harvey–Moore identity. Using the fact that the lattice Λ has rank 2, Klemm–Maulik–Pandharipande–Scheidegger then show that the invariants $r^p_{0,m}$ are uniquely determined by the relations (6.9.1) for the family $\tilde{\pi} : \tilde{X} \to C$, and the numbers $n^{\tilde{X}}_{0,(d_1,d_2)}$ and $\mathrm{NL}^{\tilde{\pi}}_{m,p,(d_1,d_2)}$ [23, Prop. 3]. The two latter sets of numbers being known by the results of (6.11) and (6.14), it is therefore enough, in order to end the proof of Theorem (4.14), to show that the numbers $r^p_{0,m}$ predicted by the Yau–Zaslow conjecture (i.e., $r^p_{0,m} = r^{m^2p-m^2+1}_{0,1}$ together with the formula of (2.1) giving the $r^p_{0,1}$'s) indeed fit in the relations (6.9.1). This takes the form of an identity between modular forms: let

$$f(z) := \frac{E_2(z)E_3(z)}{(2\pi)^{-12}\Delta(z)} = \sum_{n=-1}^{+\infty} c(n)q^n, \quad q = e^{2\pi i z};$$

what has to be proven is

$$\frac{f(z_1)E_2(z_2)}{j(z_1)-j(z_2)} = \frac{q_1}{q_1-q_2} + E_2(z_2) - \sum_{d,k,l>0} l^3 c(kl)\, q_1^{kd}q_2^{ld}.$$

This is the Harvey–Moore identity, which has been proven by Zagier, see [23, § 4.2].

References

Chapters in this Volume

[II] Th. Dedieu and E. Sernesi, *Deformations of curves on surfaces*.

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces*.

[XIII] Th. Dedieu, *Node polynomials for curves on surfaces*.

[XIV] F. Bastianelli and Th. Dedieu, *The Göttsche conjecture and the Göttsche–Yau–Zaslow formula*.

External references

[1] E. Ballico and C. Fontanari, *Finiteness of rational curves of degree 12 on a general quintic threefold*, Pure Appl. Math. Q. **11** (2015), no. 4, 537–557.

[2] W. P. Barth, K. Hulek, C. A. M. Peters and A. Van de Ven, *Compact complex surfaces*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, vol. 4, Springer-Verlag, Berlin, 2004.

[3] A. Beauville, *Variétés kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782.

[4] ———, *Counting rational curves on $K3$ surfaces*, Duke Math. J. **97** (1999), no. 1, 99–108.

[5] J. H. Bruinier, G. van der Geer, G. Harder and D. Zagier, *The 1-2-3 of modular forms*, Universitext, Springer-Verlag, Berlin, 2008, Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[6] J. Bryan and N. C. Leung, *The enumerative geometry of $K3$ surfaces and modular forms*, J. Amer. Math. Soc. **13** (2000), no. 2, 371–410 (electronic).

[7] P. Candelas, X. C. de la Ossa, P. S. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), no. 1, 21–74.

[8] X. Chen, *A simple proof that rational curves on $K3$ are nodal*, Math. Ann. **324** (2002), no. 1, 71–104.

[9] E. Cotterill, *Rational curves of degree 11 on a general quintic 3-fold*, Q. J. Math. **63** (2012), no. 3, 539–568.

[10] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Providence, RI: American Mathematical Society, 1999.

[11] J. D'Almeida, *Courbes rationnelles de degré 11 sur une hypersurface quintique générale de $\mathbf{P}^4$*, Bull. Sci. Math. **136** (2012), no. 8, 899–903.

[12] Th. Dedieu and E. Sernesi, *Equigeneric and equisingular families of curves on surfaces*, Publ. Mat. **61** (2017), no. 1, 175–212.

[13] S. Diaz and J. Harris, *Ideals associated to deformations of singular plane curves*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 433–468.

[14] B. Fantechi, L. Göttsche and D. van Straten, *Euler number of the compactified Jacobian and multiplicity of rational curves*, J. Algebr. Geom. **8** (1999), no. 1, 115–133.

[15] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics.

[16] A. Gathmann, *Gromov-Witten invariants of hypersurfaces*, Habilitation thesis, TU Kaiserslautern, 2003.

[17] ———, *The number of plane conics that are five-fold tangent to a given curve*, Compos. Math. **141** (2005), no. 2, 487–501.

[18] R. Godement, *Topologie algébrique et théorie des faisceaux*, Publ. Inst. Math. Univ. Strasbourg, No. 13, Hermann, Paris, 1958.

[19] L. Göttsche and R. Pandharipande, *The quantum cohomology of blow-ups of* $\mathbf{P}^2$ *and enumerative geometry*, J. Differential Geom. **48** (1998), no. 1, 61–90.

[20] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990), no. 1-3, 193–207.

[21] ———, *A conjectural generating function for numbers of curves on surfaces*, Comm. Math. Phys. **196** (1998), no. 3, 523–533.

[22] D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. **135** (1999), no. 1, 63–113.

[23] A. Klemm, D. Maulik, R. Pandharipande and E. Scheidegger, *Noether-Lefschetz theory and the Yau-Zaslow conjecture*, J. Am. Math. Soc. **23** (2010), no. 4, 1013–1040.

[24] J. Lee and N. C. Leung, *Counting elliptic curves in* $K3$ *surfaces*, J. Algebraic Geom. **15** (2006), no. 4, 591–601.

[25] Q. Liu, *Algebraic geometry and arithmetic curves, transl. by R. Erné*, Oxford: Oxford University Press, 2006, paperback new edition.

[26] D. Maulik, R. Pandharipande and R. Thomas, *Curves on* $K3$ *surfaces and modular forms*, J. Topol. **3** (2010), no. 4, 937–996.

[27] D. Maulik, *Stable pairs and the HOMFLY polynomial*, Invent. Math. **204** (2016), no. 3, 787–831.

[28] D. Maulik and R. Pandharipande, *Gromov-Witten theory and Noether-Lefschetz theory*, in *A celebration of algebraic geometry. A conference in honor of Joe Harris' 60th birthday, Harvard University, Cambridge, MA, USA, August 25–28, 2011*, Providence, RI: American Mathematical Society (AMS); Cambridge, MA: Clay Mathematics Institute, 2013, 469–507.

[29] A. Morelli, *Un'osservazione sulle singolarità delle trasformate birazionali di una curva algebrica*, Rend. Accad. Sci. Fis. Mat. Napoli (4) **29** (1962), 59–64.

[30] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or* $K3$ *surface*, Invent. Math. **77** (1984), no. 1, 101–116.

[31] A. Oblomkov and Z. Yun, *The cohomology ring of certain compactified Jacobians*, preprint arXiv:1710.05391.

[32] R. Pandharipande and R. P. Thomas, *13/2 ways of counting curves*, in *Moduli spaces*, London Math. Soc. Lecture Note Ser., vol. 411, Cambridge Univ. Press, Cambridge, 2014, 282–333.

[33] R. Pandharipande, *Maps, sheaves, and* $K3$ *surfaces*, Clay Lect. Notes, Oxford University Press, Oxford, 2017, 159–185.

[34] ———, *Rational curves on hypersurfaces (after A. Givental)*, Astérisque (1998), no. 252, Exp. No. 848, 5, 307–340, Séminaire Bourbaki. Vol. 1997/98.

[35] R. Pandharipande and R. P. Thomas, *Notes on the proof of the KKV conjecture*, Surv. Differ. Geom., **21** International Press, Somerville, MA, 2016, 289–311.

[36] ———, *The Katz-Klemm-Vafa conjecture for K3 surfaces*, Forum Math. Pi **4** (2016), e4, 111 pp.

[37] Séminaire Palaiseau Oct. 1981–Jan. 1982, *Géométrie des surfaces* $K3$*: modules et périodes*, Astérisque, vol. 126, Société Mathématique de France, Paris, 1985.

[38] ———, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften, vol. 334, Springer-Verlag, Berlin, 2006.

[39] J.-P. Serre, *Cours d'arithmétique*, Collection SUP: "Le Mathématicien", vol. 2, Presses Universitaires de France, Paris, 1970, english translation available.

[40] V. Shende, *Hilbert schemes of points on a locally planar curve and the Severi strata of its versal deformation*, Compos. Math. **148** (2012), no. 2, 531–547.

[41] I. Vainsencher, *Enumeration of n-fold tangent hyperplanes to a surface*, J. Algebraic Geom. **4** (1995), no. 3, 503–526.

[42] J.-L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard*, Invent. Math. **36** (1976), 295–312.

[43] C. Voisin, *Symétrie miroir*, Paris: Société Mathématique de France, 1996.

[44] S.-T. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on $K3$*, Nuclear Phys. B **471** (1996), no. 3, 503–512.

[45] Y. Zhao, *Stable maps, multiplicities, and compactified Jacobians*, preprint arXiv:2604.05659.

Lecture XII
Some classical formulæ for curves and surfaces

by Thomas Dedieu

General notation . . . 311
1 Background material . . . 312
1.1 Projective duality . . . 312
1.2 Projective duality for hypersurfaces and polarity . . . 313
1.3 Plücker formulae for plane curves . . . 314
2 Double curves of the dual of a surface in $\mathbf{P}^3$ . . . 316
2.1 Local geometry of a surface in $\mathbf{P}^3$ and its dual . . . 316
2.2 Degrees of the double curves . . . 317
3 Zero-dimensional strata of the dual surface . . . 320
3.1 Local models . . . 320
3.2 Polar cones and their cuspidal edges . . . 322
3.3 Application to the case when $X = S^\vee$ . . . 326
4 Hessian and node-couple developables . . . 329
4.1 Developables associated to curves on a surface . . . 330
4.2 The Hessian developable surface . . . 332
4.3 Ordinary double edges of the polar cone . . . 336
4.4 Application to the case when $X = S^\vee$ . . . 338
5 Formulae for surfaces projected in $\mathbf{P}^3$ . . . 339
5.1 The 3-dimensional case: surfaces with ordinary singularities . . . 340
5.2 The postulation formula . . . 343
5.3 The 2-dimensional case: multiple planes . . . 345
A Polynomial calculus . . . 346
A.1 Polarity . . . 347
A.2 Hessian and second fundamental form . . . 350
B Developable surfaces . . . 351
B.1 Synthetic description . . . 352
B.2 Analysis of contacts . . . 353
B.3 Plücker formulae . . . 357
B.4 Generalized Plücker formulae . . . 358
B.5 De Jonquières' formula . . . 361
C Biduality for asymptotic tangent lines by enumerative means . . . 362
References . . . 365

The goal of this text is to present the computation by Salmon, in the second half of the XIXth century (now a little less than two-hundred years ago), of various numbers enumerating planes with a prescribed tangency pattern with a smooth, and sufficiently general, surface S in $\mathbf{P}^3$ (or, equivalently, of hyperplane sections of S with prescribed singularities). Emblematic among these are the number $\check{t}$ of tritangent planes, given in (3.17), and the number $\check{\beta}$ of planes cutting out a curve with a tacnode, given in (3.15).

The aforementioned numbers are the degrees of various loci in the dual projective space $\check{\mathbf{P}}^3$ (the points of which correspond to planes in $\mathbf{P}^3$), among which the closures of the various equisingular strata of the dual surface $S^\vee$ (the points of the latter correspond to planes tangent to S). The results we present feature not only the computation of these degrees, but also the description of the intricate geometry of these loci and their counterparts on S. Among these loci, one finds the parabolic curve of S (the locus of points at which the tangent plane cuts out a cuspidal curve; it corresponds to the cuspidal double curve of $S^\vee$), the node-couple curve (the locus of points at which the tangent plane is also tangent at another point; it corresponds to the ordinary double curve of $S^\vee$), and the flecnodal curve (the locus of points p at which the tangent plane cuts out a curve with one local branch with a flex at p; its counterpart on $S^\vee$ is not visible in terms of equisingular strata). A fundamental aspect of this geometry is the duality relation between S and its dual, and more generally between the loci in each of them.

Salmon's arguments are specific to hypersurfaces, and based on the manipulation of polynomials and elimination theory. For instance, he determines relatively explicit polynomials cutting out the node-couple and flecnodal curves on S (the parabolic curve is cut out by the Hessian determinant). He also makes an essential use of polars, which for hypersurfaces may be defined in terms of polynomial calculus, as we recall in Appendix A.1. His method to compute the number $\check{t}$ of tritangent planes, which we present in Sections 3.2 and 3.3, is as follows. Let ϖ be a general point of $\check{\mathbf{P}}^3$, and consider the intersection of $S^\vee$ with its first and second polars with respect to ϖ. It is supported on the locus of points $\varkappa$ such that the line $\langle \varpi, \varkappa \rangle$ has intersection multiplicity at least three with $S^\vee$ at $\varkappa$, which of course comprises the triple points of $S^\vee$, and among those the points corresponding to tritangent planes. Then, a fine analysis of this intersection provides a set of relations involving the enumerative numbers we are interested in (but not only them), see Theorem (3.7), and it turns out we are able to compute sufficiently many of the quantities in these relations in some other way (for instance, the aforementioned number $\check{\beta}$ is obtained by intersecting the parabolic and flecnodal curves on S) so that we can finally find $\check{t}$.

We also present, still following Salmon, the computation of enumerative numbers related to the already mentioned loci, but not directly to the enumeration of hyperplane sections with prescribed singularities. For instance, we find the number of apparent double points of the cuspidal and ordinary double curve of $S^\vee$, namely $g_{\mathrm{He}} = \check{h}$ and $g_{\mathrm{Kn}} = \check{k}$ given in (4.17) and (4.26), respectively. In fact, we study (in Section 4) the two developable surfaces in $\mathbf{P}^3$ dual to the cuspidal and ordinary double curves of $S^\vee$, and compute their numerical characters. This involves a fine study of the local geometry of S along its parabolic curve (see Section 4.2), which complements the aforementioned description of the geometry of S and its duality with $S^\vee$. This also comprises an analysis analogous to that described above for the enumeration of tritangent planes, in which one considers this time the intersection of $S^\vee$ and its first polar with respect to ϖ with another hypersurface derived from $S^\vee$ and relative to ϖ, such that the intersection is supported on the locus of points $\varkappa$ such that the line $\langle \varpi, \varkappa \rangle$ has intersection multiplicity with $S^\vee$ greater than one both at $\varkappa$ and at some other point.

The last Section 5 is dedicated to relatively more recent formulas (from the first half of the XXth century, now a little less than one-hundred years ago), following the presentation given by Enriques in his landmark book [15]. The main novelty is that they involve new invariants from a different nature (Salmon's formulas are all in term of the degree n of the surface only), such as the arithmetic genus, and are applicable to surfaces in $\mathbf{P}^3$ with ordinary singularities, so that in practice they apply to any general three-dimensional linear subsystem of a very ample linear system on a smooth surface. This study includes in particular proofs of the Noether formula for the arithmetic genus of a smooth surface, and the classical postulation formula.

Appendix B contains a presentation of developable surfaces in $\mathbf{P}^3$ in classical terms, following Edge's beautiful book [14]. It features Plücker formulas for curves in $\mathbf{P}^3$, that are necessary

for the computations in Section 4. Perhaps more importantly, it is the occasion to describe the duality relations between a curve in $\mathbf{P}^3$ and its dual, following the guiding thread of this text. This all has been generalized, in modern times, to curves in a projective space of arbitrary dimension by Piene [36], and her results are included as well.

Appendix C proposes an application of a basic topological formula counting singular curves in a pencil, see [XI, Section 2.1], to the proof of a duality statement, namely that the orthogonality correspondence takes asymptotic tangent lines of a surface in $\mathbf{P}^3$ to asymptotic tangent lines of its dual.

For the two central Sections 3 and 4, I have followed rather closely Salmon's magnificent book [45], in its fifth edition of 1915 (I have not dug in earlier text of his, or prior versions of this book). I have done my best to add explanations and justifications wherever I felt this was needed (and this is a lot of places, if I may be so bold). Yet a blind spot of the present text is that it leaves the question of the validity of Salmon's formulas aside: they apply to a smooth surface under suitable transversality conditions, and we make all along the blanket assumption that a sufficiently general surface indeed verifies the latter transversality conditions, but do not prove it; see Paragraph (3.8) for some references on this question.

Salmon's formulas have been revisited in several modern texts already, in particular by Piene [38], using her theory of polars [37]. Besides, the title of this text is a nod at the title of [38], itself a nod at a series of papers by Roth. There is also a proof by Ronga [42], based on the use of multiple-point formulas: these ideas are discussed in [XIII]. Witaszek's thesis [53] is very complete and valuable, concentrated on the study of very general quartic surfaces. I have done my best to quote all relevant works on the topic. While I certainly have quoted all those that I took inspiration from, I claim by no means to have been encyclopedic. There are undoubtly many references missing (in particular, I have intentionally left aside some of the older material). May their authors and, more importantly, the interested readers forgive me.

Let me quote G.C. Rota to conclude this introduction: « Le teorie vanno e vengono ma le formule restano. »[1]

Thanks. I am grateful to Ragni Piene for her observations on a preliminary version of this text.

General notation. The exponent (and sometimes the superscript) $\vee$ denotes the dual, in the sense of vector spaces and projective spaces, or of projective geometry (see Section 1.1). The symbol $\perp$ denotes the orthogonal with respect to the pairing between linear functionals and vectors as well as, most often, the induced notion between points in two dual projective spaces (we will usually dodge the difference between vector spaces and projective spaces, thus trading off slight abuses for lighter notations and, hopefully, clarity). Thus, if p is a point in $\mathbf{P}^N$, then $p^\perp$ is the hyperplane in the dual projective space $\check{\mathbf{P}}^N$ whose points correspond to all hyperplanes of $\mathbf{P}^N$ that contain p; and if H is a hyperplane of $\mathbf{P}^N$, $H^\perp$ is the corresponding point of $\check{\mathbf{P}}^N$. For any subset $A \subseteq \mathbf{P}^N$, $\langle A \rangle$ denotes the span of A; thus, if p and q are two distinct points, $\langle p, q \rangle$ is the line spanned by p and q.

For a vector space V, we denote by $\mathbf{P}(V)$ the projective space of all vector lines of V. If W is a linear subspace of V, we use the shorthand $\mathbf{P}(V)/\mathbf{P}(W)$ for the projective space $\mathbf{P}(V/W)$, so that if Λ is a linear subspace of some projective space $\mathbf{P}^N$, then $\mathbf{P}^N/\Lambda$ denotes the corresponding quotient projective space. We will then usually denote by $\pi_\Lambda : \mathbf{P}^N \dashrightarrow \mathbf{P}^N/\Lambda \cong \mathbf{P}^m$ the projection from Λ. We may abuse notation and denote, for any linear subspace F of $\mathbf{P}^N/\Lambda$, by $\pi_\Lambda^{-1}(F)$ the largest linear subspace of $\mathbf{P}^N$ surjecting onto F.

[1] « The theories may come and go but the formulas remain. » Quote and translation stolen from Ottaviani's Five lectures on projective invariants.

If p is a point of a scheme X, we denote by $\mathrm{TC}_p X$ the tangent cone of X at p; if p is a smooth point of X, we denote by $T_p X$ its tangent space at p; if moreover X is seen as lying in some projective space, then $\mathbf{T}_p X$ denotes its projective tangent space at p.

If X is a degree n hypersurface in some projective space $\mathbf{P}^N$, we denote by $\mathrm{D}^{p^k} X$ its k-th polar with respect to the point p; it is a degree $n-k$ hypersurface.

The notation $O(d)$ denotes a linear combination of monomials which all have degree at least d.

The letter n is reserved throughout the text to the degree of a hypersurface, with the (hopefully only) exception of Appendix A .

1 – Background material

1.1 – Projective duality

The theme of this text is the enumeration of hyperplanes with a prescribed tangency pattern with a smooth surface in $\mathbf{P}^3$. To get started, let us recall the basics of projective duality, that we will use repeatedly in the course of the text. For a more complete introduction, see [49, Chap. 1].

(1.1) Definition. *Let X be a variety in $\mathbf{P}^N$. The* dual variety *of X is the Zariski closure $X^\vee$ in the dual projective space $\check{\mathbf{P}}^N$ of the set of points $\varpi \in \check{\mathbf{P}}^N$ such that the hyperplane $\varpi^\perp \subseteq \mathbf{P}^N$ is tangent to X at one of its smooth points.*

A central result is the following *reflexivity theorem*. For an algebraic proof, see [26] (which also features a detailed history of this result), or [21, Example 16.20]. For a proof using symplectic geometry, see [19, Chap. 1].

(1.2) Theorem. *Let X be a variety in $\mathbf{P}^N$. One has $(X^\vee)^\vee = X$. More precisely, let p and ϖ be smooth points of X and $X^\vee$ respectively: The hyperplane $\varpi^\perp \subseteq \mathbf{P}^N$ is tangent to X at p if and only if the hyperplane $p^\perp \subseteq \check{\mathbf{P}}^N$ is tangent to $X^\vee$ at ϖ.*

This may be refined in the following way. For a proof, see [25], or [52, §14.1].

(1.3) Theorem. *Suppose X is smooth and its dual is a hypersurface. A point $\varpi \in X^\vee$ is smooth if and only if the hyperplane $\varpi \subseteq \mathbf{P}^N$ is tangent to X in a unique point p, and $\varpi^\perp \cap X$ has an ordinary double point at p.*

The general philosophy is that the singularity of the dual in a point ϖ reflects the tangency scheme of the hyperplane $\varpi^\perp$ with X. I shall loosely refer to any result incarnating this philosophy as a *biduality statement*. For instance, when X itself is a smooth hypersurface one has the following result, see [12, Prop. 11.24].

(1.4) Theorem. *Let X be a smooth hypersurface in $\mathbf{P}^N$, and let $\varpi \in X^\vee$ be such that the singular locus of $\varpi^\perp \cap X$ consists of finitely many points $p_1, \ldots, p_r$. Then the tangent cone to $X^\vee$ at ϖ is the union of all hyperplanes $p_1^\perp, \ldots, p_r^\perp$, each counted with multiplicity $\mu(X, \varpi, p_i)$, the Milnor number of the singularity of $\varpi^\perp \cap X$ at p_i.*

The following result (see [49, Tm. 1.21]) says that linear projections and taking linear sections are dual operations. Recall from elementary linear algebra that if W is a linear subspace of a vector space V, then the transpose of the projection $V \to V/W$ canonically identifies with the injection $W^\perp \hookrightarrow V^\vee$.

(1.5) Proposition. *Let X be a variety in $\mathbf{P}^N$, and Λ be a general linear subspace of codimension $m+1$ in $\mathbf{P}^N$. Consider the projection from Λ,*

$$\pi_\Lambda : \mathbf{P}^N \dashrightarrow \mathbf{P}^N/\Lambda \cong \mathbf{P}^m.$$

i) *If $m > \dim(X)$ and $\pi_\Lambda|_X$ is an isomorphism to its image, then $\pi_\Lambda(X)^\vee = \Lambda^\perp \cap X^\vee$.*
ii) *If $m = \dim(X)$, let $B \subsetneq \mathbf{P}^m$ be the branch divisor of $\pi_\Lambda|_X$. One has $B^\vee = \Lambda^\perp \cap X^\vee$.*
iii) *If $m < \dim(X)$, let $\tilde{X} \to X$ be the blow-up of X at $\Lambda \cap X$, and $\tilde{\pi}_\Lambda : \tilde{X} \to \mathbf{P}^m$ be the morphism induced by π_Λ. Let $\Delta \subsetneq \mathbf{P}^m$ be the discriminant of $\tilde{\pi}_\Lambda$. One has $\Delta^\vee = \Lambda^\perp \cap X^\vee$.*

1.2 – Projective duality for hypersurfaces and polarity

When X is a hypersurface, much information on $X^\vee$ may be obtained by studying the defining equation of X using polynomial calculus. We gather here a few facts relevant for us, and refer to [13, Chap. 1] for a much more complete treatment. A key technical tool is polarity, the main results of which are included in Appendix A.1.

In this section, we always assume that X is a hypersurface in $\mathbf{P}^N$, defined by a degree n homogeneous polynomial F in a system of homogeneous coordinates $(x_0 : \ldots : x_N)$.

(1.6) Let $q = (q_0 : \ldots : q_N)$ be a smooth point of X. By Euler's formula, the equation of the tangent hyperplane to X at q is

$$\partial_0 F(q) \cdot x_0 + \cdots + \partial_N F(q) \cdot x_N = \mathrm{D}_q F(x_0, \ldots, x_N) = \mathrm{D}^{(x_0,\ldots,x_N)} F(q) = 0,$$

in the notation of (A.1). In particular, the point $p = (x_0 : \ldots : x_N)$ sits on $\mathbf{T}_q X$ if and only if q sits in the degree $n-1$ hypersurface $\mathrm{D}^p X$.

(1.7) Suppose for simplicity that X is smooth. It may be seen from (1.6) that for all $p \in \mathbf{P}^N$, the intersection $\mathrm{D}^p X \cap X$ is the locus of those points $q \in X$ such that the line $\langle p, q \rangle$ is tangent to X at q. Thus, $\mathrm{D}^p X \cap X$ is the ramification divisor of the projection $\pi_p : X \to \mathbf{P}^{N-1}$. It is called the *apparent boundary* of X as seen from the point p; I will denote it by $\mathcal{B}^p X$. The cone projecting the apparent boundary from p is called the *circumscribed cone*; I will denote it by $\mathcal{C}^p X$. When X is singular, the circumscribed cone is the closure of the set of all lines passing through p and tangent to X at a smooth point; we will use this definition in Paragraph (3.5).

(1.8) Gauss map and degree of the dual. Since there is a unique tangent hyperplane at each smooth point of X, there is a dominant rational map

$$\gamma_X : X \dashrightarrow X^\vee$$

called the *Gauss map*. If $X^\vee$ is a hypersurface, it follows from (1.3) that the Gauss map is birational.

It follows from (1.6) that the Gauss map of X is the restriction to X of the map defined on $\mathbf{P}^N$ by the N-dimensional linear system of all first polars of X, i.e., of all $\mathrm{D}^p X$, $p \in \mathbf{P}^N$. This is a linear subsystem of $|\mathcal{O}_{\mathbf{P}^N}(n-1)|$, and it is base point free if and only if X is smooth.

This gives a way to compute the degree of the dual of X, which is also called the *class* of X. If X is smooth, one finds that the degree of $X^\vee$ equals $n(n-1)^{N-1}$. If X is singular, this has to be corrected by taking into account the base locus of the linear system of first polars, which is supported on the singular locus of X. One may prove the Plücker formula (P_1) of (1.12) in this way. More generally, Teissier [48, Appendice II] has obtained a formula for the class of any hypersurface with isolated singularities; the reader may also consult [13, Section 1.2.3]. The

same formula has been proved by Laumon [33], with a method close in spirit to the topological method described in [XI, Section 2.1]. This formula has been generalized to hypersurfaces with arbitrary singularities by Piene [37].

(1.9) Assume that X is smooth. The Hessian hypersurface $\mathrm{Hess}(X)$ (see Appendix A.2) cuts out on X the locus of those points $p \in X$ such that the tangent section $\mathbf{T}_p X \cap X$ has a singularity at p worse than an ordinary double point. The intersection $X \cap \mathrm{Hess}(X)$ is the ramification divisor of the Gauss map $\gamma_X : X \to X^\vee$.

Before I close this section, let me mention that there exists a notion of polars for varieties in a projective space that are not necessarily hypersurfaces. See, for instance, [37] and [18]. Using these one may compute the class of varieties in projective space other than hypersurfaces, see, e.g., [37] and [51].

1.3 – Plücker formulae for plane curves

As a first and fundamental application of the previous results, let us explain the classical Plücker formulae for plane curves.

(1.10) Let $C \subseteq \mathbf{P}^2$ be a plane curve. It has a finite number of bitangent lines (i.e., lines tangent to C in two distinct points) and inflection points, which give rise respectively to nodes and cusp of the dual curve $C^\vee \subseteq \check{\mathbf{P}}^2$: this can be seen for instance with Theorem (1.4). In general there are no lines with a larger tangent scheme with C, so that nodes and cusps are the only singularities of $C^\vee$.

The reflexivity theorem (1.2) suggests to look for formulae linking numerical characters of C and $C^\vee$ in a symmetric fashion. In this perspective one should allow C to have nodes and cusps.

(1.11) Plückerian characters of a plane curve. We consider a plane curve $C \subseteq \mathbf{P}^2$ such that both C and its dual $C^\vee \subseteq \check{\mathbf{P}}^2$ have only ordinary double points (nodes) and ordinary cusps as singularities. We call:
— n the degree of C;
— $\check{n}$ the degree of $C^\vee$;
— β the number of bitangents of C, which is also the number of nodes of $C^\vee$;
— ι the number of flexes of C, which is also the number of cusps of $C^\vee$;
— δ the number of nodes of C, which is also the number of bitangents of $C^\vee$;
— κ the number of cusps of C, which is also the number of flexes of $C^\vee$.

(1.12) Plücker formulae for plane curves. One has the following relations for a curve C as in (1.11).

$$(P_1)\quad \check{n} = n(n-1) - 2\delta - 3\kappa; \qquad (\check{P}_1)\quad n = \check{n}(\check{n}-1) - 2\beta - 3\iota;$$
$$(P_2)\quad \iota = 3n(n-2) - 6\delta - 8\kappa; \qquad (\check{P}_2)\quad \kappa = 3\check{n}(\check{n}-2) - 6\beta - 8\iota,$$

For a more symmetric version one may prefer to write $\check{\delta}$ and $\check{\kappa}$ for β and ι, respectively.

We sketch the proof, as it is emblematic of our approach in this text, and refer to [16] for a detailed implementation.

One first computes the class of a plane curve with nodes and cusps, following (1.8): this will give the two relations on the first line by applying the computation to both C and $C^\vee$. Let $p \in \mathbf{P}^2$ be a general point. It follows from Theorem (A.10) that the first polar $\mathrm{D}^p C$ passes

through all nodes and cusps of C, is smooth at these points, and has general tangent line at the nodes and tangent line equal to the reduced tangent cone of C at the cusps. Hence $\mathrm{D}^p C$ intersects C with multiplicity $\geqslant 2$ at the nodes and $\geqslant 3$ at the cusps, and in fact these are the actual intersection multiplicities. The relation follows.

Then one computes the number of flexes of C by considering its intersection with its Hessian (see (1.9)): this will give the two relations on the second row. Again, one has to substract the contribution of $C \cap \mathrm{Hess}(C)$ corresponding to the singularities of C and not to actual inflection points. If q is a node of C, then the Hessian curve also has an ordinary double point at q, with tangent cone equal to that of C. If q is a cusp of C, then the Hessian curve has a triple point with two local branches at q, one smooth with general tangent, and one cuspidal with tangent cone equal to that of C. It follows that C intersects its Hessian with multiplicity $\geqslant 6$ at the nodes and $\geqslant 8$ at the cusps, and again these are the actual intersection multiplicities, hence the formula.

(1.13) Generators for the Plücker relations. The four relations in (1.12) generate a codimension three complete intersection ideal, i.e., the Plücker formulae reduce to three formulae only, with the Koszul type relations as the only relations among them. Indeed, the following relation holds:

$$P_1 + \check{P}_1 = \frac{1}{3}(P_2 + \check{P}_2),$$

so that any three among $P_1, P_2, \check{P}_1, \check{P}_2$ generate all Plücker relations. It is arguably an endless game to search for a preferred set of three generators for the Plücker relations; the three relations below form an appealing generating set:

$$\begin{aligned}
&(P_1') && 3n - \kappa = 3\check{n} - \iota \\
&(P_2') && n(n-2) + \check{n}(\check{n}-2) = 2\delta + 3\kappa + 2\beta + 3\iota \\
&(P_3') && 18(\delta - \beta) = (\kappa - \iota)(\kappa - \iota + 6\check{n} - 27).
\end{aligned}$$

They are indeed relations as

$$\begin{aligned}
P_1' &= 3P_1 - P_2 = -3\check{P}_1 + \check{P}_2 \\
P_2' &= P_1 + \check{P}_1 = \frac{1}{3}(P_2 + \check{P}_2) \\
P_3' &= 3\left[3(n+\check{n}) + \kappa - \iota - 3\right] P_1 - \left[3(n+\check{n}) + \kappa - \iota\right] P_2 + 9\,\check{P}_1,
\end{aligned}$$

and they indeed generate the Plücker relations as

$$\begin{pmatrix} P_1 \\ P_2 \\ \check{P}_1 \\ \check{P}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{6}(n+\check{n}) + \frac{1}{18}(\kappa - \iota) & \frac{1}{2} & -\frac{1}{18} \\ \frac{1}{2}(n+\check{n}) + \frac{1}{6}(\kappa - \iota) - 1 & \frac{3}{2} & -\frac{1}{6} \\ -\frac{1}{6}(n+\check{n}) - \frac{1}{18}(\kappa - \iota) & \frac{1}{2} & \frac{1}{18} \\ -\frac{1}{2}(n+\check{n}) - \frac{1}{6}(\kappa - \iota) + 1 & \frac{3}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} P_1' \\ P_2' \\ P_3' \end{pmatrix}.$$

(1.14) Smooth plane curves. When the plane curve C is smooth one has $\delta = \kappa = 0$, so one is left with four Plückerian characters and three independent Plücker relations. The two characters $\check{n}$ and ι are expressed in terms of n by the relations P_1 and P_2, and one finds in addition:

$$\beta = \tfrac{1}{2} n(n-2)(n-3)(n+3).$$

(1.15) The genus formula. The Plücker formulae usually come adjoined with the following relation, gotten from the fact that the two curves C and $\check{C}$, being birational, have the same geometric genus. Computing their respective geometric genera by means of the adjunction formula, one gets:

$$\tfrac{1}{2}(n-1)(n-2) - \delta - \kappa = \tfrac{1}{2}(\check{n}-1)(\check{n}-2) - \beta - \iota.$$

The latter relation may in fact be obtained from the Plücker relations: it is

$$\tfrac{1}{2}(-P_1 - 2\check{P}_1 + \check{P}_2) = -\tfrac{1}{2}\left[\tfrac{1}{3}(n+\check{n}) + \tfrac{1}{9}(\kappa - \iota) - 1\right] P_1' + \tfrac{1}{18}\, P_3'.$$

The condition that a plane curve and its dual both have nodes and cusps as their only singularities poses some delicate questions. We refer to [32] for recent results. Finally, let me mention [31], in which the author introduces a concept of virtual numbers of nodes and cusps, which allows him to give symmetric Plücker formulas valid for curves with arbitrary singularities.

2 – Double curves of the dual of a surface in $\mathbf{P}^3$

We now start our study of surfaces in $\mathbf{P}^3$. Here we give a first description of the dual of a smooth surface in $\mathbf{P}^3$, that will be refined later.

2.1 – Local geometry of a surface in $\mathbf{P}^3$ and its dual

We first take the occasion to introduce some necessary notions on the local geometry of a surface in $\mathbf{P}^3$. Let S be a surface in $\mathbf{P}^3$.

(2.1) Parabolic points. A *parabolic point* of S is a point $p \in S$ such that the Hessian of S at p vanishes (see (A.11)).

Equivalently, the polar quadric of S at p, viz. $\mathrm{D}_{p^2}S$, is singular, in the sense that it is either a singular quadric indeed, or the whole $\mathbf{P}^3$. If p is a smooth point of S, then the first (resp. second) possibility is equivalent to the tangent hyperplane section $S \cap \mathbf{T}_p S$ having a double point at p with tangent cone a double line (resp. multiplicity strictly larger than 2 at p). Thus, in general, the tangent hyperplane section of S at a parabolic point p will have a cusp at p.

If the Hessian of S does not vanish identically on S (this is the case if and only if the dual of S is a surface, see (B.5)), we shall call the curve $S \cap \mathrm{Hess}(S)$ the *parabolic curve* of S, or indifferently the *Hessian curve*, and denote it by $\mathrm{He}(S)$.

(2.2) Asymptotic tangent lines. A line $\Lambda \subseteq \mathbf{P}^3$ is an *asymptotic tangent* of S at the point $p \in S$ if Λ and S have intersection multiplicity at least 3 at p.

By Theorem (A.8), the asymptotic tangent lines at p are the lines contained in the intersection $\mathbf{T}_p S \cap \mathrm{D}_{p^2}S$ of the tangent hyperplane and the polar quadric of S at p (note that $\mathbf{T}_p S = \mathrm{D}_p S$ and $\mathrm{D}_{p^2}S$ are tangent at p by Theorem (A.10)). Thus, there are two distinct asymptotic tangent lines at p if p is not a parabolic point, and one double asymptotic tangent lines at p if p is a parabolic point and the polar quadric $\mathrm{D}_{p^2}S$ is not the whole $\mathbf{P}^3$. If $\mathrm{D}_{p^2}S$ is the whole $\mathbf{P}^3$, then all lines in $\mathbf{T}_p S$ passing through p are asymptotic tangents.

If the polar quadric $\mathrm{D}_{p^2}S$ is not the whole $\mathbf{P}^3$, then the sum of the asymptotic tangent lines at p is the tangent cone at p of the tangent hyperplane section $\mathbf{T}_p S \cap S$.

(2.2.1) Flex tangent lines. The asymptotic tangent lines may also be called *flex tangent lines*. Let $H \subseteq \mathbf{P}^3$ be a hyperplane containing $p \in S$. The curve $H \cap S$ has a flex at p if and only if the plane H contains an asymptotic tangent line of S at p, as an elementary local computation shows.

(2.2.2) A duality statement for asymptotic tangents. For all smooth $p \in S$, if $\Lambda \subseteq \mathbf{P}^3$ is an asymptotic tangent line of S at p, then the line $\Lambda^\perp \subseteq \check{\mathbf{P}}^3$ is an asymptotic tangent line of $S^\vee$ at $(\mathbf{T}_p S)^\perp$.

This is proved in [C, (3.11)], and in Appendix C; see also Proposition (4.6).

(2.3) Codimension one singular loci of the dual. Let S be a general surface in $\mathbf{P}^3$. Then its dual $S^\vee$ is singular along the two curves parametrizing, respectively, bitangent hyperplanes of S, and hyperplanes tangent to S at a parabolic point.

I will call the former the *ordinary double curve* of $S^\vee$. The surface $S^\vee$ indeed has an ordinary double point at its generic point by Theorem (1.4). The two local sheets of $S^\vee$ along the ordinary double curve correspond to deformations of bitangent hyperplanes preserving either one of the two tangency points. The *node-couple curve* $\mathrm{Kn}(S) \subseteq S$ is the closure of the locus of points $p \in S$ such that $\mathbf{T}_p S$ is a bitangent plane. Thus, the Gauss map realizes $\mathrm{Kn}(S)$ as a double cover of the ordinary double curve of $S^\vee$.

I will call *cuspidal double curve* of $S^\vee$ the image of the parabolic curve of S by the Gauss map. The surface $S^\vee$ indeed has an A_2 double point at its generic point, as follows from the fact that the ramification divisor of the birational map $S \dashrightarrow S^\vee$ is the parabolic curve, see [13, Proposition 1.2.1].

We will make throughout the text the blanket assumption that the codimension one singular loci of the dual surface $S^\vee$ amount eaxctly to its ordinary and cuspidal double curves, to which we will refer as the *double curves* of $S^\vee$. We refer to Paragraph (3.8) for a discussion of this assumption.

2.2 – Degrees of the double curves

This section is dedicated to formulae that give the degrees of the ordinary and cuspidal double curves of the dual of a surface S in $\mathbf{P}^3$; the main result is Corollary (2.7). These formulae have been given a modern proof in [24] (see, in particular, Section VII.1 therein).

The underlying idea is that the double curves of the dual $S^\vee$ amount to behaviours occuring in codimension two in the linear system of hyperplane sections of S, hence these behaviours are visible in a general codimension one linear subsystem; therefore, one should be able to find all the relevant information after projecting S from a general point.

(2.4) Setup. Let S be a smooth, degree n, surface in $\mathbf{P}^3$. For all $p \in \mathbf{P}^3$, we let $\mathcal{B}^p S$ be the apparent boundary of S from p (since S is smooth, this is the curve $S \cap \mathrm{D}^p S$), and consider the plane curve $B_p \subseteq \mathbf{P}^2$ image of $\mathcal{B}^p S$ by the projection $\pi_p : \mathbf{P}^3 \dashrightarrow \mathbf{P}^2$ from the point p; it is the branch divisor of the projection $\pi_p|_S : S \to \mathbf{P}^2$.

The main result of [10] (see also [30]) asserts that for general p, the curve B_p has only nodes and cusps as singularities (see [20] for a general result valid in all dimensions), and it is irreducible. In addition, we shall assume that the dual plane curve $B_p^\vee$ has only nodes and cusps as singularities as well: since $B_p^\vee$ is isomorphic to $p^\perp \cap S^\vee$, this is equivalent to the assumption that the codimension one singular loci of $S^\vee$ are as described in (2.3).

(2.5) Plückerian characters of the branch curve. Let ν be the degree of the branch curve $B_p \subseteq \mathbf{P}^2$, $\check{\nu}$ be its class, and δ, κ, β, ι be its other Plückerian characters as in (1.12). These

numbers may all be expressed as polynomials in the degree of S:

$$\begin{aligned} \nu &= n(n-1) & \check{\nu} &= n(n-1)^2 \\ \delta &= \tfrac{1}{2}n(n-1)(n-2)(n-3) & \beta &= \tfrac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12) \\ \kappa &= n(n-1)(n-2) & \iota &= 4n(n-1)(n-2). \end{aligned}$$

(2.6) Let me first outline a quick proof of these formulae, based on the two facts that, on the one hand, B_p is the birational projection from p of the apparent boundary $\mathcal{B}^p S$, and on the other hand, it is the plane dual of the section of $S^\vee$ by the plane $p^\perp \subseteq \check{\mathbf{P}}^3$.

This implies that: i) the degree and class of B_p equal the degrees of $\mathcal{B}^p S$ and $S^\vee$ respectively, which gives the formulae for ν and $\check{\nu}$ (see (2.8) below): ii) the geometric genus of B_p equals that of $\mathcal{B}^p S$, which is easy to compute since the latter is a complete intersection. We can then derive all remaining characters using the Plücker formulae for plane curves. This is carried out in a more general context in Section 5.3.

Since B_p is a general hyperplane section of $S^\vee$, the above formulae give the following.

(2.7) Corollary. *The numbers of binodal and cuspidal curves in a general net of hyperplane sections of S are respectively*

$$\tfrac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12) \quad \textit{and} \quad 4n(n-1)(n-2).$$

These numbers are the degrees of the ordinary and cuspidal double curves of $S^\vee$ respectively.

(2.8) Let me now give a description of the Plückerian characters in (2.5) in terms of the tangent hyperplanes to S. It gives a way to compute these characters directly (although for β I don't know of an elementary way to do it), regardless of the quick proof given in (2.6).

(2.8.1) Degree of B_p. The curve B_p is the projection of the circumscribed curve $\mathcal{B}^p S = S \cap \mathrm{D}^p S$ from the outer point p. Therefore, $\deg(B_p) = \deg(\mathcal{B}^p S) = \deg(S) \cdot \deg(\mathrm{D}^p S)$.

(2.8.2) Class of B_p. The curve $B_p^\vee \subseteq \check{\mathbf{P}}^2$ is the hyperplane section of $S^\vee$ by $p^\perp$. Indeed, for all $q \in B_p$, the hyperplane $\pi_p^{-1}(\mathbf{T}_q B_p)$ is tangent to S at some point $q' \in \mathcal{B}^p S$ and contains p; conversely, if $H \subseteq \mathbf{P}^3$ contains p and is tangent to S at some point q', then $q' \in \mathcal{B}^p S$ and $\pi_p(H)$ is the tangent line to B_p at $\pi_p(q')$.
This implies that $\deg(B_p^\vee) = \deg(S^\vee) = \deg(S) \cdot \deg(\mathrm{D}^p S)^2$.

(2.8.3) Nodes of B_p. The nodes of B_p are the points of B_p with two distinct preimages in $\mathcal{B}^p S$, hence they correspond to unordered pairs of distinct points $q, q' \in \mathcal{B}^p S$ such that p, q, q' are aligned. Since $\mathcal{B}^p S$ is the curve of points $q \in S$ such that the line $\langle p, q \rangle$ is tangent to S at q, the nodes of B_p correspond to the bitangent lines to S passing by p. The number of such lines may be computed in a natural way using discriminants; this is explained in [C, Section 4.1].

(2.8.4) Flexes of B_p. Since B_p is the branch curve of the projection of S from p, it has a flex at a point q if and only if the hyperplane $\pi_p^{-1}(\mathbf{T}_q B_p)$ cuts out a curve with a cuspidal singularity on S.

This may be seen with the following local computation. Let q be an inflection point of B_p, and $q' \in \mathcal{B}^p S$ be such that $\pi_p(q') = q$. Locally around q', S may be described as a double cover of a neighbourhood of q in $\mathbf{P}^2$, hence as a hypersurface $z^2 = f(x, y)$ in affine coordinates (x, y, z), such that the projection from p is $(x, y, z) \mapsto (x, y)$ and $f(x, y) = 0$ is an equation of B_q locally at q. Assume that the tangent line $\mathbf{T}_q B_p$ is $y = 0$. Then the section of S by its tangent plane at q' is the hypersurface $z^2 = f(x, 0)$ in affine coordinates (x, z), and since q is a

flex of B, $f(x,0)$ vanishes to the third order at 0, hence the curve $z^2 = f(x,0)$ has a cusp at the origin, as we wanted to show. The converse is proved in the same fashion.

It follows that the number of flexes of B_p is the number of hyperplanes passing through p and tangent to S at a parabolic point. This is the number of intersection points of $\mathcal{B}^p S$ with the parabolic curve, i.e., $\deg(S) \cdot \deg(\mathrm{D}^p S) \cdot \deg(\mathrm{Hess}(S))$.

(2.8.5) Cusps of B_p. The cusps of B_p correspond to the points $q \in \mathcal{B}^p S$ such that the line $\langle p, q\rangle$ is tangent to $\mathcal{B}^p S$ at q, see Lemma (B.9). By Lemma (2.9) below, this happens if and only if the line $\langle p, q\rangle$ is an asymptotic tangent of S at q. Therefore, the number of cusps of B_p is $\deg(S) \cdot \deg(\mathrm{D}^p S) \cdot \deg(\mathrm{D}^{p^2} S)$.

(2.8.6) Bitangents of B_p. A line $\Lambda \subseteq \mathbf{P}^2$ is bitangent to B_p if and only if the plane $\pi_p^{-1}(\Lambda) \subseteq \mathbf{P}^3$ is bitangent to S. In [45, §605–607] Salmon constructs from the equation of S, by rather elaborate elimination theory, a polynomial which cuts out on S the locus of tangency points of all bitangent planes of S. This polynomial has degree $(n-2)(n^3 - n^2 + n - 12)$, and from this one deduces the number of bitangent planes of S that pass through a given general point, and therefore the number of bitangent lines of B_p, see Theorem (3.15) and Corollary (3.16) in [C]. This construction by Salmon is the main topic of [C].

(2.9) Lemma. *Let $q \in \mathcal{B}^p S$. The line $\langle p, q\rangle$ is tangent to $\mathcal{B}^p S$ at q if and only if it is an asymptotic tangent of S at q.*

This may be proved in several ways that I deem interesting. One may first of all do it by a local computation: this is done in [10, Prop. 3.9], and I will leave it as an exercise. I shall now give two other proofs.

Proof of (2.9) by polarity. By assumption, $q \in \mathrm{D}^p S$, and the line $\langle p, q\rangle$ is tangent to S at q. Then, the line $\langle p, q\rangle$ is an asymptotic tangent of S at q if and only if $q \in \mathrm{D}^{p^2} S$, which by polar symmetry is equivalent to $p \in \mathrm{D}^{q^{n-2}} S$. On the other hand, being tangent to S, the line $\langle p, q\rangle$ is tangent to $\mathcal{B}^p S$ at q if and only if it is tangent to $\mathrm{D}^p S$, which is equivalent to $p \in \mathrm{D}^{q^{n-2}}(\mathrm{D}^p S)$ (note that $\mathrm{D}^p S$ is a degree $n-1$ hypersurface). Now, $\mathrm{D}^{q^{n-2}}(\mathrm{D}^p S) = \mathrm{D}^p(\mathrm{D}^{q^{n-2}} S)$, and the two conditions $p \in \mathrm{D}^{q^{n-2}} S$ and $p \in \mathrm{D}^p(\mathrm{D}^{q^{n-2}} S)$ are equivalent by the Euler formula, see (A.6). □

Proof of (2.9) by duality. We shall use the duality statement that the orthogonal of an asymptotic tangent to S at q is an asymptotic tangent to $S^\vee$ at $(\mathbf{T}_q S)^\perp$, see (2.2.2).

Thus, in the conditions of the lemma, the line $\langle p, q\rangle$ is an asymptotic tangent of S at q if and only if the line $\langle p, q\rangle^\perp$ is an asymptotic tangent of $S^\vee$ at $(\mathbf{T}_q S)^\perp$. Since the plane $p^\perp$ contains the line $\langle p, q\rangle^\perp$, this is equivalent to the hyperplane section $p^\perp \cap S^\vee$ having a flex along the line $\langle p, q\rangle^\perp$ at the point $(\mathbf{T}_q S)^\perp$. In turn, since the projection B_p of $\mathcal{B}^p S$ from p is dual as a plane curve to $p^\perp \cap S^\vee$, this is equivalent to B_p having a cusp at the projection of q from p, and the latter condition is equivalent, as we have already noted, to the line $\langle p, q\rangle$ being tangent to $\mathcal{B}^p S$ at q. □

Let $q \in \mathcal{B}^p S$ be a smooth point. We have seen above that, on the one hand, B_p has a flex at $\pi_p(q)$ if and only if q is a parabolic point of S, and then the tangent $\mathbf{T}_q(\mathcal{B}^p S)$ is the unique asymptotic tangent of S at q, and on the other hand, B_p has a cusp at $\pi_p(q)$ if and only if the line $\langle p, q\rangle$ is the tangent $\mathbf{T}_q(\mathcal{B}^p S)$, and this happens if and only if $\langle p, q\rangle$ is an asymptotic tangent at q. With this respect, it is interesting to note the following.

(2.10) Lemma. *Let $q \in \mathcal{B}^p S$ be a smooth point. The tangent line $\mathbf{T}_q(\mathcal{B}^p S)$ is an asymptotic tangent of S at q if and only if either q is a parabolic point, or $\langle p, q\rangle$ is an asymptotic tangent at q. In the latter case, $\langle p, q\rangle$ is the tangent line $\mathbf{T}_q(\mathcal{B}^p S)$.*

Proof. First note that since q is a smooth point of $\mathcal{B}^p S$, the linear spaces $\mathbf{T}_q S$ and $\mathbf{T}_q(\mathrm{D}^p S)$ intersect along a line. This implies that $\mathbf{T}_q(\mathrm{D}^p S)$ has dimension 2, and since it equals $\mathrm{D}^{q^{n-2}}\mathrm{D}^p S = \mathrm{D}^p\mathrm{D}^{q^{n-2}} S$, the polar quadric of S at q is not the whole $\mathbf{P}^3$.

Assume that q is parabolic. Then $\mathrm{D}^{q^{n-2}} S$ is a quadric cone, smooth at q by Theorem (A.10), and S has a unique flex tangent line at q, which is the line $L = \mathbf{T}_q(\mathrm{D}^{q^{n-2}} S) \cap \mathrm{D}^{q^{n-2}} S = \mathbf{T}_q S \cap \mathrm{D}^{q^{n-2}} S$ spanned by q and the vertex of the cone $\mathrm{D}^{q^{n-2}} S$. The tangent line $\mathbf{T}_q(\mathcal{B}^p S)$ equals $\mathbf{T}_q S \cap \mathbf{T}_q(\mathrm{D}^p S)$, so we only need to show that the line L is contained in $\mathbf{T}_q(\mathrm{D}^p S)$. Now, since $\mathbf{T}_q(\mathrm{D}^p S) = \mathrm{D}^p\mathrm{D}^{q^{n-2}} S$, it contains the singular point of the cone $\mathrm{D}^{q^{n-2}} S$, hence also the whole line L.

If, on the other hand, $\langle p, q\rangle$ is an asymptotic tangent at q, then it equals $\mathbf{T}_q(\mathcal{B}^p S)$ by Lemma (2.9) above, and we are done in this case.

Conversely, assume that the line $L = \mathbf{T}_q(\mathcal{B}^p S)$ is an asymptotic tangent at q, but $\langle p, q\rangle$ is not. Thus p is off L, and L is contained in the quadric $\mathrm{D}^{q^{n-2}} S$. Then, for all $r \in L$, the tangent space $\mathbf{T}_r(\mathrm{D}^{q^{n-2}} S)$ contains L, but also p because $r \in L \subseteq \mathrm{D}^p(\mathrm{D}^{q^{n-2}} S)$. It follows that the plane $\langle p, L\rangle$ is tangent to $\mathrm{D}^{q^{n-2}} S$ at all points $r \in L$, and this implies that $\mathrm{D}^{q^{n-2}} S$ is a cone, i.e., q is a parabolic point of S. □

3 – Zero-dimensional strata of the dual surface

In this section we study some zero-dimensional strata of the dual of a general smooth surface S in $\mathbf{P}^3$; in particular, we compute the number of tritangent planes of S, see (3.17). The general strategy is to look at the cuspidal edges of the polar cone of $S^\vee$ with respect to a general point of $\check{\mathbf{P}}^3$; the main result in this direction is Theorem (3.7). The ordinary double edges of this cone are studied in Section 4.

3.1 – Local models

In Section 3.2 below we shall study a batch of formulae valid for surfaces in $\mathbf{P}^3$ with a given class of singularities that are modeled on those of the dual of a general enough smooth surface. Here we describe these singularities. A discussion about why these should be the appropriate singularities may be found in Paragraph (3.8) below.

(3.1) Swallowtails. We define the *swallowtail*[2] *singularity* (also referred to in this text as "point of type β") as the singularity at the origin of the discriminant locus in the semi-universal deformation space of an A_3 curve singularity, also known as an ordinary tacnode.

The semi-universal deformation of the tacnode is the family of curves over the affine space $\mathbf{A}^3_{a,b,c}$, of dimension 3 and with coordinates (a, b, c), defined in $\mathbf{A}^2_{x,y} \times \mathbf{A}^3_{a,b,c}$ by the equation:

$$y^2 = x^4 + ax^2 + bx + c. \tag{3.1.1}$$

For more details, please see [II]. The semi-universal deformation space of the tacnode is this $\mathbf{A}^3_{a,b,c}$, and the *discriminant locus* therein is the locus over which the curves in the family defined by (3.1.1) For all (a, b, c), the curve in $\mathbf{A}^2_{x,y}$ defined by Equation (3.1.1) is singular if and only if the polynomial in x on the right-hand-side has a double root. Therefore the equation of the discriminant locus is given by the classical discriminant of a univariate degree 4 polynomial, i.e.,

$$\mathrm{Disc}(x^4 + ax^2 + bx + c) = 256c^3 - 128a^2c^2 + 144ab^2c - 27b^4 + 16a^4c - 4a^3b^2. \tag{3.1.2}$$

[2] in french, « queue d'arronde », one of seven elementary catastrophes in Thom's theory.

From this equation we see that the swallowtail singularity is a locally irreducible triple point, with tangent cone a triple plane.

Alternatively, the discriminant locus is the image of the map

$$(u, v) \longmapsto (x-u)^2(x-v)(x+2u+v)$$

(note that the absence of x^3 term on the right-hand-side of (3.1.1) is equivalent to the sum of the four roots of the polynomial in x being 0). It follows that the equation of the discriminant locus may be obtained by eliminating u and v from the equations

$$\begin{cases} a = -3u^2 - 2uv - v^2 \\ b = 2u^3 + 4u^2v + 2uv^2 \\ c = -2u^3v - u^2v^2 \end{cases}$$

gotten by computing the coefficients of $(x-u)^2(x-v)(x+2u+v)$.

Similarly, the locus in the semi-universal deformation space of the tacnode over which the curves in the family defined by (3.1.1) have a cusp is the image of the map $(u, v) \mapsto (x-u)^3(x+3u)$, hence its equations may be found by eliminating u and v from the equations

$$a = -6u^2; \quad b = 8u^3; \quad c = -3u^4.$$

This gives the ideal of equations

$$(a^2 + 12c, 8a^3 + 27b^2), \tag{3.1.3}$$

from which we see that the locus in consideration is an irreducible curve with a planar ordinary cusp at the origin, the generic point of which is a cuspidal double point of the discriminant locus. We shall call this curve the *cuspidal double curve* of the swallowtail singularity.

In turn, the locus over which the curves in the family defined by (3.1.1) have two nodes is the image of the map $(u, v) \mapsto (x-u)^2(x+u)^2$, hence its equations may be found by eliminating u and v from the equations

$$a = -2u^2; \quad b = 0; \quad c = u^4;$$

this gives the ideal of equations

$$(b, a^2 - 4c), \tag{3.1.4}$$

from which we see that the locus in consideration is a smooth irreducible curve, the generic point of which is an ordinary double point of the discriminant locus. We shall call this curve the *ordinary double curve* of the swallowtail singularity.

Finally, one finds that the tangent cone of the intersection of the cuspidal and ordinary double curves is defined by the ideal (b, c, a^2) , so these two curves intersect with multiplicity 2.

(3.2) Type γ singularity. Similarly to what we have done for the swallowtail in (3.1) above, we define the *singularity of type* γ as the singularity at the origin of the discriminant locus in the semi-universal deformation space of the multisingularity (A_1, A_2) consisting of a node and an ordinary cusp.

The semi-universal deformations of a node and a cusp are respectively given by

$$y^2 = x^2 + a \qquad \text{and} \qquad y^2 = x^3 + bx + c, \tag{3.2.1}$$

and we consider the disjoint union of their pull-back to the product $\mathbf{A}^1_a \times \mathbf{A}^2_{b,c}$. The equation of the discriminant is merely the product of the two discriminants, i.e.,

$$\text{Disc}(x^2 + a) \cdot \text{Disc}(x^3 + bx + c) = a \cdot (4b^3 + 27c^2). \tag{3.2.2}$$

At the origin, the discriminant locus has two local sheets, one smooth and one with a cuspidal double curve, and the two meet transversely.

The locus in $\mathbf{A}^1_a \times \mathbf{A}^2_{b,c}$ above which the curve defined by (3.2.1) has a cusp parametrizes those deformations of the multisingularity (A_1, A_2) along which the node smoothes and the cusp is maintained. Its has equations

$$b = c = 0.$$

It is a smooth curve, which we will call the *cuspidal double curve* of the type γ singularity.

The locus in $\mathbf{A}^1_a \times \mathbf{A}^2_{b,c}$ above which the curve defined by (3.2.1) has two nodes parametrizes those deformations of the multisingularity (A_1, A_2) along which the node is maintained and the cusp deforms to a node, which has equations

$$a = 4b^3 + 27c^2 = 0.$$

It is a curve with a planar ordinary cusp at the origin, which we will call the *ordinary double curve* of the type γ singularity. It is the intersection curve of the two sheets of the discriminant surface.

(3.3) Type t singularity. Finally, we define the *singularity of type t* as the singularity at the origin of the discriminant locus in the semi-universal deformation space of the multisingularity (A_1, A_1, A_1) consisting of three nodes.

Thus we consider three copies of the semi-universal deformations of a node,

$$y^2 = x^2 + a \qquad \text{and} \qquad y^2 = x^2 + b \qquad \text{and} \qquad y^2 = x^2 + c, \tag{3.3.1}$$

and look at the disjoint union of their pull-back to the product $\mathbf{A}^1_a \times \mathbf{A}^1_b \times \mathbf{A}^1_c$. The equation of the discriminant is the product of the three discriminants, i.e.,

$$\text{Disc}(x^2 + a) \cdot \text{Disc}(x^2 + b) \cdot \text{Disc}(x^2 + c) = abc. \tag{3.3.2}$$

The discriminant locus has three smooth local sheets meeting transversely at the origin.

The locus in $\mathbf{A}^1_a \times \mathbf{A}^1_b \times \mathbf{A}^1_c$ above which the curve defined by (3.3.1) has two nodes is the union of the three coordinate lines $\mathbf{A}^1_a \times \{0\} \times \{0\}$, $\{0\} \times \mathbf{A}^1_b \times \{0\}$, and $\{0\} \times \{0\} \times \mathbf{A}^1_c$; it is defined by the ideal of equations (ab, ac, bc), and has a triple point at the origin. Yet again, we shall call it the ordinary double curve of the type t singularity.

The locus in $\mathbf{A}^1_a \times \mathbf{A}^1_b \times \mathbf{A}^1_c$ above which the curve defined by (3.3.1) has a cusp is empty.

3.2 – Polar cones and their cuspidal edges

In this section we describe the cuspidal edges of the polar cone of a surface in $\mathbf{P}^3$ which has the same kind of singularities as the dual of a sufficiently general smooth surface. We shall call this class of singularities d-ordinary singularities, since (plain) ordinary singularities classically refer to the kind of singularities found on a sufficiently general projection to $\mathbf{P}^3$ of a smooth surface in $\mathbf{P}^5$ (or in $\mathbf{P}^N$ for any $N \geqslant 5$), see (5.3).

The central result is Theorem (3.7), which is the keystone of the computation of the number of tritangent planes in the next section.

(3.4) Surfaces with d-ordinary singularities. We say that a surface $X \subseteq \mathbf{P}^3$ has *d-ordinary singularities* if it has a curve B of ordinary double points, a curve C of simple cuspidal double points, and no other singularities except from those arising from the singularities of the curves B and C, by which we mean:
— β intersection points between B and C at which C is cuspidal (and B is smooth), at which X has a type β (i.e., swallowtail) singularity as in (3.1) above;
— γ intersection points between B and C at which B is cuspidal (and C is smooth), at which X has a type γ singularity as in (3.2) above;
— t the number of triples points of the ordinary double curve B, at which X has a type t singularity as in (3.3) above.
Moreover, we let b and c denote the degrees of the curves B and C respectively.

We now consider, for the rest of the present Section 3.2, a surface $X \subseteq \mathbf{P}^3$ with d-ordinary singularities, with the notation as in (3.4) above.

Let p be a general point of $\mathbf{P}^3$. It follows from a local computation, see Theorem (A.10), that the first polar $\mathrm{D}^p X$ contains simply the curves B and C, with tangent cone equal to that of X at the generic point of C. Therefore, the intersection $X \cap \mathrm{D}^p X$ contains B and C with multiplicities 2 and 3 respectively, and we need to refine the definition of apparent boundary given in (1.7).

(3.5) Polar cones of a surface with d-ordinary singularities. We let the *polar cone* $\mathcal{D}^p X$ of X with respect to p be the cone projecting from p over the curve $X \cap \mathrm{D}^p X$. It contains the two cones projecting B and C from p with multiplicities 2 and 3 respectively, and we let the *circumscribed cone* $\mathcal{C}^p X$ be the cone residual to these two cones in the polar cone; it is the closure of the set of all lines passing through p and tangent to X at a smooth point. Finally, we denote by $\mathcal{B}^p X$ the curve $(X \cap \mathcal{C}^p X)_{\mathrm{red}}$, which we call the *apparent boundary*.

We let a be the degree of $\mathcal{C}^p X$. Then, the degree of the polar cone $\mathcal{D}^p X$ reads

$$n(n-1) = a + 2b + 3c. \tag{3.5.1}$$

(3.6) Cuspidal edges of the polar cone. We now introduce more notation, and list various kinds of points that will appear in the intersection $X \cap \mathrm{D}^p X \cap \mathrm{D}^{p^2} X$, together with the singular points listed in (3.4). The lines joining p to the points of $X \cap \mathrm{D}^p X \cap \mathrm{D}^{p^2} X$ are the cuspidal edges of the polar cone $\mathcal{D}^p X$, in analogy with what happens when X is smooth, see Paragraph (2.8.5).

We set:
— κ the number of cuspidal edges of the circumscribed cone $\mathcal{C}^p X$;
— ρ the number of intersection points between the apparent boundary $\mathcal{B}^p X$ and the ordinary double curve B;
— σ the number of intersection points between the apparent boundary $\mathcal{B}^p X$ and the cuspidal double curve C.
We shall introduce some more notation later in (4.18), but it is not relevant for the present considerations.

Salmon gives the following set of formulae [45, §610], which describes separately the intersections with the second polar $\mathrm{D}^{p^2} X$ of the three components of the polar cone $\mathcal{D}^p X$.

(3.7) Theorem. *For $X \subseteq \mathbf{P}^3$ a surface with d-ordinary singularities and $p \in \mathbf{P}^3$ a general point, and with the notation of (3.4) and (3.6), one has:*

$$a(n-2) = \kappa + \rho + 2\sigma \ ;$$
$$b(n-2) = \rho + 2\beta + 3\gamma + 3t \ ;$$
$$c(n-2) = 2\sigma + 4\beta + \gamma.$$

He comments: « The reader can see without difficulty that the points indicated in these formulæ are included in the intersections of $[\mathrm{D}^{p^2}X$ with the three components of $\mathcal{D}^pX]$; but it is not so easy to see the reason for the numerical multipliers which are used in the formulæ. Although it is probably not impossible to account for these constants by a priori reasoning, I prefer to explain the method by which I was led to them inductively. » The latter method consists in considering the dual of a smooth cubic surface, for which all the quantities may be directly computed, and then in tuning the coefficients to make the formulae work.

The remainder of the present Section 3.2 is dedicated to the proof of the above theorem. By Theorem (A.8), the intersection

$$X \cap \mathrm{D}^pX \cap \mathrm{D}^{p^2}X = \big(\mathcal{B}^pX + 2B + 3C\big) \cap \mathrm{D}^{p^2}X$$

is supported on the locus $\mathfrak{D}$ of points q in X such that the line $\langle p, q\rangle$ has intersection multiplicity at least 3 with X at q. The proof consists in the following two points. (i) List the various kinds of points in the locus $\mathfrak{D}$; this indicates which terms (without the integer coefficients) appear on the right-hand-side of Formulae (☜); the condition that X has d-ordinary singularities leaves us with finitely many possibilities. (ii) For each of those, determine the corresponding multiplicity in the intersection cycle $X \cap \mathrm{D}^pX \cap \mathrm{D}^{p^2}X$; this will tell us the integer coefficient with which each term on the right-hand-side of (☜) appears.

The smooth points in the locus $\mathfrak{D}$ are the points of type κ, see (3.7.1) below. Some special double points of X lie in $\mathfrak{D}$, and these are the points in the two intersections $\mathcal{B}^pX \cap B$ and $\mathcal{B}^pX \cap C$. Finally, all triple points automatically lie in $\mathfrak{D}$. Thus the points appearing in $\mathfrak{D}$ are those of the types $\kappa, \rho, \sigma, \beta, \gamma, t$. The remainder of the proof consists in a case by case local study of the polars of X at all such points.

(3.7.1) Type κ. Cuspidal edges of $\mathcal{C}^pX$ appear in correspondence with points $q \in \mathcal{B}^pX$ such that the line $\langle p, q\rangle$ is tangent to $\mathcal{B}^pX$, as has been explained in (2.8.5), and such points q are simple points in the intersection $\mathcal{B}^pX \cap \mathrm{D}^{p^2}X$ as in Lemma (2.9). For general p, these points lie on neither one of the double curves of X.

(3.7.2) Type ρ. Points of type ρ are those points q lying on the ordinary double curve B such that the line $\langle p, q\rangle$ is an honest tangent to one of the two smooth local sheets of X at q; let us call X'_q this local sheet. We claim that (i) the first polar D^pX is smooth at q with the same tangent plane as X'_q, and its intersection with X locally at q consists of B and $\mathcal{C}_pX$, the latter entirely contained in X'_q, and (ii) the second polar $\mathrm{D}^{p^2}X$ is smooth at q with general tangent plane, hence it intersects both B and $\mathcal{C}_pX$ transversely at q.

Let us justify this claim by a specific local computation, as it does not directly follow from Theorem (A.10). We may assume that $q = (0:0:0:1)$ in a homogeneous system of coordinates $(x:y:z:w)$, and X is given locally at q by the equation

$$f(x,y,z) = \big[x + g_2(x,y,z) + O(3)\big]\big[y + h_2(x,y,z) + O(3)\big] = 0$$

in the affine chart $w = 1$, where g_2 and h_2 are homogeneous of degree 2, and $O(3)$ denotes

higher order terms. Then we may take $p = (1:0:0:0)$, and thus $\mathrm{D}^p = \partial_x$. Then,

$$\mathrm{D}^p f(x,y,z) = \big[x + g_2 + O(3)\big]\big[\partial_x h_2 + O(2)\big] + \big[1 + \partial_x g_2 + O(2)\big]\big[y + h_2 + O(3)\big] \\ = y + x\,\partial_x h_2 + y\,\partial_x g_2 + h_2 + O(3)$$

and part (i) of the claim follows. Finally,

$$\mathrm{D}^{p^2} f(x,y,z) = 2\,\partial_x h_2 + x\,\partial_x^2 h_2 + y\,\partial_x^2 g_2 + O(2),$$

and part (ii) follows.

(3.7.3) Type σ. Points of type σ are those points q lying on the cuspidal double curve C such that the line $\langle p, q\rangle$ lies in the tangent cone of the unique, cuspidal, local sheet of X at q. We claim that (i) the first polar $\mathrm{D}^p X$ has a double point at q, and the intersection of the tangent cones $\mathrm{TC}_q X$ and $\mathrm{TC}_q(\mathrm{D}^p X)$ is a quadruple line supported on $\mathbf{T}_q C$; moreover, (ii) the second polar $\mathrm{D}^{p^2} X$ has a simple point at q, and the intersection of its tangent plane with $[\mathrm{TC}_q X]_{\mathrm{red}}$ is the line $\mathbf{T}_q C$. By (i), since at q the intersection of $\mathrm{D}^p X$ and X is the sum of $3C$ and $\mathcal{B}^p X$, the apparent boundary $\mathcal{B}^p X$ is smooth at q, and tangent to the cuspidal edge C. Thus, (ii) tells us that the second polar $\mathrm{D}^{p^2} X$ is tangent to both curves C and $\mathcal{B}^p X$ at q, which accounts for the multiplicities 2 of σ in ($\swallow$ ı) and ($\swallow$ ı ı ı).

As in (3.7.2) above, we justify our claim by a local computation, with similar notation. We let $q = (0:0:0:1)$, and consider the following equation for X locally at q in the affine chart $w = 1$:

$$f(x,y,z) = \big[y + g_2(x,y,z) + O(3)\big]^2 + \big[x + h_2(x,y,z) + O(3)\big]^3 = 0,$$

where g_2 and h_2 are homogeneous of degree 2. We may again take $p = (1:0:0:0)$, and then

$$\mathrm{D}^p f = 2\,\partial_x g_2 \cdot y + 3x^2 + O(3), \qquad \text{and} \qquad \mathrm{D}^{p^2} f = 2\,\partial_x^2 g_2 \cdot y + 6x + O(2),$$

which proves our claim.

(3.7.4) Type t. These are the points at which X has three transverse smooth local sheets. We may directly apply Theorem (A.10), to the effect that at a type t singular point q, (i) the first polar $\mathrm{D}^p X$ has an ordinary double point at q, and (ii) the second polar $\mathrm{D}^{p^2} X$ is smooth with general tangent plane, hence intersects B transversely at q. This gives the multiplicity 3 for type t in ($\swallow$ ∖), as q is a triple point of B. On the other hand, type t points lie neither on $\mathcal{B}^p X$ nor on C.

(3.7.5) Type β. These are triple points of X, with a local equation equivalent to that in (3.1.2); in particular the tangent cone of X at such a point is a triple plane. Thus, it follows from Theorem (A.10) that if q is a type β singular point, then (i) the first polar $\mathrm{D}^p X$ has a double point at q with tangent cone a double plane supported on $(\mathrm{TC}_q X)_{\mathrm{red}}$, and (ii) the second polar $\mathrm{D}^{p^2} X$ is smooth with tangent plane equal to $(\mathrm{TC}_q X)_{\mathrm{red}}$.

The ordinary double curve B is smooth at points of type β, as we have seen in Paragraph (3.1), and the second polar is tangent to it. This accounts for the multiplicity 2 in ($\swallow$ ı). On the other hand, the cuspidal double curve C has planar ordinary cusps at points of type β, but it turns out that it intersects $\mathrm{D}^{p^2} X$ with multiplicity 4 at such points: indeed, in the notation of (3.1), C is defined by the ideal $(a^2 + 12c, 8a^3 + 27b^2)$, and the tangent cone of $\mathrm{D}^{p^2} X$ is the plane $c = 0$. This confirms the multiplicity of β in ($\swallow$ ∖), and shows that $(\mathrm{TC}_q X)_{\mathrm{red}}$ is the osculating plane of C at swallowtail points q.

(3.7.6) Type γ. These are triple points q of X, at which X has two transverse local sheets, one smooth: X_q^{s}, and the other cuspidal: X_q^{c}. The first polar $\mathrm{D}^p X$ has a double point at q, with tangent cone the sum of $[\mathrm{TC}_q(X_q^{\mathrm{c}})]_{\mathrm{red}}$ and a general plane containing the line $\mathrm{TC}_q(B)$. The second polar $\mathrm{D}^{p^2} X$ is smooth at q, with tangent plane a general plane containing the line $\mathrm{TC}_q(B)$. Therefore it intersects B, which is cuspidal at q, with multiplicity 3, and it is transverse to C, which is smooth at q. This amounts for the multiplicities 3 and 1 in ( ) and () respectively.

To clarify the situation, let us carry out the local computations proving the above assertions. We let $q = (0:0:0:1)$ and consider the two sheets X_q^s defined respectively by the equations

$$z + O(2) = 0 \qquad \text{and} \qquad y^2 + x^3 + O(4) = 0.$$

The curve B is the intersection of these two sheets, it has a cusp at q with tangent cone defined by $z = y^2 = 0$. The curve C is the singular locus of X_q^{c}, it is smooth at q with tangent line defined by $y = x = 0$. We may take $p = (u_0 : u_1 : u_2 : 0)$. Then, by Theorem (A.10) the tangent cones of $\mathrm{D}^p X$ and $\mathrm{D}^{p^2} X$ are defined respectively by

$$\mathrm{D}^p(zy^2) = y(2u_1 z + u_2 y) \qquad \text{and} \qquad \mathrm{D}^{p^2}(zy^2) = 2u_1(u_1 z + 2u_2 y),$$

which proves our claims.

The proof of Theorem (3.7) is now complete. □

3.3 – Application to the case when $X = S^\vee$

We now let S be a smooth, degree n, surface in $\mathbf{P}^3$, and call $X = S^\vee \subseteq \check{\mathbf{P}}^3$ its dual surface. We shall explore in this situation the geometric landscape described in Section 3.2 above.

(3.8) We take for granted that if S is general enough, then X has d-ordinary singularities (see (3.4)), and we assume that S is indeed general enough.

The principle behind this implication is that, since hyperplane sections of S form a 3-dimensional system, we expect that under suitable generality/transversality assumptions, only (multi-)singularities imposing at most three conditions may occur on hyperplane sections of S. The condition that a given multi-singularity imposes at most three conditions translates into the fact that its semi-universal deformation space has dimension at most three (for a general discussion of the expected codimension imposed by the presence of a multi-singularity, see [XIII, Section 4.4]). We expect moreover that, still under the aforementioned generality assumptions, for all $H^\perp \in \check{\mathbf{P}}^3$, there is a submersion from an analytic or étale neighbourhood of $H^\perp$ in $\check{\mathbf{P}}^3$ to the semi-universal deformation space of the multi-singularity of $H \cap S$; this implies that, locally around $H^\perp$, the dual $S^\vee$ looks like the discriminant locus in the deformation space of the multi-singularity of $H \cap S$.

For a proof that if S is sufficiently general, then things happen according to the above described expectations, see [6], [34], and [43]. See also [54] for a related result.

(3.9) Notation. As a rule, we use the decoration with a duality symbol to indicate that a symbol denotes a quantity associated with $X = S^\vee$. Thus, we set:
— $\check{n} = n(n-1)^2$, the degree of X.
We use the notation introduced in Section 3.2 for the singularities of $X = S^\vee$, with an additional "$\vee$" decoration.

(3.10) The double curves $\check{B}$ and $\check{C}$ are the curves in $S^\vee$ parametrizing the bitangent and parabolic tangent planes to S, respectively. The Gauss map of S induces a $2:1$ map $\mathrm{Kn}_S \to \check{B}$ from the node-couple curve $\mathrm{Kn}_S \subseteq S$, and a birational map $\mathrm{He}_S \to \check{C}$ from the Hessian curve $\mathrm{He}_S \subseteq S$. We have seen in Corollary (2.7) that the degrees of the two curves $\check{B}$ and $\check{C}$ in $\check{\mathbf{P}}^3$ are:
— $\check{b} = \frac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12)$;
— $\check{c} = 4n(n-1)(n-2)$.

(3.11) In the context of Section 3.2, we need to choose a general point $\check{p} \in \check{\mathbf{P}}^3$ to consider the polar cone $\mathcal{D}^{\check{p}}X$ and the apparent boundary $\mathcal{B}^{\check{p}}X$. This point $\check{p}$ corresponds to a hyperplane $H_{\check{p}} = \check{p}^\perp$ in $\mathbf{P}^3$. The circumscribed cone $\mathcal{C}^{\check{p}}(S^\vee) \subseteq \check{\mathbf{P}}^3$ is the dual of the plane curve $H_{\check{p}} \cap S \subseteq \mathbf{P}^3$. This gives:
— $\check{a} = n(n-1)$.
Note that with our values of $\check{a}, \check{b}, \check{c}$, Formula (3.5.1) holds indeed.

It also follows from the above description of $\mathcal{C}^{\check{p}}(S^\vee)$ that the cuspidal edges of type $\check{\kappa}$ of the polar cone $\mathcal{D}^{\check{p}}S^\vee$ are the lines orthogonals to the osculating tangent lines of the plane curve $H_{\check{p}} \cap S$ (i.e., the orthogonals of the flex tangents of $H_{\check{p}} \cap S$). Thus,
— $\check{\kappa} = 3n(n-2)$.

(3.12) The apparent boundary $\mathcal{B}^{\check{p}}X$ parametrizes points $\varpi \in X = S^\vee$ such that $\check{p} \in \mathbf{T}_\varpi X$, i.e., dually, planes $\varpi^\perp \subseteq \mathbf{P}^3$ tangent to S at a point $(\mathbf{T}_\varpi X)^\perp$ lying on $H_{\check{p}}$. It follows that the points of type $\check{\rho}$ (resp. $\check{\sigma}$) correspond to hyperplanes that are tangent to S at some point on $H_{\check{p}}$ and at some other point (resp. at some point on $H_{\check{p}}$ which is parabolic). Thus, $\check{\rho}$ (resp. $\check{\sigma}$) is the degree of the intersection of $H_{\check{p}}$ with the node-couple curve Kn_S (resp. with the Hessian curve He_S), and therefore:
— $\check{\rho} = n(n-2)(n^3-n^2+n-12)$;
— $\check{\sigma} = 4n(n-2)$
(for $\check{\rho}$, see (2.8.6); $\check{\sigma}$ follows directly from the fact that He_S is the complete intersection of S and $\mathrm{Hess}(S)$).

(3.13) Points of type $\check{\gamma}$ correspond to planes H tangent to S at two points, one of which is parabolic; correspondingly the curve $H \cap S$ has one cusp and one node. Points of type $\check{\gamma}$ are honest points on $\check{C}$, in account of the cusp in $H \cap S$; the presence of an additional, remote, node does not affect the local geometry of the curve $\hat{C}$. On the other hand, points of type $\check{\gamma}$ are cusps of $\check{B}$: if one watches a general point of $\check{B}$ move towards a point of type $\check{\gamma}$, one sees a family of hyperplane sections of S with two nodes such that one the nodes degenerates to a cusp, and this is the reason why points of type $\check{\gamma}$ are cuspidal points of $\check{B}$, much in the same way as the curve in $S^\vee$ corresponding to cuspidal tangent sections of S is a cuspidal curve of $S^\vee$. In less handwaving terms, the local geometry of $\check{B}$ and $\check{C}$ at points of type $\check{\gamma}$ is justified by the discussions in (3.2) and (3.8). The number $\check{\gamma}$ shall be derived indirectly in (3.16) below.

Points of type $\check{\beta}$ correspond to planes H tangent to S at one point, such that the section $H \cap S$ has a tacnode there. Indeed, points of type $\check{\beta}$ are cusps of $\check{C}$, so that as a general point of $\check{C}$ moves towards a point of type $\check{\beta}$, one must see a family of hyperplane sections of S with a cusp which degenerates to a tacnode. Going to a point of type $\check{\beta}$ along $\check{B}$, one merely sees two nodes tending to a common limit point, which is harmless for $\check{B}$, as indeed points $\check{\beta}$ are smooth points of $\check{B}$. The number $\check{\beta}$ will be derived in (3.15) by considering the flecnodal locus in S.

Finally, points of type $\check{t}$ correspond to the famous tritangent planes of S. Their number we shall derive using formula ( ).

(3.14) The flecnodal locus. The *flecnodal locus* is the locus of points $q \in S$ such that there exists a line having intersection multipliciy at least 4 with S at q. It is a curve on S, and for a general point q on this curve, the tangent section $\mathbf{T}_q S \cap S$ has an ordinary double point at q, such that one of the two local branches has a flex at q: the tangent line to this branch is the one line having intersection multiplicity 4 with S at q.

It has been shown by Salmon [45, §588] that the flecnodal curve is cut out on S by a polynomial of degree $11n - 24$, which we shall call the *flecnodal polynomial*. We describe his proof in [C, Section 5]. A modern proof, and a generalization to arbitrary dimension, is given in [7]. One can also find a brief account in [27, Section 8], and another generalization in [1]. Remarkably, the flecnodal polynomial vanishes modulo the equation of S if and only if S is ruled (note that the analogous vanishing property for the Hessian characterizes developable ruled surfaces): Salmon attributes this result to Cayley; for a proof, see [27, Section 8], or the references given in [7].

In [45, §589–595], Salmon gives the following explicit formula for the flecnodal polynomial, which he attributes to Clebsch. Let F be an equation for S, and denote by $\mathcal{H}$ its Hessian matrix. Set

$$\Theta = \sum_{i,j} |\mathcal{H}|_{i,j}\, \partial_i F\, \partial_j F,$$

where $|\mathcal{H}|_{i,j}$ denotes the cofactor formed by removing the i-th row and the j-th column of $\mathcal{H}$, i.e., $(-1)^{i+j}$ times the corresponding 3×3 minor. Set

$$\Phi = -\sum_{i,j} \sum_{\substack{i_1<i_2\\ j_1<j_2}} |\mathcal{H}|_{i,j}\, |\mathcal{H}|_{i_1 i_2, j_1 j_2}\, \partial_{i i_1 i_2} F\, \partial_{j j_1 j_2} F$$

where, in addition to the previous notation, $|\mathcal{H}|_{i_1 i_2, j_1 j_2}$ is $(-1)^{i_1+i_2+j_1+j_2}$ times the obvious 2×2 minor of $\mathcal{H}$. Then, the flecnodal polynomial is

$$\Theta - 4\Phi\, \mathrm{Hess}(F). \tag{3.14.1}$$

(3.15) Intersection of the parabolic and flecnodal curves. The points $q \in S$ such that the tangent section $\mathbf{T}_q S \cap S$ has a tacnode at q are the intersection points of the parabolic curve with the flecnodal curve. Indeed, a point q sits on the parabolic curve if and only if $\mathbf{T}_q S \cap S$ has a double point at q with degenerate tangent cone (or a point of higher multiplicity), and it sits on the flecnodal curve if and only if there is a line in the plane $\mathbf{T}_q S$ having intersection multiplicity at least 4 with the curve $\mathbf{T}_q S \cap S$ at q. With our blanket transversality assumption from (3.8), this happens if and only if $\mathbf{T}_q S \cap S$ has a tacnode at q.

The upshot of the above discussion is that the number $\check{\beta}$ is the number of intersection points of the parabolic and flecnodal curves, which are cut out by polynomials of degrees $4(n-2)$ and $11n-24$, respectively. There is however one last subtlety, namely that the latter two curves are everywhere tangent, see (3.15.1). Therefore, one finds:
— $\check{\beta} = 2n(n-2)(11n-24)$.

(3.15.1) The fact that the parabolic and flecnodal curves are everywhere tangent has been proved by Salmon [45, §596]; a different proof has been given by McCrory and Shifrin [34] in modern times. In fact, Salmon even proves that the two surfaces cutting out the parabolic and flecnodal curves on S are everywhere tangent.

His argument is the following. Taking into account the expression (3.14.1) for the flecnodal polynomial, it suffices to show that Θ is a square modulo the Hessian determinant $\mathrm{Hess}(F)$. Now, it follows from a determinantal identity that for all 4-tuples $(c_i) \in \mathbf{C}^4$,

$$\Big(\sum |\mathcal{H}|_{i,j}\, \partial_i F\, \partial_j F\Big)\Big(\sum |\mathcal{H}|_{i,j}\, c_i\, c_j\Big) = \Big(\sum |\mathcal{H}|_{i,j}\, \partial_i F\, c_j\Big)^2 \mod \mathrm{Hess}(F).$$

From the latter identity, Salmon deduces without further indication that Θ is indeed a square modulo $\mathrm{Hess}(F)$. I reckon that some additional arguments ought to be provided, but let me not try to do it here. Funnily enough, the squared factor on the right-hand-side of the last identity has degree $7n-15$, a quantity that will show up later in a seemingly unrelated manner, see (4.17).

(3.16) Intersection of the two double curves. We now analyze the intersection of the node-couple Kn_S and Hessian He_S curves on S. For each point q in their intersection, the tangent plane $\mathbf{T}_q S$ is either of type $\check{\gamma}$ or of type $\check{\beta}$, and we shall see that the two curves are transverse if it is of type $\check{\gamma}$, and tangent if it is of type $\check{\beta}$.

To a tangent plane of type $\check{\gamma}$ there corresponds a section of S with one node and one cusp, which gives two points on Kn_S, but only one of those lies on He_S. We add to our blanket transversality assumption that the two curves are transverse at this point.

To a tangent plane of type $\check{\beta}$ there corresponds a unique tangency point $q \in S$. On the one hand q belongs to Kn_S, and it is a ramification point of the double cover $\mathrm{Kn}_S \to \check{B}$, as $\mathbf{T}_q S \cap S$ may be seen as a bi-nodal curve with the two nodes coalesced. This implies that $T_q \mathrm{Kn}_S$ lies in the kernel of the differential of the Gauss map of S at q. On the other hand q is a ramification point of $\mathrm{He}_S \to \check{C}$ as well, since $\check{C}$ has a cusp at $(\mathbf{T}_q S)^\perp$, and this implies that $T_q \mathrm{He}_S$ lies in the kernel of the differential of the Gauss map of S at q (see also Lemma (4.9)). This implies that the two curves Kn_S and He_S are tangent at q, unless the differential of the Gauss map be 0 at q, but this is not the case by our blanket transversality assumption.

In the upshot, we find that

$$\mathrm{Kn}_S \cdot \mathrm{He}_S = 2\check{\beta} + \check{\gamma}.$$

Moreover, using the fact that both Kn_S and He_S are complete intersections of S and a hypersurface of known degree, we find

$$\mathrm{Kn}_S \cdot \mathrm{He}_S = (n-2)(n^3 - n^2 + n - 12) \cdot 4(n-2) \cdot n.$$

We may now plug in the formula for $\check{\beta}$ obtained in (3.13), and obtain:
— $\check{\gamma} = 4n(n-2)(n-3)(n^3+3n-16)$.[3]

(3.17) Formulae (𝄐) and the number of tritangent planes. First of all let me mention that with the quantities we have been able to find so far, formulae (𝄐 𝅘𝅥) and (𝄐 𝅘𝅥𝅘𝅥𝅘𝅥) are indeed verified (beware that n should be replaced by $\check{n} = n(n-1)^2$).

Formula (𝄐 𝅘𝅥𝅯) on the other hand yields the number of tritangent planes to a smooth surface of degree n in $\mathbf{P}^3$:
— $\check{t} = \frac{1}{6} n(n-2)(n^7 - 4n^6 + 7n^5 - 45n^4 + 114n^3 - 111n^2 + 548n - 960)$.

4 – Hessian and node-couple developables

In this section we determine some more enumerative numbers related to 0-dimensional strata of a smooth surface $S \subseteq \mathbf{P}^3$ and its dual, thus complementing the analysis carried out in the previous Section 3. These numbers are not immediately related to the enumeration of hyperplane sections of S with given singularities; for instance, we determine the number of flexes of the cuspidal double curve $\check{C} \subseteq S^\vee$ (this is β_{He} in (4.17)), and these correspond to planes cutting out an honest tacnodal curve on S. In general, we are concerned in this section with enumerative

[3] There is a typo in the formula for $\check{\gamma}$ in [45, §630] ($n^3 - 3n - 16$ instead of $n^3 + 3n - 16$), which curiously enough goes back and forth between the various editions of the book, as Ragni Piene has pointed out to me. The formula for $\check{\gamma}$ also appears in [45, §612], without the typo.

number associated to the Hessian and node-couple developables (for the definition, see Section 4.1 below).

There are two main aspects to this study. On the one hand we shall describe (in Section 4.3) the ordinary double edges of the polar cone of a surface with d-ordinary singularities, thus complementing the analysis of cuspidal double edges in Section 3.2 above. On the other hand we will carry out (in Section 4.2) a detailed study of the geometry of a surface along its parabolic curve, which I believe is interesting for its own sake.

4.1 – Developables associated to curves on a surface

We refer to Appendix B for the basics on developable surfaces in $\mathbf{P}^3$.

(4.1) Developables associated to curves on a surface. Let Y be a curve traced on a surface $S \subseteq \mathbf{P}^3$. Recalling that a developable surface in $\mathbf{P}^3$ merely amounts to a one-dimensional family of planes, one may consider the developable surface $\Sigma_Y \subseteq \mathbf{P}^3$ corresponding to the family of planes tangent to S at points of Y. In classical terms: this is the family of planes touching S at points of Y, and Σ_Y is the surface enveloped by the family of tangent planes to S at points of Y.

Let $\gamma_S : S \dashrightarrow S^\vee$ be the Gauss map of S. We are considering the family of planes parametrized by the curve $\gamma_S(Y) \subseteq \check{\mathbf{P}}^3$. By definition, the developable surface Σ_Y is the dual surface $\gamma_S(Y)^\vee \subseteq \mathbf{P}^3$ of the curve $\gamma_S(Y) \subseteq \check{\mathbf{P}}^3$. There exists a curve $\Gamma \subseteq \mathbf{P}^3$ such that Σ_Y is the tangential surface $\mathrm{Tan}(\Gamma)$, but there is no reason that Γ should coincide with Y, and neither that it should lie on S.

(4.2) Hessian and node-couple developables. The *Hessian and node-couple developables* associated to a smooth surface $S \subseteq \mathbf{P}^3$ are the developable surfaces Σ_Y as in (4.1) above with Y the parabolic curve $\mathrm{He}(S)$ and node-couple curve $\mathrm{Kn}(S)$, repectively. They are the dual surfaces of the cuspidal and ordinary double curves of $S^\vee$, respectively, and we will denote them by Σ_{He} and Σ_{Kn}.

(4.3) Proposition. *The generators of the node-couple developable surface are the lines joining pairs of conjugated points of the node-couple curve, i.e., the lines $\langle p, q\rangle$ such that there exists a plane which is tangent to S at both p and q.*

Proof. The node-couple developable is the dual of the curve $\gamma_S(\mathrm{Kn}(S)) \subseteq \check{\mathbf{P}}^3$, so its generators are the orthogonals of the tangents lines of $\gamma_S(K_S)$ (see Section B.1). Let $\varpi \in \gamma_S(K_S)$ be a general point. There are two points $p, q \in S$ such that the plane $H = \varpi^\perp \subseteq \mathbf{P}^3$ is tangent to S at both p and q. The tangent line $\mathbf{T}_\varpi(\gamma_S(K_S))$ is the intersection of the tangent planes to the two local sheets of $S^\vee$ at ϖ, respectively $p^\perp$ and $q^\perp$. Therefore,

$$\mathbf{T}_\varpi(\gamma_S(K_S))^\perp = (p^\perp \cap q^\perp)^\perp = \langle p, q\rangle.$$

□

We will determine the numerical characters of the node-couple developable in Section 4.4; we need the results of Section 4.3 on the ordinary double edges of the polar cone to do it.

In order to describe the generators of the developable Σ_Y for a general curve Y, we introduce the following device.

(4.4) The involution ι_p on $\mathbf{P}(T_pS)$. Let p be a smooth, non-parabolic, point of a surface $S \subseteq \mathbf{P}^3$. There is a self-map ι_p of the projective line $\mathbf{P}(T_pS)$ (indeed an involution by Proposition (4.6) below), which is defined as follows. Let $u \in \mathbf{P}(T_pS)$ be a tangent direction at p. The differential of the Gauss map sends u to a direction $\gamma_S(u)$ in $\check{\mathbf{P}}^3$ tangent to $S^\vee$ at the point $(\mathbf{T}_pS)^\perp$: thus, $\gamma_S(u) \in \mathbf{P}\big(T_{(\mathbf{T}_pS)^\perp}S^\vee\big)$. The tangent direction $\gamma_S(u)$ spans a line $\langle\gamma_S(u)\rangle$ such that

$$(\mathbf{T}_pS)^\perp \subseteq \langle\gamma_S(u)\rangle \subseteq \mathbf{T}_{(\mathbf{T}_pS)^\perp}S^\vee.$$

By biduality, $(\mathbf{T}_{(\mathbf{T}_pS)^\perp}S^\vee)^\perp \cong \{p\}$. Therefore, the orthogonal of the line $\langle\gamma_S(u)\rangle$ is such that

$$p \in \langle\gamma_S(u)\rangle^\perp \subseteq \mathbf{T}_pS,$$

hence it defines a point $v \in \mathbf{P}(T_pS)$. We define ι_p as the map $u \mapsto v$.

If one prefers, u is a point of S infinitely near p. Then $\gamma_S(u)$ is a point of $S^\vee$ infinitely near the point $(\mathbf{T}_pS)^\perp$. Therefore, $\gamma_S(u)^\perp$ is a tangent plane of S infinitely near the plane $\mathbf{T}_pS$. The intersection of these two planes is a line contained in $\mathbf{T}_pS$ and passing through p, and we let $\iota_p(u)$ be its tangent direction at p.

We will consider the map ι_p indifferently as a self-map of $\mathbf{P}(T_pS)$ or of the set of lines contained in $\mathbf{T}_pS$ and passing through p.

(4.5) Lemma. *Let Y be a curve traced on a surface $S \subseteq \mathbf{P}^3$, and $p \in Y$ be a point which is non-parabolic on S. Let Σ_Y be the developable surface defined as in (4.1). Then p sits on Σ_Y, and the generator of Σ_Y through p is the line $\iota_p(\mathbf{T}_pY)$.*

Proof. By definition, Σ_Y is the dual of the curve $\gamma_S(Y) \subseteq \check{\mathbf{P}}^3$. Therefore, by the general theory of developable surfaces (see Appendix B), the rulings of Σ_Y are the orthogonals of the tangent lines of $\gamma_S(Y)$. The lemma then follows tautologically from the definition of ι_p in (4.4). □

(4.6) Proposition. *The map ι_p is the orthogonality involution of $\mathbf{P}(T_pS)$ defined by the second fundamental form at p. In particular, its fixed points are the tangent directions of the two flex tangents of S at p.*

Since p is assumed to be non-parabolic, the second fundamental form II_p at p is a non-degenerate quadratic form on the tangent plane T_pS (up to scalar multiplication), see Appendix A.2. By definition, the orthogonality involution maps a vector line in the vector plane T_pS to its orthogonal with respect to II_p, which is another vector line in T_pS.

Proof. We consider homogeneous coordinates $(x : y : z : w)$ in $\mathbf{P}^3$, such that $p = (0 : 0 : 0 : 1)$, and $\mathbf{T}_pS$ is the plane $z = 0$. We write the equation of S in the form

$$f(x, y, z, w) = w^{n-1}z + w^{n-2}(ax^2 + 2rxy + by^2 + 2sxz + 2tyz + cz^2) + O(3) = 0.$$

It follows from the local computation in (A.13) that the second fundamental form at p is given by the matrix

$$\begin{pmatrix} a & r \\ r & b \end{pmatrix} \tag{4.6.1}$$

in the coordinates (x, y) on T_pS. On the other hand, the "large" Gauss map of S, defined on a Zariski open subset of $\mathbf{P}^3$, maps the point $(x : y : z : w)$ to the transpose of

$$\begin{pmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \\ \partial_w f \end{pmatrix} = \begin{pmatrix} w^{n-2}(2ax + 2ry + 2sz) + \cdots \\ w^{n-2}(2rx + 2by + 2tz) + \cdots \\ w^{n-1} + w^{n-2}(2sx + 2ty + 2cz) + \cdots \\ (n-1)w^{n-2}z + \cdots \end{pmatrix}.$$

We write it in the affine chart $w = 1$; its differential at the origin (i.e., at p) is then

$$\begin{pmatrix} 2a\,dx + 2r\,dy + 2s\,dz \\ 2r\,dx + 2b\,dy + 2t\,dz \\ 2s\,dx + 2t\,dy + 2c\,dz \\ (n-1)\,dz \end{pmatrix}. \tag{4.6.2}$$

Now, still in the affine chart $w = 1$, locally at p, S is given by an implicit function $z = \varphi(x, y)$ with vanishing differential at the origin. In order to compute the differential of the "small" Gauss map of S, defined on a Zariski open subset of S, we need to compose the differential at the origin of $(x, y) \mapsto (x, y, \varphi(x, y))$ with that of the "large" Gauss map. This amounts to set $dz = 0$ in (4.6.2).

The upshot is that the tangent plane to S at the point infinitely near p in the direction $(u_0 : u_1)$ is given by the equation

$$(2au_0 + 2ru_1)x + (2ru_0 + 2bu_1)y + (2su_0 + 2tu_1)z = 0.$$

Therefore its intersection with $\mathbf{T}_p S$ is given by

$$(2au_0 + 2ru_1)x + (2ru_0 + 2bu_1)y = 0,$$

hence ι_p is the involution

$$\begin{pmatrix} -r & -b \\ a & r \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{C}) \tag{4.6.3}$$

(it is indeed an involution because the trace of this matrix is 0, see [46, Chap. II, §D]). By a straightforward computation (in fact more or less the exact same computation that we have just performed), this is the orthogonality involution defined by the bilinear form given by the matrix (4.6.1), and this ends the proof. □

4.2 – The Hessian developable surface

In this section we study the family of planes in $\mathbf{P}^3$ tangent to S at its parabolic points, and the associated developable surface in $\mathbf{P}^3$. We first need to adapt the observations of (4.4) and (4.6) to the case when p is a parabolic point.

Let p be a parabolic point of S. If p is general, then the second fundamental form of S at p has rank 1, see Appendix A.2, hence the tangent section $\mathbf{T}_p S \cap S$ has a degenerate double point at p, and we call *double asymptotic (or flex) tangent line* of S at p the tangent cone $\mathrm{TC}_p(\mathbf{T}_p S \cap S)$.

(4.7) Proposition. *Let p be a parabolic point of S at which the second fundamental form does not vanish. Denote by $d\gamma_p$ the differential of the Gauss map of S at p, and by Λ the double asymptotic tangent line of S at p. Then:*
(i) the kernel of $d\gamma_p$ is the tangent space at p of Λ;
(ii) the line in $\check{\mathbf{P}}^3$ passing through the point $\mathbf{T}_p S^\perp$ and going in the direction $\mathrm{im}(d\gamma_p)$ is the orthogonal of Λ.

This tells, in particular, that if p is a general parabolic point of S, the map ι_p constructed in (4.4) is essentially a contraction of $\mathbf{P}(T_p S)$ to its point corresponding to the double flex tangent line of S at p.

Proof. This follows from the computations carried out in the proof of Proposition (4.6). Note that, in the notation introduced there, we may assume $a = r = 0$ since the second fundamental form at p has rank 1. Then, we see from (4.6.2) that the kernel of $d\gamma$ is $y = 0$, which is the reduced tangent cone of the tangent section $\mathbf{T}_pS \cap S$ at p, i.e., the tangent space at p of Λ. In addition, we see from (4.6.3) that for $u = (u_0 : u_1)$, the orthogonal of $\langle d\gamma(u)\rangle$ is $y = 0$ if $u_1 \neq 0$ (and the whole $\mathbf{T}_pS$ is $u_1 = 0$). This proves (ii). □

(4.8) Remark. Proposition (4.7) is a biduality statement, the analog of (2.2.2) for parabolic points.

Let p be a general parabolic point. The line in $\check{\mathbf{P}}^3$ passing through the point $\gamma(p) = \mathbf{T}_pS^\perp$ and going in the direction $\mathrm{im}(d\gamma_p)$ is the tangent line $\mathbf{T}_{\mathbf{T}_pS^\perp}\check{C}$ to the cuspidal double curve of $S^\vee$ at p, denoted by $\check{C}$. Therefore, the proposition tells us that the orthogonal of the double flex tangent of S at p (denoted by Λ) is the line $\mathbf{T}_{\mathbf{T}_pS^\perp}\check{C}$. We give another proof of this in Appendix C.

Note that the double flex tangent Λ is the line of maximum contact with S at p, and intersects S with multiplicity 3, and the tangent to the cuspidal edge, $\mathbf{T}_{\mathbf{T}_pS^\perp}\check{C}$, is the line of maximum contact with $S^\vee$ at the point $\mathbf{T}_pS^\perp$, and intersects $S^\vee$ with multiplicity 4 at this point (the intersection multiplicity is 2 for a general line, 3 for a general line in the tangent cone, and 4 for the tangent of the cuspidal edge).

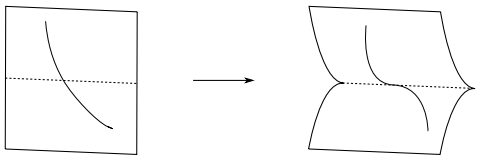


Figure 1: The Gauss map around the parabolic curve

(4.9) Proposition. *Let p be a parabolic point of S at which the second fundamental form does not vanish. The tangent line at p to the curve $S \cap \mathrm{Hess}(S)$ coincides with the double flex tangent line of S at p if and only if the tangent section $S \cap \mathbf{T}_pS$ has a tacnode at p (or a worse singularity).*

Proof. Let us write the equation of S in the form

$$f(x,y,z,w) = w^{n-1}z + w^{n-2}\left(\tfrac{1}{2}y^2 + axz + byz + \tfrac{1}{2}cz^2\right) + w^{n-3}f_3(x,y,z) + O(4) = 0,$$

in a system of homogeneous coordinates $(x : y : z : w)$ such that $p = (0 : 0 : 0 : 1)$ (n is the degree of S, and f_3 is homogeneous of degree 3). The tangent at p to the Hessian curve is obtained by intersecting $\mathbf{T}_pS$, that is, the plane $z = 0$, with the tangent plane to the Hessian, which we will find with formula (A.11.1).

In the affine chart $w = 1$, we have

$$\begin{aligned}\partial_x f &= az + \partial_x f_3 + O(3)\\ \partial_y f &= y + bz + \partial_y f_3 + O(3),\\ \partial_z f &= 1 + ax + by + cz + \partial_z f_3 + O(3)\end{aligned}$$

so that, up to a multiplicative constant, $\mathrm{Hess}(f)$ is

$$\begin{vmatrix} \frac{n}{n-1}(z + \frac{1}{2}y^2 + \cdots) & az + \partial_x f_3 + \cdots & y + bz + \partial_y f_3 + \cdots & 1 + ax + by + cz + \partial_z f_3 + \cdots \\ az + \partial_x f_3 + \cdots & \partial_x^2 f_3 + \cdots & \partial_x\partial_y f_3 + \cdots & a + \partial_x\partial_z f_3 + \cdots \\ y + bz + \partial_y f_3 + \cdots & \partial_x\partial_y f_3 + \cdots & 1 + \partial_y^2 f_3 + \cdots & b + \partial_y\partial_z f_3 + \cdots \\ 1 + ax + by + cz + \partial_z f_3 + \cdots & a + \partial_x\partial_z f_3 + \cdots & b + \partial_y\partial_z f_3 + \cdots & c + \partial_z^2 f_3 + \cdots \end{vmatrix}.$$

Plugging in $z = 0$ and keeping for each entry only the terms of order at most 1, we obtain

$$\begin{vmatrix} 0 & 0 & y & 1+ax+by \\ 0 & \partial_x^2 f_3 & \partial_x\partial_y f_3 & a+\partial_x\partial_z f_3 \\ y & \partial_x\partial_y f_3 & 1+\partial_y^2 f_3 & b+\partial_y\partial_z f_3 \\ 1+ax+by & a+\partial_x\partial_z f_3 & b+\partial_y\partial_z f_3 & c+\partial_z^2 f_3 \end{vmatrix}.$$

The only linear term in this determinant is $\partial_x^2 f_3$, so the tangent line to $S \cap \mathrm{Hess}(S)$ has equations

$$z = \partial_x^2 f_3 = 0. \tag{4.9.1}$$

Therefore, it coincides with the double flex tangent line if and only if there is no x^3 term in f_3, which is equivalent to the tangent section of S at p having an equation of the form

$$\frac{1}{2}y^2 + y(a_{21}x^2 + a_{12}xy + a_{03}y^2) + O(4) = 0,$$

which is the local form of a tacnodal singularity. □

(4.10) Remark. Let $p \in S$ be a point such that the tangent section $\mathbf{T}_pS \cap S$ has a tacnode at p (in particular p lies on the Hessian curve $\mathrm{He}(S) = S \cap \mathrm{Hess}(S)$). It follows from Proposition (4.9) that the image of the Hessian curve $\mathrm{He}(S)$ by the Gauss map (i.e., the cuspidal double curve of $S^\vee$) has a cusp at $\gamma_S(p)$. (We had seen this fact already in (3.1) and (3.7.5), arguing that $\gamma_S(p)$ is a swallowtail of $S^\vee$).

Indeed, if the second fundamental form has rank one at p, then the kernel of the differential of the Gauss map at p is the tangent space of the double flex tangent of S at p by Proposition (4.7), and this coincides with the tangent space of $S \cap \mathrm{Hess}(S)$ by Proposition (4.9). On the other hand, if the second fundamental form at p is 0, there is nothing to prove.

(4.11) Proposition. *Let $p \in S$ be a point such that the tangent section $\mathbf{T}_pS \cap S$ has a tacnode at p. The regression (i.e., cuspidal) edge of the Hessian developable Σ_{He} passes through p, and its osculating plane there is $\mathbf{T}_pS$. In particular the two surfaces Σ_{He} and S are tangent at p.*

Proof. This follows from the fact, which we have observed in (3.7.5), that $p^\perp$ is the osculating plane of the cuspidal double curve of $S^\vee$, which we will denote here by $C \subseteq \check{\mathbf{P}}^3$. Note that it has a local parametrization of the form $t \mapsto (t^2, t^3, t^4)$ at p, and has intersection multiplicity 4 with $p^\perp$ at the point $(\mathbf{T}_pS)^\perp$, see (3.1) and (3.7.5).

Let $\Gamma \subseteq \mathbf{P}^3$ be the regression edge of Σ_{He}. By the general theory of developable surfaces, see Appendix B, Γ parametrizes the osculating planes of C. Thus, since $p^\perp$ is an osculating plane of C, $p = (p^\perp)^\perp$ sits on Γ. Moreover, by the osc-duality for space curves, see again Appendix B, the osculating plane of Γ at p is the orthogonal of the point at which $p^\perp$ is osculating to C: the latter point is $\gamma_S(p) = (\mathbf{T}_pS)^\perp$, hence the osculating plane of Γ at p is $\mathbf{T}_pS$. Finally, the tangent plane to Σ_{He} at p is the osculating plane of Γ at p, which we have just seen is $\mathbf{T}_pS$. □

We may now give a direct geometric description of the Hessian developable.

(4.12) Theorem. *The generators of the Hessian developable surface are the double flex tangent lines at parabolic points of S. Its stationary planes are the tangent planes to S at points where the tangent section has a tacnode.*

Proof. The identification of the generators follows from Proposition (4.7) and Lemma (4.5) (which works fine even at parabolic points). That of the stationary planes follows from Proposition (4.9), see also Remark (4.10). □

The rest of this section is dedicated to the computation of the numerical characters of the Hessian developable.

(4.13) Lemma. *Let S, T be two surfaces in $\mathbf{P}^3$ of degrees s, t respectively. The degree of the ruled surface generated by the flex tangents of S at the points of $S \cap T$ equals $st(3s-4)$.*

Note that there is no reason that this ruled surface be developable in general, although it is indeed when $T = \mathrm{Hess}(S)$.

Proof. Let f, g be homogeneous equations of S and T respectively. The ruled surface we are interested in is the projection on the first factor of the incidence graph

$$\big\{(p,p') : p' \in S \cap T \text{ and } p \text{ sits on a flex tangent of } S \text{ at } p'\big\},$$

which is the complete intersection in $\mathbf{P}^3 \times \mathbf{P}^3$ defined by the following four bihomogeneous polynomials in the variables (p,p'): $f(p')$, $g(p')$, $\mathrm{D}^p f(p')$, $\mathrm{D}^{p^2} f(p')$, of bidegrees $(0,s)$, $(0,t)$, $(1,s-1)$, and $(2,s-2)$respectively.

Our ruled surface is therefore the hypersurface in $\mathbf{P}^3$ defined by the homogeneous equation obtained by eliminating the variable p', i.e., by the polynomial

$$\mathrm{Res}_{s,t,s-1,s-2}\big(f, g, \mathrm{D}^p f, \mathrm{D}^{p^2} f\big),$$

taken as a resultant of homogeneous polynomials in the variable p', with coefficients homogeneous polynomials in the variable p. By a standard homogeneity property of the resultant, see [C, Theorem (1.7)], it is a homogeneous polynomial of degree

$$\big[0 \times t(s-1)(s-2)\big] + \big[0 \times s(s-1)(s-2)\big] + \big[1 \times st(s-2)\big] + \big[2 \times st(s-1)\big] = st(3s-4)$$

in the variable p. □

(4.14) Corollary. *The degree of the Hessian developable surface equals $2n(n-2)(3n-4)$.*

Proof. We apply the above lemma with $T = \mathrm{Hess}(S)$. This gives the degree $4n(n-2)(3n-4)$, which however has to be divided by two, as the obtained equation is double in this case, being the two flex tangents coincident at parabolic points. □

(4.15) The class of the Hessian developable is the number of planes tangent to S at a parabolic point, and passing through a fixed general point in $\mathbf{P}^3$. It is therefore the intersection number of S, $\mathrm{Hess}(S)$, and $\mathrm{D}^{p'} S$ for a general $p' \in \mathbf{P}^3$, which equals $4n(n-1)(n-2)$.

(4.16) The number of stationary planes of the Hessian developable is, by Theorem (4.12), the number of points in the intersection of the parabolic curve $S \cap \mathrm{Hess}(S)$ with the flecnodal locus. We have seen in (3.15) that this number is $2n(n-2)(11n-24)$.

(4.17) We may now find the remaining characteristic numbers using the formulae of Appendix B. We use the notation of (B.6) and (B.17), with an additional decoration 'He' to avoid confusion; see also the warning below. We have found so far:
— $r_{\mathrm{He}} = 2n(n-2)(3n-4)$, the degree (or rank) of the developable, see Corollary (4.14);
— $n_{\mathrm{He}} = 4n(n-1)(n-2)$, its class, see (4.15);
— $\alpha_{\mathrm{He}} = 2n(n-2)(11n-24)$, its number of stationary planes, see (4.16).
Knowing three quantities is enough to derive all the others by direct applications of the formulae in (B.19) and (B.20). This way one finds, as in [45, §608]:
— $m_{\mathrm{He}} = 4n(n-2)(7n-15)$, the degree of the regression edge of Σ_{He};

— $\beta_{\mathrm{He}} = 10n(n-2)(7n-16)$, the number of flexes of the cuspidal double curve $\check{C}$ of $S^\vee$;
— $g_{\mathrm{He}} = 2n(n-2)(4n^4-16n^3+20n^2-27n+39)$, the number of apparent double points of $\check{C} \subseteq S^\vee$;
— $h_{\mathrm{He}} = 2n(n-2)(196n^4-1232n^3+2580n^2-1861n+137)$, the number of apparent double points of the regression edge of Σ_{He}.

(4.17.1) Warning. For ease of reference I have done my best to stick to Salmon's notation, as other have done before me, e.g., Piene [38], Ronga [42]. There are however some compatibility problems between the notation of (B.6) and (B.17) for developable surfaces, and that of (3.4), (3.6), and (4.20) for the singularities of the dual. For instance, $\alpha_{\mathrm{He}} = \check{\beta}$, both the number of stationary planes in the Hessian system and of swallowtails of $S^\vee$, and $g_{\mathrm{He}} = \check{h}$, the number of apparent double points of the cuspidal double curves of $S^\vee$.

(4.17.2) References. Some of these numbers have been computed by Kulikov in an interesting series of two papers, with intersection theoretic computations. In particular, the number α_{He} is computed in [28] ("number of points P_3"), and the number β_{He} is computed ("number of points type P_3") in [29]; there the regression edge of the Hessian developable is called the focal curve, and "points of type P_3" are the cusps of this curve, hence correspond to the flexes of $C = \Sigma_{\mathrm{He}}^\vee$.

Moreover, [28] provides some invariants of the flecnodal curve that we have not considered here. "Points H_3" are the ordinary double points of the flecnodal curve; they are the points $q \in S$ such that both local branches at q of the tangent section $\mathbf{T}_q S \cap S$ have a flex at q. "Points H_4" are the points where the flecnodal curve is tangent to an asymptotic tangent line of S; they are the points $q \in S$ such that the tangent section $\mathbf{T}_q S \cap S$ has a branch at q with a higher order inflection point. Kulikov obtains:
— $\#H_3 = 5n(7n^2-28n+30)$;
— $\#H_4 = 5n(n-4)(7n-12)$.
These numbers are given by Salmon as well, at [45, §604].

4.3 – Ordinary double edges of the polar cone

In this section we consider the same setup as in Section 3.2, and use the notation introduced there. In particular, we let X be a degree n surface in $\mathbf{P}^3$ with d-ordinary singularities, and call B and C its ordinary and cuspidal double curves. We consider a general point $p \in \mathbf{P}^3$, and denote by $\mathcal{D}^p X$ the polar cone of X with respect to p, and by $\mathcal{B}^p X$ the apparent boundary.

(4.18) Ordinary double edges of the polar cone. In this section we study the ordinary double edges of the polar cone $\mathcal{D}^p X$; they are the lines passing through p and intersecting X with multiplicity 2 in two distinct points. The locus of points $q \in S$ such that the line $\langle p, q\rangle$ verifies this condition is cut out on $X \cap \mathrm{D}^p X$ by a degree $(n-2)(n-3)$ hypersurface $R^p X$: this has been proved by Salmon, and his construction is explained in [C, Section 4.1]. This hypersurface has been invoked in (2.8.3) already.

We shall conduct our study of ordinary double edges by analyzing the various kinds of intersection points of this hypersurface $R^p X$ with the curve $X \cap \mathrm{D}^p X$. This strategy is similar to that followed in Section 3.2, with the hypersurface $R^p X$ playing the role that the second polar $\mathrm{D}^{p^2} X$ played there.

(4.19) Definition. *The* number of apparent double points *of a curve in* $\mathbf{P}^3$ *is the number of double points of its projection from a general point.*

The number of apparent only double points *of two curves* Y, Y' *in* $\mathbf{P}^3$ *is the number of intersection points of their projections from a general point that do not come from an actual intersection point of* Y *and* Y'*. We denote it by* $[Y, Y']^\circ$.

(4.20) Notation. We add the following notation to that introduced in Section 3.2:
— δ is the number of ordinary double edges of the circumscribed cone $\mathcal{C}_p X$ (these are the honest bitangent lines to X);
— k is the number of apparent double points of the ordinary double curve B of X;
— h is the number of apparent double points of the cuspidal double curve C of X;
— i is the number of intersection points of B and C smooth for both curves (when $X = S^\vee$ for a smooth S, this number vanishes).

The following formulae, given by Salmon without any comment, describe the intersection of the hypersurface $R^p X$ mentioned in (4.18) above with the various components of the curve

$$X \cap \mathrm{D}^p X = \mathcal{B}^p X + 2B + 3C,$$

of total degree $n(n-1)(n-2)(n-3)$. Recall that a, b, c denote the degrees of the curves $\mathcal{B}^p X, B, C$ (called A, B, C by Salmon), respectively.

(4.21) Lemma. *The following relations hold:*

$$a(n-2)(n-3) = 2\delta + 2[\mathcal{B}^p X \cdot B]^\circ + 3[\mathcal{B}^p X \cdot C]^\circ \ ;$$
$$b(n-2)(n-3) = 4k + [\mathcal{B}^p X \cdot B]^\circ + 3[B \cdot C]^\circ \ ;$$
$$c(n-2)(n-3) = 6h + [\mathcal{B}^p X \cdot C]^\circ + 2[B \cdot C]^\circ.$$

Proof. Let Y, Y' be two curves (possibly the same) among $\mathcal{B}^p X, B, C$, and denote by $m_Y, m_{Y'}$ their multiplicities in the cycle $X \cap \mathrm{D}^p X$. An intersection point between two local branches of the projections $\pi_p(Y)$ and $\pi_p(Y')$ gives a double edge of the polar cone with multiplicity $m_Y m_{Y'}$, hence two contact points $y \in Y$ and $y' \in Y'$ with X, each of which should contribute with the same multiplicity $m_Y m_{Y'}$ to the intersection $X \cap \mathrm{D}^p X \cap R^p X$. Taking into account the multplicities of Y and Y' in $X \cap \mathrm{D}^p X$, one finds that $R^p X$ should intersect Y with multiplicity $m_{Y'}$ at y and Y' with multiplicity m_Y at y'. This explains all the coefficients in the formulae. □

Some claims in the above proof arguably deserve a more rigorous treatment, but let me not try to fix this here. Note that the actual intersection points of $\mathcal{B}^p X, B, C$ do not contribute to the intersection $X \cap \mathrm{D}^p X \cap R^p X$, as they give rise to cuspidal double edges of the polar cone instead of ordinary double edges.

For two curves Y and Y' in $\mathbf{P}^3$, the total number of intersection points of the projections $\pi_p(Y)$ and $\pi_p(Y')$ is $\deg(Y)\deg(Y')$. The following set of formulae, yet again given coldbloodedly by Salmon, indicates the quantities to be subtracted from $\deg(Y)\deg(Y')$ to obtain $[Y, Y']^\circ$, in the case when Y and Y' are two components of $X \cap \mathrm{D}^p X$.

(4.22) Theorem. *The following relations hold:*

$$[\mathcal{B}^p X \cdot B]^\circ = ab - 2\rho \ ;$$
$$[\mathcal{B}^p X \cdot C]^\circ = ac - 3\sigma \ ;$$
$$[B \cdot C]^\circ = bc - 3\beta - 2\gamma - i.$$

Proof. In our setup, every actual intersection point of two curves among $\mathcal{B}^p X, B, C$ is of one of the types $\rho, \sigma, \beta, \gamma, i$, so only the coefficients are to be explained. The latter are the intersection multiplicities at the corresponding points of the projections from p of the two curves under consideration. For points of type i, it is 1 by the generality of p. The rest of the proof consists in examining one by one the other possible types.

(4.22.1) Points of type ρ. These are the points $q \in \mathcal{B}^p X \cap B$. At such a point, the surface X has two smooth local sheets X'_q and X''_q, $B = X'_q \cap X''_q$ and the circumscribed curve $\mathcal{B}^p X$ is contained in one of them, say X'_q, and transverse to B. The curve $\mathcal{B}^p X$ is the ramification curve of the projection of X'_q, hence the projections of $\mathcal{B}^p X$ and B are tangent at the projection of q, and thus intersect with multiplicity 2.

(4.22.2) Points of type σ. These are the points $q \in \mathcal{B}^p X \cap C$. At such a point, the surface X has one cuspidal local sheet X_q, and $C = \mathrm{Sing}(X_q)$. We have seen in (3.7.3) that $\mathcal{B}^p X$ and C are smooth and tangent at q, and it follows from the same argument as in (4.22.1) that their projections from p are tangent. But the claim is that, in fact, the two projections intersect with multiplicity 3.

It should be possible to prove this with a local computation, using the setup of (3.7.3), but I have not been able to do it in general. The difficulty is that, while the curve defined by the two equations f and $\mathrm{D}^p f$ splits as $\mathcal{B}^p X + 3C$ locally at q (in the notation of (3.7.3)), neither of the two summands is a Cartier divisor on X, and therefore it is difficult to find their equations. It is however doable in practice for an arbitrary, explicit, choice of the polynomials g_2 and h_2, using one's favourite computer algebra system; I have done it for several randomly chosen polynomials, using Macaulay2; I recommend to do this over a finite field. Finally, once one knows equations for C and $\mathcal{B}^p X$, one finds equations for their projections by eliminating the variable x (because $p = (1 : 0 : 0 : 0)$), and it only remains to check with one's own eyes that the intersection multiplicity at the projection of q is indeed 3.

In any event, the number $[\mathcal{B}^p X \cdot C]^\circ$ occurs only in (𓆷𓂝𓇋𓏺) and (𓆷𓂝𓇋𓏺𓏺𓏺), in which we already know all the other numbers involved. One may therefore use them to find $[\mathcal{B}^p X \cdot C]^\circ$, and thus obtain an indirect proof of Formula (𓄀𓏭).

(4.22.3) Points of type β. These are the swallowtail singularities of X. At these points B is smooth and C is cuspidal, and we can see on the local model in (3.1) that they have the same reduced tangent cone. It follows that after projection they intersect with multiplicity 3 at these points.

(4.22.4) Points of type γ. These are the points at which X has two transverse local sheets, one smooth: X^{s}_q, and the other cuspidal: X^{c}_q. There C is smooth and B is cuspidal, and these two curves have transverse reduced tangent cones. It follows that after projection they intersect with multiplicity 2 at these points.

Theorem (4.22) is now proved. □

Finally, putting together formulas (𓆷𓂝𓇋) and (𓄀) gives the following.

(4.23) Corollary.

(𓆷𓂝𓇌𓏺) $$a(n-2)(n-3) = 2\delta + 2ab + 3ac - 4\rho - 9\sigma \ ;$$

(𓆷𓂝𓇌𓏭) $$b(n-2)(n-3) = 4k + ab + 3bc - 9\beta - 6\gamma - 3i - 2\rho \ ;$$

(𓆷𓂝𓇌𓏺𓏺𓏺) $$c(n-2)(n-3) = 6h + ac + 2bc - 6\beta - 4\gamma - 2i - 3\sigma.$$

4.4 – Application to the case when $X = S^\vee$

We now proceed as in Section 3.3, i.e., we let X be the dual of a smooth, sufficiently general, degree n surface $S \subseteq \mathbf{P}^3$, and apply the formulas of Section 4.3 above to X. We maintain the setup and notation introduced in Section 3.3.

(4.24) As in (3.11), the circumscribed cone $\mathcal{C}^{\check{p}}(S^\vee)$ is the dual in $\check{\mathbf{P}}^3$ of the plane curve $\check{p}^\perp \cap S$. The number $\check{\delta}$ of ordinary edges of this cone is therefore the number of bitangents of the plane curve $\check{p}^\perp \cap S$, which by (1.14) is:
— $\check{\delta} = \frac{1}{2}n(n-2)(n-3)(n+3)$.

(4.25) The numbers $\check{h}$ and $\check{k}$ are the invariants g (introduced in (B.17)) of the Hessian and node-couple developables respectively. The former has been computed in (4.17) and the latter we don't know yet:
— $\check{h} = g_{\mathrm{He}} = 2n(n-2)(4n^4 - 16n^3 + 20n^2 - 27n + 39)$;
— $\check{k} = g_{\mathrm{Kn}}$.

(4.26) Examination of formulae (☕ 🙌) . The numbers $\check{n}, \check{a}, \check{b}, \check{c}$ have been given in Section 3.3, and the numbers $\check{\rho}, \check{\sigma}, \check{\beta}, \check{\gamma}$ have already been computed. As we mentioned earlier, the number $\check{\imath}$ vanishes. One may thus check by a direct computation that the first and third identities of (4.23) indeed hold. The second one, on the other hand, gives the formula:
— $\check{k} = \frac{1}{8}n(n-2)(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5+547n^4-1058n^3+1068n^2-1214n+1464)$.

(4.27) Numerical characters of the node-couple developable. At this point, we know the following three numerical characters of the node-couple developable: $n_{\mathrm{Kn}} = \check{b}$, the degree of the curve $\check{B} \subseteq \check{\mathbf{P}}^3$, $g_{\mathrm{Kn}} = \check{k}$, the number of apparent double points of $\check{B}$, and $\alpha_{\mathrm{Kn}} = \check{\gamma}$, the number of cusps of $\check{B}$. We may therefore derive all the others using the formulas of Section B.3; however, these ought to be suitably modified in order to take into account the presence of $\check{t}$ ordinary triple points on the curve $C_{\mathrm{Kn}} = \check{B}$. This may be taken care of in various ways, but the idea is always the same, namely that an ordinary triple point counts exactly as three nodes.

For instance, let us adapt the formula $r = n(n-1) - 2g - 3\alpha$ from (B.19) in order to compute the rank r_{Kn} of the node-couple developable. The adapted formula is

$$\begin{aligned} r_{\mathrm{Kn}} &= n_{\mathrm{Kn}}(n_{\mathrm{Kn}} - 1) - 2\big(g_{\mathrm{Kn}} + 3\#(\text{triple points})\big) - 3\alpha_{\mathrm{Kn}} \\ &= \check{b}(\check{b} - 1) - 2\check{k} - 6\check{t} - 3\check{\gamma}, \end{aligned}$$

and it gives
— $r_{\mathrm{Kn}} = n(n-2)(n-3)(n^2+2n-4)$.
We leave this task for the remaining numerical characters an exercise to the not-exhausted-yet reader.

Finally, I would like to mention the article [11], in which the authors perform an interesting study of the family of bitangent lines of a very general quartic surface, which involves in particular the node-couple developable.

5 – Formulae for surfaces projected in $\mathbf{P}^3$

In this section, we prove formulae for the enumeration of curves with prescribed singularities in a possibly incomplete linear system of dimension $r \leqslant 3$ on a smooth surface; this will take the form of enumerating curves on a surface in $\mathbf{P}^3$ with ordinary singularities, see (5.3). These formulae are from a different age with respect to those of Salmon, and involve invariants of a different nature, e.g., the arithmetic genus; by the way, beware the difference in notation with respect to the previous sections. The material presented here is taken from [15, Capitolo V].

(5.1) Setup. We consider a linear system $|V|$ of dimension $r \leqslant 3$ on a smooth surface S. We will usually make some transversality assumptions on $|V|$, which are typically granted if $|V|$ is a general subsystem of a very ample linear system. In particular, we always assume that the general member of $|V|$ is smooth and irreducible. We set:
— C the linear equivalence class of divisors in $|V|$ (and sometimes a member of $|V|$ by abuse of notation) ;
— n the degree of $|V|$, i.e., $n = C^2$;
— π the common arithmetic genus of all members of $|V|$.
We have the following first identity, given by the adjunction formula:

$$K_S \cdot C = 2\pi - 2 - n. \tag{5.1.1}$$

(5.2) Warm up: the 1-dimensional case. Let $r = 1$, and assume that $|V|$ has n simple base points, and all its singular members are 1-nodal curves. In this context, we define the class $\check{n}$ of $|V|$ as the number of its singular members. In this case, the topological formula discussed in details in [XI, Section 2.1] gives:
— $\check{n} = e(S) + n - e(C) \cdot e(\mathbf{P}^1) = c_2(S) + n + 4\pi - 4$.

Next we treat the case $r = 3$. For $r = 2$ we shall only consider a cheap version in which we assume that $|V|$ is a subsystem of a linear system of dimension at least 3.

5.1 – The 3-dimensional case: surfaces with ordinary singularities

In this section we let $r = 3$, and make the following assumption, granted by the general projection theorem, see [20], if $|V|$ is a general subssytem of a very ample linear system.

(5.3) We assume that the map defined by $|V|$ is a birational morphism from S to a degree n surface $S^\flat \subseteq \mathbf{P}^3$ with *ordinary singularities* (in the usual sense), i.e., $S^\flat$ is smooth except for an ordinary double curve D with τ pinch points and t triple points: locally at a pinch point $S^\flat$ has a Whitney umbrella singularity, i.e., it looks like the hypersurface $x^2 - zy^2 = 0$ in the affine space with coordinates x, y, z; locally at a triple point it looks like the hypersurface $xyz = 0$, so these points are non-planar triple points for D; locally at all other points of D, $S^\flat$ looks like the hypersurface $xy = 0$; in particular, D is smooth except for the triple points. For simplicity, we assume moreover that D is irreducible.

The morphism $\nu : S \to S^\flat$ is the normalization map. We let $\Delta \subseteq S$ be the pre-image of $D \subseteq S^\flat$; there is a double cover $\Delta \to D$, which ramifies exactly over the τ pinch points. By adjunction theory (see, e.g., [22, Exer. II.8.5] and [II, Section 2]), one has $\omega_S \cong \mathcal{O}_S(-\Delta) \otimes \nu^* \omega_{S^\flat}$, and $\nu_* \omega_S = \mathcal{I}_D|_{S^\flat}$, where $\mathcal{I}_D$ is the sheaf of ideals defining D in $\mathbf{P}^3$. Being $S^\flat$ a divisor in $\mathbf{P}^3$, we have $\omega_{S^\flat} \cong \omega_{\mathbf{P}^3}(S^\flat)\big|_{S^\flat} = \mathcal{O}_{S^\flat}(n-4)$ by the standard adjunction formula.

(5.4) In addition to the already introduced notation (see also (5.1)), we let:
— τ be the number of pinch points of $S^\flat$;
— t be the number of triple points of $S^\flat$;
— d be the degree of the double curve $D \subseteq \mathbf{P}^3$;
— ρ be the geometric genus of D ;
— Π be the geometric genus of $\Delta \subseteq S$.

(5.5) The curves C are represented in $\mathbf{P}^3$ as hyperplane sections of $S^\flat$, hence as degree n plane curves with d nodes. Therefore one has

$$\pi = \binom{n-1}{2} - d. \tag{5.5.1}$$

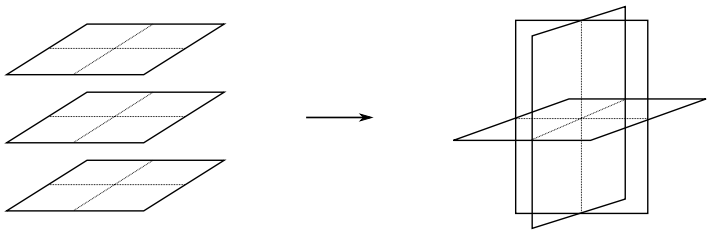


Figure 2: Normalization of $S^\flat$ at a triple point

(5.6) Effective degree of first polars. We compute the class $\check{n}$ of $S^\flat$, that is the number of singular curves in a general subpencil of $|V|$, following the method explained in (1.8).

For a general point $p \in \mathbf{P}^3$, the first polar $\mathrm{D}^p S^\flat$ is a degree $n-1$ hypersurface passing simply through D, and we define the apparent boundary $\mathcal{B}^p S^\flat$ as the Zariski closure of $(\mathrm{D}^p S^\flat \cap S^\flat) - D$, as in (3.5). The curve $\mathcal{B}^p S^\flat$ passes through all pinch points: this follows from a local computation, which we reckon the reader who has survived this far can make on his own.

Therefore, for a general pair of points $p, q \in \mathbf{P}^3$ the intersection of the two curves $\mathcal{B}^p S^\flat$ and $\mathcal{B}^q S^\flat$ consists of the τ pinch points of $S^\flat$ plus the $\check{n}$ points a such that $p, q \in \mathbf{T}_a S$. To compute this intersection number $\tau + \check{n}$, we pull-back the curves to the normalization S. They are sections of

$$\mathcal{I}_D \otimes \mathcal{O}_{S^\flat}(n-1) \cong \mathcal{I}_D \otimes \omega_{S^\flat} \otimes \mathcal{O}_{S^\flat}(3),$$

hence pull-back to curves in the class $K_S + 3C$. We thus find the identity

$$\check{n} + \tau = (K_S + 3C)^2 = K_S^2 + 3n + 12\pi - 12, \tag{5.6.1}$$

using (5.1.1) to substitute the intersection number $K_S \cdot C$.

(5.7) Genus of the curve of neutral pairs of $|V|$. One first observes that the curve Δ has three nodes above each triple point of $S^\flat$, as is best understood looking at Figure 2. Since it is otherwise smooth, its arithmetic genus is $p_a(\Delta) = p_g(\Delta) + 3t = \Pi + 3t$. On the other hand, using the already seen identity $K_S = (n-4)C - \Delta$, the adjunction formula gives

$$2(\Pi + 3t) - 2 = 2p_a(\Delta) - 2 = (K_S + \Delta) \cdot \Delta = (n-4)C \cdot \Delta = 2(n-4)d.$$

Therefore,

$$p_a(\Delta) = \Pi + 3t = (n-4)d + 1. \tag{5.7.1}$$

On the other hand, the map $\Delta \to D$ lifts to a $2:1$ covering $\bar{\Delta} \to \bar{D}$ between the normalizations of Δ and D. It is ramified at τ points, so the Riemann–Hurwitz formula gives

$$2\Pi - 2 = 2(2\rho - 2) + \tau. \tag{5.7.2}$$

(5.8) Intersections of apparent boundaries with the double curve. Let $p \in \mathbf{P}^3$ be a general point. We consider the apparent boundary $\mathcal{B}^p S^\flat$ and its intersection with the double curve D. To see the situation clearly, it is best to work on the normalization S. One has

$$\nu^* \mathcal{B}^p S^\flat \cdot \Delta = (K_S + 3C) \cdot \big((n-4)C - K_S\big) = -K_S^2 + 2\pi(n-7) + 2n^2 - 7n + 14.$$

Geometrically this intersection amounts to the τ pinch points plus the points $q \in D$ such that the tangent plane to one of the two local sheets of $S^\flat$ at q passes through p. Let a be the number

of the latter points. It may be computed by considering the intersection of the second polar $\mathrm{D}^{p^2} S^\flat$ with D. It follows from Theorem (A.8) that this intersection consists of the a points we are interested in plus the t triples points, and since those are triple for D one finds

$$a + 3t = (n-2)d.$$

Finally, we end up with

$$\tau + [(n-2)d - 3t] = -K_S^2 + 2\pi(n-7) + 2n^2 - 7n + 14. \tag{5.8.1}$$

[4]

(5.9) Postulation formula. We shall also use the following formula for the arithmetic genus of S, which we shall discuss in Section 5.2:

$$p_a(S) = \binom{n-1}{3} - (n-4)d + \rho + 2t - 1. \tag{5.9.1}$$

(5.10) Conclusion. We may now express the quantities $\check{n}$, τ, t, d, ρ, Π in terms of the four invariants n, π, $p_a(S)$, K_S^2. The formula for d is readily given by (5.5.1); we use it so substitute for d in the five equations (5.6.1), (5.7.1), (5.7.2), (5.8.1), (5.9.1), thus obtaining

$$\begin{pmatrix} 1 & & & & 1 \\ & 2 & 6 & & \\ & 2 & 0 & -4 & -1 \\ & & -3 & 0 & 1 \\ & & -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \check{n} \\ \Pi \\ t \\ \rho \\ \tau \end{pmatrix} = \begin{pmatrix} 3n + 12\pi + K_S^2 - 12 \\ (n-4)(n-1)(n-2) - 2(n-4)\pi + 2 \\ -2 \\ -\frac{1}{2}n^3 + \frac{9}{2}n^2 - 11n + 16 - K_S^2 + \pi(3n-16) \\ \frac{1}{6}(n-1)(n-2)(-2n+9) + \pi(n-4) - p_a(S) - 1 \end{pmatrix}.$$

This finally gives:

— $\check{n} = n + 4\pi + 12p_a(S) - K_S^2 + 8$;

— $d = \frac{1}{2}(n-1)(n-2) - \pi$;

— $\rho = \frac{1}{2}(n^2 - 7n) + \pi(n-12) + 9p_a(S) - 2K_S^2 + 22$;

— $\Pi = n^2 - 6n + 2\pi(n-10) + 12p_a(S) - 3K_S^2 + 33$;

— $t = \frac{1}{6}(n^3 - 9n^2 + 26n) - \pi(n-8) - 4p_a(S) + K_S^2 - 12$;

— $\tau = 2n + 8\pi - 12p_a(S) + 2K_S^2 - 20$.

(5.11) Choice of four independent invariants. I have chosen to follow Enriques' notation for ease of reference. I have however substituted the old-fashioned *linear genus* $p^{(1)}(S)$ with K_S^2. Indeed, $p^{(1)}(S)$ is defined as the *virtual* genus of canonical sections of S, that is the genus of the curves in $|K_S|$ obtained from the adjunction formula, even though $|K_S|$ may in fact be empty. In other words $p^{(1)}$ is defined by the formula

$$2p^{(1)} - 2 = (K_S + K_S) \cdot K_S,$$

hence $p^{(1)} = K_S^2 + 1$.

Nowadays the custom is to use C^2, $K_S \cdot C$, K_S^2, $c_2(S)$ rather than n, π, $p_a(S)$, K_S^2. To obtain the above formulas in this form, simply use

$$C^2 = n; \qquad K_S \cdot C + C^2 = 2\pi - 2; \qquad 1 + p_a(S) = \frac{1}{12}\big(K_S^2 + c_2(S)\big). \tag{5.11.1}$$

The former two relations we have already encountered, and the latter is the well-known Noether formula (see, e.g., [22, Example A.4.1.2]; a modern proof in classical spirit is given in [39]; we shall prove it in (5.18)).

[4] There is a typo in [15, Formula (6), p. 175], but the final formulas on p. 176 agree with ours in (5.10).

5.2 – The postulation formula

In this section we discuss the postulation formula (5.9.1), which gives an expression for the arithmetic genus $p_a(S)$ in terms of the projection $S^\flat$ in $\mathbf{P}^3$. Here, we give a first simple proof of the formula, and then explain the classical definition of the arithmetic genus and why it coincides with the modern definition, which gives another simple proof of the formula. The postulation formula is discussed in [39] as well. The interested reader may also wish to consult [55, Chap. III], in addition to [15, Cap. IV].

(5.12) Definition. *Let X be a variety. The* arithmetic genus *of X is*

$$p_a(X) := (-1)^{\dim X}\big(\chi(\mathcal{O}_X) - 1\big).$$

If X is smooth, its geometric genus *is*

$$p_g(X) := h^0(X, \omega_X).$$

If X is a degree n hypersurface in $\mathbf{P}^N$, one finds

$$p_a(X) = h^0(\mathcal{O}_{\mathbf{P}^N}(n - N - 1)) = \binom{n-1}{N}. \tag{5.12.1}$$

(5.13) Arithmetic and geometric genera of a curve. For a smooth curve there is no difference between the arithmetic and geometric genera; it is important to note that the same does not hold in higher dimension. One defines the geometric genus of an arbitrary reduced curve as the genus of its normalization. The arithmetic and geometric genera of a curve X may be related by means of the exact sequence

$$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{\bar{X}} \to \bigoplus_{p\in \mathrm{Sing}(X)} (\mathbf{C}_p)^{\oplus \delta(p)} \to 0, \tag{5.13.1}$$

where $\nu : \bar{X} \to X$ is the normalization map, and the third term is a skyscraper sheaf supported on the singular locus of X, with $\delta(p)$ an integer depending on the type of singularity, see [II, Section 2]. Taking Euler characteristics in (5.13.1), one obtains

$$p_g(X) = p_a(X) - \textstyle\sum_{p\in \mathrm{Sing}(X)} \delta(p).$$

(5.13.2) Example. Locally at a space triple point, the exact sequence (5.13.1) looks like

$$\begin{array}{ccccccc} 0 \to \mathbf{C}[x,y,z]/(xy,xz,yz) \to & \mathbf{C}[x]\oplus\mathbf{C}[y]\oplus\mathbf{C}[z] & \to & \mathbf{C}^2 & \to 0 \\ & (P,Q,R) & \mapsto & \big(P(0)-Q(0), Q(0)-R(0)\big) & \end{array}$$

so that a space triple point makes the genus drop by 2.

(5.14) Proof of the postulation formula, I. Let us apply the ideas of (5.13) in order to compute the arithmetic genus of a smooth surface S in terms of a general projection $S^\flat$ in $\mathbf{P}^3$. Thus, $S^\flat$ has ordinary singularities as described in (5.3), and we shall use the same notation as in Section 5.1. There is an exact sequence

$$\begin{array}{ccccccc} 0 \to \mathcal{O}_{S^\flat} \to \nu_*\mathcal{O}_S \to & \nu_*\mathcal{O}_\Delta & \to & \mathcal{O}_D & \to 0, \\ & (f_1, f_2) & \mapsto & f_1 - f_2 & \end{array}$$

and it gives, taking Euler characteristics:

$$p_a(S) = p_a(S^\flat) - p_a(\Delta) + p_a(D). \tag{5.14.1}$$

Now, the arithmetic genus of $S^\flat$ is that of a degree n hypersurface in $\mathbf{P}^3$, see (5.12.1), that of Δ is $(n-4)d+1$ by (5.7.1), and that of D is $\rho+2t$ by (5.13.2), so the relation (5.14.1) gives the postulation formula (5.9.1). □

To explain the classical definition of the arithmetic genus of a surface, we will use the following lemma.

(5.15) Lemma. *For all $i=0,1,2$, one has $H^i(S,\omega_S)\cong H^i(S^\flat,\mathcal{O}_{S^\flat}(n-4)\otimes\mathcal{I}_D)$.*

Proof. We use the relations between the canonical line bundles of S, $S^\flat$, and $\mathbf{P}^3$, see (5.3). By the projection formula, we have

$$\nu_*\omega_S=\nu_*\big(\nu^*\mathcal{O}_{S^\flat}(n-4)\otimes\mathcal{O}_S(-\Delta)\big)=\mathcal{O}_{S^\flat}(n-4)\otimes\mathcal{I}_D. \tag{5.15.1}$$

Moreover one has $R^i\nu_*(\omega_S)=0$ for all $i=1,2$, because ν is finite, so $H^i(S,\omega_S)\cong H^i(S^\flat,\nu_*\omega_S)$ for all $i=0,1,2$ by Leray's spectral sequence (or merely [22, Exer. III.8.1]), and the result follows. □

(5.16) Classical definition of the arithmetic genus of a surface. Let S be a smooth surface, and $S^\flat$ be a general projection of S in $\mathbf{P}^3$, as in (5.14). By Lemma (5.15), one has

$$p_g(S)=h^0(\mathbf{P}^3,\mathcal{O}_{\mathbf{P}^3}(n-4)\otimes\mathcal{I}_D),$$

i.e., the geometric genus of S is the number of independent degree $n-4$ hypersurfaces containing D; in classical words, the latter are called degree $n-4$ hypersurfaces adjoint to $S^\flat$. To compute this number, one may consider the restriction exact sequence

$$0\to\mathcal{O}_{\mathbf{P}^3}(n-4)\otimes\mathcal{I}_D\to\mathcal{O}_{\mathbf{P}^3}(n-4)\to\mathcal{O}_D(n-4)\to 0. \tag{5.16.1}$$

If the linear series cut out on D by $|\mathcal{O}_{\mathbf{P}^3}(n-4)|$ is complete and non-special, one finds that $h^0(\mathbf{P}^3,\mathcal{O}_{\mathbf{P}^3}(n-4)\otimes\mathcal{I}_D)$ equals

$$\begin{aligned} h^0\big(\mathcal{O}_{\mathbf{P}^3}(n-4)\big)-h^0\big(\mathcal{O}_D(n-4)\big)&=\tbinom{n-3}{3}-\big(1-p_a(D)+\deg\mathcal{O}_D(n-4)\big)\\ &=\tbinom{n-3}{3}-1+(\rho+2t)+(n-4)d \end{aligned} \tag{5.16.2}$$

(the first equality is given by the Riemann–Roch theorem, and the second one by the computation of $p_a(D)$ in (5.14)). One recognizes in (5.16.2) the right-hand-side of the postulation formula (5.9.1).

Classically, the arithmetic genus of S is defined as the *virtual number* of independent degree $n-4$ hypersurfaces adjoint to $S^\flat$, i.e., the value that $h^0(\mathbf{P}^3,\mathcal{O}_{\mathbf{P}^3}(n-4)\otimes\mathcal{I}_D)$ would assume if the linear series cut out on D by $|\mathcal{O}_{\mathbf{P}^3}(n-4)|$ were complete and non-special.[5] Thus, in the classical definition, the postulation formula (5.9.1) is taken as the *definition* of the arithmetic genus.

(5.17) Equivalence of the definitions: proof of the postulation formula, II. We shall now prove that the definition of the arithmetic genus given in (5.12) is equivalent to the classical definition given above. The matter is to prove the postulation formula, which we have already

[5] or "under the postulation that the linear series cut out on D by $|\mathcal{O}_{\mathbf{P}^3}(n-4)|$ is complete and non-special", hence the name "postulation formula".

done; we shall give another proof, based on the considerations of (5.16). The exact sequence (5.16.1) gives

$$\begin{aligned}\chi\big(\mathcal{O}_{\mathbf{P}^3}(n-4)\otimes\mathcal{I}_D\big) &= \chi\big(\mathcal{O}_{\mathbf{P}^3}(n-4)\big) - \chi\big(\mathcal{O}_D(n-4)\big)\\ &= \tbinom{n-3}{3} - 1 + \rho + 2t + (n-4)d,\end{aligned}$$

regardless of any assumption on the linear series cut out on D by $|\mathcal{O}_{\mathbf{P}^3}(n-4)|$. Now, I claim that

$$\chi\big(\mathcal{O}_{\mathbf{P}^3}(n-4)\otimes\mathcal{I}_D\big) = \chi(\mathcal{O}_S) - 1,$$

which will prove the equivalence of the classical and cohomological definitions of the arithmetic genus.

To prove the claim, consider the exact sequence

$$0 \to \omega_{\mathbf{P}^3} \cong \mathcal{O}_{\mathbf{P}^3}(n-4)\otimes\mathcal{I}_{S^\flat} \to \mathcal{O}_{\mathbf{P}^3}(n-4)\otimes\mathcal{I}_D \to \mathcal{O}_{S^\flat}(n-4)\otimes\mathcal{I}_D \to 0,$$

with $\mathcal{I}_{S^\flat}$ the sheaf of ideals defining $S^\flat$ in $\mathbf{P}^3$. It implies

$$\chi\big(\mathcal{O}_{\mathbf{P}^3}(n-4)\otimes\mathcal{I}_D\big) = -1 + \chi\big(\mathcal{O}_S(n-4)\otimes\mathcal{I}_D\big).$$

Now, one has $\chi(S^\flat, \mathcal{O}_{S^\flat}(n-4)\otimes\mathcal{I}_D) = \chi(S,\omega_S)$ by Lemma (5.15), and moreover $\chi(S,\omega_S) = \chi(S,\mathcal{O}_S)$ by Serre duality: this proves the claim. □

5.3 – The 2-dimensional case: multiple planes

In this section we consider a linear system $|V|$ on a smooth surface S, of dimension $r = 1$ or 2, and we assume that $|V|$ is a general linear subsystem of a 3-dimensional linear system $|V^\sharp|$ satisfying the conditions of (5.3). In particular, this is the case if $|V|$ is a general linear subsystem of a very ample linear system.

We shall apply the results obtained in Section 5.1 to $|V^\sharp|$. We will use the notation introduced there, by taking $S^\flat \subseteq \mathbf{P}^3$ to be the surface defined by $|V^\sharp|$.

(5.18) One-dimensional systems: proof of the Noether formula. Let $r = 1$. We have computed the class of $|V|$ in (5.2). On the other hand, the discriminant of $|V|$ is merely the section of that of $|V^\sharp|$ by the line $|V|$, hence the class of $|V|$ equals that of $|V^\sharp|$, which is given in (5.10). Equating these two expressions for $\check{n}$, one finds

$$c_2(S) + n + 4\pi - 4 = n + 4\pi + 12p_a(S) - K_S^2 + 8,$$

which is equivalent to the Noether formula, stated in (5.11.1). □

(5.19) We now let $r = 2$ for the rest of this section. Then $|V|$ defines an $n:1$ covering $S \to \mathbf{P}^2$ with branch curve $B \subseteq \mathbf{P}^2$, which may be incarnated as the projection of $S^\flat \subseteq \mathbf{P}^3$ from a general point p. We consider:

— N the degree of the branch curve $B \subseteq \mathbf{P}^2$;
— P the geometric genus of B ;
— κ the number of points $q \in S$ such that there exist $C, C' \in |V|$ intersecting with multiplicity $\geqslant 3$ at q ;
— ι the number of cuspidal members of $|V|$;
— δ the number of unordered pairs $q, q' \in S$ such that there exist $C, C' \in |V|$ tangent at both q and q' ;
— β the number of 2-nodal members of $|V|$.

The dual of B is the discriminant of $|V|$, so the class of B is $\check{n}$, which assumes the same value as in (5.2) and (5.10), see (5.18). Moreover, δ, κ, β, ι are respectively the numbers of nodes, cusps, bitangents, flexes of B. We take once again for granted that B and its dual have no other singularities by our generality assumption.

(5.20) Characters of the apparent boundary of $S^\flat$. The branch curve B is the image of the apparent boundary $\mathcal{B}^p S^\flat$ by the map $S^\flat \to \mathbf{P}^2$, and in fact $\mathcal{B}^p S^\flat$ is the normalization of B. This implies that N and P are respectively the degree and genus of $\mathcal{B}^p S^\flat$, and since we have computed that its class in S is $K_S + 3C$ one finds:
— $N = (K_S + 3C) \cdot C = 2n + 2\pi - 2$;
— $P = 9\pi + K_S^2 - 8$.

(5.21) Conclusion. Since we know in addition the class $\check{n}$ of B, we may derive all remaining numerical characters by applying the Plücker formulae of Section 1.3 to B. One finds:
— $\delta = 2\big[(n+\pi)^2 - 5n - 17\pi - 2K_S^2 + 6p_a(S) + 22\big]$;
— $\kappa = 3(n + 6\pi + K_S^2 - 4p_a(S) - 10)$;
— $\beta = \frac{1}{2}\big[(n + 4\pi - K_S^2 + 12p_a(S) - 1)^2 + 15n - 6\pi - 17K_S^2 + 132p_a(S) + 57\big]$;
— $\iota = 24(\pi + p_a(S))$.

(5.22) Number of cuspidal curves in a net. It is interesting to derive the quantity ι directly by projective methods. We end this section by sketching how this goes.

The parabolic curve of $S^\flat$ and the apparent boundary $\mathcal{B}^p S^\flat$ intersect at the pinch points, and at those points q on the parabolic curve such that the tangent plane at q contains p; the quantity ι is the number of points of the latter kind.

To find the intersection number of these two curves we compute the class of the pull-back of the parabolic curve to S. The degree of the Hessian is $4(n-2)$, and a local computation shows that its intersection with $S^\flat$ contains the double curve D with multiplicity 4. Thus, arguing as in (5.6), one finds that the pull-back of the parabolic curve has class $4K_S + 8C$. The class of the pull-back of $\mathcal{B}^p S^\flat$ has been computed in (5.6).

On the other hand, we claim that the two curves intersect with multiplicity 2 at each pinch point. Indeed, if q is a pinch point with preimage $q' \in S$, then the members of $|V|$ passing through q' form a pencil of curves which are all singular at q'. The condition to be cuspidal in such a pencil is that the determinant of the 2×2 matrix of second derivatives vanishes, which is a degree 2 condition. We thus find 2 cuspidal curves at q' in $|V|$, hence the claim.

Finally, we have

$$(K_S + 3C) \cdot (4K_S + 8C) = \iota + 2\tau, \tag{5.22.1}$$

which indeed gives $\iota = 24(\pi + p_a)$ with the expression for τ in (5.10).

A – Polynomial calculus

We give here a brief recap on polarity, together with a few facts about the Hessian, so that the reader unfamiliar with this may conveniently consult the relevant material. We refer the reader to [13, Chapter 1] and [5, § 5.4 and 5.6] for more details; see also [16].

A.1 – Polarity

(A.1) Notation and plea for the notation. Let $\mathbf{k}$ be a field of characteristic zero, and let $F \in \mathbf{k}[X_0, \ldots, X_n]$ be a homogeneous degree d polynomial in $n+1$ indeterminates.

For all $\mathbf{x} = (x_0, \ldots, x_n)$ and $\mathbf{y} = (y_0, \ldots, y_n)$ in $\mathbf{k}^{n+1}$, we write $\mathrm{D}_{\mathbf{x}}F$ for the differential of F at the point $\mathbf{x}$ (which is a linear functional on $\mathbf{k}^{n+1}$), and $\mathrm{D}_{\mathbf{x}}F(\mathbf{y})$ for the value of the latter at the vector $\mathbf{y}$, so that

$$\mathrm{D}_{\mathbf{x}}F(\mathbf{y}) = y_0\,\partial_0 F(\mathbf{x}) + \cdots y_n\,\partial_n F(\mathbf{x}), \tag{A.1.1}$$

where $\partial_i F$ is the polynomial $\partial F/\partial X_i$ for all $i = 0, \ldots, n$.

The quantity (A.1.1) is a function in the two variables $\mathbf{x}$ and $\mathbf{y}$, and somehow a main point in polarity theory is to consider it as a function of the variable $\mathbf{x}$ depending on the parameter $\mathbf{y}$. With this in mind we introduce the following alternative notation:

$$\big(\mathrm{D}^{\mathbf{y}}F\big)(\mathbf{x}) = \big(\mathrm{D}_{\mathbf{x}}F\big)(\mathbf{y}) = y_0\,\partial_0 F(\mathbf{x}) + \cdots y_n\,\partial_n F(\mathbf{x}). \tag{A.1.2}$$

Although it may be disturbing at first, as I have witnessed on various occasions, we shall see that it is both meaningful and useful. One could be tempted at this point to introduce $\mathrm{D}^{\mathbf{y}}_{\mathbf{x}}F$ as well, but let me refrain from doing this.

(A.2) The polar pairing. In the notation of (A.1) above, for all $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbf{k}^{n+1}$,

$$\mathrm{D}^{\mathbf{x}}F = x_0\,\partial_0 F + \cdots x_n\,\partial_n F \in \mathbf{k}[X_0, \ldots, X_n] \tag{A.2.1}$$

is a homogeneous polynomial of degree $d-1$ (if nonzero), which depends linearly on $\mathbf{x}$ (note that the X_i are indeterminates whereas the x_i are scalars). Considering this polynomial as $(\mathrm{D}^{\mathbf{x}})(F)$, where $\mathrm{D}^{\mathbf{x}}$ denotes the differential operator

$$\mathrm{D}^{\mathbf{x}} = x_0\partial_0 + \cdots x_n\partial_n,$$

for all $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbf{k}^{n+1}$, we set

$$\mathrm{D}^{\mathbf{x}_1 \cdots \mathbf{x}_k} = \mathrm{D}^{\mathbf{x}_1} \circ \cdots \circ \mathrm{D}^{\mathbf{x}_k}. \tag{A.2.2}$$

Extending (A.2.2) by $\mathbf{k}$-linearity, one obtains a perfect bilinear pairing

$$\begin{aligned} \mathrm{D} : \mathrm{Sym}^k(\mathbf{k}^{n+1}) \otimes \mathrm{Sym}^d(\check{\mathbf{k}}^{n+1}) &\longrightarrow \mathrm{Sym}^{d-k}(\check{\mathbf{k}}^{n+1}) \\ \textstyle\sum_{i,j} \Xi_i \otimes F_j &\longmapsto \textstyle\sum_{i,j} \mathrm{D}^{\Xi_i} F_j \end{aligned}$$

extending the natural pairing between $\mathbf{k}^{n+1}$ and its dual $\check{\mathbf{k}}^{n+1}$.

(A.3) The Taylor–Newton formula. In the formalism of polarity it has the following pleasant incarnation. Let $\mathbf{x} = (x_0, \ldots, x_n)$ and $\mathbf{y} = (y_0, \ldots, y_n)$ be in $\mathbf{k}^{n+1}$. As above, F is a degree d polynomial. One has:

$$\begin{aligned} F(\mathbf{x}+\mathbf{y}) &= \sum_{k=0}^{d} \sum_{k_0+\cdots+k_n=k} \frac{1}{k_0!\cdots k_n!}\, y_0^{k_0} \cdots y_n^{k_n} \left(\frac{\partial^{k_0}}{\partial X_0^{k_0}} \cdots \frac{\partial^{k_n}}{\partial X_n^{k_n}} F \right)(\mathbf{x}) \\ &= \sum_{k=0}^{d} \frac{1}{k!} \left(\left(y_0 \frac{\partial}{\partial X_0} + \cdots + y_n \frac{\partial}{\partial X_n} \right)^k F \right)(\mathbf{x}) \\ &= \sum_{k=0}^{d} \frac{1}{k!}\, \mathrm{D}^{\mathbf{y}^k} F(\mathbf{x}). \end{aligned}$$

(A.4) Polar hypersurfaces. Consider the hypersurface $Z = V(F) \subseteq \mathbf{P}^n$. For $\hat{a} = (a_0, \ldots, a_n)$ and $\mathbf{x} = (x_0, \ldots, x_n)$ in $\mathbf{k}^{n+1}$, and $k \in \mathbf{N}$,

$$\mathrm{D}^{\hat{a}^k} F(\mathbf{x}) = \big((a_0\partial_0 + \cdots + a_n\partial_n)^k F\big)(\mathbf{x})$$

is a bihomogeneous polynomial of bidegree $(k, d-k)$ in the variables $(a_0, \ldots, a_n)$ and $(x_0, \ldots, x_n)$. The hypersurface $V(\mathrm{D}^{\hat{a}^k} F)$ depends only on Z and the point $a = (a_0 : \ldots : a_n) \in \mathbf{P}^n$, see [5, Thm. 5.4.2]: we call it the *k-th polar of Z with respect to a*, and denote it by $\mathrm{D}^{a^k} Z$. It is a degree $d-k$ hypersurface. We will often abuse notation and omit the distinction between $\hat{a}$ and a. In addition, we will call $V(\mathrm{D}^{\hat{a}^k} F)$ a hypersurface even when $\mathrm{D}^{\hat{a}^k} F = 0$, and thus $V(\mathrm{D}^{\hat{a}^k} F) = \mathbf{P}^n$.

We shall also use the following notion: the *polar k-ic of Z at a*, denoted by $\mathrm{D}_{a^k} Z$, is the degree k hypersurface defined by the polynomial

$$\mathrm{D}^{\mathbf{x}^k} F(\hat{a}) = \big((x_0\partial_0 + \cdots + x_n\partial_n)^k F\big)(\hat{a})$$

in the variables $(x_0, \ldots, x_n)$.[6] In other words, we interchange the roles of $\hat{a}$ and $\mathbf{x}$ when we pass from $\mathrm{D}^{a^k} Z$ to $\mathrm{D}_{a^k} Z$; thus, by definition, for all $a, b \in \mathbf{P}^n$,

$$b \in \mathrm{D}_{a^k} Z \quad \Longleftrightarrow \quad a \in \mathrm{D}^{b^k} Z. \tag{A.4.1}$$

(A.5) Proposition (polar symmetry)**.** *Consider a degree d hypersurface $Z \subseteq \mathbf{P}^n$. Let $a, b \in \mathbf{P}^n$, and $k \in [\![1, d-1]\!]$. One has the equivalence:*

$$a \in \mathrm{D}^{b^k} Z \iff b \in \mathrm{D}^{a^{d-k}} Z.$$

In particular this says that the polar k-ic at a (i.e., $\mathrm{D}_{a^k} Z$) coincides with the $d-k$-th polar with respect to a (i.e., $\mathrm{D}^{a^{d-k}} Z$).

Proof. Let F be a homogeneous polynomial defining the hypersurface Z. Consider two sets of $n+1$ indeterminates $\mathbf{X} = (X_0, \ldots, X_n)$ and $\mathbf{Y} = (Y_0, \ldots, Y_n)$. By the Taylor–Newton Formula we have

$$F(\mathbf{X} + \mathbf{Y}) = \sum_{k=0}^{d} \frac{1}{k!} \big(\mathrm{D}^{\mathbf{Y}^k} F\big)(\mathbf{X}) \quad = \quad F(\mathbf{Y} + \mathbf{X}) = \sum_{k=0}^{d} \frac{1}{k!} \big(\mathrm{D}^{\mathbf{X}^k} F\big)(\mathbf{Y}).$$

Identifying the degree k pieces in the indeterminates $\mathbf{X}$, we find

$$\frac{1}{(d-k)!} \mathrm{D}^{\mathbf{Y}^{d-k}} F(\mathbf{X}) \;=\; \frac{1}{k!} \mathrm{D}^{\mathbf{X}^k} F(\mathbf{Y}),$$

which gives the result at once. □

(A.6) Proposition. *Let Z be a degree d hypersurface in $\mathbf{P}^n$. For all $a \in \mathbf{P}^n$ and $k = 1, \ldots, d-1$, one has:*

$$a \in Z \iff a \in \mathrm{D}^{a^k} Z.$$

Proof. Let F be an equation of Z. One has $\mathrm{D}^{a^k} F(a) = d(d-1)\cdots(d-k+1)F(a)$ by the Euler formula. □

[6] to avoid possible confusion, let me take the risk of being pedantic: consider F as a polynomial in the indeterminates $Y_0, \ldots, Y_n$, and let $X_0, \ldots, X_n$ be a second set of indeterminates; then, specializing the polynomial $\big(X_0 \frac{\partial}{\partial Y_0} + \cdots + X_n \frac{\partial}{\partial Y_n}\big)^k F(Y_0, \ldots, Y_n)$ by specializing Y_i to a_i for all i, one obtains a polynomial in the indeterminates $X_0, \ldots, X_n$: this is the defining equation of $\mathrm{D}_{a^k} Z$.

(A.7) If $b \in Z = V(F)$ is a smooth point, then $\big(\mathrm{D}_b F\big)(\mathbf{X}) = \big(\mathrm{D}^{\mathbf{X}} F\big)(b)$ is a linear homogeneous polynomial defining the tangent hyperplane to Z at b, hence polar hyperplane $D_b Z$ is the tangent hyperplane to Z at b, and moreover, for all $a \in \mathbf{P}^n$,

$$Z_{\mathrm{sm}} \cap \mathrm{D}^a Z = \{b \in Z_{\mathrm{sm}} : \mathbf{T}_b Z \ni a\},$$

where Z_{sm} denotes the smooth locus of Z. This generalizes to the following fundamental property.

For all hypersurfaces $Z \subseteq \mathbf{P}^n$ and lines $\ell \subseteq \mathbf{P}^n$, and for all $p \in \ell$, we let $i(Z, \ell)_p$ be the multiplicity with which p appears in $Z \cap \ell$.

(A.8) Theorem. *Let Z be a degree d hypersurface, $a \in Z$, and $b \in \mathbf{P}^n$. For all integer $s \geqslant 0$, one has*

$$\begin{aligned} i\big(Z, \langle a, b\rangle\big)_a \geqslant s+1 \quad &\Longleftrightarrow \quad \forall k = 1, \ldots, s, \quad b \in \mathrm{D}_{a^k} Z \\ &\Longleftrightarrow \quad \forall k = 1, \ldots, s, \quad a \in \mathrm{D}^{b^k} Z. \end{aligned}$$

Proof. Let F be a homogeneous equation for Z, and let $\hat{a}, \hat{b} \in \mathbf{k}^{n+1}$. Consider an indeterminate T. We have

$$F(\hat{a} + T\hat{b}) = \sum_{k=0}^{d} \frac{1}{k!} \big(\mathrm{D}^{(T\hat{b})^k} F\big)(\hat{a}) = \sum_{k=0}^{d} \frac{1}{k!} \big(\mathrm{D}^{\hat{b}^k} F\big)(\hat{a}) \cdot T^k$$

by the Taylor–Newton formula. This gives the result since, by definition, $i\big(Z, \langle a, b\rangle\big)_a \geqslant s+1$ if and only if T^{s+1} divides $F(\hat{a} + T\hat{b})$ (note that $\mathrm{D}^{\hat{b}^0} F(\hat{a}) = F(\hat{a}) = 0$ by hypothesis). □

It turns out that for $a \in Z$, all polars of Z with respect to a (equivalently, all polar hypersurfaces $\mathrm{D}_{a^k} Z$ at a) are tangent at a. Actually, Z and its polar k-ic at a, $\mathrm{D}_{a^k} Z$ have the same polar s-ics at a for all $s \leqslant k$, as the following identities show. Let $d = \deg(Z)$. Using polar symmetry repeatedly, one finds

$$\mathrm{D}_{a^s}\big(\mathrm{D}_{a^k} Z\big) = \mathrm{D}^{a^{k-s}}\big(\mathrm{D}_{a^k} Z\big) = \mathrm{D}^{a^{k-s}}\big(\mathrm{D}^{a^{d-k}} Z\big) = \mathrm{D}^{a^{d-s}} Z = \mathrm{D}_{a^s} Z.$$

This has the following remarkable consequence.

(A.9) Corollary. *Let Z be a hypersurface in $\mathbf{P}^n$, and a be a point of Z. The n hypersurfaces $\mathrm{D}_a Z, \ldots, \mathrm{D}_{a^n} Z$ intersect with multiplicity at least $n!$ at a (with the convention that the multiplicity is infinite if the intersection is positive dimensional at a).*

Proof. We consider only the case in which $\mathrm{D}_a Z, \ldots, \mathrm{D}_{a^n} Z$ define a local complete intersection at a, otherwise the result is trivial. Then the intersection $\mathrm{D}_a Z, \ldots, \mathrm{D}_{a^{n-1}} Z$ consists of $(n-1)!$ lines each intersecting Z with multiplicity at least n at a, as follows from Theorem (A.8). Since the polar k-ics at a of $\mathrm{D}_{a^n} Z$ are the same as those of Z, as indicated above, each of these lines also intersect $\mathrm{D}_{a^n} Z$ with multiplicity at least n at a, and the result follows. □

The polar hyperplane $\mathrm{D}_a Z$ is indeed a hyperplane only if a is a smooth point of Z, and then it is properly the tangent hyperplane to Z at a, otherwise $\mathrm{D}_a Z$ is the whole $\mathbf{P}^n$. The following result generalizes this: by (A.10.1), if Z has multiplicity exactly m at a, then $\mathrm{D}_{a^m} Z$ coincides with the tangent cone $\mathrm{TC}_a(Z)$.

(A.10) Theorem. *Let Z be a degree d hypersurface, and a be a point of Z of multiplicity m. Consider an integer $r \leqslant d$.*

(A.10.1) The polar r-ic at a, $\mathrm{D}_{a^r}Z$, is the whole $\mathbf{P}^n$ if and only if $r < m$. If $r \geqslant m$, then $\mathrm{D}_{a^r}Z$ has multiplicity m at a, and it has the same tangent cone as Z at a: $\mathrm{TC}_a(\mathrm{D}_{a^r}Z) = \mathrm{TC}_a(Z)$.
(A.10.2) Let $b \in \mathbf{P}^n - \{a\}$. If $r < m$, the r-th polar with respect to b, $\mathrm{D}^{b^r}Z$, has multiplicity at least $m - r$ at a; the multiplicity is exactly $m - r$ if b is general, and in this case the tangent cone of $\mathrm{D}^{b^r}Z$ at a equals the r-th polar with respect to b of the tangent cone $\mathrm{TC}_a(Z)$: $\mathrm{TC}_a\big(\mathrm{D}^{b^r}Z\big) = \mathrm{D}^{b^r}\big(\mathrm{TC}_a(Z)\big)$.

Note that (A.10.2) tells us that $\mathrm{D}^{b^r}Z$ contains a if $r \leqslant m-1$, and is singular at a if $r \leqslant m-2$.

Proof. Let us assume, without loss of generality, that $a = (1\!:\!0\!:\!\ldots\!:\!0)$. Then

$$F = X_0^{d-m}F_m(X_1,\ldots,X_n) + \cdots + F_d(X_1,\ldots,X_n)$$

where each F_k is homogeneous of degree k, and F_m is non-zero. One has $\mathrm{D}_{a^r}Z = \mathrm{D}^{a^{d-r}}Z$ by polar symmetry, and $\mathrm{D}^{a^{d-r}} = (\partial/\partial X_0)^{d-r}$, hence $\mathrm{D}_{a^r}Z$ is defined by the equation

$$\big(\tfrac{\partial}{\partial X_0}\big)^{d-r}F = \tfrac{(d-m)!}{(r-m)!}X_0^{r-m}F_m + \cdots + \tfrac{(d-r)!}{0!}F_r;$$

it is zero if and only if $r < m$, and if $r \geqslant m$ we see that $\mathrm{D}_{a^r}Z$ has multiplicity m at a with tangent cone defined by F_m, which proves (A.10.1).

Now let $b = (b_0, b_1, \ldots, b_n)$ be such that the corresponding point in $\mathbf{P}^n$ is distinct from a. We set $b^\natural = (0, b_1, \ldots, b_n)$ (it is non-zero because b and a are distinct in $\mathbf{P}^n$), and $b^r = (b^\natural)^r + \Xi$. Then $\mathrm{D}^{b^r} = \mathrm{D}^{(b^\natural)^r} + \mathrm{D}^{\Xi}$, where $\mathrm{D}^{(b^\natural)^r}$ is a polynomial in $\partial/\partial X_1, \ldots, \partial/\partial X_n$ only, while D^{Ξ} is a polynomial in $\partial/\partial X_0, \partial/\partial X_1, \ldots, \partial/\partial X_n$ in which all terms involve $\partial/\partial X_0$ with a positive exponent. Therefore, the piece of $\mathrm{D}^{b^r}F$ with the highest degree in X_0 is

$$\mathrm{D}^{(b^\natural)^r}(X_0^{d-m}F_m) = X_0^{d-m}\,\mathrm{D}^{(b^\natural)^r}(F_m) = X_0^{d-m}\,\mathrm{D}^{b^r}(F_m).$$

It follows that $\mathrm{D}^{b^r}Z$ has multiplicity at least $m-r$ at a; this multiplicity is exactly $m-r$ if and only if $\mathrm{D}^{b^r}(F_m)$ is non-zero, and in this case $\mathrm{D}^{b^r}(F_m)$ defines the tangent cone of $\mathrm{D}^{b^r}Z$ at a, so that the latter equals $\mathrm{D}^{b^r}(\mathrm{TC}_a(Z))$. Now $\mathrm{D}^{b^r}(F_m) = \mathrm{D}^{(b^\natural)^r}(F_m)$ is non-zero for general b, and (A.10.2) is proven. □

A.2 – Hessian and second fundamental form

Let Z be a hypersurface in $\mathbf{P}^n$, defined by the homogeneous polynomial $F \in \mathbf{k}[X_0, \ldots, X_n]$.

(A.11) Hessian of a hypersurface. The *Hessian hypersurface* of Z is the hypersurface $\mathrm{Hess}(Z)$ defined by the homogeneous polynomial

$$\mathrm{Hess}(F) = \det\big(\partial_i\partial_j F\big)_{0\leqslant i,j\leqslant n}.$$

It has degree $(n+1)(n-2)$ (if non-zero).

Using the Euler formula and the standard properties of determinants, one may obtain the alternative expression

$$\mathrm{Hess}(F) = \tfrac{(d-1)^2}{X_0^2}\begin{vmatrix} \frac{d}{d-1}\cdot F & \partial_1 F & \cdots & \partial_n F \\ \partial_1 F & \partial_1^2 F & \cdots & \partial_1\partial_n F \\ \vdots & \vdots & \ddots & \vdots \\ \partial_n F & \partial_1\partial_n F & \cdots & \partial_n^2 F \end{vmatrix}, \tag{A.11.1}$$

which is useful to study the intersection of Z and $\mathrm{Hess}(Z)$.

(A.12) Second fundamental form. Let p be a smooth point of Z. The *second fundamental form* of Z at p, denoted by $I\!I_p$, is a quadric in the tangent space T_pZ which is defined as follows. Analytically locally around p, and in an affine chart in which we identify p with $(0,\dots,0)$ and T_pZ with $\mathbf{A}^{n-1}\times\{0\}$, Z is the image of a map

$$\varphi : (u_1,\dots,u_{n-1}) \mapsto \big(u_1,\dots,u_{n-1},\varphi(u_1,\dots,u_{n-1})\big)$$

defined on a neighbourhood of the origin in the tangent space T_pZ, such that the order 0 and order 1 terms in the Taylor expansion of φ vanish. The second fundamental form is the order 2 term of the Taylor expansion of φ modulo multiplication by a non-zero scalar; it does not depend on the choice of the parametrization.

(A.13) Local forms. Assume F is in the form

$$X_0^{d-1}X_n + X_0^{d-2}\Big(\sum_{1\leqslant i,j\leqslant n} a_{ij}X_iX_j\Big) + O(3),$$

so that $p=(1:0:\ldots:0)\in Z$, and $\mathbf{T}_pZ=\{X_n=0\}$. We assume that the matrix $(a_{ij})_{1\leqslant i,j\leqslant n}$ is symmetric. By computing the first terms of the Taylor expansion of

$$F\big(1,u_1,\dots,u_{n-1},\varphi(u_1,\dots,u_{n-1})\big),$$

one sees that the second fundamental form of Z at p is the quadratic form on $T_pZ\simeq\mathbf{C}^{n-1}$ defined by the symmetric matrix $(a_{ij})_{1\leqslant i,j\leqslant n-1}$, modulo scalar multiplication.

Now let us show the relation between the determinant of this matrix and the Hessian determinant. We will assume $n=3$ in order to have smaller matrices, but it is straightforward to do the same computation for arbitrary n. By direct computation, one finds that the Hessian determinant of F at $p=(1:0:0:0)$ is

$$\begin{vmatrix} 0 & 0 & 0 & d-1\\ 0 & 2a_{11} & 2a_{12} & 2a_{13}\\ 0 & 2a_{12} & 2a_{22} & 2a_{23}\\ d-1 & 2a_{13} & 2a_{23} & 2a_{33}\end{vmatrix} = \begin{vmatrix} 0 & d-1\\ d-1 & 2a_{33}\end{vmatrix}\cdot\begin{vmatrix} 2a_{11} & 2a_{12}\\ 2a_{12} & 2a_{22}\end{vmatrix}.$$

(A.14) Corollary. *The second fundamental form at p is non-degenerate if and only if the Hessian determinant is non-zero.*

(A.15) Second fundamental form and polar quadric. Still in the notation of (A.13) above, the polar quadric of Z at p, $\mathrm{D}_{p^{d-2}}Z$, is defined by the equation

$$\mathrm{D}^{p^{d-2}}F = \big(\tfrac{\partial}{\partial X_0}\big)^{d-2}F = \tfrac{(d-1)!}{1!}X_0X_n + \tfrac{(d-2)!}{0!}\Big(\sum_{1\leqslant i,j\leqslant n} a_{ij}X_iX_j\Big).$$

We thus see that the intersection of the polar quadric $\mathrm{D}_{p^{d-2}}Z$ with the projective tangent hyperplane $\mathbf{T}_pZ=V(X_n)$ is the projective completion of the second fundamental form $I\!I_p$.

Note that $I\!I_p$ is an affine quadric in T_pZ which is defined by a homogenous degree 2 polynomial, hence it is singular at the origin, i.e., at p. On the other hand, we have seen in Theorem (A.10) that $\mathbf{T}_pZ=\mathbf{T}_p(\mathrm{D}_{p^{d-2}}Z)$, which explains why $\mathrm{D}_{p^{d-2}}Z\cap\mathbf{T}_pZ$ is singular at p.

B – Developable surfaces

In this appendix we review the basic theory of developable surfaces in the projective space of dimension 3, following the beautiful [14, Chap. V].

B.1 – Synthetic description

(B.1) Definition. A *developable surface* $S \subseteq \mathbf{P}^3$ is a ruled surface such that the tangent plane to S is constant along the lines of the ruling. In other words, for all lines Λ of the ruling, and for all $p, p' \in \Lambda$, one has $\mathbf{T}_p S = \mathbf{T}_{p'} S$. An important aspect of developable surfaces is that they are the surfaces that can be unfolded into the plane with no distortion; we will however not consider this idea any farther in this text.

In classical terminology, if S is a ruled surface, the lines of the rulings are called *generators*, and a generator Λ is said to be *torsal* if the tangent plane to S is constant along Λ. Thus developable surfaces are those ruled surfaces for which all generators are torsal.

(B.2) Developable surfaces in $\mathbf{P}^3$ are exactly those surfaces in $\mathbf{P}^3$ that have their dual which is not a hypersurface, see [49, Thm. 1.18]. It therefore follows from biduality that any non-degenerate developable surface is projective dual to a curve C in $\check{\mathbf{P}}^3$. In particular it has an ordinary double curve, corresponding to bitangent planes to C, and a cuspidal double curve, corresponding to osculating planes to C (unless C is degenerate, in which case S is a cone).

This description may be refined as follows.

(B.3) The following fact refines the description of (B.2); for a proof, see [17, Section 0.5], or [5, 5.8.14]. Also [23, §30] is very enlightening. Let S be a developable surface in $\mathbf{P}^3$. On every generator there is a singular point at which S does not have a well-defined tangent plane (beware that there are other singular points as well, see Proposition (B.14)). If S is not a cone, the locus of the singular points on the various generators is a curve Γ, called the *regression edge*, which is none other than the cuspidal double curve of $C^\vee$ mentioned in (B.2). At a general point $p \in \Gamma$, the tangent line $\mathbf{T}_p\Gamma$ is the generator of S through p, and the osculating plane $\mathrm{Osc}_p\Gamma$ is the plane tangent to S along the generator $\mathbf{T}_p\Gamma$.

This tells us that S is the *tangential surface* of Γ, namely

$$S = \mathrm{Tan}(\Gamma) := \overline{\bigcup\nolimits_{p \in \Gamma_{\mathrm{sm}}} \mathbf{T}_p\Gamma}.$$

Moreover, the curve $C = S^\vee \subseteq \check{\mathbf{P}}^3$ is what I call the *oscdual*[7] curve of $\Gamma \subseteq \mathbf{P}^3$, namely

$$C = \overline{\bigcup\nolimits_{p \in \Gamma^\circ} (\mathrm{Osc}_p\Gamma)^\perp},$$

where Γ° denotes the open subset of Γ of points p such that $\mathrm{Osc}_p\Gamma$ is a plane, i.e., Γ and $\mathbf{T}_p\Gamma$ intersect with multiplicity exactly 2 at p.

The dual of Γ is a developable surface in $\check{\mathbf{P}}^3$, the generators of which are the lines $\mathbf{T}_p\Gamma^\perp$ for all smooth points $p \in \Gamma$. It follows from biduality and the above description that this is the tangential surface of the curve C. In particular, Γ is the oscdual curve of C: this is a reflexivity statement for oscduality.

(B.4) While we shall not prove here that a developable surface in $\mathbf{P}^3$ is either a cone or the tangential surface $\mathrm{Tan}(\Gamma)$ of a curve, it is easy and instructive to see why tangential surfaces are developable. Assume Γ is parametrized locally by a map $\phi : u \mapsto \big(\alpha(u), \beta(u), \gamma(u)\big)$ in some affine chart of $\mathbf{P}^3$. Then $\mathrm{Tan}(\Gamma)$ is parametrized locally by

$$\Phi : (u, s) \longmapsto \begin{pmatrix} \alpha(u) \\ \beta(u) \\ \gamma(u) \end{pmatrix} + s \begin{pmatrix} \alpha'(u) \\ \beta'(u) \\ \gamma'(u) \end{pmatrix},$$

[7] Piene has introduced, more generally, the concept of m-dual varieties in [40]; in her notation, the oscdual of Γ is the 2-dual variety $\Gamma_2^\vee$. See also Pohl's [41].

and the differential of this parametrization at the point (u_0, s_0) is determined by

$$\frac{\partial}{\partial u} \longmapsto \begin{pmatrix} \alpha'(u_0) \\ \beta'(u_0) \\ \gamma'(u_0) \end{pmatrix} + s_0 \begin{pmatrix} \alpha''(u_0) \\ \beta''(u_0) \\ \gamma''(u_0) \end{pmatrix} \qquad \text{and} \qquad \frac{\partial}{\partial s} \longmapsto \begin{pmatrix} \alpha'(u_0) \\ \beta'(u_0) \\ \gamma'(u_0) \end{pmatrix}.$$

We thus see that, if Γ doesn't have an inflectionary behaviour at the point $\phi(u_0)$, the differential of Φ fails to be of maximal rank if and only if $s_0 = 0$, and otherwise its image is constant as s_0 moves, equal to the plane spanned by $\phi'(u_0)$ and $\phi''(u_0)$. The upshot is that the tangent plane to $\mathrm{Tan}(\Gamma)$ is constant along its generator $\mathbf{T}_{\phi(u_0)}\Gamma$, equal to the osculating plane $\mathrm{Osc}_{\phi(u_0)}\Gamma$, except at the very point $\phi(u_0)$ where it is not well-defined, and thus Γ is a cuspidal curve of $\mathrm{Tan}(\Gamma)$.

(B.5) Finally, although we shall not use it, note that developable surfaces are characterized by the property that they are contained in the zero locus of their associated Hessian determinant. This is an instance of a result valid for hypersurfaces of any dimension, which has been proved by B. Segre [47]; see also [44, Lemma 7.2.7] for a proof attributed to Kollár by the author.

B.2 – Analysis of contacts

We carry on our analysis of developable surfaces, building upon the description given in the previous section, and following the general philosophy of biduality. Recall that $S = \mathrm{Tan}(\Gamma) \subseteq \mathbf{P}^3$ is the tangential surface of the curve $\Gamma \subseteq \mathbf{P}^3$, and $C = S^\vee \subseteq \mathbf{P}^3$ is the dual curve of the developable surface S, as well as the oscdual of Γ. We leave aside for the moment the peculiar behaviour that the curve $\Gamma \subseteq \mathbf{P}^3$ may have at isolated points in some special situations.

(B.6) Characteristic numbers, I. We let:
— $m = \deg \Gamma$;
— $n = \deg C$;
— $r = \deg\big(\mathrm{Tan}(\Gamma)\big)$.

Classically, developable surfaces were considered as systems packaging together three 1-dimensional families in $\mathbf{P}^3$ of planes, points, and lines respectively; these are the planes parametrized by points of C, the points of Γ, and the tangent lines of Γ. In this perspective, m is the number of points of the system contained in a general plane of $\mathbf{P}^3$, n is the number of planes of the system passing through a general point of $\mathbf{P}^3$, and r is the number of lines of the system meeting a general line of $\mathbf{P}^3$. The integers m, n, r are called respectively the order, class, and rank of the system.

(B.7) Knowing the family of planes given by $C \subseteq \check{\mathbf{P}}^3$, one may find the generators of the developable surface $\mathrm{Tan}(\Gamma) \subseteq \mathbf{P}^3$ by taking the intersections in $\mathbf{P}^3$ of pairs of infinitely near members of C.

Indeed, the generators of $\mathrm{Tan}(\Gamma)$ are the orthogonals of the tangent lines of C. Now for $p, q \in C$, the orthogonal of the line $\langle p, q\rangle \subseteq \check{\mathbf{P}}^3$ is the intersection of the two planes $p^\perp$ and $q^\perp$ in $\mathbf{P}^3$. Letting q tend to p yields the asserted statement.

(B.8) Let o and H be a general point and a general plane in $\mathbf{P}^3$. The duality between projections and hyperplanes sections (see Proposition (1.5)) tells us that the projection of Γ from o is dual as a plane curve to the hyperplane section of $\Gamma^\vee = \mathrm{Tan}(C)$ by $o^\perp$ and, dually, that the section of $S = \mathrm{Tan}(\Gamma)$ by H is dual to the projection of C from the point $H^\perp$. This may be established using the following elementary statement, see Proposition (B.11).

(B.9) Lemma. *Let $o \in \mathbf{P}^3$ be a point off Γ, and consider the projection $\pi_o : \mathbf{P}^3 \dashrightarrow \mathbf{P}^3/o$. Let $p \in \Gamma$ be a smooth point such that the line $\langle o, p\rangle$ intersects Γ only at p.*
(i) *If the line $\langle o, p\rangle$ is tangent to Γ at p, then the projected curve $\pi_o(\Gamma)$ has a cusp at $\pi_o(p)$.*
(ii) *If the line $\langle o, p\rangle$ is not tangent to Γ at p, there is a unique plane H containing o and tangent to Γ at p, and its projection $\pi_o(H)$ is the tangent line to $\pi_o(\Gamma)$ at $\pi_o(p)$. If H osculates Γ to the order at least 3, then $\pi_o(H)$ is a flex tangent.*

This lemma is pretty clear from the differential point of view, given that the kernel of the differential of π_o at the point p is the tangent direction of the line $\langle o, p\rangle$.

Another interesting interpretation of this result is in terms of intersection multiplicities. Call $\Gamma' = \pi_o(\Gamma)$, $p' = \pi_o(p)$, and $H' = \pi_o(H)$, assuming that H is a hyperplane in $\mathbf{P}^3$ containing p, so that H' is a line in $\mathbf{P}^2$. Then, for local intersection multiplicities at p and p', the projection formula tells us that

$$(H \cdot \Gamma)_p = (H' \cdot \Gamma')_{p'}.$$

Thus, if H is an ordinary (resp. osculating) tangent plane of Γ at p, then H' is an ordinary (resp. flex) tangent of Γ' at p'. Moreover, if the line $\langle o, p\rangle$ is tangent to Γ at p, then for all planes H passing through o and p one has $(H \cdot \Gamma)_p > 1$, hence $(H' \cdot \Gamma')_{p'} > 1$ for all lines passing through p': this implies that Γ' is singular at p', and it has a cusp there because it is unibranch, as follows from our assumption that $\langle o, p\rangle$ intersects Γ only at p.

Finally, the lemma may also be proved by local computations. Let me now give a sample of these. Assume $p = (1 : 0 : 0 : 0)$ and Γ is parametrized locally at p by $t \mapsto (1 : t : t^2 : t^{3+k})$ for some non-negative integer k. Let $o = (a : b : c : d)$, and assume that $b \neq 0$ (the other possibilities may be considered analogously). Then the projection from o writes

$$(x : y : z : w) \mapsto (x - \tfrac{a}{b}y : z - \tfrac{c}{b}y : w - \tfrac{d}{b}y),$$

so that $\Gamma' = \pi_o(\Gamma)$ is locally parametrized at $p' = \pi_o(p)$ by

$$t \mapsto \Big(1 : \frac{t^2 - \frac{c}{b}t}{1 - \frac{a}{b}t} : \frac{t^{3+k} - \frac{d}{b}t}{1 - \frac{a}{b}t}\Big).$$

If $d = 0$, then o sits on the osculating plane of Γ at p. If moreover $c \neq 0$, then Γ' has a local parametrization at p' of the form $t \mapsto (1 : -\frac{c}{b}t + \sum_{m \geqslant 2} \alpha_m t^m : t^{3+k} + \sum_{m \geqslant 4+k} \beta_m t^m)$, hence it has a flex at p'. If $d = c = 0$, then o sits on the tangent line to Γ at p, and the local parametrization shows that Γ' has cusp at p': an ordinary cusp (i.e., an A_2 double point) if $k = 0$, a ramphoid cusp (i.e., an A_4 double point) if $k = 1$, and so on.

(B.10) If we project a smooth curve $\Gamma \subseteq \mathbf{P}^3$ from a general point o, its image is a plane nodal curve Γ', and the nodes correspond to the finitely many secant lines of Γ (i.e., lines of the form $\langle p, q\rangle$ with $p, q \in \Gamma$) passing through o.[8] The projection Γ' has moreover finitely many flexes (in addition to the original flexes of Γ), corresponding to the finitely many osculating hyperplanes of Γ passing through o (i.e., to the finitely intersection points of the oscdual curve $C = \mathrm{Tan}(\Gamma)^\vee$ with the hyperplane $o^\perp$).

Assume now that we project from a general point o of the tangential surface $\mathrm{Tan}(\Gamma)$. Then the projection Γ' has one cusp, corresponding to the unique generator of $\mathrm{Tan}(\Gamma)$ passing through o, and one less node than in the general situation, because the geometric genus of Γ' must always be the same as that of Γ (another interpretation of this is that one of the secant lines of Γ in the general situation has become a tangent line in the present situation). This implies in particular

[8] see [22, IV, Theorem 3.10] for a complete proof, and [20] for a vast generalization.

that the class of the plane curve Γ' in the present situation is one less than in the general situation, by the Plücker formula (P_1), see (1.12).

Moreover, the projection Γ' from a point $o \in \mathrm{Tan}(\Gamma)$ has two less flexes than the projection from a general point. This follows from the Plücker formula (P_2), arguing once again that a node has become a cusp. From a more geometric point of view, we have seen in (B.7) that the generators of $\mathrm{Tan}(\Gamma)$ are the intersections of two infinitely near osculating planes of Γ; these are the two osculating planes of Γ through o that no longer give flexes of Γ'. Note that these two infinitely near osculating planes of Γ correspond to the tangency point of the oscdual curve $C = \mathrm{Tan}(\Gamma)^\vee$ with the plane $o^\perp$.

(B.11) Proposition. *The section of* $\mathrm{Tan}(\Gamma)$ *by a plane* H *is the dual curve of the plane curve* $\pi_{H^\perp}(C)$.

Proof. For all $p \in \mathrm{Tan}(\Gamma) \cap H$, there is a point $q \in \Gamma$ such that $p = H \cap \mathbf{T}_q\Gamma$. By duality, the orthogonal line $(\mathbf{T}_q\Gamma)^\perp$ is the tangent to C at the point $(\mathrm{Osc}_q\Gamma)^\perp$. It follows that the plane $p^\perp$ is tangent to C at $(\mathrm{Osc}_q\Gamma)^\perp$, as it contains $(\mathbf{T}_q\Gamma)^\perp$; moreover, as it passes through the point $H^\perp$, its projection from $H^\perp$ is a tangent line of $\pi_{H^\perp}(C)$. Conversely, all tangent lines to $\pi_{H^\perp}(C)$ are obtained in this way. □

As an illustration of the proposition, note that the plane curve $H \cap \mathrm{Tan}(\Gamma)$ has cusps at the points where H meets the cuspidal edge of $\mathrm{Tan}(\Gamma)$, i.e., Γ itself, while the dual plane curve $\pi_{H^\perp}(C)$ correspondingly has flex points given by the osculating planes to C passing through $H^\perp$, i.e., the planes $q^\perp$ for $q \in H \cap \Gamma$.

(B.12) Proposition. *The two tangential surfaces* $\mathrm{Tan}(\Gamma)$ *and* $\mathrm{Tan}(C)$ *have the same degree.*

Proof. As we have seen in (B.6), the degree of $\mathrm{Tan}(\Gamma)$ is the number of lines $\mathbf{T}_p\Gamma$ meeting a given, general, line $\Lambda \subseteq \mathbf{P}^3$. Since the orthogonal $(\mathbf{T}_p\Gamma)^\perp$ is the tangent line to C at the point $(\mathrm{Osc}_p\Gamma)^\perp$, this is also the number of lines $\mathbf{T}_{p'}C$ meeting the line $\Lambda^\perp$, i.e., the degree of $\mathrm{Tan}(C)$. □

(B.12.1) Remark. Proposition (B.12) is a twisted version of the fact that the class of a (non-developable) ruled surface equals its degree.

To compare the two situations, let's recall the proof. Let $S \subseteq \mathbf{P}^3$ be a ruled surface, and assume $\check{S}$ is a surface. The class of S is the number of tangent planes of S containing a given, general, line $\Lambda \subseteq \mathbf{P}^3$. Since $\check{S}$ is a surface, the planes tangent to S are exactly the planes that contain a ruling of S. It follows that the class of S is the number of rulings of S that meet Λ. But the latter number also equals the degree of S.

If S is developable, it is no longer true that its tangent planes are those containing a ruling (indeed there is only one tangent plane per ruling), and the number of planes tangent to S containing a line Λ is zero, as $S^\vee$ actually has codimension 2 in $\check{\mathbf{P}}^3$.

It is useful to analyze the sections of the developable surface by planes containing a generator.

(B.13) Lemma. *Let the hyperplane* $H \subseteq \mathbf{P}^3$ *contain the line* $\mathbf{T}_p\Gamma$ *for some* $p \in \Gamma$*, and write* $H \cap \mathrm{Tan}(\Gamma)$ *as* $\mathbf{T}_p\Gamma + R$*. In general, the residual section* R *has one more flex and two less cusps than the general plane section of* $\mathrm{Tan}(\Gamma)$*. The extra flex point is* p*, and* $\mathbf{T}_pR = \mathbf{T}_p\Gamma$.

Proof. The point $H^\perp \in \check{\mathbf{P}}^3$ sits on the line $(\mathbf{T}_p\Gamma)^\perp$ which is tangent to C at the point $(\mathrm{Osc}_p\Gamma)^\perp$, so the curve $\pi_{H^\perp}(C)$ has one more cusp and two less flexes than the projection of C from a general point, as explained in Paragraph (B.10). Dually, this gives the first assertion about R. In addition, note that the class of $\pi_{H^\perp}(C)$ is one less than that of a general projection, agreeing with the fact that $\deg(R) = \deg(\mathrm{Tan}\,\Gamma) - 1$.

The tangent cone to $\pi_{H^\perp}(C)$ at its extra cusp is the projection from $H^\perp$ of the plane $\mathrm{Osc}_{(\mathrm{Osc}_p\Gamma)^\perp}C$; its orthogonal is the extra flex point of R, and it is the point p. Moreover, the tangent line to R at p is the orthogonal of the cuspidal point of $\pi_{H^\perp}(C)$; the latter is the projection from $H^\perp$ of the line $(\mathbf{T}_p\Gamma)^\perp$, hence $\mathbf{T}_pR$ equals $\mathbf{T}_p\Gamma$. □

We may now prove the following. Before we start, note that being $\mathrm{Tan}(\Gamma)$ ruled, its double points lie on two generators: if they are distinct then the double point is on the ordinary double curve, otherwise it is on the cuspidal edge. Given a generator $\mathbf{T}_p\Gamma$, p is the only point of $\mathbf{T}_p\Gamma$ lying on the cuspidal edge, unless $\mathbf{T}_p\Gamma$ is tangent to Γ at some other point.

(B.14) Proposition. *The degree of the ordinary double curve of* $\mathrm{Tan}(\Gamma)$ *equals* $r-4$.

Proof. Consider the section of $\mathrm{Tan}(\Gamma)$ by a plane H containing a generator $\mathbf{T}_p\Gamma$ as in Lemma (B.13): it consists of the line line $\mathbf{T}_p\Gamma$ and a residual curve R of degree $r-1$. They meet with multiplicity 3 at p, and at $r-4$ further points $p_1,\dots,p_{r-4}$, which are the intersection points of $\mathbf{T}_p\Gamma$ and the ordinary double curve of $\mathrm{Tan}(\Gamma)$. □

Note that, in the notation of the above proof, since $p_1,\dots,p_{r-4}$ are the intersection points of $\mathbf{T}_p\Gamma$ and the ordinary double curve, they are independent of the choice of H in the pencil of planes containing $\mathbf{T}_p\Gamma$.

For all $i=1,\dots,r-4$, there is a point $q_i\in\Gamma$ such that $p_i=\mathbf{T}_p\Gamma\cap\mathbf{T}_{q_i}\Gamma$. It follows that the plane $p_i^\perp\subseteq\check{\mathbf{P}}^3$ is tangent to C at the two points $(\mathrm{Osc}_p\Gamma)^\perp$ and $(\mathrm{Osc}_{q_i}\Gamma)^\perp$, which explains why it should be an ordinary double point of the surface $\mathrm{Tan}(\Gamma)=C^\vee$ according to the general principles of duality.

(B.15) Lemma. *Let* $p\in\Gamma$. *The section of* $\mathrm{Tan}(\Gamma)$ *by its tangent plane* $\mathrm{Osc}_p(\Gamma)$ *decomposes as* $2\mathbf{T}_p\Gamma+R$, *where the curve* R *is tangent to* $\mathbf{T}_p\Gamma$ *at* p, *and passes through all the ordinary double points* $p_1,\dots,p_{r-4}$ *of* $Tan(\Gamma)$ *lying on* $\mathbf{T}_p\Gamma$.

(B.16) Remark. The pencil of curves cut out on $\mathrm{Tan}(\Gamma)$ by the planes containing $\Lambda=\mathbf{T}_p\Gamma$ is a family of plane curves with prescribed contact $3p+p_1+\cdots+p_{r-4}$ with Λ, which degenerates to a curve containing Λ itself when the plane tends to the tangent plane $\mathrm{Osc}_p(\Gamma)$. This is exactly the situation considered by Caporaso and Harris in [8] and studied in the first half of this volume.

Proof of Lemma (B.15). Let $H=\mathrm{Osc}_p\Gamma$. The section $H\cap\mathrm{Tan}(\Gamma)$ must decompose as $2\mathbf{T}_x\Gamma+R$, with R the plane dual of the projected curve $\pi_{H^\perp}(C)$. This time the point $H^\perp\in\check{\mathbf{P}}^3$ sits on the curve C, and the projection $\pi_{H^\perp}(C)$ is smooth at the point image of $H^\perp$, namely $\pi_{H^\perp}(\mathbf{T}_{H^\perp}C)$, with tangent line $\pi_{H^\perp}(\mathrm{Osc}_{H^\perp}C)$ there. It follows that $p=(\mathrm{Osc}_{H^\perp}C)^\perp$ is on the plane dual R, with tangent $\mathbf{T}_pR=(\mathbf{T}_{H^\perp}C)^\perp=\mathbf{T}_p\Gamma$. This proves the first assertion.

For the second, letting once again $p_i=\mathbf{T}_p\Gamma\cap\mathbf{T}_{q_i}\Gamma$, we have seen that the plane $p_i^\perp$ is tangent to C at the two points $H^\perp$ and $(\mathrm{Osc}_{q_i}\Gamma)^\perp$. Thus $\pi_{H^\perp}(p_i^\perp)$ is the tangent line to $\pi_{H^\perp}(C)$ at $\pi_{H^\perp}((\mathrm{Osc}_{q_i}\Gamma)^\perp)$, hence $p_i\in R$ since the latter is the plane dual of $\pi_{H^\perp}(C)$. □

Finally, let us check by duality that the degree of R is indeed $r-2$ in the above context. Let g be the genus of C. The projection of C from a general point of $\check{\mathbf{P}}^3$ is a nodal degree n plane curve of genus g, which thus has $\delta=\frac{1}{2}(n-1)(n-2)-g$ nodes, hence class $r=n(n-1)-2\delta$. If however it gets projected from a general point on itself, it still gives a genus g nodal curve, but with degree $n-1$, so that the number of nodes becomes $\delta'=\frac{1}{2}(n-2)(n-3)-g=\delta-(n-2)$. The class of this curve is then $(n-1)(n-2)-2\delta'$, which is $r-2$ as required.

B.3 – Plücker formulae

We now explain how the Plücker formulae for plane curves may be used to derive relations between the numerical characters of a developable surface, following Cayley [9]. As for plane curves, it turns out to be appropriate to let the curves Γ and C have various kinds of singularities which are stable under oscduality. We thus consider a slightly more general situation than what we did in Subsection B.2, by allowing the stationary behaviours indicated below.

(B.17) Characteristic numbers, II. In addition to the quantities introduced in (B.6), we let
— α the number of ordinary cusps of C, i.e., points at which C has a local parametrization of the form $t \mapsto (t^2, t^3, t^4)$;
— β the number of ordinary cusps of Γ ;
— x the degree of the ordinary double curve of $\mathrm{Tan}(\Gamma)$;
— y the degree of the ordinary double curve of $\mathrm{Tan}(C)$;
— g the number of apparent double points of C;
— h the number of apparent double points of Γ.
The quantities α and β are classically referred to as the number of stationary planes and points of the system. The quantity x is the number of points in a general plane $H \subseteq \mathbf{P}^3$ that lie on two distinct tangent lines of Γ, see Proposition (B.14) and the comment before it; dually, y is the number of planes passing through a general point, and containing two distinct tangent lines of Γ. The quantity h is the number of lines meeting Γ in two points (i.e., secants of Γ) passing through a general point of $\mathbf{P}^3$; dually, g is the number of lines lying in a general plane $H \subseteq \mathbf{P}^3$ that are contained in two osculating planes of Γ.

(B.18) Lemma. *Consider the osculatory Gauss map $\Gamma \to C$, mapping a general point $p \in \Gamma$ to the point $(\mathrm{Osc}_p\Gamma)^\perp \in \check{\mathbf{P}}^3$. It sends a point where Γ has a local parametrization of the form $t \mapsto (t, t^2, t^4)$ (resp. $t \mapsto (t^2, t^3, t^4)$) to a point where C has a local parametrization of the form $t \mapsto (t^2, t^3, t^4)$ (resp. $t \mapsto (t, t^2, t^4)$).*

The upshot is that the osculatory Gauss map turns points of types α and β of Γ to points of types β and α of C, respectively.

Proof. This is a local computation: to a local branch of Γ parametrized by $t \in \mathbf{C} \mapsto f(t) \in \mathbf{C}^4$, the osculatory Gauss maps associates the local branch of C parametrized by $t \in \mathbf{C} \mapsto \big(f \wedge \partial_t f \wedge \partial_t^2 f\big)(t) \in \bigwedge^3 \mathbf{C}^4 \cong \check{\mathbf{C}}^4$. In the former case, the result follows from the identity

$$\begin{pmatrix} 1 \\ t \\ t^2 \\ t^4 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 2t \\ 4t^3 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 2 \\ 12t^2 \end{pmatrix} = \begin{pmatrix} -6t^4 & 16t^3 & -12t^2 & 2 \end{pmatrix}.$$

We leave the computation in the latter case to the reader. □

(B.19) Let us apply the Plücker formulae to a general hyperplane section $H \cap \mathrm{Tan}(\Gamma)$; as a plane curve, it is dual to $\pi_{H^\perp}(C)$. The former is a curve of degree r, with x nodes and m cusps, and the latter has degree n and g nodes; for general H, its only cusps come from cusps of C, which are in the number α. We thus get the following identities:

$$\begin{aligned} n &= r(r-1) - 2x - 3m; & r &= n(n-1) - 2g - 3\alpha; \\ \alpha &= 3r(r-2) - 6x - 8m; & m &= 3n(n-2) - 6g - 8\alpha, \end{aligned}$$

hence

$$\alpha - m = 3(n - r); \qquad 2(g - x) = (n - r)(n + r - 9).$$

(B.20) Dually, one may apply the Plücker formulae to the pair of dual plane curves $\pi_o(\Gamma)$ and $o^\perp \cap \mathrm{Tan}(C)$ for a general point $o \in \mathbf{P}^3$. This gives:

$$\begin{aligned} r &= m(m-1) - 2h - 3\beta; & m &= r(r-1) - 2y - 3n; \\ n &= 3m(m-2) - 6h - 8\beta; & \beta &= 3r(r-2) - 6y - 8n, \end{aligned}$$

hence

$$n - \beta = 3(r - m) \qquad 2(y - h) = (r - m)(r + m - 9).$$

(B.21) Eventually, one may combine the identities from (B.19) and (B.20) to obtain:

$$\begin{gathered} \alpha - \beta = 2(n - m), \qquad x - y = n - m, \\ 2(g - h) = (n - m)(n + m - 7). \end{gathered}$$

(B.22) It seems natural to allow the system to have stationary lines as well, i.e., the curve Γ to have points at which it is locally parametrized by $t \mapsto (t, t^3, t^4)$. I have chosen not to do it from the start, in order to give a more straightforward application of the Plücker formulae for plane curves. Let me now add two statements with which it is possible to adapt the above arguments in order take the possibility of stationary lines in consideration, which I will leave to the reader.

First, a local computation as in the proof of (B.18) shows that the osculatory Gauss map sends a point at which Γ has a stationary tangent line to a point at which C has a stationary tangent line. The second statement is the following.

(B.23) Lemma. *The stationary lines of the system give cuspidal lines to both tangential surfaces* $\mathrm{Tan}(\Gamma)$ *and* $\mathrm{Tan}(C)$.

Proof. Let $p \in \Gamma$ be a point at which Γ is locally parametrized by $t \mapsto (t, t^3, t^4)$, and $\Lambda = \mathbf{T}_p\Gamma$ be the corresponding (stationary) generator. We shall prove that for a general plane $H \subseteq \mathbf{P}^3$, the section $H \cap \mathrm{Tan}(\Gamma)$ has a cusp at the point $H \cap \Lambda$, or equivalently the curve $\pi_{H^\perp}(C)$ has a flex at $\pi_{H^\perp}\big((\mathrm{Osc}_p\Gamma)^\perp\big)$ along the line $\pi_{H^\perp}(\Lambda^\perp)$.

As we have seen, the curve C has a local parametrization of the form $t \mapsto (t, t^3, t^4)$ at the point $(\mathrm{Osc}_p\Gamma)^\perp$. Thus after projection from the general point $H^\perp$, the curve $\pi_{H^\perp}(C)$ has a local parametrization of the form $t \mapsto (t, t^3)$. The result follows since $\Lambda^\perp$ is the tangent line to C at $(\mathrm{Osc}_p\Gamma)^\perp$. □

B.4 – Generalized Plücker formulae

In this section I outline the generalization of the previous formulae to curves in a projective space of arbitrary dimension, which is due to Piene [36]. This is not used in the course of the text, but it would be a pity not to mention it. Note that there also exists a duality between the i-th ranks of an arbitrary variety in projective space and those of its dual, which is due to Urabe [50], see also [26], and [2], but this I will be reasonable and not talk about it here.

(B.24) So far we have only considered extrinsic invariants of the curve $\Gamma \subseteq \mathbf{P}^3$. Adding its geometric genus p_g in the picture, one finds:

$$r = 2m + 2p_g - 2 - \beta \tag{B.24.1}$$

(recall the notation from (B.6) and (B.17)).

Proof. Let $\Lambda \subseteq \mathbf{P}^3$ be a general line. We want to compute the number of tangent lines to Γ meeting Λ, or equivalently the number of planes tangent to Γ and containing Λ. To this end we consider the projection from Λ; it induces a degree m covering $\bar{\Gamma} \to \mathbf{P}^1$, where $\bar{\Gamma}$ is the normalization of Γ. By the Riemann–Hurwitz formula, it has $2m + 2p_g - 2$ ramification points, of which β come from the ramification points of the normalization map of Γ. The other ones correspond to planes tangent to Γ and containing Λ. □

(B.24.2) Remark. The same reasoning may be applied to the curve C; one gets in this way the identity

$$r = 2n + 2p_g - 2 - \alpha.$$

Indeed, the osculatory Gauss map gives a birationality between Γ and C, so these two curves have the same geometric genus, and r is the degree of both $\mathrm{Tan}(C)$ and $\mathrm{Tan}(\Gamma)$ by Proposition (B.12).

Putting the two identities together, one finds the formula $2(n-m) = \alpha - \beta$ which we have already gotten in (B.21).

(B.25) New setup. From now on, we consider a curve $\Gamma \subseteq \mathbf{P}^N$ of degree m and geometric genus p_g. Let $\bar{\Gamma} \to \Gamma$ be its normalization, and $f : \bar{\Gamma} \to \mathbf{P}^N$ its composition with $\Gamma \hookrightarrow \mathbf{P}^N$. For all $p \in \Gamma$ one may choose a local parameter t at p on $\bar{\Gamma}$ and a system of homogeneous coordinates $(x_0 : x_1 : \ldots : x_N)$ on $\mathbf{P}^N$ such that $f : \bar{\Gamma} \to \mathbf{P}^N$ is given locally by

$$t \mapsto \left(1 : t^{1+s_1} : \ldots : t^{N+s_1+\cdots+s_N}\right)$$

up to higher order terms (i.e., the i-th coordinate of f is a formal power series in t whose leading term is $t^{i+s_1+\cdots+s_i}$), and $s_1, \ldots, s_N$ are non-negative integers depending only on p. The point p is *ordinary* if $s_1 = \cdots = s_N = 0$, and *stationary* or *hyperosculating* otherwise.

For all $i = 0, \ldots, N-1$, the i-th osculating subspace of Γ at $p \in \bar{\Gamma}$, denoted $\mathrm{Osc}^i_p(\Gamma)$, is the $(N-i)$-plane in $\mathbf{P}^N$ defined by the equations $x_N = \cdots = x_{N-i+1} = 0$ in a system of homogeneous coordinates as above. We shall consider the i-th *osculating variety* of Γ, denoted by $\mathrm{Osc}^i(\Gamma)$, that is the union of the i-th osculating subspaces of Γ at all the points of $\bar{\Gamma}$. For $i = 0$ this is Γ itself, for $i = 1$ this is the tangential surface of Γ that we have considered in the previous subsections; for $i = N-1$ this fills the whole $\mathbf{P}^N$, hence it is best seen as a curve $C \subseteq \check{\mathbf{P}}^N$ which I call the *osc-dual* curve of Γ.

(B.26) The i-th ranks. For all $i = 0, \ldots, N-1$, let r_i be the degree of the i-th osculating variety of Γ; it is called the *i-th rank* of Γ. For $i = N-1$, this is by definition the degree of the osc-dual curve $C \subseteq \check{\mathbf{P}}^N$. We define integers k_i by

$$k_i := \sum\nolimits_{p \in \Gamma} s_i(p),$$

where $s_i(p)$ is defined as in (B.25).

A standard computation on the Chern classes of the bundles of principal parts of the embedding of $\bar{\Gamma}$ in $\mathbf{P}^N$, see [13, Prop. 10.4.13], gives the following formulae:

$$r_i = (i+1)\big(m + i(p_g - 1)\big) - \sum\nolimits_{j=1}^{i} \big((i-j+1)k_j\big) \qquad \text{for all } i = 0, \dots, N-1 \tag{B.26.1}$$

$$\sum\nolimits_{j=1}^{N} \big((N-j+1)k_j\big) = (N+1)\big(m + N(p_g - 1)\big). \tag{B.26.2}$$

Formula (B.26.1) for $i = 0$ is trivial, as $r_0 = \deg(\Gamma) = m$ by definition. Formula (B.26.2) has to be understood as the incarnation for $i = N$ of formula (B.26.1), setting $r_N = 0$. For $i = 1$, one gets a formula which is exactly (B.24.1). Eventually, note that formula (B.26.2) may be interpreted as counting the stationary points of Γ, with the multiplicity

$$s_N + 2s_{N-1} + \cdots + (N-1)s_2 + Ns_1.$$

(B.27) Generalized Plücker formulae. As a straightforward corollary of the formulas in (B.26), one has

$$r_{i-1} - 2r_i + r_{i+1} = 2p_g - 2 - k_{i+1}, \qquad \text{for all } i = 0, \dots, N-1, \tag{B.27.1}$$

with the convention that $r_N = r_{-1} = 0$.

(B.27.2) Plane curves. For $N = 2$, we are considering a plane curve Γ of degree m. The number k_1 (resp. k_2) counts, with multiplicities, the cusps (resp. the flexes) of Γ, so we will call it κ (resp. ι) as usual. Formula (B.27.1) gives the two identities

$$-2m + \check{m} = 2p_g - 2 - \kappa \; ; \qquad\qquad m - 2\check{m} = 2p_g - 2 - \iota.$$

The second formula is merely the first applied to C, which in this case is the plain dual of Γ. Assuming Γ has only δ nodes and κ (ordinary) cusps as singularities, these may be recovered from the usual formulae

$$\check{m} = m(m-1) - 2\delta - 3\kappa \; ; \qquad\qquad 2p_g - 2 = m(m-3) - 2\delta - 2\kappa$$

(the first one is the computation of the class using polars for a singular curve as in (1.12), and the second one is the adjunction formula for a (possibly) singular plane curve. One gets the first formula by writing

$$\check{m} - 2m = m(m-1) - 2\delta - 3\kappa - 2m = m(m-3) - 2\delta - 3\kappa = 2p_g - 2 - \kappa.$$

(B.27.3) Space curves. For $N = 3$ we are considering a space curve $\Gamma \subseteq \mathbf{P}^3$ as in the previous subsections. In the notation therein, the ranks r_0, r_1, r_2 are respectively the degree m, the rank r, and the class n. On the other hand, k_1, k_2, k_3 are respectively the number β of cusps of Γ, the number of stationary lines (let us call it γ), and the number α of cusps of C. Formula (B.27.1) gives the three identities

$$-2m + r = 2p_g - 2 - \beta \; ; \qquad m - 2r + n = 2p_g - 2 - \gamma \; ; \qquad r - 2n = 2p_g - 2 - \alpha.$$

(B.28) Piene duality [36, Thm. 5.1]**.** *Let $\Gamma \subseteq \mathbf{P}^N$ be a curve, and $C \subseteq \check{\mathbf{P}}^N$ its osc-dual curve. The osc-dual curve of C is Γ, and the following relations hold for all $i = 0, \dots, N-1$:*

$$r_i(C) = r_{N-i-1}(\Gamma) \tag{B.28.1}$$

$$k_i(C) = k_{N-i-1}(\Gamma). \tag{B.28.2}$$

Formula (B.28.1) generalizes the fact that for $N = 3$ the two tangential surfaces $\mathrm{Tan}(\Gamma)$ and $\mathrm{Tan}(C)$ have the same degree, see Proposition (B.12), and Formula (B.28.2) the fact that the osc-Gauss map turns stationary planes, lines and points of Γ into stationary points, lines, and planes of C respectively, see Lemma (B.18).

(B.29) Interpretation as strata of the dual hypersurface. Underlying the merely numerical relations of (B.28) is the following geometric duality, which generalizes (B.3).

For all $i > 0$, we say that a hyperplane $H \subseteq \mathbf{P}^N$ is *i-osculating* to Γ at $p \in \bar{\Gamma}$ if it contains $\mathrm{Osc}^i_p(\Gamma)$. If $p \in \bar{\Gamma}$ is an ordinary point of Γ, this is equivalent to the divisor $f^*H - (i+1)p$ on $\bar{\Gamma}$ being effective. We let $\Gamma^{\vee_i}$ be the closed subset parametrizing hyperplanes i-osculating to Γ at some point on $\bar{\Gamma}$; thus $\Gamma^{\vee_1}$ is the plain dual hypersurface $\Gamma^\vee$, and $\Gamma^{\vee_{N-1}}$ is the osc-dual curve C.

For all $i = 0, \dots, N-1$, one has $\Gamma^{\vee_i} = \mathrm{Osc}^{N-1-i}(C)$, and moreover $\mathrm{Osc}^i(\Gamma)^\vee = \Gamma^{\vee_{i+1}}$. In fact, for all $p \in \bar{\Gamma}$, the linear space $\mathrm{Osc}^i_p(\Gamma)^\perp$ is the $(N-1-i)$-th osculating space of C at p (note that the map $p \in \bar{\Gamma} \mapsto [\mathrm{Osc}^{N-1}_p(\Gamma)] \in \check{\mathbf{P}}^N$ is the normalization of C).

For all $i = 0, \dots, N-1$, let $f_i : \bar{\Gamma} \to \mathbf{G}(\mathbf{P}^N, i)$ and $f^{\vee_i} : \bar{\Gamma} \to \mathbf{G}(\check{\mathbf{P}}^N, i))$ be the maps $p \mapsto \mathrm{Osc}^i_p(\Gamma)$ and $p \mapsto \mathrm{Osc}^i_p(C)$ respectively. The map $f^{\vee_{N-1-i}}$ equals the composition of f_i with the orthogonality map $\Lambda \in \mathbf{G}(\mathbf{P}^N, i) \mapsto \Lambda^\perp \in \mathbf{G}(\check{\mathbf{P}}^N, N-1-i)$. It follows that the two curves $f_i(\bar{\Gamma})$ and $f^{\vee_{N-1-i}}(\bar{\Gamma})$ have the same degree under the Plücker embeddings of $\mathbf{G}(\mathbf{P}^N, i)$ and $\mathbf{G}(\check{\mathbf{P}}^N, N-1-i)$ respectively; these degrees are $r_i(\Gamma)$ and $r_{N-1-i}(C)$ respectively (see, e.g., [21, Example 19.11]), so we find Formula (B.28.1).

B.5 – De Jonquières' formula

The next result is even more general than the previous ones. See [4, Chap. VIII] for a proof and pointers to the literature.

(B.30) Theorem (de Jonquières formula)**.** *Let m be a positive integer, and $m_1, \dots, m_a$ be non–negative integers such that $m = \sum_{1 \leqslant s \leqslant a} s \cdot m_s$. The virtual number of divisors having m_s points of multiplicity s for all $s = 1, \dots, a$ in a given linear series of degree m and dimension $i = \sum_{1 \leqslant s \leqslant a} (s-1) \cdot m_s$ on a smooth, genus g curve is the coefficient of the $t_1^{m_1} \cdots t_a^{m_a}$ term in the formal power series*

$$\left(1 + 1^2 t_1 + 2^2 t_2 + \cdots + a^2 t_a\right)^g \cdot \left(1 + 1\, t_1 + 2 t_2 + \cdots + a t_a\right)^{m-i-g}.$$

(B.31) Link with the i-th ranks. We now show how the de Jonquières formula may be used to derive the i-th ranks of a curve $\Gamma \subseteq \mathbf{P}^N$ of degree m and geometric genus g as in (B.25), thus recovering formula (B.26.1).

By definition, $r_i(\Gamma) = r_{N-1-i}(C) = \deg(\mathrm{Osc}^i(\Gamma))$ is the number of hyperplanes i-osculating to Γ in a general i-dimensional linear subsystem of $|\mathcal{O}_\Gamma(1)|$. By de Jonquières' formula, the number of hyperplanes H such that $f^*H - (i+1)p$ is effective for some $p \in \bar{\Gamma}$ in a general

i-dimensional linear subsystem of $|\mathcal{O}_\Gamma(1)|$ is the coefficient of the monomial $t_1^{m-i-1}t_{i+1}$ in

$$\begin{aligned}\big(1+1^2t_1+(i+1)^2t_{i+1}\big)^g\cdot\big(1+1\,t_1+(i+1)t_{i+1}\big)^{m-i-g}&\\ =\big(1+1^2t_1+[(i+1)+i(i+1)]t_{i+1}\big)^g\cdot\big(1+1\,t_1+(i+1)t_{i+1}\big)^{m-i-g}&\\ =\sum_{j=0}^{g}\binom{g}{j}\big(1+t_1+(i+1)t_{i+1}\big)^{(g-j)+m-i-g}\cdot\big(i(i+1)t_{i+1}\big)^j,&\end{aligned}$$

which is also the coefficient of $t_1^{m-i-1}t_{i+1}$ in

$$\big(1+t_1+(i+1)t_{i+1}\big)^{m-i}+i(i+1)t_{i+1}\cdot g\big(1+t_1+(i+1)t_{i+1}\big)^{m-i-1},$$

that is

$$(m-i)(i+1)+i(i+1)\cdot g=(i+1)\big(m+(g-1)i\big). \tag{B.31.1}$$

To obtain r_i from the number (B.31.1), one ought to subtract the excess contributions of the various stationary points of Γ. For $i=1$ for instance, for all cusp p of Γ (i.e., for all $p\in\bar\Gamma$ such that $s_1(p)>0$), the whole $p^\perp\subseteq\check{\mathbf{P}}^N$ gives rise to hyperplanes contributing to the number (B.31.1), so that the latter actually is the degree of the reducible hypersurface sum of $\Gamma^{\vee_1}$ and the various $p^\perp$ for all cusps p of Γ. Taking this into consideration, one recovers Formula (B.24) for r_1.

For $i=2$, also the points $p\in\bar\Gamma$ such that $s_1(p)=0$ and $s_2(p)>0$ contribute to (B.31.1), to the effect that $(\mathrm{Osc}^1_p(\Gamma))^\perp$ occurs together with $\Gamma^{\vee_2}$. In general, all $p\in\bar\Gamma$ such that $s_j(p)>0$ for some $j\leqslant i$ contribute to (B.31.1), and to conclude we need to show that each contributes with the multiplicity

$$s_i+2\,s_{i-1}+\cdots+i\,s_1$$

in de Jonquières' formula. I have not found this particular computation in the literature, and will leave this as an open exercise to the reader.

C – Biduality for asymptotic tangent lines by enumerative means

Let Λ be an asymptotic tangent line to a surface $S\subseteq\mathbf{P}^3$ at a smooth point p. In this appendix, we show that the line $\Lambda^\perp$ is an asymptotic tangent line to the dual surface $S^\vee$ at the point $(\mathbf{T}_pS)^\perp$, by computing the intersection multiplicity of $\Lambda^\perp$ and $S^\vee$ at this point. We also prove several variants, see Theorem (C.2).

To compute this intersection multiplicity, we consider the pencil of hyperplane sections of S parametrized by $\Lambda^\perp$, and count its singular members with the topological formula involving Euler numbers described in [XI, Section 2.1]. The intersection multiplicity we are looking for is the multiplicity of the tangent section $\mathbf{T}_pS\cap S$ as a singular member of this pencil (indeed the singular members of $\Lambda^\perp$ correspond to the intersection points of $\Lambda^\perp$ with $S^\vee$). The upshot is the following lemma.

(C.1) Lemma. *Let $S\subseteq\mathbf{P}^3$ be a smooth surface, Λ be a line, and H be a hyperplane containing Λ. Let $\tilde S\to S$ be the minimal sequence of blow-ups at reduced points such that the proper transform $\tilde\Lambda$ of the pencil $\Lambda^\perp$ is base-point-free. Let $\tilde C_H$ and $\tilde C_{\mathrm{gen}}$ be the members of $\tilde\Lambda$ corresponding to $C_H=H\cap S$ and a general member C_{gen} of $\Lambda^\perp$, respectively. The intersection multiplicity of $\Lambda^\perp$ and $S^\vee$ at $H^\perp$ equals the difference in Euler numbers $e(\tilde C_H)-e(\tilde C_{\mathrm{gen}})$.*

The assumption that S is smooth is only to simplify matters: one may adapt the lemma to a surface with given singularities, using the same method. The lemma follows from a direct application of [XI, Lemma (2.3)] to the morphism $\tilde{S} \to \mathbf{P}^1$ given by $\tilde{\Lambda}$. The same idea appears in [35] and [3]. With this tool at hand, we may now prove the following biduality statement.

(C.2) Theorem. *Let $S \subseteq \mathbf{P}^3$ be a smooth surface, and $p \in S$. The orthogonal line to*
(a) a tangent at p,
(b) one of the two asymptotic tangents at a non-parabolic p,
(c) the asymptotic tangent at a general parabolic point p,
(d) the asymptotic tangent at a parabolic point p of type β,
respectively, is
(a) a tangent at $(\mathbf{T}_p S)^\perp$,
(b) one of the two asymptotic tangents at $(\mathbf{T}_p S)^\perp$,
(c) the tangent at $(\mathbf{T}_p S)^\perp$ of the cuspidal double curve of $S^\vee$,
(d) the common reduced tangent cone at $(\mathbf{T}_p S)^\perp$ of both the ordinary and cuspidal double curves of $S^\vee$.

Proof. We proceed with a case by case analysis, and apply Lemma (C.1) in all cases. We use similar notation (and abuse of notation) in all cases, which we introduce in the first case, and then use freely in the others.

(a) Let Λ be a general tangent line to S, and call p the contact point. Then all members of $\Lambda^\perp$ are tangent at p, and we have to blow-up twice at p to remove the base points: we let $\varepsilon_1 : S_1 \to S$ be the blow-up at p, with exceptional divisor E_1, and $\varepsilon_2 : S_2 \to S_1$ be the blow-up at the point $p_1 \in E_1$ corresponding to the tangent direction at $p \in S$ defined by Λ, with exceptional divisor E_2.

Locally over $p \in S$, the $\tilde{S} \to S$ of Lemma (C.1) is $\varepsilon = \varepsilon_2 \varepsilon_1 : S_2 \to S$. Let C be the hyperplane class on S. Forgetting about the base points of $\Lambda^\perp$ away from p, the base-point-free pencil $\tilde{\Lambda}$ of Lemma (C.1) is the complete linear system $|\varepsilon^* C - \varepsilon_2^* E_1 - E_2|$. Our task now is to compute the transform $\tilde{C}_0$ of the curve $C_0 = S \cap \mathbf{T}_p S$ in this pencil.

Since $C_0 \subseteq S$ has a double point at p, we have

$$\varepsilon_1^* C_0 = C_0 + 2E_1,$$

where by abuse of notation C_0 also denotes the proper transform of C_0 in S_1. Next, one has

$$\varepsilon^* C_0 = \varepsilon_2^*(C_0 + 2E_1) = C_0 + 2E_1 + 2E_2$$

with the same abuse of notation, this time for both C_0 and E_1. Finally we find

$$\tilde{C}_0 = \varepsilon^* C_0 - \varepsilon_2^* E_1 - E_2 = \varepsilon^* C_0 - (E_1 + E_2) - E_2 = C_0 + E_1.$$

This is a reduced curve with two ordinary double points, hence it counts with multiplicity 2 as a singular member of $\Lambda^\perp$ (see [XI, subsec. 2.1] for the details of the computation). It follows that $\Lambda^\perp$ is tangent to $S^\vee$ at $(\mathbf{T}_p S)^\perp$.

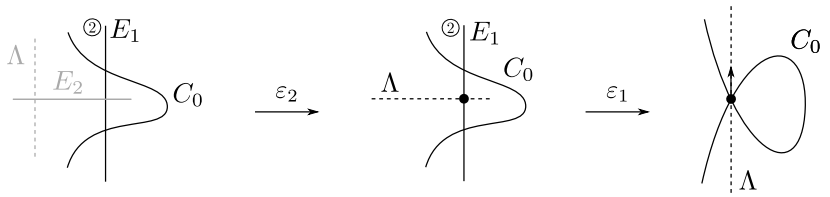


Figure 3: Pencil with base locus an ordinary tangent line Λ

(b) Let Λ be an asymptotic tangent at a general point p. Then the members of $\Lambda^\perp$ are all osculating to the order 3 at p (in fact, we know they all have a flex along Λ), and we need to blow-up three times to remove the base points, with exceptional divisors E_1, E_2, E_3. The base-point-free pencil $\tilde{\Lambda}$ is now

$$|\varepsilon^* C - \varepsilon_3^*\varepsilon_2^* E_1 - \varepsilon_3^* E_2 - E_3| = |\varepsilon^* C - E_1 - 2E_2 - 3E_3|.$$

The curve $C_0 = S \cap \mathbf{T}_p S$ has an ordinary double point at p, with one local branch simply tangent to Λ. We thus have successively

$$\begin{aligned}
\varepsilon_1^* C_0 &= C_0 + 2E_1 \\
\varepsilon_2^*\varepsilon_1^* C_0 &= C_0 + 2E_1 + 3E_2 \\
\varepsilon_3^*\varepsilon_2^*\varepsilon_1^* C_0 &= C_0 + 2E_1 + 3E_2 + 3E_3,
\end{aligned}$$

hence the transform $\tilde{C}_0$ of $C_0 \subseteq S$ in $\tilde{\Lambda}$ is

$$\tilde{C}_0 = \varepsilon^* C_0 - E_1 - 2E_2 - 3E_3 = C_0 + E_1 + E_2.$$

It is reduced with 3 nodes, hence counts with multiplicity 3 as a singular member of $\Lambda^\perp$. This shows that $\Lambda^\perp$ is an asymptotic tangent to $S^\vee$.

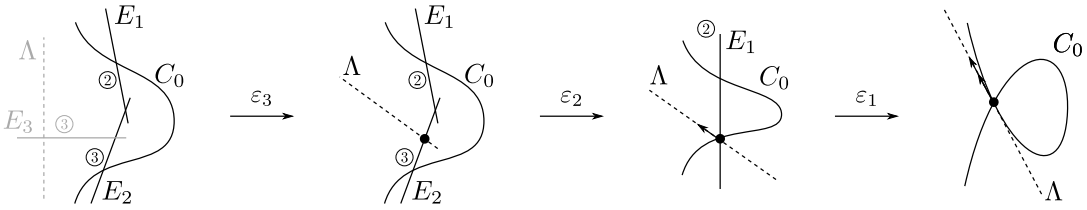


Figure 4: Pencil with base locus an asymptotic tangent line Λ

(c) Let Λ be an asymptotic tangent at a general parabolic point p. The situation is the same as in case (b), except that the tangent section $C_0 = S \cap \mathbf{T}_p S$ now has a cusp at p, with tangent cone supported on Λ. We have

$$\begin{aligned}
\varepsilon_1^* C_0 &= C_0 + 2E_1 \\
\varepsilon_2^*\varepsilon_1^* C_0 &= C_0 + 2E_1 + 3E_2 \\
\varepsilon_3^*\varepsilon_2^*\varepsilon_1^* C_0 &= C_0 + 2E_1 + 3E_2 + 3E_3,
\end{aligned}$$

and the transform of C_0 in the base-point-free pencil $\tilde{\Lambda}$ is

$$\tilde{C}_0 = \varepsilon^* C_0 - E_1 - 2E_2 - 3E_3 = C_0 + E_1 + E_2.$$

It is reduced with a triple point, hence it counts with multiplicity 4 by [XI, Lemma (2.5)].

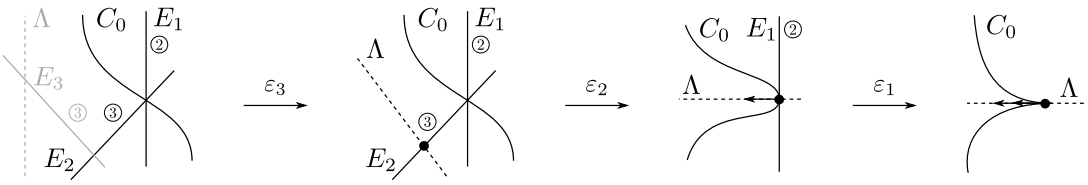


Figure 5: Pencil with base locus the double asymptotic tangent Λ at a parabolic point

The point $(\mathbf{T}_pS)^\perp$ lies on the cuspidal double curve of $S^\vee$. A general line through this point intersects $S^\vee$ with multiplicity 2, a general line in the tangent cone of $S^\vee$ with multiplicity 3, and the tangent line of the cuspidal double curve with multiplicity 4. Hence $\Lambda^\perp$ is the latter line.

(d) Lastly, let p be a point such that $C_0 = S \cap \mathbf{T}_pS$ has a tacnode at p, and Λ be the asymptotic tangent of S at p, which is also the support of the tangent cone of C_0 at p. The base-point-free pencil is defined as in cases (b) and (c). The computations now give

$$\begin{aligned}\varepsilon_1^*C_0 &= C_0 + 2E_1\\ \varepsilon_2^*\varepsilon_1^*C_0 &= C_0 + 2E_1 + 4E_2\\ \varepsilon_3^*\varepsilon_2^*\varepsilon_1^*C_0 &= C_0 + 2E_1 + 4E_2 + 4E_3\end{aligned}$$

so that

$$\tilde{C}_0 = \varepsilon^*C_0 - E_1 - 2E_2 - 3E_3 = C_0 + E_1 + 2E_2 + E_3.$$

This time the curve is not reduced. However this is not a reason for the topological formula behind Lemma (C.1) to fail, and the latter is still valid. The only caution is that the computation of $e(\tilde{C}_0)$ is no longer liable to [XI, Lemma (2.5)]; we compute it directly as the Euler number of $\tilde{C}_{0,\mathrm{red}}$, as indeed the topological formula doesn't see multiplicities, i.e., $e(\tilde{C}_0) = e(\tilde{C}_{0,\mathrm{red}})$.

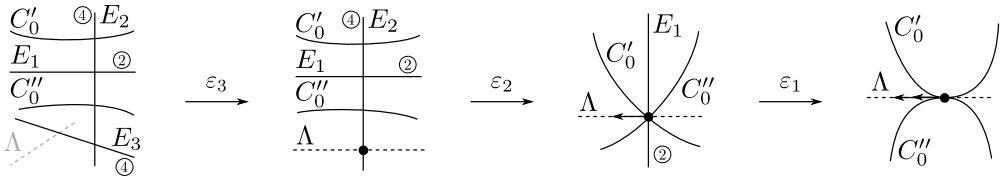


Figure 6: Pencil with base locus the double asymptotic tangent Λ at a swallowtail point

Since the curve $C_0 \subseteq S$ has a tacnode, its proper transform in $\tilde{S}$ has genus $g-2$, where g stands for the genus of a general member of $\Lambda^\perp$. It then follows from the additivity of the Euler number as in the proof of [XI, Lemma (2.5)], that

$$\begin{aligned}e(C_0 + E_1 + E_2 + E_3) &= e(C_0) + e(E_1) + e(E_2) + e(E_3) - 4\\ &= \big(2 - 2(g-2)\big) + 2 + 2 + 2 - 4 = e(C_{\mathrm{gen}}) + 6.\end{aligned}$$

One then argues as in case (c), remembering the local swallowtail model of $S^\vee$ at $(\mathbf{T}_pS)^\perp$ given in (3.1). The point $(\mathbf{T}_pS)^\perp$ is triple for $S^\vee$, and the only line through this point intersecting $S^\vee$ with multiplicity strictly larger than 4 is the common support of the respective tangent cones of both double curves of $S^\vee$ at $(\mathbf{T}_pS)^\perp$. □

(C.2.1) Remark. In the swallowtail model of (3.1), the support of the tangent cones of the double curves of $S^\vee$ is the line $b = c = 0$, which is downright contained in the discriminant hypersurface (3.1.2). Of course it is not the case in general that, in the notation of (d), the line $\Lambda^\perp$ is contained $S^\vee$.

References

Chapters in this Volume

[II] Th. Dedieu and E. Sernesi, *Deformations of curves on surfaces.*

[XI] Th. Dedieu, *Enumerative geometry of K3 surfaces.*

[C] L. Busé and Th. Dedieu, *Generalized weight properties of resultants and discriminants, and applications to projective enumerative geometry.*

[XIII] Th. Dedieu, *Node polynomials for curves on surfaces.*

External References

[1] V. Alexeev and P. Engel, *The flex divisor of a $K3$ surface*, Int. Math. Res. Not. **2023** (2023), no. 10, 8356–8370.

[2] P. Aluffi, *Projective duality and a Chern-Mather involution*, Trans. Amer. Math. Soc. **370** (2018), no. 3, 1803–1822.

[3] P. Aluffi and F. Cukierman, *Multiplicities of discriminants*, Manuscripta Math. **78** (1993), no. 3, 245–258.

[4] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften, vol. 267, Springer-Verlag, New York, 1985.

[5] M. Beltrametti, E. Carletti, D. Gallarati and G. Monti Bragadin, *Lectures on curves, surfaces and projective varieties – a classical view of algebraic geometry*, EMS Textbooks in Mathematics, European Mathematical Society, 2009, Original version (in italian) issued in 2003.

[6] J. W. Bruce, *The duals of generic hypersurfaces*, Math. Scand. **49** (1981), no. 1, 36–60.

[7] L. Busé, C. D'Andrea, M. Sombra and M. Weimann, *The geometry of the flex locus of a hypersurface*, Pacific J. Math. **304** (2020), no. 2, 419–437.

[8] L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392.

[9] A. Cayley, *Mémoire sur les courbes à double courbure et les surfaces développables*, Journal de mathématiques pures et appliquées 1re série **10** (1845), 245–250.

[10] C. Ciliberto and F. Flamini, *On the branch curve of a general projection of a surface to a plane*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3457–3471.

[11] P. Corvaja and F. Zucconi, *The surface of gauss double points*, Eur. J. Math. **8** (2022), no. 3, 871–892.

[12] A. Dimca, *Topics on real and complex singularities*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1987, An introduction.

[13] I. V. Dolgachev, *Classical algebraic geometry*, Cambridge University Press, Cambridge, 2012, A modern view.

[14] W. L. Edge, *The theory of ruled surfaces*, Cambridge University Press, 1931, `http://www.maths.ed.ac.uk/cheltsov/edge2013/read.html`.

[15] F. Enriques, *Le superficie algebriche*, N. Zanichelli, Bologna, 1949.

[16] G. Fischer, *Plane algebraic curves*, Student Mathematical Library, vol. 15, American Mathematical Society, Providence, RI, 2001, Translated from the 1994 German original by Leslie Kay.

[17] G. Fischer and J. Piontkowski, *Ruled varieties*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 2001, An introduction to algebraic differential geometry.

[18] A. G. Flores and B. Teissier, *Local polar varieties in the geometric study of singularities*, Ann. Fac. Sci. Toulouse Math. (6) **27** (2018), no. 4, 679–775.

[19] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008, Reprint of the 1994 edition.

[20] L. Gruson and C. Peskine, *On the smooth locus of aligned Hilbert schemes, the k-secant lemma and the general projection theorem*, Duke Math. J. **162** (2013), no. 3, 553–578.

[21] J. Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course.

[22] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[23] D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie.*, vol. 37, Springer, Berlin, 1932, engl. transl. available as "Geometry and the Imagination".

[24] B. Iversen, *Numerical invariants and multiple planes*, Amer. J. Math. **92** (1970), 968–996.

[25] N. M. Katz, *Pinceaux de Lefschetz: théorèmes d'existence*, SGA 7 II, Exposé XVII.

[26] S. L. Kleiman, *Concerning the dual variety*, in *18th Scandinavian Congress of Mathematicians (Aarhus, 1980)*, Progr. Math., vol. 11, Birkhäuser, Boston, MA, 1981, 386–396.

[27] J. Kollár, *Szemerédi-Trotter-type theorems in dimension 3*, Adv. Math. **271** (2015), 30–61.

[28] Val. S. Kulikov, *Calculus of singularities of imbedding of a general algebraic surface in the projective space* $\mathbf{P}^3$, Funktsional. Anal. i Prilozhen. **17** (1983), no. 3, 15–27.

[29] ———, *The number of singular focal centers of projection of an algebraic surface*, Uspekhi Mat. Nauk **39** (1984), no. 5(239), 245–246.

[30] Val. S. Kulikov and Vik. S. Kulikov, *Generic coverings of the plane with A-D-E-singularities*, Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), no. 6, 65–106.

[31] Vik. S. Kulikov, *A remark on classical Pluecker's formulae*, Ann. Fac. Sci. Toulouse Math. (6) **25** (2016), no. 5, 959–967.

[32] Vik. S. Kulikov and E. Shustin, *Duality of planar and spacial curves: new insight*, Eur. J. Math. **1** (2015), no. 3, 462–482.

[33] G. Laumon, *Degré de la variété duale d'une hypersurface à singularités isolées*, Bull. Soc. Math. France **104** (1976), no. 1, 51–63.

[34] C. McCrory and T. Shifrin, *Cusps of the projective Gauss map*, J. Differential Geom. **19** (1984), no. 1, 257–276.

[35] A. Parusiński, *Multiplicity of the dual variety*, Bull. London Math. Soc. **23** (1991), no. 5, 429–436.

[36] R. Piene, *Numerical characters of a curve in projective n-space*, in *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, 1977, 475–495.

[37] ———, *Polar classes of singular varieties*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 2, 247–276.

[38] ———, *Some formulas for a surface in* $\mathbf{P}^3$, in *Algebraic geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977)*, Lecture Notes in Math., vol. 687, Springer, Berlin, 1978, 196–235.

[39] ———, *A proof of Noether's formula for the arithmetic genus of an algebraic surface*, Compositio Math. **38** (1979), no. 1, 113–119.

[40] ———, *A note on higher order dual varieties, with an application to scrolls*, in *Singularities, Part 2 (Arcata, Calif., 1981)*, Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, 335–342.

[41] W. F. Pohl, *Differential geometry of higher order*, Topology **1** (1962), 169–211.

[42] F. Ronga, *On the number of planes triply tangent to a surface in projective space*, in *Proceedings of the 1984 Vancouver conference in algebraic geometry*, CMS Conf. Proc., vol. 6, Amer. Math. Soc., Providence, RI, 1986, 389–395.

[43] ———, *On the singularities of the surface reciprocal to a generic surface in projective space*, J. Differ. Geom. **29** (1989), no. 1, 105–112.

[44] F. Russo, *On the geometry of some special projective varieties*, Lecture Notes of the Unione Matematica Italiana, vol. 18, Springer, Cham; Unione Matematica Italiana, Bologna, 2016.

[45] G. Salmon, *A Treatise on the Analytic Geometry of Three Dimensions, Vol. II*, Chelsea Publishing Company, New York, 1965, reprint of the fifth (latest) edition.

[46] P. Samuel, *Géométrie projective*, Presses Universitaires de France, Paris, 1986, English translation available.

[47] B. Segre, *Bertini forms and Hessian matrices*, J. London Math. Soc. **26** (1951), 164–176.

[48] B. Teissier, *Résolution simultanée II*, in *Séminaire sur les singularités des surfaces*, Lect. Notes Math., vol. 777, Springer, 1980, 82–146.

[49] E. A. Tevelev, *Projective duality and homogeneous spaces*, Encyclopaedia of Mathematical Sciences, vol. 133, Springer-Verlag, Berlin, 2005, Invariant Theory and Algebraic Transformation Groups, IV.

[50] T. Urabe, *Duality of numerical characters of polar loci*, Publ. Res. Inst. Math. Sci. **17** (1981), 331–345.

[51] ———, *Generalized Pluecker formulae*, Publ. Res. Inst. Math. Sci. **17** (1981), 347–362.

[52] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours spécialisés, vol. 10, Société Mathématique de France, 2002, English translation available.

[53] J. Witaszek, *The geometry of smooth quartics*, MSc thesis Universität Bonn, 2014.

[54] Geng Xu, *Subvarieties of general hypersurfaces in projective space*, J. Differential Geom. **39** (1994), no. 1, 139–172.

[55] O. Zariski, *Algebraic surfaces*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, With appendices by S. S. Abhyankar, J. Lipman and D. Mumford, Preface to the appendices by Mumford, Reprint of the second (1971) edition.

Appendix C
Generalized weight properties of resultants and discriminants, and applications to projective enumerative geometry

by Laurent Busé and Thomas Dedieu

1 Reduced resultant 371
1.1 Inertia forms and the ordinary multivariate resultant 371
1.2 The reduced resultant 375
1.3 Generalized weight properties 377
2 Reduced discriminant and Salmon formula 379
2.1 Discriminant of a homogeneous polynomial 380
2.2 The reduced discriminant 382
2.3 Application to the Salmon Formula 391
3 Computation of the node-couple degree by elimination 393
3.1 An elimination trick 393
3.2 Study of a pencil of planes orthogonal to a tangent line 396
3.3 Generalization to hypersurfaces of arbitrary dimension 399
4 Number of bitangent lines and generalizations 402
4.1 Number of bitangent lines 402
4.2 A generalization in arbitrary dimension 403
4.3 A dual generalization in arbitrary dimension 404
5 The flecnodal polynomial 406
References 407

In his wonderful book [17], first published in 1862, Salmon casually gives the leading term of some Taylor expansion of the discriminant of a plane algebraic curve. Consider the curve $\mathrm{V}(F) \subseteq \mathbf{P}^2$; in a suitable system of homogeneous coordinates $(X:Y:Z)$, its equation F has the form

$$F(X,Y,Z) = TZ^{d-1}Y + \frac{1}{2}Z^{d-2}(AX^2 + 2BXY + CY^2) + \sum_{k\geqslant 3} Z^{d-k}F_k(X,Y) = 0,$$

where each polynomial $F_k(X,Y)$ is homogeneous of degree k in X, Y, and T, A, B, C are coefficients of F. Salmon states, without any explanation or reference, that the discriminant of F is of the form

(🔔) $$\mathrm{Disc}(F) = T^2A(B^2-AC)^2\,\Phi + T^3\,\Psi,$$

where Φ is "the discriminant when T vanishes" (and Ψ is some polynomial in the coefficients of F) [17, § 605]. This note arose as an attempt to understand and prove this formula.

We introduce the *reduced discriminant* of a hypersurface (or homogeneous polynomial), of which Φ is an instance, it turns out. It is deeply linked with the *reduced resultant*, introduced by Zariski in [20], in 1936 long after Salmon. Of course, when T is zero, $\mathrm{V}(F)$ is singular at the point $(0:0:1)$, no matter the other coefficients of F, and correspondingly $\mathrm{Disc}(F)$ vanishes identically. The polynomial Φ, in fact, vanishes at those values of the coefficients of F for which the curve $\mathrm{V}(F)$ has some additional singularity besides the point $(0:0:1)$. The reduced discriminant generalizes this to the case when F has a singular point of arbitrary multiplicity s. Somehow built in the theory is a general form of Formula (). We show how to get () specifically (in fact with a normalization factor $-\frac{1}{2}$, which is probably irrelevant for Salmon), and generalize the formula for hypersurfaces of arbitrary dimension, see Section 2.3. This is certainly what Salmon has in mind, although arguably only for $s = 2$, and possibly only empirically. It has since then been apparently completely forgotten; we have only found a faint trace of these ideas in Salmon's works, see Paragraph (2.19). We point out that the reduced discriminant is a particular instance of the theory of toric discriminants studied in [11], which is more general in terms of the family of hypersurfaces that can be considered. The methodology that we develop here in the specific setting of the reduced discriminant is different from the theory of toric discriminants, and we found it particularly well adapted, if not necessary, to tackle Salmon's formula.

Salmon then uses Formula () to derive various enumerative quantities for surfaces $S \subseteq \mathbf{P}^3$. In particular, he computes the number of bitangent planes passing through a fixed general point $p \in \mathbf{P}^3$. We explain his method, which involves a remarkable trick in elimination theory, and generalize it for hypersurfaces of arbitrary dimension. Salmon's strategy is to consider a pencil of planes with center a line tangent to the surface S at some point p'; this pencil contains a finite number of planes tangent to S, among which the tangent plane at p' counts with multiplicity 2 in general, and with greater multiplicity if it has some special feature, e.g., if it is a bitangent plane. In an appropriate setting, this multiplicity is the order in T of the polynomial in (), and the game is to understand the conditions on the point p' that make this multiplicity jump. In the upshot we get, indirectly from the reduced discriminant Φ, a curve in S parametrizing those points p' such that the tangent plane at p' is a bitangent plane; following Salmon, we call it the *node-couple curve*. There are other well-known ways to compute the number of bitangent planes to a surface in $\mathbf{P}^3$ (see, e.g., [XII]). The present method has the advantage of being core level: it is only a study of homogeneous polynomials and their discriminants.

The present text includes two other projective enumerative computations taken from Salmon's book; they both fit in the framework of what we would call *reduced elimination theory*. The first one gives the number of bitangent lines to a surface in $\mathbf{P}^3$ through a general point. We generalize it in two ways: to the enumeration of lines in $\mathbf{P}^n$ having contact of the form $2p + (n-1)q + \cdots$ with a hypersurface on the one hand, and to the enumeration of bitangent codimension 2 linear spaces on the other hand. While Salmon's computation and the first generalization do not involve openly the reduced discriminant (thanks to the existence of a cheap substitute in the two indeterminates case), it is essential for the latter generalization. The second computation is that of the degree of the *flecnodal curve* of a surface $S \subseteq \mathbf{P}^3$, which is the locus of points p at which the tangent plane cuts out a curve with one of its two branches at p that has an inflection point (equivalently, there is a line having contact order with S at p at least 4). We present here Salmon's ideas, noting that this computation is carried out in modern standards and in arbitrary dimension in [2].

We also give a synthetic account of the theory of resultants and discriminants, in its modern form worked out by Jouanolou, see, e.g., [14]. On the one hand this is necessary for us to develop the theory of the reduced discriminant (the reduced resultant had already been treated by Ould Mohamdi in [15], under the direction of Jouanolou; note, however, that passing from

the resultant to the discriminant is not straightforward, see, e.g., , the comments after Definition (2.10)). On the other hand, we believe this text is a good occasion to make Jouanolou's formalism accessible to the XXIst century classical algebraic geometer.

We put particular focus on the homogeneity properties of the resultants and discriminants, with respect to various gradings. They are indeed essential for the enumerative applications, and we give numerous illustrations, including a proof of Bezout's theorem and the computation of the degree of the dual of a smooth hypersurface. Beyond that, they are central in the theory of reduced resultants and discriminants, see Theorem (1.19) and Corollary (2.20), and indeed the key to Formula (🔔) and its generalizations.

ThD thanks Ragni Piene and Israel Vainsencher for their interest, and for suggesting he talk with LB; he also thanks François Lê for sharing some insights on Salmon's work. LB is grateful to Alexandru Dimca for useful discussions on Milnor number and for indicating the reference [10]. We both thank the referees for their helpful observations.

(0.1) Notation. Let $\mathbf{k}$ be a commutative ring, $n \geqslant 1$ be an integer, and $\mathbf{X} := (X_0, X_1, \ldots, X_n)$ be a sequence of indeterminates. Given a multi-index $\boldsymbol{\alpha} := (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, we denote by $\mathbf{X}^{\boldsymbol{\alpha}}$ the monomial $X_0^{\alpha_0} X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, and set $|\boldsymbol{\alpha}| = \sum_{i=0}^n \alpha_i$.

(0.2) Definition. *Given a grading* $\mathbf{k}[\mathbf{X}] = \bigoplus_{d \geqslant 0} \mathbf{k}[\mathbf{X}]_d$*, the* order *of a polynomial* $F \in \mathbf{k}[\mathbf{X}]$ *is the largest integer* e *such that* $F \in \bigoplus_{d \geqslant e} \mathbf{k}[\mathbf{X}]_d$*.*

We will usually consider gradings obtained by assigning a weight w_i *to each indeterminate* X_i*: the degree* d *homogeneous piece* $\mathbf{k}[\mathbf{X}]_d$ *is then generated by all monomials* $\mathbf{X}^{\boldsymbol{\alpha}}$ *such that* $\sum_{i=0}^n \alpha_i w_i = d$*.*

Given a subset $I \subseteq \{1, \ldots, n\}$*, the* order *of a polynomial* F in the indeterminates X_i, $i \in I$*, is the order of* F *with respect to the grading obtained by assigning to* X_i *the weight* 1 *if* $i \in I$*, and the weight* 0 *if* $i \notin I$*.*

Beware that $\mathbf{k}$ itself will often be a polynomial ring, and we will consider gradings and orders both on $\mathbf{k}[\mathbf{X}]$ and on $\mathbf{k}$ itself.

1 – Reduced resultant

Let $F_1, \ldots, F_{n+1}$ be $n+1$ homogeneous polynomials in the indeterminates $X_0, X_1, \ldots, X_n$. They define a collection of $n+1$ hypersurfaces in the projective space $\mathbf{P}^n$, and the intersection of these $n+1$ hypersurfaces is empty if $F_1, \ldots, F_{n+1}$ are general. In fact, the emptiness of this intersection is characterized by the non-vanishing of the resultant $\mathrm{Res}(F_1, \ldots, F_{n+1})$ of these polynomials (see §1.1). Thus, the resultant characterizes those collections of homogeneous polynomials that have a common non-trivial root. The purpose of the *reduced resultant* is similar: one still considers $n+1$ homogeneous polynomials $F_1, \ldots, F_{n+1}$, but this time one assumes that they have a common non-trivial root; then the reduced resultant will characterize those collections of polynomials having an additional common non-trivial root. It has been introduced by Zariski [20]; a more complete and modern treatment is given in [15].

In what follows, we use the resultant of multivariate homogeneous polynomials as developed in [14] (see also [6, 7, 11]). An introduction and a brief recap are given in Section 1.1 below.

1.1 – Inertia forms and the ordinary multivariate resultant

We follow the beautiful presentation in [19, Chapter XI] (beware that this Chapter on Elimination Theory has disappeared in later editions).

(1.1) Saturation of a homogeneous ideal. We recall the following for the convenience of the reader; see, e.g., [12, Lecture 5] or [13, Exercise II.5.10] for more details. The *saturation* of a homogeneous ideal $I \subseteq \mathbf{k}[\mathbf{X}]$ is the homogeneous ideal

$$\begin{aligned} \bar{I} &= \big\{F \in \mathbf{k}[\mathbf{X}] : \forall i = 0, \dots, n,\ \exists N_i \text{ such that } X_i^{N_i} F \in I\big\} \\ &= I : (X_0, \dots, X_n)^{\infty}. \end{aligned} \tag{1.1.1}$$

For sufficiently large m, the graded pieces I_m and $\bar{I}_m$ are equal. Moreover, for two homogeneous ideals I and J, the following three propositions are equivalent:
(i) $\bar{I} = \bar{J}$;
(ii) $I_m = J_m$ for sufficiently large m;
(iii) $I \cdot \mathbf{k}[\mathbf{X}, \frac{1}{X_i}] = J \cdot \mathbf{k}[\mathbf{X}, \frac{1}{X_i}]$ for all $i = 0, \dots, n$.
In other words, a subscheme $V \subseteq \mathbf{P}^n_{\mathbf{k}}$ is defined (scheme-theoretically) by a homogeneous ideal $I \subseteq \mathbf{k}[\mathbf{X}]$ if and only if the saturation $\bar{I}$ equals the homogeneous ideal I_V of V. For example, for all $F \in \mathbf{k}[\mathbf{X}]$, the two ideals (F) and $(X_0F, \dots, X_nF)$ define the same subcheme of $\mathbf{P}^n_{\mathbf{k}}$ (and we recommend that the reader check it for themselves).

In particular, if $\mathbf{k}$ is a field, a subscheme $V \subseteq \mathbf{P}^n_{\mathbf{k}}$ defined by a homogeneous ideal I is empty if and only if the degree 0 piece $\bar{I}_0$ is non-zero: this is the homogeneous Nullstellensatz! (see also [5]). On the other hand, the non-emptiness of V is equivalent to the existence of a point in $V(\mathbf{k}')$ for some finite field extension $\mathbf{k}'$ of $\mathbf{k}$.

When $\mathbf{k}$ is an arbitrary commutative integral domain,[1] the vanishing of $\bar{I}_0$ is equivalent to the scheme-theoretic image of the map $V \to \mathrm{Spec}(\mathbf{k})$ being equal to the whole $\mathrm{Spec}(\mathbf{k})$. Indeed, the subscheme of $\mathrm{Spec}(\mathbf{k})$ defined by $\bar{I}_0$ coincides as a set with the image of $V \to \mathrm{Spec}(\mathbf{k})$ — this is the proof that projective morphisms are closed —, and moreover $\bar{I}_0$ defines the scheme-theoretic image of $V \to \mathrm{Spec}(\mathbf{k})$ (see, e.g., [14, §1] for more details).

(1.2) Let $d_1, \dots, d_r$ be positive integers. For all $j = 1, \dots, r$, we consider the *generic* homogeneous degree d_j polynomial in the indeterminates $\mathbf{X} = (X_0, \dots, X_n)$

$$F_j := \sum_{|\boldsymbol{\alpha}| = d_j} U_{j,\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}.$$

"Generic" refers to the fact that the coefficients of the F_j's are indeterminates; indeed, F_j is an incarnation of the generic point, in the sense of schemes, of the affine space parametrizing homogeneous degree d_j polynomials.

We let $A_{\mathbf{Z}}$ be the algebra generated by the indeterminates coefficients of the F_j's, i.e., we set $A_{\mathbf{Z}} := \mathbf{Z}\big[U_{j,\boldsymbol{\alpha}}\big]_{j=1,\dots,r,\ |\boldsymbol{\alpha}|=d_j}$. Thus, $F_j \in A_{\mathbf{Z}}[\mathbf{X}]$ for all $j = 1, \dots, r$.

Let $\mathbf{k}$ be a commutative ring, and let $u_{j,\boldsymbol{\alpha}} \in \mathbf{k}$ for all $j = 1, \dots, r$ and $|\boldsymbol{\alpha}| = d_j$. For all $T \in A_{\mathbf{Z}}[\mathbf{X}]$, we denote by $T(u_{j,\boldsymbol{\alpha}}) \in \mathbf{k}[\mathbf{X}]$ the polynomial obtained by evaluating the indeterminates $U_{j,\boldsymbol{\alpha}}$ at $u_{j,\boldsymbol{\alpha}} \in \mathbf{k}$ for all $j = 1, \dots, r$ and $|\boldsymbol{\alpha}| = d_j$ in the polynomial T.

(1.3) Definition. *An* inertia form *for the polynomials $F_1, \dots, F_r$ is an element $T \in A_{\mathbf{Z}}[\mathbf{X}]$ such for all $i = 0, \dots, n$, there exists $N_i \in \mathbf{N}$ such that $X_i^{N_i} T \in (F_1, \dots, F_r)$.*

In other words, the inertia forms for $F_1, \dots, F_r$ are the elements of the saturation of the ideal $(F_1, \dots, F_r)$ in $A_{\mathbf{Z}}[\mathbf{X}]$. We denote by $\mathfrak{I}_{\mathbf{Z}}$ the ideal of all degree 0 inertia forms for $F_1, \dots, F_r$; thus, in the notation introduced in (1.1.1),

$$\mathfrak{I}_{\mathbf{Z}} = \big(I : (X_0, \dots, X_n)^{\infty}\big) \cap A_{\mathbf{Z}}.$$

Note that this is a homogeneous ideal with respect to the standard grading on $A_{\mathbf{Z}}$. In view of the above considerations, we have the following.

[1] this arguably unconventional notation is used ubiquitously by Jouanolou.

(1.4) Theorem. *Suppose that* $\mathbf{k}$ *is a field, and let* $u_{j,\boldsymbol{\alpha}} \in \mathbf{k}$ *for for all* $j = 1, \dots, r$ *and* $|\boldsymbol{\alpha}| = d_j$. *Consider the polynomials* $F_1(u_{1,\boldsymbol{\alpha}}), \dots, F_r(u_{r,\boldsymbol{\alpha}}) \in \mathbf{k}[\mathbf{X}]$. *The two following propositions are equivalent:*
(i) the ideal $\big(F_1(u_{1,\boldsymbol{\alpha}}), \dots, F_r(u_{r,\boldsymbol{\alpha}})\big)$ *defines a non-empty subscheme of* $\mathbf{P}^n_{\mathbf{k}}$*;*
(ii) for all $T \in \mathfrak{I}_{\mathbf{Z}}$, $T(u_{j,\boldsymbol{\alpha}}) = 0$.

This tells us that a given specialization to a field of the polynomials F_j defines a non-empty subscheme if and only if all the constants in the saturation of $(F_1, \dots, F_r)$ vanish in this specialization (see also [5]). Note that for all $T \in \mathfrak{I}_{\mathbf{Z}}$, one has $T(u_{j,\boldsymbol{\alpha}}) \in \mathbf{k}$, since T has degree 0 in $\mathbf{X}$.

(1.5) Example. Set $n = 1$, and let $F_1 = U_{1,0}X_0 + U_{1,1}X_1$ and $F_2 = U_{2,0}X_0 + U_{2,1}X_1$. Then $R = U_{1,0}U_{2,1} - U_{1,1}U_{2,0}$ is a degree 0 inertia form of F_1 and F_2, as indeed

$$X_0 R = U_{2,1}F_1 - U_{1,1}F_2 \quad \text{and} \quad X_1 R = U_{1,0}F_2 - U_{2,0}F_1.$$

We thus recover the usual determinant of two linear forms. We recommend that the reader come back to this example when we introduce various gradings on $A_{\mathbf{Z}}$ later on.

We emphasize that in general the subscheme of $\mathrm{Spec}(\mathbf{k})$ defined by the specialization of $\mathfrak{I}_{\mathbf{Z}}$ coincides only set-theoretically with the scheme-theoretic image of $V \to \mathrm{Spec}(\mathbf{k})$ (see [7, §3, Remarque 1] and [14, §1]), which is the reason why we assume that $\mathbf{k}$ is a field in Theorem (1.4). If $\mathbf{k}$ is an arbitrary commutative ring, what is indeed true is that the subscheme defined by the $F_j(u_{j,\alpha})$'s surjects onto $\mathrm{Spec}(\mathbf{k})$ as a set if and only if $\mathfrak{I}_{\mathbf{Z}} \otimes_{A_{\mathbf{Z}}} \mathbf{k}$ is contained in the nilradical $\sqrt{(0)}$ of $\mathbf{k}$, but this says nothing more than Theorem (1.4).

(1.6) Theorem (see [4, §2.1])**.** *The ideal of inertia forms for* $F_1, \dots, F_r$ *is prime, and so is the ideal* $\mathfrak{I}_{\mathbf{Z}} \subseteq A_{\mathbf{Z}}$ *of inertia forms of degree 0.*

The *resultant situation* is when we consider $n+1$ homogeneous equations in $\mathbf{P}^n$.

(1.7) Theorem. *If* $r = n+1$, *the ideal* $\mathfrak{I}_{\mathbf{Z}}$ *is principal. Up to sign it has a unique generator, which is an irreducible element of* $A_{\mathbf{Z}}$*; we denote it by* $\mathrm{Res}_{d_1,\dots,d_{n+1}} \in A_{\mathbf{Z}}$.

Moreover, for all $k \in [\![1, n+1]\!]$, $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ *is homogeneous of degree* $\prod_{j\neq k} d_j$ *with respect to the coefficients of the polynomial* F_k*, i.e., with respect to the indeterminates* $U_{k,\boldsymbol{\alpha}}$, $|\boldsymbol{\alpha}| = d_k$ *(all assumed to have weight one).*

Let $f_1, \dots, f_{n+1} \in \mathbf{k}[\mathbf{X}]$ be polynomials of respective degrees $d_1, \dots, d_{n+1}$. They are specializations $F_1(u_{1,\boldsymbol{\alpha}}), \dots, F_r(u_{r,\boldsymbol{\alpha}})$ of $F_1, \dots, F_{n+1} \in A_{\mathbf{Z}}[\mathbf{X}]$ respectively, for an appropriate canonical choice of $u_{j,\boldsymbol{\alpha}} \in \mathbf{k}$, for all $j = 1, \dots, n+1$ and $|\boldsymbol{\alpha}| = d_k$. We let $\mathrm{Res}(f_1, \dots, f_{n+1}) \in \mathbf{k}$ (or $\mathrm{Res}_{d_1,\dots,d_{n+1}}(f_1, \dots, f_{n+1}) \in \mathbf{k}$, if we want to emphasize the dependency on the degrees) be the corresponding specialization of $\mathrm{Res}_{d_1,\dots,d_{n+1}} \in A_{\mathbf{Z}}$. The multi-homogeneity property stated in the above theorem may then be rephrased as follows: for all $\lambda \in \mathbf{k}$,

$$\mathrm{Res}_{d_1,\dots,d_{n+1}}(f_1, \dots, \lambda f_k, \dots, f_{n+1}) = \lambda^{d_1 \cdots d_{k-1} d_{k+1} \cdots d_{n+1}} \mathrm{Res}_{d_1,\dots,d_{n+1}}(f_1, \dots, f_k, \dots, f_{n+1}).$$

The sign indeterminacy in the definition of $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ is usually removed by imposing the normalizing equality $\mathrm{Res}(X_0^{d_0}, \dots, X_n^{d_n}) = 1$.

(1.8) Divisibility property (see, e.g., [14, §5.6])**.** Let $F_1, \dots, F_{n+1}$ and $G_1, \dots, G_{n+1}$ be two sequences of homogeneous polynomials in $\mathbf{k}[\mathbf{X}]$ such that we have the inclusion of ideals of $\mathbf{k}[\mathbf{X}]$,

$$(G_1, \dots, G_{n+1}) \subseteq (F_1, \dots, F_{n+1}).$$

Then, $\mathrm{Res}(F_1,\dots,F_{n+1})$ divides $\mathrm{Res}(G_1,\dots,G_{n+1})$ in $\mathbf{k}$.

Besides its ordinary multi-homogeneity property given in Theorem (1.7), the resultant has other homogeneous structures that we call "weight properties" to emphasize that the grading of the coefficient ring $A_{\mathbf{Z}}$ is not the standard one.

(1.9) Proposition (see [14, §5.13.2])**.** *In the notation of Paragraph (1.2), let k be an integer in $[\![0,n]\!]$, and consider the grading on $A_{\mathbf{Z}} = \mathbf{Z}[U_{j,\boldsymbol{\alpha}}]_{j=1,\dots,n+1,\,|\boldsymbol{\alpha}|=d_j}$ defined by*

$$\mathrm{weight}(U_{j,\boldsymbol{\alpha}}) = \alpha_k. \tag{1.9.1}$$

In this grading, the resultant $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ is homogeneous of degree $d_1 d_2 \cdots d_{n+1}$.

(1.10) Bezout Theorem, which counts the number of roots of a finite complete intersection scheme in a projective space, and thus is the mother of all statements in projective enumerative geometry, can be deduced from this property. This goes as follows.

Consider n homogeneous polynomials $F_1,\dots,F_n \in \mathbf{k}[X_0,\dots,X_n]$, of degrees $d_1,\dots,d_n$ respectively. The idea is to project $\mathbf{P}^n$ to $\mathbf{P}^1$ from a codimension 2 linear space, in order to reduce to plain polynomials in one indeterminate only, i.e., homogeneous polynomials in two indeterminates. We thus consider

$$\tilde{F}_j\,(T_0,T_1,\dots,T_{n-1}) = F_j\left(T_0X_0,\,T_0X_1,\,T_1,\dots,T_{n-1}\right) \in \mathbf{k}[X_0,X_1][T_0,\dots,T_{n-1}] \tag{1.10.1}$$

for all $j=1,\dots,n$. A point $(x_0:x_1) \in \mathbf{P}^1_{\mathbf{k}}$ sits in the projection of $\mathrm{V}(F_1,\dots,F_n) \subseteq \mathbf{P}^n$ if and only if the specializations of $\tilde{F}_1,\dots,\tilde{F}_n$, obtained by specializing X_0 and X_1 to x_0 and x_1 respectively, have a common zero in $\mathbf{P}^{n-1}$. We thus consider the resultant of $\tilde{F}_1,\dots,\tilde{F}_n$ with respect to the indeterminates $T_0,\dots,T_{n-1}$,

$$\mathrm{Res}\big(\tilde{F}_1,\dots,\tilde{F}_n\big) \in \mathbf{k}[X_0,X_1].$$

Writing $F_j = \sum_{|\boldsymbol{\alpha}|=d_j} u_{j,\boldsymbol{\alpha}}\mathbf{X}^{\boldsymbol{\alpha}}$, one has

$$\tilde{F}_j = \sum_{|\boldsymbol{\alpha}|=d_j} \left(u_{j,\boldsymbol{\alpha}}X_0^{\alpha_0}X_1^{\alpha_1}\right) T_0^{\alpha_0+\alpha_1}T_1^{\alpha_2}\cdots T_{n-1}^{\alpha_n}.$$

Thus, $\tilde{F}_j$ has degree d_j in $T_0,\dots,T_{n-1}$, and its coefficient in the monomial $T_0^{\tilde{\alpha}_0}T_1^{\tilde{\alpha}_1}\cdots T_{n-1}^{\tilde{\alpha}_{n-1}}$ is a degree $\tilde{\alpha}_0$ homogeneous polynomial in X_0,X_1.

Then, it follows from the above Proposition (1.9) that

$$\mathrm{Res}_{d_1,\dots,d_n}\big(\tilde{F}_1,\dots,\tilde{F}_n\big) \in \mathbf{k}[X_0,X_1]$$

is homogeneous of degree $d_1\cdots d_n$. Therefore, $\mathrm{V}\big(\mathrm{Res}(\tilde{F}_1,\dots,\tilde{F}_n)\big) \subseteq \mathbf{P}^1_{\mathbf{k}}$ consists of $d_1\cdots d_n$ points, counted with multiplicities. One may then conclude that $\mathrm{V}(F_1,\dots,F_n) \subseteq \mathbf{P}^n_{\mathbf{k}}$ itself consists of $d_1\cdots d_n$ points, by the classical arguments used to prove Bezout's theorem for the intersection of two plane curves, see e.g., [1, §4.5].

The following is a parent of the weight property given in Proposition (1.9) above. It is proven in [14, §5.13] as well.

(1.11) Proposition. *In the same situation as in Proposition (1.9) above, consider the grading on $A_{\mathbf{Z}}$ defined by*

$$\mathrm{weight}(U_{j,\boldsymbol{\alpha}}) = d_j - \alpha_k. \tag{1.11.1}$$

In this grading, the resultant $\mathrm{Res}_{d_1,\dots,d_{n+1}} \in A_{\mathbf{Z}}$ is homogeneous of degree $nd_1d_2\cdots d_{n+1}$.

One may also use the following corollary, based on the standard homogeneity of the resultant in Theorem (1.7), to obtain additional weight properties from (1.9) and (1.11).

(1.12) Corollary. *Assume that the resultant* $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ *is homogeneous of degree* δ *for the grading on* $A_{\mathbf{Z}}$ *defined by* $\mathrm{weight}(U_{j,\boldsymbol{\alpha}}) = w_{j,\boldsymbol{\alpha}}$. *Let* $r_1,\dots,r_{n+1} \in \mathbf{Z}$. *For the new grading on* $A_{\mathbf{Z}}$ *defined by* $\mathrm{weight}(U_{j,\boldsymbol{\alpha}}) = w_{j,\boldsymbol{\alpha}} + r_j$, *the resultant* $\mathrm{Res}_{d_1,\dots,d_{n+1}}$ *is homogeneous of degree*

$$\delta + \sum_{1\leqslant k\leqslant n+1} \Big(r_k \prod\nolimits_{j\neq k} d_j \Big).$$

Proof. Let $k \in [\![1, n+1]\!]$. Since the resultant is homogeneous of degree $\prod_{j\neq k} d_j$ with respect to the indeterminates $(U_{k,\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}|=d_k}$ (for the standard grading), a shift by r_k in the weights of all the indeterminates $(U_{k,\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}|=d_k}$ induces a shift by $r_k \prod_{j\neq k} d_j$ in the degree of the resultant. □

1.2 – The reduced resultant

We shall now explain how to adapt the ideas of the previous paragraph to develop the theory of the reduced resultant. We refer to [20] and [15] for the details and proofs. Somehow, this is a generalization of the following toy example.

(1.13) Example (projection of a complete intersection from one of its points)**.** Let $F, G \in \mathbf{k}[\mathbf{X}]$ be two homogeneous polynomials of degrees a and b, defining a complete intersection $V \subseteq \mathbf{P}^n$, and suppose one wants to project V from a point $p_0 \in \mathbf{P}^n$. Assume for simplicity that $\mathbf{k}$ is an algebraically closed field. We may take $p_0 = (1 : 0 : \dots : 0)$. Then, we consider the two polynomials

$$\begin{aligned} & F(T, X_1, \dots, X_n) = F_0 T^a + F_1 T^{a-1} + \cdots + F_a \\ \text{and} \quad & G(T, X_1, \dots, X_n) = G_0 T^b + G_1 T^{b-1} + \cdots + G_b \end{aligned} \tag{1.13.1}$$

in $\mathbf{k}[X_1,\dots,X_n][T]$. We are abusing notation here, as one should consider instead the two polynomials $F(T, S\,X_1, \dots, S\,X_n)$ and $G(T, S\,X_1, \dots, S\,X_n)$ that are homogeneous in the couple of indeterminates (S, T). Moreover, beware that (1.13.1) above is written "in reverse order" with respect to what we did when discussing Bezout Theorem, see (1.10.1).

If $p_0 \notin V$, the point $(x_1 : \dots : x_n) \in \mathbf{P}^{n-1}_{\mathbf{k}}$ belongs to the projection of V from p_0 if and only if the two polynomials in (1.13.1) have a common root in $\mathbf{P}^1$, hence the equation of the projection is given by

$$\mathrm{Res}_{a,b}(F, G) \in \mathbf{k}[X_1, \dots, X_n],$$

which is homogeneous of degree ab in the indeterminates $(X_1,\dots,X_n)$ by Proposition (1.11), as the polynomials F_i are homogeneous of degree i in $\mathbf{k}[X_1,\dots,X_n]$.

On the other hand, if $p_0 \in V$ then, letting a' and b' be the respective multiplicities of p_0 in the hypersurfaces $\mathrm{V}(F)$ and $\mathrm{V}(G)$, one has

$$F_0 = \cdots = F_{a'-1} = G_0 = \cdots = G_{b'-1} = 0,$$

so that (1.13.1) becomes

$$\begin{aligned} & F(T, X_1, \dots, X_n) = F_{a'} T^{a-a'} + \cdots + F_a = {}^\flat F \\ \text{and} \quad & G(T, X_1, \dots, X_n) = G_{b'} T^{b-b'} + \cdots + G_b = {}^\flat G. \end{aligned} \tag{1.13.2}$$

It follows that the equation of the projection of V from p_0 is given by

$$\mathrm{Res}_{a-a',b-b'}({}^\flat F, {}^\flat G) \in \mathbf{k}[X_1, \dots, X_n].$$

We shall see later on that this polynomial is the *reduced resultant* of F and G truncated at orders $a-a'$ and $b-b'$ respectively, as polynomials in the indeterminate T. It is a homogeneous polynomial of degree $ab-a'b'$ in $(X_1,\dots,X_n)$: indeed, the coefficient of ${}^\flat F$ (respectively, ${}^\flat G$) in T^i is a homogeneous polynomial in $(X_1,\dots,X_n)$ of degree $a-i=\deg_T({}^\flat F)-i+a'$ (respectively, $b-i=\deg_T({}^\flat G)-i+b'$). Therefore, Corollary (1.12) applied to the grading of Proposition (1.11) gives that $\mathrm{Res}_{a-a',b-b'}({}^\flat F,{}^\flat G)$ is homogeneous of degree

$$(a-a')(b-b')+a'(b-b')+b'(a-a')=ab-a'b',$$

as we had announced. This weight property is an instance of that given in Corollary (1.20), which applies to reduced resultants in general.

(1.14) Let $d_1,\dots,d_{n+1}$ be positive integers. For all $j\in[\![1,n+1]\!]$, we consider the generic homogeneous degree d_j polynomial F_j, which we write as

$$F_j=\sum_{|\boldsymbol{\alpha}|=d_j} U_{j,\boldsymbol{\alpha}}\mathbf{X}^{\boldsymbol{\alpha}}=\sum_{k=0}^{d_j} X_0^{d_j-k}\,F_{j,k}(X_1,\dots,X_n).$$

Thus, for all $k=0,\dots,d_j$, the polynomial $F_{j,k}$ is a degree k homogeneous element of $A_{\mathbf{Z}}[X_1,\dots,X_n]$. We fix integers $s_j\in[\![1,d_j]\!]$ for all $j=1,\dots,n+1$. The *truncation* of F_j at order d_j-s_j with respect to X_0 is the polynomial

$$H_j=\sum_{k=s_j}^{d_j} X_0^{d_j-k}F_{j,k}=X_0^{d_j-s_j}F_{j,s_j}+\cdots+X_0F_{j,d_j-1}+F_{j,d_j}\in A_{\mathbf{Z}}[\mathbf{x}].$$

This definition gives a special role to the indeterminate X_0, and to the point $(1:0:\dots:0)\in\mathbf{P}^n$. One may thus think of the latter as the (chosen) origin in the affine space $\mathbf{P}^n-\mathrm{V}(X_0)$, and of $\mathrm{V}(X_0)$ as the hyperplane at infinity. We may occasionally use this terminology.

The purpose of reduced elimination theory is to study inertia forms of the truncations $H_1,\dots,H_{n+1}$ defined above. Note that, for all $j=1,\dots,n+1$, the truncation H_j is the generic homogeneous polynomial of degree d_j with a multiplicity s_j zero in the origin $(1:0:\dots:0)$. The wish for the reduced resultant is that it is a polynomial in the coefficients of $H_1,\dots,H_{n+1}$, which vanishes if and only if $H_1,\dots,H_{n+1}$ have a non-trivial common root in addition to that of multiplicity $s_1\cdots s_{n+1}$ at the origin.

This can be done with essentially the same strategy as in the classical case, which we have reviewed in Section 1.1. We will only consider the "resultant situation", when the number of polynomials is $n+1$.

(1.15) Theorem ([20, Theorem 6 and §8] and [15, Theorem II.0.5 and §IV.0])**.** *Assume that $d_j>s_j$ for some $j\in[\![1,n+1]\!]$. The ideal of reduced inertia forms of degree 0*

$$\mathcal{Q}_{\mathbf{Z}}=\big((H_1,\dots,H_{n+1}):(X_1,\dots,X_n)^\infty\big)\cap A_{\mathbf{Z}}$$

is a prime and principal ideal of $A_{\mathbf{Z}}$. The reduced resultant*, denoted*

$$\mathrm{redRes}^{s_1,\dots,s_{n+1}}_{d_1,\dots,d_{n+1}}\in A_{\mathbf{Z}},$$

is defined, up to sign, as the generator of $\mathcal{Q}_{\mathbf{Z}}$; it is therefore an irreducible element of $A_{\mathbf{Z}}$.

Moreover, if $d_j > s_j$ for at least two distinct integers $j, j' \in [\![1, n+1]\!]$, then for all $i \in [\![1, n+1]\!]$ the reduced resultant is a homogeneous polynomial of degree

$$\frac{d_1 d_2 \cdots d_{n+1}}{d_i} - \frac{s_1 s_2 \cdots s_{n+1}}{s_i}$$

with respect to the coefficients of the polynomial H_i, i.e., with respect to the indeterminates $U_{i,\boldsymbol{\alpha}}$ such that $|\boldsymbol{\alpha}| = d_i$ and $\alpha_0 \leqslant d_i - s_i$.

If there is only one integer $j \in [\![1, n+1]\!]$ such that $d_j > s_j$, then the reduced resultant is equal to the resultant of the polynomials $H_1, \ldots, H_{j-1}, H_{j+1}, \ldots, H_{n+1}$, as polynomials in $X_1, \ldots, X_n$ only.[2]

The sign indeterminacy in the definition of the reduced resultant can be removed by means of Theorem (1.19) below, once the sign of the ordinary resultant has been chosen.

Note that, in the above statement, the ideal $(H_1, \ldots, H_{n+1})$ is saturated with respect to $(X_1, \ldots, X_n)$, which is the defining ideal of the point $(1 : 0 : \ldots : 0)$, whereas for the plain resultant we considered instead the saturation with respect to the irrelevant ideal $(X_0, X_1, \ldots, X_n)$. Beware moreover that the polynomials $H_1, \ldots, H_{n+1}$ are not homogeneous in the set of indeterminates $(X_1, \ldots, X_n)$.

The reduced resultant depends, of course, only on the coefficients of the generic truncated polynomials $H_1, \ldots, H_{n+1}$, and not on all the coefficients of the polynomials $F_1, \ldots, F_{n+1}$. We will often denote it by $\mathrm{redRes}(H_1, \ldots, H_{n+1})$ without printing the integers d_i and s_i, that are implicitly given by the polynomials $H_1, \ldots, H_{n+1}$. We may also use the notation $\mathrm{redRes}(F_1, \ldots, F_{n+1})$, to avoid giving particular names to the truncation.

The reduced resultant of a collection of polynomials $h_1, \ldots, h_{n+1} \in \mathbf{k}[\mathbf{X}]$ with zeros of respective multiplicities at least $s_1, \ldots, s_{n+1}$ at the origin is defined as the corresponding specialization of the generic reduced resultant; it is an element in $\mathbf{k}$, denoted by $\mathrm{redRes}(h_1, \ldots, h_{n+1})$, or possibly $\mathrm{redRes}(f_1, \ldots, f_{n+1})$ depending on the context.

(1.16) Vanishing of the reduced resultant. The reduced resultant $\mathrm{redRes}(H_1, \ldots, H_{n+1})$ is a polynomial in the coefficients of the polynomials H_j, $j = 1, \ldots, n+1$, i.e., an element of the ring

$$\mathbf{Z}\big[U_{i,\boldsymbol{\alpha}}\big]_{i=1,\ldots,n+1,\, |\boldsymbol{\alpha}|=d_i,\, \alpha_0 \leqslant d_i - s_i} \subseteq A_{\mathbf{Z}}.$$

Its vanishing on an algebraically closed field $\mathbf{k}$ characterizes those collections of hypersurfaces of $\mathbf{P}^n_{\mathbf{k}}$ defined by $h_1, \ldots, h_{n+1}$ that have a further intersection point, infinitely near or not, besides the origin $p_0 = (1 : 0 : \ldots : 0)$, i.e., those collections such that one of the two following conditions holds:

(a) the hypersurfaces defined by $h_1, \ldots, h_{n+1}$ intersect at a point which is different from p_0;
(b) the polynomials f_{j,s_j}, $j = 1, \ldots, n+1$, have a common root in $\mathbf{P}^{n-1}_{\mathbf{k}}$, which means that the tangent cones of the hypersurfaces $\mathrm{V}(h_1), \ldots, \mathrm{V}(h_{n+1})$ at p_0 have a line in common; equivalently, the proper transforms of $\mathrm{V}(h_1), \ldots, \mathrm{V}(h_{n+1})$ in the blow-up of $\mathbf{P}^n_{\mathbf{k}}$ at p_0 have a point in common which lies on the exceptional divisor.

This property of the reduced resultant is proved in [20, Theorem 3.1, 3.2 and 3.3], and in [15, Proposition I.1].

1.3 – Generalized weight properties

In [20] Zariski showed that the reduced resultant can be computed from the corresponding resultant. To obtain this property, he introduced a generalization of the grading (1.9.1) and,

[2] if $s_j = d_j$, then $H_j = F_{j,d_j}$ is a polynomial in $X_1, \ldots, X_n$ only; geometrically, the polynomial H_j defines a cone in $\mathbf{P}^n$ over the hypersurface defined by F_{j,d_j} in $\mathbf{P}^{n-1}$.

although the resultant is not homogeneous with respect to this new grading, he proved that its graded piece of smallest degree is connected to the reduced resultant.
We maintain the notation of Section 1.2.

(1.17) The Zariski grading. We define a grading on $A_{\mathbf{Z}} = \mathbf{Z}[U_{j,\boldsymbol{\alpha}}]$ by assigning for all j

$$\operatorname{weight}(U_{j,\boldsymbol{\alpha}}) = \begin{cases} 0 & \text{if } \alpha_0 < d_j - s_j \\ \alpha_0 - d_j + s_j & \text{otherwise,} \end{cases}$$

and weight 0 to the constants. We find it helpful to visualize this definition as follows:

$$F_j = \underbrace{X_0^{d_j} F_{j,0}}_{\text{coeffs have weight } s_j} + \cdots + \underbrace{X_0^{d_j-s_j+1} F_{j,s_j-1}}_{\text{coeffs have weight } 1} + \underbrace{X_0^{d_j-s_j} F_{j,s_j} + \cdots + F_{j,d_j}}_{\text{coeffs have weight } 0}. \tag{1.17.1}$$

Note in particular that the indeterminates $U_{j,\boldsymbol{\alpha}}$ whose weight equals 0 in this grading are exactly the coefficients of the truncation H_j of the polynomial F_j. The grading (1.9.1) introduced in Proposition (1.9) is a particular case of a Zariski grading (corresponding to $s_j = d_j$ for all j), which explains the terminology "generalized weight properties".

The main property of the Zariski grading is that it allows the computation of the reduced resultant of $H_1, \ldots, H_{n+1}$ (the truncations of $F_1, \ldots, F_{n+1}$ at the orders $d_1 - s_1, \ldots, d_{n+1} - s_{n+1}$, respectively) from the resultant of $F_1, \ldots, F_{n+1}$. We need one more piece of notation to see how this goes.

(1.18) For all $j = 1, \ldots, n+1$, we let G_j be the quotient of the Euclidean division of F_j by $X_0^{d_j-s_j}$ in $A_{\mathbf{Z}}[X_1, \ldots, X_n][X_0]$, i.e.,

$$\begin{aligned} G_j \;=\; \frac{1}{X_0^{d_j-s_j}} \sum_{k \leqslant s_j} X_0^{d_j-k}\, F_{j,k} \;&=\; \sum_{k=0}^{s_j} X_0^{s_j-k}\, F_{j,k} \\ &= X_0^{s_j}\, F_{j,0} + \cdots + X_0\, F_{j,s_j-1} + F_{j,s_j}. \end{aligned}$$

The polynomial G_j is a generic degree s_j homogeneous polynomial in the set of variables $\mathbf{X}$. Beware that F_j does not equal $H_j + X_0^{d_j-s_j} G_j$, as F_{j,s_j} appears in both H_j and G_j:

$$F_j = \underbrace{X_0^{d_j}\, F_{j,0} + \cdots + \underbrace{X_0^{d_j-s_j}\, F_{j,s_j} + \cdots + F_{j,d_j}}_{= H_j}}_{= X_0^{d_j-s_j}\, G_j}. \tag{1.18.1}$$

Also, we advise the reader to compare Displays (1.17.1) and (1.18.1).

(1.19) Theorem (Zariski Formula, [20, Theorem 5.1 and Theorem 5.2] and [15, Lemme IV.1.6])**.** *The nonzero homogeneous piece of lowest degree with respect to the Zariski grading of* $\operatorname{Res}(F_1, \ldots, F_{n+1})$ *has degree* $s_1 s_2 \cdots s_{n+1}$*. Denote it by* $\big[\operatorname{Res}(F_1, \ldots, F_{n+1})\big]_{s_1 s_2 \cdots s_{n+1}}$.
(a) If $s_j < d_j$ *for at least two distinct integers* $j = j_1, j_2 \in [\![1, n+1]\!]$*, then*

$$\big[\operatorname{Res}(F_1, \ldots, F_{n+1})\big]_{s_1 s_2 \cdots s_{n+1}} = \operatorname{Res}(G_1, \ldots, G_{n+1}) \cdot \operatorname{redRes}(H_1, \ldots, H_{n+1}).$$

(b) If there exists $j_0 \in [\![1, n+1]\!]$ *such that* $s_{j_0} < d_{j_0}$ *and* $s_j = d_j$ *for all* $j \neq j_0$*, then*

$$\big[\operatorname{Res}(F_1, \ldots, F_{n+1})\big]_{s_1 s_2 \cdots s_{n+1}} = \operatorname{Res}(G_1, \ldots, G_{n+1}) \cdot \operatorname{redRes}(H_1, \ldots, H_{n+1})^{d_{j_0}-s_{j_0}}.$$

Note that despite appearances, there is a Zariski Formula in all cases, because if none of the two conditions in (a) and (b) of the above statement is verified, then the reduced resultant is actually an ordinary resultant.

Observe that, in the Zariski grading: (i) the coefficients of $H_1, \dots, H_{n+1}$ all have weight 0, hence $\mathrm{redRes}(H_1, \dots, H_{n+1})$ is homogeneous of degree 0; (ii) for all $j = 1, \dots, n+1$, the coefficient of G_j in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight α_0 hence, by Proposition (1.9), $\mathrm{Res}(G_1, \dots, G_{n+1})$ is homogeneous of weight $s_1 s_2 \cdots s_{n+1}$. Thus, indeed,

$$\mathrm{Res}(G_1, \dots, G_{n+1}) \cdot \mathrm{redRes}(H_1, \dots, H_{n+1})^e$$

is homogeneous of weight $s_1 s_2 \cdots s_{n+1}$ for all e.

(1.20) Corollary. *Consider the grading of $A_{\mathbf{Z}}$ defined by*

$$\mathrm{weight}(U_{j,\boldsymbol{\alpha}}) = d_j - \alpha_0.$$

With respect to this grading, $\mathrm{redRes}(H_1, \dots, H_{n+1})$ *is homogeneous of degree*

$$n(d_1 \cdots d_{n+1} - s_1 \cdots s_{n+1})$$

if there are at least two integers $j = j_1, j_2$ such that $s_j < d_j$, and homogeneous of degree

$$n\, d_1 \dots d_{n+1}/d_{j_0}$$

if there exists j_0 such that $s_{j_0} < d_{j_0}$ and $s_j = d_j$ for all $j \neq j_0$.

This is the homogeneity property of the reduced resultant that particularizes to give the degree of $\mathrm{Res}(^\flat F, {}^\flat G)$ in Example (1.13).

Proof. The grading of $A_{\mathbf{Z}}$ we are considering is that of Proposition (1.11), with $k = 0$; thus for all $j = 1, \dots, n+1$ the coefficients of $F_{j,l}$ all have weight l, for $l = 0, \dots, d_j$. Then we know by Proposition (1.11) that the resultant of $F_1, \dots, F_{n+1}$ is homogeneous of degree $n\, d_1 \cdots d_{n+1}$, and by the same argument the resultant of $G_1, \dots, G_{n+1}$ is homogeneous of degree $n\, s_1 \cdots s_{n+1}$. Therefore, we deduce from Theorem (1.19) that the reduced resultant of $H_1, \dots, H_{n+1}$ is homogeneous, of degree

$$n\, d_1 \cdots d_{n+1} - n\, s_1 \cdots s_{n+1}$$

if $s_j < d_j$ for two distinct indices $j = j_1, j_2$, and of degree

$$\frac{n\, d_1 \cdots d_{n+1} - n\, s_1 \cdots s_{n+1}}{d_{j_0} - s_{j_0}} = n\, \frac{d_1 \cdots d_{n+1}}{d_{j_0}}\, \frac{d_{j_0} - s_{j_0}}{d_{j_0} - s_{j_0}}$$

if $s_j = d_j$ for all $j \neq j_0$ and $s_{j_0} < d_{j_0}$. □

2 – Reduced discriminant and Salmon formula

In this Section we give a rigorous proof of formula (🔭). This is done by introducing the concept of reduced discriminant. We begin with a quick recap on the ordinary discriminant of a hypersurface, following [4, §4]; see also [7] and [11, Chapter 13, §D].

2.1 – Discriminant of a homogeneous polynomial

(2.1) Let d be a positive integer, and consider the generic homogeneous degree d polynomial $F = \sum_{|\boldsymbol{\alpha}|=d} U_\alpha \mathbf{X}^{\boldsymbol{\alpha}}$ in $n+1$ indeterminates $\mathbf{X} = (X_0, \ldots, X_n)$. We set $A_{\mathbf{Z}} = \mathbf{Z}[U_{\boldsymbol{\alpha}}]_{|\boldsymbol{\alpha}|=d}$. For all $i = 0, \ldots, n$ we let ∂_i denote derivation with respect to the indeterminate X_i.

(2.2) Definition. *There is a unique element* $\mathrm{Disc}_d(F) \in A_{\mathbf{Z}}$ *(often simply denoted by* $\mathrm{Disc}(F)$*) such that*

$$d^{a(n,d)}\mathrm{Disc}_d(F) = \mathrm{Res}\big(\partial_0 F, \ldots, \partial_n F\big) \tag{2.2.1}$$

in $A_{\mathbf{Z}}$*, where* $a(n,d) = \frac{(d-1)^{n+1}-(-1)^{n+1}}{d} \in \mathbf{Z}$*. It is homogeneous of degree* $(n+1)(d-1)^n$ *with respect to the coefficients of the polynomial* F*, i.e., with respect to the indeterminates* $U_{\boldsymbol{\alpha}}$*,* $|\boldsymbol{\alpha}| = d$*.*

For a homogeneous degree d *polynomial* $f \in \mathbf{k}[\mathbf{X}]$*, we define the* discriminant $\mathrm{Disc}(f) \in \mathbf{k}$ *of* f *as the specialization* $\sigma(\mathrm{Disc}(F)) \in \mathbf{k}$*, where* $\sigma : A_{\mathbf{Z}} \to \mathbf{k}$ *is the unique specialization morphism mapping* F *to* f*.*

We emphasize that the factor $d^{a(n,d)}$ in Equation (2.2.1) is here to make the discriminant a universal object (i.e., to make it behave well under specialization) which yields the expected smoothness criterion. An alternative way of defining the discriminant is to use Formula (2.5.1) below. Either way there is an annoying parasitic factor. The issue comes from the fact that, while the resultant characterizes the existence of a non-trivial common zero in $\mathbf{P}^n$ for $n+1$ polynomials, the wish for the discriminant is that it will characterize the existence of a non-trivial common zero for the $n+2$ polynomials $F, \partial_0 F, \partial_1 F, \ldots, \partial_n F$ which are linked by the Euler Formula. The next result tells us that with Definition (2.2), the discriminant grants our wish.

(2.3) Proposition. *The ideal of inertia forms*

$$(\partial_0 F, \partial_1 F, \ldots, \partial_n F, F) : (X_0, \ldots, X_n)^\infty$$

is a prime and principal ideal in $A_{\mathbf{Z}}$*. It is generated by the discriminant* $\mathrm{Disc}(F)$*, which is therefore an irreducible polynomial in* $A_{\mathbf{Z}}$*.*

In fact, what we really want is that the (non-)vanishing of the discriminant characterizes smoothness. By the Jacobian criterion for smoothness, this is equivalent to the non-existence of a common root $F, \partial_0 F, \partial_1 F, \ldots, \partial_n F$. Thus, Proposition (2.3) gives the following.

(2.4) Theorem (smoothness criterion)**.** *Suppose* $\mathbf{k}$ *is an algebraically closed field, and consider a degree* d *homogeneous polynomial* $f \in \mathbf{k}[\mathbf{X}]$*. The following are equivalent:*
(i) the hypersurface $\mathrm{V}(f) \subseteq \mathbf{P}^n_{\mathbf{k}}$ *is smooth;*
(ii) $\mathrm{Disc}(f) \neq 0$*.*

The following formula gives an alternative way of defining the discriminant. This is the path we shall follow to define the reduced discriminant. As we shall see, however, there will be some additional technical difficulty to overcome, which did not occur when passing from the ordinary resultant to its reduced form either.

(2.5) Proposition. *Let* $\bar{F}$ *be the polynomial* $F(0, X_1, \ldots, X_n) \in A_{\mathbf{Z}}[X_1, \ldots, X_n]$*. We have the following identity in* $A_{\mathbf{Z}}$*:*

$$\mathrm{Res}\big(\partial_1 F, \ldots, \partial_n F, F\big) = \mathrm{Disc}(F) \cdot \mathrm{Disc}(\bar{F}). \tag{2.5.1}$$

The polynomial $\bar{F}$ is the equation of the hypersurface in $\mathbf{P}^{n-1}$ cut out by $\mathrm{V}(F)$ on the hyperplane $\mathrm{V}(X_0) \subseteq \mathbf{P}^n$, of which we think as the hyperplane at infinity. If $\mathbf{k}$ is an algebraically closed field and $f \in \mathbf{k}[\mathbf{X}]$ is a degree d homogeneous polynomial, then, by Theorem (2.4), the vanishing of $\mathrm{Disc}(\bar{f})$ is equivalent to the hyperplane section at infinity $\mathrm{V}(f) \cap \mathrm{V}(X_0)$ being singular. For a general f such that $\mathrm{Disc}(\bar{f}) = 0$, the hypersurface $\mathrm{V}(f)$ is non-singular and tangent to the hyperplane $\mathrm{V}(X_0)$.

Proof of Proposition (2.5). We shall prove the following identity in $A_{\mathbf{Z}}$:

$$d^{(d-1)^n}\mathrm{Res}\big(\partial_1 F, \dots, \partial_n F, F\big) = \mathrm{Res}\big(\partial_0 F, \dots, \partial_n F\big) \cdot \mathrm{Res}\big(\partial_1 \bar{F}, \dots, \partial_n \bar{F}\big). \tag{2.5.2}$$

We claim that it is equivalent to Formula (2.5.1). To see why, first note that at the right-hand-side of Display (2.5.2), the first (respectively second) factor is the resultant of $n+1$ (respectively n) polynomials in $n+1$ (respectively n) indeterminates. Since $a(n,d) + a(n-1,d) = (d-1)^n$, it follows from Formula (2.2.1) that the identities (2.5.2) and (2.5.1) are indeed equivalent.

Now, the identity (2.5.2) may be derived from the Euler Formula, as follows. The latter formula says that $d \cdot F$ is congruent to $X_0\, \partial_0 F$ modulo the ideal $(\partial_1 F, \dots, \partial_n F)$, hence

$$\mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, d \cdot F\big) = \mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, X_0\, \partial_0 F\big) \tag{2.5.3}$$

by [14, §5.9, Transformations élémentaires]. The left-hand-side of (2.5.3) equals that of (2.5.2) by the elementary homogeneity properties of the resultant, see Theorem (1.7). Its right-hand-side, on the other hand, equals

$$\mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, X_0\big) \cdot \mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, \partial_0 F\big)$$

by multiplicativity of the resultant, see [14, §5.7]. Eventually,

$$\mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, X_0\big) = (-1)^{n(d-1)^n}\mathrm{Res}\big(\,\partial_1 F|_{X_0=0}\,, \dots,\, \partial_n F|_{X_0=0}\big)$$

by [14, (5.13.5)][3], and

$$\mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, \partial_0 F\big) = (-1)^{n(d-1)^{n+1}}\mathrm{Res}\big(\partial_0 F,\, \partial_1 F, \dots, \partial_n F\big)$$

by [14, §5.8, Effet d'une permutation des polynômes]. The upshot is that

$$\mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, X_0\, \partial_0 F\big) = (-1)^{nd(d-1)^n}\, \mathrm{Res}\big(\partial_0 F, \dots, \partial_n F\big) \cdot \mathrm{Res}\big(\partial_1 \bar{F}, \dots, \partial_n \bar{F}\big),$$

which gives the result since $d(d-1)$ is always even. □

[3] the $(-1)^{n(d-1)^n}$ factor is due to the fact that we need to exchange the roles of X_0 and X_n, which is not indifferent since by definition the resultant is normalized by imposing $\mathrm{Res}(X_0^{d_0}, \dots, X_n^{d_n}) = 1$, see right after Theorem (1.7).

To work out the claimed identity, one may first perform a base change by exchanging X_0 and X_n which, by [14, (5.13.1)], gives

$$\mathrm{Res}\big(\partial_1 F, \dots, \partial_n F,\, X_0\big) = (-1)^{(d-1)^n}\mathrm{Res}\big(\partial_1 F(X_n, X_1, \dots, X_{n-1}, X_0),\, \dots,\, X_n\big);$$

then, apply [14, (5.13.5)] to the resultant on the right-hand-side, and eventually perform another base change to go from $(X_1, \dots, X_{n-1}, X_0)$ to $(X_0, X_1, \dots, X_{n-1})$, which introduces the additional factor $(-1)^{(n-1)(d-1)^n}$, hence in total the factor $(-1)^{n(d-1)^n}$ indeed.

An easier way to get the correct sign in general is to consider the normalizing condition, once one knows that there exists $c \in \mathbf{Z}$ such that

$$\mathrm{Res}_{d_1,\dots,d_n,1}(F_1, \dots, F_n, X_0) = c\,\mathrm{Res}_{d_1,\dots,d_n}(\,F_1|_{X_0=0}\,, \dots,\, F_n|_{X_0=0}).$$

Specializing $F_1, \dots, F_n$ to $X_1^{d_1}, \dots, X_n^{d_n}$ respectively, one gets

$$\mathrm{Res}_{d_1,\dots,d_n,1}(X_1^{d_1}, \dots, X_n^{d_n}, X_0) = c\,\mathrm{Res}_{d_1,\dots,d_n}(X_1^{d_1}, \dots, X_n^{d_n}),$$

where the left-hand-side equals $(-1)^{nd_1\cdots d_n}$ by [14, §5.8], and the right-hand-side equals c.

Similarly to the resultant, the discriminant is homogeneous under the two gradings of the coefficient ring $A_{\mathbf{Z}}$ introduced in Propositions (1.9) and (1.11).

(2.6) Proposition. *Let k be an integer in $[\![0, n]\!]$.*
(2.6.1) In the grading of $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = \alpha_k$*, the discriminant* $\mathrm{Disc}(F)$ *is homogeneous of degree $d(d-1)^n$.*
(2.6.2) In the grading of $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = d - \alpha_k$*, the discriminant* $\mathrm{Disc}(F)$ *is homogeneous of degree $nd(d-1)^n$.*

One may use (2.6.1) to compute the degree of the dual to a smooth hypersurface in $\mathbf{P}^{n+1}$, see Paragraph (3.7).

Proof. The idea is to consider (2.2.1) and the corresponding weight properties of the resultant. We may assume $k \neq 0$, for otherwise we may write a formula analogous to (2.5.1) with respect to an indeterminate other than X_0 in order to reduce to this case.

First consider the grading of (2.6.1). For $j \neq k$ the coefficient of $\partial_j F$ in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight α_k, whereas for $j = k$ the coefficient of $\partial_k F$ in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight $\alpha_k + 1$. We may thus apply Corollary (1.12) to the homogeneity property (1.9), with $r_k = 1$ and $r_j = 0$ for all $j \neq k$, which gives that $\mathrm{Res}\big(\partial_0 F, \ldots, \partial_n F\big)$ is homogeneous of degree

$$(d-1)^{n+1} + (d-1)^n.$$

Using (2.2.1), this gives (2.6.1).

Similarly, for the grading of (2.6.2): for $j \neq k$ the coefficient of $\partial_j F$ in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight $d - \alpha_k = (d-1) - \alpha_k + 1$, whereas for $j = k$ the coefficient of $\partial_k F$ in $\mathbf{X}^{\boldsymbol{\alpha}}$ has weight $d - (\alpha_k + 1) = (d-1) - \alpha_k$. We thus apply Corollary (1.12) to the homogeneity property (1.11), with $r_j = 1$ for $j \neq k$ and $r_k = 0$, which gives that $\mathrm{Res}\big(\partial_0 F, \ldots, \partial_n F\big)$ is homogeneous of degree

$$n(d-1)^{n+1} + n(d-1)^n.$$

Again this gives the wanted result by (2.2.1). □

In turn, one may reproduce the argument given in (1.12) to deduce further weight properties from the two latter results and the standard homogeneity property of the discriminant stated in Definition (2.2).

(2.7) Corollary. *Assume that the discriminant* $\mathrm{Disc}_d \in A_{\mathbf{Z}}$ *is homogeneous of degree δ for the grading on $A_{\mathbf{Z}}$ defined by* $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = w_{\boldsymbol{\alpha}}$*. Let $r \in \mathbf{Z}$. For the new grading on $A_{\mathbf{Z}}$ defined by* $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = w_{\boldsymbol{\alpha}} + r$*, the discriminant* Disc_d *is homogeneous of degree*

$$\delta + r(n+1)(d-1)^n.$$

2.2 – The reduced discriminant

(2.8) We write the generic homogeneous degree d polynomial as

$$F = X_0^d F_0 + \cdots + X_0^{d-s} F_s + \cdots + F_d$$

where each F_k is homogeneous of degree k in the indeterminates $X_1, \ldots, X_n$. Given an integer $s \in [\![2, d-1]\!]$, we consider the truncation H of F at order $d - s$ with respect to X_0, defined as in Paragraph (1.14), and set G as in Paragraph (1.18); thus,

$$\begin{aligned} H &= X_0^{d-s} F_s + X_0^{d-s-1} F_{s+1} + \cdots + F_d, \\ G &= X_0^s F_0 + X_0^{s-1} F_1 + \cdots + F_s. \end{aligned}$$

The truncation H is the generic degree d polynomial with a multiplicity s zero at the point $(1:0:\ldots:0)$. The wish for the reduced discriminant is that it is a polynomial in the coefficients of H, the vanishing of which characterizes the existence of an additional singular non-trivial zero of H.

The polynomial H is homogeneous of degree d, and has order s in $X_1, \ldots, X_n$ (recall Definition (0.2)). Moreover, its partial derivatives with respect to the indeterminates $X_1, \ldots, X_n$ all have degree $d-1$, and order $s-1$ in $X_1, \ldots, X_n$. Therefore, the reduced resultant

$$\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H) = \mathrm{redRes}^{s-1,\ldots,s-1,s}_{d-1,\ldots,d-1,d}(\partial_1 H, \ldots, \partial_n H, H)$$

is well defined.

(2.9) Proposition. *With the above notation, the discriminants* $\mathrm{Disc}(F_s)$ *and* $\mathrm{Disc}(F_d)$ *both divide the reduced resultant* $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$.

Proof. Both discriminants $\mathrm{Disc}(F_s)$ and $\mathrm{Disc}(F_d)$ are irreducible as elements of $A_{\mathbf{Z}}$, and therefore coprime because they do not depend on the same coefficients of F. To prove the proposition, it thus suffices to prove that the vanishing of $\mathrm{Disc}(F_s)$ or $\mathrm{Disc}(F_d)$ implies the vanishing of the reduced resultant. This follows from (1.16), as we shall now explain.

The vanishing of $\mathrm{Disc}(F_d)$ at f (a specialization of F with coefficients in an algebraically closed field $\mathbf{k}$) implies the existence of a common root at infinity ($x_0 = 0$) of the polynomials $h, \partial_1 h, \ldots, \partial_n h$ (with h the specialization of H corresponding to f). Indeed, if $\mathrm{Disc}(F_d)$ vanishes at f, then by Theorem (2.4) there exists a non-trivial zero ${}^\flat x \in \mathbf{k}^n$ of $f_d, \partial_1 f_d, \ldots \partial_n f_d$, and then $(0, {}^\flat x) \in \mathbf{k}^{n+1}$ is a non-trivial zero of $h, \partial_1 h, \ldots, \partial_n h$, hence the reduced resultant vanishes at f by (a) of (1.16).

Similarly, the vanishing of $\mathrm{Disc}(F_s)$ at f implies the existence of a common non-trivial zero of $h, \partial_1 h, \ldots, \partial_n h$ infinitely near the point $p_0 = (1:0:\ldots:0)$: indeed, $f_s, \partial_1 f_s, \ldots, \partial_n f_s$ define the respective tangent cones at p_0 of the hypersurfaces $V(h), V(\partial_1 h), \ldots, V(\partial_n h)$, and the vanishing of $\mathrm{Disc}(F_s)$ at f implies the existence of a non-trivial zero ${}^\flat x \in \mathbf{k}^n$ of $f_s, \partial_1 f_s, \ldots \partial_n f_s$. Thus, by (b) of (1.16), the reduced resultant vanishes at f if $\mathrm{Disc}(F_s)$ vanishes at f. □

Alternatively, this proposition can be proved by means of inertia forms, as follows. By Theorem (1.15), the reduced resultant $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$ belongs to the ideal of inertia forms $(\partial_1 H, \ldots, \partial_n H, H) : (X_1, \ldots, X_n)^\infty$. Therefore, for all integer $i = 1, \ldots, n$, there exists an integer N_i such that

$$X_i^{N_i}\, \mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H) \in (\partial_1 H, \ldots, \partial_n H, H); \tag{2.9.1}$$

specializing the variable X_0 to 0 in (2.9.1), we get that

$$X_i^{N_i}\, \mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H) \in (\partial_1 F_d, \ldots, \partial_n F_d, F_d).$$

It follows that $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$ belongs to the ideal of inertia forms $(\partial_1 F_d, \ldots, \partial_n F_d, F_d) : (X_1, \ldots, X_n)^\infty$. By Proposition (2.3), this ideal is generated by the discriminant of the polynomial F_d, hence $\mathrm{Disc}(F_d)$ divides $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$. A similar argument, albeit slightly more technical, can be used to show that $\mathrm{Disc}(F_s)$ divides $\mathrm{redRes}(\partial_1 H, \ldots, \partial_n H, H)$: see [15, Lemme I.1.3].

In the notation of (2.5), $F_d = \bar{F}$ and $F_s = \bar{G}$. Observe that F_d and F_s are generic homogeneous polynomials of degrees d and k respectively, in the indeterminates $(X_1, \ldots, X_n)$. Proposition (2.9) then leads to the following definition.

(2.10) Definition. *The* reduced discriminant *of F with respect to the truncation at order $d-s$ for the indeterminate X_0, denoted by* $\mathrm{redDisc}_d^s(H)$*, or simply* $\mathrm{redDisc}(H)$*, is defined by the equality*

$$\mathrm{Disc}(F_d)\,\mathrm{Disc}(F_s)\,\mathrm{redDisc}(H) = \mathrm{redRes}\big(\partial_1 H,\dots,\partial_n H, H\big) \in A_{\mathbf{Z}}. \tag{2.10.1}$$

For a degree d polynomial $h \in \mathbf{k}[\mathbf{X}]$ with a singularity of order s at the origin, $\mathrm{redDisc}_d^s(h)$ is defined as the specialization of $\mathrm{redDisc}_d^s(H)$ with respect to the unique specialization $A_{\mathbf{Z}} \to \mathbf{k}$ mapping H to h.

Definition (2.10) implies that $\mathrm{redDisc}(H)$ is a primitive polynomial in $A_{\mathbf{Z}}$ (i.e., the greatest common divisor of its coefficients equals 1), because $\mathrm{redRes}(\partial_1 H,\dots,\partial_n H, H)$ is primitive by [4, Proposition 4.24].

The identity (2.10.1) should be compared to (2.5.1). Beware that the reduced discriminant is not merely the reduced resultant of all the partial derivatives, because of the factor $\mathrm{Disc}(F_s)$ in (2.10.1). The factor $\mathrm{Disc}(F_d)$ is an artefact of our definition by considering $\mathrm{redRes}\big(\partial_1 H,\dots,\partial_n H, H\big)$ and not plainly $\mathrm{redRes}\big(\partial_0 H, \partial_1 H,\dots,\partial_n H\big)$, in order to define the reduced discriminant as a primitive polynomial with integer coefficients without dealing with possible constant factors, similar to $d^{a(n,d)}$ in (2.2.1). The factor $\mathrm{Disc}(F_s)$ on the other hand is geometrically meaningful, as we will see in (2.12) below.

According to (2.2.1), the classical discriminant of a homogeneous polynomial can be computed from the resultant of all its partial derivatives, up to the extraneous integer factor $d^{a(n,d)}$. It turns out that a similar formula holds for the reduced discriminant.

(2.11) Proposition. *The following identity in $A_{\mathbf{Z}}$ holds:*

$$d^{a(n,d)-(s-1)^n}\,\mathrm{Disc}(F_s)\,\mathrm{redDisc}(H) = \mathrm{redRes}(\partial_0 H,\dots,\partial_n H). \tag{2.11.1}$$

Proof. To prove this formula, we consider the reduced resultant $\mathcal{R} := \mathrm{redRes}(\partial_1 H,\dots,\partial_n H, dH)$. First, from the homogeneity property (see Theorem (1.15)), we have

$$\mathcal{R} := d^{(d-1)^n-(s-1)^n}\,\mathrm{redRes}(\partial_1 H,\dots,\partial_n H, H).$$

Next, the Euler Identity gives $\mathcal{R} := \mathrm{redRes}(\partial_1 H,\dots,\partial_n H, \sum_{i=0}^n X_i\partial_i H)$, and we claim that

$$\mathcal{R} = \mathrm{redRes}(\partial_1 H,\dots,\partial_n H, X_0\partial_0 H) \tag{2.11.2}$$

$$= (-1)^{n(d-1)^n}\,\mathrm{Res}(\partial_1 F_d,\dots,\partial_n F_d)\,\mathrm{redRes}(\partial_1 H,\dots,\partial_n H,\partial_0 H) \tag{2.11.3}$$

(observe that $\partial_0 H$ has degree $d-1$ and order s in $X_1,\dots,X_n$, and $H|_{X_0=0} = F_d$). We conclude the proof of (2.11.1) before justifying the two above equalities. The Zariski formula together with the property of resultants under permutation of the polynomials [14, §5.8] yields the equality:

$$\mathrm{redRes}(\partial_1 H,\dots,\partial_n H,\partial_0 H) = (-1)^{n(d-1)^{n+1}-ns(s-1)^n}\,\mathrm{redRes}(\partial_0 H,\dots,\partial_n H).$$

In addition, $\mathrm{Res}(\partial_1 F_d,\dots,\partial_n F_d) = d^{a(n-1,d)}\mathrm{Disc}(F_d)$, so putting everything together, including (2.10.1), we deduce that

$$d^{(d-1)^n-(s-1)^n-a(n-1,d)}\mathrm{Disc}(F_s)\,\mathrm{redDisc}(H) = (-1)^N\,\mathrm{redRes}(\partial_0 H,\dots,\partial_n H),$$

where $N = n(d-1)^n + n(d-1)^{n+1} - n(s-1)^n s = nd(d-1)^n - n(s-1)^n s$. Now, N is an even integer, and $(d-1)^n - a(n-1,d) = a(n,d)$, hence the claimed formula.

To justify (2.11.2), we prove the following invariance property of the reduced resultant under some elementary transformations. Let $F_1, \dots, F_{n+1}$ be generic polynomials of degree $d-1, \dots, d-1, d$ respectively, as well as generic linear forms $L_1, \dots, L_n$ in the variables $X_1, \dots, X_n$. Then, from the definition of the reduced resultant as a generator of the ideal of inertia forms in the generic setting, we deduce that

$$\mathrm{redRes}^{s-1,\dots,s-1,s}_{d-1,\dots,d-1,d}(H_1, \dots, H_{n+1}) \text{ divides } \mathrm{redRes}^{s-1,\dots,s-1,s}_{d-1,\dots,d-1,d}(H_1, \dots, H_{n+1} + \sum_{i=1}^{n} L_i H_i).$$

We notice that this divisibility property remains valid under any specialization, in particular under the specialization sending F_{n+1} to $F_{n+1} - \sum_{i=1}^{n} L_i F_i$, and leaving $F_1, \dots, F_n$ invariant. This implies that the above divisibility property also holds in the other direction, hence these two reduced resultant are equal, up to sign. Applying Zariski formula and the invariance of the resultant under elementary transformations [14, §5.9], we deduce that this sign equals 1.

The equality (2.11.3), on the other hand, is a consequence of the following multiplicativity property of the reduced resultant:

$$\begin{aligned}(2.11.4)\quad &\mathrm{redRes}^{s_1,\dots,s_{n+1}}_{d_1,\dots,d_{n+1}+1}(H_1, \dots, H_n, X_0 H_{n+1}) \\ &\qquad = (-1)^{d_1 \cdots d_n} \mathrm{Res}(\bar{F_1}, \dots, \bar{F_n}) \cdot \mathrm{redRes}^{s_1,\dots,s_{n+1}}_{d_1,\dots,d_{n+1}}(H_1, \dots, H_n, H_{n+1}),\end{aligned}$$

which follows from the Zariski formula. Indeed, let $F_1, \dots, F_{n+1}$ be generic polynomials as above, and denote by H_i and G_i their decompositions corresponding to the truncation at order $d_i - s_i$ for all i. On the one hand, we have (observe that F_{n+1} and $X_0 F_{n+1}$ have the same "G", namely G_{n+1}):

$$\begin{aligned}\mathrm{Res}(F_1, \dots, F_n, X_0 F_{n+1}) = \mathrm{Res}(G_1, \dots, G_n, G_{n+1})\, &\mathrm{redRes}(H_1, \dots, H_n, X_0 H_{n+1}) \\ &+ \text{ terms of higher weight.}\end{aligned}$$

On the other hand, by properties of the classical resultant,

$$\mathrm{Res}(F_1, \dots, F_n, X_0 F_{n+1}) = (-1)^{d_1 \dots, d_n} \mathrm{Res}(\bar{F_1}, \dots, \bar{F_n}) \mathrm{Res}(F_1, \dots, F_n, F_{n+1}),$$

hence, by the Zariski formula,

$$\begin{aligned}&\mathrm{Res}(F_1, \dots, F_n, X_0 F_{n+1}) = \\ &(-1)^{d_1 \dots, d_n} \mathrm{Res}(\bar{F_1}, \dots, \bar{F_n}) \mathrm{Res}(G_1, \dots, G_{n+1})\, \mathrm{redRes}(H_1, \dots, H_{n+1}) + \text{ terms of higher weight.}\end{aligned}$$

The comparison of the two above expressions of $\mathrm{Res}(F_1, \dots, F_n, X_0 F_{n+1})$ yields the expected equality. □

(2.12) Vanishing of the reduced discriminant. By definition of the truncation H, the generic hypersurface $\mathrm{V}(H) \subseteq \mathbf{P}^n$ has an ordinary s-fold point at the origin $(1:0:\dots:0)$. By Paragraph (1.16), the vanishing of $\mathrm{redRes}\big(\partial_1 h, \dots, \partial_n h, h\big)$ for some specialization h of H in an algebraically closed field $\mathbf{k}$ corresponds to one of the two following properties:

(a) the existence of a common zero for the polynomials $\partial_1 h, \dots, \partial_n h, h$ which is not the origin;
(b) the existence of a common zero for the polynomials $\partial_1 f_s, \dots, \partial_n f_s, f_s$, equivalently a common root for the polynomials $\partial_1 f_s, \dots, \partial_n f_s$ by Euler Identity, assuming for simplicity that the degree d is non-zero in $\mathbf{k}$.

Property (b) is equivalent to the vanishing of $\mathrm{Disc}(f_s)$, i.e., to the tangent cone of the hypersurface $\mathrm{V}(h)$ at the origin being a cone over a singular degree s hypersurface (with the convention that a hypersurface of degree $s' > s$ is a singular degree s hypersurface); in other words, in the blow-up of $\mathrm{V}(h)$ at the origin, the exceptional divisor is singular. This is equivalent to the singularity of $V(h)$ at the origin being worse than an ordinary s-fold point, but does not imply that the blow-up of $V(h)$ is singular, see Example (2.13) below.

Property (a) is implied by the vanishing of $\mathrm{Disc}(f_d)$, as we have seen in the proof of Proposition (2.9). The latter condition is equivalent to the hypersurface $\mathrm{V}(h)$ being tangent to the hyperplane at infinity $\mathrm{V}(X_0)$.

In Definition (2.10) we discard the two factors $\mathrm{Disc}(F_s)$ and $\mathrm{Disc}(F_d)$; the upshot is that the vanishing of the reduced discriminant $\mathrm{redDisc}(H)$ defines an (irreducible[4]) divisor which has as a dense subset the locus of those hypersurfaces $\mathrm{V}(h)$ that have a singularity off the origin. This implies, since smoothness is an open property, that if $\mathrm{redDisc}(H)$ vanishes at h, then the proper transform[5] of $V(h)$ in the blow-up of the origin is singular, in other words $V(h)$ has an additional singularity, possibly infinitely near, besides its s-fold point at the origin. In fact the converse holds as well, as we shall see in [3]. Thus, $\mathrm{redDisc}(H)$ vanishes at h if and only $V(h)$ has an additional singularity, possibly infinitely near.

Note that we make a difference between having a singularity worse than an ordinary s-fold point and having an additional singularity infinitely near the ordinary s-fold point, see the following example and (2.15) below for more details.

(2.13) Example. Set $n = 2$ and $s = 2$. In this case, $\mathrm{Disc}(F_2)$ vanishes at h if and only if f_2 is a square, equivalently, h may be written as

$$h = X_0^{d-2}\,X^2 + X_0^{d-3}\left(a_3Y^3 + a_2XY^2 + a_1X^2Y + a_0X^3\right) + \cdots$$

after a suitable change of coordinates $(X, Y) = \phi(X_1, X_2)$. Let $\varepsilon : \tilde{\mathbf{P}}^2 \to \mathbf{P}^2$ be the blow-up at the origin. The relevant place to study the proper transform of $V(h)$ over the origin $(1\!:\!0\!:\!0) \in \mathbf{P}^2$ is the affine plane with coordinates (U, Y) in $\tilde{\mathbf{P}}^2$ which dominates the affine plane $(X_0 = 1)$ in $\mathbf{P}^2$, and such that $\varepsilon^*(X, Y) = (UY, Y)$. In this affine chart, the pull-back ε^*h is given by

$$U^2Y^2 + \left(a_3Y^3 + a_2UY^3 + a_1U^2Y^3 + a_0U^3Y^3\right) + \cdots$$

and the proper transform of $V(h)$ is defined by the equation obtained by factoring out Y^2 from ε^*h, i.e.,

$$U^2 + \left(a_3Y + a_2UY + a_1U^2Y + a_0U^3Y\right) + \cdots$$

This shows that the proper transform of $V(h)$ is singular at some point lying on the exceptional divisor of ε (defined by $Y = 0$ in this affine chart) if and only if $a_3 = 0$.

The generic h such that $\mathrm{Disc}(F_2)$ vanishes defines a plane curve with an ordinary cusp at the origin (local equation $x^2 + y^3 = 0$); the proper transform of this curve in the blow-up at the origin is smooth (and tangent to the exceptional divisor), hence $\mathrm{redDisc}(H)$ does not vanish at h even though $V(h)$ has a singularity worse than an ordinary double point at the origin. On the other hand, the generic h such that $V(h)$ has an additional singularity infinitely near the origin (which implies that $\mathrm{redDisc}(H)$ vanishes) defines a plane curve with a tacnode at the

[4] the proof of the irreducibility of the reduced discriminant will appear in [3], but this property also follows from the theory of toric discriminants (see [11, Chapter 9]), of which the reduced discriminants are particular instances.

[5] here we take the following slightly abusive definition for the proper transform: let $\varepsilon : \tilde{\mathbf{P}}^n \to \mathbf{P}^n$ be the blow-up at the origin, with exceptional divisor E; the *proper transform* of $V(h)$ is the divisor $\varepsilon^*V(h) - sE$. If $V(h)$ has an ordinary $(s+1)$-fold at the origin, its genuine proper transform is $\hat{V}(h) = \varepsilon^*V(h) - (s+1)E$, which is smooth along E, whereas its proper transform in our sense is $\hat{V}(h) + E$, which is singular along E.

origin (local equation $x^2+y^4=0$, two smooth local branches tangent to each other); the proper transform of such a curve has an ordinary double point located on the exceptional divisor.

(2.14) Remark. In the reduced resultant situation, the condition that the $n+1$ hypersurfaces $\mathrm{V}(h_1),\dots,\mathrm{V}(h_{n+1})$ have an infinitely near additional intersection point is not divisorial, i.e., it does not correspond to an irreducible element factoring out of $\mathrm{redRes}(H_1,\dots,H_{n+1})$. The reason is that this condition has codimension larger than one: indeed it amounts to the $n+1$ leading terms $f_{1,s_1},\dots,f_{n+1,s_{n+1}}$ having a non-trivial common zero, in other words the $n+1$ corresponding hypersurfaces in $\mathbf{P}^{n-1}$ have non-trivial intersection: these are $n+1$ hypersurfaces in $\mathbf{P}^{n-1}$, hence the condition has codimension larger than one.

In contrast, in the reduced discriminant situation the condition that the hypersurface $\mathrm{V}(h)$ has a singularity worse than an ordinary s-fold point is divisorial, as it corresponds to the vanishing of $\mathrm{Disc}(F_s)$ at the leading term f_s of h. The vanishing of $\mathrm{redDisc}(H)$ at h is not implied by the latter condition, but by the stronger condition that $V(h)$ has an additional singularity infinitely near its s-fold point.

(2.15) We may use the Milnor number (see [II, Section 1] for the definition) to give a precise meaning to the condition of having a singlarity worse than an ordinary s-fold point. Assume that h defines a hypersurface with an isolated singularity at the origin, so that its Milnor number at the origin $\mu_0(h)$ is well-defined. If the singularity of h at the origin is an ordinary s-fold point, then $\mu_0(h)=(s-1)^n$, and condition (b) in Paragraph (2.12) above is equivalent to the condition that $\mu_0(h)>(s-1)^n$ by [10, Theorem 1].

(2.16) Lemma. *In the standard grading of $A_{\mathbf{Z}}$, given by* $\mathrm{weight}(U_{\boldsymbol{\alpha}})=1$ *for all $\boldsymbol{\alpha}$, the reduced discriminant* $\mathrm{redDisc}_d^s$ *is homogeneous of degree*

$$(n+1)\big[(d-1)^n-(s-1)^n\big]-2n(s-1)^{n-1}. \tag{2.16.1}$$

Proof. This follows from the definition by a direct computation, since the degrees of the other quantities in Display (2.10.1) are known. It is arguably safer, however, to carry this out explicitly.

The two plain discriminants $\mathrm{Disc}(F_d)$ and $\mathrm{Disc}(F_s)$ are homogeneous of respective degrees $n(d-1)^{n-1}$ and $n(s-1)^{n-1}$, see Definition (2.2). Moreover, it follows from Theorem (1.15) that

$$\mathrm{redRes}^{s-1,\dots,s-1,s}_{d-1,\dots,d-1,d}(\partial_1 H,\dots,\partial_n H,H)$$

is homogeneous of degree

$$n\big(d(d-1)^{n-1}-s(s-1)^{n-1}\big)+\big((d-1)^n-(s-1)^n\big).$$

It thus follows from the equality in Display (2.10.1) that $\mathrm{redDisc}(H)$ is homogeneous, of degree

$$\begin{aligned}\big[nd(d-1)^{n-1}+(d-1)^n-n(d-1)^{n-1}\big]-\big[ns(s-1)^{n-1}+(s-1)^n+n(s-1)^{n-1}\big]\\ =(n+1)(d-1)^n-(s-1)^{n-1}\big((n+1)s+n-1\big),\end{aligned}$$

and the result follows. □

We now turn to the generalized weight properties of the reduced discriminant.

(2.17) Example. As an illustrative example, we consider the case $n=1$ and set

$$F=\sum\nolimits_{i=0}^{d}U_iX_0^{d-i}X_1^i=X_0^dF_0+X_0^{d-1}F_1+\cdots+X_0^{d-s}F_s+\cdots F_d,$$

with $F_i = U_i X_1^i$ for all $i = 0, \dots, d$. We consider

$$H = \sum\nolimits_{i=s}^{d} U_i X_0^{d-i} X_1^i = \sum\nolimits_{i=s}^{d} X_0^{d-i} F_i = X_1^s \big(U_s X_0^{d-s} + U_{s+1} X_0^{d-s-1} X_1 + \cdots + U_d X_1^{d-s}\big),$$

the truncation of F at order $d - s > 0$ with respect to X_0. Then, setting $H = X_1^s \cdot {}^\flat H$ it is easy to check that $\operatorname{redDisc}_d^s(H) = \pm \operatorname{Disc}({}^\flat H)$; it is therefore an irreducible polynomial of degree $2(d - s - 1)$ in the coefficients of H (compare with Formula (2.16.1) for the degree).

Moreover, using the weight properties in Proposition (2.6), we can deduce weight properties of $\operatorname{redDisc}(H)$. Suppose that $A_{\mathbf{Z}}$ is graded with the rule $\operatorname{weight}(U_i) = \max(0, i - s)$; then the reduced discriminant $\operatorname{redDisc}_d^s(H)$ is homogeneous of degree

$$(d - s)(d - s - 1) \tag{2.17.1}$$

by Proposition (2.6), as ${}^\flat H$ has degree $d - s$. Similarly, if $A_{\mathbf{Z}}$ is graded with the rule $\operatorname{weight}(U_i) = d - i$ then the same conclusion holds (note that $d - i = (d - s) - (i - s)$).

We can generalize this following Corollary (1.12). Let r be an integer and consider the grading of $A_{\mathbf{Z}}$ defined by the rule $\operatorname{weight}(U_i) = i - s + r$ if $i \geqslant s$, and $\operatorname{weight}(U_i) = 0$ otherwise, then $\operatorname{redDisc}_d^s(H)$ is homogeneous of degree $(d - s + 2r)(d - s - 1)$. In particular, if $r = s$, i.e., $\operatorname{weight}(U_i) = i$ if $i \geqslant s$ and $\operatorname{weight}(U_i) = 0$ otherwise, we get that $\operatorname{redDisc}_d^s(H)$ is homogeneous of degree

$$(d + s)(d - s - 1) = d(d - 1) - s(s + 1). \tag{2.17.2}$$

In fact, since $\operatorname{redDisc}_d^s(H)$ a polynomial in $U_s, \dots, U_d$ only, the weights of $U_0, \dots, U_{s-1}$ don't matter. Similarly, in the grading of $A_{\mathbf{Z}}$ is defined by the rule $\operatorname{weight}(U_i) = d - i + r$ if $i \geqslant s$ and $\operatorname{weight}(U_i) = 0$ otherwise, $\operatorname{redDisc}_d^s(H)$ is homogeneous of degree $(d - s + 2r)(d - s - 1)$.

The following result is similar to Theorem (1.19). It is the key to the generalized Salmon formula for the discriminant.

(2.18) Theorem ([4, Theorem 4.25])**.** *Suppose that the ring $A_{\mathbf{Z}}$ is graded by means of the Zariski grading (1.17), i.e.,* $\operatorname{weight}(U_{\boldsymbol{\alpha}}) = \max(\alpha_0 - d + s, 0)$. *Then* $\operatorname{Disc}(F)$ *has order $s(s-1)^n$, and its homogeneous part D_0 in this degree satisfies the following equality in $A_{\mathbf{Z}}$:*

$$D_0 \operatorname{Disc}(F_d) = \operatorname{Disc}(G) \operatorname{Disc}(F_s) \operatorname{redRes}\big(\partial_1 H, \dots, \partial_n H, H\big). \tag{2.18.1}$$

Let us point out that the three elements $\operatorname{Disc}(F_d)$, $\operatorname{Disc}(F_s)$, and $\operatorname{redRes}(\partial_1 H, \dots, \partial_n H, H)$ have degree 0 with respect to the Zariski grading, while $\operatorname{Disc}(G)$ is homogeneous of degree $s(s-1)^n$ by (2.6.1); note that F_d and F_s are generic homogeneous polynomials of respective degrees d and s in the indeterminates $(X_1, \dots, X_n)$, while G is generic homogeneous of degree s in $(X_0, \dots, X_n)$.

(2.19) Remark. The only trace towards the reduced discriminant in Salmon's work that we have found is in [18, §117], where he proves, in the above language, that for $s = 2$, $\operatorname{Disc}(F)$ has order at least 2.

He obtains this as a direct consequence of the fact that if f is a polynomial which is singular at a point x, then the tangent space of the discriminant hypersurface V(Disc) at $[f]$ contains the hyperplane of polynomials vanishing at x. Thus, for the polynomial

$$f = \sum_{a>1} F_a X_0^{d-a} + T \cdot \big(U_1' X_1 + \cdots + U_n' X_n\big) X_0^{d-1},$$

one finds that $\mathrm{Disc}(f)$ is divisible by T^2 (here, F is as in (2.1) and (2.8), and $T, U'_1, \dots, U'_n$ are new indeterminates). It follows that $\mathrm{Disc}(F)$ sits in the ideal

$$\big(U_{d,0,\dots,0}\big) + \big(U_{d-1,1,\dots,0},\ \dots\ , U_{d-1,0,\dots,1}\big)^2 \subseteq A_{\mathbf{Z}},$$

which means that it has order at least 2 with respect to the Zariski grading for $s = 2$.

Proof of Theorem (2.18). The idea is to specialize Theorem (1.19), part (a), and in particular the formula

$$\big[\mathrm{Res}(F_1, \dots, F_{n+1})\big]_{s_1 s_2 \cdots s_{n+1}} = \mathrm{Res}(G_1, \dots, G_{n+1}) \cdot \mathrm{redRes}(H_1, \dots, H_{n+1}),$$

to the discriminant situation, i.e., $F_1, \dots, F_{n+1}$ specialize to $\partial_1 F, \dots, \partial_n F, F$ respectively, and the G_j's and H_j's specialize according to the truncations corresponding to $s_1, \dots, s_{n+1} = s - 1, \dots, s-1, s$, as indicated in Paragraph (2.8). Thus, we consider the following specializations:

$$\begin{aligned}
\mathrm{Res}(F_1, \dots, F_{n+1}) &\rightsquigarrow \mathrm{Res}(\partial_1 F, \dots, \partial_n F, F) \\
\mathrm{Res}(G_1, \dots, G_{n+1}) &\rightsquigarrow \mathrm{Res}(\partial_1 G, \dots, \partial_n G, G) \\
\mathrm{redRes}(H_1, \dots, H_{n+1}) &\rightsquigarrow \mathrm{redRes}(\partial_1 H, \dots, \partial_n H, H)
\end{aligned}$$

On the other hand, we have

$$\mathrm{Res}(\partial_1 F, \dots, \partial_n F, F) = \mathrm{Disc}(F)\,\mathrm{Disc}(\bar{F}) \tag{2.19.1}$$

$$\mathrm{Res}(\partial_1 G, \dots, \partial_n G, G) = \mathrm{Disc}(G)\,\mathrm{Disc}(\bar{G}) \tag{2.19.2}$$

by Proposition (2.5). By definition, $\bar{F} = F_d$ and $\bar{G} = G_s = F_s$, and the result follows. We refer to [4] for more details. □

The following corollary is the general form of Salmon's formula. As we will see in the next section, Salmon's formula is a particular form of the corollary in the case $s = 2$.

(2.20) Corollary. *In the Zariski grading of $A_{\mathbf{Z}}$ as in Theorem (2.18), the discriminant* $\mathrm{Disc}(F)$ *has order $s(s-1)^n$, and can be written as*

$$\mathrm{Disc}(F) = \mathrm{Disc}(G)\,\mathrm{Disc}(F_s)^2\ \mathrm{redDisc}(H) + \big(\text{terms of Zariski weight} > s(s-1)^n\big). \tag{2.20.1}$$

Proof. The only novelty with respect to the previous theorem is the expression for D_0 (in the notation of Theorem (2.18)) in terms of the reduced discriminant. By Definition (2.10), one has

$$\mathrm{redRes}\big(\partial_1 H, \dots, \partial_n H, H\big) = \mathrm{Disc}(F_d)\,\mathrm{Disc}(F_s)\ \mathrm{redDisc}(H).$$

Thus, Equation (2.18.1) writes

$$D_0\,\mathrm{Disc}(F_d) = \mathrm{Disc}(G)\,\mathrm{Disc}(F_s)\,\mathrm{Disc}(F_d)\,\mathrm{Disc}(F_s)\ \mathrm{redDisc}(H)$$

which gives

$$D_0 = \mathrm{Disc}(G)\,\mathrm{Disc}(F_s)^2\,\mathrm{redDisc}(H)$$

as we wanted. □

We point out that, in the above, the factor $\mathrm{Disc}(F_s)$ comes once from Formula (2.18.1), where it came from Formula (2.19.2) as $\mathrm{Disc}(\bar{G})$, and once from the definition of $\mathrm{redDisc}(H)$, which explains the square in Formula (2.20.1). In the discussion following Definition (2.10) above, we explained that, in that definition, $\mathrm{Disc}(F_s)$ is a geometrically meaningful term, as F_s is the

equation of the tangent cone of $\mathrm{V}(H)$ at the origin, whereas $\mathrm{Disc}(F_d)$ is a technical artefact. It is thus fortunate that the latter term does not appear in Formula (2.20.1). Note that, indeed, the factor $\mathrm{Disc}(F_d)$ in the definition of $\mathrm{redDisc}(H)$ cancels out with the same factor coming from Formula (2.19.1) as $\mathrm{Disc}(\bar{F})$.

Corollary (2.20) provides an interesting connection between ordinary and reduced discriminants. As a first illustration of its interest, we give the following generalized weight properties of the reduced discriminant, which generalize the computations of Example (2.17) to arbitrary n.

(2.21) Proposition.
(2.21.1) In the grading of $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = \alpha_0$, *the reduced discriminant* $\mathrm{redDisc}_d^s(F)$ *is homogeneous of degree*

$$d(d-1)^n - (s-1)^{n-1}\big[d\big((n+1)s+n-1\big) - ns(s+1)\big].$$

(2.21.2) In the grading of $A_{\mathbf{Z}}$ defined by $\mathrm{weight}(U_{\boldsymbol{\alpha}}) = d-\alpha_0$, *the reduced discriminant* $\mathrm{redDisc}_d^s(F)$ *is homogeneous of degree*

$$n\big[d(d-1)^n - s(s+1)(s-1)^{n-1}\big].$$

Proof. In the two gradings of (2.21.1) and (2.21.2), the plain discriminant $\mathrm{Disc}(F)$ is homogeneous. Thus all its summands are homogeneous, and in particular its piece of lowest degree in the Zariski grading is homogeneous. We shall see that $\mathrm{Disc}(G)$ and $\mathrm{Disc}(F_s)$ are homogeneous as well. Therefore, it follows from Formula (2.20.1) that $\mathrm{redDisc}(H)$ is homogeneous. Then it is only a matter of computing its degree.

In the grading of (2.21.1), $\mathrm{Disc}_d(F)$ has weight $d(d-1)^n$ by Proposition (2.6). One has $G = X_0^s F_0 + X_0^{s-1}F_1 + \cdots + F_s$ and the coefficients of F_i have weight $d-i = (s-i)+(d-s)$, hence $\mathrm{Disc}_d(G)$ has weight

$$s(s-1)^n + (n+1)(d-s)(s-1)^n$$

by Proposition (2.6) and Corollary (2.7). The coefficients of F_s all have weight $d-s$, hence $\mathrm{Disc}_s(F_s)$ has weight

$$n(d-s)(s-1)^{n-1}$$

by the homogeneity property of the discriminant in Definition (2.2) and Corollary (2.7) (note that $\mathrm{Disc}_s(F_s)$ is a discriminant for polynomials in n indeterminates, i.e., one less than $\mathrm{Disc}_d(F)$ and $\mathrm{Disc}_d(G)$). One thus finds, by the above Corollary (2.20), that $\mathrm{redDisc}_d^s(F)$ is homogeneous of degree

$$\begin{aligned} d(d-1)^n - s(s-1)^n - (n+1)(d-s)(s-1)^n - 2n(d-s)(s-1)^{n-1} \\ &= d(d-1)^n - (s-1)^{n-1}\big[s(s-1) + (n+1)(d-s)(s-1) + 2n(d-s)\big] \\ &= d(d-1)^n - (s-1)^{n-1}\big[d\big((n+1)s+(n-1)\big) - ns(s+1)\big]. \end{aligned}$$

In the grading of (2.21.2), $\mathrm{Disc}(F)$ is homogeneous of degree $nd(d-1)^n$ by Proposition (2.6), and $\mathrm{Disc}(G)$ is homogeneous of degree $ns(s-1)^n$ by the same result. In addition, all the coefficients of F_s have weight s in this grading hence, by the homogeneity property given in Definition (2.2), $\mathrm{Disc}(F_s)$ is homogeneous of degree $ns(s-1)^{n-1}$ (note that F is a homogeneous polynomial in n variables only). Then the conclusion follows by a direct computation. □

For $n = 1$, the degree in (2.21.1) is

$$d(d-1) - \big[2sd - s(s+1)\big] = d^2 - (2s+1)d + s(s+1),$$

which agrees with (2.17.1), and the degree in (2.21.2) agrees with (2.17.2).

In the above paragraphs we have introduced the reduced discriminant and provided some first properties that are sufficient for our purposes. A more detailed and complete study of this new eliminant polynomial, including for instance its irreducibility and the geometric meaning of its vanishing, in particular its connection to Milnor number, will appear in [3].

2.3 – Application to the Salmon Formula

We shall now see that Salmon's formula (🔈) is a particular case of the decomposition formula given in Corollary (2.20). We will then be able to generalize it to the case of a hypersurface in arbitrary dimension.

(2.22) Original Salmon Formula. Let $F(X, Y, Z)$ be the generic homogeneous polynomial of degree $d \geqslant 3$, and set

$$F = X^d F_0 + X^{d-1} F_1 + X^{d-2} F_2 + \cdots + X F_{d-1} + F_d,$$

$$F_0 = U, \qquad F_1(Y, Z) = SY + TZ, \qquad F_2(Y, Z) = \frac{1}{2}(AY^2 + 2BYZ + CZ^2);$$

(we introduce a rational number in the definition of F_2 to follow Salmon's notation; this is natural in terms of the Taylor–Newton Formula of [XII, Paragraph (A.3)]).

We consider the truncation of F at order $d-2$ with respect to X, and the corresponding Zariski grading. So, U has degree 2, S and T have degree 1, and the coefficients of the F_k's with $k \geqslant 2$ have degree 0 (this includes the coefficients A, B, C). We let

$$H = X^{d-2} F_2 + \cdots + X F_{d-1} + F_d \quad \text{and} \quad G = X^2 F_0 + X F_1 + F_2.$$

Then, Corollary (2.20) tells us that

$$\operatorname{Disc}(F) = \operatorname{Disc}(G) \operatorname{Disc}(F_2)^2 \operatorname{redDisc}(H) + \big(\text{terms of Zariski weight} \geqslant 3\big). \tag{2.22.1}$$

Discriminants of quadratic forms are readily computed, see, e.g., [7, n°5 Exemple 6, p. 363], which is based on Identity (2.2.1); one has

$$\operatorname{Disc}(F_2) = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2,$$

and

$$\operatorname{Disc}(G) = \frac{1}{2} \cdot \begin{vmatrix} 2U & S & T \\ S & A & B \\ T & B & C \end{vmatrix} = \frac{1}{2}\left(2ACU - AT^2 - 2B^2U + 2BST - CS^2\right).$$

Thus, Display (2.22.1) reads

$$\operatorname{Disc}(F) = \frac{1}{2}\left(2ACU - AT^2 - 2B^2U + 2BST - CS^2\right) \cdot \left(AC - B^2\right)^2 \cdot \operatorname{redDisc}(H) \mod \big((S,T)^3 + U(S,T) + (U^2)\big).$$

Then, the specialization $U = S = 0$ yields

$$\operatorname{Disc}(F) = -\frac{1}{2} A T^2 \cdot \left(AC - B^2\right)^2 \cdot \operatorname{redDisc}(H) \mod (T^3); \tag{2.22.2}$$

this is Salmon's Formula (🔈), with the normalization factor $-\frac{1}{2}$, and with $\Phi = \operatorname{redDisc}(H)$.

(2.23) Salmon Formula in arbitrary dimension. The Salmon Formula for the discriminant of a plane curve can be generalized to the case of a hypersurface in a projective space of arbitrary dimension as follows. In a suitable system of homogeneous coordinates, any hypersurface $\mathrm{V}(F) \subseteq \mathbf{P}^n$ has an equation of the form

$$F(X_0, X_1, \dots, X_n) = TX_0^{d-1}X_n + \sum_{k=2}^{d} X_0^{d-k} F_k(X_1, \dots, X_n), \tag{2.23.1}$$

where for all $k = 2, \dots, d$, the polynomial F_k is homogeneous of degree k in the indeterminates $X_1, \dots, X_n$. This normal form merely imposes that the hypersurface $\mathrm{V}(F)$ goes through the point $(1 : 0 : \cdots : 0)$ and that its tangent hyperplane at this point is given by $X_n = 0$. Applying Corollary (2.20) as above, and setting $H = \sum_{k=2}^{d} X_0^{d-k} F_k$, we find that

$$\mathrm{Disc}(F) = \mathrm{Disc}(TX_0X_n + F_2)\,\mathrm{Disc}(F_2)^2\,\mathrm{redDisc}(H) \mod T^3.$$

Let $\bar{F}_2(X_1, \dots, X_{n-1})$ be the homogeneous polynomial of degree 2 in the indeterminates $X_1, \dots, X_{n-1}$ defined as

$$\bar{F}_2(X_1, \dots, X_{n-1}) = F_2(X_1, \dots, X_{n-1}, 0)$$

(beware the difference in notation with (2.5)). We have:

$$\begin{aligned}
2^{a(n,2)}\,&\mathrm{Disc}(TX_0X_n + F_2) \\
&= \mathrm{Res}\big(\partial_0(TX_0X_n + F_2),\, \partial_1(TX_0X_n + F_2),\, \dots,\, \partial_n(TX_0X_n + F_2)\big) \\
&= \mathrm{Res}\big(TX_n,\, \partial_1F_2,\, \dots,\, \partial_{n-1}F_2,\, TX_0 + \partial_nF_2\big) \\
&= -\mathrm{Res}\big(TX_0 + \partial_nF_2,\, \partial_1F_2,\, \dots,\, \partial_{n-1}F_2,\, TX_n\big) && (2.23.2) \\
&= -T \cdot \mathrm{Res}\big(TX_0 + \partial_nF_2|_{X_n=0},\, \partial_1F_2|_{X_n=0},\, \dots,\, \partial_{n-1}F_2|_{X_n=0}\big) && (2.23.3) \\
&= -T^2 \cdot \mathrm{Res}(\partial_1\bar{F}_2,\, \dots,\, \partial_{n-1}\bar{F}_2) = -2^{a(n-2,2)}\,T^2 \cdot \mathrm{Disc}(\bar{F}_2), && (2.23.4)
\end{aligned}$$

where:
— the exponents $a(n, 2)$ and $a(n-2, 2)$ are as defined in (2.2);
— (2.23.2) is obtained by [14, §5.8, Effet d'une permutation des polynômes];
— (2.23.3) is obtained by [14, (5.13.5)];
— (2.23.4) is obtained by [14, §5.10, Formule de Laplace].
Alternatively, since the resultants above are resultants of homogeneous polynomials of degree 1 they may be computed as determinants, see [14, §5.3, Cas des formes linéaires], to the effect that

$$\mathrm{Res}\big(TX_n,\, \partial_1F_2,\, \dots,\, \partial_{n-1}F_2,\, TX_0 + \partial_nF_2\big) = \begin{vmatrix} 0 & 0 & \cdots & \cdots & 0 & T \\ 0 & [& & \partial_1F_2 & &] \\ \vdots & & & \vdots & & \\ 0 & [& & \partial_{n-1}F_2 & &] \\ T & [& & \partial_nF_2 & &] \end{vmatrix}, \tag{2.23.5}$$

where each "$[\ \partial_iF_2\ ]$" denotes the line containing the n coefficients of ∂_iF_2 in the indeterminates $X_1, \dots, X_n$; from formula (2.23.5) one may easily retrace the previous computations.

Either way, the upshot is that $\mathrm{Disc}(TX_0X_n + F_2) = -T^2\,\mathrm{Disc}(\bar{F}_2)$ (note that $a(n, 2) = a(n-2, 2)$), and eventually we obtain the following generalized Salmon formula:

$$\mathrm{Disc}(F) = -T^2\,\mathrm{Disc}(\bar{F}_2)\,\mathrm{Disc}(F_2)^2\,\mathrm{redDisc}(H) \mod T^3. \tag{2.23.6}$$

In the situation of the original Salmon Formula above, one has $F_2(Y, Z) = \frac{1}{2}(AY^2 + 2BYZ + CZ^2)$ and $\bar{F}_2(Y) = \frac{1}{2}AY^2$, hence $\mathrm{Disc}(F_2) = AC - B^2$ and $\mathrm{Disc}(\bar{F}_2) = \frac{1}{2}A$, and thus the two formulas (2.22.2) and (2.23.6) are coherent.

3 – Computation of the node-couple degree by elimination

In [17, §605–607], Salmon sets up the following strategy to compute the number of 2-nodal curves in a general net of hyperplane sections of a smooth (hyper)surface $S \subseteq \mathbf{P}^3$. For $p' \in S$ and $p'' \in \mathbf{T}_{p'}S - \{p'\}$, consider the pencil $\langle p', p''\rangle^\perp \subseteq \check{\mathbf{P}}^3$ of (hyper)planes containing p' and p''. It cuts out $d^\vee = \deg(S^\vee)$ points on the dual surface $S^\vee$, counted with multiplicities, which correspond to planes tangent to S. Among these, $\mathbf{T}_{p'}S$ counts doubly if it is a plain tangent plane, and triply if it is plainly bitangent.

Indeed, the line $\langle p', p''\rangle^\perp$ is contained in the plane $(p')^\perp$, hence tangent to $S^\vee$ at the point $(\mathbf{T}_{p'}S)^\perp \in S^\vee$. If the plane $\mathbf{T}_{p'}S$ is plainly bitangent to S, then $(\mathbf{T}_{p'}S)^\perp$ is a general point on the ordinary double curve of $S^\vee$, and the line $\langle p', p''\rangle^\perp$ is tangent to one of the two transverse sheets of $S^\vee$ at $(\mathbf{T}_{p'}S)^\perp$. The idea is then to determine first the conditions on p' for $\mathbf{T}_{p'}S$ to count with multiplicity greater than 2 in $\langle p', p''\rangle^\perp$, and then to sort out the various corresponding geometric situations. A key element to carry this out is the famous formula (🔏); another one is the elimination procedure (3.2).

We work out Salmon's procedure in subsection 3.2, and in subsection 3.3 we show how it carries over for hypersurfaces in a projective space of arbitrary dimension. From now on, we work over an algebraically closed field $\mathbf{k}$ of characteristic 0. We use freely the theory of polarity, on which there is a recap in the previous chapter [XII, Appendix A].

3.1 – An elimination trick

This section is dedicated to an elimination trick due to Salmon, see [17, §606]. It is the main technical device that he uses to conduct his study of pencils of planes orthogonal to a tangent line, which eventually gives the node-couple degree. We advise the reader to skip this section in first reading and move on to Section 3.2 where the general picture of Salmon's approach is given, and to come back here only when needed.

(3.1) Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, and consider a smooth, degree d, surface $S \subseteq \mathbf{P}^3$ defined by a homogeneous polynomial $F \in \mathbf{k}[X, Y, Z, W]$; set

$$\mathcal{T}_S = \big\{(p', p'') : p' \in S \text{ and } p'' \in \mathbf{T}_{p'}S\big\} \subseteq \mathbf{P}^3 \times \mathbf{P}^3.$$

Let $\Phi_{\hat{p}', \hat{p}''}(\hat{p})$ be a trihomogeneous polynomial in the sets of variables $(\hat{p}', \hat{p}'', \hat{p}) \in (\mathbf{k}^4)^3$, of tridegree (λ, μ, μ) for some non-negative integers λ, μ. Assume that for all $(p', p'') \in \mathcal{T}_S$, $p' \neq p''$, the hypersurface $\mathrm{V}(\Phi_{\hat{p}', \hat{p}''}) \subseteq \mathbf{P}^3$ consists of μ planes, counted with multiplicities, all containing the line $\langle p', p''\rangle$.

(3.2) Theorem. *Let p' be a point of S. The following assertions are equivalent:*
(i) there exists $p'' \in \mathbf{T}_{p'}S - \{p'\}$ such that the tangent plane $\mathbf{T}_{p'}S$ is a component of $\mathrm{V}(\Phi_{\hat{p}', \hat{p}''})$;
(ii) for all $p'' \in \mathbf{T}_{p'}S - \{p'\}$, $\mathbf{T}_{p'}S$ is a component of $\mathrm{V}(\Phi_{\hat{p}', \hat{p}''})$.
Moreover, there exists a homogeneous polynomial $G \in \mathbf{k}[X, Y, Z, W]$ of degree $\lambda + (d-2)\mu$ such that (i) and (ii) are equivalent to:
(iii) p' lies on the hypersurface $\mathrm{V}(G) \subseteq \mathbf{P}^3$.

We need some preparation for the proof. We begin with the following characterization of the non-emptiness of the intersection of two lines in $\mathbf{P}^3$, one given parametrically and the other defined by equations.

(3.3) Lemma. *Let $L', L'' \in \mathbf{k}[X,Y,Z,W]$ be two linear functionals, and $\hat{p}', \hat{p}'' \in \mathbf{k}^4$. The intersection of the two lines $\mathrm{V}(L', L'')$ and $\langle p', p''\rangle$ is non-empty if and only if*

$$D_{L',L''}(p',p'') \stackrel{def.}{=} \begin{vmatrix} L'(\hat{p}') & L''(\hat{p}') \\ L'(\hat{p}'') & L''(\hat{p}'') \end{vmatrix}$$

vanishes.

Proof. The line $\langle p', p''\rangle$ is the image of the map $\mathbf{P}^1 \to \mathbf{P}^3$ defined by the linear map $(u,v) \in \mathbf{k}^2 \mapsto u\hat{p}' + v\hat{p}'' \in \mathbf{k}^4$. Therefore, the lines $\mathrm{V}(L', L'')$ and $\langle p', p''\rangle$ intersect if and only if the two polynomials $L'(u\hat{p}' + v\hat{p}'')$ and $L''(u\hat{p}' + v\hat{p}'')$ share a common root in $\mathbf{P}^1$. These polynomials are linear forms in u, v, and the polynomial $D_{L',L''}(p', p'')$ is the determinant of the corresponding linear system. □

For all L', L'', $D_{L',L''}$ is a bihomogeneous polynomial in the two sets of variables (p', p''), of bidegree $(1,1)$ and anti-symmetric; this implies that it is irreducible.

(3.4) Lemma. *For general linear forms L', L'', the intersection in $\mathbf{P}^3 \times \mathbf{P}^3$ of $\mathcal{T}_S$ and the hypersurface $\mathrm{V}(D_{L',L''})$ is integral.*

Proof. A natural idea is to try and prove the lemma as an application of Bertini's theorem. It is however delicate to proceed this way, as for instance the map defined on $\mathbf{P}^3 \times \mathbf{P}^3$ by the linear system of anti-symmetric forms of bidegree $(1,1)$ is certainly not generically finite, and even less so the map defined by the system of all $D_{L',L''}$. We will thus rather prove the result by hand.

Let us first note that as L', L'' range through all linear forms, the bihomogeneous forms $D_{L',L''}$ cover all rank 2, bilinear, anti-symmetric forms. This is a direct calculation: for

$$L' = A'X + B'Y + C'Z + D'W \quad \text{and} \quad L'' = A''X + B''Y + C''Z + D''W,$$

$D_{L',L''}$ is the bilinear form

$$\left(\begin{pmatrix} X' \\ Y' \\ Z' \\ W' \end{pmatrix}, \begin{pmatrix} X'' \\ Y'' \\ Z'' \\ W'' \end{pmatrix} \right) \longmapsto (X', Y', Z', W') \begin{pmatrix} 0 & U_{01} & U_{02} & U_{03} \\ -U_{01} & 0 & U_{12} & U_{13} \\ -U_{02} & -U_{12} & 0 & U_{23} \\ -U_{03} & -U_{13} & -U_{23} & 0 \end{pmatrix} \begin{pmatrix} X'' \\ Y'' \\ Z'' \\ W'' \end{pmatrix}$$

where the coefficients U_{ij} are the six 2×2 minors of the matrix

$$\begin{pmatrix} A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \end{pmatrix}.$$

It follows that the only relation between the coefficients U_{ij} is the vanishing of the Pfaffian of the anti-symmetric matrix (U_{ij}), i.e.,

$$U_{01}U_{23} - U_{02}U_{13} + U_{03}U_{12} = 0$$

(this is the Plücker equation of the Grassmannian $\mathrm{Gr}(2,4)$ in $\mathbf{P}^5$).

Now to the lemma. Projection on the first factor in $\mathbf{P}^3 \times \mathbf{P}^3$ makes $\mathcal{T}_S$ a $\mathbf{P}^2$ bundle over S: for all $p' \in S$, the fibre over p' is the zero locus of the linear form $\mathrm{D}_{p'}F$ (the differential of F, a defining equation for S, at the point p'; it is non-zero as we assume that S is smooth). Similarly, for all $p' \in \mathbf{P}^3$ the fibre $\mathrm{pr}_1^{-1}(p') \cap \mathrm{V}(D_{L',L''})$ is the zero locus of the linear form $D_{L',L''}(p', \_)$, which is either a hyperplane or the whole $\mathbf{P}^3$. We claim that for general L', L'', the intersection $\mathcal{T}_S \cap \mathrm{V}(D_{L',L''})$ is the closure of a $\mathbf{P}^1$-bundle over an open subset of S, which implies at once that

it is irreducible. Moreover, this implies that the intersection $\mathcal{T}_S \cap \mathrm{V}(D_{L',L''})$ is fibrewise over S a transverse intersection of hyperplanes, and therefore it is reduced. Thus the claim proves the lemma. To prove the claim it suffices to show that, for general L', L'', the set of points $p' \in S$ such that the linear form $D_{L',L''}(p', \_)$ is a multiple of $\mathrm{D}_{p'}F$ has dimension 0; in particular, the intersection $\mathcal{T}_S \cap \mathrm{V}(D_{L',L''})$ cannot have any irreducible component generically a $\mathbf{P}^2$-bundle over a curve in S.

The claim follows from explicit computations. The two linear forms $\mathrm{D}_{p'}(F)$ and $D_{L',L''}(p', \_)$ are, in coordinates,

$$\big(\partial_X F(p'),\, \partial_Y F(p'),\, \partial_Z F(p'),\, \partial_W F(p')\big) \quad \text{and} \quad \begin{pmatrix} X'' \\ Y'' \\ Z'' \\ W'' \end{pmatrix}^{\mathsf{T}} \cdot \begin{pmatrix} 0 & U_{01} & U_{02} & U_{03} \\ -U_{01} & 0 & U_{12} & U_{13} \\ -U_{02} & -U_{12} & 0 & U_{23} \\ -U_{03} & -U_{13} & -U_{23} & 0 \end{pmatrix}.$$

Both have the line $p' \subseteq \mathbf{k}^4$ in their kernel, so we consider the corresponding linear forms on the quotient $\mathbf{k}^4/p'$, given by line matrices with three entries; up to a linear change of coordinates we may assume that $p' = (0:0:0:1)$, in which case we are simply forgetting the last entry in the two above line matrices. The claim then boils down to the fact that the three 2×2 minors of the 2×3 matrix obtained by concatenating the two line matrices corresponding to $\mathrm{D}_{p'}(F)$ and $D_{L',L''}(p', \_)$ define a locus of codimension at least two on S, for general L', L''. Since everything is explicit, one may observe that for a general choice of L', L'', any two of these three minors have no common factor, which proves the claim. □

(3.5) Proof of Theorem (3.2). The proof is based on the fact that $\mathbf{T}_{p'}S$ is a component of $\mathrm{V}(\Phi_{p',p''})$ if and only if, for all line $\Lambda \subseteq \mathbf{P}^3$ the intersection $L \cap \mathbf{T}_{p'}S \cap \mathrm{V}(\Phi_{p',p''})$ is non-empty; a moment of thought should convince the reader that this fact is indeed true. Then the idea is to consider a suitable resultant to express this fact, and to factor a power of $D_{L',L''}(p', p'')$ out of this resultant.

Consider $T_{p'} = \mathrm{D}^p F(p')$ as a trihomogeneous polynomial of tridegree $(d-1, 0, 1)$ in (p', p'', p). Let $L', L'' \in \check{\mathbf{P}}^3$ be two generic linear forms in p, and consider the line $\Lambda = \mathrm{V}(L', L'')$ in $\mathbf{P}^3$. It intersects $\mathbf{T}_{p'}S \cap \mathrm{V}(\Phi_{p',p''})$ if and only if the resultant

$$R = \mathrm{Res}(T_{p'}, \Phi_{p',p''}, L', L''),$$

with respect to the set of variables p, vanishes. It follows from the homogeneity properties of the resultant, see Theorem (1.7), that R is a multi-homogeneous polynomial in the sets of variables (L', L'', p', p''), of multi-degree $(\mu, \mu, \mu(d-1)+\lambda, \mu)$.

For a given $(p', p'') \in \mathcal{T}_S$, the polynomial $\Phi_{p',p''}$ splits as the product of μ linear forms $\Phi^i_{p',p''}$, $i = 1, \ldots, \mu$, that all vanish along the line $\langle p', p'' \rangle$. Although this splitting may not exist globally over $\mathcal{T}_S$ (in other words the factors $\Phi^i_{p',p''}$ may not be polynomials in (p', p'')), there exists a global splitting over a finite cover $\tilde{\mathcal{T}}_S$ of $\mathcal{T}_S$[6] (in other words the factors $\Phi^i_{p',p''}$ have their

[6] let us briefly recap how such a finite cover may be constructed. The family of planes defined over $\mathcal{T}_S$ by $\Phi_{p',p''}$ corresponds to a subscheme $\mathcal{P}$ of $\mathcal{T}_S \times \check{\mathbf{P}}^3$ with the first projection inducing a degree μ map $\mathcal{P} \to \mathcal{T}_S$. Our goal is to show that there exists a base change $\tilde{\mathcal{T}}_S \to \mathcal{T}_S$ such that $\mathcal{P} \times_{\mathcal{T}_S} \tilde{\mathcal{T}}_S$ splits as μ copies of $\tilde{\mathcal{T}}_S$. As a first step, we consider the following (somewhat absurd non-sensical) base change.

$$\begin{array}{ccc} \mathcal{P} \times_{\mathcal{T}_S} \mathcal{P} & \to & \mathcal{P} \\ \downarrow & & \downarrow \\ \mathcal{P} & \to & \mathcal{T}_S \end{array}$$

The diagonal Δ provides a section of the degree μ map $\mathcal{P} \times_{\mathcal{T}_S} \mathcal{P} \to \mathcal{P}$, hence $\mathcal{P} \times_{\mathcal{T}_S} \mathcal{P}$ splits as $\Delta + \mathcal{P}'$ where

coefficients in a finite extension of $\mathbf{k}[p', p'']$). Then, by the multiplicativity property, see [14, § 5.7, p. 154], the resultant R splits as the product of the μ resultants $R_i = \mathrm{Res}(T_{p'}, \Phi^i_{p',p''}, L', L'')$. By Lemma (3.3), for all i the resultant R_i vanishes wherever $D_{L',L''}$ vanishes on $\mathcal{T}_S$, or rather on $\tilde{\mathcal{T}}_S$, as both forms $T_{p'}$ and $\Phi^i_{p',p''}$ vanish along the line $\langle p', p''\rangle$. By Lemma (3.4), this implies that $D_{L',L''}$ divides R_i as a function of $(p', p'') \in \tilde{\mathcal{T}}_S$. Eventually, the upshot is that $(D_{L',L''})^\mu$ divides R, and since both $D_{L',L''}$ and R are well-defined as functions of $(p', p'') \in \mathcal{T}_S$, so is the quotient. Therefore, there exists a multi-homogeneous polynomial R' in the sets of variables (L', L'', p', p'') such that after restriction to $\check{\mathbf{P}}^3 \times \check{\mathbf{P}}^3 \times \mathcal{T}_S$ one has $R = (D_{L',L''})^\mu R'$. Computing degrees, one finds that R' has multi-degree $(0, 0, \mu(d-2)+\lambda, 0)$ in (L', L'', p', p''), which proves the theorem. □

3.2 – Study of a pencil of planes orthogonal to a tangent line

(3.6) General setup. We recall that $\mathbf{k}$ is assumed to be an algebraically closed field of characteristic 0. We let $S \subseteq \mathbf{P}^3$ be a smooth surface of degree d, defined by a homogeneous polynomial $F(X, Y, Z, W)$. We consider generic 4-tuples $\hat{p}' = (X', Y', Z', W')$, $\hat{p}'' = (X'', Y'', Z'', W'')$, $\hat{p} = (X, Y, Z, W)$, and call p', p'', p the corresponding points in $\mathbf{P}^3$ (beware the unusual distribution of the prime decorations). The choice of $\hat{p}', \hat{p}'', \hat{p}$ defines a system of homogeneous coordinates $(\alpha : \beta : \gamma)$ on the plane $\langle p', p'', p\rangle$ generated by p', p'' and p. We consider

$$
\begin{aligned}
f(\alpha,\beta,\gamma) &\overset{\text{def.}}{=} F(\alpha\hat{p}' + \beta\hat{p}'' + \gamma\hat{p}) \\
&= \alpha^d \cdot F(\hat{p}') + \alpha^{d-1} \cdot \mathrm{D}^{\beta\hat{p}''+\gamma\hat{p}} F(\hat{p}') + \frac{1}{2}\alpha^{d-2} \cdot \mathrm{D}^{(\beta\hat{p}''+\gamma\hat{p})^2} F(\hat{p}') \mod (\beta,\gamma)^3 \\
&= \alpha^d \cdot F(\hat{p}') + \alpha^{d-1}\big(\beta \mathrm{D}^{\hat{p}''} F(\hat{p}') + \gamma \mathrm{D}^{\hat{p}} F(\hat{p}')\big) \\
&\quad + \frac{1}{2}\alpha^{d-2}\big(\beta^2 \mathrm{D}^{\hat{p}''^2} F(\hat{p}') + 2\beta\gamma \mathrm{D}^{\hat{p}''\hat{p}} F(\hat{p}') + \gamma^2 \mathrm{D}^{\hat{p}^2} F(\hat{p}')\big) \mod (\beta,\gamma)^3.
\end{aligned}
\tag{3.6.1}
$$

Considered as a homogeneous polynomial in the variables (α, β, γ), this is the equation of the hyperplane section of S by $\langle p', p'', p\rangle$.

(3.7) Degree of the dual surface. As a warm-up, let us derive the degree of the dual surface $S^\vee \subseteq \check{\mathbf{P}}^3$ of S from the relevant homogeneity property of the discriminant. We specialize p'', p to two general points of $\mathbf{P}^3_{\mathbf{k}}$. Then the coefficients of $f(\alpha, \beta, \gamma)$ are polynomials in the set of indeterminates $\hat{p}' = (X', Y', Z', W')$, and so is the discriminant $\mathrm{Disc}(f)$ with respect to the variables α, β, γ. The latter vanishes at a point $p' \in \mathbf{P}^3_{\mathbf{k}}$ if and only if the plane $\langle p', p'', p\rangle$ is tangent to S. Thus the zero locus $\mathrm{V}\big(\mathrm{Disc}(f)\big) \subseteq \mathbf{P}^3_{\mathbf{k}}$ is the union of all planes tangent to S and containing the line $\langle p'', p\rangle$; in particular the degree of $\mathrm{Disc}(f)$ (as a polynomial in p', what else?) is the number of tangent planes to S containing the line $\langle p'', p\rangle$, which is the degree of the dual surface $S^\vee \subseteq \check{\mathbf{P}}^3$. This is also the number of points in the intersection of $S^\vee$ with the line $(p'', p)^\perp \subseteq \check{\mathbf{P}}^3$, the latter parametrizing planes passing through p'' and p.

The coefficient of $f(\alpha, \beta, \gamma)$ in $\alpha^a\beta^b\gamma^c$ is a polynomial of degree a in the set of indeterminates $\hat{p}'$, as can be seen on the Taylor–Newton expansion, the first terms of which are given in Display (3.6.1). Therefore, by (2.6.1), the discriminant $\mathrm{Disc}(f)$ is homogeneous of degree $d(d-1)^2$ in $\hat{p}'$, and the dual surface $S^\vee$ has degree $d^\vee = d(d-1)^2$ in $\check{\mathbf{P}}^3$.

$\mathcal{P}'$ comes with a degree $\mu - 1$ map to $\mathcal{P}$. Then, by induction on μ, one may construct a finite cover $\tilde{\mathcal{P}} \to \mathcal{P}$ such that $\mathcal{P}' \times_{\mathcal{P}} \tilde{\mathcal{P}}$ splits as $\mu - 1$ copies of $\tilde{\mathcal{P}}$, hence $\mathcal{P} \times_{\mathcal{T}_S} \tilde{\mathcal{P}}$ splits as μ copies of $\tilde{\mathcal{P}}$. Thus $\tilde{\mathcal{P}}$ is suitable as the $\tilde{\mathcal{T}}_S$ we were looking for. We point out the (non-accidental) similarity of this construction with that of a splitting field for a given polynomial P in one indeterminate X. The first base change in our construction is the analogue of the field extension $\mathbf{k}[X]/(P)$. Also note that, in general, the map $\tilde{\mathcal{T}}_S \to \mathcal{T}_S$ will have degree μ!.

(3.8) Remark. If we let all three points (or 4-tuples) p', p'', p be generic, then $\mathrm{Disc}(f(\alpha,\beta,\gamma))$ is tri-homogeneous with respect to the three sets of indeterminates p', p'', p, of tri-degree

$$(d^\vee, d^\vee, d^\vee) = (d(d-1)^2, d(d-1)^2, d(d-1)^2),$$

as follows from the above analysis. Note that obviously p', p'', p play symmetric roles in the definition of f.

In order to enumerate bitangent planes, we will take points p', p'' such that the line $\langle p', p''\rangle$ is tangent to S at p', and consider a reduced discriminant of the corresponding $f(\alpha,\beta,\gamma)$. Thus we need to specialize our setup a little.

(3.9) The "tangent line" setup. In the situation set-up in Paragraph (3.6) above, we specialize p' to the general point of S, and p'' to the general point of $\mathbf{T}_{p'}S$. Then $F(\hat{p}') = \mathrm{D}^{\hat{p}''}F(\hat{p}') = 0$, so (3.6.1) reduces to

$$f(\alpha,\beta,\gamma) = T\alpha^{d-1}\gamma + \frac{1}{2}\alpha^{d-2}(A\beta^2 + 2B\beta\gamma + C\gamma^2) \mod (\beta,\gamma)^3, \tag{3.9.1}$$

where $T = \mathrm{D}^{\hat{p}}f(\hat{p}')$, $A = \mathrm{D}^{\hat{p}''^2}f(\hat{p}')$, $B = \mathrm{D}^{\hat{p}''\hat{p}}f(\hat{p}')$, $C = \mathrm{D}^{\hat{p}^2}f(\hat{p}')$. Note in particular that, as homogeneous polynomials in the set of indeterminates $\hat{p} = (X, Y, Z, W)$, T and C are the equations of the tangent plane $\mathbf{T}_{p'}S$, and the polar quadric $\mathrm{D}^2S(p')$ of S at p', respectively.

We consider the discriminant of the plane curve $S \cap \langle p', p'', p\rangle$, i.e., the discriminant of $f \in A[\alpha,\beta,\gamma]$ with $A = \mathbf{k}[X,Y,Z,W]$; by Formula ( ), or rather Formula (2.22.2), it writes

$$\mathrm{Disc}(f) = -\frac{1}{2}T^2\big(A(B^2-AC)^2\,\Phi + T\,\Psi\big), \quad \text{with} \quad \Phi = \mathrm{redDisc}^2_d(f). \tag{3.9.2}$$

As explained in the previous paragraphs, it vanishes if and only if the plane $\langle p', p'', p\rangle$ is tangent to S; It is a trihomogeneous polynomial in the variables $\hat{p}'$, $\hat{p}''$ and $\hat{p}$; as a polynomial in p, it vanishes along the $d^\vee$ planes (counted with multiplicities) tangent to S in the pencil $(p', p'')^\perp \subseteq \check{\mathbf{P}}^3$ of planes containing the line $\langle p', p''\rangle \subseteq \mathbf{P}^3$.

(3.10) Remark. The fact that T^2 factors out of $\mathrm{Disc}(f)$ gives an algebraic proof of the fact that the tangent plane $\mathbf{T}_{p'}S$ appears with multiplicity $\geqslant 2$ in the scheme $(p', p'')^\perp \cap S^\vee$; or rather, the point $(\mathbf{T}_{p'}S)^\perp \in \check{\mathbf{P}}^3$ appears with multiplicity $\geqslant 2$ in the scheme $(p', p'')^\perp \cap S^\vee$.

In other words, if a line $\Lambda \subseteq \mathbf{P}^3$ is tangent to S at some point p, then the line $\Lambda^\perp \subseteq \check{\mathbf{P}}^3$ is tangent to the dual surface $S^\vee$ at the point $(\mathbf{T}_pS)^\perp$.

Salmon's idea is that if the tangent plane $\mathbf{T}_{p'}S$ is also tangent to S at some point different from p', then it should appear with multiplicity > 2 in $(p', p'')^\perp \cap S^\vee$; equivalently, $\mathrm{Disc}(f)$ should be divisible by T^3 in the above setup.

Therefore, we shall now discuss the various possibilities for T to divide $A(B^2-AC)^2\,\Phi$, in the notation of Equation (3.9.2).

(3.11) Vanishing of A and biduality for inflectional tangents. The polynomial A has degree 0 in p, therefore it is divisible by T if and only if it is identically zero. By definition $A = \mathrm{D}^{\hat{p}''^2}f(\hat{p}')$, so its vanishing is equivalent to the point p'' being on the polar quadric $\mathrm{D}_{p'^2}S$ of S at p'. Since $p'' \in \mathbf{T}_{p'}S$, this in turn is equivalent to the line $\langle p', p''\rangle$ being one of the two inflectional tangents of S at p' (i.e., the two lines Λ having intersection multiplicity with S at p' at least 3; in symbols, $i(\langle p', p''\rangle, S)_{p'} \geqslant 3$).

We thus find an algebraic proof of the fact that

$$\text{if} \quad i\big(\langle p', p''\rangle, S\big)_{p'} \geqslant 3, \quad \text{then} \quad i\big((p', p'')^\perp, S^\vee\big)_{(\mathbf{T}_{p'}S)^\perp} \geqslant 3,$$

which is a manifestation of biduality. Geometrically, if a line $\Lambda \subseteq \mathbf{P}^3$ is an inflectional tangent to S at p' (i.e., L is the tangent line to one of the two local branches of $\mathbf{T}_{p'}S \cap S$ at p'), then its orthogonal $\Lambda^\perp \subseteq \check{\mathbf{P}}^3$ is an inflectional tangent to $S^\vee$ at the point $(\mathbf{T}_{p'}S)^\perp$.

As a side remark note that when $A = 0$, all the curves cut out on S by a member of the pencil $(p', p'')^\perp$ have an inflection point at p'.

(3.12) We shall analyze the divisibilities of $B^2 - AC$ and Φ by T using Theorem (3.2). To see that the latter result indeed applies, we note that for $(p', p'') \in \mathcal{T}_S$ (see the notation in Paragraph (3.1)), the discriminant in Display (3.9.2), $T^2A(B^2 - AC)^2\Phi + T^3\Psi$, defines as a homogeneous polynomial in the variable p a hypersurface consisting of $d^\vee$ planes, counted with multiplicities, all containing the line $\langle p', p''\rangle$. This implies that so does its homogeneous piece of lowest degree with respect to any grading, as the piece of lowest degree of a product is the product of the pieces of lowest degree. For the Zariski grading, see subsection 2.3, the piece of lowest degree of the discriminant is $T^2A(B^2 - AC)^2\Phi$. The upshot is that $T^2A(B^2 - AC)^2\Phi$, as a polynomial in p, defines a sum of planes, all containing the line $\langle p', p''\rangle$, which implies that so do all the factors $T, A, B^2 - AC, \Phi$.

Note that this is obvious for T, which defines the plane $\mathbf{T}_{p'}S$. The polynomial A, on the other hand, is independent on the variable p, hence defines either the whole space, or the empty set.

(3.13) Divisibility of $B^2 - AC$ by T and biduality at parabolic points. The polynomial $B^2 - AC$ is tri-homogeneous of tri-degree $(2(d-2), 2, 2)$ in the variables (p', p'', p). By Theorem (3.2), arguing as in Paragraph (3.12) above, there exists a homogeneous polynomial H of degree $4(d-2)$ in the variable p', with constant coefficients, such that for fixed $p' \in S$ and $p'' \in \mathbf{T}_{p'}S$, T divides $B^2 - AC$ as polynomials in p if and only if $H(p') = 0$.

The polynomial H is, in fact, the Hessian determinant. Indeed, it follows from (3.9.1) that the tangent cone of the section of S by its tangent hyperplane at p', $S \cap \mathbf{T}_{p'}S$, is defined by the quadratic form given by the symmetric matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}.$$

Thus, $B^2 - AC$ is zero modulo T if and only if the curve $S \cap \mathbf{T}_{p'}S$ has a degenerate tangent cone at p', i.e., p' is a parabolic point of S.

Geometrically this is explained as follows. If p' is a parabolic point of S, then the point $(\mathbf{T}_{p'}S)^\perp \in \check{\mathbf{P}}^3$ sits on the cuspidal double curve of $S^\vee$, and for $p'' \in \mathbf{T}_{p'}S - \{p'\}$, the line $(p', p'')^\perp \subseteq \check{\mathbf{P}}^3$ is contained in the tangent cone of $S^\vee$ at the point $(\mathbf{T}_{p'}S)^\perp \in \check{\mathbf{P}}^3$, hence $i((p', p'')^\perp, S^\vee)_{(\mathbf{T}_{p'}S)^\perp} \geqslant 3$.

(3.14) Divisibility of the reduced discriminant Φ by T. The discriminant in Display (3.9.2) is tri-homogeneous in the variables (p', p'', p) with respect to the standard grading, of degree $d^\vee = d(d-1)^2$ with respect to all three variables, as we have observed above. This implies that also the homogeneous piece of (3.9.2) of lowest degree with respect to the Zariski grading is tri-homogeneous of tri-degree $(d^\vee, d^\vee, d^\vee)$ for the standard grading. Moreover, T, A, and $B^2 - AC$ have respective tri-degrees $(d-1, 0, 1)$, $(d-2, 2, 0)$, and $(2(d-2), 2, 2)$ in (p', p'', p). Thus, computing degrees, one finds that Φ has tri-degree (λ, μ, μ), with

$$\lambda = (d-2)(d^2-6) \quad \text{and} \quad \mu = d^3 - 2d^2 + d - 6.$$

Since Φ is $\mathrm{redDisc}_d^2(f)$ with respect to the indeterminate α, and the coefficient of $f(\alpha, \beta, \gamma)$ in $\alpha^a\beta^b\gamma^c$ is a polynomial of degree a in the indeterminates p', the fact that Φ is homogeneous

in p' and its degree follow directly from (2.21.1); to wit, in our situation the degree in (2.21.1) reads

$$d(d-1)^2 - \big[7d-12\big] = d^3 - 2d^2 - 6d + 12 = (d-2)(d^2-6).$$

The homogeneity and degrees in p and p'' however do not directly follow from Proposition (2.21), as for instance the coefficients in $A\beta^2 + 2B\beta\gamma + C\gamma^2$ do not all have the same weights in p or p''. It is noticeable that $T^2A(B^2-AC)^2$ is homogeneous in p and p'' because the weights of T^2 and A balance each other; this is a very elementary reason why T "should" appear with a square in Formula (3.9.2). The degree of $\mathrm{redDisc}_d^2(f)$ with the weights as in (2.21.2) is

$$2\big[d(d-1)^2 - 6\big] = 2\big[d^3 - 2d^2 + d - 6\big],$$

and one observes that its degrees in p and p'' respectively are one half of this, thus this degree "distributes equally" between p and p''.

In conclusion, it follows as in Paragraph (3.13) from Theorem (3.2) that there exists a homogeneous polynomial K of degree $(d-2)(d^3-d^2+d-12)$ in the variable p', with constant coefficients, such that for fixed $p' \in S$ and $p'' \in \mathbf{T}_{p'}S$, T divides Φ if and only if $K(p') = 0$.

We have thus arrived at the following statement.

(3.15) Theorem. *Let S be a smooth, degree d, surface in $\mathbf{P}^3$. There is a hypersurface $V(K)$ of degree $(d-2)(d^3-d^2+d-12)$, the intersection of which with S is the locus of tangency points of planes bitangent to S.*

It has to be recognized that the proof given above of the latter statement rests on the not fully justified fact that T divides $\Phi = \mathrm{redDisc}(f)$ if and only if the tangent plane $\mathbf{T}_{p'}S$ is also tangent to S at some additional point. The "if" part is established by [XII, Theorem (1.4)], which is a result due to Dimca [8], but what is really needed is the "only if" part. Evidence in favour of that assertion is provided by the irreducibility of the reduced discriminant Φ (which, as already mentioned, follows from the fact that the reduced discriminant is a toric discriminant; see also [3]).

(3.16) Corollary. *The ordinary double curve of the dual surface $S^\vee$ has degree $\frac{1}{2}d(d-1)(d-2)(d^3-d^2+d-12)$.*

Proof. Let $p'' \in \mathbf{P}^3$ be a general point. The locus of those points $p' \in S$ such that there exists a plane through p'' tangent to S at p' is the apparent boundary $\mathrm{D}^{p''}S \cap S$. Therefore, by Theorem (3.15), the locus of points $p' \in S$ such that $\mathbf{T}_{p'}S$ is bitangent and passes through p'' is $\mathrm{D}^{p''}S \cap S \cap V(K)$. Now for each bitangent plane there are two tangency points p', so the number of bitangent planes passing through p'' is

$$\frac{1}{2} \cdot \deg(S) \cdot \deg(\mathrm{D}^{p''}S) \cdot \deg K.$$

□

3.3 – Generalization to hypersurfaces of arbitrary dimension

In fact, Salmon's procedure works in arbitrary dimension, using the generalization (2.23.6) of formula (🔔). The arguments are direct generalizations, so we are going to be sketchy.

(3.17) Let $V = \mathrm{V}(F)$ be a smooth hypersurface of degree d in $\mathbf{P}^n$, and consider n points $p_1, \ldots, p_{n-1}, p$ (or rather tuples $\hat{p}_1, \ldots, \hat{p}_{n-1}, \hat{p} \in \mathbf{k}^{n+1}$). The points $p_1, \ldots, p_{n-1}$ (if in linear general position, i.e., if $\hat{p}_1, \ldots, \hat{p}_{n-1}$ are linearly independent) define a pencil of hyperplanes in $\mathbf{P}^n$, and for all p (not in $\langle p_1, \ldots, p_{n-1}\rangle$), the tuples $\hat{p}_1, \ldots, \hat{p}_{n-1}, \hat{p}$ define a system of homogeneous coordinates $(\alpha_1 : \ldots : \alpha_{n-1} : \alpha)$ on the member H_p of the pencil determined by p, i.e., $\langle p_1, \ldots, p_{n-1}, p\rangle$. In this system of coordinates, the hyperplane section $H_p \cap V$ is the zero locus of the polynomial

$$\begin{aligned} f(\alpha_1, \ldots, \alpha_{n-1}, \alpha) &\stackrel{\text{def.}}{=} F(\alpha_1\hat{p}_1 + \cdots + \alpha_{n-1}\hat{p}_{n-1} + \alpha\hat{p}) \\ &= \sum_{k=0}^{d} \frac{1}{k!}\, \alpha_1^{d-k}\, \mathrm{D}^{(\alpha_2\hat{p}_2 + \cdots + \alpha\hat{p})^k} F(\hat{p}_1). \end{aligned}$$

We shall assume that $p_1 \in V$ and $p_2, \ldots, p_{n-1} \in \mathbf{T}_{p_1}V$, equivalently

$$F(\hat{p}_1) = \mathrm{D}^{\hat{p}_2} F(\hat{p}_1) = \cdots = \mathrm{D}^{\hat{p}_{n-1}} F(\hat{p}_1) = 0.$$

Then,

$$f(\alpha_1, \ldots, \alpha_{n-1}, \alpha) = T\alpha_1^{d-1}\alpha + \frac{1}{2}\alpha_1^{d-2} f_2(\alpha_2, \ldots, \alpha_{n-1}, \alpha) \mod (\alpha_2, \ldots, \alpha_{n-1}, \alpha)^3$$

where $T = \mathrm{D}^{\hat{p}} F(\hat{p}_1)$ and $f_2(\alpha_2, \ldots, \alpha_{n-1}, \alpha) = \mathrm{D}^{(\alpha_2\hat{p}_2 + \cdots + \alpha\hat{p})^2} F(\hat{p}_1)$. By Formula (2.23.6), the generalization of (3.11), one has

$$\mathrm{Disc}(f) = T^2 \cdot \mathrm{Disc}\big(\frac{1}{2}\bar{f}_2\big) \cdot \mathrm{Disc}\big(\frac{1}{2} f_2\big)^2 \cdot \Phi \mod T^3,$$

with $\bar{f}_2(\alpha_2, \ldots, \alpha_{n-1}) = f_2(\alpha_2, \ldots, \alpha_{n-1}, 0)$ and $\Phi = \mathrm{redDisc}_d^2(f)$. Our task is to analyze the divisibilities of $\mathrm{Disc}(\frac{1}{2}f_2)$ and Φ by T.

Note that $\bar{f}_2$ has degree 0 in p, hence T divides $\mathrm{Disc}(\frac{1}{2}\bar{f}_2)$ if and only if $\mathrm{Disc}(\frac{1}{2}\bar{f}_2) = 0$. This happens if and only if the section of V by $\langle p_1, \ldots, p_{n-1}\rangle$, which is always singular at p_1, either has multiplicity two at p_1 with singular tangent cone, or has multiplicity greater than two at p_1.

(3.18) Theorem (3.2) (Salmon's elimination trick) generalizes as follows. Given $n-1$ linear forms $L_1, \ldots, L_{n-1} \in \mathbf{P}^n$, the line $\mathrm{V}(L_1, \ldots, L_{n-1})$ intersects the $(n-2)$-dimensional linear space $\langle p_1, \ldots, p_{n-1}\rangle$ if and only if

$$D_{L_1,\ldots,L_{n-1}}(p_1, \ldots, p_{n-1}) \stackrel{\text{def.}}{=} \det\big(L_i(p_j)\big)_{1 \leqslant i,j \leqslant n-1}$$

vanishes.

Let

$$\mathcal{T}_V := \{(p_1, \ldots, p_{n-1}) \in (\mathbf{P}^n)^{n-1} : p_1 \in V \text{ and } p_2, \ldots, p_{n-1} \in \mathbf{T}_{p_1}V\},$$

and consider a multihomogeneous polynomial $\Phi_{p_1,\ldots,p_{n-1}}(p)$ of multidegree $(\lambda, \mu, \ldots, \mu)$ in the sets of variables $p_1, \ldots, p_{n-1}, p$, such that for all $(p_1, \ldots, p_{n-1}) \in \mathcal{T}_V$, the hypersurface $\mathrm{V}(\Phi_{p_1,\ldots,p_{n-1}})$ consists of μ hyperplanes, counted with multiplicities, all containing the $(n-2)$-plane $\langle p_1, \ldots, p_{n-1}\rangle$.

Then, the locus of those $p_1 \in V$ such that there exist $p_2, \ldots, p_{n-1} \in \mathbf{T}_{p_1}V$ such that $\mathbf{T}_{p_1}X$ is a component of $\mathrm{V}(\Phi_{p_1,\ldots,p_{n-1}})$ is cut out on V by a hypersurface of degree $\lambda + \mu(d-2)$.

One may then argue as in (3.12) to analyze the divisibilities of $\mathrm{Disc}(\frac{1}{2}f_2)$ and Φ by T.

(3.19) For $\mathrm{Disc}(\frac{1}{2}f_2)$, we find as before that it is divisible by T if and only if p_1 lies on the Hessian hypersurface of V, since $\frac{1}{2}f_2 \pmod T$ defines the tangent cone at p_1 of the tangential hyperplane section $\mathbf{T}_{p_1}V \cap V$, hence $\mathrm{Disc}(\frac{1}{2}f_2)$ vanishes modulo T if and only if the latter tangent cone is singular, i.e., if and only if p_1 is a parabolic point of V.

This is consistent with the output of Salmon's elimination trick: by [7, n°5 Exemple 6, p. 363], one has

$$\mathrm{Disc}\Big(\frac{1}{2}f_2\Big) = \frac{1}{2^e}\det\begin{pmatrix} \mathrm{D}^{(\hat{p}_2)^2}F(\hat{p}_1) & \cdots & \mathrm{D}^{\hat{p}_2\hat{p}_{n-1}}F(\hat{p}_1) & \mathrm{D}^{\hat{p}_2\hat{p}}F(\hat{p}_1) \\ \vdots & \ddots & \vdots & \vdots \\ \mathrm{D}^{\hat{p}_{n-1}\hat{p}_2}F(\hat{p}_1) & \cdots & \mathrm{D}^{(\hat{p}_{n-1})^2}F(\hat{p}_1) & \mathrm{D}^{\hat{p}_{n-1}\hat{p}}F(\hat{p}_1) \\ \mathrm{D}^{\hat{p}\hat{p}_2}F(\hat{p}_1) & \cdots & \mathrm{D}^{\hat{p}\hat{p}_{n-1}}F(\hat{p}_1) & \mathrm{D}^{\hat{p}^2}F(\hat{p}_1) \end{pmatrix}$$

where e equals 1 if n is even and 0 if n is odd. This polynomial is multihomogeneous of multidegree $\big((n-1)(d-2), 2, \ldots, 2\big)$ in $p_1, \ldots, p_{n-1}, p$, so the argument of (3.18) produces a homogeneous polynomial H of degree

$$(n-1)(d-2) + 2(d-2) = (n+1)(d-2)$$

in the variable p_1.

(3.20) The analysis of the divisibility of the reduced discriminant Φ by T, on the other hand, gives rise to the node-couple polynomial K.

Let us first compute the multidegree of Φ in the variables $p_1, \ldots, p_{n-1}, p$. This goes as in Paragraph (3.14). First of all, $\mathrm{Disc}(f)$ is n-homogeneous of n-degree $(d^\vee, \ldots, d^\vee)$ in $p_1, \ldots, p_{n-1}, p$, with $d^\vee = d(d-1)^{n-1}$, hence so is its homogeneous piece of lowest degree with respect to the Zariski grading

$$T^2\,\mathrm{Disc}(\tfrac{1}{2}\bar{f}_2)\,\mathrm{Disc}(\tfrac{1}{2}f_2)^2\,\Phi.$$

On the other hand, T has n-degree $(d-1, 0, \ldots, 0, 1)$, while $\mathrm{Disc}(\frac{1}{2}f_2)$ has n-degree $((n-1)(d-2), 2, \ldots, 2)$ as we saw in Paragraph (3.19) above. The same computation gives the n-degree of $\mathrm{Disc}(\frac{1}{2}\bar{f}_2)$, namely $((n-2)(d-2), 2, \ldots, 2, 0)$. One thus finds that Φ has degrees

$$d\big[(d-1)^{n-1} - 1\big] - 3(n-1)(d-2) \quad \text{and} \quad d(d-1)^{n-1} - 6$$

in p_1, and $p_2, \ldots, p_{n-1}, p$, respectively (note that the former degree is divisible by $d-2$). As in Paragraph (3.14), the degree in p_1 of $\mathrm{redDisc}_d^2(f)$ is given by (2.21.1), while the degree with respect to the grading of (2.21.2) is equally distributed between the degrees in $p_2, \ldots, p_{n-1}, p$ respectively.

Finally, by the result in Paragraph (3.18), there exists a polynomial K in p_1, homogeneous of degree

$$d\big[(d-1)^n - 1\big] - 3(n+1)(d-2) = (d-2)\Big(d \cdot \frac{(d-1)^n - 1}{d-2} - 3(n+1)\Big),$$

such that T divides $\Phi = \mathrm{redDisc}_d^2(f)$ if and only if $p_1 \in V$ lies on the hypersurface $\mathrm{V}(K)$. We call K the node-couple polynomial. One thus obtains the following result.

(3.21) Theorem. *Let V be a smooth degree d hypersurface in $\mathbf{P}^n$, $n > 1$. The number of hyperplanes bitangent to V passing through $n-2$ fixed general points in $\mathbf{P}^n$ is*

$$\frac{1}{2}d(d-1)^{n-2}(d-2)\Big(d \cdot \frac{(d-1)^n - 1}{d-2} - 3(n+1)\Big).$$

For $n = 3$ one recovers Theorem (3.15), and for $n = 2$ the number of bitangents to a smooth plane curve of degree d, viz.

$$\frac{1}{2}(d^\vee - 1)(d^\vee - 2) - 3d(d-2) - \frac{1}{2}(d-1)(d-2) = \frac{1}{2}d(d-2)(d-3)(d+3).$$

4 – Number of bitangent lines and generalizations

In this section, we elaborate on the computation by Salmon of the number of bitangent lines to a surface in $\mathbf{P}^3$ passing through a general point. We give his proof in Section 4.1, and a generalization to hypersurfaces in $\mathbf{P}^n$ in the next Section 4.2. These proofs only involve the reduced discriminant of polynomials in two indeterminates, which is easily expressed as a plain discriminant, see Example (2.17). We give however another generalization in Section 4.3, for which it is necessary to consider reduced discriminants of polynomials with an arbitrary number of indeterminates. The two statements of Sections 4.1 and 4.2 are instances of a vast generalization by Fehér and Juhász [9], which is described in [XIII, Theorem (7.10)].

4.1 – Number of bitangent lines

The Grassmannian of lines in $\mathbf{P}^3$ has dimension 4. Passing through a fixed point imposes 2 conditions to a line in $\mathbf{P}^3$, and being tangent to a surface (at an unprescribed point) imposes 1 condition, so one expects finitely many bitangent lines to a surface passing through a general point in $\mathbf{P}^3$. In this subsection we prove the following, along the lines of [16, §279].

(4.1) Theorem. *Let S be a smooth surface of degree d in $\mathbf{P}^3$, and $p \in \mathbf{P}^3$ a general point. The number of lines bitangent to S and passing through p is*

$$\frac{1}{2}d(d-1)(d-2)(d-3). \tag{4.1.1}$$

Salmon's strategy is similar in spirit to that exposed in subsection 3.2. The fundamental fact is the following. Consider as before

$$\mathcal{T}_S = \big\{(p', p'') \in \mathbf{P}^3 \times \mathbf{P}^3 : p' \in S \text{ and } p'' \in \mathbf{T}_{p'}S - \{p'\}\big\};$$

equivalently, (p', p'') is in $\mathcal{T}_S$ if and only if $\langle p', p''\rangle$ is a line, tangent to S at p'.

(4.2) Proposition. *There exists a polynomial R, bihomogeneous in p', p'' with respective degrees $(d-2)(d-3)$ and $(d+2)(d-3)$, such that the locus of points $(p', p'') \in \mathcal{T}_S$ for which the line $\langle p', p''\rangle$ is bitangent to S is cut out on $\mathcal{T}_S$ by $V(R)$.*

Proof. Let F be an equation of $S \subseteq \mathbf{P}^3$. For $(p', p'') \in \mathcal{T}_S$, we consider the homogeneous polynomial in (α, β)

$$\begin{aligned} f(\alpha,\beta) &= F(\alpha p' + \beta p'') \\ &= \alpha^d F(p') + \alpha^{d-1}\beta \mathrm{D}^{p''}F(p') + \frac{1}{2}\alpha^{d-2}\beta^2 \mathrm{D}^{(p'')^2}F(p') + \cdots + \beta^d F(p'') \\ &= \beta^2\Big(\frac{1}{2}\alpha^{d-2}\mathrm{D}^{(p'')^2}F(p') + \cdots + \beta^{d-2}F(p'')\Big) \end{aligned} \tag{4.2.1}$$

(this is an abuse of notation: actually one should consider two liftings $\hat{p}', \hat{p}'' \in \mathbf{k}^4$ of p' and p'' respectively, and $F(\alpha\hat{p}' + \beta\hat{p}'')$); the last equality comes from the fact that $\langle p', p''\rangle$ is tangent to S at p'. Let ${}^\flat f(\alpha, \beta)$ be the homogeneous polynomial of degree $d-2$ between parentheses

at the right-hand-side of (4.2.1). The line $\langle p', p''\rangle$ is a bitangent to S if and only if ${}^\flat f$ has a multiple root, so the polynomial R we are looking for is merely the discriminant of ${}^\flat f$ which is none other, up to sign, than the reduced discriminant of f, as noticed in (2.17).

Now, the polynomial f is of the form

$$f(\alpha, \beta) = \beta^2 \left(a_2\, \alpha^{d-2} + a_3\, \alpha^{d-3}\beta + \cdots a_d\, \beta^{d-2}\right)$$

where the coefficients a_i are homogeneous in p' and p'' of degrees $d-i$ and i respectively. Therefore, we deduce from Example (2.17) (see also Proposition (2.21)) that $\mathrm{redDisc}(f)$ has degrees $(d-2)(d-3)$ and $(d+2)(d-3)$ in p' and p'' respectively, which concludes the proof. □

(4.3) Remark. The degree of $\mathrm{redDisc}(f)$ in p'' may be computed alternatively as follows. The plane curve $C_{p'} := \mathbf{T}_{p'}S \cap S$ has in general a double point at p', and what we want is the number of lines in $\mathbf{T}_{p'}S$ passing through p' and tangent to $C_{p'}$ at some other point. This is the number of ramification points of the projection of $C_{p'}$ from p'; the latter is a $(d-2):1$ map $\bar{C}_{p'} \to \mathbf{P}^1$, where $\bar{C}_{p'}$ denotes the normalization of $C_{p'}$ at p', so it follows from the Riemann–Hurwitz formula that the number of ramification points equals

$$2g(\bar{C}_{p'}) - 2 + 2(d-2) = \big((d-1)(d-2) - 2\big) - 2 + 2(d-2) = (d+2)(d-3)$$

as required.

We were not able to find, on the other hand, a geometric argument to compute the degree in p' in a comparable fashion. We wonder wether there is an explanation to why this degree $(d-2)(d-3)$ is so nice, in particular in its role in (4.1.1). It is conceivable that it has something to do with the degree of the dual to a smooth plane curve of degree $d-2$.

Proof of Theorem (4.1). Let $p \in \mathbf{P}^3$ be a general point. The locus of those points $p' \in S$ such that $(p', p) \in \mathcal{T}_S$ (equivalently, $p \in \mathbf{T}_{p'}S$) is the apparent boundary $\mathrm{D}^p S \cap S$. Among these points p', the locus of those p' for which the line $\langle p, p'\rangle$ is a bitangent to S is cut out by $V(R^p)$, where $R^p = R(\_, p)$ with R the polynomial of Proposition (4.2); it is therefore a complete intersection in $\mathbf{P}^3$ of type $(d, d-1, (d-2)(d-3))$. One concludes by observing that there are two points p' for every bitangent line to S passing through p. □

4.2 – A generalization in arbitrary dimension

The following is a fairly direct generalization of Theorem (4.1); as we will see the proof is almost identical to that of Salmon for $n = 2$ described in the previous subsection.

(4.4) Theorem. *Consider an integer $n > 3$. Let V be a smooth hypersurface of degree $d \geqslant n+1$ in $\mathbf{P}^n$, and p be a general point of $\mathbf{P}^n$. The number of lines passing through p and with two contact points with V of respective orders 2 and $n-1$ is*

$$\prod_{k=0}^{n}(d-k). \tag{4.4.1}$$

To be precise, the lines under consideration in the above statement are those lines L in $\mathbf{P}^n$ such that the intersection scheme of L with V has the form

$$L \cap V = 2p_1 + (n-1)p_2 + p_3 + \cdots + p_{d-n+1}.$$

For $n = 3$ this would be bitangent lines as in Theorem (4.1); the reason why we assume $n > 3$ above is because if $n = 3$ the number (4.4.1) actually counts ordered pairs of points (p_1, p_2) such that $\langle p_1, p_2\rangle$ is a bitangent line, so that the number of bitangent lines is only one half of the number (4.4.1), as we have seen indeed in Theorem (4.1).

The family of lines in $\mathbf{P}^n$ passing through p has dimension $n-1$: indeed, a line containing p is uniquely determined by the datum of a point in the projective quotient $\mathbf{P}^n/p$.[7] On the other hand an order 2 contact (i.e., an ordinary tangency) is 1 condition, and an order $n-1$ contact is $n-2$ conditions. It is thus expected that the number of lines as in the theorem be finite.

Proof of the theorem. Consider

$$\mathcal{T}_V = \{(p', p'') \in \mathbf{P}^n \times \mathbf{P}^n : \quad p' \in V, \quad p'' \in \mathrm{D}_{p'} V \cap \cdots \cap \mathrm{D}_{p'^{n-2}} V, \quad \text{and} \quad p'' \neq p'\} :$$

the pair (p', p'') is in $\mathcal{T}_V$ if and only if $\langle p', p''\rangle$ is a line, with contact of order $n-1$ with V at p', see [XII, Theorem (A.8)]. For $(p', p'') \in \mathcal{T}_S$, we consider the homogeneous polynomial in (α, β)

$$\begin{aligned} f(\alpha,\beta) &= F(\alpha p' + \beta p'') \\ &= \alpha^d F(p') + \alpha^{d-1}\beta \mathrm{D}^{p''} F(p') + \frac{1}{2}\alpha^{d-2}\beta^2 \mathrm{D}^{p''^2} F(p') + \cdots + \beta^d F(p'') \\ &= \beta^{n-1}\Big(\frac{1}{(n-1)!}\alpha^{d-n+1}\mathrm{D}^{p''^{n-1}} F(p') + \cdots + \beta^{d-n+1} F(p'')\Big), \end{aligned} \tag{4.4.2}$$

and denote by ${}^\flat f(\alpha, \beta)$ the degree $d-n+1$ polynomial between parentheses in the last expression. The line $\langle p', p''\rangle$ has an additional tangent point with V if and only if the discriminant of ${}^\flat f$ vanishes. For all $i = n-1, \ldots, d$, $\mathrm{D}^{p''^i} F(p')$ has degree i in p'' and $d-i$ in p'. Therefore, by Example (2.17), $\mathrm{Disc}({}^\flat f)$ has degree $(d-n+1)(d-n)$ in p'.

Now, the locus of those points $p' \in V$ such that $(p', p) \in \mathcal{T}_V$, i.e., such that the line $\langle p, p'\rangle$ has contact of order $n-1$ with V at p', is the intersection $V \cap \mathrm{D}^p V \cap \cdots \cap \mathrm{D}^{p^{n-2}}$ (see [XII, Theorem (A.8)]), which is the complete intersection of $n-1$ hypersurfaces of degrees $d, d-1, \ldots, d-n+2$ respectively. By the first part of the proof, those p' such that the line $\langle p, p'\rangle$ has an additional contact of order 2 are cut out in this complete intersection by an n-th equation, of degree $(d-n+1)(d-n)$. One thus gets a 0-dimensional complete intersection of degree

$$d\,(d-1)\cdots(d-n+2)\big[(d-n+1)(d-n)\big],$$

which is isomorphic to the family of lines we are interested in. □

4.3 – A dual generalization in arbitrary dimension

The family of codimension 2 linear subspaces $\Lambda \subseteq \mathbf{P}^n$ passing through $n-2$ points $p_1, \ldots, p_{n-2}$ in linear general position is isomorphic to $\mathbf{P}^2$: indeed, such a linear subspace Λ is uniquely determined by the datum of a point in the 2-dimensional projective quotient

$$\mathbf{P}^n/\langle p_1, \ldots, p_{n-2}\rangle$$

(i.e., $\mathbf{P}\big(\mathbf{k}^{n+1}/\langle \hat{p}_1, \ldots, \hat{p}_{n-2}\rangle$, see footnote 7). Besides, the Grassmannian of codimension 2 linear subspaces in $\mathbf{P}^n$ has dimension $2(n-1)$, and in this codimension passing through a given point is given by two independent linear conditions.

In $\mathbf{P}^n/\langle p_1, \ldots, p_{n-2}\rangle \cong \mathbf{P}^2$, those Λ that are tangent to V at one point are parametrized by a curve C, and those that are bitangent to V (i.e., tangent at two points) are finitely many. The degree of C is readily computed; the theorem below gives the number of bitangent codimension 2 linear subspaces.

In Paragraph (4.6) below, we explain how the degree of C may be found using homogeneity properties of the discriminant; we shall see that the number of bitangent codimension 2 subspaces involves analogously a homogeneity property of the reduced discriminant.

[7] this abusive but suggestive notation has the following meaning: if $\mathbf{P}^n = \mathbf{P}(\mathbf{k}^{n+1})$, and $\hat{p} \in \mathbf{k}^{n+1}$ represents $p \in \mathbf{P}^n$, then $\mathbf{P}^n/p$ stands for the $(n-1)$-dimensional projective space $\mathbf{P}\big(\mathbf{k}^{n+1}/\langle \hat{p}\rangle\big)$.

(4.5) Theorem. *Consider an integer $n \geqslant 3$. Let V be a general hypersurface of degree $d \geqslant 4$ in $\mathbf{P}^n$, and $p_1, \ldots, p_{n-2}$ be points in linear general position in $\mathbf{P}^n$. Then the number of codimension 2 linear subspaces $\Lambda \subseteq \mathbf{P}^n$ passing through $p_1, \ldots, p_{n-2}$ and bitangent to V is*

$$\frac{1}{2}\, d(d-1)^{n-2}(d-2)\, \frac{d(d-1)^{n-2} - (3n-5)d + 6(n-2)}{d-2}.$$

The peculiar way in which the formula is written is meant to emphasize that 2 is a root of the polynomial $d(d-1)^{n-2} - (3n-5)d + 6(n-2)$.

The transversality condition that must be satisfied by V for the formula to be valid is the following: V must be smooth and have finitely many bitangent codimension 2 subspaces through general points $p_1, \ldots, p_{n-2}$, each being tangent at only finitely many points. If it happens that some bitangent subspace has more than two ordinary tangency points, then it has to be counted with the appropriate multiplicity.

We will not insist on verifying that the open subset of the linear space of all degree d hypersurfaces defined by this transversality condition is indeed non-empty, but we will indicate where it is needed in the proof.

(4.6) The degree of the curve $C \subseteq \mathbf{P}^2 \cong \mathbf{P}^n/\langle p_1, \ldots, p_{n-2}\rangle$ of simply tangent codimension 2 subspaces is the number of its intersection points with a given line in $\mathbf{P}^2$; it is also the number of $(n-2)$-planes $\Lambda \subseteq \mathbf{P}^n$ tangent to V, passing through $p_1, \ldots, p_{n-2}$, and contained in a fixed hyperplane H which itself passes through $p_1, \ldots, p_{n-2}$.

The degree of C may be computed using the homogeneity properties of the (plain) discriminant. Let $F \in \mathbf{k}[X_0, \ldots, X_n]$ be an equation of V. We consider the polynomial $f \in \mathbf{k}[A, A_1, \ldots, A_{n-2}]$, depending on the (implicit) parameters $p, p_1, \ldots, p_{n-2} \in \mathbf{P}^n$, defined by

$$f(A, A_1, \ldots, A_{n-2}) = F(A.p + A_1.p_1 + \cdots + A_{n-2}.p_{n-2})$$

(with the usual abuse of notation that we do not distinguish between $p, p_1, \ldots, p_{n-2} \in \mathbf{P}^n$ and their representatives $\hat{p}, \hat{p}_1, \ldots, \hat{p}_{n-2} \in \mathbf{k}^{n+1}$). Then,

$$f(A, A_1, \ldots, A_{n-2}) = \\ A^d F(p) + A^{d-1}\mathrm{D}^{A_1.p_1+\cdots+A_{n-2}.p_{n-2}}F(p) + \cdots + \frac{1}{d!}\mathrm{D}^{(A_1.p_1+\cdots+A_{n-2}.p_{n-2})^d}F(p)$$

by the Taylor–Newton formula, and from this expression we see that the coefficient of f in the monomial $A^\alpha A_1^{\alpha_1} \cdots A_{n-2}^{\alpha_{n-2}}$ is homogeneous of degree α in the parameter p. Therefore, the discriminant $\mathrm{Disc}(f)$ is homogenous of degree $d(d-1)^{n-2}$ in the parameter p, by Proposition (2.6.1).

The hypersurface in $\mathbf{P}^n$ defined by this homogeneous polynomial is the cone of vertex $\langle p_1, \ldots, p_{n-2}\rangle$ swept out by those $(n-2)$-planes $\langle p, p_1, \ldots, p_{n-2}\rangle$ that are tangent to V. It projects from $\langle p_1, \ldots, p_{n-2}\rangle$ to the curve C in $\mathbf{P}^n/\langle p_1, \ldots, p_{n-2}\rangle \cong \mathbf{P}^2$, hence C has degree $d(d-1)^{n-2}$.

As we will see in the proof of Theorem (4.5), the tangency points with V of the $(n-2)$-planes parametrized by C form the complete intersection curve $V \cap \mathrm{D}^{p_1}V \cap \cdots \cap \mathrm{D}^{p_{n-2}}V$, which has degree $d(d-1)^{n-2}$ and projects birationally to C from $\langle p_1, \ldots, p_{n-2}\rangle$; this gives another way of computing the degree of C.

(4.7) Proof of Theorem (4.5). We continue with the setup introduced in (4.6) above, but now we assume that p sits on V and is such that the $(n-2)$-plane $\langle p, p_1, \ldots, p_{n-2}\rangle$ is tangent to

V at p itself; equivalently, p sits on the complete intersection curve $V \cap \mathrm{D}^{p_1}V \cap \cdots \cap \mathrm{D}^{p_{n-2}}V$.[8] Under this assumption, the polynomial f takes the form

$$f(A, A_1, \ldots, A_{n-2}) = \\ \frac{1}{2}A^{d-2}\mathrm{D}^{(A_1.p_1+\cdots+A_{n-2}.p_{n-2})^2}F(p) + \cdots + \frac{1}{d!}\mathrm{D}^{(A_1.p_1+\cdots+A_{n-2}.p_{n-2})^d}F(p),$$

and its coefficient in the monomial $A^\alpha A_1^{\alpha_1}\cdots A_{n-2}^{\alpha_{n-2}}$ is still homogeneous of degree α in the parameter p, if non-zero.

The $(n-2)$-plane $\langle p, p_1, \ldots, p_{n-2}\rangle$ is bitangent to V if and only if the reduced discriminant $\mathrm{redDisc}_d^2(f)$ vanishes. Thus, the locus of points $p \in V$ such that $\langle p, p_1, \ldots, p_{n-2}\rangle$ is tangent to V in two points including p is the complete intersection of the curve $V \cap \mathrm{D}^{p_1}V \cap \cdots \cap \mathrm{D}^{p_{n-2}}V$ with the hypersurface defined by $\mathrm{redDisc}_d^2(f)$ as a polynomial in p.[9] The latter is homogeneous of degree $d(d-1)^{n-2} - (3n-5)d + 6(n-2)$ by (2.21.1). Therefore, the locus of tangency points of bitangent $(n-2)$-planes through $p_1, \ldots, p_{n-2}$ has degree

$$d(d-1)^{n-2}\big[d(d-1)^{n-2} - (3n-5)d + 6(n-2)\big].$$

The result follows, since each bitangent $(n-2)$-plane is tangent in two points. □

5 – The flecnodal polynomial

This is carried out by Salmon in [17, §588], with [17, §473] as a fundamental tool. This has already been revisited in modern standards in [2], and actually extended there to hypersurfaces in a projective space of arbitrary dimension, so we are going to be brief.

(5.1) The problem. Let S be a smooth surface in $\mathbf{P}^3$ of degree $d > 1$. For a general point $p \in S$, there are two lines having intersection multiplicity at p with S strictly greater than 2: these are the tangent lines to the two smooth branches at p of the curve $\mathbf{T}_pS \cap S$, which has an ordinary double point at p; thus these two lines intersect S with multiplicity 3 at p.

We shall see that those points $p \in S$ such that there is a line intersecting S with multiplicity strictly greater than 3 at p is a curve $Fl(S)$, cut out on S by a polynomial of degree $11d - 24$. We call this curve (respectively polynomial) the *flecnodal* curve (respectively polynomial) of S.

At a general point p of the flecnodal curve, the section of S by its tangent hyperplane $\mathbf{T}_pS$ is a curve with a non-degenerate double point at p (i.e., a double point with tangent cone of maximal rank), such that one of its two local branches has an inflection point at p. In general, the tangent line to the latter branch meets S with multiplicity 4 at p. Those points $p \in S$ such that the curve $\mathbf{T}_pS \cap S$ has a tacnode (i.e., a double point with local equation $y^2 = x^4$) also belong to the flecnodal curve: they are its intersection points with the Hessian of S, and they are cuspidal points of the cuspidal double curve of $S^\vee$ (the latter curve parametrizes those hyperplanes that cut out a cuspidal curve on S).

The following statement is in the spirit of reduced elimination theory, with the polynomial R (in the notation of the proposition) playing the role of the reduced resultant. Here we are not

[8] this equivalence needs the smoothness of V; the fact that this is a complete intersection is equivalent to the fact that tangent subspaces through $p_1, \ldots, p_{n-2}$ form a curve.

[9] again, the fact that this is indeed a complete intersection is equivalent to the fact that bitangent subspaces through $p_1, \ldots, p_{n-2}$ are finitely many, and each of those is tangent to V at only finitely many points; multiple points in this complete intersection will occur if some bitangent subspace has more than two ordinary tangency points with V.

prescribing individual multiplicities at the point q for the polynomials F, G, H as in the reduced resultant situation, but rather a multiplicity for the ideal they generate; then R vanishes when the subscheme defined by this ideal is larger than what is imposed by the hypothesis, as does redRes in the reduced resultant situation.

(5.2) Proposition ([17, §473])**.** *Let $F_q(p), G_q(p), H_q(p)$ be three bi-homogeneous polynomials in $p, q \in \mathbf{P}^3$, of bi-degrees (λ, μ), (λ', μ'), (λ'', μ'') respectively. We assume that for the general point $q \in \mathbf{P}^3$,*

$$\mathrm{mult}_q\big(\mathrm{V}(F_q, G_q, H_q)\big) = \lambda\lambda'\lambda''.$$

The locus of those $q \in \mathbf{P}^3$ such that $\mathrm{V}(F_q, G_q, H_q)$ contains a point in addition to q counted with multiplicity $\lambda\lambda'\lambda''$ is the zero locus of a homogeneous polynomial R of degree

$$\lambda'\lambda''\mu + \lambda\lambda''\mu' + \lambda\lambda'\mu'' - \lambda\lambda'\lambda''.$$

Of course, the condition that the scheme $\mathrm{V}(F_q, G_q, H_q)$ contains a point in addition to q is equivalent to its having positive dimension. Salmon claims that it is equivalent to the fact that $\mathrm{V}(F_q, G_q, H_q)$ contains a line; we have not been able to prove it, but this is not needed for the application.

Proof. We want to characterize when the scheme $\mathrm{V}(F_q, G_q, H_q)$ has positive dimension. The idea is that this is equivalent to its having non-empty intersection with any hyperplane. So let L be a non-zero linear form in p, and consider the resultant $\mathrm{Res}(L, F_q, G_q, H_q)$. It follows from the Poisson formula (see, e.g., [2, Prop. 2.2] and the references therein) and our assumption on F, G, H that there exists a polynomial R such that

$$\mathrm{Res}(L, F_q, G_q, H_q) = L(q)^{\lambda\lambda'\lambda''} \cdot R(L, F_q, G_q, H_q). \tag{5.2.1}$$

Computing degrees, one sees that R is homogeneous of degree 0 in the coefficients of L, i.e., it does not depend on L. It follows that $V(F_q, G_q, H_q)$ has positive dimension if and only if $R(F_q, G_q, H_q) = 0$. Eventually, one computes the degrees of $R(F_q, G_q, H_q)$ using the identity (5.2.1). □

(5.3) Theorem. *Let S be a smooth surface in $\mathbf{P}^3$ of degree d. There exists a homogeneous polynomial Fl of degree $11d - 24$, such that the locus of points $p \in S$ such that there is a line intersecting S with multiplicity at least 4 in p is cut out on S by $\mathrm{V}(Fl)$.*

Proof. Let $p \in S$. It follows from [XII, Thm. (A.8)] that there is a line intersecting S with multiplicity at least 4 in p if and only if the three polar hypersurfaces $\mathrm{D}_p S, \mathrm{D}_{p^2} S, \mathrm{D}_{p^3} S$ (respectively, the tangent plane, the polar quadric, and the polar cubic of S at p) have a common point besides p, and that this in turn is equivalent to their having a whole line in common.

On the other hand, $\mathrm{D}_p S, \mathrm{D}_{p^2} S, \mathrm{D}_{p^3} S$, which have degrees $1, 2, 3$ respectively, intersect with multiplicity 6 at p by [XII, Cor. (A.9)]. We are therefore in a position to apply Proposition (5.2), with (λ, μ), (λ', μ'), (λ'', μ'') equal to $(1, d-1)$, $(2, d-2)$, $(3, d-3)$ respectively. The result follows. □

References

Chapters in this Volume

[II] Th. Dedieu and E. Sernesi, *Deformations of curves on surfaces.*

[XII] Th. Dedieu, *Some formulas for curves and surfaces in $\mathbf{P}^3$.*

[XIII] Th. Dedieu, *Node polynomials for curves on surfaces.*

External References

[1] M. Beltrametti, E. Carletti, D. Gallarati and G. Monti Bragadin, *Lectures on curves, surfaces and projective varieties – a classical view of algebraic geometry*, EMS Textbooks in Mathematics, European Mathematical Society, 2009, Original version issued in 2003.

[2] L. Busé, C. D'Andrea, M. Sombra and M. Weimann, *The geometry of the flex locus of a hypersurface*, Pacific Journal of Mathematics **304** (2020), no. 2, 419–437.

[3] L. Busé and Th. Dedieu, *Discriminants of hypersurfaces with prescribed ordinary singularities*, in preparation.

[4] L. Busé and J.-P. Jouanolou, *On the discriminant scheme of homogeneous polynomials*, Math. Comput. Sci. **8** (2014), no. 2, 175–234.

[5] P. Cartier and J. Tate, *Simple proof of the main theorem of elimination theory in algebraic geometry*, Enseign. Math. (2) **24** (1978), no. 3-4, 311–317.

[6] D. A. Cox, J. B. Little and D. O'Shea, *Using algebraic geometry*, Graduate texts in mathematics, Springer, New York, 1998.

[7] M. Demazure, *Résultant, discriminant*, Enseign. Math. (2) **58** (2012), no. 3-4, 333–373.

[8] A. Dimca, *Topics on real and complex singularities*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1987, An introduction.

[9] L. M. Fehér and A. P. Juhász, *Plücker formulas using equivariant cohomology of coincident root strata*, arXiv:2312.06430.

[10] M. Furuya and M. Tomari, *A characterization of semi-quasihomogeneous functions in terms of the Milnor number*, Proc. Amer. Math. Soc. **132** (2004), no. 7, 1885–1890.

[11] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008, Reprint of the 1994 edition.

[12] J. Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course.

[13] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[14] J.-P. Jouanolou, *Le formalisme du résultant*, Adv. Math. **90** (1991), no. 2, 117–263.

[15] E. K. Ould Mohamdi, *Élimination réduite*, Thèse, Université de Strasbourg, 1988.

[16] G. Salmon, *A Treatise on the Analytic Geometry of Three Dimensions, Vol. I*, Chelsea Publishing Company, New York, 1958, reprint of the seventh (latest) edition.

[17] ———, *A Treatise on the Analytic Geometry of Three Dimensions, Vol. II*, Chelsea Publishing Company, New York, 1965, reprint of the fifth (latest) edition.

[18] ———, *Lessons introductory to the modern higher algebra*, Hodges, Foster and Co., Dublin, 1876, third edition.

[19] B. L. van der Waerden, *Modern Algebra. Vol. II*, Frederick Ungar Publishing Co., New York, 1950, Translated from the third German edition.

[20] O. Zariski, *Generalized weight properties of the resultant of $n+1$ polynomials in n indeterminates*, Trans. Amer. Math. Soc. **41** (1937), no. 2, 249–265.

Lecture XIII
Node polynomials for curves on surfaces

by Thomas Dedieu

1 Introduction and statement of results 409
2 The inductive procedure 414
2.1 The induced pairs 414
2.2 The recursive formula 417
2.3 The computation — Proof of the main theorem 420
2.4 Comparison with Vainsencher's computation 424
3 Applications 427
3.1 Curves in a linear system on a fixed surface 427
3.2 Plane curves in a solid in $\mathbf{P}^4$ 428
4 Equisingularity classes 431
4.1 Infinitely near points and proximity 431
4.2 Multiplicities and proximity equalities 432
4.3 Equisingularity and Enriques diagrams 433
4.4 Equisingular loci 437
5 Regularity conditions for curves in a fixed linear system 441
5.1 Application of equisingular deformation theory 442
5.2 Application of Bertini's theorem 443
5.3 Conclusion 444
5.4 Particular cases 446
6 Inclusion-exclusion and the structure of node polynomials 447
6.1 The principle and an example 447
6.2 Multiplicative structure of the lattice of polydiagonals 449
6.3 Further considerations 454
7 Kazarian's Thom polynomials 455
A Basics on Bell polynomials 461
References 463

1 – Introduction and statement of results

The main theme of this text is the following set of formulae, first found by Vainsencher [39] (for $\delta \leqslant 6$) and shortly after improved by Kleiman and Piene [21, 22, 23, 24], counting δ-nodal curves in a complete linear system on a smooth surface, if $\delta \leqslant 8$ and the corresponding line bundle is sufficiently positive.

(1.1) The formulae involve the *complete exponential Bell polynomials* $P_r \in \mathrm{Z}[X_1, \ldots, X_r]$, $r \in \mathbf{N}$, defined by the formal identity

$$\sum_{r \geqslant 0} P_r \frac{T^r}{r!} = \exp\left(\sum_{q \geqslant 1} X_q \frac{T^q}{q!} \right)$$

in $\mathbf{Q}(X_1, \ldots,)[[T]]$. Thus $P_0 = 1$, $P_1 = X_1$, $P_2 = X_1^2 + X_2$, $P_3 = X_1^3 + 3X_2X_1 + X_3$, and so forth. For all r, the polynomial P_r is weighted homogeneous of degree r, if one assigns each X_i the weight i. For convenience we also set $P_r = 0$ for all negative integers r.

(1.2) Let S be a smooth irreducible complex projective surface, and L be a line bundle on S. Consider the four Chern numbers

$$d = L^2 \ ; \quad k = L \cdot K_S \ ; \quad s = K_S^2 \ ; \quad x = c_2(S),$$

and set

$$\begin{aligned}
a_1 &= 3d + 2k + x \ ; \\
a_2 &= -42d - 39k - 6s - 7x \ ; \\
a_3 &= 1380d + 1576k + 376s + 138x \ ; \\
a_4 &= -72360d - 95670k - 28842s - 3888x \ ; \\
a_5 &= 5225472d + 7725168k + 2723400s + 84384x \ ; \\
a_6 &= -481239360d - 778065120k - 308078520s + 7918560x \ ; \\
a_7 &= 53917151040d + 93895251840k + 40747613760s - 2465471520x \ ; \\
a_8 &= -7118400139200d - 13206119880240k - 6179605765200s + 516524964480x.
\end{aligned}$$

(1.3) Theorem (Vainsencher, Kleiman–Piene)**.** *Let $\delta \leqslant 8$ be an integer, and assume that L may be written as $L = mA + B$ for an integer $m \geqslant 3\delta$ and line bundles A and B, such that A is very ample and B is globally generated.*[1] *Fix $n - \delta$ general points $p_1, \ldots, p_{n-\delta} \in S$, where n is the dimension of the complete linear system $|L|$. The number of δ-nodal curves among members of $|L|$ passing through $p_1, \ldots, p_{n-\delta}$ is given by the polynomial expression in d, k, s, x:*

$$N_\delta = \frac{1}{\delta!} P_\delta(a_1, \ldots, a_\delta). \tag{1.3.1}$$

These formulae are obtained as an application of a more general result, that I shall now describe.

(1.4) Let $\pi : F \to Y$ be a smooth projective family of surfaces, where the base Y is equidimensional and Cohen–Macaulay. Let D be a relative effective divisor on F/Y. Consider the three Chern classes

$$v = c_1\big(\mathcal{O}_F(D)\big) \quad \text{and} \quad w_j = c_j\big(\Omega^1_{F/Y}\big) \text{ for } j = 1, 2,$$

and set

$$\begin{aligned}
x_2 &= v^3 + v^2 w_1 + v w_2 \ ; \\
x_3 &= v^6 + 4v^5 w_1 + 5v^4(w_1^2 + w_2) + v^3(2w_1^3 + 11 w_1 w_2) + v^2(6w_1^2 w_2 + 4w_2^2) + 4v w_1 w_2^2 \ ; \\
x_4 &= v^{10} + 10 v^9 w_1 + v^8(40 w_1^2 + 15 w_2) + v^7(82 w_1^3 + 111 w_1 w_2) \\
&\quad + v^6(91 w_1^4 + 315 w_1^2 w_2 + 63 w_2^2) + v^5(52 w_1^5 + 29 w_1^3 w_2 + 324 w_1 w_2^2) \\
&\quad + v^4(12 w_1^6 + 282 w_1^4 w_2 + 593 w_1^2 w_2^2 + 85 w_2^3) + v^3(72 w_1^5 w_2 + 464 w_1^3 w_2^2 + 259 w_1 w_2^3) \\
&\quad + v^2(132 w_1^4 w_2^2 + 246 w_1^2 w_2^3 + 36 w_2^4) + v(72 w_1^3 w_2^3 + 36 w_1 w_2^4).
\end{aligned}$$

[1] if $\delta = 8$, one needs to add the technical assumption that among curves in $|L|$ with an ordinary 4-tuple point, the analytic type of the 4-tuple point is not constant, i.e., the cross-ratio of the four tangents varies.

We shall use these polynomial expressions to define recursively eight Chern classes $b_1, \dots, b_8$ on F. To do so we define, for all polynomial expressions $R(v, w_1, w_2)$ and integers i, a new polynomial expression $Q_{R,i}(v, w_1, w_2)$ as follows: (i) consider a new variable e, and form the polynomial expression

$$R(v - ie,\, w_1 + e,\, w_2 - e^2)\ ;$$

(ii) perform the Euclidean division of this polynomial by $e^3 + w_1 e^2 + w_2 e$; (iii) consider the coefficient in e^2 of the remainder of this division, and define $Q_{R,i}(v, w_1, w_2)$ to be its opposite. The geometric meaning of these operations will be apparent later in the text. Then, we define for all $s = 0, \dots, 7$:

$$\begin{aligned} b_{s+1} &= P_s\big(Q_{b_1,2},\, \dots,\, Q_{b_s,2}\big) \cdot x_2 \\ &\quad - s(s-1)(s-2) \cdot P_{s-3}\big(Q_{b_1,3},\, \dots,\, Q_{b_{s-3},3}\big)\big) \cdot x_3 \\ &\quad + 3281 \cdot 7! \cdot P_{s-7} \cdot x_4. \end{aligned} \tag{1.4.1}$$

In this definition, the polynomials P_r are as defined in (1.1) above; note that for $s = 0, 1, 2$, only the first line is involved (as the other two are zero), and for $s = 3, 4, 5, 6$, only the first two lines are involved; for $s = 7$, P_{s-7} is merely $P_0 = 1$.

(1.5) In order to state the general result leading to (1.3), we need to consider *(multi-)equisingularity classes*. Informally this is a collection of planar singularity types, e.g., $\delta \mathrm{A}_1$ denotes a collection of δ nodes, $(\delta - 1)\mathrm{A}_1 + \mathrm{A}_2$ denotes a collection of $\delta - 1$ nodes and one cusp, and so on. Roughly speaking a singularity type is the same as the name of a singularity; for instance "ordinary quadruple point" is a singularity type, and all ordinary quadruple points are of this type, regardless of the cross-ratio of the tangents of their four local branches. A precise definition will be given in Section 4, where we will see in particular that equisingularity classes may be encoded by Enriques diagrams.

Let S be a multi-equisingularity class. In the setup of (1.4), we can form the subset $Y(\mathsf{S}) \subseteq Y$ consisting of those y for which the curve D_y in the surface F_y has singularities of the type defined by S. It is a constructible set in Y, which has an expected codimension depending only on S, and that we shall denote by $\mathrm{expcod}(\mathsf{S})$; thus, $\mathrm{codim}_Y(Y(\mathsf{S}))$ is smaller or equal than $\mathrm{expcod}(\mathsf{S})$, and if equality holds we say that $Y(\mathsf{S})$ has the expected codimension in Y. For convenience, we agree that the empty set has the expected codimension in Y.

Moreover, we denote by $Y(\infty)$ the subset of Y consisting of those points y such that the curve D_y is non-reduced. It is closed in Y.

(1.6) Theorem (Vainsencher, Kleiman–Piene)**.** *Consider the setup of (1.4). Let $\delta \leqslant 8$ be a non-negative integer, and assume that the two following conditions hold:*[2]
(i) $\mathrm{codim}\big(Y(\infty)\big) \geqslant \delta + 1$*;*
(ii) for all multi-equisingularity classes S, $\mathrm{codim}_Y\big(Y(\mathsf{S})\big) \geqslant \min\big(\delta + 1, \mathrm{expcod}(\mathsf{S})\big)$.

Then $Y(\delta \mathrm{A}_1)$, the locus of δ-nodal curves in Y, has pure codimension δ (if non-empty); its closure $\overline{Y(\delta \mathrm{A}_1)}$ is the support of a natural nonnegative cycle $U(D, \delta)$, and the following equality in the Chow ring of Y holds:

$$\big[U(D, \delta)\big] = \frac{1}{\delta!}\, P_\delta\big(\pi_* b_1,\, \dots,\, \pi_* b_\delta\big), \tag{1.6.1}$$

where P_δ is the δ-th Bell polynomial, and $b_1, \dots, b_\delta$ are the rational equivalence classes on F defined in (1.4).

[2] for $\delta = 8$, add the following assumption: (iii) for S the equisingularity class of an ordinary quadruple point, the analytic type of D_y at its quadruple point is non-constant along $Y(\mathsf{S})$.

To save the reader from a possible moment of confusion, let me emphasize that the Bell polynomials appear twice in this story, first in the definition of the classes $b_1, \dots, b_\delta$, and second in the expression of $[U(D,\delta)]$ in terms of these very $b_1, \dots, b_\delta$.

In setup (1.4) one considers a family of curves in a possibly non-constant family of surfaces. This is indeed necessary for the approach of Vainsencher and Kleiman–Piene to Theorem (1.3), in which one considers, given a fixed surface S, the family $\pi : F \to Y = S$ such that for all $s \in S$, the surface F_s is the blow-up of S at the point s. This also gives the possibility of further applications, including the following important one.

(1.7) Theorem (Vainsencher, Kleiman–Piene)**.** *Let V be a general threefold in $\mathbf{P}^4$ of degree m. If $m \geq 4$, then V contains precisely the following number of 6-nodal plane curves of degree m:*

$$\begin{aligned}(1.7.1)\quad \frac{5}{6!}m\big(&m^{17} - 12m^{16} + 24m^{15} + 155m^{14} - 405m^{13} + 1082m^{12} - 18469m^{11} + 66446m^{10}\\ &- 192307m^9 + 1242535m^8 - 4049006m^7 + 11129818m^6 - 53664614m^5\\ &+ 166756120m^4 - 415820104m^3 + 1293514896m^2 - 2517392160m + 1781049600\big).\end{aligned}$$

For $m = 5$, Vainsencher and Kleiman–Piene have refined this formula by computing the number of irreducible 6-nodal plane quintics in V (the refinement resides in the irreducibility), see Theorem (3.5). This computation has been important for the understanding of the enumeration of rational curves in Calabi–Yau threefolds, see [XI, Section 4].

In the situation of Theorem (1.7), the family of surfaces $\pi : F \to Y$ to be considered is the universal family of planes in $\mathbf{P}^4$ over the Grassmannian $\mathbf{G}(2,4)$, and the relative divisor is cut out by the hypersurface V: for all $y = [\Pi] \in Y = \mathrm{Gr}(3,5)$, F_y is the plane $\Pi \subseteq \mathbf{P}^4$ itself, and D_y is the plane curve $V \cap \Pi$.

(1.8) One may of course expect that formulae analogous to (1.3.1) and (1.6.1) hold for all δ. The existence of formulae analogous to (1.3.1) without the requirement that they be structured in terms of Bell polynomials, i.e., the existence of polynomials T_δ in the Chern classes d, k, s, x which enumerate (under suitable ampleness assumptions) δ-nodal curves in a complete linear system on a fixed surface was the object of the Göttsche conjecture. The polynomials T_δ are called node polynomials, or universal polynomials. The existence of formulae analogous to (1.6.1) without the Bell polynomials structure, i.e., the existence of universal polynomials in the Chern classes v, w_1, w_2, is a family version of the Göttsche conjecture. The Göttsche conjecture has been proved by Tzeng [38], and another proof by Kool–Shende–Thomas [27] has appeared shortly after. The family version of the conjecture has been proved by Laarakker [28] (then a student of Kool), by elaborating on the proof of [27]. We report on the Göttsche conjecture in [XIV]; in this text, we focus rather on giving explicit formulae.

It had been proved by Göttsche that if his conjecture holds true, then the node polynomials T_δ satisfy some structure conditions encoded in the so-called Göttsche–Yau–Zaslow formula, see [XIV, Theorem (1.3)]. Qviller [31] (then a student of Piene) has shown that this formula implies the Bell polynomial structure of the node polynomials T_δ as in Formula (1.3.1). Once established the existence of formulae analogous to (1.3.1) for all δ, one may compute the polynomials $a_i(d,k,s,x)$ by interpolation from known cases. In fact, in [31, Algorithm 2.1] Qviller explains how to extract the a_i from the Göttsche–Yau–Zaslow formula. He gives them up to $i = 15$ in [32, Table 2] (only up to $i = 11$ in [31]), which we reproduce below; following his notation, we give $\tilde{a}_i = \frac{1}{(i-1)!}a_i$:

$$\tilde{a}_9 = 26842726680d + 52612204910k + 26239943207s - 2617350984x \ ;$$
$$\tilde{a}_{10} = -513240952752d - 1055936555124k - 556487181661s + 62064807888x \ ;$$
$$\tilde{a}_{11} = 9861407170992d + 21186861410508k + 11720114258490s - 1410986931936x \ ;$$
$$\tilde{a}_{12} = -190244562607008d - 425029422316200k - 245491696730341s + 31230909182592x \ ;$$
$$\tilde{a}_{13} = 3682665360521280d + 8525631885908256k + 5119580760611226s - 678769122880224x \ ;$$
$$\tilde{a}_{14} = -71494333556133600d - 171005998538392560k - 106382292871378404s - 14560213534363728x \ ;$$
$$\tilde{a}_{15} = 1391450779290676680d + 3429957097334083248k + 2203960837196658328s - 309288199242633956x.$$

In the particular case of the projective plane, the Göttsche conjecture implies the existence of univariate polynomials R_δ such that for large enough d, $R_\delta(d)$ is the number of δ-nodal, degree d plane curves passing through the appropriate number of fixed general points: this was known as the Di Francesco–Itzykson conjecture, and has been proved by Fomin and Mikhalkin [12] using floor diagrams. Building on their work, Block [5] has developed an algorithm to compute the polynomials R_δ; he gives them for all $\delta \leqslant 14$. In [6] he has extended these results to the relative setting, i.e., to the enumeration of plane curves with prescribed tangency pattern with a fixed line.

(1.9) The last section presents a brief account on Kazarian's Thom polynomials, see [19] (anecdotally, my last words during the final session of our seminar in Tor Vergata were for this very topic). In a differential framework, they give (explicit!) formulae enumerating curves with virtually any kind of singularities, thus providing a vast generalization of the formulae discussed in detail in this text. In particular, one can use Kazarian's Thom polynomials to recover virtually all the formulae for surfaces in $\mathbf{P}^3$ that have been given in [XII]. On the latter application, see also [34] by Sasajima and Ohmoto, which deals with threefolds in $\mathbf{P}^4$ as well; for considerations in the same circle of ideas by the same authors, see [35].

We also review applications of Kazarian's Thom polynomials to the enumeration of linear spaces with prescribed tangency pattern with a hypersurface.

(1.10) It is natural to wonder why the universal enumerative formulae for δ-nodal curves may be structured in terms of Bell polynomials (besides their miraculous fitness for the recursive computation in Paragraph (2.15)). An explanation has been provided by Qviller [32], which resides in a computation by inclusion-exclusion with respect to the lattice of polydiagonals in some appropriate fiber product, and the fact that the coefficients of the Bell polynomials may be expressed in terms of numbers of partitions. We shall see this in Section 6.

(1.11) The Göttsche universal polynomials are enumerative as soon as the line bundle L is δ-very ample as is proved in [27], see [XIV, Theorem (1.2)]. It follows that so are the formulae analogous to (1.3.1) for all δ; in particular, Theorem (1.3) holds under the weaker assumption that L is δ-very ample. See also [25] for even better validity conditions in case S is a rational surface; when S is the projective plane, this is the result described in Paragraph (5.12).

I have nevertheless chosen to give the original proof by Kleiman and Piene of the validity of Formula (1.3.1) for $\delta \leqslant 8$ and for 3δ-very ample line bundles, as I reckon it displays interesting and instructive techniques. The workaround to avoid Kleiman and Piene's arguments, which indeed necessitate the full assumptions of Theorem (1.3), would be to extract a_i for $i = 1, \ldots, 8$ from the Göttsche–Yau–Zaslow formulae, and then observe that they coincide with the a_i given in (1.2); then Theorem (1.3) with the weaker assumptions is a mere corollary of [XIV, Theorem (1.2)]. Note that while δ-ampleness gives sufficient control on the codimensions in $|L|$ of the loci of bad curves (i.e., curves with singularities worse than nodes) in order to prove [XIV, Theorem (2.1)], it does not give the full condition (ii) of Theorem (1.6).

The organization of the text is as follows. We directly give the proof in Section 2 of the main Theorem (1.6); this uses the theory of equisingularity, although it is only treated in Section 4. In Section 3 we do the computations giving Formulae (1.3.1) and (1.7.1), leaving aside the question of their enumerativity; the latter question is studied in Section 5, where we thus complete the proof of Theorem (1.3). Section 6 is devoted to Qviller's study of the Bell polynomials structure of formulae (1.3.1) and (1.6.1), and Section 7 to Kazarian's Thom polynomials. There is an appendix providing the basic results on Bell polynomials.

Thanks. I am grateful to Ragni Piene for her observations on a preliminary version of this text.

2 – The inductive procedure

This section is dedicated to the proof of Theorem (1.6), following Kleiman and Piene [24]; we discuss the relation with Vainsencher's ideas [39] in the last Section 2.4.

The idea is the following. Consider the situation set up in (1.4). Using sheaves of principal parts, also known as jet bundles, it is possible to compute the rational equivalence in F of the locus X_2 along which the relative divisor D/Y has a double point; then, pushing the class $[X_2]$ down to Y should give the class $[U(D,1)]$ of the (closure of the) locus in Y of points y such that the divisor D_y has one node. To add one more node, one considers the pull-back to X_2 of the family $\pi : F \to Y$, and modify it so that the surface over the point $x \in X_2$ is the blow-up of $F_{\pi(x)}$ at x; then one considers the proper transform of the relative divisor pulled-back from D, and repeat the procedure.

In section 2.1 we explain the construction of the pairs $(F_i/X_i, D_i)$ induced by the pair $(F/Y, D)$ we start with; this gives the general setup for the proof of Theorem (1.6). In section 2.2 we state and prove the recursive formula which is the engine powering Theorem (1.6). After that there only remains a (substantial) computation to be made in order to obtain Theorem (1.6): we do it in section 2.3.

2.1 – The induced pairs

(2.1) Let, as in (1.4), $\pi : F \to Y$ be a smooth projective family of surfaces, where Y is equidimensional and Cohen–Macaulay, and let D be a relative effective divisor on F/Y. Consider the product $F \times_Y F$, and denote by $\mathrm{pr}_1, \mathrm{pr}_2 : F \times_Y F \to F$ the two projections, by $\Delta \subseteq F \times_Y F$ the diagonal, and by $\mathcal{I}_\Delta$ its ideal. Then, for all non-negative integers i, the i-th *bundle of principal parts* of D is

$$\mathcal{P}^i_{F/Y}(D) = \mathrm{pr}_{2*}\big(\mathrm{pr}_1^*\mathcal{O}_F(D)\big/\,\mathcal{I}_\Delta^{i+1}\big).$$

The fibre of $\mathcal{P}^i_{F/Y}(D)$ over a point $x \in F$ identifies with the space of global sections of the restriction of $\mathcal{O}_F(D)$ to the i-th order infinitesimal neighbourhood of x in F. You may consult [9, Section 7.2] for more details.

The relative divisor D is defined by a global section s of $\mathcal{O}_F(D)$, and the latter induces a section s_i of $\mathcal{P}^i_{F/Y}(D)$ for all i. For all $i \geqslant 1$, let $X_i = \mathrm{V}(s_{i-1}) \subseteq F$ be the scheme of zeroes of the section s_{i-1}. As a set, X_i is the locus of points $x \in F$ such that the divisor $D_{\pi(x)}$ has a point of multiplicity at least i at x. In particular, $X_1 = D$.

(2.2) Next, let $\varepsilon\colon F' \to F \times_Y F$ be the blow-up along the diagonal Δ, and denote its exceptional divisor by E. Let π' be the composed map $\mathrm{pr}_2\varepsilon$; then $\pi'\colon F' \to F$ is another smooth family of surfaces, with fibre over a point $x \in F$ the blow-up of the surface $F_{\pi(x)}$ at the point x.

For all $i \geqslant 2$, set $F_i := {\pi'}^{-1}(X_i)$, and denote by $\pi_i \colon F_i \to X_i$ the restriction of π'. Eventually, let ρ_i and ρ_i' be the closed immersions $X_i \hookrightarrow F$ and $F_i \hookrightarrow F'$ respectively. This gives the following commutative diagram.

(2.2.1)
$$\begin{array}{ccccccc} F & \xleftarrow{\mathrm{pr}_1} & F \times_Y F & \xleftarrow{\varepsilon} & F' & \xleftarrow{\rho_i'} & F_i \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi_i} \\ Y & \xleftarrow{\pi} & F & = & F & \xleftarrow{\rho_i} & X_i \end{array}$$

By the definition of X_i, the pull-back of D to F_i by the composed map $\rho_i' \varepsilon \mathrm{pr}_1$ contains the exceptional divisor ${\rho_i'}^* E$ with multiplicity i; we set

$$D_i = (\rho_i' \, \varepsilon \, \mathrm{pr}_1)^* D - i \cdot {\rho_i'}^* E.$$

It is a relative effective divisor on F_i/X_i. See [24, Lemma 2.2] for more information.

(2.3) The sheaf $\mathcal{P}^{i-1}_{F/Y}(D)$ fits into an exact sequence

(2.3.1)
$$0 \longrightarrow \mathcal{O}_F(D) \otimes \mathrm{Sym}^{i-1}\big(\Omega^1_{F/Y}\big) \longrightarrow \mathcal{P}^{i-1}_{F/Y}(D) \longrightarrow \mathcal{P}^{i-2}_{F/Y}(D) \longrightarrow 0,$$

where the first term is the symmetric power of the sheaf of relative differentials twisted by $\mathcal{O}_F(D)$, see, e.g., [9, Theorem 7.2]. This implies in particular that $\mathcal{P}^{i-1}_{F/Y}(D)$ is locally free of rank $\binom{i+1}{2}$. Therefore, at each scheme point $x \in X_i$, the codimension of X_i in F is at most $\binom{i+1}{2}$, and if equality holds then X_i is a local complete intersection in F at x, hence equidimensional and Cohen–Macaulay, so that the assumptions of (1.4) hold for the new family $\pi_i : F_i \to X_i$.

Moreover, if X_i has the expected codimension $\binom{i+1}{2}$ in F, then its class $[X_i]$ equals the top Chern class of the vector bundle $\mathcal{P}^{i-1}_{F/Y}(D)$. It is then an exercise in Chern classes computations to express $[X_i]$ in terms of

(2.3.2)
$$v = c_1\big(\mathcal{O}_F(D)\big) \quad \text{and} \quad w_j = c_j\big(\Omega^1_{F/Y}\big) \text{ for } j = 1, 2,$$

using the above exact sequence for all $i' \leqslant i$ (see [9, Section 7.3.2] for details). We shall use (compare with (1.4)):

(2.3.3)
$$\begin{aligned} [X_2] &= v^3 + w_1 v^2 + w_2 v; \\ [X_3] &= v^6 + 4w_1 v^5 + (5w_1^2 + 5w_2)v^4 + (2w_1^3 + 11 w_1 w_2)v^3 + (6w_2 w_1^2 + 4w_2^2)v^2 + 4v w_1 w_2^2; \\ [X_4] &= v^{10} + 10 w_1 v^9 + (15 w_2 + 40 w_1^2)v^8 + (82 w_1^3 + 111 w_1 w_2)v^7 \\ &\quad + (91 w_1^4 + 315 w_2 w_1^2 + 63 w_2^2)v^6 + (52 w_1^5 + 429 w_2 w_1^3 + 324 w_1 w_2^2)v^5 \\ &\quad + (12 w_1^6 + 282 w_2 w_1^4 + 593 w_2^2 w_1^2 + 85 w_2^3)v^4 + (72 w_2 w_1^5 + 464 w_2^2 w_1^3 + 259 w_1 w_2^3)v^3 \\ &\quad + (132 w_2^2 w_1^4 + 246 w_2^3 w_1^2 + 36 w_2^4)v^2 + (72 w_2^3 w_1^3 + 36 w_1 w_2^4)v. \end{aligned}$$

(2.4) We will want to apply the above formulae to the pairs $(F_i/X_i, D_i)$. Let $e = [E]$ be the class of the exceptional divisor on F'. By a fairly common and convenient abuse of notation, we let e, v, w_1, w_2 also denote their own pull-backs by whichever map is appropriate. We have the following relation on F':

(2.4.1)
$$e^3 + w_1 e^2 + w_2 e = 0,$$

see [9, Theorem 13.14] or [13, Remark 3.2.4, p. 55].

By definition of D_i,

(2.4.2)
$$c_1(\mathcal{O}_{F_i}(D_i)) = v - ie.$$

(2.5) Lemma. *The relative Chern classes of F'/F are*

$$c_1(\Omega^1_{F'/F}) = w_1 + e \quad \textit{and} \quad c_2(\Omega^1_{F'/F}) = w_2 - e^2. \tag{2.5.1}$$

Pulling-back by ρ_i and ρ'_i, we obtain

$$c_1(\Omega^1_{F_i/X_i}) = w_1 + e \quad \text{and} \quad c_2(\Omega^1_{F_i/X_i}) = w_2 - e^2$$

as well.

Proof. For this proof it is necessary to indicate explicitly some of the pull-backs, so I will indicate those which I feel are relevant. Besides, I will drop the superscript in 'Ω' to improve readability. The map $\pi' : F' \to F$ is $\mathrm{pr}_2\varepsilon$, so we have an exact sequence

$$0 \to \mathrm{pr}_2^*\Omega_F \to \Omega_{F'} \to \Omega_{F'/F} \to 0.$$

Notice for instance that I dropped the ε^* in the leftmost term of the sequence; also, Ω_F and $\Omega_{F'}$ should be understood as $\Omega_{F/Y}$ and $\Omega_{F'/Y}$. From this sequence we get that

$$c_1(\Omega_{F'/F}) = c_1(\Omega_{F'}) - \mathrm{pr}_2^* c_1(\Omega_F) \tag{2.5.2}$$

$$c_2(\Omega_{F'/F}) = c_2(\Omega_{F'}) - \mathrm{pr}_2^* c_2(\Omega_F) - \mathrm{pr}_2^* c_1(\Omega_F) \cdot c_1(\Omega_{F'/F}). \tag{2.5.3}$$

On the other hand, from [13, Example 15.4.3], we have

$$c_1(\Omega_{F'}) = \varepsilon^* c_1(\Omega_{F^2}) + e \tag{2.5.4}$$

$$c_2(\Omega_{F'}) = \varepsilon^* c_2(\Omega_{F^2}) + j_* {\varepsilon_E}^* c_1(\Omega_\Delta) - e^2, \tag{2.5.5}$$

where F^2 stands for $F \times_Y F$, $\varepsilon_E : E \to \Delta$ is the restriction of $\varepsilon : F' \to F$, and j is the inclusion map $E \hookrightarrow F'$. There is an isomorphism $\Omega_{F^2} \cong \mathrm{pr}_1^*\Omega_F \oplus \mathrm{pr}_2^*\Omega_F$, hence

$$c_1(\Omega_{F^2}) = \mathrm{pr}_1^* c_1(\Omega_F) + \mathrm{pr}_2^* c_1(\Omega_F) \tag{2.5.6}$$

$$c_2(\Omega_{F^2}) = \mathrm{pr}_1^* c_2(\Omega_F) + \mathrm{pr}_2^* c_2(\Omega_F) + \mathrm{pr}_1^* c_1(\Omega_F) \cdot \mathrm{pr}_2^* c_1(\Omega_F). \tag{2.5.7}$$

From (2.5.2), (2.5.4), and (2.5.6), one gets

$$c_1(\Omega_{F'/F}) = \mathrm{pr}_1^* c_1(\Omega_F) + e, \tag{2.5.8}$$

which is the first formula we wanted to prove. Plugging it in (2.5.3), we get

$$c_2(\Omega_{F'/F}) = c_2(\Omega_{F'}) - \mathrm{pr}_2^* c_2(\Omega_F) - \mathrm{pr}_2^* c_1(\Omega_F) \cdot \mathrm{pr}_1^* c_2(\Omega_F) - \mathrm{pr}_2^* c_1(\Omega_F) \cdot e, \tag{2.5.9}$$

hence, with (2.5.5) and (2.5.7),

$$\begin{aligned} c_2(\Omega_{F'/F}) &= \mathrm{pr}_1^* c_2(\Omega_F) + \mathrm{pr}_2^* c_2(\Omega_F) + \mathrm{pr}_1^* c_1(\Omega_F) \cdot \mathrm{pr}_2^* c_1(\Omega_F) \\ &\quad + j_* {\varepsilon_E}^* c_1(\Omega_\Delta) - e^2 \\ &\quad - \mathrm{pr}_2^* c_2(\Omega_F) - \mathrm{pr}_2^* c_1(\Omega_F) \cdot \mathrm{pr}_1^* c_2(\Omega_F) - \mathrm{pr}_2^* c_1(\Omega_F) \cdot e \\ &= \mathrm{pr}_1^* c_2(\Omega_F) - e^2 + j_* {\varepsilon_E}^* c_1(\Omega_\Delta) - \mathrm{pr}_2^* c_1(\Omega_F) \cdot e. \end{aligned}$$

Since Δ is the diagonal, the second projection pr_2 realizes an isomorphism $\Delta \simeq F$, and thus the two terms $j_* {\varepsilon_E}^* c_1(\Omega_\Delta)$ and $\mathrm{pr}_2^* c_1(\Omega_F) \cdot e$ are equal (note that there is an omitted ε^* in the latter term); this gives the second formula we wanted to prove. □

2.2 – The recursive formula

(2.6) Definition. *Let $\pi : F \to Y$ be a smooth projective family of surfaces with a relative effective divisor D as in (1.4). Let δ be a non-negative integer. If assumption (ii) of Theorem (1.6) holds, then we define $U(D,\delta)$ to be the closure of $Y(\delta \mathrm{A}_1)$ in Y, i.e., $U(D,\delta)$ is the closure of the locus of points $y \in Y$ such that the singularities of the curve D_y are exactly δ ordinary nodes. Note that $U(D,0) = Y$. For negative δ, we set $U(D,\delta) = 0$ by convention.*

(2.7) Warning. The above definition of $U(D,\delta)$ is a simplified version of the "natural nonnegative cycle $U(D,\delta)$" introduced by Kleiman and Piene in [24, Definition 5.1]. Let us denote the latter by $\tilde{U}(D,\delta)$ in order to distinguish between the two definitions. In reality, Theorem (1.6) holds for the cycles $\tilde{U}(D,\delta)$, i.e., Formula (1.6.1) should be read as $[\tilde{U}(D,\delta)] = \frac{1}{\delta!} P_\delta(\pi_* b_1, \ldots, \pi_* b_\delta)$. However, we will only prove the theorem under the simplifying assumption that the cycles $\tilde{U}(D,\delta)$ and $U(D,\delta)$ coincide, see Condition $(\star)$ on p.420. We will give sufficient conditions for this simplifying assumption to hold in the context of Theorem (1.3), in Remark (5.8).

We will define $\tilde{U}(D,\delta)$ in Remark (4.28). For the moment, let us only comment on the differences between the various definitions.

If assumption (ii) of Theorem (1.6) holds, the difference between $U(D,\delta)$ and $\tilde{U}(D,\delta)$ is that the cycle $\tilde{U}(D,\delta)$ includes multiplicities, whereas $U(D,\delta)$ does not, being merely defined as a set-theoretic closure. If assumption (ii) of Theorem (1.6) does not hold, it is also possible that for some irreducible components U of $\tilde{U}(D,\delta)$, the fibre of D over the general point of U is not δ-nodal.

For the record, we also note that [24, Definition 5.1] differs from the older definition in [21, Section 4]: the older definition is without multiplicities, and moreover it does not exclude curves with multiple components.

(2.8) The idea of the proof of Theorem (1.6) is to relate $U(D,\delta)$ with $U(D_2,\delta-1)$, where D_2 is the relative divisor of the family F_2/X_2 defined in Section 2.1. As we will see, the pair $(F_2/X_2, D_2)$ inherits good properties from assumption (ii) for $(F/Y, D)$. Still, there are components of $U(D_2,\delta-1)$ that do not come from $U(D,\delta)$ and therefore should be discarded.

Let us consider the case $\delta = 4$ to understand the issue: $U(D,4)$ is the (closure of the) family of 4-nodal curves, and has pure codimension 4 in Y by assumption (ii). In turn, X_2 is the locus in F where D has a point of multiplicity at least 2, and D_2 is $D - 2E$ (with our convention that pull-backs are implicit) in the family F_2/X_2 of surfaces blown-up at a point of X_2. For $y \in Y(4\mathrm{A}_1)$, the curve D_y gives rises to four 3-nodal curves in the family D_2, one over each of the four points in X_2 corresponding to the four nodes of D_y. This gives a locus of pure codimension 3 in X_2 of 3-nodal curves, which is the one we care about.

It turns out however that there is another locus of pure codimension 3 of 3-nodal curves in X_2. Consider indeed the locus $Y(\mathrm{D}_4)$ [3] of curves with one ordinary triple point in Y; it has pure codimension 4 by assumption (ii) as well; surely, a curve with one ordinary triple point and no further singularity is not a limit of 4-nodal curves in Y. For $y \in Y(\mathrm{D}_4)$, the curve D_y gives rise to one 3-nodal curve in the family D_2, namely the fibre of D_2 over the triple point $x \in F_y$ of D_y, which belongs to X_2; since $D_2 = D - 2E$, the curve $(D_2)_y$ is the proper transform of D_y in the blow-up of D_y at x plus the exceptional divisor E_y with multiplicity 1. Since $Y(\mathrm{D}_4)$ has pure codimension 4 in Y and is entirely contained in X_2, the corresponding component of $U(D_2,3)$ has pure codimension 3 in X_2.

An illustrative way of phrasing this, although arguably unconventional, is that ordinary triple points may be considered as double points with three infinitely near double points.

[3] we use roman capitals to denote singularity classes, so that for instance D_4 does not get confused with the divisor D_4 on the surface F_4/X_4.

In Proposition (2.10) below we use the inclusion-exclusion principle in order to take this phenomenon into account, and come up with a recursive formula for the class $[U(D,\delta)]$. Before we can state it we need to discuss the properties inherited by the pairs $(F_i/X_i, D_i)$ from assumption (ii) of Theorem (1.6) on the initial pair $(F/Y, D)$.

Given $\delta \geqslant 1$ and $i \geqslant 2$, we set

$$\delta_i = \delta - \binom{i+1}{2} + 2; \tag{2.8.1}$$

we also set $\delta_1 = \delta$ by convention. Note that $\binom{i+1}{2} - 2$ is the expected codimension for the equisingular locus in Y of fibres of D with an ordinary i-fold point (this is a standard dimension count, which is generalized in Proposition (4.23)).

(2.9) Proposition. *Let $\pi : F \to Y$ be a smooth projective family of surfaces with a relative effective divisor D as in (1.4). Let δ be a non-negative integer, and assume that condition (ii) of Theorem (1.6) holds. Then, for all $i \geqslant 2$:*
(i) condition (ii) of Theorem (1.6) holds for the pair $(F_i/X_i, D_i)$ in codimension δ_i, i.e., for all multi-equisingularity class S, $\operatorname{codim}_{X_i}\big(X_i(\mathsf{S})\big) \geqslant \min\big(\delta_i + 1, \operatorname{expcod}(\mathsf{S})\big)$;
(ii) for all $s \leqslant \delta_i$, each component of $U(D_i, s)$ is naturally birational to a component of either the equisingular locus in Y of curves with an ordinary i-fold point and s nodes, or that of curves with an ordinary $(i+1)$-fold point and $s - i - 1$ nodes.

In order to prove this proposition properly, it is best to have an accurate description of equisingularity classes in terms of arrangements of infinitely near multiple points. For expository reasons I have chosen to describe this later in the text (in Section 4), so that I could directly give the essential ideas underlying Theorem (1.6), in the hope that the reader would have an intuitive understanding of these matters. Persevering with this hope, I will prove the proposition now, but feel free to read Section 4 first.

Proof. Let S' be an equisingularity class. The equisingular locus $X_i(\mathsf{S}')$ is the locus of points $x \in X_i$ such that the curve $(D_i)_x = (D - iE)_x$ in F'_x, the blow-up of $F_{\pi(x)}$ at x, has singularities of type S'. In particular, the curve $(D_i)_x$ is reduced, hence it either does not contain the exceptional divisor E_x, or has it as a multiplicity 1 component. Thus in the blow-down $F_{\pi(x)}$, the curve $D_{\pi(x)}$ has multiplicity either i or $i+1$ at x. Let S be its equisingularity class S; it is determined by the multiplicity at x, the equisingularity class S', and the intersection pattern of $(D_i)_x$ with the exceptional divisor E_x. In particular the following inequality holds,

$$\operatorname{expcod}(\mathsf{S}) \geqslant \operatorname{expcod}(\mathsf{S}') + \binom{i+1}{2} - 2, \tag{2.9.1}$$

see Section 4.4. Moreover, for any irreducible component Z of $X_i(\mathsf{S}')$ containing x, there is a dense subset $U \subseteq Z$ such that $\pi(U) \subseteq Y(\mathsf{S})$. Then,

$$\operatorname{codim}_Y\big(\pi(U)\big) \geqslant \min\big(\delta + 1, \operatorname{expcod}(\mathsf{S})\big)$$

by condition (ii) of Theorem (1.6) for the pair $(F/Y, D)$. Since F has dimension 2 over Y,

$$\operatorname{codim}_F(Z) = \operatorname{codim}_Y\big(\pi(U)\big) + 2.$$

In addition, X_i has codimension at most $\binom{i+1}{2}$ in F, see (2.3), so eventually

$$\operatorname{codim}_{X_i}(Z) \geqslant \min\big(\delta + 1, \operatorname{expcod}(\mathsf{S})\big) + 2 - \binom{i+1}{2},$$

and, taking (2.9.1) into account, assertion (i) of the proposition follows.

To prove assertion (ii) we need to show that if x is general in $X_i(\mathsf{S}')$ (or rather in the irreducible component Z), then the curve $D_{\pi(x)}$ in the blow-down has an ordinary singularity at x. Consider first the case when $(D_i)_x$ does not contain E_x; then $D_{\pi(x)}$ has a point of multiplicity i at x, and it is ordinary if and only if $(D_i)_x$ intersects E_x transversely. If the singularity at x were not ordinary, then we would have a strict inequality in (2.9.1), and thus Z would give by projection by π a family of too large dimension in $Y(\mathsf{S})$, given that (ii) of Theorem (1.6) holds for $(F/Y, D)$; note in particular that this condition implies that X_i has codimension exactly $\binom{i+1}{2}$ in F (we are now tacitly assuming $\delta_i \geqslant 0$, for otherwise $U(D_i, s) = 0$ by definition). The proof in the case when $(D_i)_x$ contains E_x is similar: then $D_{\pi(x)}$ has a point of multiplicity $i+1$ at x, the inequality

$$\operatorname{expcod}(\mathsf{S}) \geqslant \operatorname{expcod}(\mathsf{S}') + \binom{i+2}{2} - 2$$

holds instead of (2.9.1), and it is strict if and only if the singularity at x is non-ordinary. The conclusion follows as in the first case, noting that X_{i+1} has codimension $\binom{i+2}{2}$ in Y. □

We are now ready to state the recursive formula for $[U(D, \delta)]$.

(2.10) Proposition. *Consider the situation of Theorem (1.6). Then the following formula holds:*

$$[U(D,\delta)] = \frac{1}{\delta}\,\pi_*\Big(\sum_{i\geqslant 2}(-1)^i \rho_{i*}[U(D_i,\delta_i)]\Big), \tag{2.10.1}$$

where $\rho_i \colon X_i \to F$ denotes the inclusion, see Diagram (2.2.1)*, and the integers δ_i are as defined in* (2.8.1)*.*

The recursive formula (2.10.1) holds for the natural cycles $\tilde{U}(D, \delta)$ as well (in the notation of Warning (2.7)), see [24, Proposition 5.3]. The proof of the formula for the $\tilde{U}(D, \delta)$ requires a scheme-theoretic version of assertion (ii) of Proposition (2.9), which is proven in [24, Section 3].

Besides, while we implicitly work over the field of complex numbers in the present text, Proposition (2.9), (ii) (and therefore Formula (2.10.1)) may not hold in positive characteristic: in this case it is indeed not sufficient to prove bijectivity between two dense open subsets to obtain birationality.

Proof. The proof is along the lines of the discussion in paragraph (2.8) above. There is a $\delta : 1$ covering of $Y(\delta \mathrm{A}_1)$ in $X_2((\delta-1)\mathrm{A}_1) = X_2(\delta_2 \mathrm{A}_1)$, corresponding to the fact that each δ-nodal curve of the family D/Y has δ different partial normalisations appearing as members of D_2/X_2. By Proposition (2.9), the components of $X_2(\delta_2 \mathrm{A}_1)$ are of two kinds: those of the first kind sum up as the total space of the above mentioned $\delta : 1$ covering, whereas those of the second kind amount to curves of the family D/Y with a triple point and $\delta_2 - 3 = \delta_3$ nodes. In sum,

$$[U(D_2,\delta_2)] = \sum_{\substack{Z\subseteq X_2(\delta_2\mathrm{A}_1)\\ \text{type I}}} [Z] + \sum_{\substack{Z\subseteq X_2(\delta_2\mathrm{A}_1)\\ \text{type II}}} [Z] \qquad \text{and} \qquad \pi_*\Big(\sum_{\substack{Z\subseteq X_2(\delta_2\mathrm{A}_1)\\ \text{type I}}} [Z]\Big) = \delta[U(D,\delta)].$$

In turn, again by Proposition (2.9), the components of $X_3(\delta_3 \mathrm{A}_1)$ are of two kinds: those of the first kind are birational to the components of type II of $X_2(\delta_2 \mathrm{A}_1)$, whereas those of the second kind amount to curves of the family D/Y with a quadruple point and $\delta_3 - 4 = \delta_4$ nodes. In sum,

$$[U(D_3,\delta_3)] = \sum_{\substack{Z\subseteq X_3(\delta_3\mathrm{A}_1)\\ \text{type I}}} [Z] + \sum_{\substack{Z\subseteq X_3(\delta_3\mathrm{A}_1)\\ \text{type II}}} [Z] \quad \text{and} \quad \pi_*\Big(\sum_{\substack{Z\subseteq X_3(\delta_3\mathrm{A}_1)\\ \text{type I}}} [Z]\Big) = \pi_*\Big(\sum_{\substack{Z\subseteq X_2(\delta_2\mathrm{A}_1)\\ \text{type II}}} [Z]\Big).$$

Thus, continuing until i is large enough so that $\delta_i < 0$, we find

$$\delta\,[U(D,\delta)] = \sum_{\substack{i\geqslant 2\\ \delta_i\geqslant 0}} (-1)^i\,[U(D_i,\delta_i)],$$

which is the asserted formula. □

(2.11) Both Propositions (2.9) and (2.10) hold without any limitation on δ. The reason why one assumes $\delta \leqslant 8$ in Theorem (1.6) is that, although condition (ii) of this very Theorem (1.6) propagates indefinitely to the induced pairs $(F_i/X_i, D_i)$ with δ replaced by δ_i (this is assertion (i) of Proposition (2.9)), this is not the case for condition (i) for $\delta \geqslant 8$.[4]

To explain the issue, set $\delta = 8$ and assume conditions (i) and (ii) of Theorem (1.6) hold for $(F/Y, D)$. The equisingular stratum $Y(8\mathrm{A}_1)$ has codimension 8 in Y, and the idea of the theorem is to consider the pair $(F_2/X_2, D_2)$ and the equisingular stratum $X_2(7\mathrm{A}_1)$, which has codimension 7 in X_2, in order to understand $Y(8\mathrm{A}_1)$. Now X_4 (the scheme of points of multiplicity at least 4 in the divisor D) is contained in X_2, by definition of the X_i's, and it too has codimension 7 in X_2, as X_2 and X_4 have codimension 3 and 10 in F respectively. Since D_2 is the pull-back of D minus twice the exceptional divisor of ε, the fibres of D_2 over those points of X_2 that lie in X_4 contain the exceptional divisor with multicipity 2. Thus $X_4 \subseteq X_2(\infty)$, and condition (i) for $\delta_2 = \delta - 1 = 7$ does not hold for $(F_2/X_2, D_2)$.

This creates trouble because if we form the "X_2" of the pair $(F_2/X_2, D_2)$, it will contain the whole exceptional divisor over X_4, i.e., $\pi_2^{-1}X_4 \cap E$, which has codimension 8 in F_2, whereas the singular points of the fibres over $X_2(7\mathrm{A}_1)$ have codimension 9 in F_2. The upshot is that $\pi_2^{-1}X_4 \cap E$ will give an excess contribution when one computes the class of $X_2(7\mathrm{A}_1)$, and this has to be controlled if one is willing to prove Theorem (1.6) for $\delta = 8$.

Before we proceed, let me emphasize that the reason why Proposition (2.10) holds regardless of this issue is that, by definition, the general point in any component of the support of $U(D_i, \delta_i)$ corresponds to a curve of the family D_i/X_i which is δ_i-nodal, i.e., its singularities are exactly δ_i ordinary double points, hence non-reduced curves are completely disregarded in Formula (2.10.1).

2.3 – The computation — Proof of the main theorem

In this section we compute the class $[U(D,\delta)]$ by induction using formula (2.10.1), which will prove Theorem (1.6). We do it in (2.15) below, but before that, let us start by working out in a pedestrian way the formulae for the very first values of δ.

We make the following simplifying assumption for the whole Section 2.3 (see Warning (2.7) and the notation therein):

$(\star)$ *for all $i = 1, \ldots, \delta$, the cycle $\tilde{U}(D_i, \delta_i)$ coincides with $U(D_i, \delta_i)$.*

If assumption (ii) of Theorem (1.6) holds, this boils down to the cycles $\tilde{U}(D_i, \delta_i)$ being reduced. We will only prove Theorem (1.6) under this simplifying assumption. Note yet that $(\star)$ holds often enough; in particular, see Remark (5.8) for conditions implying it in the context of Theorem (1.3).

(2.12) Example. For $\delta = 1$ we have $[U(D,1)] = \pi_*[X_2]$ thanks to assumptions (i) and (ii) in Theorem (1.6), and the simplifying assumption $(\star)$ above. Thus, the formula for $[X_2]$ given in

[4] In fact the issue arises for all $\delta \geqslant 8$; for $\delta = 8$, Kleiman and Piene are able to overcome the issue, see Paragraph (2.16).

(2.3.3) gives us the formula we are looking for. Explicitly, we set

$$\begin{aligned} b_1(v, w_1, w_2) &= x_2(v, w_1, w_2) \\ &= v^3 + w_1 v^2 + w_2 v \end{aligned}$$

(compare with the definition in Display (1.4.1)), and then

$$[U(D, 1)] = \pi_*\big(b_1(v, w_1, w_2)\big). \tag{2.12.1}$$

Next, for $\delta = 2$, we get

$$[U(D, 2)] = \frac{1}{2}\pi_*\rho_{2*}[U(D_2, 1)]$$

from the recursive formula (2.10.1), with the the notation as in Diagram (2.2.1). Applying formula (2.12.1) to the pair $(F_2/X_2, D_2)$, and using (2.4.2) and (2.5.1) to get an expression for the classes in F_2 analogous to v, w_1, w_2 in F, we get

$$[U(D_2, 1)] = \pi_{2*}\big(\, b_1(v - 2e,\, w_1 + e,\, w_2 - e^2)\,\big). \tag{2.12.2}$$

We will need to push this class forward to F via ρ_2 to derive the formula for $[U(D, 2)]$ as π_* of some class in F. We do it in (2.14) below using the following general formula.

(2.13) Lemma. *Let R be a polynomial expression in v, w_1, w_2. Consider an integer $i \geqslant 2$ and classes v, w_1, w_2, e as in* (2.3.2) *and (2.4). Then*

$$\pi_{i*}\, \rho_i'^{*}\, R(v - ie, w_1 + e, w_2 - e^2) = \rho_i^*\, \big[\pi^*\pi_* R(v, w_1, w_2) + Q_{R,i}(v, w_1, w_2)\big],$$

in the notation of Diagram (2.2.1)*, and with $Q_{R,i}$ as defined in (1.4).*

Proof. By commutativity of Diagram (2.2.1), one has

$$\begin{aligned} \pi_{i*}\, \rho_i'^{*}\, R(v - ie, w_1 + e, w_2 - e^2) &= \rho_i^*\, \pi'_*\, R(v - ie, w_1 + e, w_2 - e^2) \\ &= \rho_i^*\, \mathrm{pr}_{2*}\, \varepsilon_*\, R(v - ie, w_1 + e, w_2 - e^2). \end{aligned}$$

We shall then use the two relations

$$\varepsilon_* e = 0 \quad \text{and} \quad \varepsilon_* e^2 = -[\Delta], \tag{2.13.1}$$

where Δ is the diagonal in $F \times_Y F$, see [13, Corollary 4.2.2] or [9, Proposition 13.12]. We expand the polynomial expression $R(v - ie, w_1 + e, w_2 - e^2)$ and reduce it modulo the relation (2.4.1); expanding a notation shorthand, we obtain

$$R(v - ie, w_1 + e, w_2 - e^2) = \varepsilon^* R(v, w_1, w_2) + e \cdot \varepsilon^* S_{R,i}(v, w_1, w_2) - e^2 \cdot \varepsilon^* Q_{R,i}(v, w_1, w_2)$$

for some polynomial $S_{R,i}$, and with $Q_{R,i}$ as defined in (1.4) (the minus sign is a feature, not a bug). Thus, with the relations (2.13.1), we get the equality of classes in $F \times_Y F$,

$$\varepsilon_*\, R(v - ie, w_1 + e, w_2 - e^2) = R(v, w_1, w_2) + Q_{R,i}(v, w_1, w_2) \cdot [\Delta].$$

In the above, $R(v, w_1, w_2)$ is a shorthand for $\mathrm{pr}_1^* R(v, w_1, w_2)$ so, since $\mathrm{pr}_{2*}\mathrm{pr}_1^* = \pi^*\pi_*$ by the commutativity of Diagram (2.2.1),

$$\mathrm{pr}_{2*}\, \varepsilon_*\, \mathrm{pr}_1^* R(v, w_1, w_2) = \pi^*\pi_* R(v, w_1, w_2).$$

On the other hand,

$$\mathrm{pr}_{2*}\big(\, \mathrm{pr}_1^* Q_{R,i}(v, w_1, w_2) \cdot [\Delta]\,\big) = Q_{R,i}(v, w_1, w_2)$$

since Δ is the diagonal. Summing up,

$$\mathrm{pr}_{2*}\, \varepsilon_*\, R(v - ie, w_1 + e, w_2 - e^2) = \pi^*\pi_* R(v, w_1, w_2) + Q_{R,i}(v, w_1, w_2),$$

which proves the lemma. □

(2.14) Example (2.12) continued. Expanding a notation shorthand in (2.12.2), we have

$$\begin{aligned}[U(D_2,1)] &= \pi_{2*}{\rho'_2}^*\big(\, b_1(v-2e,\, w_1+e,\, w_2-e^2)\,\big)\\ &= \rho_2^*\big[\pi^*\pi_* b_1(v,w_1,w_2) + Q_{b_1,2}(v,w_1,w_2)\big]\end{aligned}$$

as an application of Lemma (2.13). Then, as ρ_2 is the closed immersion $X_2 \hookrightarrow F$,

$$\begin{aligned}\rho_{2*}[U(D_2,1)] &= \rho_{2*}\rho_2^*\big[\pi^*\pi_* b_1(v,w_1,w_2) + Q_{b_1,2}(v,w_1,w_2)\big]\\ &= \big[\pi^*\pi_* b_1(v,w_1,w_2) + Q_{b_1,2}(v,w_1,w_2)\big]\cdot[X_2],\end{aligned}$$

after what

$$\begin{aligned}\pi_*\rho_{2*}[U(D_2,1)] &= \pi_*\Big(\,\big[\pi^*\pi_* b_1(v,w_1,w_2) + Q_{b_1,2}(v,w_1,w_2)\big]\cdot[X_2]\,\Big)\\ &= \pi_* b_1(v,w_1,w_2)\cdot\pi_*[X_2] + \pi_*\big(\,Q_{b_1,2}(v,w_1,w_2)\cdot[X_2]\,\big)\end{aligned}$$

where we used the projection formula for the first summand. Finally, recalling that

$$[X_2] = b_1(v,w_1,w_2) = x_2(v,w_1,w_2),$$

we get

$$\pi_*\rho_{2*}[U(D_2,1)] = \pi_* b_1(v,w_1,w_2)\cdot\pi_* b_1(v,w_1,w_2) + \pi_*\big(\,Q_{b_1,2}(v,w_1,w_2)\cdot x_2(v,w_1,w_2)\,\big);$$

we set

$$b_2(v,w_1,w_2) = Q_{b_1,2}(v,w_1,w_2)\cdot x_2(v,w_1,w_2),$$

(compare with Display (1.4.1)), and then

$$\begin{aligned}\pi_*\rho_{2*}[U(D_2,1)] &= (\pi_* b_1)^2 + \pi_* b_2\\ &= P_2\,(\pi_* b_1,\, b_2),\end{aligned}$$

as we wanted to show.

(2.15) Proof of Theorem (1.6) when $\delta \leqslant 7$, under the simplifying assumption $(\star)$. The argument is by induction on δ. For $\delta = 1$, assumptions (i) and (ii) imply that X_2 has pure codimension 3 in F, and the general point in any component of X_2 is the only singular point of a 1-nodal curve in the family D. Indeed, on the one hand X_2 has codimension at most 3 in F by its description in Section 2.1, and on the other hand the locus of 1-nodal curves in D has pure codimension 1 in Y, whereas all other equisingular loci as well as $Y(\infty)$ have codimension strictly greater; thus only the families of 1-nodal curves in D are large enough for the corresponding singular points to fill up a dense open subset in a component of X_2, and these very components have codimension exactly 3 in F. It follows that $[U(D,1)] = \pi_*[X_2]$, thanks to the simplifying assumption $(\star)$; this proves the theorem for $\delta = 1$.

Assume next that $\delta \geqslant 2$ and that the theorem holds for all families verifying the hypotheses of the theorem with δ replaced by $\delta' < \delta$. The recursive formula (2.10.1) gives

$$\delta[U(D,\delta)] = \pi_*\Big(\,\rho_{2*}[U(D_2,\delta-1)] - \rho_{3*}[U(D_3,\delta-4)]\,\Big) \tag{2.15.1}$$

since $\delta_i = 0$ for all $i > 3$, because $\delta \leqslant 7$. We want to apply the theorem by induction to the induced pairs $(F_i/X_i, D_i)$ for $i = 2, 3$, in order to compute $[U(D,\delta_i)]$. That condition (ii) in the theorem holds for these two pairs, for the integers δ_2 and δ_3 instead of δ, is assertion (i) of

Proposition (2.9). Condition (i) on the other hand is not granted a priori, and we will need to remove some negligible loci in Y to make it hold. By the definition of D_2 and D_3, $X_2(\infty)$ and $X_3(\infty)$ are

$$\big(\pi^{-1}Y(\infty)\cap X_2\big)\cup X_4 \quad\text{and}\quad \big(\pi^{-1}Y(\infty)\ \cap X_3\big)\cup X_5,$$

respectively. By condition (i) for the inital pair $(F/Y, D)$, $Y(\infty)$ has codimension at least $\delta+1$ in Y, hence we may replace Y by $Y - Y(\infty)$. Moreover, since $\delta \leqslant 7$, by condition (ii) the locus $Y(4\text{-tuple})$ (over which the fibres of D have a point of multiplicity at least 4) has codimension at least $\delta+1$ as well, so we may further replace Y by $Y - Y(4\text{-tuple})$. The upshot is that condition (i) holds for the pairs $(F_i/X_i, D_i)$, $i = 2, 3$, as well. Finally, note that X_2 and X_3 are local complete intersections in F, as they have the expected codimension everywhere; it follows that they are Cohen–Macaulay, since F is, as it is smooth over the Cohen–Macaulay Y. Thus, by induction, formula (2.15.1) gives

(2.15.2)

$$\delta[U(D,\delta)] = \pi_*\Big(\rho_{2*}\frac{1}{(\delta-1)!}P_{\delta-1}\big(\pi_{2*}b_1(D_2),\dots\big) - \rho_{3*}\frac{1}{(\delta-4)!}P_{\delta-4}\big(\pi_{3*}b_1(D_3),\dots\big)\Big).$$

Then, by Lemma (2.13), and arguing as in paragraph (2.14), we get for $i = 2, 3$:

$$\begin{aligned}\rho_{i*}P_{\delta_i}\big(\pi_{i*}b_1(D_i),\dots\big) &= \rho_{i*}\rho_i{}^*\big[P_{\delta_i}\big(\pi^*\pi_*b_1(D) + Q_{b_1,i}(D),\ \dots\big)\big]\\ &= \big[P_{\delta_i}\big(\pi^*\pi_*b_1(D) + Q_{b_1,i}(D),\ \dots\big)\big]\cdot x_i(D).\end{aligned}$$

In turn, using identity (A.7.2) about Bell polynomials, this gives:

$$\rho_{i*}P_{\delta_i}\big(\pi_{i*}b_1(D_i),\dots\big) = \sum_{s=0}^{\delta_i}\binom{\delta_i}{s}P_s\big(\pi^*\pi_*b_1(D),\ \dots\big)\,P_{\delta_i-s}\big(Q_{b_1,i}(D),\ \dots\big)\cdot x_i(D),$$

hence, by the projection formula,

$$\pi_*\rho_{i*}P_{\delta_i}\big(\pi_{i*}b_1(D_i),\dots\big) = \sum_{s=0}^{\delta_i}\binom{\delta_i}{s}P_s\big(\pi_*b_1(D),\ \dots\big)\cdot\pi_*\Big(P_{\delta_i-s}\big(Q_{b_1,i}(D),\ \dots\big)\cdot x_i(D)\Big).$$

Thus, returning to Display (2.15.2),

$$\begin{aligned}\delta!\,[U(D,\delta)] &= \sum_{s=0}^{\delta-1}\binom{\delta-1}{s}P_s\big(\pi_*b_1(D),\ \dots\big)\cdot\pi_*\Big(P_{\delta-1-s}\big(Q_{b_1,2}(D),\ \dots\big)\cdot x_2(D)\Big)\\ &\quad - \frac{(\delta-1)!}{(\delta-4)!}\sum_{s=0}^{\delta-4}\binom{\delta-4}{s}P_s\big(\pi_*b_1(D),\ \dots\big)\cdot\pi_*\Big(P_{\delta-4-s}\big(Q_{b_1,3}(D),\ \dots\big)\cdot x_3(D)\Big)\\ &= \sum_{s=0}^{\delta-1}\binom{\delta-1}{s}P_s\big(\pi_*b_1(D),\ \dots\big)\cdot\pi_*\Big(P_{\delta-1-s}\big(Q_{b_1,2}(D),\ \dots\big)\cdot x_2(D)\\ &\quad - (\delta-1-s)(\delta-2-s)(\delta-3-s)P_{\delta-4-s}\big(Q_{b_1,3}(D),\ \dots\big)\cdot x_3(D)\Big)\\ &= \sum_{s=0}^{\delta-1}\binom{\delta-1}{s}P_s\big(\pi_*b_1(D),\ \dots\big)\cdot\pi_*b_{\delta-s}(D),\end{aligned}$$

where the last equality follows directly from the definition of $b_{\delta-s}(D)$, see Display (1.4.1). By identity (A.7.1) for Bell polynomials, this reads

$$\delta!\,[U(D,\delta)] = P_\delta\big(\pi_*b_1(D),\ \dots\big),$$

which is the formula we wanted to prove. □

(2.16) Discussion of Theorem (1.6) when $\delta = 8$. For $\delta \geqslant 8$ non-reduced curves inevitably enter the picture, and with them excess contributions, see Paragraph (2.11). For $\delta = 8$ Kleiman and Piene are able to make the appropriate correction, along the following lines. First apply Proposition (2.10) as before, to the effect that

$$8\,[U(D,8)] = \pi_*\Big(\,\rho_{2*}[U(D_2,7)] - \rho_{3*}[U(D_3,4)] + \rho_{4*}[U(D_4,0)]\,\Big).$$

For $i = 3, 4$, the induced pairs $(F_i/X_i, D_i)$ do satisfy the conditions of the theorem for $\delta = \delta_i$ (shrinking Y appropriately), hence the two last summands may be computed using the theorem for $\delta' < 8$. Namely, we have

$$[U(D_3,4)] = \frac{1}{4!}P_4\big(\pi_{3*}b_1(D_3),\ldots,\pi_{3*}b_4(D_3)\big) \quad \text{and} \quad [U(D_4,0)] = [X_4].$$

For $i = 2$ however, condition (i) does not hold as $X_2(\infty) = X_4$ has codimension 7 in X_2 as we have seen in Paragraph (2.11). Still, if we remove the troublesome $X_2(\infty)$, then Formula (1.6.1) does give the class of the locus of 7-nodal curves in $X_2 \setminus X_4$; the upshot is that

$$\frac{1}{7!}P_7\big(\pi_{2*}b_1(D_2),\ldots,\pi_{2*}b_7(D_2)\big)$$

decomposes as the sum of $[U(D_2,7)]$ and a term supported on X_4.

Kleiman and Piene [24] show that in fact, provided the additional condition in Footnote 2 holds, the latter term supported on X_4 is actually $C.[X_4]$ for some integer C independent of the family $(F/Y, D)$. It is then possible to compute the constant C if one is able to enumerate the number of 8-nodal curves in one single example; Kleiman and Piene [22, Example 3.8] compute by hand the number of 8-nodal plane quintics passing through 12 general points (one could also have used the recursive formula of Caporaso and Harris), and thus find that $C = -3280$. The upshot is that

$$\begin{aligned}&8\,[U(D,8)]\\ &\quad= \pi_*\Big(\rho_{2*}\Big(\frac{1}{7!}P_7\big(\pi_{2*}b_1(D_2),\,\ldots\big) + 3280\,x_4(D)\Big) - \rho_{3*}\frac{1}{4!}P_4\big(\pi_{3*}b_1(D_3),\,\ldots\big) + \rho_{4*}x_4(D)\Big),\end{aligned}$$

from which Formula (1.6.1) follows by the same computations as before. □

The proof that the correction term is indeed $C.[X_4]$ for some integral constant C is fairly elaborate, and I will not report on it. Please see [24, Sections 6–7].

2.4 – Comparison with Vainsencher's computation

The idea of Vainsencher's proof of his version of Theorem (1.6), see [39], is essentially equivalent to that of Kleiman and Piene which I have been describing above. Yet I believe it is interesting to compare the two, as it sheds some light on the recursive formula (2.10.1).

(2.17) Vainsencher makes a direct iterated use of the formulae for the class $[X_2]$: he first considers the X_2 of the pair $(F/Y, D)$ and the induced pair $(F_2/X_2, D_2)$ as in Section 2.1, then he considers "X_2" of $(F_2/X_2, D_2)$, $X_{2,2}$ say, with the corresponding induced pair $(F_{2,2}/X_{2,2}, D_{2,2})$, and so on.

He coined the kind of singularities one directly enumerates with this procedure, with the possibility of using any X_i instead of X_2 at any stage: given a divisor D in a smooth variety V, a *singularity sequence of type* $(m_1,\ldots,m_r)$ on D is an ordered sequence of (possibly infinitely near) points $(x_1,\ldots,x_r)$ on D such that:

(i) x_1 is a point of multiplicity at least m_1 of D, and
(ii) in the blow-up $\varepsilon_1 : V_1 \to V$ at x_1, with exceptional divisor E_1, the sequence $(x_2, \dots, x_r)$ is a singularity sequence of type $(m_2, \dots, m_r)$ of the divisor $\varepsilon_1^* D - m_1 E_1$.

Then, given a pair $(F/Y, D)$ of arbitrary relative dimension n, satisfying conditions analogous to those of Theorem (1.6), the locus in Y representing members of the family D with a singularity sequence of type $(m_1, \dots, m_r)$ is closed in Y of the expected codimension $\sum_{i=1}^{r} \binom{n+m_i-1}{n} - r\,n$, and it is the support of a natural nonnegative cycle, namely the push-forward by the composition of natural maps

$$X_{m_1,\dots,m_r} \subseteq F_{m_1,\dots,m_{r-1}} \longrightarrow \cdots \longrightarrow X_{m_1,m_2} \subseteq F_{m_1} \longrightarrow X_{m_1} \subseteq F \longrightarrow Y$$

of the fundamental class $[X_{m_1,\dots,m_r}]$, which may be computed as in Section 2.1.

(2.18) Denote the type of singularity sequence $(m, \dots, m)$ with r repetitions by (m^r) for short. Given a δ-nodal curve C, each ordering of its nodes yields a singularity sequence of type (2^δ). However not all singularity sequences of type (2^δ) are of this kind: one may of course have any kind of double points instead of nodes, but these are not a problem as non-ordinary double points have expected codimension larger than that of ordinary double points; the non-nodal singularity sequences of type (2^δ) one needs to be careful about are those arising from a point of multiplicity larger than 2: for instance, an ordinary triple point has expected codimension 4 and gives rise to 3! singularity sequences of type (2^4), as explained in Paragraph (2.8).

Thus, the enumeration of curves with a singularity sequence of type (2^δ) in a family D as in Paragraph (2.17) above amounts to the enumeration of curves with any of the (finitely many) singularity types that give rise to singularity sequences of type (2^δ) and have expected codimension δ, with weights corresponding to the possible orderings. Thus, after one has enumerated curves with a singularity sequence of type (2^δ), one is left with the task of isolating the contribution of δ-nodal curves in this enumeration.

It is worth noting that the recursive formula (2.10.1) of Kleiman and Piene provides an automated process for selecting only the contribution of δ-nodal curves among all curves with a singularity sequence of type (2^δ).

(2.19) For $\delta \leqslant 7$, the list of the singularity types giving rise to singularity sequences of type (2^δ) and of the appropriate expected codimesion δ has been worked out by Vainsencher [39] and Kleiman–Piene [21]. To do so it is convenient to use Enriques diagrams, which are presented in Section 4; this is the approach adopted in [21]. The list is given in Table 1, see Section 4 for the (standard) notation.

The cases $\delta \leqslant 4$ have already been discussed above (note that D_4 is the equisingularity class of ordinary triple points). For $\delta = 5$ nothing really new happens, but let me enumerate the number of sequences arising from an ordinary triple point and a node: one may either let the genuine node come first, and then we have 3! orderings for the rest as we know from the $\delta = 4$ case, or let the triple point come first, after what we have four nodes in the divisor "$\varepsilon^* D - 2E$" (three of wich at the intersection of the proper transform with the exceptional divisor), which gives 4! possible orderings; the total number of singularity sequences is therefore $3! + 4! = 30$. We leave it as a fun exercise to the reader to count the number of singularity sequences of type (2^6) and (2^7) arising from singularities $\mathrm{D}_4 + 2\mathrm{A}_1$ and $\mathrm{D}_4 + 3\mathrm{A}_1$, respectively.

The singularity D_6 is a triple point with three local branches, two of which are tangent, or in more classical terms, a triple point with an infinitely near double point (see Section 4 for more details). Thus, after blowing-up the ambient surface at such a singularity, when taking "$\varepsilon^* D - 2E$", one arrives at one ordinary double point and an ordinary triple point, both at the intersection of the proper transform with the exceptional divisor; this is a collection of

δ	type of singularity	number of sequences
$\leqslant 3$	$\delta \mathrm{A}_1$	$\delta!$
4	$4\mathrm{A}_1$	4!
	D_4	3!
5	$5\mathrm{A}_1$	5!
	$\mathrm{D}_4 + \mathrm{A}_1$	30
6	$6\mathrm{A}_1$	6!
	$\mathrm{D}_4 + 2\mathrm{A}_1$	180
	D_6	30
7	$7\mathrm{A}_1$	7!
	$\mathrm{D}_4 + 3\mathrm{A}_1$	1260
	$\mathrm{D}_6 + \mathrm{A}_1$	210
	E_7	30

Table 1: Singularity types giving rise to singularity sequences of type (2^r), and with expected codimension r

singularities of type $\mathrm{D}_4 + \mathrm{A}_1$, hence the number 30 of singularity sequences of type (2^6) arising from a D_6 singularity.

The singularity E_7 is a triple point with two local branches, one ordinary cusp and one smooth branch, with their tangent cones supported on the same line; in other words, it is a triple point with an infinitely near tacnode. After blowing-up and taking "$\varepsilon^* D - 2E$", one arrives at a singularity of type D_6, with the two tangent local branches given by the exceptional divisor and the proper transform of the cuspidal branch respectively. Hence the number of singularity sequences of type (2^7) arising from a E_7 singularity is yet again 30. □

From the classification given in Table 1 one deduces, under the assumptions of Theorem (1.6), the following decompositions of the cycles $[U(D, 2^r)]$ enumerating curves with a singularity sequence of type (2^r) in D, obtained as the push-forward of $[X_{2^r}]$ as in Paragraph (2.17) above: $[U(D, 2^\delta)] = \delta!\,[U(D, \delta\mathrm{A}_1)]$ for all $\delta \leqslant 3$, and

$$
\begin{aligned}
\big[U(D, 2^4)\big] &= 4!\,\big[U(D, 4\mathrm{A}_1)\big] + 3!\,\big[U(D, \mathrm{D}_4)\big] \\
\big[U(D, 2^5)\big] &= 5!\,\big[U(D, 5\mathrm{A}_1)\big] + 30\,\big[U(D, \mathrm{D}_4 + \mathrm{A}_1)\big] \\
\big[U(D, 2^6)\big] &= 6!\,\big[U(D, 6\mathrm{A}_1)\big] + 180\,\big[U(D, \mathrm{D}_4 + 2\mathrm{A}_1)\big] + 30\,\big[U(D, \mathrm{D}_6)\big] \\
\big[U(D, 2^7)\big] &= 7!\,\big[U(D, 7\mathrm{A}_1)\big] + 1260\,\big[U(D, \mathrm{D}_4 + 3\mathrm{A}_1)\big] + 210\,\big[U(D, \mathrm{D}_6 + \mathrm{A}_1)\big] + 30\,\big[U(D, \mathrm{E}_7)\big].
\end{aligned}
\tag{2.19.1}
$$

In these formulae, for all (multi)equisingularity class S, $U(D, \mathsf{S})$ denotes the closure in Y of the constructible set $Y(\mathsf{S})$ of all points $y \in Y$ such that the singularities of the curve D_y are of the type S; thus, $U(D, \delta\mathrm{A}_1)$ is what was denoted by $U(D, \delta)$ before.

(2.20) To conclude this section let me outline, following [21], how to compute, under the assumptions of Theorem (1.6) for $\delta \leqslant 7$, the non-nodal terms in the decompositions of Display (2.19.1) (it would be outrageous to call them parasitic), so that it is possible to obtain $[U(D, \delta)]$ from $[U(D, 2^\delta)]$.

First of all, for $\delta \leqslant 3$, it is straightforward to obtain $[U(D, \delta)]$ from $[U(D, 2^\delta)]$, hence to compute $[U(D, \delta)]$ from the procedure in Paragraph (2.17) (correspondingly, there is only one non-trivial summand in the recursive formula (2.10.1) in this case).

Then, it is possible to perform the same computations for the pair $(F_3/X_3, D_3)$; note that the conditions of Theorem (1.6) still hold for $\delta \leqslant 3$ for this pair. Moreover, thanks to these very conditions, the members of the family D_3 generically correspond to the members of the

initial family D with an ordinary triple point (i.e., with a singularity of type D_4: once again I apologize for this regrettable notation). Thus, it is possible to compute $[U(D, \mathrm{D}_4 + \delta\mathrm{A}_1)]$ for all $\delta = 0, 1, 2, 3$:

$$\big[U(D, \mathrm{D}_4 + \delta\mathrm{A}_1)\big] = \pi_*\big[U(D_3, \delta\mathrm{A}_1)\big].$$

We will see now how to compute $[U(D, \mathrm{D}_6)]$, $[U(D, \mathrm{D}_6 + \mathrm{A}_1)]$, and $[U(D, \mathrm{E}_7)]$. When this is done we will know all cycle classes in the formulae of Display (2.19.1) except $[U(D, \delta\mathrm{A}_1)]$ for $\delta = 4, 5, 6, 7$, so that it is possible to invert the system of linear relations to find them. The idea is once again to apply the formulae we know so far to an aptly chosen induced pair.

As we have seen in Paragraph (2.19) above, a singularity D_6 in D gives a D_4 and a node in D_2, with the two latter singularity on the exceptional divisor. Of course it is also possible to obtain $\mathrm{D}_4 + \mathrm{A}_1$ in D_2 from a curve in D with singularities of type $\mathrm{D}_4 + 2\mathrm{A}_1$, in two ways as one may blow-up at either one of the two nodes. In fact, the following relation holds:

$$\pi_*\big[U(D_2, \mathrm{D}_4 + \mathrm{A}_1)\big] = 2\left[U(D, \mathrm{D}_4 + 2\mathrm{A}_1)\right] + \big[U(D, \mathrm{D}_6)\big]$$

(see [21, p. 233] for more details). Since we already know $[U(D', \delta\mathrm{A}_1)]$ and $[U(D', \mathrm{D}_4 + \delta\mathrm{A}_1)]$ for $\delta \leqslant 3$ for any family D', we obtain a general formula for $[U(D, \mathrm{D}_6)]$. In a similar fashion, we have the relation:

$$\pi_*\big[U(D_2, \mathrm{D}_4 + 2\mathrm{A}_1)\big] = 3\left[U(D, \mathrm{D}_4 + 3\mathrm{A}_1)\right] + \big[U(D, \mathrm{D}_6 + \mathrm{A}_1)\big],$$

which gives a general formula for $[U(D, \mathrm{D}_6 + \mathrm{A}_1)]$.

Finally, since a singularity E_7 gives a D_6 in the family D_2, see Paragraph (2.19), we have the analogous relation:

$$\pi_*\big[U(D_2, \mathrm{D}_6)\big] = \big[U(D, \mathrm{D}_6 + \mathrm{A}_1)\big] + \big[U(D, \mathrm{E}_7)\big].$$

It gives a general formula for $[U(D, \mathrm{E}_7)]$, and thus we have completed our program.

3 – Applications

In this section we spell out the computations for various applications of Theorem (1.6), essentially Theorems (1.3) and (1.7). The question of their validity is left aside for the moment, and will be considered later in Section 5, after the necessary material in equisingularity theory is treated.

3.1 – Curves in a linear system on a fixed surface

In this section I show how to get Formula (1.3.1) from (1.6.1). The setup is thus as in Theorem (1.3).

(3.1) Let S be a smooth irreducible complex projective surface, and L be a line bundle on S; set

$$d = L^2 \ ; \quad k = L \cdot K_S \ ; \quad s = K_S^2 \ ; \quad x = c_2(S).$$

In order to apply Theorem (1.6), we let Y be the complete linear system $|L|$, and set $F = S \times Y$; we take $\pi : F \to Y$ and $p : F \to S$ to be the second and first projections, respectively. We let the relative divisor D in F/Y be the universal family of the linear system $|L|$.

Explicitly, let $s_0, \ldots, s_n$ be a basis of $H^0(S, L)$, and consider the corresponding homogeneous coordinates $(a_0 : \ldots : a_n)$ on the projective space $|L|$. Then D is defined in $F = S \times Y$ by the equation

$$a_0 s_0 + \cdots + a_n s_n = 0.$$

Thus we see that $\mathcal{O}_F(D) = p^*L \otimes \pi^*\mathcal{O}_Y(1)$, hence

$$v = c_1\big(\mathcal{O}_F(D)\big) = p^*l + \pi^*h, \quad \text{where} \quad l = c_1(L) \quad \text{and} \quad h = c_1\big(\mathcal{O}_Y(1)\big).$$

Also, $\Omega^1_{F/Y}$ is simply the pull-back $p^*\Omega^1_S$, hence

$$w_j = c_j\big(\Omega^1_{F/Y}\big) = p^*c_j\big(\Omega^1_S\big) \quad \text{for} \quad j = 1, 2.$$

(3.2) Lemma. *Let α, β_1, β_2 be non-negative integers such that $\alpha + \beta_1 + 2\beta_2 = \delta + 2$. Then:*

$$\pi_*\big(v^\alpha w_1^{\beta_1} w_2^{\beta_2}\big) = \begin{cases} \binom{\delta+2}{2} d.h^\delta & \text{if } (\alpha, \beta_1, \beta_2) = (\delta+2, 0, 0) \\ (\delta+1)k.h^\delta & \text{if } (\alpha, \beta_1, \beta_2) = (\delta+1, 1, 0) \\ s.h^\delta & \text{if } (\alpha, \beta_1, \beta_2) = (\delta, 2, 0) \\ x.h^\delta & \text{if } (\alpha, \beta_1, \beta_2) = (\delta, 0, 1) \\ 0 & \text{otherwise,} \end{cases}$$

with d, k, s, x as in (3.1) above.

Proof. For dimension reasons, one has $w_1^{\beta_1} = 0$ if $\beta_1 > 2$ and $w_2^{\beta_2} = 0$ if $\beta_2 > 1$; similarly, when expanding $v^\alpha = (p^*l + \pi^*h)^\alpha$, all terms involving p^*l to a power striclty larger than 2 vanish. Moreover, for integers $\alpha', \alpha'', \beta_1, \beta_2$,

$$\begin{aligned} \pi_*\big(\,\pi^*h^{\alpha'} \cdot p^*(l^{\alpha''} w_1^{\beta_1} w_2^{\beta_2})\,\big) &= h^{\alpha'} \cdot \pi_* p^*(l^{\alpha''} w_1^{\beta_1} w_2^{\beta_2}) \\ &= \begin{cases} l^{\alpha''} w_1^{\beta_1} w_2^{\beta_2}.h^{\alpha'} & \text{if } \alpha'' + \beta_1 + 2\beta_2 = 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then the result follows from standard algebraic manipulations. □

Finally, applying the recipes in (1.4) with the above incarnations of v, w_1, w_2 to compute

$$a_i = \pi_* b_i(D, F/Y) \quad \text{for} \quad i = 1, \ldots, 8,$$

one gets the formulae in (1.2), and thus (1.3.1), as desired.

3.2 – Plane curves in a solid in $\mathbf{P}^4$

In this section I show how to get Formula (1.7.1) from (1.6.1), and then how to derive the number of irreducible 6-nodal plane quintics lying in a general quintic threefold.

(3.3) Computations for Formula (1.7.1)**.** Let $Y = \mathrm{Gr}(3, 5)$ be the Grassmannian of 2-planes $\Pi \subseteq \mathbf{P}^4$, and F be the corresponding universal family: thus

$$F = \mathbf{P}(\mathcal{Q}) \subseteq Y \times \mathbf{P}^4,$$

where $\mathcal{Q}$ is the tautological rank 3 quotient bundle of $\mathcal{O}_Y^5$. Let $\pi : F \to Y$ and $p : F \to \mathbf{P}^4$ be the two projections.

Let $V \subseteq \mathbf{P}^4$ be a general threefold of degree m, and define

$$D = F \cap (Y \times V).$$

If $m \geqslant 2$, then V cannot contain a plane, and thus for all $y \in Y$, the fibre D_y, which identifies with the intersection $F_y \cap V$ in $\mathbf{P}^4$, is a degree m plane curve; the upshot is that D is a relative effective divisor on F/Y.

We may then apply Theorem (1.6) to derive the class in Y of the locus of δ-nodal curves in the family D. The Grassmannian Y has dimension 6, so this computation makes sense only if $\delta \leqslant 6$. For all $\delta = 1, \dots, 6$, Kleiman and Piene show in [22, Section 4] that assumptions (i) and (ii) of Theorem (1.6) hold in our situation if $m \geqslant 4$. In fact, this follows without too much difficulty from the verification of these very assumptions in the case of plane curves, i.e., the situation of Section 3.1 in the particular case when $S = \mathbf{P}^2$ and $L = \mathcal{O}_{\mathbf{P}^2}(m)$, carried out in [22, Section 3]. I will report on the latter analysis in Section 5, but skip the details of the adaptation to our current situation.

I will now sketch the computations needed to apply the recipes of (1.4) in the case under consideration, leaving aside the particulars of intersection theory on Grassmannians. Since the divisor D is defined in $Y \times \mathbf{P}^4$ by the equation of $V \subseteq \mathbf{P}^4$, constant along Y, one has $\mathcal{O}_F(D) = p^*\mathcal{O}_{\mathbf{P}^4}(m)$, hence

$$v = c_1\big(\mathcal{O}_F(D)\big) = m.p^*h, \quad \text{where} \quad h = c_1\big(\mathcal{O}_{\mathbf{P}^4}(1)\big).$$

To compute the Chern classes $w_j = c_j(\Omega^1_{F/Y})$, one uses the Euler exact sequence

$$0 \longrightarrow \Omega^1_{F/Y}(1) \longrightarrow \pi^*\mathcal{Q} \longrightarrow \mathcal{O}_F(1) \longrightarrow 0,$$

and finds

$$w_1 = \pi^*q_1 - 3p^*h \quad \text{and} \quad w_2 = \pi^*q_2 - 2p^*h \cdot \pi^*q_1, \quad \text{where} \quad q_j = c_j(\mathcal{Q}) \quad \text{for} \quad j = 1, 2.$$

To compute with these classes note that, for dimension reasons, $h^j = 0$ for $j > 4$, and $\pi_*p^*h^j = 0$ for $j = 0, 1$; moreover,

$$\pi_*p^*h^2 = [Y], \qquad \pi_*p^*h^3 = q_1, \qquad \pi_*p^*h^4 = q_1^2 - q_2.$$

Then, it is algorithmic to derive the $a_i = \pi_*b_i(D, F/Y)$, $i = 1, \dots, 6$, with the notation as in (1.4). After that, Formula (1.6.1) gives an expression of the classes $[U(D, \delta)]$, $\delta = 1, \dots, 6$, in the Chow ring of the Grassmannian $Y = \mathrm{Gr}(3, 5)$, as polynomials in the integer m and the classes q_1 and q_2, if $m \geqslant \delta/2 + 1$.

For $\delta = 6$, this class has degree $6 = \dim(Y)$, and one finds Formula (1.7.1) by applying the identities describing the intersection product in the Chow ring of the $Y = \mathrm{Gr}(3, 5)$, namely

$$q_1^6 = 5, \qquad q_1^4q_2 = 3, \qquad q_1^2q_2^2 = 2, \qquad q_2^3 = 1.$$

For lower values of δ, one may use the expression of $[U(D, \delta)]$ to compute other enumerative numbers. Kleiman and Piene give the example of

$$[U(D, 3)] \cdot q_1^3 = \frac{m}{6}\left(5\,m^8 - 30\,m^7 + 33\,m^6 + 23\,m^5 + 102\,m^4 + 359\,m^3 - 2330\,m^2 + 2048\,m + 240\right),$$

which is the number of 3-nodal plane sections of V whose span meets three given lines in general position in $\mathbf{P}^4$.

(3.4) Example. For $\delta = 6$ and $m = 4$, one finds that in a general quartic solid there are 5600 6-nodal plane quartic curves; each of those is the sum of four coplanar lines, since plane quartics have arithmetic genus 3.

(3.5) Theorem (Katz, Vainsencher, Kleiman–Piene)**.** *Let V be a general quintic threefold in $\mathbf{P}^4$; it contains exactly:*
(i) 2875 *lines;*
(ii) 609 250 *smooth conics;*
(iii) 17 601 000 *irreducible* 6*-nodal plane quintics.*

For the first two numbers, see [18, p. 175] and, for more details on the number of lines, [9, Section 6.5]. I will now explain how to compute the third from the first two and Formula (1.7.1), following Vainsencher and Kleiman–Piene; again, I will skip the verification of the validity of the computation, for which I refer to [22, Proof of Theorem 4.3].

(3.6) Maintain the notation of Paragraph (3.3), and assume $m = 5$. Applying Formula (1.7.1) one finds the number of 6-nodal plane quintics in V, 21 617 125. Among them the reducible ones are either the sum of a line and a 2-nodal quartic curve, or the sum of a conic and cubic.

Curves of the latter type are the easier to enumerate. Let C be a smooth conic contained in V. Then C spans a plane, which intersects V residually in a cubic C'; for general V it is proven in [22] that for all smooth conics C in V the cubic C' is smooth and intersects C transversely. Thus the number of 6-nodal quintics sum of a conic and a cubic in V equals the number of smooth conics in V, namely 609 250.

Now let L be a line in V. Planes in $\mathbf{P}^4$ containing L form a projective plane in the Grassmannian $Y = \mathrm{Gr}(3, 5)$, and each of those cuts out V residually in a plane quartic. We will compute the number of 2-nodal curves in this family by applying Theorem (1.6) yet again (taking for granted that the quartics intersect L transversely). Then we will need to subtract 2875 times this number from the number of 6-nodal plane quintics in V.

Denote by $Y_L \subseteq Y$ the variety of planes $\Pi \subseteq \mathbf{P}^4$ containing L, and set $F_L = F \times_Y Y_L$, so that $\pi_L = \pi|_{F_L} : F_L \to Y_L$ is the two-dimensional family of planes we want to consider. Then we set

$$D_L = F_L \cap (Y_L \times V) \quad \text{and} \quad D'_L = D_L - (Y_L \times L);$$

thus D_L is the restriction to F_L of the universal family D of plane quintics in V, and D'_L is the universal family of plane quartics residual to L. We want to apply Theorem (1.6) to the pair $(D'_L, F_L/Y_L)$.

The two Chern classes $w'_j = c_j(\Omega^1_{F_L/Y_L})$ are simply the restrictions of the two classes $w_j = c_j(\Omega^1_{F/Y})$, as $F_L = \pi^{-1}(F_L)$. Now let us compute the class of D'_L. The class of D_L is the restriction of $v = c_1(\mathcal{O}_F(D)) = 5p^*h$ to F_L, and we want to subtract the class of the divisor $Y_L \times L$. The inclusion of this divisor in F_L corresponds to a quotient map $\mathcal{Q}|_{Y_L} \twoheadrightarrow \mathcal{O}^{\oplus 2}_{Y_L}$, and the sheaf of ideals of $Y_L \times L$ in F_L is $\pi_L^*\mathcal{K}_L(-1)$, where $\pi : F_L \to Y_L$ is the projection and $\mathcal{K}_L$ is the locally free $\mathcal{O}_{Y_L}$-module fitting in the exact sequence

$$0 \longrightarrow \mathcal{K}_L \longrightarrow \mathcal{Q}|_{Y_L} \longrightarrow \mathcal{O}^{\oplus 2}_{Y_L} \longrightarrow 0.$$

Thus,

$$v' = c_1\big(\mathcal{O}_{F_L}(Y_L \times L)\big) = -c_1\big(\pi_L^*\mathcal{K}_L(-1)\big) = (p^*h - \pi^*q_1)|_{F_L}\,.$$

Finally we compute, in the notation of Paragraph (1.4),

$$\begin{aligned} a_i &= \pi_{L*} b_i\,(v', w'_1, w'_2) \\ &= \pi_{L*} b_i\,\big({\rho'_L}^*(p^*h - \pi^*q_1),\, {\rho'_L}^* w_1,\, {\rho'_L}^* w_2\big) \\ &= \rho_L^*\pi_* b_i\,\big(p^*h - \pi^*q_1,\, w_1,\, w_2\big) \\ &= [Y_L] \cdot \pi_* b_i\,\big(p^*h - \pi^*q_1,\, w_1,\, w_2\big), \end{aligned}$$

where ρ_L and ρ'_L are the two closed immersions in the following Cartesian diagram.

$$\begin{array}{ccc} F_L & \xhookrightarrow[\rho'_L]{} & F \\ {\scriptstyle \pi_L}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y_L & \xhookrightarrow[\rho_L]{} & Y \end{array}$$

Using the formula

$$[Y_L] = (q_1^2 - q_2)^2$$

for the class of the Schubert cell Y_L in the Grassmannian Y, one completes the computation and finds that the number of 2-nodal quartics in the family D'_L is 1185 by Formula (1.6.1).

The upshot is that the number of irreducible 6-nodal quintics in V is

$$21617125 - 609250 - 2875 \times 1185 = 17601000$$

as asserted. □

4 – Equisingularity classes

This section is devoted to the definition and study of equisingularity classes. The first two sections 4.1 and 4.2 contain preliminaries on infinitely near points and their configurations, while the main definitions (including those of equisingularity classes and Enriques diagrams) are given in section 4.3. In the last section 4.4 we discuss the expected codimension of the locus of curves belonging to a prescribed equisingularity class in a given family of curves.
In this section, unless explicit indication of the contrary, curves are assumed to be reduced.

4.1 – Infinitely near points and proximity

(4.1) Definition (infinitely near points)**.** *Let p be a (closed) point on a smooth surface S. The* points in the first infinitesimal neighbourhood *of p are the (closed) points of the exceptional divisor of the blow-up of S at p. Then by induction, for all $i > 1$, the* points in the i-th infinitesimal neighbourhood *of p are the points in the first infinitesimal neighbourhood of some point in the $(i-1)$-th infinitesimal neighbourhood of p.*

Points infinitely near *to p are those points lying in the i-th infinitesimal neighbourhood of p for some $i > 0$. By contrast, points lying on S itself are called* proper points*. An infinitely near point of S is a point infinitely near to some proper point of S.*

(4.2) Definition (precedence)**.** *Let p, q be two points, proper or infinitely near, on S. We say that p* precedes *q if q is infinitely near to p. When this is the case, we write $p < q$. We will also use the usual variants, like $p \leqslant q$ and so on.*

(4.3) Definition (proximity)**.** *Let p, q be two points, proper or infinitely near, on S. We say that q is* proximate *to p if it belongs to the exceptional divisor E_p of the blow-up of S at p, or to the proper transform of E_p in some further blow-up of S. When this is the case, we write $p \prec q$. We will also use the usual variants, like $p \preccurlyeq q$ and so on.*

For example, the points in the first and second infinitesimal neighbourhood of the origin of the affine plane lying on the proper transforms of the curve $\mathrm{V}(y^2 - x^3)$ are both proximate to the origin.

Observe that a point proximate to p lies either in the first infinitesimal neighbourhood of p or in the first infinitesimal neighbourhood of a point proximate to p (this follows directly from the definition; checking this is a good test of your understanding of the definition).

(4.4) Lemma. *Let p, q be two points on S, proper or infinitely near, and assume that $p < q$. Then q is proximate to either one or two points infinitely near or equal to p.*

Proof. Let $\varepsilon : S' \to S$ be a composition of blow-ups

$$S' = S_n \xrightarrow{\varepsilon_n} S_{n-1} \xrightarrow{\varepsilon_{n-1}} \cdots \xrightarrow{\varepsilon_2} S_1 \xrightarrow{\varepsilon_1} S_0 = S$$

such that q is a proper point of S'; we may assume that each ε_i is the blow-up at some (proper) point $p_i \in S_{i-1}$. The exceptional divisor of ε is a disjoint union of trees of smooth rational curves, the irreducible components of which are the proper transforms $E_1, \dots, E_n$ of the exceptional divisors of $\varepsilon_1, \dots, \varepsilon_n$ respectively.

Thus the point q lies on either one or two of the curves $E_1, \dots, E_n$, and correspondingly it is proximate to either one or two points infinitely near or equal to p. □

(4.5) Definition (free and satellite points)**.** *Let p be a proper point of S, and q a point infinitely near to p. The point q is* free *if it is proximate to exactly one point infinitely near or equal to p; otherwise it is a* satellite*. One says that a satellite point is a* satellite of *the last free point that precedes it.*

Thus, in the setup of the proof of Lemma (4.4) above, the point q is free if it is a smooth point of the exceptional divisor of ε, and it is satellite otherwise. Since the exceptional divisor is a union of trees, a satellite point is proximate to exactly two points infinitely near or equal to p.

All points in the first infinitesimal neighbourhood of a proper point are free. The first infinitesimal neighbourhood of a free (infinitely near) point contains exactly one satellite point, whereas that of a satellite point contains exactly two satellite points.

4.2 – Multiplicities and proximity equalities

(4.6) Let C be a curve on a smooth surface S, and q be an infinitely near point on S. One defines the multiplicity of C at q as follows. Let $\varepsilon : S' \to S$ be a composition of blow-ups such that q is a proper point of S', and let C_q be the proper transform of C in S'. The multiplicity of C at q is the multiplicity of C_q at q. We denote it by $\mathrm{mult}_q(C)$.

We say that the point q lies on C if $\mathrm{mult}_q(C)$ is positive; it is a multiple point of C if $\mathrm{mult}_q(C) > 1$, and a simple point if $\mathrm{mult}_q(C) = 1$.

The following result will arguably feel very natural. I will not prove it; if necessary, you can find it as [7, Theorem 3.3.1].

(4.7) Proposition (Noether's formula)**.** *Let A and B be two curves on a smooth surface S, and p be a proper point on S. The intersection multiplicity of A and B at p (possibly infinite) is*

$$(A \cdot B)_p = \sum_{p \leqslant q} \mathrm{mult}_q(A)\,\mathrm{mult}_q(B) \tag{4.7.1}$$

(the sum is on all points infinitely near or equal to p).

The sequence of multiplicities of a curve at infinitely near points satisfies the following relations.

(4.8) Proposition (Proximity equalities)**.** *Let C be a curve on a smooth surface S, and p be a point on S, proper or infinitely near. Then,*

$$\mathrm{mult}_p(C) = \sum_{p \prec q} \mathrm{mult}_q(C) \tag{4.8.1}$$

(the sum is on all points q proximate to p).

Proof. Let $\varepsilon : S' \to S$ be a composition of blow-ups such that p is a proper point of S', and let $\varepsilon' : S'' \to S'$ be the blow-up at p, with exceptional divisor E_p. Let C_p and C'_p be the proper transforms of C in S' and S'' respectively. Then,

$$\mathrm{mult}_p(C) = \mathrm{mult}_p(C_p) = \sum_{q \in E_p} \left(E_p \cdot C'_p\right)_q,$$

where the first equality is by definition. Then one obtains the relation by applying Noether's formula (4.7.1) to each summand, noting that E_p is smooth hence has multiplicity one at all its points, proper or infinitely near. □

(4.9) Remark. As a consequence of the proximity equalities, if some satellite point q lies on a curve C, then it is proximate to a multiple point of C. Indeed, let p be the immediate predecessor of q, and p_0 be the other point q is proximate to. Then both p and q are proximate to p_0 hence, by the proximity equalities,

$$\mathrm{mult}_{p_0}(C) \geqslant \mathrm{mult}_p(C) + \mathrm{mult}_q(C),$$

and the sum is greater than one since both p and q lie on C.

4.3 – Equisingularity and Enriques diagrams

(4.10) Definition (singular points of a curve)**.** *Let C be a curve on a smooth surface S. A point p of C, proper or infinitely near, is a* singular point *of C if one of the following conditions holds:*
(a) p is a multiple point of C;
(b) p is a satellite point;
(c) p precedes a satellite point on C.

Note that there may be simple and free points on a curve C preceding a satellite point still on C, and these are singular points of C. For example, this is the case of the point in the first infinitesimal neighbourhood of the origin on the affine curve $\mathrm{V}(y^2 - x^3)$. Nervertheless, a proper point of C may precede a satellite point on C only if it is multiple by Remark (4.9), hence the proper singular points of C are the proper multiple points of C.

The following alternative characterization of singular points is helpful in grasping the concept (so let me leave the proof as an exercise). Let p be an infinitely near point on C, and consider $\pi : S' \to S$ the minimal series of blow-ups giving rise to p i.e., the minimal series of blow-ups such that p is a proper point of S'. Call E the exceptional divisor of π; it is a tree of smooth rational curves. Call C' the proper transform of C in S'. Then p is a singular point of C if and only if the intersection multiplicity of C' with E at p is larger than 1. Thus, if p is a simple point of C (equivalently, of C'), either C' is tangent to E at p, or p is a double point of E.

(4.11) In order to define equisingularity of two curves with planar singularities, we need the following definitions. Let C be a reduced curve on a smooth surface S. The *cluster of singular points* of C is the set of all singular points of C, proper or infinitely near, endowed with the two order relations of precedence and proximity; we denote it by $\mathrm{Sing}(C)$. The *augmented cluster of singular points* of C is the set of all singular points of C, together with those non-singular points of C that lie in the first infinitesimal neighbourhood of a singular point, endowed with the two order relations of precedence and proximity; we denote it by $\mathrm{Sing}^+(C)$. It should be noted that, by resolution of singularities for curves, both sets $\mathrm{Sing}(C)$ and $\mathrm{Sing}^+(C)$ are finite.

The set underlying the augmented cluster of singular points may be defined alternatively as follows. Let $p_1, \ldots, p_r$ be the singular points of C which are proper on S. For all $i = 1, \ldots, r$,

let $\gamma_{i,1}, \dots, \gamma_{i,s_i}$ be the local branches of C at p_i. For all $i = 1, \dots, r$ and $j = 1, \dots, s_i$, let $q_{i,j}$ be the first point (for the precedence order relation) on the branch $\gamma_{i,j}$ which is non-singular in the sense of (4.10). Then the set underlying $\mathrm{Sing}^+(C)$ is the set of all proper or infinitely near points on C preceding or equal to a point $q_{i,j}$.

(4.12) Definition (equisingularity)**.** *Let $C \subseteq S$ and $D \subseteq T$ be two reduced curves on two smooth surfaces respectively. The two curves C and D are* equisingular *if there exists a bijection $\varphi : \mathrm{Sing}^+(C) \to \mathrm{Sing}^+(D)$ such that both φ and φ^{-1} preserve the two order relations of precedence and proximity.*

We do not require anything on the multiplicities of C and D at their singular points in the above definition. In fact, it comes automatically that φ preserves multiplicities.

(4.13) Proposition. *Let $\varphi : \mathrm{Sing}^+(C) \to \mathrm{Sing}^+(D)$ be a bijection as in Definition (4.12), realizing the equisingularity of C and D. Then, for all points $p \in \mathrm{Sing}^+(C)$:*

$$\mathrm{mult}_p(C) = \mathrm{mult}_{\varphi(p)}(D).$$

Proof. The proof is by decreasing induction with respect to the precedence order on $\mathrm{Sing}^+(C)$. If p is a singular point of C maximal in $\mathrm{Sing}^+(C)$ for precedence, then so is $\varphi(p)$, and thus p and $\varphi(p)$ are simple points of C and D respectively, in particular they both have multiplicity 1. Now if p is non-maximal in $\mathrm{Sing}^+(C)$, then

$$\mathrm{mult}_p(C) = \sum_{p \prec q} \mathrm{mult}_q(C)$$

by the proximity equality (4.8.1), hence

$$\mathrm{mult}_p(C) = \sum_{\varphi(p) \prec \varphi(q)} \mathrm{mult}_{\varphi(q)}(D)$$

by induction, and thus $\mathrm{mult}_p(C) = \mathrm{mult}_{\varphi(p)}(D)$ by the proximity equality. □

Let me remark that the above proposition is false if one considers plain clusters of singular points instead of augmented clusters. Indeed the cluster associated to an ordinary singularity of any multiplicity m consists of a single point. So with Sing instead of Sing^+ in Definition (4.12), all ordinary singularities would be equisingular, regardless of the multiplicity, which is certainly not the definition we want.

If one prefers, it is possible to define equisingularity by considering the clusters Sing and adding the condition that the bijection φ preserves multiplicities.

(4.14) It follows from Definition (4.12) that equisingularity is an equivalence relation; we call *equisingularity classes* the equivalence classes under this equivalence relation. My generic notation for equisingularity classes is S. By definition, an equisingularity class S may be represented by a finite set Sing^+ equipped with the two order relations of precedence and proximity; as it will not cause any trouble that this set is defined only up to bijection as in Definition (4.12), I will abuse notation and denote it by $\mathrm{Sing}^+(\mathsf{S})$. By removing the maximal points of $\mathrm{Sing}^+(\mathsf{S})$ with respect to precedence, one finds the set $\mathrm{Sing}(\mathsf{S})$. Moreover, by Proposition (4.13), all points $p \in \mathrm{Sing}^+(\mathsf{S})$ come with an assigned multiplicity $\mathrm{mult}_p(\mathsf{S})$.

I will say loosely that a reduced curve C in a smooth surface S, or merely a reduced curve C with planar singularities, *has singularities of type* S if it belongs to the equisingularity class S.

One may prefer to reserve the term equisingularity class for equivalence classes of germs of curves under equisingularity, so that an equisingularity class is the datum of one singularity only. From this perspective, it would be better to name the notion of equisingularity class used in the present text multi-equisingularity class, as it is equivalent to the data of fintely many equisingular classes of germs.

(4.15) Other definitions of equisingularity. Equinsigularity had originally been defined by Zariski, in different terms, but with a similar flavour. The definition given here is equivalent to Zariski's, see [7, Section 3.8]. Also, see [8, Section 1.1] for a discussion of Teissier's version of Zariski's definition and more references.

Equisingularity is also equivalent to topological equivalence, where two germs of isolated planar curve singularities $(C_1, 0) \subseteq (\mathbf{C}^2, 0)$ and $(C_2, 0) \subseteq (\mathbf{C}^2, 0)$ are *topologically equivalent* if there exists a homeomorphism $(\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ mapping $(C_1, 0)$ to $(C_2, 0)$. For references about this statement, again see [7, Section 3.8] and [8, Section 1.1].

(4.16) Enriques diagrams. I now describe a graphic representation of equisingularity classes, designed to show the singular points and the precedence and proximity orders between them. It has been introduced by Enriques, see [10, Libro Quarto, Capitolo I (to be found in volume II)], and these representations are now called *Enriques diagrams*. Let S be an equisingularity class.

To materialize the precedence relation, we see $\mathrm{Sing}^+(\mathsf{S})$ as the set of vertices of an oriented graph, with the following definition: for all $p, q \in \mathrm{Sing}(\mathsf{S}^+)$, there is an edge from p to q if q is in the first infinitesimal neighbourhood of p. Thus, p precedes q if and only if there is an oriented chain of edges joining p to q. This makes $\mathrm{Sing}(\mathsf{S}^+)$ a disjoint union of (oriented) trees. For a tree, an orientation is equivalent to the choice of an origin; thus we will indicate the orientation simply by indicating the respective origins of the various trees composing $\mathrm{Sing}(\mathsf{S}^+)$, which are the vertices corresponding to proper singular points.

The proximity relation on the other hand is represented by drawing the edges in two possible ways, either straight or curved, according to the following rules:

(i) if q is in the first infinitesimal neighbourhood of p, and q is free (equivalently, q is proximate only to p), then the edge from p to q is a smooth curved edge with (if p is not a proper point) the same tangent at p as the edge ending at p (there is a unique such edge since the graph is a union of trees);

(ii) if q is in the first infinitesimal neighbourhood of p, then all points infinitely near to q and proximate to p (all such points are satellite), as well as the edges joining them, are represented on a straight half-line starting at q and orthogonal to the edge from p to q.

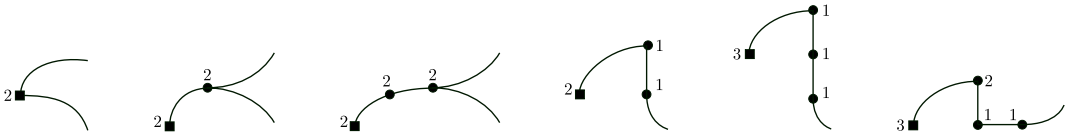
To avoid self-intersections in the diagram, half-lines as in rule (ii) above are alternately oriented to the right and to the left of the preceding edge.

Although this is not necessary (by Proposition (4.13)), it is often helpful to indicate at each vertex of the diagram the multiplicity of the corresponding point. Kleiman and Piene have chosen to represent only $\mathrm{Sing}(\mathsf{S})$ in their version of Enriques diagrams, which they call minimal Enriques diagrams; this causes no trouble if the multiplicities are indicated, as we have seen in the discussion after the proof of Proposition (4.13). Their choice is more economic, but representing the whole $\mathrm{Sing}^+(\mathsf{S})$ makes the diagram more suggestive: it has the advantage of immediately showing all local branches, and it lets the diagram do a better job in representing the geometric difference between free and satellite points. An enlightening discussion of the nice geometric features of Enriques diagrams is included in [7, p. 99–100].

Graphic convention. The origins of the trees (i.e., proper singular points) are represented by square black dots. Infinitely near singular points are represented by round black dots. Non-singular points (i.e., points of $\mathrm{Sing}^+(\mathsf{S}) \setminus \mathrm{Sing}(\mathsf{S})$) are represented by simple points (thus, they

are just the endpoints of the edge pointing toward them, without any further inking). The multiplicities of points in $\mathrm{Sing}(\mathsf{S})$ are indicated, but not those of points in $\mathrm{Sing}^+(\mathsf{S}) \setminus \mathrm{Sing}(\mathsf{S})$ (because the latter are all simple).

(4.17) Example. The six diagrams below are the respective Enriques diagrams of affine plane curves, defined by the indicated equation. Thus the first three are the diagrams of a node, an ordinary tacnode, and an ordinary oscnode, respectively; the last three are the diagrams of an ordinary cusp of order two, and two different cusps of order three.



$y^2 - x^2$ $y^2 - x^4$ $y^2 - x^6$ $y^2 - x^3$ $y^3 - x^4$ $y^3 - x^5$

For the two cusps of order three, name p_0, p_1, p_2, p_3 the points in S, following the order of precedence; then, for $\mathrm{V}(y^3 - x^4)$, the two points p_2, p_3 are proximate to p_0 (in addition to their respective immediate predecessors), and for $\mathrm{V}(y^3 - x^5)$, the point p_2 is proximate to p_0 whereas the point p_3 is proximate to p_1.

You may want to check the proximity equalities of (4.8) against these examples. More examples are given in Paragraph (4.29) below.

(4.18) Notation. In addition to the data mentioned in Paragraph (4.14), an equisingularity class S carries the following intrinsic characteristics:
— $r(\mathsf{S})$, the number of points in $\mathrm{Sing}^+(\mathsf{S}) \setminus \mathrm{Sing}(\mathsf{S})$;
— $\mathrm{roots}(\mathsf{S})$, the number of roots in $\mathrm{Sing}^+(\mathsf{S})$, i.e., singular points with no predecessor;
— $\mathrm{free}(\mathsf{S})$, the number of free singular points, i.e., free points in $\mathrm{Sing}(\mathsf{S})$ (by our Definition (4.5), free points are always infinitely near; beware that this is not the definition adopted by Kleiman and Piene).

Thus, $\mathrm{roots}(\mathsf{S})$ is the number of trees of which $\mathrm{Sing}^+(\mathsf{S})$ is the disjoint union; this is the number of proper singular points of the equisingularity class S. On the other hand, $r(\mathsf{S})$ is the total number of local branches of curves in S (i.e., the sum of the number of local branches at the various proper singular points), see Paragraph (4.11).

(4.19) Proposition. *Let C be a curve in the equisingularity class S. Then the difference $p_a(C) - g(C)$ depends on S only, and equals*

$$\delta(\mathsf{S}) = \sum_{p \in \mathrm{Sing}(\mathsf{S})} \binom{\mathrm{mult}_p}{2}. \tag{4.19.1}$$

Proof. The proof is by induction on the number of singular points. If there is none, the result is trivial. Implicitly, the curve C lies on a smooth surface S. Let p be a proper singular point of C, and consider the blow-up $\varepsilon : S' \to S$ at p, with exceptional divisor E. Then, the proper transform of C in S' is $C' = \varepsilon^* C - \mathrm{mult}_p.E$. By comparing the applications of the adjunction formula to $C \subseteq S$ and $C' \subseteq S'$ respectively, one finds

$$p_a(C') = p_a(C) - \binom{\mathrm{mult}_p}{2}.$$

The result then follows by induction, since the equisingularity class of C' is obtained by removing p from S. □

Given a reduced curve C in a smooth surface S, we define its *total Milnor number* as

$$\mu(C) = h^0(S, \mathcal{O}_S/\mathcal{J}_{C,S}),$$

where $\mathcal{J}_{C,S}$ is the Jacobian ideal of C in S, locally generated by the partial derivatives of the equation of C in S, see [II, Section 1]. Then, together with the celebrated Milnor–Jung formula, see for instance [15, Chapter I, Proposition 3.35], the above proposition gives the following. Beware that the Milnor–Jung formula is not always true in positive characteristic, see [3] for a recent account.

(4.20) Corollary. *The total Milnor number of C depends on S only, and equals*

$$\mu(\mathsf{S}) = 2\delta(\mathsf{S}) - r(\mathsf{S}) + \mathrm{roots}(\mathsf{S}). \tag{4.20.1}$$

(4.21) Remark. The definition of $r(\mathsf{S})$ in [21] is different from ours, but the two are equivalent. Indeed, the total number of local branches may be computed as follows:

$$r(\mathsf{S}) = \sum_{p \in \mathrm{Sing}(\mathsf{S})} \Big(\mathrm{mult}_p - \sum_{\substack{q \in \mathrm{Sing}(\mathsf{S}) \\ p \prec q}} \mathrm{mult}_q \Big).$$

This follows from the proximity equalities, and I leave the proof as an exercise.

4.4 – Equisingular loci

In this section we study, given a family $(F/Y, D)$ of curves on smooth surfaces as in Paragraph (1.4), the locus $Y(\mathsf{S}) \subseteq Y$ of curves belonging to a given equisingularity class S. The results are taken from Kleiman and Piene [21, 23]. In particular, we show that it has an expected codimension, and that its actual codimension is everywhere smaller or equal to the expected codimension. The main statement is Proposition (4.23). We shall need the following numerical characters attached to an equisingularity class, in addition to those introduced in (4.18) above.

(4.22) Let S be an equisingularity class, as in (4.14). We set:

$$\begin{aligned} \dim(\mathsf{S}) &= 2\,\mathrm{roots}(\mathsf{S}) + \mathrm{free}(\mathsf{S}); \\ \deg(\mathsf{S}) &= \sum_{p \in \mathrm{Sing}(\mathsf{S})} \binom{\mathrm{mult}_p + 1}{2}; \\ \mathrm{expcod}(\mathsf{S}) &= \deg(\mathsf{S}) - \dim(\mathsf{S}). \end{aligned}$$

Now let F/Y be a smooth projective family of surfaces over a Noetherian base, and D be a relative effective divisor D on F. Denote by $Y(\infty)$ the locus of points $y \in Y$ such that the curve D_y, i.e., the fibre of D over y, is non-reduced. Moreover, for all equisingularity classes S, denote by $Y(\mathsf{S})$ the locus of points $y \in Y$ such that the curve D_y belongs to S.

(4.23) Proposition (Kleiman–Piene)**.** *In the above setup, the following hold:*
(i) The set $Y(\infty)$ is closed in Y.
(ii) There are only finitely many equisingularity classes S such that $Y(\mathsf{S})$ is non-empty.
(iii) For all equisingularity classes S, the set $Y(\mathsf{S})$ is constructible in Y, and at all point $y \in Y(\mathsf{S})$ we have

$$\mathrm{codim}_Y \big(Y(\mathsf{S})\big)_y \leqslant \mathrm{expcod}(\mathsf{S}).$$

The idea of the proof is to elaborate on the following classical dimension count for nodal curves on a fixed surface S: for all $p \in S$, imposing that curves in some linear system $|L|$ have multiplicity at least 2 at p amounts to 3 linear conditions on $|L|$; since p moves in 2 dimensions along S, the locus of curves with one node has codimension at most one (if non-empty). The definition of equisingularity in the previous section makes it clear that any type of singularity may be understood as a collection of infinitely near multiple points, with appropriate precedence and proximity relations among them. So we will apply a similar dimension count to such collections.

(4.24) The first step is the construction of the locus $H(\mathsf{S})$ in the Hilbert scheme of F/Y which parametrizes collections of infinitely near points with the configuration determined by the equisingularity class S.

In order to understand this construction, let us see how to construct one such configuration on a given surface S. First choose pairwise distinct points on S in correspondence with the roots of S, and blow-up S at these points. Next, for each root p, choose pairwise distinct points on the irreducible component E_p of the exceptional divisor over p, in correspondence with the immediate successors of p in $\mathrm{Sing}(\mathsf{S})$, and blow-up at all these newly chosen points. Then we want to consider those points in $\mathrm{Sing}(\mathsf{S})$ the second infinitesimal neighbourhood of the roots; the novelty is that from now on there may be satellite points. Let q be one of the points chosen in the previous step; we choose pairwise distinct points on the irreducible component E_q of the exceptional divisor over q, in correspondence with the immediate successors of q in $\mathrm{Sing}(\mathsf{S})$, with the following new rule: for all immediate successors r of q:

— if r is free, the corresponding point on E_q must be chosen off the intersection with the proper transforms of the previous exceptional divisors;
— if r is proximate to some $p < q$, then the corresponding point must be the intersection point of E_q with the proper transform of E_p.

We then repeat this last step until all points in $\mathrm{Sing}(\mathsf{S})$ have been chosen an incarnation as an infinitely near point on S.

In this construction, representatives of roots may be chosen virtually anywhere on S, representatives of free singular points may be chosen each virtually anywhere on an irreducible rational curve, and there is only one possible choice for satellite singular points. Thus the number of degrees of freedom in the construction is $\dim(\mathsf{S})$.

Now we must pass from a configuration of infinitely near points to a subscheme of S. This goes as follows. Let $\varepsilon : S' \to S$ be the sequence of all blow-ups considered above; include for convenience also the blow-ups of the last points of $\mathrm{Sing}(\mathsf{S})$ (with respect to precedence). For all point $p \in \mathrm{Sing}(\mathsf{S})$, which we identify with its incarnation, let E_p be the irreducible exceptional divisor over p (it dwells in some surface between S and S'), and denote by abuse of notation by $\varepsilon^* E_p$ the pull-back of E_p to S' by those blow-ups coming after the blow-up of q; thus $\varepsilon^* E_p$ is a possibly reducible exceptional divisor of ε, with self-intersection -1. Then we set

$$\mathcal{I} = \varepsilon_* \mathcal{O}_{S'}\Big(- \sum_{p \in \mathrm{Sing}(\mathsf{S})} \mathrm{mult}_p . \, \varepsilon^* E_p\Big).$$

It defines a 0-dimensional subscheme of S, supported at the points we have chosen to incarnate the roots of S, and of degree $\deg(\mathsf{S})$: the latter fact is a generalization of the fact for $p \in S$ with maximal ideal $\mathfrak{m}_p$, the ideal $\mathfrak{m}_p^{\mathrm{mult}_p}$ defines a 0-dimensional subscheme of length $\binom{\mathrm{mult}_p+1}{2}$. For a complete proof, using the framework of complete ideals, see [7, Section 8.3].

In sum, the above construction provides a locally closed subset $H(\mathsf{S})$ of dimension $\dim(\mathsf{S})$ of the Hilbert scheme of 0-dimensional, degree $\deg(\mathsf{S})$ subschemes of S, parametrizing collections of infinitely near points in the configuration determined by S. In general, the following holds.

(4.25) Proposition. *Let* S *be an equisingularity class, and* F/Y *be a smooth projective family of surfaces. The family of all collections of infinitely near points in the members of* F/Y *in the configuration determined by* S *exists as a locally closed subset* $H_{F/Y}(\mathsf{S})$ *of* $\mathrm{Hilb}^{\deg(\mathsf{S})}_{F/Y}$*, the relative Hilbert scheme of* 0*-dimensional subschemes of degree* $\deg(\mathsf{S})$*. Moreover,* $H_{F/Y}(\mathsf{S})$ *is smooth over* Y*, of pure (relative) dimension* $\dim(\mathsf{S})$*.*

The complete proof of this result is the object of [23], see Corollaries 5.5 and 5.8 in that paper. In fact Kleiman and Piene prove a more precise result, valid in arbitrary characteristic; they study infinitely near points from a functorial point of view, prove the existence of a so-to-speak moduli space $Q(\mathsf{S})$ of infinitely near points on F/Y in the configuration determined by S and of a universal injection $\Psi : Q(\mathsf{S}) \to \mathrm{Hilb}^{\deg(\mathsf{S})}_{F/Y}$, which in characteristic 0 is an embedding. Please see the introduction of [23] for references to previous works by various authors in the direction of Proposition (4.25) above.

(4.26) Consider now, in addition to the objects in Proposition (4.25), a relative effective divisor D on F/Y. To relate $H_{F/Y}(\mathsf{S})$ to the study of members of the family D in the equisingularity class S, we form the intersection[5]

$$Z(\mathsf{S}) = H_{F/Y}(\mathsf{S}) \cap \mathrm{Hilb}^{\deg(\mathsf{S})}_{D/Y}.$$

The points of $Z(\mathsf{S})$ are in bijection with the collections Σ of infinitely near points on a surface F_y, $y \in Y$, in the configuration determined by S, and such that the 0-dimensional subscheme Σ is contained in the divisor D_y.

The following is an important point for the proof of Proposition (4.23): by [1, Proposition 4], the Hilbert scheme $\mathrm{Hilb}^{\deg(\mathsf{S})}_{D/Y}$ is a subscheme of $\mathrm{Hilb}^{\deg(\mathsf{S})}_{F/Y}$ locally cut out by $\deg(\mathsf{S})$ equations. Thus, for all $z \in Z(\mathsf{S})$,

$$\mathrm{codim}_{H_{F/Y}(\mathsf{S})}\big(Z(\mathsf{S})\big)_z \leqslant \deg(\mathsf{S}). \tag{4.26.1}$$

(4.27) We now prove Proposition (4.23).

Proof of assertion (i). In the notation of Section 2.1, the set $Y(\infty)$ is the image in Y of the set of those $x \in X_2$ at which the fibre of X_2/Y has positive dimension. The latter set is closed in X_2, which itself is closed in F as explained in Section 2.1. The result follows since F/Y is projective. □

Proof of assertion (ii). By flatness of D/Y, the curves D_y have the same arithmetic genus for all $y \in Y$. Moreover, the number of components of their normalizations $\bar{D}_y$ is bounded from above (again by flatness, the degree of the D_y's with respect to some relative polarization is constant along Y), so the geometric genus of the D_y's is bounded from below. This implies that the δ-invariants of the D_y's, $\delta(D_y) = p_a(D_y) - g(D_y)$ is bounded from above. By Proposition (4.19), this bounds from above the number of multiple points, proper or infinitely near, of each D_y, as well as their multiplicities.

It remains to show that for each of these finitely many possible distributions of multiple points, there may be only finitely many simple singular points. Argueing local branch by local branch, the possible number of which is bounded from above by the previous considerations, this follows from [7, Theorem 3.5.8]. □

[5] if one is to work with the moduli space $Q(\mathsf{S})$, this has to be substituted with the fibre product $G(\mathsf{S}) = Q(\mathsf{S}) \times_{\mathrm{Hilb}^{\deg(\mathsf{S})}_{F/Y}} \mathrm{Hilb}^{\deg(\mathsf{S})}_{D/Y}$, see [24, Paragraph 2.5].

Proof of assertion (iii). Let S be an equisingularity class. By construction, $Y(\mathsf{S})$ is contained in $f(Z(\mathsf{S}))$, where $Z(\mathsf{S})$ is as in Paragraph (4.26), and f denotes the structure morphism $\mathrm{Hilb}^{\deg(\mathsf{S})}_{D/Y} \to Y$. The rest of $f(Z(\mathsf{S}))$ consists of points $y \in Y$ such that either D_y is non-reduced, or it is reduced with singularities worse than S. In the latter case, let S' be the corresponding equisingularity class. Then, again by construction, there is an injection

$$j : \mathrm{Sing}(\mathsf{S}) \hookrightarrow \mathrm{Sing}(\mathsf{S}')$$

such that for all $p \in \mathrm{Sing}(\mathsf{S})$, $j(p)$ has multiplicity at least that of p. Then, $\deg(\mathsf{S}')$ is larger than $\deg(\mathsf{S})$. The upshot is that

$$Y(\mathsf{S}) = f\big(Z(\mathsf{S})\big) \setminus \Big(Y(\infty) \cup \bigcup\nolimits_{\deg(\mathsf{S}')>\deg(\mathsf{S})} f\big(Z(\mathsf{S}')\big)\Big),$$

where the union ranges over the finitely many S' such that $Y(\mathsf{S}')$ is non-empty and $\deg(\mathsf{S}') > \deg(\mathsf{S})$. Since $Z(\mathsf{S})$ and the $Z(\mathsf{S}')$ are locally closed by Proposition (4.25), this proves that $Y(\mathsf{S})$ is constructible.

It remains to prove the bound on the codimension. Let $y \in Y(\mathsf{S})$. By construction there is a unique point $z \in Z(\mathsf{S})$ over y. Since $H_{F/Y}(\mathsf{S})$ is smooth of dimension $\dim(\mathsf{S})$ over Y by Proposition (4.25), one has

$$\mathrm{codim}_Y\big(Y(\mathsf{S})\big)_y = \mathrm{codim}_{H_{F/Y}(\mathsf{S})}\big(Z(\mathsf{S})\big)_z - \dim(\mathsf{S}),$$

and by (4.26.1), this is smaller than or equal to $\deg(\mathsf{S}) - \dim(\mathsf{S}) = \mathrm{expcod}(\mathsf{S})$. □

(4.28) Remark. We may now give the correct definition of the "natural cycles $U(D,\delta)$" appearing in Theorem (1.6), i.e., the cycles $\tilde{U}(D,\delta)$ in the notation of Warning (2.7).

Consider the multi-equisingularity class $\mathsf{S} = \delta\mathrm{A}_1$ of δ ordinary double points. Form the parameter scheme $Z(\delta\mathrm{A}_1)$ of 0-dimensional subschemes made of δ disjoint subschemes isomorphic to $\mathrm{Spec}\big(\mathbf{C}[x_1,x_2]/(x_1,x_2)^2\big)$, and contained in a divisor of the family D/Y, as in (4.26). The cycle $\tilde{U}(D,\delta)$ is the cycle obtained from the push-forward $f_*[Z(\delta\mathrm{A}_1)]$ of the fundamental cycle $[Z(\delta\mathrm{A}_1)]$ to the base Y by removing the components supported in $Y(\infty)$.

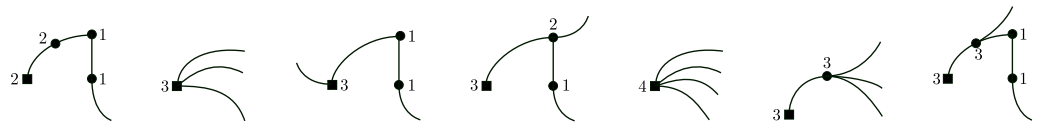
(4.29) Classification of singularities of small codimension. In [21, Section 2], Kleiman and Piene show how one can use Enriques diagrams in order to classify plane curve singularities using combinatorial methods. In particular, they give the complete list of plane curve singularities with expected codimension at most 10 (as they point out, this may also be deduced from Arnold's classification results in [2]). As a sideremark, note that these are the singularities one expects to find on hyperplane sections of a sufficiently general surface $S \subseteq \mathbf{P}^N$ for $N \leqslant 10$, hence those to be considered in order to study the corresponding stratification of the dual $S^\vee \subseteq \check{\mathbf{P}}^N$, as we have done for surfaces in $\mathbf{P}^3$ in [XII].

We give their list in Table 2, in increasing order of expected codimension. It is limited to equisingularity classes with only one root, for any other class is the combination of several of those. Note that $\mathrm{X}_{1,0}$ is the ordinary quadruple point, and has expected codimension 8. Singularities with expected codimension at most 8 are those in the first column (except A_k and D_k for $k > 8$).

Below, we give the Enriques diagrams for a sample of them (you may find all of them, in Kleiman–Piene's minimal form, in [21]). Note that we have already given the diagrams for singularities A_1, A_3, A_5, A_2, E_6, and E_8 (in that order) in Example (4.17).

type		normal form		type	normal form	
A_k	$(1 \leqslant k \leqslant 10)$	$y^2 + x^{k+1}$		$J_{2,0}$	$x^3 + ax^2y^2 + y^6$	$(4a^3 + 27 \neq 0)$
D_k	$(4 \leqslant k \leqslant 10)$	$x^2y + y^{k-1}$		$X_{1,1}$	$x^4 + x^2y^2 + ay^5$	$(a \neq 0)$
E_6		$x^3 + y^4$		$J_{2,1}$	$x^3 + x^2y^2 + ay^7$	$(a \neq 0)$
E_7		$x^3 + xy^3$		$X_{1,2}$	$x^4 + x^2y^2 + ay^6$	$(a \neq 0)$
E_8		$x^3 + y^5$		Z_{11}	$x^3y + y^5 + axy^4$	
$X_{1,0}$		$x^4 + ax^2y^2 + y^4$	$(a^2 \neq 4)$	$Y_{1,1}$	$x^5 + ax^2y^2 + y^5$	$(a \neq 0)$

Table 2: Plane curve singularities with expected codimension at most 10



A_4 D_4 D_5 E_7 $X_{1,0}$ $J_{2,0}$ $J_{2,1}$

You may observe that $J_{2,0}$ is a triple point followed by D_4 (i.e., two infinitely near triple points), and $J_{2,1}$ is a triple point followed by D_5.

5 – Regularity conditions for curves in a fixed linear system

In this section we concentrate on the situation of Theorem (1.3), with the goal of providing sufficient conditions for the enumerative validity of the formulae in that theorem. Thus we will consider a smooth irreducible complex projective surface S, a line bundle L on S, and curves in the fixed linear system $|L|$, which we will denote by Y. We denote by D the universal divisor in $F = S \times Y$, see Section 3.1.

Before we start, let me state the following little lemma (see [21, Lemma (4.7)]), which is often overlooked. It asserts that imposing the passing through c general points on S defines a sufficiently general (for our enumerative purposes) linear subsystem of $|L|$ of codimension c. Set $n = \dim Y$, and let r be an integer such that $0 \leqslant r \leqslant n$. For all $\mathbf{p} = (p_1, \ldots, p_{n-r}) \in S^{n-r}$, denote by $\Lambda(\mathbf{p})$ the r-dimensional linear subsystem of Y of members of Y passing through $p_1, \ldots, p_{n-r}$.

(5.1) Lemma. *Let U be a reduced subscheme of Y of pure codimension r, and ∂U its boundary, i.e., $\partial U = \overline{U} \setminus U$. Assume that for all $y \in U$, the corresponding curve D_y is reduced. Then there exists a non-empty open subset $\mathcal{S}^\circ$ of S^{n-r} with the following property: for all $\mathbf{p} \in \mathcal{S}^\circ$, $\Lambda(\mathbf{p}) \cap U$ is finite and reduced, and $\Lambda(\mathbf{p}) \cap \partial U$ is empty.*

Proof. Let D° be the smooth locus in D of the projection $D \to Y$, and form the fibered product $\mathcal{D}^\circ = D^\circ \times_Y \cdots \times_Y D^\circ$; it is smooth over Y. Hence $\mathcal{D}^\circ \times_Y U$ is reduced. Moreover, it is dense in $D^{n-r} \times_Y U$, where D^{n-r} is the fibered product $D \times_Y \cdots \times_Y D$. Also, $D^{n-r} \times_Y U$ is of pure dimension $2(n-r)$. Consider the natural map,

$$D^{n-r} \times_Y U \to S^{n-r},$$

where S^{n-r} is the plain product $S \times \cdots \times S$. By Sard's theorem, its fibers are finite and reduced over a non-empty open subset $\mathcal{S}_0$ of S^{n-r}. These fibers are simply the $\Lambda(\mathbf{p}) \cap U$.

On the other hand, $D^{n-r} \times_Y \partial U$ is of dimension at most $2(n-r)-1$, so the map,

$$D^{n-r} \times_Y \partial U \to S^{n-r}$$

cannot be surjective. Hence, its fibers are empty over a nonempty subset $\mathcal{S}_0$ of $\mathcal{S}_1$. These fibers are the $\Lambda(\mathbf{p}) \cap \partial U$. Thus we may take $\mathcal{S}^\circ = \mathcal{S}_1$ and the lemma is proved. □

We shall need the following definition throughout.

(5.2) Definition. *Let k be a non-negative integer. The line bundle L is* k-very ample *if for all 0-dimensional, length $k+1$, subscheme Z of S, the restriction map*

$$H^0(S, L) \to H^0(Z, L|_Z)$$

is surjective.

5.1 – Application of equisingular deformation theory

(5.3) Proposition. *Let S be an equisingularity class such that $Y(\mathsf{S})$ is non-empty. If L is $(\mu(\mathsf{S})-1)$-very ample, then $Y(\mathsf{S})$ is locally closed in Y, smooth, and of pure codimension*

$$\operatorname{codim}_Y\big(Y(\mathsf{S})\big) = \operatorname{expcod}(\mathsf{S}).$$

(5.4) In order to prove the proposition, we shall consider the deformation theory of the singularities of the members of $Y(\mathsf{S})$. We refer to [II] on this subject, and recall the following necessary facts.

Let $y \in Y(\mathsf{S})$, and consider the corresponding curve D_y. Then D_y has isolated singularities, and we may consider a semi-universal deformation space B for these singularities. The space B is smooth, with tangent space at the origin

$$H^0(S, \mathcal{O}_S/\mathcal{T}_{D_y}),$$

where $\mathcal{T}_{D_y}$ is the Tjurina ideal of D_y in S, locally generated by the equation of D_y and its partial derivatives. Thus $\mathcal{T}_{D_y}$ contains the Jacobian ideal of D_y, hence its degree is at most the total Milnor number $\mu(D_y) = \mu(\mathsf{S})$, see Corollary (4.20) for this equality.

There is a natural map $\phi : U \to B$ defined on an analytic neighbourhood of y in Y, and its differential at y is the (quotient of the) restriction map

$$H^0(S, L)/H^0(S, \mathcal{O}_S) \longrightarrow H^0\big(S,\, L/\mathcal{T}_{D_y}L\big).$$

Since L is $(\mu(\mathsf{S})-1)$-very ample, this differential is surjective, hence the map $\phi : U \to B$ is smooth (up to shrinking U if necessary).

(5.5) In the deformation space B, there is a smooth closed subset B^{es}, whose points represent the equisingular deformations of the singularities of D_y; the codimension of B^{es} in B is

$$\operatorname{codim}_B(B^{\mathrm{es}}) = \operatorname{expcod}(\mathsf{S}). \tag{5.5.1}$$

I give some indications about this below, but before that let us conclude with the proposition.

Proof of Proposition (5.3). Via the map ϕ defined in Paragraph (5.4) above,

$$Y(\mathsf{S}) \cap U = \phi^{-1}(B^{\mathrm{es}}).$$

Thus, $Y(\mathsf{S})$ is locally closed and smooth since B^{es} is closed and smooth in B, and

$$\operatorname{codim}_Y\big(Y(\mathsf{S})\big)_y = \operatorname{codim}_B(B^{\mathrm{es}})_0 = \operatorname{expcod}(\mathsf{S}).$$

□

The two books [15] and [16] provide a fairly thorough presentation of equisingular deformation theory. See [16, Remark 2.2.44.1, (ii), p. 196] for a concise discussion of the smoothness of B^{es} and its codimension in B; Equality (5.5.1) then amounts to [16, Corollary 1.1.64]. The bottom line is that B^{es} is equal to the stratum B^{μ} in B of deformations with constant Milnor number [15, Chapter II, Corollary 2.68], and the latter is smooth by [15, Chapter II, Theorem 2.61] and [15, Chapter II, Theorem 2.38].

The expression of the codimension of B^{es} in terms of $\mathrm{expcod}(\mathsf{S})$ is attributed to Shustin [36] by Kleiman and Piene; to the best of my knowledge, it follows from the theory only in a rather roundabout way. The argument is given in [21, p. 222–223], and goes as follows. First, for large enough d, there exists plane curves of degree d in the equisingularity class S, see [16, Theorem 4.5.13]. Next, if d is large enough, the stratum $V_d^{\mathsf{S}} \subseteq |\mathcal{O}_{\mathbf{P}^2}(d)|$ parametrizing degree d plane curves in the equisingularity class S is smooth of the expected dimension by [16, Theorem 2.2.40] or [16, Theorem 4.3.2]; in particular it has codimension $\mathrm{codim}_B(B^{\mathrm{es}})$ in $|\mathcal{O}_{\mathbf{P}^2}(d)|$. Finally, one computes as in the next Section 5.2 (see also the proof of [16, Corollary 1.1.64]) that the codimension of V_d^{S} in $|\mathcal{O}_{\mathbf{P}^2}(d)|$ equals $\mathrm{expcod}(\mathsf{S})$, for d large enough. This altogether proves Equality (5.5.1).

5.2 – Application of Bertini's theorem

Let S be an equisingularity class, and recall the following notation from Section 4.4 (in the setup of the present section, $F = S \times Y$ as indicated above):

— $H_{F/Y}(\mathsf{S})$ is the parameter space in $\mathrm{Hilb}^{\deg(\mathsf{S})}_{F/Y}$ of infinitely near multiple points in the configuration determined by S;
— $Z(\mathsf{S})$ is the intersection $H_{F/Y}(\mathsf{S}) \cap \mathrm{Hilb}^{\deg(\mathsf{S})}_{D/Y}$
— f is the structure map $\mathrm{Hilb}^{\deg(\mathsf{S})}_{D/Y} \to Y$.

By construction, $Y(\mathsf{S})$ is contained in $f(Z(\mathsf{S}))$. Here we consider moreover

$$Z_0(\mathsf{S}) = \left(f^{-1}Y(\mathsf{S})\right) \cap Z(\mathsf{S}).$$

Then f induces a bijection $Z_0(\mathsf{S}) \to Y(\mathsf{S})$, since by definition, for $y \in Y(\mathsf{S})$ the singularities of the curve D_y gives exactly one configuration of infinitely near multiple points as determined by S.

In the present situation, being $F = S \times Y$, $H_{F/Y}(\mathsf{S})$ is the product $H_S(\mathsf{S}) \times Y$, where $H_S(\mathsf{S}) \subseteq \mathrm{Hilb}^{\deg(\mathsf{S})}_S$ is the parameter space of configurations of infinitely near points in the fixed surface S. The projection $p : F \to S$ induces a map $Z(\mathsf{S}) \to H_S(\mathsf{S})$: despite appearances it is simple, indeed it maps a divisor containing a configuration of multiple infinitely near points to that configuration in the Hilbert scheme of S.

(5.6) Proposition. *Assume that L is $\deg(\mathsf{S})$-very ample. Then the following hold:*
(i) $Z(\mathsf{S})/H_S(\mathsf{S})$ is a bundle of projective spaces of codimension $\deg(\mathsf{S})$ in $|L|$;
(ii) $Z_0(\mathsf{S})$ is a dense open subset of $Z(\mathsf{S})$.

Proof. Let us consider a point of $H_S(\mathsf{S})$: this is a configuration of infinitely near points on S as determined by S, materialized as a 0-dimensional subscheme of length $\deg(\mathsf{S})$ of S; denote by $\mathcal{I}$ the ideal of this subscheme. Then the fibre of $Z(\mathsf{S})$ over this point of $H_S(\mathsf{S})$ is merely the linear subsystem $\mathbf{P}H^0(S, \mathcal{I}L)$ of Y, parametrizing those divisors containing the chosen configuration infinitely near points.

Since L is $(\deg(\mathsf{S}) - 1)$-very ample, the restriction map $H^0(S, L) \to H^0(S, L/\mathcal{I}L)$ is surjective, hence $H^0(S, \mathcal{I}L)$ has constant codimension $\deg(\mathsf{S})$ in $H^0(S, L)$ as $\mathcal{I}$ varies. Thus the fibres

of $Z(\mathsf{S})/H_S(\mathsf{S})$ are all projective spaces of the same dimension, since $H^0(S,\, L/\mathcal{I}L) = \deg(\mathsf{S})$ for all $\mathcal{I}$, and assertion (i) follows.

Consider next the minimal blow-up $\varepsilon : S' \to S$ on which our infinitely near points are realized as proper points. Then on S', $\varepsilon^{-1}\mathcal{I}\cdot\varepsilon^*L$ is an invertible sheaf which we will denote by L', and the two linear systems $\mathbf{P}H^0(S,\, \mathcal{I}L)$ and $\mathbf{P}H^0(S', L')$ are the same. By definition, members of this linear system that are smooth and transverse to every component of the exceptional divisor of ε map on S to curves in the equisingularity class S. Now, since L is $\deg(\mathsf{S})$-very ample, the linear system $\mathbf{P}H^0(S', L')$ is free from base points and thus, by Bertini's theorem, smooth members transverse to the exceptional divisor form a dense open subset of $\mathbf{P}H^0(S', L')$. Assertion (ii) follows. □

(5.7) Corollary. *If L is $\deg(\mathsf{S})$-very ample, then $Y(\mathsf{S})$ is a non-empty, dense, open subset of $f(Z(\mathsf{S}))$, and it has pure codimension $\operatorname{expcod}(\mathsf{S})$ in Y.*

Proof. By the proposition, the closures of the images $f(Z_0(\mathsf{S}))$ and $f(Z(\mathsf{S}))$ in Y are equal. Since $Z_0(\mathsf{S})$ maps bijectively to $Y(\mathsf{S})$, $Y(\mathsf{S})$ is dense in $f(Z(\mathsf{S}))$. Now $Z(\mathsf{S})$ is a bundle of projective spaces of codimension $\deg(\mathsf{S})$ in $|L| = Y$ over $H_S(\mathsf{S})$, and the latter is smooth of dimension $\dim(\mathsf{S})$ by Proposition (4.25); the assertion on the codimension follows. □

(5.8) Remark. We may now prove that if L is 3δ-very ample then, in the context of Theorem (1.3), the simplifying assumption $(\star)$ made for the proof of Theorem (1.6) in Section 2.3 indeed holds.

For any $i = 1, \ldots, \delta$, consider the multi-equisingularity class $i\mathrm{A}_1$ of i nodes. The parameter scheme $H_S(i\mathrm{A}_1) \subseteq \operatorname{Hilb}^{3i}_S$ is smooth by Proposition (4.25) (in fact, for $\mathsf{S} = i\mathrm{A}_1$, it is isomorphic to the parameter scheme of sets of i pairwise distinct points on S). Therefore, by Proposition (5.6), (i), $Z(i\mathrm{A}_1)$ is smooth as well, hence the fundamental cycle $[Z(i\mathrm{A}_1)]$ is reduced.

On the other hand, the map f induces a bijection $Z_0(i\mathrm{A}_1) \to Y(i\mathrm{A}_1)$, and $Z_0(i\mathrm{A}_1)$ is a dense open subset of $Z(i\mathrm{A}_1)$ by Proposition (5.6), (ii). In characteristic zero[6], this implies that the restriction of f to $Z(i\mathrm{A}_1)$ is birational onto $Y(i\mathrm{A}_1)$. It follows that the natural cycle $\tilde{U}(D, i)$ equals the closure of $Y(i\mathrm{A}_1)$ as a scheme, as we wanted to prove (see Remark (4.28) for the definition of $\tilde{U}(D, i)$).

5.3 – Conclusion

In this section we still consider a line bundle L on a fixed surface S, and the corresponding pair $(F/Y, D)$ where Y is the complete linear system $|L|$, $F = S \times Y$, and D is the universal divisor. We shall use the results of the two preceding Sections 5.1 and 5.2 to prove the following.

(5.9) Proposition. *Let $1 \leqslant r \leqslant 8$, and assume that L is $3r$-very ample.*
(i) Let S be an equisingularity class such that $\operatorname{expcod}(\mathsf{S}) \leqslant r$. Then $Y(\mathsf{S})$ is locally closed in Y, smooth, of pure codimension $\operatorname{expcod}(\mathsf{S})$; moreover, $Y(\mathsf{S})$ is open and dense in $f(Z(\mathsf{S}))$.
(ii) Conditions (i) and (ii) of Theorem (1.6) hold.

Let us first prove the first assertion, which is a direct application of the results in the two preceding sections.

Proof of assertion (i). By looking at the classification[7] in Paragraph (4.29) above, one finds the

[6] in characteristic two, the map $Z_0(i\mathrm{A}_1) \to Y(i\mathrm{A}_1)$ is totally inseparable of degree two, see, e.g., [17].

[7] Kleiman and Piene [21, p. 228] state that the second inequality holds without the assumption $\operatorname{expcod}(\mathsf{S}) \leqslant 8$, and announce a proof in a forthcoming (at the time of [21]) paper; I have not been able to find it in their series of papers, my best guess (besides my own short-sightedness) is that it got lost in the flow of versions the whole body of their work has come through.

two following inequalities hold:

$$\mu(\mathsf{S}) \leqslant \mathrm{expcod}(\mathsf{S}) + 1 \quad \text{and} \quad \deg(\mathsf{S}) \leqslant 3\mathrm{expcod}(\mathsf{S}).$$

Then, we may apply Propositions (5.3) and (5.6), and the assertion follows. □

To prove assertion (ii) however, we need to prove that $\mathrm{codim}_Y\big(Y(\mathsf{S})\big) \geqslant \min\big(r+1, \mathrm{expcod}(\mathsf{S})\big)$ for all equisingularity class S, and in general our positivity assumption on L is not sufficient in order to apply the results of the previous sections to all possible S (even though those S for which $Y(\mathsf{S})$ is non-empty are finitely many). The trick to get control on all $Y(\mathsf{S})$ is to find for all S an equisingularity subclass to which one can apply the previous results.

Recall from Section 4.3 that an equisingularity class may be seen as a finite set $\mathrm{Sing}(\mathsf{S})$ endowed with the two order relations of precedence and proximity, and with a multiplicity (or weight) attached to each element.

(5.10) Definition. *An* equisingularity subclass S' *of* S *is an equisingularity class corresponding to a subset of* $\mathrm{Sing}(\mathsf{S})$*, endowed with the induced relations of precedence and proximity, such that the multiplicities for* S' *are lower or equal to those for* S*.*

Thus for instance, A_{2i-1} is a subclass of A_{2i} and of A_{2i+1}, but A_{2i} is not a subclass of A_{2i+1}; also, D_{2i} is a subclass of D_{2i+1} and of D_{2i+2}, but D_{2i+1} is not a subclass of D_{2i+2}. The result we shall use is the following.

(5.11) Lemma. *Let* $1 \leqslant r \leqslant 8$*, and* S *an equisingularity class. If* $\mathrm{expcod}(\mathsf{S}) \geqslant r+1$*, then* S *contains a subclass* S' *such that* $\mathrm{expcod}(\mathsf{S}') \geqslant r$ *and* $\deg(\mathsf{S}') \leqslant 3r$*.*

This is [21, Lemma (4.4)]: the proof given there is combinatorial and I will skip it, although I genuinely don't take combinatorial to imply either easy or boring. The authors there announce a proof valid without any restriction on r in a forthcoming paper, but I could not find it, see footnote 7 again.

Proof of assertion (ii). Let us first prove that condition (ii) of Theorem (1.6) holds. Let S be an equisingularity class with $\mathrm{expcod}(\mathsf{S}) > r$; we need to show that $Y(\mathsf{S})$ has codimension at least $r+1$ at every point. To this end, let us consider S' as in Lemma (5.11) above. Being S' a subclass of S, we have $Z(\mathsf{S}) \subseteq Z(\mathsf{S}')$, and thus $Y(\mathsf{S}) \subseteq f(Z(\mathsf{S}'))$. Since $\deg(\mathsf{S}') \leqslant 3r$, the class S' is liable to Proposition (5.6) and Corollary (5.7); in particular, $Y(\mathsf{S}')$ is dense in the image of $Z(\mathsf{S}')$ in Y, and has pure codimension $\mathrm{expcod}(\mathsf{S}')$ in Y. If $\mathrm{expcod}(\mathsf{S}') > r$, then $Y(\mathsf{S}')$ is already small enough for the inclusion $Y(\mathsf{S}) \subseteq f(Z(\mathsf{S}'))$ to grant $\mathrm{codim}_Y(Y(\mathsf{S}))$ at all points of $Y(\mathsf{S})$.

It may happen however that $\mathrm{expcod}(\mathsf{S}') = r$. Then S' is a strict subclass of S, as we assume $\mathrm{expcod}(\mathsf{S}) > r$, and we will argue that $Y(\mathsf{S})$ is contained in a proper closed subset of the closure of $Y(\mathsf{S}')$, which implies that $Y(\mathsf{S})$ indeed has codimension strictly greater than r at all points. Since $\mathrm{expcod}(\mathsf{S}') = r$, the class S' is liable to assertion (i), and thus $Y(\mathsf{S}')$ is locally closed in Y. This implies that $Y(\mathsf{S}')$ is a dense open subset of the closure of $f(Z(\mathsf{S}'))$. Since also $Y(\mathsf{S})$ is contained in $f(Z(\mathsf{S}'))$, it must be contained in a proper closed subset of $f(Z(\mathsf{S}'))$, and our claim is proved.

We will now prove that condition (i) of Theorem (1.6) holds with a similar argument. Consider the equisingularity class $\mathsf{S} = r\mathrm{A}_1$ of r ordinary double points. Then the locus of non-reduced curves, $Y(\infty)$, lies in $f(Z(r\mathrm{A}_1))$ because we can find r distinct double points schematically contained in a non-reduced curve. As above, $Y(r\mathrm{A}_1)$ is locally closed and dense in the closure $f(Z(r\mathrm{A}_1))$, of the expected codimension r; $Y(\infty)$ is then contained in a proper closed subset of the closure of $f(Z(r\mathrm{A}_1))$, hence it has codimension strictly greater than r as we wanted to show. □

We have now proved the two assertions of Proposition (5.9). This completes the proof of Theorem (1.3): the proposition says that the hypotheses of Theorem (1.6) are verified in the conditions of Theorem (1.3); we may thus apply Theorem (1.6) as explained in Section 3.1, which gives the wanted result, thanks also to the little Lemma (5.1).

5.4 – Particular cases

(5.12) Plane curves. For plane curves of degree d with δ nodes, Kleiman and Piene prove that Formula (1.3.1) is valid if

$$d \geqslant \tfrac{1}{2}\,\delta + 1,$$

see [22, Theorem 3.1] and in particular [22, Lemma 3.3]. To do so, they use in a crucial way the following result of Greuel and Lossen [14].

Let Y be the complete linear system of plane curves of degree d, and let S be an equisingularity class. Let C be a plane curve with singularities of type S. The equisingular deformations of C in the plane are governed by the so-called *equisingular ideal* which we denote by I^{es}, see [II, Section 6]. Call τ^{es} the co-length of this ideal; this is the codimension of B^{es} in B in the notation of Section 5.1 above. Then, by [14, Corollary 3.9], if C is not the union of d lines through a point, and if

$$4d > 4 + \tau^{\mathrm{es}},$$

then at the point $[C]$, $Y(\mathsf{S})$ is smooth and has codimension τ^{es} in Y.

(5.13) Curves on $K3$ surfaces. If S is a $K3$ or Enriques surface, the results of Knutsen in [26] give necessary and sufficient conditions for k-very ampleness of the line bundle L, hence sufficient conditions for the validity of Formula (1.3.1).

In particular, if S is a $K3$ surface with Picard number 1, denoting by L_1 the positive generator of the Picard group, then for all $k \geqslant 0$:
— $L = L_1$ is k-very ample if $k \leqslant \frac{1}{4}L^2$;
— for all $m > 1$, $L = mL_1$ is k-very ample if $k + 1 < \frac{m-1}{m^2}L^2$.

(5.14) Curves on Abelian surfaces. In this case, Kleiman and Piene provide the following sufficient conditions, in [22, Lemma 5.3]. Let S be an Abelian surface with Picard number 1, and denote by L_1 the positive generator of the group of divisors modulo numerical equivalence. Let L be an effective line bundle on S. Then for all $k \geqslant 0$:
— if $L \equiv L_1$, then L is k-very ample if $k + 1 \leqslant \frac{1}{4}L^2$;
— if $L \equiv mL_1$ for some $m > 1$, then L_1 is k-very ample if $k + 1 < \frac{m-1}{m^2}L^2$.

These conditions are derived by Kleiman and Piene directly from the Beltrametti–Sommese theorem [4]; this is also the basic idea underlying [26], but Knutsen's results are much more precise, valid without restriction on the Picard number.[8]

In [22, Section 5], Kleiman and Piene carry out an enumeration of curves on Abelian surfaces that goes beyond the situation of Theorem (1.3) and of the present section, as it is about curves in a given homology class instead of a given linear equivalence class. This can be done thanks to the flexibility of the setup of Theorem (1.6): then, one takes the parameter space Y to be a projective bundle over the dual Abelian surface $\widehat{A}$, with fibre over the point $[L] \in \widehat{A}$ the complete linear system $|L|$.

[8] Beware that the application of Knutsen's theorem to Enriques surfaces given in [22, Remark 5.5] is unfortunately a statement about an empty class of examples, since all Enriques surfaces have Picard number 10. Similarly, [22, Lemma 5.3] does not apply to bielliptic surfaces, since the latter all have Picard number 2.

6 – Inclusion-exclusion and the structure of node polynomials

In this section I explain, following Qviller [32], the structure of node polynomials in terms of the Bell polynomials. The main point is to give an intersection theoretic definition of, say, the numbers a_i appearing in Formula (1.3.1), but we need to set things up before we can do so.

6.1 – The principle and an example

(6.1) Introduction. Let $\pi : F \to Y$ be a smooth projective family of surfaces, and let D be a relative effective divisor on F/Y. The starting point is, as in the Kleiman–Piene approach, to consider the critical locus X of the map $D \to Y$ induced by π: this is the X_2 from Section 2.1, but since here we will not consider any X_i with $i > 2$, we denote it simply by X. The next steps, however, are different.

Whereas Kleiman–Piene consider an induced family over X, namely $(F_2/X_2, D_2)$ in the notation of Section 2.1, the critical locus of this new family, and then repeat this by induction, Qviller seeks to consider the self-intersection of X in the sense of topologists, also known as the double locus in classical algebraic geometric terms. Concretely, he accesses it by means of the intersection

$$p_1^{-1}X \cap p_2^{-1}X \quad \text{in} \quad F \times_Y F$$

where p_1 and p_2 denote the two projections of $F \times_Y F$ to its factors; equivalently, this is the fibre product $X \times_Y X$. Of course, one considers the double locus when one is interested in 2-nodal curves, in general one ought to consider finer self-intersections, by means of

$$p_1^{-1}X \cap \cdots \cap p_\delta^{-1}X \quad \text{in} \quad F \times_Y \cdots \times_Y F,$$

or, equivalently, $X \times_Y \cdots \times_Y X$.

There are however contributions in this intersection that we need to get rid of in order to isolate that of δ-nodal curves in the family D: there are the various polydiagonals in the product $X \times_Y \cdots \times_Y X$, which give excess contributions, as well as other contributions supported on various proper closed subsets of the polydiagonals; Example (6.2) below should give a good idea of what is going on.

Then the point is that the various polydiagonals are organized as a lattice, which makes it possible to take care of the unwanted contributions by a method of inclusion-exclusion, which finally leads to a formula with Bell polynomials.

(6.2) Example. Let me describe the situation when one is interested in 2-nodal curves, assuming everything happens as expected. This is taken from [32, Example 2.5], which is written in the context of curves in a linear system on a fixed surface.

The critical locus X has dimension -1 over Y (as it has codimension 3 in F, and F has dimension 2 over Y), hence the fibre product $X \times_Y X$ is expected to have dimension -2 over Y. There are three loci of dimension at least -2 over Y which appear as the support of a primary component of the scheme $X \times_Y X$, namely:
— the locus of ordered pairs of nodes, which projects onto the locus of binodal curves in Y;
— the diagonal in $X \times_Y X$, which projects onto the locus of 1-nodal curves;
— the locus of cusps.
The locus of ordered pairs of nodes has the expected dimension -2 over Y; the diagonal has dimension -1 over Y, hence it gives an excess contribution; the locus of cusps is contained in the diagonal, but has the expected dimension -2 over Y and gives a proper contribution, with

the multiplicity 2. We shall compute the contributions of the two latter loci to the intersection product $p_1^{-1}X \cdot p_2^{-1}X$; this will give us the contribution of ordered pairs of nodes, in a form that relates to Formula (1.6.1), as we shall see.

Let us first work out Formula (1.6.1) explicitly; the expressions are in terms, as before, of

$$v = c_1\big(\mathcal{O}_F(D)\big) \quad \text{and} \quad w_j = c_j\big(\Omega^1_{F/Y}\big) \text{ for } j = 1, 2.$$

The class of X in F is

$$b_1 = v^3 + w_1 v^2 + w_2 v.$$

From the algorithm in Paragraph (1.4), we find

$$Q_{b_1,2} = -7v - 6w_1,$$

and

$$\begin{aligned} b_2 &= (-7v - 6w_1) \cdot (v^3 + w_1 v^2 + w_2 v) \\ &= -7\,v^4 - 13\,v^3 w_1 - 6\,v^2 w_1^2 - 7\,v^2 w_2 \end{aligned}$$

(there is a term in $w_1 w_2$ which vanishes for degree/dimension reasons). The upshot is that binodal curves are enumerated by the following class in Y: $\frac{1}{2}\big((\pi_* b_1)^2 + \pi_* b_2\big)$.

The class $(\pi_* b_1)^2$ is the push-forward (or push-down) to Y of the intersection product $p_1^{-1}X \cdot p_2^{-1}X$. We shall now see that the class $\pi_* b_2$ is the correction term relative to the unwanted contributions. Let us first work out the excess contribution of the diagonal, as this will also serve to introduce some notation. We apply [13, Proposition 9.1.1]: both $p_1^{-1}X$ and $p_2^{-1}X$ have codimension 3 in $F \times_Y F$, hence the proper (excess) contribution of the diagonal $\Delta_X \subseteq X \times_Y X$ to the intersection product is a class of dimension $n-2$, where $n = \dim(Y)$, namely

$$\Big\{ c\big(p_1^* N_{X/F}\big|_{\Delta_X}\big) \cdot c\big(p_2^* N_{X/F}\big|_{\Delta_X}\big) \cdot c\big(N_{\Delta_X/F^2}\big)^{-1} \cap [\Delta_X] \Big\}_{n-2},$$

where F^2 denotes the fibre product $F \times_Y F$; also, c denotes the total Chern class and $\{\_\}_k$ denotes the k-dimensional part of the class between braces; as usual, the inverse of a class is defined by taking the inverse in the sense of formal power series. Now, the embedding $\Delta_X \hookrightarrow F^2$ is the composition of the two regular embeddings $\Delta_X \hookrightarrow \Delta_F \hookrightarrow F^2$ (where $\Delta_F \subseteq F \times_Y F$ denotes the diagonal), with normal bundles $N_{X/F} \cong \mathcal{P}^1_{F/Y}(D)$ and $T_{F/Y}$ respectively; thus the class in the display above equals

$$\Big\{ c\big(\mathcal{P}^1_{F/Y}(D)\big) \cdot c\big(T_{F/Y}\big)^{-1} \cap [X] \Big\}_{n-2},$$

a class in the Chow group $\mathrm{CH}^1(X)$. By the exact sequence (2.3.1), we find this is

$$\Big\{ c\big(\mathcal{O}_F(D) \otimes \Omega^1_{F/Y}\big) \cdot c\big(\mathcal{O}_F(D)\big) \cdot c\big(T_{F/Y}\big)^{-1} \cap [X] \Big\}_{n-2}. \tag{6.2.1}$$

(6.3) Notation. Following [19, Section 8], we let

$$\begin{aligned} \alpha &= 1 + \alpha_1 + \alpha_2 + \cdots \\ &= c\big(\mathcal{O}_F(D) \otimes \Omega^1_{F/Y}\big) \cdot c\big(T_{F/Y}\big)^{-1}, \end{aligned}$$

see Paragraph (7.2). Thus, with the notation v, w_1, w_2 recalled above,

$$\begin{aligned} 1 + \alpha_1 + \alpha_2 + \cdots &= \frac{(1+v)^2 + (1+v)w_1 + w_2}{1 - w_1 + w_2} \\ &= \big((1+v)^2 + (1+v)w_1 + w_2\big)\big(1 + (w_1 - w_2) + (w_1 - w_2)^2\big), \end{aligned}$$

hence

$$\begin{aligned}\alpha_1 &= 2(v+w_1)\,;\\ \alpha_2 &= (v+w_1)(v+2w_1)\,;\\ \alpha_3 &= v(vw_1+3w_1^2-2w_2)\,;\\ \alpha_4 &= v^2(w_1^2-w_2)\,;\end{aligned}$$

and $\alpha_k = 0$ for all $k > 4$ (note that several terms vanish for degree reasons in these computations).

(6.4) Example (6.2) continued. We now resume the computation of the proper (excess) contribution of Δ_X to $p_1^{-1}X \cdot p_2^{-1}X$. Since X itself has pure dimension $n-1$, we need to consider the degree 1 homogeneous part of the class capped with $[X]$ in (6.2.1); with the notation above, it is

$$\alpha_1 + v = 3v + 2w_1,$$

hence the proper contribution of the diagonal is

$$(3v+2w_1)\cdot(v^3+w_1v^2+w_2v).$$

This contribution does not include the contribution of the cuspidal locus, which is the support of a different component of the normal cone $\mathrm{NC}_{X\times_Y X}(F\times_Y F)$. To compute the latter, we rely on a formula of Kazarian's (in the case of a linear system on a fixed surface, one can use [XII, (5.22)] instead); these are presented in the following Section 7. Considering the class R_{A_2} at [19, p. 711] and Theorem 8.4 in that same article, one finds that the (codimension 2) class on Y enumerating 1-cuspidal curves is the push-down to Y of

$$\alpha_1\cdot(v^3+w_1v^2+w_2v) = 2(v+w_1)\cdot(v^3+w_1v^2+w_2v).$$

Since we agreed that the contribution to $p_1^{-1}X\cdot p_2^{-1}X$ of the 1-cuspidal locus has to be counted with the multiplicity 2, we find that the total of the contributions to be discarded is the push-down to Y of

$$\big((\alpha_1+v)+2\alpha_1\big)\cdot(v^3+w_1v^2+w_2v) = (7v+6w_1)\cdot(v^3+w_1v^2+w_2v)\ :$$

this is $-b_2$ from the first part of the example. The upshot is that the contribution to $p_1^{-1}X\cdot p_2^{-1}X$ of the locus of 2-nodal curves is $(\pi_*b_1)^2+\pi_*b_2$, which has to be divided by two as each binodal curve comes with two possible orderings of its nodes.

For $\delta > 2$ we don't quite have (yet?) as accurate a description of all contributions supported on some diagonal, as we shall see in Section 6.3. Still, these various contributions organize within the lattice of polydiagonals, and this is sufficient to understand the structure of node polynomials in terms of the Bell polynomials, as we will now see.

6.2 – Multiplicative structure of the lattice of polydiagonals

In this section we will state and prove (partially) Qviller's intersection theoretic definition of the classes $a_i = \pi_*b_i$ (in the notation of Theorem (1.6)), thus explaining the appearance of the Bell polynomials in Formula (1.6.1).

(6.5) Partitions and polydiagonals. We shall consider the various diagonals in the fibre product

$$X^r = X \times_Y \cdots \times_Y X.$$

They are in bijection with the (non-singleton) partitions ϖ of $[\![1,r]\!] = \{1,\dots,r\}$:[9] associated to $\varpi = \coprod_{1\leqslant i\leqslant k} J_i$ is the diagonal $\Delta_\varpi \subseteq X^r$ defined as follows:

$$\Delta_\varpi = \big\{(x_1,\dots,x_r) : \forall i = 1,\dots,j,\ \forall s,t \in J_i,\ x_s = x_t\big\}$$

(if ϖ is the singleton partition, this gives X^r itself, which is not considered a diagonal). We denote by Π_r the set of all partitions of $[\![1,r]\!]$, and by Π_r° the set of all non-singleton partitions of $[\![1,r]\!]$.

(6.6) Definition. *Let $r \geqslant 1$. For all non-singleton partition $\varpi \in \Pi_r^\circ$, we let $B_\varpi \in \mathrm{CH}_{n-r}(\Delta_\pi)$ be the part of the intersection product*

$$p_1^*X \cdot \ldots \cdot p_r^*X$$

(where $p_1,\dots,p_r$ are the various projections $X^r \to X$) suported in the diagonal Δ_ϖ, in the sense of [13, Definition 6.1.2]: thus B_ϖ is the sum of the contributions of the various distinguished subvarieties contained in Δ_ϖ.

For instance, in the situation of Example (6.2) above, for the single block partition $\varpi = [\![1,2]\!]$, B_ϖ is the sum of both the excess contribution of the diagonal itself and the locus of 1-cuspidal curves.

(6.7) Definition. *Moreover, we let*

$$a_r(F/Y,D) = (-1)^{r-1}(r-1)!\,\pi_* B_{[\![1,r]\!]} \in \mathrm{CH}_{n-r}(Y),$$

where $[\![1,r]\!]$ denotes the single block partition. Here we use by abuse of notation the same symbol for both $\pi : F \to Y$ and its restriction to the critical locus $X \subseteq F$.

Note that $\Delta_{[\![1,r]\!]}$ is the small diagonal

$$\big\{(x_1,\dots,x_r) :\ x_1 = \cdots = x_r\big\} \ \subseteq\ X^r\,;$$

thus, it may be identified with X, and it makes sense in the above definition to consider the push-down of $B_{[\![1,r]\!]} \in \mathrm{CH}(\Delta_{[\![1,r]\!]})$ to Y by π.

(6.8) Theorem (Qviller)**.** *Let F/Y be a family of surfaces and D be a relative effective divisor. Let $\delta \geqslant 1$ and assume that the pair $(F/Y,D)$ is sufficiently generic for δ-nodal curves (see below). Then, the class of the locus of δ-nodal curves in Y is*

$$\frac{1}{\delta!}\,P_\delta\big(a_1(F/Y,D),\ \dots,\ a_\delta(F/Y,D)\big) \tag{6.8.1}$$

where P_δ is the δ-th Bell polynomial as defined in Paragraph (1.1).

The condition that the pair $(F/Y,D)$ be sufficiently generic for δ-nodal curves means that the following conditions hold:

(i) the locus $Y(\delta\mathrm{A}_1)$ of δ-nodal curves has codimension δ in Y at all its points;

[9] in case you're wondering, 'ϖ' is 'variant π'.

(ii) for all equisingularity classes S with at least δ proper singular points,[10] if $\mathsf{S} \neq \delta \mathrm{A}_1$, then the locus $Y(\mathsf{S})$ has codimension in Y strictly larger than δ at all its points;
(iii) the locus $Y(\infty)$ of non-reduced curves has codimension in Y strictly larger than δ at all its points.

In the situation of a linear system $|L|$ on a fixed surface, i.e., $F = S \times |L|$ and D is the universal divisor, cf. Paragraph (3.1) or Theorem (1.3), this sufficient genericity condition for δ-nodal curves holds if L is r-very ample by Kool–Shende–Thomas's proof of the Göttsche conjecture, see [XIV, Theorem (2.1)].

The following lemma captures the combinatorial structure which makes the Bell polynomials show up.

(6.9) Lemma. *Denote by I_δ the part of the intersection product $p_1^*X \cdot \ldots \cdot p_\delta^*X$ that is not supported on the polydiagonals. Then*

$$I_\delta = \sum_{\varpi \in \Pi_\delta} m_\varpi B_\varpi \tag{6.9.1}$$

where, for all partitions ϖ of $[\![1, \delta]\!]$, the coefficient m_ϖ is defined as follows: for all $i = 1, \ldots, \delta$, denote by $s_i(\varpi)$ the number of blocks of size i in ϖ; then

$$m_\varpi = \prod_{i=1}^{\delta} \big((-1)^{i-1}(i-1)!\big)^{s_i(\pi)}. \tag{6.9.2}$$

Proof. By definition,

$$I_\delta = p_1^*X \cdot \ldots \cdot p_\delta^*X - \sum_{Z \subseteq \bigcup_{\varpi \in \Pi_\delta^\circ} \Delta_\varpi} (p_1^*X \cdot \ldots \cdot p_\delta^*X)_Z,$$

where the sum ranges over all distinguished varieties Z (in the sense of [13, Definition 6.1.2]) of the intersection $p_1^*X \cdot \ldots \cdot p_\delta^*X$ contained in some diagonal of the fibre product $X \times_Y \cdots \times_Y X$, and for all Z, $(p_1^*X \cdot \ldots \cdot p_\delta^*X)_Z$ denotes the contribution of Z to the intersection product,[11] i.e., the sum of those terms corresponding to an irreducible component of the normal cone with support equal to Z.

By definition the distinguished subvarieties are irreducible, so we may rearrange the second summand in the above display in the form

$$-\sum_{Z \subseteq \bigcup_{\varpi \in \Pi_\delta^\circ} \Delta_\varpi} (p_1^*X \cdot \ldots \cdot p_\delta^*X)_Z = \sum_{\varpi \in \Pi_\delta^\circ} \sum_{Z \subseteq \Delta_\varpi} m_\varpi . (p_1^*X \cdot \ldots \cdot p_\delta^*X)_Z, \tag{6.9.3}$$

where the m_ϖ's are arranged so that for all Z, the relevant m_ϖ's sum up to -1. Thus, the m_ϖ's are defined by induction on Π_δ° with respect to the partial order by refinement, as we will now see explicitly. Also, we will prove that they verify Equality (6.9.2).

For two partitions ϖ and ϖ' of $[\![1, \delta]\!]$, we say that ϖ *refines* ϖ', and we write $\varpi \prec \varpi'$,[12] if every block of ϖ is contained in a block of ϖ'. We let $\hat{0}$ and $\hat{1}$ denote the singleton and the single-block partition of $[\![1, \delta]\!]$ respectively; thus $\hat{0}$ is a refinement of every partition, and every

[10] in the terminology of Paragraph (4.18), this means that $\mathrm{Sing}(\mathsf{S})$ has at least δ distinct roots, equivalently the Enriques diagram of S has at least δ connected components.

[11] we avoid the notation $(p_1^*X \cdot \ldots \cdot p_\delta^*X)^Z$, which has a different meaning in [13, Definition 6.1.2].

[12] agreed, this is the same notation as for the proximity relation, but this should not create any trouble.

partition is a refinement of $\hat{1}$. Moreover, for all ϖ, we denote by $|\varpi|$ the number of its blocks; thus $|\hat{0}| = \delta$ and $|\hat{1}| = 1$.

Now to the definition of the m_ϖ's. Let us first consider the largest diagonals, i.e., those corresponding to a partition ϖ with only one non-trivial block of size 2, equivalently those ϖ with $|\varpi| = \delta - 1$. We set

$$m_\varpi = -1 \quad \text{if} \quad |\varpi| = \delta - 1, \tag{6.9.4}$$

so that those distinguished varieties Z contained in one only of the largest diagonals (and therefore in no smaller diagonal) occur with the correct coefficient -1 in the summation (6.9.3). Now assume Z is contained in some smaller diagonals, and let Δ_ϖ be the smallest diagonal in which it is contained; then a diagonal $\Delta_{\varpi'}$ contains Z if and only if ϖ' is a refinement of ϖ. Thus we impose the recursive relation

$$m_\varpi = -1 - \sum_{\substack{\varpi' \in \Pi_\delta^\circ \\ \varpi' \prec \varpi}} m_{\varpi'} \quad \text{for all} \quad \varpi \in \Pi_\delta^\circ, \tag{6.9.5}$$

so that each Z contained in some diagonal occurs with total coefficient -1 in the summation (6.9.3). The two conditions (6.9.4) and (6.9.5) are sufficient to define m_ϖ for all $\varpi \in \Pi_\delta^\circ$ by induction, and with such a definition the relation (6.9.3) holds.

Now let us set in addition $m_{\hat{0}} = 1$ for the singleton partition. As we have observed in Paragraph (6.5), one has $\Delta_{\hat{0}} = X^\delta$ (which strictly speaking is not a diagonal), hence

$$p_1^* X \cdot \ldots \cdot p_\delta^* X = \sum_{Z \subseteq \Delta_{\hat{0}}} m_{\hat{0}} . (p_1^* X \cdot \ldots \cdot p_\delta^* X)_Z,$$

and therefore

$$I_\delta = \sum_{\varpi \in \Pi_\delta} \sum_{Z \subseteq \Delta_\varpi} m_\varpi . (p_1^* X \cdot \ldots \cdot p_\delta^* X)_Z$$

(note that the summation ranges over $\Pi_\delta = \Pi_\delta^\circ \cup \{\hat{0}\}$). It is thus sufficient to prove Formula (6.9.2) for all $\varpi \in \Pi_\delta$ in order to complete the proof of the lemma.

Observe that m_ϖ is defined for all $\varpi \in \Pi_\delta$ by the two conditions

$$m_{\hat{0}} = 1, \qquad \text{and} \qquad m_\varpi = - \sum_{\substack{\varpi' \in \Pi_\delta \\ \varpi' \prec \varpi}} m_{\varpi'} \quad \text{for all} \quad \varpi \in \Pi_\delta, \tag{6.9.6}$$

as they imply the two conditions (6.9.4) and (6.9.5). It follows that

$$m_\varpi = \mu(\hat{0}, \varpi)$$

where μ is the Möbius function of the partially ordered set Π_δ, see [37, Section 3.7 and Display (3.15)].

To compute the relevant values of this Möbius function, consider the partially ordered sets Π_r for all r, with Möbius function μ_r, and with singleton and single-block partitions $\hat{0}_r$ and $\hat{1}_r$ respectively. The induction relations (6.9.6) imply that

$$\mu_r(\hat{0}_r, \hat{1}_r) = -(r-1)\, \mu_{r-1}(\hat{0}_{r-1}, \hat{1}_{r-1}),$$

hence

$$\mu_r(\hat{0}_r, \hat{1}_r) = (-1)^{r-1}(r-1)!\,.$$

Now, if ϖ is a partition $J_1 \coprod \cdots \coprod J_k$ with each block J_j of size r_j, one has

$$\mu(\hat{0}, \varpi) = \prod_{j=1}^{k} \mu_{r_j}(\hat{0}_{r_j}, \hat{1}_{r_j}) = \prod_{j=1}^{k} \big((-1)^{r_j-1}(r_j-1)!\big),$$

hence finally

$$m_\varpi = \mu(\hat{0}, \varpi) = \prod_{i=1}^{r} \big(\mu_i(\hat{0}_i, \hat{1}_i)\big)^{s_i(\varpi)},$$

and Formula (6.9.2) holds, which proves the lemma. For more details on the computation of the Möbius function, please see [37, Example 3.10.4]. □

In order to prove the theorem, we will also need the following lemma. It explains why it is sufficient to understand the classes $B_{[\![1,r]\!]}$ (or $B_{\hat{1}_r}$ depending on one's favourite notation), corresponding to the smallest diagonals $\Delta_{[\![1,r]\!]} \subseteq X^r$ for all $r = 2, \ldots, \delta$, to understand the classes B_ϖ of an arbitrary diagonal $\Delta_\varpi \subseteq X^\delta$. For example, the part of $p_1^*X \cdot \ldots \cdot p_6^*X$ suported in the diagonal

$$\Delta_{1|23|456} = \big\{(x_1, \ldots, x_6) :\ x_2 = x_3 \quad \text{and} \quad x_4 = x_5 = x_6\big\}$$

may be computed from the parts of $p_1^*X \cdot p_2^*X$ and $p_1^*X \cdot p_2^*X \cdot p_3^*X$ in X^2 and X^3 supported in the small diagonals $X_{[\![1,2]\!]} \subseteq X^2$ and $X_{[\![1,3]\!]} \subseteq X^3$. This explains the relevance of Definition (6.7) and its role in Formula (6.8.1).

(6.10) Lemma. *For all partitions $\varpi \in \Pi_\delta$, the following equality of classes in Y holds:*

$$\gamma_{\delta *} B_\varpi = \prod_{i=1}^{\delta} \big(\gamma_{i*} B_{[\![1,i]\!]}\big)^{s_i(\varpi)},$$

where γ_i denotes the projection $F^i = F \times_Y \cdots \times_Y F \to Y$ for all i. Recall also that $s_i(\varpi)$ is the number of blocks of size i in the partition ϖ.

Let me omit the proof of this lemma, as I believe the statement is intuitive and the proof is essentially technical and doesn't add much to one's global understanding of the story. Please refer to [32, Proposition 3.10] for details. Now, let us prove the main result of this section.

Proof of Theorem (6.8). The genericity assumption ensures that the push-down to Y of the class I_δ of Lemma (6.9) enumerates δ-nodal curves, with an ordering of the nodes, in the family D. Indeed, curves with strictly less than δ proper singular points correspond in X^δ to loci that are contained in the polydiagonals, hence their contribution to the intersection product $p_1^*X \cdot \ldots \cdot p_\delta^*X$ is discarded in the definition of I_δ. On the other hand, among curves with at least δ singular points, those with more singularities than δ nodes occur in codimension at least $\delta + 1$ in Y, hence they don't contribute to $p_1^*X \cdot \ldots \cdot p_\delta^*X$. The upshot is that δ-nodal curves in Y are enumerated by the class $\frac{1}{\delta!}\gamma_{\delta *} I_\delta$, with γ_δ the projection $F^\delta \to Y$ as in Lemma (6.10).

Now, one has $I_\delta = \sum_{\varpi \in \Pi_\delta} m_\varpi B_\varpi$ as in Lemma (6.9). For all partitions $\varpi \in \Pi_\delta$, by Lemma (6.10) and the definitions of the coefficients m_ϖ in Lemma (6.10) and the classes $a_i(F/Y, D)$ in Definition (6.7), the following equality of classes in Y holds,

$$\gamma_{\delta *}(m_\varpi B_\varpi) = \prod_{i=1}^{\delta} a_i(F/Y, D)^{s_i(\varpi)}.$$

It follows that

$$\gamma_{\delta *} I_\delta = \sum_{s_1+\cdots+\delta s_\delta=\delta} e_{s_1,\ldots,s_\delta} \Big(\prod_{i=1}^{\delta} a_i(F/Y,D)^{s_i} \Big)$$

where $e_{s_1,\ldots,s_\delta}$ is the number of partitions $\varpi \in \Pi_\delta$ with s_1 blocks of size 1, …, s_δ blocks of size δ. By Corollary (A.4) on Bell polynomials, the above equality reads

$$\gamma_{\delta *} I_\delta = P_\delta\big(a_1(F/Y,D),\, \ldots,\, a_\delta(F/Y,D)\big),$$

which proves the theorem. □

6.3 – Further considerations

Theorem (6.8) presents the issue, as acknowledged by Qviller himself, that although the classes $a_i(F/Y,D)$ are given an intersection theoretic definition, they are not quite directly computable. In particular, in view of the Göttsche conjecture, it would be desirable to have a direct proof that, in the context of a linear system on a fixed surface (i.e., $F = S \times |L|$ and D is the universal divisor over $|L|$), the classes $a_i(F/Y,D)$ are linear combinations of the four Chern numbers of the polarized surface (S,L), i.e., $d = L^2$, $k = L \cdot K_S$, $s = K_S^2$, $x = c_2(S)$. Qviller gives a number of hints and conjectures pointing in this direction in [32, Sections 4–5], that I will leave aside in the present text.

Also, the proof of Theorem (6.8) does not provide as precise a picture as that gotten in Example (6.2) when we computed the class of the locus of binodal curves. Here I will very briefly report on Qviller's considerations in this direction.

(6.11) Excess contribution of the polydiagonals. It is interesting to isolate, in the intersection product $p_1^* X \cdot \ldots \cdot p_\delta^* X$ in $F \times_Y \cdots \times_Y F$, the contributions of those distinguished varieties of the correct dimension $n - \delta$, where $n = \dim Y$. In the context of Example (6.2), this means the contribution of both the binodal locus and the cuspidal locus; note that the latter is contained in the diagonal.

This amounts to the computation of the (excess) contributions of all the diagonals Δ_ϖ in $X^\delta = X \times_Y \cdots \times_Y X$. To do this in a sensible way, one would try to express the result in terms of the contributions of the small diagonals $\Delta_{[1,i]}$ in X^i for all $i = 2, \ldots, \delta$, as has been done for Theorem (6.8). One would thus build up a definition similar to Definition (6.7) for the $a_i(F/Y,D)$, in which the class $B_{[1,i]}$ (the sum of the contributions of the various distinguished subvarieties contained in $\Delta_{[1,i]}$) would be substituted by the mere contribution of the sole distinguished subvariety $\Delta_{[1,i]}$.

It is indeed possible to express the contribution of an arbitrary polydiagonal Δ_ϖ in these terms, as the appropriate result analogous to Lemma (6.10) holds, see [32, Theorem 4.3]. This doesn't work directly, however, to express the sum of the contributions of all the polydiagonals: the problem is that these excess contributions do not satisfy the principle of inclusion-exclusion, which is the guiding line of Lemma (6.9) above. The basic issue is that Segre classes do not satisfy the principle of inclusion-exclusion.

Let me try to make this intelligible on an example, taken from [32, Example 4.4]. Consider $\delta = 3$. Then there are four diagonals to be considered, the small diagonal Δ_{123} and the three large diagonals $\Delta_{12|3}$, $\Delta_{1|23}$, and $\Delta_{2|13}$. All three large diagonals contain the small diagonal. Thus, according to the principle of inclusion-exclusion, if one sums the contributions of the three large diagonals, then the contribution of the small diagonal is counted three times instead of one only, hence the correct total contribution should be the sum of the contributions of the three large diagonals minus twice the contribution of the small diagonal. It turns out however, that this does not give the correct answer.

This being observed, Qviller suggests to compute correction terms to the formula predicted by the inclusion-exclusion principle. To do this end, he uses the following multiple point formulae.

(6.12) Multiple point formulae. The set of r-fold points of a map of schemes $f : V \to W$ is the set $M_r \subseteq V$ of points x such that there exists r distinct points (possibly infinitely near each other) in $f^{-1}f(x)$. It is endowed with a natural scheme structure. The object we are interested in in the previous paragraph, namely the sum of the contributions to $p_1^*X \cdot \ldots \cdot p_\delta^*X$ in $F \times_Y \cdots \times_Y F$ of those distinguished varieties of the correct dimension $n - \delta$, may be understood as the scheme of δ-fold points of the map $\pi : X \to Y$. For instance, in the situation of Example (6.2), the scheme of 2-fold points of $X \to Y$ consists of the locus of binodal curves plus the locus of cuspidal curves, the latter with multiplicity 2.

In certain cases there exist formulae which give the class of M_r in the Chow ring of V in terms of the Chern classes of f, where the total Chern class of f is defined by

$$c(f) = \frac{f^*c(T_W)}{c(T_V)}.$$

For instance, Kleiman gives in [20] the following formulae, valid for the map $\pi : X \to Y$ we are considering (the c_i's are the Chern classes of π):

$$\begin{aligned} m_2 &= \pi^*\pi_*[X] - c_1 \cap [X]\,; \\ m_3 &= \pi^*\pi_* m_2 - 2c_1 \cap m_2 + 2c_2 \cap m_1\,; \\ m_4 &= \pi^*\pi_* m_3 - 3c_1 \cap m_3 + 6c_2 \cap m_2 - 6(c_1c_2 + 2c_3) \cap m_1. \end{aligned}$$

Qviller uses the latter formulae to derive the following expressions of the push-down to Y of the contributions to $p_1^*X \cdot \ldots \cdot p_\delta^*X$ of distinguished varieties of the correct dimension for $\delta = 2, 3, 4$:

$$\begin{aligned} \pi_* m_2 &= P_2(d_1, d_2)\,; \\ \pi_* m_3 &= P_3(d_1, d_2, d_3 + d_3')\,; \\ \pi_* m_4 &= P_4(d_1, d_2, d_3 + d_3', d_4 + d_4')\,; \end{aligned}$$

here, P_2, P_3, P_4 are the Bell polynomials, for all i the class d_i is defined by the contribution of the small diagonal $\Delta_{[\![1,i]\!]}$ to $p_1^*X \cdot \ldots \cdot p_i^*X$, with a coefficient $(-1)^{i-1}(i-1)!$ dictated by the principle of inclusion-exclusion, and the classes d_i' are ad hoc correction terms compensating the failure of the principle of inclusion-exclusion for the polydiagonals.

The strategy in [20] to derive the above multiple point formulae is similar in spirit to the inductive strategy of Kleiman and Piene described in Section 2: to compute m_3, one identifies M_3 with the locus of double points of the map $M_2 \to Y$, and so forth.

7 – Kazarian's Thom polynomials

In this section I give some formulae by Kazarian [19] which generalize those of Theorems (1.3) and (1.6) to arbitrary multisingularities. The existence of such formulae had been previously proven by Li and Tzeng [29], see also Rennemo [33], and generalizes the predictions of the Göttsche conjecture.

(7.1) Thom polynomials. The original result proved by Thom (see [19, Section 1] and the references therein) is the following. Let $f : M \to N$ be a holomorphic map between non-singular complex varieties. Let S be a class of local singularities of maps: this is a type of singularity

supported at one point, a typical example is the class of germs of maps with dimension of the kernel of the differential at least r; see [19, Section 1] for a precise definition. There exists a universal polynomial, nowadays called Thom polynomial, in the Chern classes of the map f, defined by

$$c(f) = 1 + c_1(f) + \cdots = \frac{f^* c(T_N)}{c(T_M)},$$

which gives the class $[M(\mathsf{S})]$ of the locus of points in M at which f has a singularity of type S, if f is a generic map.

This is generalized by Kazarian to multisingularities in [19]. In this context, a multisingularity is an arbitrary finite ordered set of local singularities $\mathsf{S} = (\mathsf{S}_1, \ldots, \mathsf{S}_r)$, and the corresponding locus

$$M(\mathsf{S}) \subseteq M \times_N \cdots \times_N M$$

is the set of r-tuples $(x_1, \ldots, x_r)$ such that the x_i's are pairwise distinct and f has a singularity of type S_i at x_i for all $i = 1, \ldots, r$. Denote by $\tilde{m}_\mathsf{S}$ and $\tilde{n}_\mathsf{S}$ the push-forward of the class $[M(\mathsf{S})]$ to M and N, by the first projection p_1 and by fp_1 respectively. They come with multiplicities determined by the repetitions of the various local singularity classes in S: if S is some ordering of k_1 singularities of type T_1, …, k_s singularities of type T_s, with the T_j's pairwise distinct, set

$$\mathrm{aut}(\mathsf{S}) = k_1! \cdots k_s!;$$

then there are integral classes m_S and n_S such that

$$\tilde{m}_\mathsf{S} = \mathrm{aut}(\mathsf{S}^\flat).m_\mathsf{S} \quad \text{and} \quad \tilde{n}_\mathsf{S} = \mathrm{aut}(\mathsf{S}).n_\mathsf{S},$$

where $\mathsf{S}^\flat = (\mathsf{S}_2, \ldots, \mathsf{S}_r)$; with these definitions, one has $n_\mathsf{S} = f_* m_\mathsf{S}$.

Kazarian proves the existence of a universal polynomial with integral coefficients computing, for a generic proper map, the class of $\tilde{m}_\mathsf{S}$ in terms of the Chern class of the map f and the so-called Landweber–Novikov classes: by definition the latter are the following classes in N,

$$s_I(f) = f_*(c_1(f)^{i_1} c_2(f)^{i_2} \cdots)$$

for all $I = (i_1, i_2, \ldots)$, but by abuse of notation we also consider them as classes on M, namely as the pull-back

$$f^* f_*(c_1(f)^{i_1} c_2(f)^{i_2} \cdots).$$

The article [19] concentrates on the case when $\dim(M) < \dim(N)$. In the application we are interested in, $f : M \to N$ will be the map $\pi : X \to Y$ where X is the critical locus of a family of curves D in a family of surfaces $\pi : F \to Y$.

Kazarian's proof of the existence of Thom polynomials for multisingularities is based on a topological argument using complex cobordism theory. More recently, Ohmoto [30] has proposed a proof in the context of algebraic geometry over an algebraically closed field of characteristic zero, using as a key ingredient intersection theory on relative Hilbert schemes of points and its relation with the Thom–Mather theory of singularities of mappings. Once the existence of Thom polynomials is proved, there are various possibilities to compute them. Kazarian has offered very many such computations, see [19] as well as the references therein. Here I will give a very short selection of those.

(7.2) Legendrian singularities. The particular Thom polynomials we are interested in concern a special class of maps: a holomorphic map $f : M \to N$ is a *Legendrian map* if it can be written as a composed map $M \to \mathbf{P}(\Omega^1_N) \to N$, where the second map is the projection and the first map is a Legendrian immersion, meaning that $\dim M = \dim N - 1$, and the natural contact

form[13] on $\mathbf{P}(\Omega^1_N)$ pulls back to 0 on M. The Legendrian maps form a special class of maps whose typical singularities differ from the typical singularities of the generic maps. We will be interested in a particular instance of Legendrian maps that I will describe shortly. Before that, let me mention that Thom polynomials for Legendrian maps are given in terms of a particular set of classes, namely

$$v = c_1\big(\mathcal{O}_{\mathbf{P}(\Omega^1_N)}(1)\big) \quad \text{and} \quad \alpha_i = c_i\big(f^*T_N - T_M \otimes \mathcal{O}_{\mathbf{P}(\Omega^1_N)}(1)\big); \tag{7.2.1}$$

the latter description means that the α_i are defined by the following formal identity giving the total Chern class α,

$$\alpha = \frac{c\big(f^*T_N\big)}{c\big(T_M \otimes \mathcal{O}_{\mathbf{P}(\Omega^1_N)}(1)\big)}.$$

The following is a Legendrian map, and it should look familiar if you came here after reading previous parts of the text. Consider a smooth family $\pi : F \to Y$, and D a smooth hypersurface of the total space F. Denote by $X \subseteq D$ the subset at which D is tangent to the fibres of π, equivalently, at which the fiber of D is singular. Under a suitable transversality condition, X is smooth and the restriction $\pi : X \to Y$ is a Legendrian map; the map $X \to \mathbf{P}(\Omega^1_Y)$ is defined by mapping a point x to the hyperplane $\pi_* T_{D,x}$ of $T_{Y,\pi(x)}$. In this particular example, the classes in Display (7.2.1) are

$$v = c_1\big(\mathcal{O}_F(D)\big) \quad \text{and} \quad \alpha = c\big(\mathcal{O}_F(D) \otimes \Omega^1_{F/Y}\big) \cdot c\big(T_{F/Y}\big)^{-1}. \tag{7.2.2}$$

Kazarian's result on the existence of Thom polynomials in this context is the following.

(7.3) Theorem (Kazarian)**.** *For all sequences $\mathsf{T} = (\mathsf{T}_1, \dots, \mathsf{T}_r)$ of classes of isolated hypersurface singularities, there exists a universal "residual" polynomial R_{T} in the classes v and α_i, independent of the order of the sequence T, and such that for all sequences $\mathsf{S} = (\mathsf{S}_1, \dots, \mathsf{S}_r)$ and all generic Legendrian map $f : M \to N$, the classes m_{S} and n_{S}, defined as in Paragraph (7.1), are given by the following formulae:*

$$m_{\mathsf{S}} = \sum_{J_1 \sqcup \dots \sqcup J_k = [\![1,r]\!]} R_{\mathsf{S}_{J_1}} \cdot f^* f_* R_{\mathsf{S}_{J_2}} \cdots f^* f_* R_{\mathsf{S}_{J_k}};$$
$$n_{\mathsf{S}} = \sum_{J_1 \sqcup \dots \sqcup J_k = [\![1,r]\!]} f_* R_{\mathsf{S}_{J_1}} \cdot f_* R_{\mathsf{S}_{J_2}} \cdots f_* R_{\mathsf{S}_{J_k}},$$

where the summation ranges over all possible partitions of the set $[\![1,r]\!]$ in a disjoint union $J_1 \sqcup \dots \sqcup J_k$ of an arbitrary number of unordered subsets, such that J_1 is always the subset containing 1; for all subsets $J = \{j_1, \dots, j_s\}$, R_{S_J} denotes the universal residual polynomial attached to the sequence $(\mathsf{S}_{j_1}, \dots, \mathsf{S}_{j_s})$, in any order.

(7.4) On the structure of the formulae. In the theorem, the term "residual" is used for the following reason. Consider a sequence $(\mathsf{S}_1, \mathsf{S}_2)$ consisting of two classes of singularities. In order to enumerate, in N say, the fibres of f with a singularity of type S_1 and a singularity of type S_2, the naive way to proceed is to intersect the two classes $f_* R_{\mathsf{S}_1}$ and $f_* R_{\mathsf{S}_2}$, which enumerate fibers with a singularity S_1 and fibers with a singularity S_2, respectively. The problem however is that there is a lot of unwanted contributions in this intersection product, as has been observed

[13] this is a 1-form η on $\mathbf{P}(\Omega^1_N)$, locally defined up to scaling (or rather a global section of $\Omega^1_{\mathbf{P}(\Omega^1_N)} \otimes \mathcal{O}_{\mathbf{P}(\Omega^1_N)}(1)$), such that the form $\eta \wedge (d\eta)^{\dim(N)-1}$ is nowhere vanishing; it is deduced from the Liouville form on Ω^1_N, which in local coordinates writes $\sum_{i=1}^n q_i dx_i$, with q_i the local fibre coordinates of Ω^1_N.

at length in the previous Section 6. The residual polynomial $R_{\mathsf{S}_1,\mathsf{S}_2}$ is designed exactly to compensate for these unwanted contributions.

In fact, if one considers a sequence $\mathsf{S} = (r\mathsf{S}_0)$ consisting of r times the same singularity S_0, then the formulae of Theorem (7.3) may be phrased in terms of the Bell polynomials, namely

$$n_{r\mathsf{S}} = P_r(f_*R_{\mathsf{S}_0}, f_*R_{2\mathsf{S}_0}, \ldots, f_*R_{r\mathsf{S}_0})$$

with P_r the r-th Bell polynomial. This is observed by Kazarian at [19, p. 683], and the computations leading to this formulation are exactly the same as those in the proof of Theorem (6.8). In brief, for $\mathsf{S} = (r\mathsf{S}_0)$, for all subsets $J \subseteq [\![1, r]\!]$, $R_{\mathsf{S}_J} = R_{|J|.\mathsf{S}_0}$, hence

$$\sum_{J_1 \sqcup \cdots \sqcup J_k = [\![1,r]\!]} f_*R_{\mathsf{S}_{J_1}} \cdot f_*R_{\mathsf{S}_{J_2}} \cdots f_*R_{\mathsf{S}_{J_k}} = \sum_{s_1 + \cdots + rs_r = r} e_{s_1,\ldots,s_r} \Big(\prod_{i=1}^{r} f_*R_{i\mathsf{S}_0} \Big)$$

where $e_{s_1,\ldots,s_r}$ is the number of partitions of $[\![1, r]\!]$ with s_1 subsets of size 1, …, s_r subsets of size r. By Corollary (A.4) on Bell polynomials, the right-hand-side term in the above display is

$$P_r(f_*R_{\mathsf{S}_0}, f_*R_{2\mathsf{S}_0}, \ldots, f_*R_{r\mathsf{S}_0}),$$

as asserted.

In particular, when S_0 is the ordinary double point singularity A_1, Kazarian's residual classes $R_{\mathrm{A}_1}, \ldots, R_{r\mathrm{A}_1}$ coincide with the classes $b_1, \ldots, b_r$ in Theorem (1.6). You may verify this directly with example in the tables below, see also the computations in Example (6.2). For instance, from the formula in Paragraph (7.5) and the expressions in Paragraph (6.3), one finds

$$\begin{aligned} R_{\mathrm{A}_1^2} &= -v - 3\alpha_1 \\ &= -7v - 6w_1, \end{aligned}$$

which is $Q_{b_1,2}$ from Paragraph (1.4), as we have computed in Example (6.2).

(7.5) Residual classes of codimension at most 4 of Legendrian maps. Reproduced below is the set of explicit formulae of [19, Section 9] for residual classes of Legendrian maps in the context of Theorem (7.3) above. Let me point out that the use of the classes v and α_i of Display (7.2.1) makes the formulae rather compact. For explicit formulae of the α_i in terms of our usual classes v, w_1, w_2 in case F has dimension 2 over Y, see Paragraph (6.3). Also, note that we are considering singularities of codimension at most 4 for the critical locus X, hence singularitites of codimension at most 5 for the original family of hypersurfaces D. The multisingularity sequences are written in multiplicative notation.

$$\begin{aligned} R_{A_1} &= 1 \\ R_{A_2} &= \alpha_1 \\ R_{A_1^2} &= -v - 3\alpha_1 \\ R_{A_3} &= v\alpha_1 + 3\alpha_2 \\ R_{A_1A_2} &= -6(v\alpha_1 + 2\alpha_2) \\ R_{A_1^3} &= 2(v^2 + 19v\alpha_1 + 30\alpha_2) \\ R_{A_4} &= v^2\alpha_1 + 4v\alpha_2 + 3\alpha_1\alpha_2 + 6\alpha_3 \\ R_{D_4} &= -v\alpha_2 + \alpha_1\alpha_2 - 2\alpha_3 \\ R_{A_1A_3} &= -4(2v^2\alpha_1 + 5v\alpha_2 + 6\alpha_1\alpha_2 + 3\alpha_3) \\ R_{A_2^2} &= -3(3v^2\alpha_1 + 8v\alpha_2 + 7\alpha_1\alpha_2 + 6\alpha_3) \\ R_{A_1^2A_2} &= 24(3v^2\alpha_1 + 7v\alpha_2 + 6\alpha_1\alpha_2 + 3\alpha_3) \end{aligned}$$

$$
\begin{aligned}
R_{A_1^4} &= -6(v^3 + 111v^2\alpha_1 + 239v\alpha_2 + 171\alpha_1\alpha_2 + 78\alpha_3)\\
R_{A_5} &= v^3\alpha_1 - 4v^2\alpha_2 + 16v\alpha_1\alpha_2 - 12v\alpha_3 + 27\alpha_1\alpha_3 + 6\alpha_4\\
R_{D_5} &= -2(2v^2\alpha_2 - 2v\alpha_1\alpha_2 + 7v\alpha_3 - 3\alpha_1\alpha_3 + 6\alpha_4)\\
R_{A_1A_4} &= -10(v^3\alpha_1 - 4v^2\alpha_2 + 14v\alpha_1\alpha_2 - 16v\alpha_3 + 21\alpha_1\alpha_3 - 6\alpha_4)\\
R_{A_1D_4} &= 4(5v^2\alpha_2 - 5v\alpha_1\alpha_2 + 16v\alpha_3 - 6\alpha_1\alpha_3 + 12\alpha_4)\\
R_{A_2A_3} &= -6(2v^3\alpha_1 - 10v^2\alpha_2 + 28v\alpha_1\alpha_2 - 39v\alpha_3 + 39\alpha_1\alpha_3 - 18\alpha_4)\\
R_{A_1^2A_3} &= 2(56v^3\alpha_1 - 220v^2\alpha_2 + 684v\alpha_1\alpha_2 - 951v\alpha_3 + 891\alpha_1\alpha_3 - 522\alpha_4)\\
R_{A_1A_2^2} &= 18(7v^3\alpha_1 - 20v^2\alpha_2 + 74v\alpha_1\alpha_2 - 96v\alpha_3 + 95\alpha_1\alpha_3 - 50\alpha_4)\\
R_{A_1^3A_2} &= -48(28v^3\alpha_1 - 55v^2\alpha_2 + 250v\alpha_1\alpha_2 - 318v\alpha_3 + 300\alpha_1\alpha_3 - 180\alpha_4)\\
R_{A_1^5} &= 24(v^4 + 671v^3\alpha_1 - 701v^2\alpha_2 + 4863v\alpha_1\alpha_2 - 5844v\alpha_3 + 5490\alpha_1\alpha_3 - 3420\alpha_4)
\end{aligned}
$$

(7.6) Curves in a linear system on a fixed surface. Theorem (7.3) may be applied to enumerate curves in a complete linear system $|L|$ on a surface S, arguing as in Section 3.1. The result then takes the following form, see [19, Theorem 10.1], in terms of the classes

$$d = L^2\ ;\quad k = L\cdot K_S\ ;\quad s = K_S^2\ ;\quad x = c_2(S).$$

(7.7) Theorem (Kazarian)**.** *For all sequences* $\mathsf{T} = (\mathsf{T}_1, \dots, \mathsf{T}_r)$ *of classes of isolated planar curve singularities, there exists a universal integral linear combination* a_{T} *of the numbers* d, k, s, x*, independent of the order of the sequence* T*, and such that for all sequences* $\mathsf{S} = (\mathsf{S}_1, \dots, \mathsf{S}_r)$*, and for all sufficiently generic complete linear systems* $|L|$ *on a surface, the number* N_{S} *of curves in* $|L|$ *with singularities of type* S*, and passing through* $\dim|L| - \operatorname{codim}(\mathsf{S})$ *general points on the surface, is given by the formula*

$$N_{\mathsf{S}} = \frac{1}{\operatorname{aut}(\mathsf{S})} \sum_{J_1\sqcup\cdots\sqcup J_k = [\![1,r]\!]} a_{\mathsf{S}_{J_1}} a_{\mathsf{S}_{J_2}} \cdots a_{\mathsf{S}_{J_k}},$$

as in Theorem (7.3).

The number $\operatorname{aut}(\mathsf{S})$ is defined in Paragraph (7.1) above, and the expected codimension of a sequence of singularities is the same as that defined in Paragraph (4.22); see [19, Definition 1.1] for a definition in the more general context considered by Kazarian.

The genericity condition is the following. Denote by Y_{reg} the locus in $|L|$ of curves C that are reduced, and such that the deformations of C in $|L|$ induce a system of semi-universal deformations of the multigerm of singularities of C. Then the condition is that the complement of Y_{reg} in $|L|$ has codimension strictly larger than $\operatorname{codim}(\mathsf{S})$. This condition may be verified in practice with the techniques described in Section 5.1.

Now let us give the explicit formulae for the linear forms a_{S} for singularities of codimension up to 5, reproducing the table in [19, Section 10]. They are readily computed from those given above in Paragraph (7.5), argueing as in Section 3.1, but I felt it would be nice to include them for ease of reference. For $\mathsf{S} = A_1^r$, $a_{A_1^r}$ coincides with a_r given in Paragraph (1.2) (fortunately).

$$
\begin{aligned}
a_{A_1} &= 3d + 2k + x & a_{A_1^4} &= -72360d - 95670k - 28842s - 3888x\\
a_{A_2} &= 12d + 12k + 2s + 2x & a_{A_5} &= 630d + 1140k + 498s - 60x\\
a_{A_1^2} &= -42d - 39k - 6s - 7x & a_{D_5} &= 84d + 132k + 44s + 20x\\
a_{A_3} &= 50d + 64k + 17s + 5x & a_{A_1A_4} &= -5460d - 9240k - 3740s + 200x\\
a_{A_1A_2} &= -240d - 288k - 72s - 24x & a_{A_1D_4} &= -420d - 624k - 196s - 100x\\
a_{A_1^3} &= 1380d + 1576k + 376s + 138x & a_{A_2A_3} &= -6300d - 10332k - 4044s + 60x
\end{aligned}
$$

$$a_{A_4} = 180d + 280k + 100s \qquad a_{A_1^2A_3} = 52920d + 84180k + 31816s + 240x$$
$$a_{D_4} = 15d + 20k + 5s + 5x \qquad a_{A_1A_2^2} = 53676d + 84456k + 31716s + 72x$$
$$a_{A_1A_3} = -1260d - 1820k - 596s - 60x \qquad a_{A_1^3A_2} = -505008d - 770112k - 279792s - 5616x$$
$$a_{A_2^2} = -1260d - 1800k - 588s - 48x \qquad a_{A_1^5} = 5225472d + 7725168k + 2723400s + 84384x$$
$$a_{A_1^2A_2} = 9000d + 12360k + 3864s + 456x$$

(7.8) Contacts of hyperplanes with hypersurfaces. Arguing as in Section 3.2, Kazarian gives applications of his Theorem (7.3) to the enumeration of hyperplanes satisfying tangency conditions with a fixed hypersurface in projective space.

In this case the Legendrian map is as follows. We consider a smooth, degree d, hypersurface $V \subseteq \mathbf{P}^n$. Then, we let $Y = \check{\mathbf{P}}^n$ be the dual projective space, parametrizing hyperplanes in $\mathbf{P}^n$, and $F \subseteq \mathbf{P}^n \times \check{\mathbf{P}}^n$ be the point-hyperplane incidence correspondence, i.e.,

$$F = \big\{(x, H) \in \mathbf{P}^n \times \check{\mathbf{P}}^n : \ x \in H\big\};$$

in Grothendieck notation, F identifies with the projective bundle $\mathbf{P}(T_{\mathbf{P}^n})$. The map $\pi : F \to Y$ is the second projection. Next, we consider the divisor $D = V \times \check{\mathbf{P}}^n$ in F, and the critical locus X of the map $\pi : D \to Y$. In this case, X is the conormal variety of V, i.e.,

$$X = \big\{(x, H) \in V \times \check{\mathbf{P}}^n : \ H \text{ is tangent to } V \text{ at } x\big\},$$

and it identifies with the projectivization $\mathbf{P}(N_V)$ of the normal bundle of V in $\mathbf{P}^n$.[14] Let me quote the two following sets of formulae from [19, p. 716–717], as they have been considered in this volume in [XII, C].

(7.8.1) Contacts of lines with a plane curve. Given a generic, smooth, degree d, curve $C \subseteq \mathbf{P}^2$,

$$N_{A_2} = 3d(d-2) \qquad N_{A_1^2} = \frac{1}{2}d(d-3)(d-2)(d+3)$$

are the number of flex and bitangent lines to C respectively. These numbers are contained in the Plücker formulae, see [XII, Section 1.3].

(7.8.2) Contacts of planes with a surface in three-space. Given a generic, smooth, degree d, surface $S \subseteq \mathbf{P}^3$,

$$N_{A_3} = 2d\,(11d - 24)(d-2) \qquad N_{A_1A_2} = 4d(d-3)(d-2)(d^3 + 3d - 16)$$
$$N_{A_1^3} = \frac{1}{6}d(d-2)(d^7 - 4d^6 + 7d^5 - 45d^4 + 114d^3 - 111d^2 + 548d - 960)$$

are the number of planes cutting out on S, respectively, a curve with a tacnode, a curve with one node and one cusp, and a curve with three nodes. These numbers have been computed by Salmon, see [XII, Section 4.4]: N_{A_3} is the number of intersection points of the flecnodal and parabolic curves on S, $2N_{A_3} + N_{A_1A_2}$ is the number of intersection points of the node-couple and the parabolic curves, and $N_{A_1^3}$ is the number of tritangent planes.

(7.9) Contacts of linear spaces with hypersurfaces. Kazarian gives more generally some formulae for the number of linear spaces of arbitrary dimension with prescribed contact conditions with a hypersurface in projective space.

In this case, the parameter space Y is the Grassmannian $\mathbf{G}(k, n)$ of k-planes in $\mathbf{P}^n$ with the universal family $\pi : F \to Y$, the divisor is cut out by $V \times Y$ on $F \subseteq \mathbf{P}^n \times Y$, and

$$X = \big\{(x, \Lambda) \in V \times \mathbf{G}(k, n) : \ \Lambda \subseteq T_{X,x}\big\}$$

is a Grassmann bundle over V. I quote the following set of formulae from [19, p. 719].

[14] Actually, it is better to see it as $\mathbf{P}\,(N_V(-1))$, so that the map $V \to V^\vee \subseteq \check{\mathbf{P}}^n$ is given by a linear subsystem of $|\mathcal{O}_{\mathbf{P}(N_V(-1))}(1)|$.

(7.9.1) Contacts of lines with a surface in three-space. The Grassmannian of lines in $\mathbf{P}^3$ has dimension 4, and the following formulae encompass those contact conditions that directly give finitely many curves, without any further Schubert-type condition. Let S be a generic, smooth, degree d, surface in $\mathbf{P}^3$.

$$N_{A_4} = 5d\,(7d-12)(d-4) \qquad N_{A_1^2A_2} = \frac{1}{2}d(d-6)(d-5)(d-4)(d^3+9d^2+20d-60)$$
$$N_{A_1A_3} = 2d\,(d-5)(3d-5)(d-4)(d+6) \qquad N_{A_1^4} = \frac{1}{12}d(d-7)(d-6)(d-5)(d-4)(d^3+6d^2+7d-30)$$
$$N_{A_2^2} = \frac{1}{2}d\,(d-5)(d-4)(d^3+3d^2+29d-60)$$

On the other hand, we have seen in [XII, Paragraph (2.8)] and [C, Theorem (4.1)] that the number of bitangent and flex-tangent lines to S passing through a general point of $\mathbf{P}^3$ are respectively

$$d(d-1)(d-2) \quad \text{and} \quad \frac{1}{2}d(d-1)(d-2)(d-3);$$

they correspond to the singularities A_1^2 and A_2 respectively. Also, we have seen in [C, Theorem (4.4)] that the number of lines in $\mathbf{P}^n$ $(n>3)$ passing through a general point, and with two contacts of orders 2 and $n-1$ respectively (singularities A_1 and A_{n-2}) with a degree d hypersurface, is $\prod_{k=0}^{n}(d-k)$. The following more general formula has appeared in a recent preprint.

(7.10) Theorem (Fehér–Juhász, [11, Theorem 4.7])**.** *Let V be a generic, smooth, degree d hypersurface in $\mathbf{P}^n$, and consider a general point $v \in \mathbf{P}^n$. For all $e_1,\dots,e_m$ such that $\sum_{k=1}^{m} ke_k = n-1$, the number of lines passing through v and with tangency to V corresponding to the singularity $A_1^{e_1}\cdots A_m^{e_m}$ equals*

$$\frac{1}{\prod_{k=1}^{m} e_k!}d\,(d-1)\cdots(d-\textstyle\sum_{k=1}^{m}(k+1)\,e_k+1).$$

In the context of the theorem, consider the Grassmannian of lines in $\mathbf{P}^n$: it has dimension $2(n-1)$; passing through a point is $n-1$ conditions, and the tangency conditions amount to another $n-1$ conditions.

In the same article, Fehér and Juhász give many more formulae, all built upon their main result, that for arbitrary $e_1,\dots,e_m$, the number of lines with tangency to V corresponding to $A_1^{e_1}\cdots A_m^{e_m}$ and intersecting a general linear subspace of $\mathbf{P}^n$ of codimension $\sum_{k=1}^{m} k\,e_k+1$ is polynomial of degree $\sum_{k=1}^{m}(k+1)\,e_k$ in d.

A – Basics on Bell polynomials

(A.1) Complete and partial Bell polynomials. Consider indeterminates X_i, $i \in \mathbf{N}^*$, and T. The *complete (exponential) Bell polynomials* P_r, $r \in \mathbf{N}$, are defined by the formal identity

$$\sum_{r\geqslant 0} P_r\frac{T^r}{r!} = \exp\left(\sum_{q\geqslant 1} X_q\frac{T^q}{q!}\right)$$

in $\mathbf{Q}(X_1,\dots,)[[T]]$. The *partial Bell polynomials* $B_{r,k}$, $r,k \in \mathbf{N}$, $r \geqslant k$, on the other hand are defined by the formal identities

$$\sum_{r\geqslant k} B_{r,k}\frac{T^r}{r!} = \frac{1}{k!}\left(\sum_{q\geqslant 1} X_q\frac{T^q}{q!}\right)^k.$$

It follows directly that for all $r \in \mathbf{N}$, one has

$$P_r = \sum_{k \leqslant r} B_{r,k}.$$

(A.2) Lemma. *The polynomial $B_{r,k}$ involves only the indeterminates $X_1, \dots, X_{r-k+1}$, and*

$$B_{r,k} = \sum_{\substack{i_1+\cdots+i_{r-k+1}=k \\ i_1+\cdots+(r-k+1)i_{r-k+1}=r}} \frac{r!}{i_1! \cdots i_{r-k+1}!} \frac{1}{\big(1!\big)^{i_1} \cdots \big((r-k+1)!\big)^{i_{r-k+1}}} X_1^{i_1} \cdots X_{r-k+1}^{i_{r-k+1}}.$$

Proof. By Newton's multinomial formula,

$$\frac{1}{k!}\Big(\sum_{q \geqslant 1} X_q \frac{T^q}{q!}\Big)^k = \sum_{i_1+i_2+\cdots=k} \frac{1}{i_1!\, i_2! \cdots} \Big(X_1 \frac{T}{1!}\Big)^{i_1} \Big(X_2 \frac{T^2}{2!}\Big)^{i_2} \cdots$$

where the sum ranges over all sequences $(i_j)_{j \in \mathbf{N}^*}$ of non-negative integers such that $\sum_{j \geqslant 1} i_j = k$. The sequences that contribute to the T^r term are those verifying in addition $\sum_{j \geqslant 1} j.i_j = r$; this implies $i_j = 0$ for all $j > r-k+1$. Then the formula for $B_{r,k}$ follows directly. □

(A.3) Lemma. *Let $r, k \in \mathbf{N}$ such that $k \leqslant r$. For all non-negative integers $i_1, \dots, i_{r-k+1}$ such that $i_1 + \cdots + i_{r-k+1} = k$ and $i_1 + \cdots + (r-k+1)i_{r-k+1} = r$, the number*

$$\frac{r!}{i_1! \cdots i_{r-k+1}!} \frac{1}{\big(1!\big)^{i_1} \cdots \big((r-k+1)!\big)^{i_{r-k+1}}}$$

is the number of unordered partitions of $[\![1, r]\!]$ into k unordered blocks, of which i_1 have size 1, …, i_{r-k+1} have size $r-k+1$.

Proof. First, the number

$$\frac{r!}{\big(1!\big)^{i_1} \cdots \big((r-k+1)!\big)^{i_{r-k+1}}}$$

is the number of partitions of $[\![1, r]\!]$ into i_1 blocks of size 1, …, i_{r-k+1} blocks of size $r-k+1$, where there is an ordering of the blocks, but each block does not have an inner ordering. Indeed, there is a bijection between permutations of $[\![1, r]\!]$ and ordered partitions of $[\![1, r]\!]$ into i_1 ordered blocks of size 1, …, i_{r-k+1} ordered blocks of size $r-k+1$: for each permutation, the ordered partition is simply obtained by putting the first term in the first block and so forth until i_1 blocks of size 1 have been chosen, then the next two terms in that order in the first block of size 2, and so on. Then, forgetting the inner ordering of each block, one obtains a total of

$$\frac{r!}{\big(1!\big)^{i_1} \cdots \big((r-k+1)!\big)^{i_{r-k+1}}}$$

ordered partitions in unordered blocks. Finally, forgetting the ordering of the blocks, one obtains the asserted number. □

The two above lemmas (A.2) and (A.3) have the following direct corollaries.

(A.4) Corollary. *The complete Bell polynomial P_r involves only the indeterminates $X_1, \dots, X_r$, and for all $(i_1, \dots, i_r)$, the coefficient of its $X_1^{i_1} \cdots X_r^{i_r}$ term is the number of partitions of $[\![1, r]\!]$ into i_1 blocks of size 1, …, i_r blocks of size r.*

(A.5) Corollary. *The partial Bell polynomial $B_{r,k}$ has non-negative integer coefficients. It is homogeneous of degree k with respect to the standard grading on $\mathrm{Z}[X_1, \dots, X_{r-k+1}]$. It is homogeneous of degree r with respect to the grading defined by assigning degree i to X_i for all i.*

(A.6) Corollary. *The complete Bell polynomial $P_{r,k}$ has non-negative integer coefficients. It is homogeneous of degree r with respect to the grading defined by assigning degree i to X_i for all i.*

Finally, let us prove the following indentities.

(A.7) Proposition. *Let $r \geqslant 0$ be an integer, and consider indeterminates $X_1, \dots, X_{r+1}$ and $Y_1, \dots, Y_{r+1}$. The two following identitities hold:*

$$P_{r+1}\big(X_1, \dots, X_{r+1}\big) = \sum_{s=0}^{r} \binom{r}{s} X_{r-s+1}\, P_s; \tag{A.7.1}$$

$$P_r\big(X_1 + Y_1, \dots, X_r + Y_r\big) = \sum_{s=0}^{r} \binom{r}{s} P_s\big(X_1, \dots\big)\, P_{r-s}\big(Y_1, \dots\big). \tag{A.7.2}$$

Proof. To prove the first identity, let us differentiate with respect to the indeterminate T the formal identity,

$$\sum_{r\geqslant 0} P_r \frac{T^r}{r!} = \exp\bigg(\sum_{q\geqslant 1} X_q \frac{T^q}{q!}\bigg);$$

one finds

$$\begin{aligned}\sum_{r\geqslant 0} P_{r+1}\frac{T^r}{r!} &= \bigg(\sum_{q\geqslant 0} X_{q+1}\frac{T^q}{q!}\bigg)\,\exp\bigg(\sum_{q\geqslant 1} X_q\frac{T^q}{q!}\bigg)\\ &= \bigg(\sum_{q\geqslant 0} X_{q+1}\frac{T^q}{q!}\bigg)\bigg(\sum_{r\geqslant 0} P_r\frac{T^r}{r!}\bigg)\\ &= \sum_{r\geqslant 0}\bigg(\sum_{p+q=r}\frac{1}{p!q!}\,X_{q+1}P_p\bigg)T^r,\end{aligned}$$

and (A.7.1) follows. For the other one, one computes

$$\begin{aligned}\exp\bigg(\sum_{q\geqslant 1}(X_q+Y_q)\frac{T^q}{q!}\bigg) &= \exp\bigg(\sum_{q\geqslant 1} X_q\frac{T^q}{q!}\bigg)\cdot\exp\bigg(\sum_{q\geqslant 1} Y_q\frac{T^q}{q!}\bigg)\\ &= \bigg(\sum_{r\geqslant 0} P_r(X_1,\dots)\frac{T^r}{r!}\bigg)\bigg(\sum_{r\geqslant 0} P_r(Y_1,\dots)\frac{T^r}{r!}\bigg)\\ &= \sum_{r\geqslant 0}\bigg(\sum_{p+q=r}\frac{1}{p!q!}\,P_p\big(X_1,\dots\big)P_q\big(Y_1,\dots\big)\bigg)T^r,\end{aligned}$$

and (A.7.2) follows. □

References

Chapters in this Volume

[II] E. Sernesi, *Deformations of curves on surfaces.*

[XI] Th. Dedieu, *Enumerative geometry of $K3$ surfaces*.

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces*.

[C] L. Busé and Th. Dedieu, *Generalized weight properties of resultants and discriminants, and applications to projective enumerative geometry*.

[XIV] F. Bastianelli and Th. Dedieu, *Göttsche conjecture and Göttsche–Yau–Zaslow formula*.

External References

[1] A. B. Altman, A. Iarrobino and S. Kleiman, *Irreducibility of the compactified jacobian*, in *Real and Complex Singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, 1–12.

[2] V. I. Arnold, *Local normal forms of functions*, Invent. Math. **35** (1976), 87–109.

[3] E. Artal Bartolo and P. Cassou-Noguès, *Milnor number of plane curve singularities in arbitrary characteristic*, arXiv:2409.13520.

[4] M. Beltrametti and A. J. Sommese, *Zero cycles and k-th order embeddings of smooth projective surfaces*, in *Problems in the theory of surfaces and their classification. Papers from the meeting held at the Scuola Normale Superiore, Cortona, Italy, October 10-15, 1988*, London: Academic Press; Rome: Istituto Nazionale di Alta Matematica Francesco Severi, 1991, Appendix by Lothar Göttsche: Identification of very ample line bundles on $S^{(r)}$, 33–48; appendix: 44–48.

[5] F. Block, *Computing node polynomials for plane curves*, Math. Res. Lett. **18** (2011), no. 4, 621–643.

[6] ———, *Relative node polynomials for plane curves*, J. Algebraic Combin. **36** (2012), no. 2, 279–308.

[7] E. Casas-Alvero, *Singularities of plane curves*, Lond. Math. Soc. Lect. Note Ser., vol. 276, Cambridge: Cambridge University Press, 2000.

[8] Th. Dedieu and E. Sernesi, *Equigeneric and equisingular families of curves on surfaces*, Publ. Mat., Barc. **61** (2017), no. 1, 175–212.

[9] D. Eisenbud and J. Harris, *3264 and all that*, Cambridge: Cambridge University Press, 2016.

[10] F. Enriques and O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni abgebriche.*, Zanichelli, Bologna, 1934.

[11] L. M. Fehér and A. P. Juhász, *Plücker formulas using equivariant cohomology of coincident root strata*, arXiv:2312.06430.

[12] S. Fomin and G. Mikhalkin, *Labeled floor diagrams for plane curves*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 6, 1453–1496.

[13] W. Fulton, *Intersection theory.*, 2nd ed. ed., Ergeb. Math. Grenzgeb., 3. Folge, vol. 2, Berlin: Springer, 1998.

[14] G.-M. Greuel and C. Lossen, *Equianalytic and equisingular families of curves on surfaces*, Manuscr. Math. **91** (1996), no. 3, 323–342.

[15] G.-M. Greuel, C. Lossen and E. Shustin, *Introduction to singularities and deformations*, Springer Monogr. Math., Berlin: Springer, 2007.

[16] ———, *Singular algebraic curves*, Springer Monogr. Math., Berlin: Springer, 2018, with an appendix by O. Viro.

[17] N. M. Katz, *Pinceaux de Lefschetz : théorème d'existence*, in *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, vol. Vol. 340, Springer-Verlag, Berlin-New York, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz, 212–253.

[18] S. Katz, *Rational curves on Calabi-Yau threefolds*, in *Essays on mirror manifolds*, Cambridge, MA: International Press, 1992, 168–180.

[19] M. È. Kazaryan, *Multisingularities, cobordisms, and enumerative geometry.*, Russ. Math. Surv. **58** (2003), no. 4, 665–724.

[20] S. Kleiman, *Multiple-point formulas. I: Iteration*, Acta Math. **147** (1981), 13–49.

[21] S. Kleiman and R. Piene, *Enumerating singular curves on surfaces*, in *Algebraic geometry: Hirzebruch 70. Proceedings of the algebraic geometry conference in honor of F. Hirzebruch's 70th birthday, Stefan Banach International Mathematical Center, Warszawa, Poland, May 11–16, 1998*, Providence, RI: American Mathematical Society, 1999, 209–238.

[22] ———, *Node polynomials for families: Methods and applications*, Math. Nachr. **271** (2004), 69–90.

[23] ———, *Enriques diagrams, arbitrarily near points, and Hilbert schemes*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. **22** (2011), no. 4, 411–451, with Appendix B by I. Tyomkin.

[24] ———, *Node polynomials for curves on surfaces*, SIGMA, Symmetry Integrability Geom. Methods Appl. **18** (2022), paper 059, 23.

[25] S. Kleiman and V. V. Shende, *On the Göttsche threshold (with an appendix by Ilya Tyomkin).*, in *A celebration of algebraic geometry. A conference in honor of Joe Harris' 60th birthday, Harvard University, Cambridge, MA, USA, August 25–28, 2011*, Providence, RI: American Mathematical Society (AMS); Cambridge, MA: Clay Mathematics Institute, 2013, 429–449.

[26] A. L. Knutsen, *On kth-order embeddings of K3 surfaces and Enriques surfaces*, Manuscr. Math. **104** (2001), no. 2, 211–237.

[27] M. Kool, V. Shende and R. P. Thomas, *A short proof of the Göttsche conjecture*, Geom. Topol. **15** (2011), no. 1, 397–406.

[28] T. Laarakker, *The Kleiman-Piene conjecture and node polynomials for plane curves in* $\mathbb{P}^3$, Sel. Math., New Ser. **24** (2018), no. 5, 4917–4959.

[29] J. Li and Y.-J. Tzeng, *Universal polynomials for singular curves on surfaces*, Compos. Math. **150** (2014), no. 7, 1169–1182.

[30] T. Ohmoto, *Universal polynomials for multi-singularity loci of maps*, 2024, preprint arXiv:2406.12166.

[31] N. Qviller, *The di Francesco-Itzykson-Göttsche conjectures for node polynomials of* $\mathbb{P}^2$, Int. J. Math. **23** (2012), no. 4, 19, Id/No 1250049.

[32] ———, *Structure of node polynomials for curves on surfaces*, Math. Nachr. **287** (2014), no. 11-12, 1394–1420.

[33] J. V. Rennemo, *Universal polynomials for tautological integrals on Hilbert schemes*, Geom. Topol. **21** (2017), no. 1, 253–314.

[34] T. Sasajima and T. Ohmoto, *Classical formulae on projective surfaces and 3-folds with ordinary singularities, revisited*, Saitama Math. J. **31** (2017), 141–159.

[35] ———, *Thom polynomials in $\mathcal{A}$-classification I: counting singular projections of a surface*, in *Schubert varieties, equivariant cohomology and characteristic classes—IMPANGA 15*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2018, 237–259.

[36] E. I. Shustin, *On manifolds of singular algebraic curves*, Sel. Math. Sov. **10** (1983), no. 1, 27–37.

[37] R. P. Stanley, *Enumerative combinatorics. Vol. 1.*, 2nd ed., Camb. Stud. Adv. Math., vol. 49, Cambridge: Cambridge University Press, 2012.

[38] Y.-J. Tzeng, *A proof of the Göttsche–Yau–Zaslow formula*, J. Differential Geom. **90** (2012), 439–472.

[39] I. Vainsencher, *Enumeration of n-fold tangent hyperplanes to a surface*, J. Algebr. Geom. **4** (1995), no. 3, 503–526.

Lecture XIV
The Göttsche conjecture and the Göttsche–Yau–Zaslow formula

by Francesco Bastianelli and Thomas Dedieu

1 Introduction 467
2 Kool-Shende-Thomas' proof of Göttsche conjecture 470
2.1 Sufficiently ample line bundles 471
2.2 Euler characteristics of Hilbert schemes of points over curves . . . 474
2.3 Proof of the Göttsche conjecture 476
2.4 The Ellingsrud–Göttsche–Lehn recursion 479
3 Tzeng's proof of the Göttsche–Yau–Zaslow formula 481
3.1 The algebraic cobordism group of surfaces and line bundles 482
3.2 Göttsche's enumerative integrals 484
3.3 Degeneration of enumerative integrals 485
3.4 Conclusion 490
References 491

1 – Introduction

The main theme of this note is the set of enumerative conjectures stated and discussed by Göttsche in [7]. We retrace and summarize the proofs of the Göttsche conjecture given first by Tzeng [28], and shortly after by Kool, Shende and Thomas [15]. We also discuss the proof of the Göttsche–Yau–Zaslow formula given by Tzeng. Accordingly, [7], [15] and [28] are the main sources for this note, and the proofs given here are directly inspired by those included in the original papers.

Let S be a smooth complex projective surface, and let δ be a positive integer. A δ-*nodal* curve on S is a reduced (possibly reducible) curve $C \subseteq S$ having exactly δ nodes (i.e., ordinary double points), and no other singularities. Given a line bundle L on S, we denote by $a_\delta(S, L)$ the number of δ-nodal curves contained in a general linear subsystem $\Lambda \subseteq |L|$ of dimension δ. When L is sufficiently ample, δ-nodal curves in $|L|$ indeed occur in codimension δ, so that $a_\delta(S, L)$ is a finite number; moreover, it coincides with the number of δ-nodal curves passing through $\dim |L| - \delta$ general points of S, see [13, p.234], reproduced in [XIII, Lemma (5.1)]. The study of $a_\delta(S, L)$ is a very classical theme, and the main topic of this note. We refer the reader to [14, 15, 28], and [XIII] in this volume, for a detailed bibliography and a treatise of the background underlying the results we are going to discuss.

The first result we would like to present is the so-called Göttsche conjecture [7, Conjecture 2.1]. In the light of [29, 13] (described in [XIII]) where formulas for the numbers $a_\delta(S, L)$ were given for all $\delta \leqslant 8$, it predicts that if the line bundle L on S is sufficiently ample with respect to δ, then the number $a_\delta(S, L)$ is computed by a universal polynomial of degree δ in the numbers $L^2, LK_S, K_S^2, c_2(L)$ (universal means that the same polynomial works for all pairs

(S, L)). Göttsche's conjecture has been proved in terms of the following positivity notion for a line bundle L on S.

(1.1) Definition. *Let L be a line bundle on a smooth projective surface S, and let $k \geqslant 0$ be an integer. We say that L is* k-very ample *if for any 0-dimensional subscheme $Z \subseteq S$ of length $k+1$, the restriction map $H^0(S, L) \to H^0(Z, L \otimes \mathcal{O}_Z)$ is surjective.*

The Göttsche conjecture was first proved by Tzeng [28]. Shortly after, Kool, Shende and Thomas proved the following version (see [15, Theorem 4.1]), with weaker positivity assumptions on L.

(1.2) Theorem (Göttsche conjecture)**.** *For all positive integers δ, there exists a polynomial $T_\delta(x, y, z, t)$ of degree δ with the following property: for any smooth complex projective surface S and for any δ-very ample line bundle L on S, a general δ-dimensional linear subsystem $\Lambda \subseteq |L|$ contains exactly $T_\delta(L^2, LK_S, K_S^2, c_2(S))$ δ-nodal curves, i.e.,*

$$a_\delta(S, L) = T_\delta(L^2, LK_S, K_S^2, c_2(S)). \tag{1.2.1}$$

It is worth noting that the coefficients of the polynomial $T_\delta(x, y, z, t)$ may be computed for all δ. Indeed, the numbers $a_\delta(S, L)$ can be determined for all line bundles on $\mathbf{P}^2$ by [2, Theorem 1.1], and for all primitive line bundles on a $K3$ surface by [1, Theorem 1.1]; moreover, the 4-tuple of topological invariants $(L^2, LK_S, K_S^2, c_2(L))$ is $(n^2, -3n, 9, 3)$ if $(S, L) = \big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(n)\big)$, and $(L^2, 0, 0, 24)$ if L is a primitive line bundle on a $K3$ surface S; it follows that the coefficients of each $T_\delta(x, y, z, t)$ can be computed by solving a system of linear equations obtained from (1.2.1) using different values of n and L^2.

We will discuss the two proofs of Theorem (1.2) given by Tzeng [28], and Kool, Shende, and Thomas [15], respectively. There is also another proof due to Liu [23, 24], but we will not report on it. The Kool–Shende–Thomas proof is close in spirit to the work of Vainsencher and Kleiman–Piene on node polynomials, which is described in [XIII]. In particular it enables one to compute explicitly the node polynomials T_δ although, in its authors' own words, it is not terribly efficient. Tzeng's proof, on the other hand, involves an abstract version of the degeneration methods constituting the backbone of this volume, see, in particular, [I] and [VI].

We retrace the Kool–Shende–Thomas proof in Section 2. It relies on BPS calculus on Hilbert schemes of points on (arbitrary reduced) curves on a surface S, see [26, 27], the calculation of tautological integrals on Hilbert schemes of points on S by means of the recursion introduced in [5], and the deformation theory of plane singularities of curves, see, e.g., [3] and [II]. The main idea underlying the proof is to describe the numbers $a_\delta(S, L)$ of δ-nodal curves in a general linear subsystem $\Lambda \subseteq |L|$ in terms of the Euler characteristics of relative Hilbert schemes of points associated to the universal curve $\mathcal{C} \to \Lambda$. This is what BPS calculus is used for; after that, the proof consists in (non-trivial) intersection theoretic considerations. We refer to the introduction of Section 2 for a more detailed outline.

We survey Tzeng's proof in Section 3. Ultimately, it relies on the fact, proved in [16], that the algebraic cobordism group of smooth, complex, projective surfaces is generated by (the classes of) $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$. Tzeng's proof is formulated in the framework of [17, 18] for algebraic cobordism, which is based on the notion of (extended) double point relation: roughly speaking, one mods out the free abelian group generated by all smooth surfaces by relations corresponding to degenerations of one surface to the transverse union of two surfaces. Tzeng's main result is a suitable version of Jun Li's celebrated degeneration formula in this context, which involves Li and Wu's construction of a moduli space of stable ideal sheaves in [21]. The degeneration formula enables her to express the numbers a_δ for the pair (S, L) (actually, the better suited variant $d_\delta(S, L)$ introduced by Göttsche) in terms of the numbers a_δ for the generators of the algebraic cobordism group, which then leads to the Göttsche conjecture.

The other results we will discuss concern the structure of the generating function of the universal polynomials $T_\delta(L^2, LK_S, K_S^2, c_2(S))$, that is the following formal power series in x,

$$T(S,L) := \sum_{\delta \geqslant 0} T_\delta(L^2, LK_S, K_S^2, c_2(S))\, x^\delta$$

(we set $T_0(x,y,z,t) = 1$). In [7] Göttsche proved that if the numbers $a_\delta(S,L)$ are indeed computed by universal polynomials, then the function $T(S,L)$ must satisfy rather strong conditions. In particular, it has the following multiplicative structure by [7, Proposition 2.3], which we may now state unconditionally since the Göttsche conjecture is proved.

(1.3) Theorem. *There exist universal (invertible) power series $A_1, A_2, A_3, A_4 \in \mathbf{Q}[[x]]$ such that*

$$T(S,L) = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)}.$$

Göttsche's proof of this fact consists in considering a pair (S,L) obtained as the disjoint union of pairs of the form $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(n))$ and $(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(n,n))$, and using the additivity properties of the 4-tuples $(L^2, LK_S, K_S^2, c_2(S))$. We will give Tzeng's proof in Section 3.4; it formulates the same idea in the somehow taylor-made theory of algebraic cobordism.

We further point out that, as was the case for the coefficients of the universal polynomials $T_\delta(x,y,z,t)$, the coefficients of the universal power series A_1, A_2, A_3, A_4 may be determined by the recursive formulas describing the numbers $a_\delta(S,L)$ for certain classes of pairs (S,L) (see, e.g., the aforementioned [2, Theorem 1.1] for line bundles on $\mathbf{P}^2$ and [30, Theorem 6.7] for the case of Hirzebruch surfaces).

Another important result concerning the generating function $T(S,L)$ is the Göttsche–Yau–Zaslow formula, which was conjectured by Göttsche [7] based on the formulas in [29, 13], as a generalization of the Yau–Zaslow formula for rational curves on $K3$ surfaces [32], see [XI]. In particular, the Göttsche–Yau–Zaslow formula expresses the universal power series $T(S,L)$ in terms of five universal generating functions in one variable q: three explicit quasimodular forms, and two unknown universal power series, whose coefficients may nevertheless be determined by the recursion formula of [2].

We refer the reader to [11] for some preliminary notions on modular and quasimodular forms. Let $\tau \in \mathbb{C}$ be a complex number with positive imaginary part, and set $q := e^{2\pi i \tau}$. Consider the *discriminant form*

$$\Delta(\tau) := q \prod_{n>0} (1-q^n)^{24}$$

and the *second Eisenstein series*

$$G_2(\tau) := -\frac{1}{24} + \sum_{n>0} \left(\sum_{k|n} k \right) q^n,$$

which are a modular form and a quasimodular form, respectively. Furthermore, given the differential operator $D := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$, we have that for any quasimodular form f, the derivative Df is again a quasimodular form. Then the following holds, see [7, Conjecture 2.4] and [28, Theorem 1.2].

(1.4) Theorem (Göttsche-Yau–Zaslow formula)**.** *There exist universal power series $B_1(q)$ and $B_2(q)$ such that*

$$\sum_{\delta \geqslant 0} T_\delta(L^2, LK_S, K_S^2, c_2(S)) \cdot DG_2(\tau)^\delta = \frac{(DG_2(\tau)/q)^{\chi(L)}\, B_1(q)^{K_S^2} B_2(q)^{LK_S}}{(\Delta(\tau) D^2 G_2(\tau)/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

We give the proof in Section 3.4. An essential input in it is Bryan and Leung's main result in [1], see [XI].

We note that, according to [7, Remark 2.6] and [28, Corollary 4.4], the Göttsche–Yau–Zaslow formula can be reformulated as follows. For a smooth projective surface S and for any $l, m, r \in \mathbf{Z}$, we define

$$N_r^S(l,m) := T_{l+\chi(\mathcal{O}_S)-1-r}(2l+m, m, K_S^2, c_2(S)).$$

Then Theorem (1.4) gives that

$$\sum_{l\in\mathbf{Z}} N_r^S(l,m)q^l = B_1(q)^{K_S^2} B_2(q)^m (DG_2(\tau))^r \frac{D^2G_2(\tau)}{(\Delta(\tau)D^2G_2(\tau)/q^2)^{\chi(\mathcal{O}_S)/2}},$$

and Theorem (1.2) ensures that if L is a δ-very ample line bundle on S with $\delta = \chi(L) - 1 - r$, then the coefficient $N_r^S\left(\frac{L^2-LK_S}{2}, LK_S\right)$ counts the number of δ-nodal curves in a general linear subsystem $W \subseteq |L|$ of codimension r. Analogously, setting $g = \frac{L^2+LK_S}{2} + 1 - \delta$, the coefficient

$$M_g^S\left(\tfrac{L^2-LK_S}{2}, LK_S\right) := N_{g-m-2+\chi(\mathcal{O}_S)}\left(\tfrac{L^2-LK_S}{2}, LK_S\right)$$

computes the number of nodal curves of geometric genus g in a general linear subsystem of $|L|$ having codimension $r = g - LK_S - 2 + \chi(\mathcal{O}_S)$.

Acknowledgements. The first author would like to thank Giulio Codogni, Lothar Göttsche, Manfred Lehn, Donatella Iacono, Margarida Melo, and Filippo Viviani, for helpful discussions and explanations during the preparation of a series of lectures on this topic and during the writing of this note.

The second author thanks Laurent Manivel and Adrien Sauvaget for their help with the Ellingsrud–Göttsche–Lehn recursion, and the first author for everything.

(1.5) Notation. We work throughout over the field $\mathbf{C}$ of complex numbers. By *curve* we mean a connected complete reduced algebraic curve, unless otherwise stated.

Given a projective variety X, we say that a property holds for a *general* point $x \in X$ if it holds on an open non-empty subset of X. In particular, if L is a line bundle on a smooth surface S, by *general* linear subsystem $\Lambda \subseteq |L|$ of dimension δ we mean that Λ is parameterized over an open subset of the Grassmannian of δ-dimensional linear subspaces of $|L| \simeq \mathbf{P}^{h^0(L)-1}$.

For a positive integer k, we denote by $X^{[k]}$ the k-fold Hilbert scheme of points on a variety X, parameterizing 0-dimensional subschemes of S of length k, and by $\mathrm{Hilb}^k(\mathcal{X}/B)$ the k-fold relative Hilbert scheme of points on the fibres of a family $\mathcal{X} \to B$ of varieties.

The *Euler number* (sometimes Euler characteristic) of a topological space X is the number $e(X) = \sum_i (-1)^i \dim H_c^i(X, \mathbf{Z})$, where the subscript c means cohomology with compact support. We refer to [XI, Section 2.1] for some basic properties of the Euler number, including its additivity, and an expression when X is a fibration.

2 – Kool-Shende-Thomas' proof of Göttsche conjecture

In this section, we present the proof of Theorem (1.2) given by Kool, Shende, and Thomas [15]. So, we consider a line bundle L on a smooth surface S, and we assume that L is δ-very ample for some integer $\delta \geqslant 1$.

In §2.1, we show that a general, δ-dimensional, linear subsystem Λ of $|L|$ contains a finite number of δ-nodal curves, each occurring with multiplicity 1, and that all the other elements of

Λ are reduced curves having larger geometric genus. This is an application of the deformation theory of plane curve singularities, using in an essential way the δ-very ampleness of L.

In §2.2, we first review some facts concerning the generating functions of the Euler characteristics $e(C^{[k]})$ of Hilbert schemes of points on curves C, for any reduced curve C contained in S; this is the so-called BPS calculus on Hilbert schemes of points. We then use these facts to express the number $a_\delta(S,L)$ of δ-nodal curves in a general $\Lambda \subseteq |L|$ as a linear combination of the Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$, $k = 0, \dots, \delta$, of the relative Hilbert schemes of points on the fibres of the universal curve $\mathcal{C} \to \Lambda$. This is done by stratifying Λ according to the topological types of its members, and using the additivity of the Euler number.

In §2.3, we conclude the proof by showing that each term appearing in this linear combination is indeed a polynomial in L^2, LK_S, K_S^2 and $c_2(L)$. To this aim, we retrace Kool–Shende–Thomas' computation of the Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$ in terms of certain tautological integrals, first on the Hilbert schemes $S^{[k]}$, and then on the products S^k.

The passing from integrals on $S^{[k]}$ to integrals on S^k relies on the recursion of Ellingsrud, Göttsche and Lehn [5, Sections 3 and 4], which we survey in §2.4.

(2.1) We point out that the idea underlying this count of δ-nodal curves generalizes the following argument for the case $\delta = 1$ (it is explained in more details in [XI, Section 2.1]). Assume that L is a very ample line bundle and $\Lambda \subseteq |L|$ is a general pencil, and denote by p the genus of the smooth fibres of the universal curve $\mathcal{C} \to \Lambda \cong \mathbf{P}^1$.

The quantity $e(C) - (2-2p)$ equals 1 if C is 1-nodal, and 0 if C is smooth. Therefore, summing this quantity over $\Lambda \cong \mathbf{P}^1$, we obtain the identity

$$a_1(S,L) = e(\mathcal{C}) - (2-2p)\, e(\mathbf{P}^1).$$

To see this formula as a particular case of the Kool–Shende–Thomas' count, we rewrite it in terms of linear combinations of Euler characteristics of Hilbert schemes of points: one has

$$e(C) - (2-2p) = e(C^{[1]}) - (2-2p)\, e_0(C^{[0]})$$

and, correspondingly,

$$e(\mathcal{C}) - (2-2p)\, e(\mathbf{P}^1) = e(\mathrm{Hilb}^1(\mathcal{C}/\Lambda)) - (2-2p)\, e(\mathrm{Hilb}^0(\mathcal{C}/\Lambda)).$$

In §2.2, we give a similar expression of $a_\delta(S,L)$ as a sum over all members C of $\Lambda \cong \mathbf{P}^\delta$ of some linear combination of the Euler characteristics $e(C^{[k]})$, $k = 1, \dots, \delta$, which takes value 1 for δ-nodal curves, and 0 for curves having higher geometric genus. So, in order to prove the Göttsche conjecture, it remains to express the corresponding linear combination of the Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$, $k = 0, \dots, \delta$, in terms of the topological invariants L^2, LK_S, K_S^2 and $c_2(S)$; this is the purpose of §2.3.

For $\delta = 1$, this goes simply as follows: $e(\mathcal{C}) = c_2(S) + L^2$ as $\mathcal{C}$ is the blow-up of S at L^2 points, $2p - 2 = L(L + K_S)$ by the adjunction formula, and $e(\mathbf{P}^1) = 2$, so that we obtain

$$a_1(S,L) = c_2(S) + 3L^2 + 2LK_S,$$

which is indeed a polynomial of degree 1 in the topological numbers $L^2, LK_S, K_S^2, c_2(S)$.

2.1 – Sufficiently ample line bundles

Let L be a line bundle on a smooth projective surface S. We follow [15, Section 2] to show how the δ-very ampleness condition on L ensures that the singularities of curves in a general

dimension δ linear subsystem $\Lambda \subseteq |L|$ behave sufficiently well for the count of δ-nodal curves in Λ by means of the Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$ in subsection 2.2 to make sense, see Theorem (2.10). Roughly speaking, the singularities of the members of Λ behave as expected up to codimension δ in $|L|$. The precise statement is the following.

(2.2) Theorem. *Assume L is δ-very ample for some positive integer δ. Let p denote the common arithmetic genus of all members of $|L|$, and consider a general linear subsystem $\Lambda \subseteq |L|$ of dimension δ. Then all members of Λ are reduced curves, and have geometric genus $g \geqslant p-\delta$. Moreover, there are finitely many members of genus $g = p - \delta$; they are all δ-nodal curves, and appear with multiplicity* 1.

Let us clarify the last sentence. Recall that the locus $V^\delta(\Lambda)$ parametrizing δ-nodal members of Λ is well-defined as a subscheme of Λ, see [4] for a discussion and the relevant references; the locus $V_g(\Lambda)$ parametrizing genus g members of Λ, however, is only defined set-theoretically (it is a locally closed subset of Λ). Now, the last sentence means that (i) the set $V_{p-\delta}(\Lambda)$ has dimension 0 and coincides with the support of $V^\delta(\Lambda)$, and (ii) the scheme $V^\delta(\Lambda)$ is reduced.

(2.3) Remark. Note that the theorem implies that all members of Λ have at most δ singular points, and all members with δ singular points are δ-nodal curves. Indeed, each singular point makes the geometric genus drop by at least one.

The rest of Section 2.1 is dedicated to the proof of Theorem (2.2). We will use extensively the deformation theory of planar curve singularities, for which we refer to [II] in this volume, and to [3, 8] for extended treatises; the reader may also consult [4]. The punchline is that the δ-very ampleness property ensures that the maps from the relevant equigeneric and equisingular subloci of $|L|$ to the corresponding semi-universal deformation spaces of singularities are smooth, because their differentials are restriction maps, and the δ-very ampleness grants their surjectivity.

(2.4) Let us first show that the subscheme $V^\delta(|L|) \subseteq |L|$ of δ-nodal curves in $|L|$ is smooth of codimension δ. By Bertini's theorem this implies that $V^\delta(\Lambda) = V^\delta(|L|) \cap \Lambda$ is smooth of dimension 0, i.e., δ-nodal curves in Λ are finitely many and appear with multiplicity 1.

Let $[C] \in |L|$ be a δ-nodal curve, and J be its Jacobian ideal. The vector space $B := H^0(C, \mathcal{O}_C/J)$ is the product of the semi-universal deformation spaces of the various singularities of C. We thus have a map of pointed germs $\varphi : (|L|, [C]) \to (B, 0)$, and its differential is the restriction map

$$\rho_J : T_{|L|,[C]} \cong H^0(S, L)/\langle s_C \rangle \longrightarrow H^0(Z, L \otimes \mathcal{O}_C/J),$$

where $Z \subseteq S$ is the singular subscheme of C, defined in C by the ideal J, and $s_C \in H^0(S, L)$ is a section with zero divisor equal to C. Since C is δ-nodal, its singular subscheme has length δ, hence the restriction map $H^0(S, L) \to H^0(Z, L \otimes \mathcal{O}_Z)$ is surjective by the positivity property of L. This implies that the map ρ_J is surjective, hence φ is smooth.

The semi-universal deformation space of a node is a line, in which the equisingular locus is only the origin. Thus the locus of δ-nodal curves in B is only the origin, which is smooth and has codimension δ in B. By smoothness of φ, the same holds for $V^\delta(|L|)$ around $[C]$, which proves our claim.

(2.5) Consider a non-negative integer $g' < p - \delta$, and let $V_{g'}(|L|) \subseteq |L|$ be the locally closed subset of reduced curves having geometric genus g'. We now show that $V_{g'}(|L|)$ has codimension (strictly) greater than δ in $|L|$, which implies that Λ does not contain any curve of genus g'.

Let $[C] \in V_{g'}(|L|)$. We consider its conductor ideal A, and let Y be the subscheme of C defined by A. By [II, Proposition 3.1] the tangent cone of $V_{g'}(|L|)$ at $[C]$ is contained in the

kernel of the restriction map

$$\rho_Y : H^0(S, L)/\langle s_C\rangle \longrightarrow H^0\left(Y, L \otimes \mathcal{O}_C/A\right),$$

where s_C is again a section with zero divisor equal to C.

Since the co-genus $p - g'$ of C in $|L|$ is greater than δ, we may find a subscheme $Z \subseteq Y$ of length $\delta + 1$. Then the positivity property of L implies that the restriction map

$$\rho_Z : H^0(S, L)/\langle s_C\rangle \longrightarrow H^0\left(Z, L \otimes \mathcal{O}_Z\right)$$

is surjective, hence its kernel has codimension $\delta+1$. Since $\rho_Z = \pi \circ \rho_Y$, where π is the restriction map $H^0(L \otimes \mathcal{O}_Y) \to H^0(L \otimes \mathcal{O}_Z)$, this implies that the kernel of ρ_Y has codimension at least $\delta + 1$, hence $V_{g'}(|L|)$ has codimension at least $\delta + 1$ in $|L|$ as required.

(2.6) Let τ be a collection of topological types of planar singularities, which does not consist of δ nodes, and such that a member of $|L|$ with singularities of type τ has genus $p - \delta$. We now show that the subscheme $V^\tau(|L|) \subseteq |L|$ parametrizing curves with topological type τ has codimension (strictly) greater than δ in $|L|$.

Let $[C] \in V^\tau(|L|)$, and I be its equisingular ideal. By [II, Proposition 3.1], the tangent space to $V^\tau(|L|)$ at $[C]$ is the kernel of the restriction map

$$\rho_Y : H^0(S, L)/\langle s_C\rangle \longrightarrow H^0\left(Y, L \otimes \mathcal{O}_C/I\right),$$

where Y is the subscheme of C defined by I. The assumptions on τ imply that we may find a subscheme $Z \subseteq Y$ of length $\delta + 1$, see [II, Lemma 2.4]. Then, as in (2.5), the δ-very ampleness of L implies that the kernel of ρ_Y has codimension at least $\delta+1$, hence $V^\tau(|L|)$ has codimension at least $\delta + 1$ in $|L|$ as claimed.

(2.7) Finally, we proceed to show that the locus $V^{\mathrm{nr}}(|L|) \subseteq |L|$ parametrizing non-reduced curves has codimension (strictly) greater than δ.

We claim that if C is a non-reduced member of $|L|$, then the tangent cone to $V^{\mathrm{nr}}(|L|)$ at $[C]$ is contained in the kernel of the restriction map

$$\rho : H^0(S, L)/\langle s_C\rangle \longrightarrow H^0\left(C_{\mathrm{red}}, L \otimes \mathcal{O}_{C_{\mathrm{red}}}\right).$$

When we know this, we conclude that $V^{\mathrm{nr}}(|L|)$ has codimension greater than δ always along the same lines: we pick a 0-dimensional subscheme $Z \subseteq C_{\mathrm{red}}$ of length $\delta + 1$, and the kernel of ρ is contained in the kernel of the restriction to Z, which has codimension $\delta + 1$ since L is δ-very ample.

To prove our claim, we argue as in [31, §2.1] and consider the incidence variety

$$\mathcal{D} := \left\{\left([C], p\right) \in |L| \times S : C \text{ is singular at } p\right\}.$$

It is smooth: indeed since L is very ample and S is smooth, $\mathcal{D}$ is a projective bundle over S. On the other hand, if $[C] \in V^{\mathrm{nr}}(|L|)$ then $([C], p) \in \mathcal{D}$ for all $p \in C_{\mathrm{red}}$. It follows that the tangent cone to $V^{\mathrm{nr}}(|L|)$ at $[C]$ is contained in the image of the tangent space to $\mathcal{D}$ at $([C], p)$ by the projection $T_{|L|\times S} \to T_{|L|}$ for all $p \in C_{\mathrm{red}}$. We shall show that the latter image is contained in the kernel of the restriction

$$H^0(S, L)/\langle s_C\rangle \longrightarrow H^0\big(p, L|_p\big),$$

and this will prove our claim.

The equations of $\mathcal{D}$ in $|L| \times S$ are as follows. Let $([C], p) \in |L| \times S$; we choose a local system of coordinates (x, y) on S around p, and see $|L|$ around $[C]$ as a finite dimensional affine space of polynomials $f \in \mathbf{C}[x, y]$. In this representation, $\mathcal{D}$ is defined by the equations

$$f(x, y) = df(x, y) = 0.$$

By differentiation, the latter equations give the equations of the tangent space of $\mathcal{D}$ in $T_{|L|} \times T_S$. In particular, if $(h, v) \in T_{|L|,f} \times T_{S,(x,y)}$ sits in $T_{\mathcal{D}}$, then

$$df(x, y)(v) + h(x, y) = 0,$$

which is equivalent to $h(x, y) = 0$ since $df(x, y) = 0$. In other words, if $([C], p) \in \mathcal{D}$, then the projection of $T_{\mathcal{D},([C],p)}$ to $T_{|L|,[C]}$ is contained in the kernel of the restriction map

$$H^0(S, L)/\langle s_C \rangle \longrightarrow H^0\big(p,\, L|_p\big),$$

as we wanted to prove. We have thus proved our claim, hence that $V^{\mathrm{nr}}(|L|)$ has codimension greater than δ.

The results proved in (2.4), (2.5), (2.6), and (2.7) together amount to Theorem (2.2). The proof of the latter is therefore complete. □

2.2 – Euler characteristics of Hilbert schemes of points over curves

We consider a smooth projective surface S endowed with a δ-very ample line bundle L on S, as in Theorem (2.2). We follow [15, Section 3] in order to express the number $a_\delta(S, L)$ of δ-nodal curves in a general δ-dimensional linear subsystem $\Lambda \subseteq |L|$ in terms of Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$ of relative Hilbert schemes, where $\mathcal{C} \to \Lambda$ is the universal curve.

To this aim, we recall a couple of preliminary results describing the generating series of the Euler characteristics $e(C^{[k]})$ of k-fold Hilbert schemes of points on curves $C \subseteq S$, see [15, Propositions 3.1 and 3.2]. Both results were first proved in [26] in the setting of stable pairs, whereas [27, Section 2] includes proofs using only Hilbert schemes.

(2.8) Proposition. *Let S be a smooth surface and let $C \subseteq S$ be a curve having arithmetic genus p and geometric genus g. There exists a unique sequence of integers $(n_{C,g}, n_{C,g+1}, \ldots, n_{C,p})$, depending only on the topological type of C, such that the generating series of the Euler characteristics $e(C^{[k]})$ satisfies*

$$\sum_{k=0}^{\infty} e(C^{[k]}) q^k = \sum_{i=g}^{p} n_{C,i} q^{p-i} (1 - q)^{2i-2}.$$

Writing the generating series $\sum e(C^{[k]}) q^k$ as a sum $\sum n_{C,i} q^{p-i} (1 - q)^{2i-2}$ with i ranging between $-\infty$ and p is granted by a formal property of power series, see [27, Proposition 1]. However, the geometry of C is involved in the proof that $n_{C,i} = 0$ for all $i < g$; this is deduced from [27, Proposition 8], see [27, Corollary 11].

When the curve C is δ-nodal, the generating series of Euler characteristics $e(C^{[k]})$ is governed by the following, see [27, Corollary 12].

(2.9) Proposition. *Let C be a δ-nodal curve with arithmetic genus p. Then the coefficients $n_{C,i}$ appearing in Proposition (2.8) are*

$$n_{C,i} = \tbinom{\delta}{p-i}$$

for all $i = p - \delta, \ldots, p$.

When C is a smooth curve this was proved by Macdonald [25], in order to give a new proof of the de Jonquières formulas discussed in [XII]. In this case one finds

$$\sum_{k=0}^{\infty} e\big(C^{[k]}\big)q^k = (1-q)^{2p-2} = \sum_{k=0}^{2p-2} (-1)^k \tbinom{2p-2}{k} q^k.$$

While $e(C) = 2-2p$, and $e(C^{[k]}) = 0$ if $k > 2p-2$ because then $C^{[k]} \to J^k(C)$ gives a structure of projective bundle over a torus, the values of $e(C^{[k]})$ for $1 < k \leqslant 2p-2$ are not obvious to us.

In the general case of Proposition (2.9), Kool, Shende, and Thomas interpret the number $\binom{\delta}{p-i}$ as the number of partial normalizations of C at $p-i$ of its δ nodes. This is the reason why Proposition (2.9) is considered a statement in BPS calculus, see [XI].

We may now state the main result of this section, see [15, Theorem 3.4]. It follows from Propositions (2.8), (2.9) and Theorem (2.2)

(2.10) Theorem. *Let S be a smooth projective surface equipped with a line bundle L, and denote by p be the arithmetic genus of all members of $|L|$. Let δ be a positive integer such that L is δ-very ample, and consider a general linear subsystem $\Lambda \subseteq |L|$ of dimension δ. Let $\mathcal{C} \to \Lambda$ be the universal curve over Λ. There are integers $n_{\Lambda,p-\delta}, \dots, n_{\Lambda,\delta}$ such that the following identity of formal power series holds:*

$$\sum_{k=0}^{\infty} e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))q^k = \sum_{i=p-\delta}^{p} n_{\Lambda,i} q^{p-i}(1-q)^{2i-2}. \tag{2.10.1}$$

Moreover, the number $a_\delta(S,L)$ of δ-nodal curves in Λ equals $n_{\Lambda,p-\delta}$.

Proof. Let T be the (finite) set indexing the topological types of the members of Λ. For all $\tau \in T$, let $V^\tau_{\mathrm{red}} = V^\tau(\Lambda)_{\mathrm{red}}$ be the locus of curves having topological type τ, and let g_τ denote the common geometric genus of these curves. Note that $g_\tau \geqslant p-\delta$ by Theorem (2.2). Let $\mathcal{C}_\tau \to V^\tau_{\mathrm{red}}$ be the universal curve.

For all $k \geqslant 0$, the relative Hilbert scheme $\mathrm{Hilb}^k(\mathcal{C}_\tau/V^\tau_{\mathrm{red}})$ is a topological fibre bundle, so $e(\mathrm{Hilb}^k(\mathcal{C}_\tau/V^\tau_{\mathrm{red}})) = e(C^{[k]}_\tau)e(V^\tau_{\mathrm{red}})$ for any membre C_τ of Λ of topological type τ. For all $i = g_\tau, \dots, p$, we denote by $n_{\tau,i}$ the number $n_{C_\tau,i}$ as in Proposition (2.8) (it depends only on τ); we set $n_{\tau,i} = 0$ if $i < g_\tau$. Then, by the additivity of the Euler number, and using Proposition (2.8), we have

$$\begin{aligned}\sum_{k=0}^{\infty} e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))q^k &= \sum_{k=0}^{\infty}\Big(\sum_{\tau\in T} e(C^{[k]}_\tau)e(V^\tau_{\mathrm{red}})\Big)q^k = \sum_{\tau\in T}\Big(\sum_{k=0}^{\infty} e(C^{[k]}_\tau)q^k\Big)e(V^\tau_{\mathrm{red}}) = \\ &= \sum_{\tau\in T}\Big(\sum_{i=g_\tau}^{p} n_{\tau,i}q^{p-i}(1-q)^{2i-2}\Big)e(V^\tau_{\mathrm{red}}) = \sum_{i=p-\delta}^{p} n_{\Lambda,i}q^{p-i}(1-q)^{2i-2},\end{aligned}$$

with $n_{\Lambda,i} := \sum_{\tau\in T} n_{\tau,i}e(V^\tau_{\mathrm{red}})$: this proves the identity (2.10.1).

Theorem (2.2) ensures that δ-nodal curves are the only curves in Λ having geometric genus $p-\delta$, they are finitely many, and they appear with multiplicity 1. Therefore $n_{\Lambda,p-\delta} = a_\delta(S,L)\cdot n_{\delta A_1,p-\delta}$, where δA_1 denotes the topological type of δ-nodal curves members of $|L|$. Since $n_{\delta A_1,p-\delta} = 1$ by Proposition (2.9), one has $n_{\Lambda,p-\delta} = a_\delta(S,L)$ as we wanted. □

(2.11) Corollary. *The number $a_\delta(S,L)$ is a linear combination of the numbers $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$, $k = 0,\dots,\delta$, in which the coefficient of $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$ is a polynomial in p of degree at most $\delta-k$ for all k, and the coefficient of $e(\mathrm{Hilb}^\delta(\mathcal{C}/\Lambda))$ is 1.*

Proof. Let $e_{\Lambda,k} = e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$ for all $k = 0, \dots, \delta$. Expanding $(1-q)^{2i-2}$ for all i in (2.10.1) gives

$$\sum_{k=0}^{\infty} e_{\Lambda,k} q^k = \sum_{i=p-\delta}^{p} n_{\Lambda,i} \sum_{m=0}^{2i-2} \tbinom{2i-2}{m} (-1)^m q^{p-i+m}.$$

Then, by comparing the coefficients of q^k for $k = 0, \dots, \delta$, one sees with standard algebraic manipulations that

$$e_{\Lambda,k} = \sum_{m=0}^{k} (-1)^m \tbinom{2(p-k+m)-2}{m} n_{\Lambda,p-k+m}. \tag{2.11.1}$$

Let us now show by induction on $k = 0, \dots, \delta$ that

$$n_{\Lambda,p-k} = P_{k,0}(p)\, e_{\Lambda,0} + \cdots + P_{k,k-1}(p)\, e_{\Lambda,k-1} + e_{\Lambda,k},$$

where each $P_{k,s}$ is a polynomial of degree at most $k-s$, independent of Λ. The case $k = \delta$ gives the wanted result, since $a_\delta(S,L) = n_{\Lambda,p-\delta}$ by Theorem (2.10).

For $k = 0$ one has $n_{\Lambda,p} = e_{\Lambda,0}$ by (2.11.1), as required. Suppose now that k is positive. For all $m = 1, \dots, k$, we set $Q_{k,m}(p) = (-1)^m \binom{2(p-k+m)-2}{m}$, a polynomial of degree m in p. By (2.11.1) and our recursion hypothesis, we have

$$e_{\Lambda,k} = n_{\Lambda,p-k} + \sum_{m=1}^{k} Q_{k,m}(p) \sum_{s=0}^{k-m} P_{k-m,s}(p) e_{\Lambda,s} = n_{\Lambda,p-k} + \sum_{s=0}^{k-1} \sum_{m=1}^{k-s} Q_{k,m}(p)\, P_{k-m,s}(p) e_{\Lambda,s}$$

(setting $P_{m,m} = 1$). This gives an explicit expression for $P_{k,s}$, which ends the proof since

$$\deg(Q_{k,m} P_{k-m,s}) \leqslant m + (k-m-s) = k-s.$$

□

2.3 – Proof of the Göttsche conjecture

In this section we prove Theorem (1.2), following [15, Section 4]. With Corollary (2.11) at hand, the remaining task is to compute the Euler characteristics $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$, expressing them as polynomials of degree k in the numbers $L^2, LK_S, K_S^2, c_2(L)$; this is done with the recursion of [5].

Proof of Theorem (1.2). Consider a smooth projective surface S and a δ-very ample line bundle L on S. Let p be the arithmetic genus of the curves in $|L|$, and let $\Lambda \subseteq |L|$ be a general linear subsystem of dimension δ with universal curve $\mathcal{C} \to \Lambda$. Corollary (2.11) ensures that the number $a_\delta(S,L)$ of δ-nodal curves in Λ is a linear combination of the Euler numbers $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$, $k = 0, \dots, \delta$, where the coefficient of each $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$ is a polynomial of degree at most $\delta - k$ in p, and the coefficient of $e(\mathrm{Hilb}^\delta(\mathcal{C}/\Lambda))$ equals 1. Moreover, these polynomials are independent of the pair (S,L) and, since $p = \frac{1}{2}L^2 + \frac{1}{2}K_S L + 1$ by the adjunction formula, the coefficient of $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$ is a polynomial of degree at most $\delta - k$ in L^2 and $K_S L$. The Göttsche conjecture therefore follows from Proposition (2.12) below. □

(2.12) Proposition. *The Euler characteristics $e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda))$ is a polynomial of degree exactly k in the numbers $L^2, LK_S, K_S^2, c_2(L)$.*

Proof. The case $k = 0$ follows from the following equalities:

$$e(\mathrm{Hilb}^0(\mathcal{C}/\Lambda)) = e(\Lambda) = e(\mathbf{P}^\delta) = \delta + 1.$$

Let us now assume that $1 \leqslant k \leqslant \delta$. We divide the proof in three steps.

Step 1. We consider the Hilbert scheme $S^{[k]}$ and the universal subscheme $Z_k \subseteq S \times S^{[k]}$, endowed with the projections $\pi_1 \colon Z_k \to S$ and $\pi_2 \colon Z_k \to S^{[k]}$. We define a vector bundle $L^{[k]}$ of rank k on $S^{[k]}$,

$$L^{[k]} := {\pi_2}_* \pi_1^* L.$$

Letting $\mathbf{P} := \mathbf{P}\left(H^0(S,L)\right) = |L|$, we consider the line bundle $L \boxtimes \mathcal{O}_{\mathbf{P}}(1) := \pi_S^* L \otimes \pi_{\mathbf{P}}^* \mathcal{O}_{\mathbf{P}}(1)$ on $S \times \mathbf{P}$, where π_S and $\pi_{\mathbf{P}}$ are the two projections. In view of the standard identification $H^0(S,L) \otimes H^0(S,L)^* \cong \mathrm{Hom}\left(H^0(S,L), H^0(S,L)\right)$, we may consider the canonical section $\mathrm{id}_{H^0(S,L)} \in H^0(S,L) \otimes H^0(S,L)^*$ of the line bundle $L \boxtimes \mathcal{O}_{\mathbf{P}}(1)$. By taking the pullback of $\mathrm{id}_{H^0(S,L)}$ to $S \times S^{[k]} \times \mathbf{P}$, and pushing down to $S^{[k]} \times \mathbf{P}$ its restriction to $Z_k \times \mathbf{P} \subseteq S \times S^{[k]} \times \mathbf{P}$, we obtain a tautological section of the vector bundle $M^{[k]}$ on $S^{[k]} \times \mathbf{P}$ defined as

$$M^{[k]} = L^{[k]} \boxtimes \mathcal{O}_{\mathbf{P}}(1) := \pi_{S^{[k]}}^* L^{[k]} \otimes \pi_{\mathbf{P}}^* \mathcal{O}_{\mathbf{P}}(1).$$

By abuse of notation, we denote by $\mathcal{C} \to \mathbf{P}$ the universal curve over the complete linear system $|L|$. The relative Hilbert scheme $\mathrm{Hilb}^k(\mathcal{C}/\mathbf{P})$ is a smooth subvariety of $S^{[k]} \times \mathbf{P}$, of dimension $k + \dim(\mathbf{P})$ hence codimension k in $S^{[k]} \times \mathbf{P}$. Since it coincides moreover with the vanishing locus of the tautological section of $M^{[k]}$ corresponding to $\mathrm{id}_{H^0(S,L)}$, its class in the Chow ring of $S^{[k]} \times \mathbf{P}$ is Poincaré dual to the Chern class $c_k(M^{[k]})$, that is $\left[\mathrm{Hilb}^k(\mathcal{C}/\mathbf{P})\right] = c_k(M^{[k]}) \cap \left[S^{[k]} \times \mathbf{P}\right]$ (see, e.g., [6, Example 3.2.16]). We note furthermore that the fibre of the projection $\mathrm{Hilb}^k(\mathcal{C}/\mathbf{P}) \to S^{[k]}$ over any 0-dimensional scheme $Z \in S^{[k]}$ is given by pairs $(Z, [s]) \in S^{[k]} \times \mathbf{P}$ such that s vanishes at Z, i.e., $s \in \ker\left(H^0(S,L) \xrightarrow{\gamma_Z} H^0\left(Z, L \otimes \mathcal{O}_Z\right)\right)$. As L is $(k-1)$-very ample for all $k = 1, \ldots, \delta + 1$, the map γ_Z is surjective, so that the tautologial section of $M^{[k]}$ vanishes transversally with respect to the fibration $\mathrm{Hilb}^k(\mathcal{C}/\mathbf{P}) \to S^{[k]}$, whose fibres therefore have constant dimension. Moreover, since the linear subsystem $\Lambda \subseteq \mathbf{P}$ is general, the restriction $\mathrm{Hilb}^k(\mathcal{C}/\Lambda) \subseteq S^{[k]} \times \Lambda$ is smooth by Bertini's Theorem. Therefore, since the class $\left[\mathrm{Hilb}^k(\mathcal{C}/\Lambda)\right]$ is Poincaré dual in the Chow ring of $S^{[k]} \times \Lambda$ to the k-th Chern class of the restriction of $M^{[k]}$, the Euler characteristic of $\mathrm{Hilb}^k(\mathcal{C}/\Lambda)$ satisfies

$$\begin{aligned} e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda)) &= \int_{\mathrm{Hilb}^k(\mathcal{C}/\Lambda)} c_k(T_{\mathrm{Hilb}^k(\mathcal{C}/\Lambda)}) = \int_{\mathrm{Hilb}^k(\mathcal{C}/\Lambda)} c_\bullet(T_{\mathrm{Hilb}^k(\mathcal{C}/\Lambda)}) \\ &= \int_{\mathrm{Hilb}^k(\mathcal{C}/\Lambda)} \frac{c_\bullet(T_{S^{[k]} \times \Lambda})}{c_\bullet(M^{[k]})} = \int_{S^{[k]} \times \Lambda} c_k(M^{[k]}) \frac{c_\bullet(T_{S^{[k]} \times \Lambda})}{c_\bullet(M^{[k]})}. \end{aligned}$$

Step 2. Let us now spell out the various terms in the last integral above. Let $\omega = c_1(O_\Lambda(1))$ be the hyperplane class on $\Lambda \simeq \mathbf{P}^\delta$. One has $c_\bullet(T_\Lambda) = \sum_{k=0}^{\delta} \binom{\delta+1}{k} \omega^k = (1+\omega)^{\delta+1}$ (see, e.g., [10, p.414]), hence $c_\bullet(T_{S^{[k]} \times \Lambda}) = c_\bullet(T_{S^{[k]}}) c_\bullet(T_\Lambda) = c_\bullet(T_{S^{[k]}})(1+\omega)^{\delta+1}$.

On the other hand, since $M^{[k]}$ is the tensor product of the rank k vector bundle $\pi_{S^{[k]}}^* L^{[k]}$ with the line bundle $\pi_{\mathbf{P}}^* \mathcal{O}_{\mathbf{P}}(1)$, its restriction to Λ satisfies $c_k(M^{[k]}) = \sum_{i=0}^{k} \omega^i c_{k-i}(L^{[k]})$ (see, e.g., [6, p. 55]). Finally, [6, Example 3.2.2] yields

$$\begin{aligned} c_\bullet(M^{[k]}) &= \sum_{p=0}^{k} c_p(M^{[k]}) = \sum_{p=0}^{k} \left(\sum_{i=0}^{p} \tbinom{k-i}{p-i} c_i(L^{[k]}) \omega^{p-i} \right) = \\ &= \sum_{i=0}^{k} c_i(L^{[k]}) \sum_{p=i}^{k} \tbinom{k-i}{p-i} \omega^{p-i} = \sum_{j=0}^{k} c_{k-j}(L^{[k]})(1+\omega)^j. \end{aligned}$$

Thus,

$$e(\mathrm{Hilb}^k(\mathcal{C}/\Lambda)) = \int_{S^{[k]}\times\Lambda} \frac{c_\bullet(T_{S^{[k]}})(1+\omega)^{\delta+1}\sum_{i=0}^k \omega^i c_{k-i}(L^{[k]})}{\sum_{j=0}^k c_{k-j}(L^{[k]})(1+\omega)^j} \tag{2.12.1}$$

$$= \int_{S^{[k]}} \mathrm{Coeff}_{\omega^\delta}\Big(\frac{c_\bullet(T_{S^{[k]}})(1+\omega)^{\delta+1}\sum_{i=0}^k \omega^i c_{k-i}(L^{[k]})}{\sum_{j=0}^k c_{k-j}(L^{[k]})(1+\omega)^j}\Big). \tag{2.12.2}$$

Step 3. We now end the proof with the recursion procedure developed in [5], see Proposition (2.14). The latter, applied k times, turns the above integral over $S^{[k]}$ into the integral over S^k of a polynomial in the Chern classes of $\mathrm{pr}_i^* L$, $\mathrm{pr}_i^* T_S$, and $\mathrm{pr}_{ij}^* T_\Delta$ $(i \neq j)$, where $\Delta \subseteq S \times S$ is the diagonal, $\mathrm{pr}_i : S^k \to S$ is the i-th projection, and $\mathrm{pr}_{ij} = (\mathrm{pr}_i, \mathrm{pr}_j)$. The latter integral results (with some non-trivial work, see, for instance, [5, Proposition 0.5 and its proof]) in a polynomial in L^2, LK_S, K_S^2, and $c_2(S)$, of degree at most k.

It remains to prove that this degree is exactly k. To do so, we concentrate on the $c_2(S)^k$ term. One looks the explicit formulas for the recursion straight in the eyes, viz. (2.14.3), (2.14.4), (2.14.5), (2.14.6), (2.14.7), and finds that the only contribution to the $c_2(S)^k$ term comes from the $c_0(L^{[k]})$ term. One may thus, for our present purpose, replace the fraction in (2.12.2) by

$$\frac{c_\bullet(T_{S^{[k]}})(1+\omega)^{\delta+1}\omega^k}{(1+\omega)^k} = c_\bullet(T_{S^{[k]}})\omega^k(1+\omega)^{\delta-k+1},$$

the coefficient of which in ω^δ is $(\delta-k+1)\,c_\bullet(T_{S^{[k]}})$. The corresponding summand of the integral (2.12.2) is $(\delta-k+1)\,c_{2k}(T_{S^{[k]}})$, which contributes by $(\delta-k+1)/k!$ to the $c_2(S)^k$ term, the factorial coming from the k successive applications of (2.14.2). □

(2.13) Example. Let us spell out this computation for the number of 1-nodal curves in a pencil, and see how it relates with the standard way to do it, which we recalled in the introduction to the present Section 2. In this case we have $\Lambda \simeq \mathbf{P}^1$. Equation (2.11.1) gives

$$e(\mathrm{Hilb}^1(\mathcal{C}/\Lambda)) = n_{\Lambda,p-1} - (2p-2)n_{\Lambda,p} \quad \text{and} \quad e(\mathrm{Hilb}^0(\mathcal{C}/\Lambda)) = n_{\Lambda,p}.$$

We have $e(\mathrm{Hilb}^0(\mathcal{C}/\Lambda)) = e(\Lambda) = 2$, and we concentrate on the computation of $e(\mathrm{Hilb}^1(\mathcal{C}/\Lambda))$ as the integral (2.12.2). The numerator of the integrand of (2.12.1) is

$$\begin{aligned}(1-K+c_2)(1+\omega)^2(L+\omega) &= (1-K+c_2)(1+2\omega)(L+\omega)\\ &= \big[L-KL\big] + \big[1-K+2L+c_2-2KL\big]\omega,\end{aligned}$$

and on the other hand

$$\begin{aligned}\frac{1}{1+L+\omega} &= 1-(L+\omega)+(L+\omega)^2-(L+\omega)^3\\ &= 1-(L+\omega)+(L^2+2L\omega)-3L^2\omega = \big[1-L+L^2\big]+\big[-1+2L-3L^2\big]\omega.\end{aligned}$$

The coefficient in ω of the integrand of (2.12.1) is therefore

$$(L-KL)(-1+2L-3L^2)+(1-K+2L+c_2-2KL)(1-L+L^2) = 1-K+L^2+c_2;$$

integrated against the fundamental class $[S]$, it gives L^2+c_2, as required, see (2.1).

2.4 – The Ellingsrud–Göttsche–Lehn recursion

We now describe the inductive computation of [5]. Our aim is to provide the right amount of detail so that: i) one may make sense of the arguments at the end of the proof of Proposition (2.12); and ii) one should be able to make it work in practice. We refer to the original article [5] for the proofs.

Let S be a smooth projective surface, equipped with a line bundle $L \to S$. We consider the Hilbert schemes $S^{[n]}$ and the vector bundles $L^{[n]} \to S^{[n]}$. We call T_n the tangent sheaf of $S^{[n]}$, and $\mathcal{I}_n$ the ideal sheaf of the universal subscheme $Z_n \subseteq S^{[n]} \times S$. For any sequence I of indices in $\{0, \dots, m\}$ we let pr_I be the projection from $S^{[n]} \times S^m = S^{[n]} \times S \times \cdots \times S$ to the factors indexed by I, $S^{[n]}$ being the 0-th factor; beware that we will (abusively) use this notation both for $S^{[n+1]} \times S^m$ and $S^{[n]} \times S^{m+1}$. We let Δ denote the diagonal in S^2.

(2.14) Proposition. *Let f be a polynomial in the Chern classes of the following sheaves on $S^{[n+1]} \times S^m$:*

$$\mathrm{pr}_0^* T_{n+1},\ \mathrm{pr}_{0i}^* \mathcal{I}_{n+1},\ \mathrm{pr}_{ij}^* \mathcal{O}_\Delta,\ \mathrm{pr}_i^* T_S,\ \mathrm{pr}_0^* L^{[n+1]},\ \mathrm{pr}_i^* L \qquad (\textit{for } 1 \leqslant i < j \leqslant m).$$

There is a polynomial $f^\flat$ in the Chern classes of the analogously defined sheaves on $S^{[n]} \times S^{m+1}$, depending only on f, and such that

$$\int_{S^{[n+1]} \times S^m} f = \int_{S^{[n]} \times S^{m+1}} f^\flat.$$

Key to the recursive procedure is the consideration of the incidence variety $S^{[n,n+1]} \subseteq S^{[n]} \times S^{[n+1]}$ of all pairs (Z, Z') such that $Z \subseteq Z'$. There is an identification $S^{[n,n+1]} \cong \mathbf{P}(\mathcal{I}_n)$, coming from the fact that if Z' is obtained by extending Z at the closed point $x \in S$, then the ideal sheaf defining Z in S is an extension of the skyscraper sheaf $\mathbf{C}_x$ by the ideal sheaf of Z'.

We let $\mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{I}_n)}(1)$. and consider the following maps: ϕ and ψ are the two projections of $\mathbf{P}(\mathcal{I}_n)$ to $S^{[n]}$ and $S^{[n+1]}$, respectively, and $\rho : \mathbf{P}(\mathcal{I}_n) \to S$ maps a pair (Z, Z') to the point $x \in S$ at which Z and Z' differ.

$$\begin{array}{ccccc} & & \mathbf{P}(\mathcal{I}_n) = S^{[n,n+1]} & & \\ & \swarrow^{\phi=\mathrm{pr}_1} & \downarrow{\scriptstyle \rho} & \searrow^{\psi=\mathrm{pr}_2} & \\ S^{[n]} & & S & & S^{[n+1]} \end{array}$$

Moreover, let $j = (\mathrm{id}, \rho) : \mathbf{P}(\mathcal{I}_n) \to \mathbf{P}(\mathcal{I}_n) \times S$, $\phi_S = \phi \times \mathrm{id}_S : \mathbf{P}(\mathcal{I}_n) \times S \to S^{[n]} \times S$, and $\psi_S = \psi \times \mathrm{id}_S : \mathbf{P}(\mathcal{I}_n) \times S \to S^{[n+1]} \times S$. We have the following exact sequence of sheaves on $S^{[n,n+1]} \times S$:

$$0 \to \psi_S^* \mathcal{I}_{n+1} \to \phi_S^* \mathcal{I}_n \to j_* \mathcal{L} \to 0. \tag{2.14.1}$$

Note that $j_* \mathcal{L} = p^* \mathcal{L} \otimes \rho_S^* \mathcal{O}_\Delta$, where p is the projection $S^{[n,n+1]} \times S \to S^{[n,n+1]}$, and $\rho_S = \rho \times \mathrm{id}_S : S^{[n,n+1]} \times S \to S^2$.

Proof of Proposition (2.14). We want to relate integrals on $S^{[n+1]} \times S^m$ to integrals on $S^{[n]} \times S^{m+1}$. To this end we consider the product $\Sigma = S^{[n,n+1]} \times S^m$, together with the maps $\Psi = \psi \times \mathrm{id}_{S^m} : \Sigma \to S^{[n+1]} \times S^m$ and $\Phi = (\phi, \rho) \times \mathrm{id}_{S^m} : \Sigma \to S^{[n]} \times S \times S^m$. Moreover, we call $p_\Sigma : \Sigma = S^{[n,n+1]} \times S^m \to S^{[n,n+1]}$ the projection on the first factor.

$$\begin{array}{ccccc} & & \Sigma = S^{[n,n+1]} \times S^m & & \\ & \swarrow^{\Psi=\psi\times\mathrm{id}} & \downarrow{\scriptstyle p_\Sigma} & \searrow^{\Phi=(\phi,\rho)\times\mathrm{id}} & \\ S^{[n+1]} \times S^m & & S^{[n,n+1]} & & S^{[n]} \times S \times S^m \end{array}$$

The first step is to lift our integral to Σ, using the fact that Ψ is generically finite of degree $n+1$. This gives:

$$\int_{S^{[n+1]}\times S^m} f = \frac{1}{n+1}\int_{\Sigma} \Psi^* f. \tag{2.14.2}$$

We aim to find an expression for $\Psi^* f$ of the form $\sum_{\nu\geqslant 0}\Phi^* f_\nu \cdot p_\Sigma^*\big(c_1(\mathcal{L})^\nu\big)$, in the Grothendieck ring $K(\Sigma)$ of coherent sheaves on Σ. Then we will conclude by applying the projection formula.

We first have the following two straightforward formulas (in the Grothendieck ring, as will be the case for all other formulas of the same kind), which display on simple instances the index shift resulting from the additional S factor induced by the ρ component of Φ.

$$\Psi^*\mathrm{pr}_i^* T_S = \Phi^*\mathrm{pr}_{i+1}^* T_S \tag{2.14.3}$$

$$\Psi^*\mathrm{pr}_{ij}^* \mathcal{O}_\Delta = \Phi^*\mathrm{pr}_{i+1,j+1}^* \mathcal{O}_\Delta. \tag{2.14.4}$$

Beware that, by abuse of notation, the 'pr' on the left-hand-side refer to projections from $S^{[n+1]}\times S^m$ whereas those on the right-hand-side refer to projections from $S^{[n]}\times S^{m+1}$. The first non-trivial formula in the induction procedure is the following, and comes from (2.14.1) together with the given expression for $j_*\mathcal{L}$:

$$\Psi^*\mathrm{pr}_{0i}^* \mathcal{I}_{n+1} = \Phi^*\mathrm{pr}_{0,i+1}^* \mathcal{I}_n - p_\Sigma^*\mathcal{L}\cdot\Phi^*\mathrm{pr}_{1,i+1}^*\mathcal{O}_\Delta. \tag{2.14.5}$$

Next we have the following relation, which is a consequence of the computation in the Grothendieck group of $S^{[n,n+1]}$ carried out in [5, §2]:

$$\begin{aligned}\Psi^*\mathrm{pr}_0^* T_{n+1}&\\ = \Phi^*\mathrm{pr}_0^* T_n + p_\Sigma^*\mathcal{L}\cdot\Phi^*\mathrm{pr}_{01}^*\mathcal{I}_n^\vee + p_\Sigma^*\mathcal{L}^\vee\cdot\Phi^*\mathrm{pr}_{01}^*\mathcal{I}_n\cdot\Phi^*\mathrm{pr}_1^*\omega_S^\vee - \Phi^*\mathrm{pr}_1^*(\mathcal{O}_S - T_S + \omega_S^\vee).\end{aligned} \tag{2.14.6}$$

The last relation follows from [5, Lemma 2.1]:

$$\Psi^*\mathrm{pr}_0^* L^{[n+1]} = \Phi^*\mathrm{pr}_0^* L^{[n]} + p_\Sigma^*\mathcal{L}\cdot\Phi^*\mathrm{pr}_1^* L. \tag{2.14.7}$$

Now, applying formulas (2.14.3), (2.14.4), (2.14.5), (2.14.6), and (2.14.7) to transform $\Psi^* f$, and plugging the result in (2.14.2), one obtains an identity of the form

$$\int_{S^{[n+1]}\times S^m} f = \int_\Sigma \Big(\sum\nolimits_{\nu\geqslant 0}\Phi^* f_\nu\cdot p_\Sigma^*\big(c_1(\mathcal{L})^\nu\big)\Big) \tag{2.14.8}$$

as we wanted. By the projection formula, the right-hand-side equals

$$\int_{S^{[n]}\times S^{m+1}} \Phi_*\Big(\sum\nolimits_{\nu\geqslant 0}\Phi^* f_\nu\cdot p_\Sigma^*\big(c_1(\mathcal{L})^\nu\big)\Big) = \int_{S^{[n]}\times S^{m+1}}\Big(\sum\nolimits_{\nu\geqslant 0}(-1)^\nu f_\nu\cdot c_\nu(-\mathrm{pr}_{01}^*\mathcal{I}_n)\Big),$$

where the last equality is given by the identity $(\phi,\rho)_*\big(c_1(\mathcal{L})^\nu\big) = (-1)^\nu c_\nu(-\mathcal{I}_n)$ in the Chow ring of $S^{[n]}\times S$, see [5, Lemma 1.1].[1] This ends the proof. □

(2.15) Example. To illustrate the recursion, let us compute $\int_{S^{[2]}} c_2(L^{[2]})^2$. We consider the diagram

$$\begin{array}{ccccc} & & \Sigma = S^{[1,2]}\times\{\mathrm{pt}\} & & \\ & \swarrow^{\Psi} & \downarrow{\scriptstyle p} & \searrow^{\Phi} & \\ S^{[2]}\times\{\mathrm{pt}\} & & S^{[1,2]} & & S^{[1]}\times S\times\{\mathrm{pt}\}\end{array}$$

[1] By definition the Chern classes $c_\nu(-\mathcal{F})$ are the coefficients of the multiplicative inverse of the Chern polynomial $c_t(\mathcal{F})$, i.e., the Chern classes $c_\nu(-\mathcal{F})$ are the Segre classes $s_\nu(\mathcal{F})$.

in which {pt} denotes the point. This factor may of course be forgotten, except for compatibility with the general notation; thus, the map p is merely the identity. Also, the pair $(S^{[1]}, L^{[1]})$ may be identified with (S, L).

We start from the identity $\int_{S^{[2]}} c_2(L^{[2]})^2 = \frac{1}{2}\int_{S^{[1,2]}} \Psi^* c_2(L^{[2]})^2$. Next, we have $\Psi^* L^{[2]} = \Phi^*\mathrm{pr}_0^* L + p^*\mathcal{L}\cdot\Phi^*\mathrm{pr}_1^* L$ by (2.14.7). In general, $c_2(A+B) = c_2(A) + c_2(B) + c_1(A)c_1(B)$; here, $\Psi^* L^{[2]}$ is expressed as a sum of line bundle, so we simply have

$$c_2(\Psi^* L^{[2]}) = c_1(\Phi^*\mathrm{pr}_0 L)\cdot\big(c_1(p^*\mathcal{L}) + c_1(\Phi^*\mathrm{pr}_1^* L)\big)$$

(we have also used that $c_1(AB) = c_1(A) + c_1(B)$), hence

$$c_2(\Psi^* L^{[2]})^2 = \Phi^*\mathrm{pr}_0^* c_1(L)^2 \cdot \big(\underbrace{p^* c_1(\mathcal{L})^2}_{\nu=2} + \underbrace{2p^* c_1(\mathcal{L})\cdot\Phi^*\mathrm{pr}_1^* c_1(L)}_{\nu=1} + \underbrace{\Phi^*\mathrm{pr}_1^* c_1(L)^2}_{\nu=0}\big)$$

(ν is the index that appears at the end of the proof of Proposition (2.14), from (2.14.8) on).

Now, the universal subscheme $Z_1 \subseteq S^{[1]}\times S$ identifies with the diagonal Δ in $S\times S$, and we thus have $\mathcal{O}_{S\times S} = \mathcal{I}_1 + \iota_*\mathcal{O}_\Delta$, where ι denotes the inclusion map $\Delta \hookrightarrow S\times S$. Therefore, $c_1(\mathcal{I}_1) = -c_1(\iota_*\mathcal{O}_\Delta)$, and $c_2(\mathcal{I}_1) = -c_2(\iota_*\mathcal{O}_\Delta)$. By the Riemann–Roch formula without denominators, we have

$$c_1(\iota_*\mathcal{O}_\Delta) = 0 \quad\text{and}\quad c_2(\iota_*\mathcal{O}_\Delta) = -[\Delta],$$

see [6, Example 15.3.1]. Thus,

$$\frac{1}{1 + c_1(\mathcal{I}_1) + c_2(\mathcal{I}_1) + \cdots} = \frac{1}{1 + [\Delta] + \cdots} = 1 - [\Delta] + \cdots,$$

hence

$$\Phi_* p^* c_1(\mathcal{L}) = -c_1(-\mathcal{I}_1) = 0 \quad\text{and}\quad \Phi_* p^* c_1(\mathcal{L})^2 = c_2(-\mathcal{I}_1) = -[\Delta].$$

In the upshot,

$$\int_{S^{[1,2]}} \Phi^*\mathrm{pr}_0^* c_1(L)^2\cdot\Phi^*\mathrm{pr}_1^* c_1(L)^2 = \int_S c_1(L)^2\cdot\int_S c_1(L)^2 = (L^2)^2,$$

$$\int_{S^{[1,2]}} \Phi^*\mathrm{pr}_0^* c_1(L)^2\cdot 2p^* c_1(\mathcal{L})\cdot\Phi^*\mathrm{pr}_1^* c_1(L) = -2\int_{S\times S}\mathrm{pr}_0^* c_1(L)^2\cdot\mathrm{pr}_1^* c_1(L)\cdot c_1(-\mathcal{I}_1) = 0,$$

$$\int_{S^{[1,2]}} \Phi^*\mathrm{pr}_0^* c_1(L)^2\cdot p^* c_1(\mathcal{L})^2 = \int_{S\times S}\mathrm{pr}_0^* c_1(L)^2\cdot c_2(-\mathcal{I}_1) = -\int_{S\times S}\mathrm{pr}_0^* c_1(L)^2\cdot[\Delta] = -L^2,$$

and therefore

$$\int_{S^{[2]}} c_2(L^{[2]})^2 = \frac{1}{2}\int_{S^{[1,2]}}\Psi^* c_2(L^{[2]})^2 = \frac{1}{2}L^2(L^2-1).$$

3 – Tzeng's proof of the Göttsche–Yau–Zaslow formula

In this section, we outline the proof of Theorem (1.4) given by Tzeng [28]. The plan of the proof is the following. In §3.1, we follow [18] in order to define the algebraic cobordism group $(\omega_{2,1}, +)$ of equivalence classes of pairs $[S, L]$ of surfaces and line bundles with respect to the extended double point relation (cf. Definition (3.3)), endowed with the sum induced by disjoint union of surfaces. Then we recall the description of $\omega_{2,1}\otimes_{\mathbf{Z}}\mathbf{Q}$ as a four dimensional vector space over $\mathbf{Q}$ spanned by $[\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}]$, $[\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)]$, $[\mathbf{P}^1\times\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}]$, and $[\mathbf{P}^1\times\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0)]$.

In §3.2 we introduce Göttsche's enumerative integrals $d_\delta(S, L)$ of a line bundle L on a smooth surface S, and we show, following Göttsche, that if L is sufficiently ample, then $d_\delta(S, L)$ equals the number $a_\delta(S, L)$ of δ-nodal curves in a general δ-dimensional linear subsystem of $|L|$.

Section 3.3 is dedicated to Tzeng's degeneration formula for the numbers $d_\delta(S,L)$. It can be understood as saying that the map $\phi : (S,L) \mapsto \sum d_\delta(S,L)x^\delta$ is a homomorphism of $\mathbf{Q}$-vector spaces between the algebraic cobordism group $\omega_{2,1} \otimes_{\mathbf{Z}} \mathbf{Q}$ and the space of invertible power series $(\mathbf{Q}[[x]]^\times, \cdot)$.

Finally, §3.4 contains the proofs of Theorems (1.3) and (1.4). The proof of Theorem (1.3) mostly consists in putting together the results of the previous sections; after that, Theorem (1.4) is obtained by plugging in the Bryan–Leung formula in addition.

3.1 – The algebraic cobordism group of surfaces and line bundles

We now review the construction of the algebraic cobordism group $(\omega_{2,1}, +)$ of line bundles on surfaces, and the results of [18] giving a basis of $\omega_{2,1} \otimes_{\mathbf{Z}} \mathbf{Q}$ as a $\mathbf{Q}$-vector space.

Let $(\mathcal{M}_2, +)$ denote the free abelian group generated by isomorphism classes $[S]$ of smooth projective surfaces. One may think of the sum of two distinct isomorphism classes $[S_1]$ and $[S_2]$ in $\mathcal{M}_2$ as the isomorphism class of the disjoint union $S_1 \sqcup S_2$.

(3.1) Definition. *We say that four smooth projective surfaces $S_0, \dots, S_3$ satisfy a* double point relation $[S_0] - [S_1] - [S_2] + [S_3]$ *if there exists a family $\mathcal{X} \xrightarrow{\pi} \mathbf{P}^1$ of surfaces such that:*

(i) the total space $\mathcal{X}$ is smooth of pure dimension 3;
(ii) the fibre over 0 is S_0;
(iii) S_1 and S_2 are smooth divisors of $\mathcal{X}$ intersecting transversely, and the fibre over ∞ is the union $S_1 \cup S_2$;
(iv) let $D = S_1 \cap S_2$; the surface S_3 is the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O}_D \oplus N_{D/S_1}) \cong \mathbf{P}(N_{D/S_2} \oplus \mathcal{O}_D)$.

In the above, $N_{D/S}$ denotes the normal bundle of D in S. Note that the surfaces S_1, S_2 and the curve D are allowed to be disconnected and empty.

(3.2) Remark. The isomorphism $\mathbf{P}(\mathcal{O}_D \oplus N_{D/S_1}) \cong \mathbf{P}(N_{D/S_2} \oplus \mathcal{O}_D)$ in (iv) above follows from the fact that $\mathcal{O}_D(S_1 + S_2) \cong \mathcal{O}_D$. Indeed, it implies $N_{D/S_1} \otimes N_{D/S_2} \cong \mathcal{O}_D$, hence $\mathcal{O}_D \oplus N_{D/S_1} \cong (N_{D/S_1} \otimes N_{D/S_2}) \oplus (N_{D/S_1} \otimes \mathcal{O}_D) \cong N_{D/S_1} \otimes (N_{D/S_2} \oplus \mathcal{O}_D)$. See also the discussion of the triple point formula and its variants in [I, Section 3].

Analogously, let $(\mathcal{M}_{2,1}, +)$ be the free abelian group generated by isomorphism classes $[S, L]$, where S is a smooth projective surfaces and L is a line bundle on S, and extend Definition (3.1) as follows.

(3.3) Definition. *For $i = 0, \dots, 3$, let $[S_i, L_i]$ be the class of a pair of a smooth projective surface S_i and a line bundle L_i on it. We say that*

$$[S_0, L_0] - [S_1, L_1] - [S_2, L_2] + [S_3, L_3] \tag{3.3.1}$$

is an extended double point relation *if $[S_0] - [S_1] - [S_2] + [S_3]$ is a double point relation via a family $\mathcal{X} \xrightarrow{\pi} \mathbf{P}^1$ of surfaces as above, and there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $L_i = \mathcal{L}|_{S_i}$ for $i = 0, 1, 2$, and $L_3 = \eta^*(\mathcal{L}|_D)$, where η is the structure map*

$$\eta : S_3 = \mathbf{P}(\mathcal{O}_D \oplus N_{D/S_1}) \longrightarrow D.$$

The following fundamental example is closely related to the degeneration to the normal cone. For $\mathbf{P}^2$, it is the degeneration to $\mathbf{P}^2 \cup_{\mathbf{P}^1} \mathbf{F}_1$ used in [2], see [VI].

(3.4) Example (see [28, Lemma 2.4])**.** Let us consider a smooth projective surface S and a line bundle L on S. Given a smooth curve $C \subseteq S$, let $\mathcal{X} \xrightarrow{\mathrm{bl}} S \times \mathbf{P}^1$ be the blow-up of the

product $S \times \mathbf{P}^1$ along the curve $C \times \{\infty\}$, and let $E \cong \mathbf{P}(\mathcal{O}_C \oplus N_{S/C})$ be its exceptional divisor endowed with the $\mathbf{P}^1$-bundle map $E \xrightarrow{\eta} C$. The following is an extended double point relation for all integers k,

$$\big[S, L\big] - \big[S, L(-kC)\big] - \big[E, \eta^*(L|_C) \otimes \mathcal{O}_E(kC)\big] + \big[E, \eta^* L(-kC)|_C)\big].$$

To see this fact, let $\mathrm{pr}_1 \colon S \times \mathbf{P}^1 \to S$ and $\mathrm{pr}_2 \colon S \times \mathbf{P}^1 \to \mathbf{P}^1$ denote the natural projections. Consider the family $\mathcal{X} \xrightarrow{\pi} \mathbf{P}^1$ given by $\pi := \mathrm{pr}_2 \circ \mathrm{bl}$, and define a line bundle $\mathcal{L}$ on $\mathcal{X}$ as

$$\mathcal{L} := (\mathrm{bl}^* \mathrm{pr}_1^* L) \otimes \mathcal{O}_{\mathcal{X}}(-kC).$$

The morphism π is a smooth over $\mathbf{P}^1 - \{\infty\}$, with fibres $\pi^{-1}(0) \cong S$ and $\pi^{-1}(\infty) \cong S \cup_C E$. Finally, defining $S_0, S_1 \cong S$ and $S_2, S_3 \cong E$ as in Definition (3.1), the line bundle $\mathcal{L}$ is such that $\mathcal{L}|_{S_0} = L$, $\mathcal{L}|_{S_1} = L \otimes \mathcal{O}_S(-kC)$, $\mathcal{L}|_{S_2} = \eta^*(L|_C) \otimes \mathcal{O}_E(kC)$, and $\mathcal{L}|_{S_3} = \eta^*((\mathcal{O}_S(-kC) \otimes L)|_C)$.

We may now define the algebraic cobordism group of line bundles on surfaces.

(3.5) Definition. *Let $\mathcal{R}$ denote the subgroup of $\mathcal{M}_{2,1}$ generated by all extended double point relations. The* algebraic cobordism group *of surfaces and line bundles is the quotient*

$$\omega_{2,1} := \mathcal{M}_{2,1}/\mathcal{R}.$$

One may define more generally the algebraic cobordism group $\omega_{n,r}$ of isomorphism classes of pairs of n-dimensional varieties and rank r-vector bundles. Lee and Pandharipande [18] described the structure of the $\mathbf{Q}$-vector space $\omega_{n,r} \otimes_{\mathbf{Z}} \mathbf{Q}$, by exhibiting a basis and showing that Chern invariants respect algebraic cobordism. In the case of surfaces and line bundles, [18, Theorems 1 and 4] give the following, see [28, Theorem 2.3].

(3.6) Theorem. *The map $[S, L] \mapsto (L^2, LK_S, K_S^2, c_2(S))$ is an isomorphism $\omega_{2,1} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}^4$ of $\mathbf{Q}$-vector spaces. The family*

$$\big([\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}],\ [\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)],\ [\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}],\ [\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,0)] \big) \tag{3.6.1}$$

is a basis of the vector space $\omega_{2,1} \otimes_{\mathbf{Z}} \mathbf{Q}$, and for any pair $[S, L]$, we have

$$\big[S, L\big] = a_1 \big[\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}\big] + a_2 \big[\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)\big] + a_3 \big[\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}\big] + a_4 \big[\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,0)\big],$$

with $a_1 = -L^2 + \frac{K_S^2 - 2c_2(S)}{3}$, $a_2 = L^2$, $a_3 = \frac{3L^2 + LK_s}{2} - \frac{K_S^2 - 3c_2(S)}{4}$ and $a_4 = -\frac{3L^2 + LK_S}{2}$.

The first assertion is the content of [18, Theorems 1 and 4], the proof of which we will not discuss. A fundamental input behind it is [16, Theorem 1.2.19].

This being established, proving that the family in (3.6.1) is a basis is elementary: it is enough to check that the 4-tuples $(L^2, LK_S, K_S^2, c_2(S)) \in \mathbf{Q}^4$ corresponding to the basis vectors are linearly independent, which may be seen by computing them explicitly; one finds $(0, 0, 9, 3)$, $(1, -3, 9, 3)$, $(0, 0, 8, 4)$, and $(0, -2, 8, 4)$, respectively. Then, one obtains the values of $a_1, \ldots, a_4$ for $[S, L]$ by solving for them in the equality $(L^2, LK_S, K_S^2, c_2(S)) = a_1(0, 0, 9, 3) + \cdots + a_4(0, -2, 8, 4)$.

(3.7) Remark. For future use, note that a family $([S_i, L_i])_{i=1,\ldots,4}$ gives a basis of $\omega_{2,1} \otimes_{\mathbf{Z}} \mathbf{Q}$ if and only if the corresponding vectors $(L_i^2, L_i K_{S_i}, K_{S_i}^2, c_2(S_i)) \in \mathbf{Q}^4$ are linearly independent over $\mathbf{Q}$.

(3.8) Remark. Note that the data of the 4-tuples $(L^2, LK_S, K_S^2, c_2(S))$ and $(\chi(L), LK_S, K_S^2, \chi(\mathcal{O}_S))$ are equivalent since, by Noether's formula and the Riemann–Roch theorem,

$$\chi(\mathcal{O}_S) = \frac{1}{12}\left(K_S^2 + c_2(S)\right) \quad \text{and} \quad \chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2}\left(L^2 - LK_S\right).$$

Observe moreover that the above formulas ensure that $L^2 - LK_S$ is even and $K_S^2 + c_2(S)$ is divisible by 3 and 4, so that the coefficients $a_1, \ldots, a_4$ in Theorem (3.6) turn out to be integers, as expected.

(3.9) Example. Let $[S_1, L_1]$ and $[S_2, L_2]$ be two classes consisting of a $K3$ surface endowed with a primitive line bundle. Then the 4-tuples $((\chi(L), LK_S, K_S^2, \chi(\mathcal{O}_S))$ corresponding to the family

$$\left([\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}],\ [\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)],\ [S_1, L_1],\ [S_2, L_2]\right) \tag{3.9.1}$$

are $(1, 0, 9, 1)$, $(3, -3, 9, 1)$, $\left(2 + \frac{L_1^2}{2}, 0, 0, 2\right)$ and $\left(2 + \frac{L_2^2}{2}, 0, 0, 2\right)$. This implies that the family in (3.9.1) is a basis of $\omega_{2,1} \otimes_{\mathbf{Z}} \mathbf{Q}$ if and only if $L_1^2 \neq L_2^2$.

3.2 – Göttsche's enumerative integrals

In this section we introduce, following Göttsche [7, Section 5], some intersection numbers $d_\delta(S, L)$ on Hilbert schemes of points on S which will be used as a substitute for the numbers $a_\delta(S, L)$ of δ-nodal curves in a general δ-dimensional linear subsystem of $|L|$; this is possible thanks to Theorem (3.11), due to Göttsche as well, which asserts that $a_\delta(S, L)$ and $d_\delta(S, L)$ coincide if L is sufficiently positive.

This substitution circumvents the issue that the numbers $a_\delta(S, L)$ behave rather poorly under degeneration. For instance, we have seen in the first chapters of this volume, e.g., [I] and [VI], that in a degeneration of S to a transverse union of two surfaces S_1 and S_2, in order to find the number $a_\delta(S, L)$ on $S_1 \cup S_2$ one ought to count not only curves with nodes on S_1 and S_2, but also curves with tacnodes along the double curve $S_1 \cap S_2$. In contrast, Tzeng's results presented in the next section establish good properties of the numbers $d_\delta(S, L)$.

For any integer $k > 0$, let us consider the Hilbert scheme $S^{[k]}$ parameterizing 0-dimensional subschemes of S of length k, and let $Z_k \subseteq S \times S^{[k]}$ be the universal subscheme, with projections $\pi_1 \colon Z_k \to S$ and $\pi_2 \colon Z_k \to S^{[k]}$. As in §2.3, we denote by $L^{[k]}$ the vector bundle of rank k on $S^{[k]}$ given by $L^{[k]} := {\pi_2}_* {\pi_1}^* L$.

(3.10) Definition (Göttsche's enumerative integrals)**.** *Let $\mathring{W}^{3\delta} \subseteq S^{[3\delta]}$ be the locally closed subset parametrizing all closed subschemes of the form $\coprod_{i=1}^{\delta} \operatorname{Spec}(\mathcal{O}_{S,x_i}/\mathfrak{m}_{S,x_i}^2)$, where $x_1, \ldots, x_\delta$ are pairwise distinct points of S. Let $W^{3\delta} \subseteq S^{[3\delta]}$ be the closure of $\mathring{W}^{3\delta}$; it is birational to $S^{[\delta]}$. We let*

$$d_\delta(S, L) := \int_{W^{3\delta}} c_{2\delta}(L^{[3\delta]}),$$

and write the associated generating function

$$\phi(S, L) := \sum_{\delta \geqslant 0} d_\delta(S, L) x^\delta.$$

In [7, Proposition 5.2], Göttsche proved that if L is sufficiently ample with respect to δ, then the number $d_\delta(S, L)$ coincides with the number $a_\delta(S, L)$ of δ-nodal curves in a general δ-dimensional linear subsystem of $|L|$. Having now Theorem (2.2) at hand, we can slightly weaken the positivity condition and simplify the proof.

(3.11) Theorem. *Let S be a smooth projective surface, and let L be a $(3\delta-1)$-very ample line bundle on S, for some $\delta > 0$. Then a general linear subsystem of $|L|$ of dimension δ contains exactly $d_\delta(S,L)$ curves having exactly δ nodes as singularities, i.e., $a_\delta(S,L) = d_\delta(S,L)$.*

Proof. We will prove that a general linear subsystem $\Lambda \subseteq |L|$ of dimension δ contains $d_\delta(S,L)$ curves (counted with multiplicities) having at least δ singularities. Then, the result will follow directly from Theorem (2.2), see Remark (2.3).

Let $(s_1,\dots,s_{\delta+1})$ be a basis of the linear subspace of $H^0(S,L)$ corresponding to $\Lambda \subseteq \mathbf{P}(H^0(S,L))$, and consider the global sections $\pi_{2*}\pi_1^* s_1,\dots,\pi_{2*}\pi_1^* s_{\delta+1}$ of the vector bundle $L^{[3\delta]}$. Moreover, let $Y \subseteq S^{[3\delta]}$ be the locus of points $[Z] \in S^{[3\delta]}$ such that the vectors $\pi_{2*}{\pi_1}^* s_1(Z),\dots,\pi_{2*}{\pi_1}^* s_{\delta+1}(Z)$ are linearly dependent, i.e., such that there exists a divisor D belonging to Λ that contains Z.

Since L is $(3\delta-1)$-very ample, the restriction map $H^0(S,L) \to H^0(Z, L\otimes\mathcal{O}_Z)$ is surjective for all $[Z] \in S^{[3\delta]}$, hence the evaluation map $H^0(S,L)\otimes\mathcal{O}_{S^{[3\delta]}} \to L^{[3\delta]}$ is surjective as well, and $L^{[3\delta]}$ is globally generated. Thus, by [6, Example 14.3.2] applied to the restriction of $L^{[3\delta]}$ to $W^{3\delta}$, we find that $Y\cap W^{3\delta}$ has codimension 2δ in $W^{3\delta}$, and class $[Y\cap W^{3\delta}] = c_{2\delta}(L^{[3\delta]})\cap[W^{3\delta}]$ by the Thom–Porteous formula. Moreover, since the dimension of $W^{3\delta}\setminus \mathring{W}^{3\delta}$ is smaller than 2δ, the same argument applied to the restriction of $L^{[3\delta]}$ to $W^{3\delta}\setminus\mathring{W}^{3\delta}$ gives that $Y\cap W^{3\delta}$ lies entirely in $\mathring{W}^{3\delta}$.

We conclude that the number of members of Λ (counted with multiplicities) containing a point of $\mathring{W}^{3\delta}$ is $d_\delta(S,L) := \int_{W^{3\delta}} c_{2\delta}(L^{[3\delta]})$. Now a curve $C\subseteq S$ contains a scheme $Z\in\mathring{W}^{3\delta}$ with $Z_{\mathrm{red}} = x_1+\dots+x_\delta$ if and only if it is singular at all points $x_1,\dots,x_\delta\in S$. Therefore, the number of curves in Λ having at least δ singularities is $d_\delta(S,L)$, and this ends the proof. □

3.3 – Degeneration of enumerative integrals

We now follow [28, Section 3] in order to show how the generating function $\phi(S,L) = \sum d_\delta(S,L)x^\delta$ of the Göttsche enumerative integrals of a pair (S,L) respects extended double point relations. The result below is Tzeng's main contribution to the proof of the Göttsche conjecture. It rests on Proposition (3.15), which may be seen as a version of Jun Li's celebrated degeneration formula [19].

(3.12) Theorem. *If $[S_0,L_0]-[S_1,L_1]-[S_2,L_2]+[S_3,L_3]$ is an extended double point relation in $\mathcal{M}_{2,1}$, then*

$$\phi(S_0,L_0) = \frac{\phi(S_1,L_1)\phi(S_2,L_2)}{\phi(S_3,L_3)}.$$

Equivalently, the map $\phi\colon \big(\omega_{2,1}\otimes_{\mathbf{Z}}\mathbf{Q},+\big)\longrightarrow\big(\mathbf{Q}[[x]]^\times,\times\big)$ is a $\mathbf{Q}$-linear group homomorphism.

Let us consider a family $\mathcal{X}\xrightarrow{\pi}\mathbf{P}^1$ of surfaces and a line bundle $\mathcal{L}$ inducing the extended double point relation $[S_0,L_0]-[S_1,L_1]-[S_2,L_2]+[S_3,L_3]$ appearing in the statement, in the same notation as in Definitions (3.1) and (3.3). For all $t\in\mathbf{P}^1$, we let X_t be the fibre $\pi^{-1}(t)$. The fibre X_0 is S_0, and the fibre X_∞ is the union of S_1 and S_2 meeting transversally along a smooth curve D.

Let δ be a positive integer; for all smooth surfaces S we let $W^{3\delta}_S\subseteq S^{[3\delta]}$ be as in Definition (3.10). The strategy of [28] to prove Theorem (3.12) is to study the degeneration of $W^{3\delta}_{X_t}$ at $t=\infty$ by means of the good degeneration of Hilbert schemes constructed by Li and Wu [21], see also [20, Section 3.4].

Let $U\subseteq\mathbf{P}^1$ be the largest open subset containing ∞ and such that π is smooth over $U-\{\infty\}$, and let us still denote by $\mathcal{X}\xrightarrow{\pi}U$ the pullback of the original family over $\mathbf{P}^1$. For all $n\geqslant 1$, Li and Wu construct a family $\pi^{[n]}\colon\mathcal{X}^{[n]}\to U$ which is a proper and separated Deligne–Mumford

stack of finite type over U, such that for all $t \neq \infty$, the fibre of $\mathcal{X}^{[n]}$ over t is the Hilbert scheme $X_t^{[n]}$ of n points on the smooth surface X_t; the fibre over ∞ is different, however, as we will see.

The first step in the construction is to replace $\mathcal{X} \xrightarrow{\pi} U$ by an alternative family $\pi[n]\colon \mathcal{X}[n] \to U$ of surfaces, called an *expansion*, equal to $\mathcal{X} \to U$ over $U - \{\infty\}$, and with fibre over ∞ a semistable model of $S_1 \cup S_2$, viz.

$$\mathcal{X}[n]_\infty := S_1 \cup \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_{n-1} \cup S_2$$

where, for all $i = 1, \ldots, n-1$, the surface Δ_i is a $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O}_D \oplus N_{S_1/D})$ over D with two distinguished sections D_i^- and D_i^+, and $\mathcal{X}[n]_\infty$ is obtained by identifying $D_i^+ \subseteq \Delta_i$ with $D_{i+1}^- \subseteq \Delta_{i+1}$, $D \subseteq S_1$ with $D_1^- \subseteq \Delta_1$, and $D_{n-1}^+ \subseteq \Delta_{n-1}$ with $D \subseteq S_2$.

Li and Wu introduce the notion of *stable ideal sheaves*, see [20, Definition 3.53], and let $\pi^{[n]}\colon \mathcal{X}^{[n]} \to U$ be the moduli space of stable ideal sheaves of the family $\mathcal{X}[n] \to U$. For all $t \neq \infty$, the fibre ${\pi^{[n]}}^{-1}(t)$ is the Hilbert scheme $X_t^{[n]}$ parametrizing 0-dimensional subschemes of X_t of length n, whereas the fibre ${\pi^{[n]}}^{-1}(\infty)$ is the union

$$\mathcal{X}_\infty^{[n]} := \bigcup_{k=0}^{n} \big((S_1/D)^{[k]} \times (S_2/D)^{[n-k]}\big), \tag{3.12.1}$$

where $(S_1/D)^{[k]}$ (resp. $(S_2/D)^{[k]}$) parametrizes 0-dimensional schemes of length k supported on the smooth locus of $S_1 \cup \Delta_1 \cup \cdots \cup \Delta_i$ for some $i \leqslant k$ (resp. $\Delta_j \cup \cdots \cup \Delta_{n-1} \cup S_2$ for some $j \leqslant n-k$);[2] by convention, $(S_1/D)^{[0]} = (S_2/D)^{[0]} = \{\mathrm{pt}\}$.

We now set $n = 3\delta$, and consider the expanded family of surfaces $\pi[3\delta]\colon \mathcal{X}[3\delta] \to U$, and the moduli space of stable ideal sheaves $\pi^{[3\delta]}\colon \mathcal{X}^{[3\delta]} \to U$.[3] The following lemma describes the good degeneration of $W^{3\delta}_{X_t}$ in $\mathcal{X}^{[3\delta]}$ at $t = \infty$. Let $\mathcal{W}^{3\delta}$ be the the closure in $\mathcal{X}^{[3\delta]}$ of the set

$$\mathring{\mathcal{W}}^{3\delta} := \coprod_{t \neq \infty} \mathring{W}^{3\delta}_{X_t} = \left\{ \Big[\coprod_{i=1}^{\delta} \mathrm{Spec}\left(\mathcal{O}_{X_t, x_i}/\mathfrak{m}_{x_i}^2\right) \Big] : t \neq \infty \text{ and } x_1, \ldots, x_\delta \in X_t \text{ pairwise distinct} \right\}.$$

For all $k = 1, \ldots, \delta$ and $i = 1, 2$, we let $W^{3k}_{S_i/D}$ be the closure in $(S_i/D)^{[3k]}$ of the locus parametrizing unions of k disjoint subschemes of the form $\mathrm{Spec}(\mathcal{O}_x/\mathfrak{m}_x^2)$; by convention, $W^0_{S_i/D} := \{\mathrm{pt}\}$.

(3.13) Lemma ([28, Lemma 3.8])**.** *The map $\mathcal{W}^{3\delta} \to U$ is flat, its fibre over $t \neq \infty$ is $W^{3\delta}_{X_t}$, and its fibre over ∞ is such that for all $m = 0, \ldots, 3\delta$,*

$$\mathcal{W}^{3\delta} \cap \big((S_1/D)^{[m]} \times (S_2/D)^{[3\delta - m]}\big) = \begin{cases} W^{3k}_{S_1/D} \times W^{3(\delta-k)}_{S_2/D} & \text{if } m = 3k \text{ for some } k \in \mathbf{N} \\ \varnothing & \text{otherwise.} \end{cases}$$

Let us now briefly survey the definitions of a universal subscheme over $\mathcal{X}^{[n]}$, and of a vector bundle $\mathcal{L}^{[n]}$ on $\mathcal{X}^{[n]}$ from the line bundle $\mathcal{L}$ on $\mathcal{X}$, see [28, Definition-Proposition 3.4]. Let n be a positive integer. We consider the universal closed subscheme $Z_0^{[n]} \subseteq S_0 \times S_0^{[n]}$. For $i = 1, 2$ and $0 \leqslant k \leqslant n$, we may also introduce universal closed subschemes $Z_i^{[k]} \subseteq (S_i/D) \times (S_i/D)^{[k]}$;

[2] For a subscheme Z supported on $S_1 \cup \Delta_1 \cup \cdots \cup \Delta_i$ (resp. $\Delta_j \cup \cdots \cup \Delta_{n-1} \cup S_2$), we require moreover that all components $\Delta_1, \ldots, \Delta_i$ (resp. $\Delta_j, \ldots, \Delta_{n-1}$) contain a point of the support of Z (the components not supporting any piece of Z ought to be contracted).

[3] beware the unfortunate proximity of the two otherwise standard notations.

beware that S_i/D denotes a suitable expansion of S_i along the curve D. Let

$$\begin{array}{ccc} Z_0^{[n]} & \xrightarrow{q_0^{[n]}} & S_0^{[n]} \\ {\scriptstyle p_0^{[n]}}\downarrow & & \\ S_0 & & \end{array} \quad \text{and} \quad \begin{array}{ccc} Z_i^{[k]} & \xrightarrow{q_i^{[k]}} & (S_i/D)^{[k]} \\ {\scriptstyle p_i^{[k]}}\downarrow & & \\ (S_i/D) & & \end{array}$$

be the corresponding projections, and define the vector bundles of rank $3k$

$$L_i^{[k]} := (q_i^{[k]})_*(p_i^{[k]})^* L.$$

for all $i = 0, 1, 2$. We denote by $\pi_1^{[k,n]}$ and $\pi_2^{[k,n]}$ the natural projections

$$\begin{array}{ccc} (S_1/D)^{[k]} \times (S_2/D)^{[n-k]} & \xrightarrow{\pi_2^{[k,n]}} & (S_2/D)^{[n-k]} \\ {\scriptstyle \pi_1^{[k,n]}}\downarrow & & \\ (S_1/D)^{[k]}. & & \end{array}$$

With this notation, there exists a universal closed subscheme $\mathcal{Z} \subseteq \mathcal{X}[n] \times \mathcal{X}^{[n]}$ with projections

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{Q} & \mathcal{X}^{[n]} \\ {\scriptstyle P}\downarrow & & \\ \mathcal{X}[n]. & & \end{array}$$

Its fibres over 0 and ∞ are $Z_0^{[n]}$ and

$$\mathcal{Z}_\infty := \left(\bigcup_{k=0}^{n} \left(Z_1^{[k]} \times (S_2/D)^{[n-k]} \right) \right) \cup \left(\bigcup_{k=0}^{n} \left((S_1/D)^{[k]} \times Z_2^{[n-k]} \right) \right),$$

respectively.

There is a line bundle $\mathcal{L}[n]$ on $\mathcal{X}[n]$ obtained by pull-back from the line bundle $\mathcal{L}$ on $\mathcal{X}$. Setting $\mathcal{L}^{[n]} := Q_* P^* \mathcal{L}[n]$, the following holds.

(3.14) Lemma ([28, Lemmas 3.6 and 3.7])**.** *The sheaf $\mathcal{L}^{[n]}$ is a vector bundle of rank $3n$ over $\mathcal{X}^{[n]}$, such that*

$$\mathcal{L}^{[n]}|_{S_0} = L_0^{[n]} \quad \textit{and} \quad \mathcal{L}^{[n]}|_{(S_1/D)^{[k]} \times (S_2/D)^{[n-k]}} = (\pi_1^{[k,n]})^* L_1^{[k]} \oplus (\pi_2^{[k,n]})^* L_2^{[n-k]}.$$

In order to study degenerations of the Göttsche enumerative numbers, we define for $i = 1, 2$ the *relative Göttsche integrals*

$$d_k(S_i/D, L_i) = \begin{cases} 1 & \text{for } k = 0 \\ \displaystyle\int_{W^{3k}_{S_i/D}} c_{2k}(L_i^{[3k]}) & \text{for } k \geqslant 1, \end{cases}$$

and their *relative generating function*

$$\phi(S_i/D, L_i) = \sum_{k \geqslant 0} d_k(S_i/D, L_i) x^k.$$

Note that the relative Göttsche integrals are defined independently of the family $\mathcal{X} \to U$, and depend only on S_i and $D \subseteq S_i$: to define $(S_i/D)^{[3k]}$ and $L_i^{[3k]}$, one considers the degeneration to the normal cone as in Example (3.4).

We may now finally state the degeneration formula which lies at the heart of Tzeng's proof of the Göttsche conjecture.

(3.15) Proposition ([28, Proposition 3.9])**.** *If* $[S_0, L_0] - [S_1, L_1] - [S_2, L_2] + [S_3, L_3]$ *is an extended double point relation in* $\mathcal{M}_{2,1}$*, then*

$$d_\delta\,(S_0, L_0) = \sum_{k=0}^{\delta} d_k\,(S_1/D, L_1)\, d_{\delta-k}\,(S_2/D, L_2)\,, \tag{3.15.1}$$

hence

$$\phi(S_0, L_0) = \phi(S_1/D, L_1) \cdot \phi(S_2/D, L_2). \tag{3.15.2}$$

We emphasize that while $d_\delta(S_0, L_0)$ is a genuine Göttsche integral, as defined in Section 3.2, the $d_k(S_i/D, L_i)$ are relative, i.e., they are defined in terms of good degenerations à la Jun Li et al.

Proof. Formula (3.15.1) follows from the good degeneration results surveyed above, after fairly mechanical computations. The moduli space of stable ideal sheaves $\mathcal{X}^{[3\delta]} \to U$ provides a degeneration of the Hilbert scheme $S_0^{[3\delta]}$ to a moduli space defined on an expansion of $S_1 \cup_D S_2$, see (3.12.1). Within this degeneration, the family $\mathcal{W}^{3\delta} \to U$ provides a flat degeneration of $W_{S_0}^{3\delta}$ to the fibre $\mathcal{W}_\infty^{3\delta}$ over $\infty \in U$, see Lemma (3.13), and the vector bundle $\mathcal{L}^{[3\delta]}$ provides a degeneration of $L_0^{[3\delta]}$, see Lemma (3.14). Formula (3.15.1) will follow from the equality

$$c_{2\delta}(\mathcal{L}^{[3\delta]})\big[\mathcal{W}_0^{[3\delta]}\big] = c_{2\delta}(\mathcal{L}^{[3\delta]})\big[\mathcal{W}_\infty^{[3\delta]}\big]\,:$$

the left-hand-side plainly writes

$$c_{2\delta}(\mathcal{L}^{[3\delta]})\big[\mathcal{W}_0^{[3\delta]}\big] = c_{2\delta}(\mathcal{L}^{[3\delta]})\big[W_{S_0}^{[3\delta]}\big] = \int_{W_{S_0}^{3\delta}} c_{2\delta}(L_0^{[3\delta]}) = d_\delta(S_0, L_0),$$

and we will see that the right-hand-side equals the right-hand-side of (3.15.1), which will end the proof.

By Lemmas (3.13) and (3.14), we have

$$\begin{aligned} c_{2\delta}(\mathcal{L}^{[3\delta]})\big[\mathcal{W}_\infty^{[3\delta]}\big] &= c_{2\delta}(\mathcal{L}^{[3\delta]})\Big[\bigcup\nolimits_{k=0}^{\delta} W_{S_1/D}^{3k} \times W_{S_2/D}^{3(\delta-k)}\Big] \\ &= \sum_{k=0}^{\delta} c_{2\delta}\big(\,(\pi_1^{[3k,3\delta]})^* L_1^{[3k]} \oplus (\pi_2^{[3k,3\delta]})^* L_2^{[3(\delta-k)]}\,\big)\,\big[W_{S_1/D}^{3k} \times W_{S_2/D}^{3(\delta-k)}\big]. \end{aligned}$$

Now,

$$c_{2\delta}\big((\pi_1^{[3k,3\delta]})^* L_1^{[3k]} \oplus (\pi_2^{[3k,3\delta]})^* L_2^{[3(\delta-k)]}\big) = \sum_{l=0}^{2\delta} c_l\big((\pi_1^{[3k,3\delta]})^* L_1^{[3k]}\big)\, c_{2\delta-l}\big((\pi_2^{[3k,3\delta]})^* L_2^{[3(\delta-k)]}\big),$$

so that after capping with $\left[W^{3k}_{S_1/D} \times W^{3(\delta-k)}_{S_2/D}\right]$ the only non-zero term is obtained for $l = \dim W^{3k}_{S_1/D} = 2k$. Therefore, we finally obtain

$$\begin{aligned} c_{2\delta}(\mathcal{L}^{[3\delta]})\left[\mathcal{W}^{[3\delta]}_{\infty}\right] &= \sum_{k=0}^{\delta} c_{2k}(L_1^{[3k]})\left[W^{3k}_{S_1/D}\right] c_{2(\delta-k)}(L_2^{[3(\delta-k)]})\left[W^{3(\delta-k)}_{S_2/D}\right] \\ &= \sum_{k=0}^{\delta} d_k\left(S_1/D, L_1\right) d_{\delta-k}\left(S_2/D, L_2\right), \end{aligned}$$

which ends the proof of Formula (3.15.1). Formula (3.15.2) follows at once. □

With Proposition (3.15) at hand, the proof of Theorem (3.12) consists in using degenerations to the normal cone (and Proposition (3.15) itself) in order to replace all relative generating functions by absolute ones in the degeneration formula (3.15.2). We will thus use several times the following lemma.

(3.16) Lemma. *Let S be a smooth surface, and $C \subseteq S$ be a smooth curve with normal bundle $N_{C/S}$. Let $E \cong \mathbf{P}(\mathcal{O}_C \oplus N_{C/S})$, and $\eta : E \to C$ be the $\mathbf{P}^1$-bundle map. For all line bundles L on S, we have*

$$\phi(S, L) = \phi(S/C, L) \cdot \phi(E/C, \eta^*(L|_C)). \tag{3.16.1}$$

Proof. Let us consider the degeneration of S to $S \cup_C E$ constructed in Example (3.4), endowed with the line bundle $\mathcal{L}$ obtained by pulling L back (this is the case $k = 0$ in Example (3.4)). On the degenerate fibre $S \cup_C E$, one has $\mathcal{L}|_S = L$ and $\mathcal{L}|_E = \eta^* L|_C$. Therefore, the degeneration formula (3.15.2) applied to this situation gives the required Formula (3.16.1). □

(3.17) *Proof of Theorem (3.12).* We have

$$\phi(S_0, L_0) = \phi(S_1/D, L_1) \cdot \phi(S_2/D, L_2). \tag{3.17.1}$$

by Proposition (3.15).

By Lemma (3.16), we have

$$\phi(S_i, L_i) = \phi(S_i/D, L_i) \cdot \phi(E_i/D_i^-, \eta_i^*\, L_i|_D) \tag{3.17.2}$$

for $i = 1, 2$, where $E_i := \mathbf{P}(\mathcal{O}_D \oplus N_{D/S_i})$, $\eta_i : E_i \to D$ is the $\mathbf{P}^1$-bundle map, and $D_i^- := \mathbf{P}(N_{X_i/D})$ and $D_i^+ := \mathbf{P}(\mathcal{O}_D)$ are the two distinguished sections of η_i. Now, the assumed extended double point relation tells us that $S_3 \simeq E_i$, and $L_3 \simeq \eta_i^*\, \mathcal{L}|_D$ for $i = 1, 2$, in the notation introduced after Theorem (3.12). Since $\mathcal{L}|_{S_i} = L_i$, we have $L_3 \simeq \eta_i^*\, L_i|_D$, so that (3.17.2) may be rewritten as

$$\phi(S_i, L_i) = \phi(S_i/D, L_i) \cdot \phi(E_i/D_i^-, L_3). \tag{3.17.3}$$

Finally, we apply Lemma (3.16) to the curve D_1^- in the surface E_1; in this case, the degeneration to the normal cone considered in the proof of the lemma is a degeneration of E_1 to $E_1 \cup_D E_1$, where the curve D is embedded as D_1^- in the first E_1, and as D_1^+ in the second E_1. We obtain the identity

$$\phi(E_1, L_3) = \phi(E_1/D_1^-, L_3) \cdot \phi(E_1/D_1^+, L_3). \tag{3.17.4}$$

Since $E_1 \simeq S_3$ on the one hand, and on the other hand the isomorphism $(E_1, \eta_1^*\ L_1|_D)) \simeq (E_2, \eta_2^*\ L_2|_D)$ maps D_1^+ to D_2^- (see Remark (3.2)), (3.17.4) may be rewritten as

$$\phi(S_3, L_3) = \phi(E_1/D_1^-, L_3) \cdot \phi(E_2/D_2^-, L_3). \tag{3.17.5}$$

In conclusion, by (3.17.1), (3.17.3), and (3.17.5), we have

$$\phi(S_0, L_0) = \frac{\phi(S_1, L_1)}{\phi(E_1/D_1^-, \eta_1^*\ L_1|_D)} \cdot \frac{\phi(S_2, L_2)}{\phi(E_2/D_2^-, \eta_2^*\ L_2|_D)} = \frac{\phi(S_1, L_1) \cdot \phi(S_2, L_2)}{\phi(S_3, L_3)},$$

as we wanted. □

3.4 – Conclusion

This final section is dedicated to the proofs of Theorems (1.3) and (1.4). The substantial work has already been done, and it remains only to assemble the various pieces.

We first prove the analogue of Theorem (1.3) for the generating function $\phi(S, L) = \sum d_\delta(S, L)x^\delta$, and describe explicitly the universal power series involved. Theorem (1.3) itself will follow rather directly, thanks to Theorem (3.11).

(3.18) Proposition. *For all pairs (S, L), one has*

$$\phi(S, L) = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)},$$

with

$$A_1 = \phi(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2})^{-1}\phi(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))\phi(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1})^{\frac{3}{2}}\phi(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0))^{-\frac{3}{2}},$$
$$A_2 = \phi(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1})^{\frac{1}{2}}\phi(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0))^{-\frac{1}{2}},$$
$$A_3 = \phi(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2})^{-\frac{1}{3}}\phi(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1})^{-\frac{1}{4}},$$
$$A_4 = \phi(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2})^{-\frac{2}{3}}\phi(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1})^{\frac{3}{4}}.$$

Proof. Given a class $[S, L] \in \omega_{2,1}$, Theorem (3.6) tells us that

$$[S, L] = a_1[\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}] + a_2[\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)] + a_3[\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}] + a_4[\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0)],$$

with $a_1 = -L^2 + \frac{K_S^2 - 2c_2(S)}{3}$, $a_2 = L^2$, $a_3 = \frac{3L^2+LK_s}{2} - \frac{K_S^2-3c_2(S)}{4}$ and $a_4 = -\frac{3L^2+LK_S}{2}$. Then, Theorem (3.12) yields

$$\phi(S, L) = \phi(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2})^{a_1}\phi(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))^{a_2}\phi(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1})^{a_3}\phi(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0))^{a_4},$$

from which the assertion follows by replacing $a_1, \ldots, a_4$ with their expressions in terms of L^2, LK_S, K_S^2, and $c_2(S)$. □

(3.19) Remark. According to Remark (3.7), we may easily determine an alternative basis of $\omega_{2,1}\otimes_{\mathbf{Z}}\mathbf{Q}$, and the corresponding coefficients of an arbitrary class $[S, L] \in \omega_{2,1}$ may be otained as in the proof of Theorem (3.6). Consequently, the universal power series $A_1, A_2, A_3, A_4 \in \mathbf{Q}[[x]]$ could be described in terms of any such basis.

(3.20) Remark. Tzeng proved the Göttsche conjecture as follows (see [28, Theorem 1.1]). She observed that by expanding the series A_1, A_2, A_3, A_4 in Proposition (3.18), the coefficient $d_\delta(S, L)$ of x^δ in the generating function $\phi(S, L)$ is a universal polynomial T_δ of degree δ in the variables $L^2, LK_S, K_S^2, c_2(S)$ (cf. [28, Corollary 4.2]), i.e., the analogue of the Göttsche conjecture holds for the numbers $d_\delta(S, L)$, without any positivity assumption on L. Then, the conjecture for the numbers $a_\delta(S, L)$ follows by Theorem (3.11), under a suitable positivity asumption (Tzeng needed $(5\delta-1)$-very ampleness, but we now know that $(3\delta-1)$-very ampleness suffices, thanks to [15], see Theorem (2.2)).

(3.21) *Proof of Theorem (1.3).* Consider the generating function $T(S,L)$ of the universal polynomials $T_\delta(L^2, LK_S, K_S^2, c_2(S))$; the existence of the latter is guaranteed by Theorem (1.2). If L is $(3\delta-1)$-very ample, then Theorems (1.2) and (3.11) ensure that $T_\delta(L^2, LK_S, K_S^2, c_2(S)) = a_\delta(S,L) = d_\delta(S,L)$. Thus the generating functions $T(S,L)$ and $\phi(S,L)$ coincide, and we deduce from Proposition (3.18) that $T(S,L) = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)}$. □

Finally, let us prove the Göttsche–Yau–Zaslow formula.

(3.22) *Proof of Theorem (1.4).* Let us consider the generating function

$$\gamma(S,L)(q) := \sum_{\delta \geqslant 0} T_\delta(L^2, LK_S, K_S^2, c_2(S)) \left(DG_2(\tau)\right)^\delta$$

appearing in the right-hand side of Göttsche–Yau–Zaslow formula. Given two integers $g \geqslant \delta \geqslant 0$, let (S,L) consist of a very general $K3$ surface S endowed with a primitive line bundle L such that $L^2 = 2(g+\delta-1)$, and let $M_g(\delta)$ denote the number of curves in $|L|$ having geometric genus g and passing through g general points of S (or, equivalently, contained in a general linear subsystem of $V \subseteq |L|$ of codimension g and dimension δ). Bryan and Leung [1]) (see [XI, Theorem (3.5)]) proved that the generating function of the numbers $M_g(\delta)$ satisfies the formula

$$\sum_{\delta \geqslant 0} M_g(\delta) q^{g+\delta-1} = \frac{DG_2(\tau)^g}{\Delta(\tau)}$$

which, according to [7, Remark 2.6], can be reformulated as

$$\gamma(S,L)(q) = \frac{(DG_2(\tau)/q)^{\chi(L)}}{\Delta(\tau) D^2 G_2(\tau)/q^2}. \tag{3.22.1}$$

On the other hand, we recall that the 4-tuples $(L^2, LK_S, K_S^2, c_2(S))$ and $(K_S^2, LK_S, \chi(L), \chi(\mathcal{O}_S))$ determine each other through linear relations (cf. Remark (3.8)). Since $\gamma(S,L)(q)$ coincides with the generating function $\phi(S,L)$ computed at $x = DG_2(\tau)$, we deduce from Proposition (3.18) that there exist four universal power series $B_1(q), \ldots, B_4(q) \in \mathbf{Q}[[q]]$ such that

$$\gamma(S,L)(q) = B_1(q)^{K_S^2} B_2(q)^{LK_S} B_3(q)^{\chi(L)} B_4(q)^{\chi(\mathcal{O}_S)}. \tag{3.22.2}$$

When S is a $K3$ surface, equation (3.22.2) becomes $\gamma(S,L)(q) = B_3(q)^{\chi(L)} B_4(q)^2$. Moreover, as we consider different pairs (S,L) such that $L^2 = 2(g+\delta-1)$, the latter equality must be satisfied for any positive integer $\chi(L) = g+\delta+1$. Comparing $\gamma(S,L)(q) = B_3(q)^{\chi(L)} B_4(q)^2$ and (3.22.1), we then conclude that $B_3(q) = DG_2(\tau)/q$ and $B_4(q) = \left(\Delta(\tau) D^2 G_2(\tau)/q^2\right)^{-1/2}$. Therefore (3.22.2) leads to the Göttsche–Yau–Zaslow formula

$$\gamma(S,L)(q) = B_1(q)^{K_S^2} B_2(q)^{LK_S} \frac{(DG_2(\tau)/q)^{\chi(L)}}{\left(\Delta(\tau) D^2 G_2(\tau)/q^2\right)^{\chi(\mathcal{O}_S)/2}}.$$

□

References

Chapters in this Volume

[I] C. Ciliberto and Th. Dedieu, *Limits of nodal curves: generalities and examples.*

[II] Th. Dedieu, E. Sernesi, *Deformations of curves on surfaces.*

[VI] Th. Dedieu, C. Galati, *Degenerations and deformations of generalized Severi varieties.*

[XI] Th. Dedieu, *Enumerative geometry of $K3$ surfaces.*

[XII] Th. Dedieu, *Some classical formulæ for curves and surfaces.*

[XIII] Th. Dedieu, *Node polynomials for curves on surfaces.*

External References

[1] J. Bryan, N. C. Leung, The enumerative geometry of K3 surfaces and modular forms, *J. Amer. Math. Soc.* **13** (2000), 371–410.

[2] L. Caporaso, J. Harris, Counting plane curves of any genus, *Invent. Math.* **131**(2) (1998), 345–392.

[3] S. Diaz, J. Harris, Ideals associated to deformations of singular plane curves, *Trans. Amer. Math. Soc.* **309** (1988), 433–468.

[4] Th. Dedieu, E. Sernesi, Equigeneric and equisingular families of curves on surfaces, *Publ. Mat.* **61** (2017), no. 1, 175–212.

[5] G. Ellingsrud, L. Göttsche, M. Lehn, On the cobordism class of the Hilbert scheme of a surface, *J. Algebraic Geom.* **10** (2001), 81–100.

[6] W. Fulton, *Intersection theory*, Second edition, Ergeb. Math. Grenzgeb. (3) **2**, Springer-Verlag, Berlin (1998).

[7] L. Göttsche, A conjectural generating function for numbers of curves on surfaces, *Comm. Math. Phys.* **196** (1998), 523–533.

[8] G.-M. Greuel, C. Lossen, E. Shustin, *Introduction to singularities and deformations*, Springer Monographs in Mathematics, Springer, Berlin (2007).

[9] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York-Heidelberg (1977).

[10] P. A. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics, Wiley Interscience, New York (1978).

[11] M. Kaneko, D. Zagier, A generalized Jacobi theta function and quasimodular forms, in: *The moduli space of curves (Texel Island, 1994)*, 165–172, Progr. Math. **129**, Birkhäuser Boston, Boston, MA (1995).

[12] S. Kleiman, The transversality of a general translate, *Compositio Math.* **38** (1974), 287–297.

[13] S. Kleiman, R. Piene, Enumerating singular curves on surfaces, in: *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)*, 209–238, Contemp. Math. **241**, Amer. Math. Soc., Providence, RI (1999).

[14] ———, Node polynomials for families: methods and applications, *Math. Nachr.* **271** (2004), 69–90.

[15] M. Kool, V. Shende, R. P. Thomas, A short proof of the Göttsche conjecture, *Geom. Topol.* **15** (2011), 397–406.

[16] M. Levine and F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[17] M. Levine, R. Pandharipande, Algebraic cobordism revisited, *Invent. Math.* **176** (2009), 63–130.

[18] Y.-P. Lee, R. Pandharipande, Algebraic cobordism of bundles on varieties, *J. Eur. Math. Soc. (JEMS)* **14** (2012), 1081–1101.

[19] J. Li, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293.

[20] ———, *Good degenerations of moduli spaces*, in *Handbook of moduli. Vol. II*, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, 299–351.

[21] J. Li, B. Wu, Good degeneration of Quot-schemes and coherent systems, *Comm. Anal. Geom.* **23** (2015), 841–921.

[22] J. Li, Y.-J. Tzeng, Universal polynomials for singular curves on surfaces, *Compos. Math.* **150** (2014), 1169–1182.

[23] A.-K. Liu, Family blowup formula, admissible graphs and the enumeration of singular curves, I, *J. Differential Geom.* **56** (2000), 381–579.

[24] ———, The algebraic proof of the universality theorem, preprint, arXiv:math.AG/0402045 (2004).

[25] I. G. Macdonald, Symmetric products of an algebraic curve, *Topology* **1** (1962), 319–343.

[26] R. Pandharipande, R. P. Thomas, Stable pairs and BPS invariants, *J. Amer. Math. Soc.* **23** (2010), 267–297.

[27] V. Shende, Hilbert schemes of points on a locally planar curve and the Severi strata of its versal deformation, *Compos. Math.* **148** (2012), 531–547.

[28] Y.-J. Tzeng, A proof of the Göttsche–Yau–Zaslow formula, J. Differential Geom. **90** (2012), 439–472.

[29] I. Vainsencher, Enumeration of n-fold tangent hyperplanes to a surface, *J. Algebraic Geom.* **4** (1995), 503–526.

[30] R. Vakil, Counting curves on rational surfaces, *Manuscripta Math.* **102** (2000), 53–84.

[31] C. Voisin, *Hodge theory and complex algebraic geometry II*, Cambridge Studies in Advanced Mathematics **77**, Cambridge University Press, Cambridge (2007).

[32] S. T. Yau, E. Zaslow, BPS states, string duality, and nodal curves on K3, *Nuclear Phys. B* **471** (1996), 503–512.