A note on Severi varieties of nodal curves on Enriques surfaces

Ciro Ciliberto, Thomas Dedieu, Concettina Galati and Andreas Leopold Knutsen

Abstract Let $|L|$ be a linear system on a smooth complex Enriques surface S whose general member is a smooth and irreducible curve of genus p, with $L^2 > 0$, and let $V_{|L|,\delta}(S)$ be the Severi variety of irreducible δ-nodal curves in $|L|$. We denote by $\pi : X \to S$ the universal covering of S. In this note we compute the dimensions of the irreducible components V of $V_{|L|,\delta}(S)$. In particular we prove that, if C is the curve corresponding to a general element $[C]$ of V, then the codimension of V in $|L|$ is δ if $\pi^{-1}(C)$ is irreducible in X and it is $\delta - 1$ if $\pi^{-1}(C)$ consists of two irreducible components.

1 Introduction

Let S be a smooth complex projective surface and L a line bundle on S such that the complete linear system $|L|$ contains smooth, irreducible curves (such a line bundle, or linear system, is often called a *Bertini system*). Let

$$p := p_a(L) = \frac{1}{2} L \cdot (L + K_S) + 1,$$

Ciro Ciliberto
Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 00133 Roma, Italy, e-mail: cilibert@axp.mat.uniroma2.it

Thomas Dedieu
Institut de Mathématiques de Toulouse, UMR5219. Université de Toulouse, CNRS. UPS IMT, F-31062 Toulouse Cedex 9, France. e-mail: thomas.dedieu@math.univ-toulouse.fr

Concettina Galati
Dipartimento di Matematica e Informatica, Università della Calabria, via P. Bucci, cubo 31B, 87036 Arcavacata di Rende (CS), Italy, e-mail: galati@mat.unical.it

Andreas Leopold Knutsen
Department of Mathematics, University of Bergen, Postboks 7800, 5020 Bergen, Norway e-mail: andreas.knutsen@math.uib.no

be the arithmetic genus of any curve in $|L|$.

For any integer $0 \leq \delta \leq p$, consider the locally closed, functorially defined subscheme of $|L|$

$$V_{|L|,\delta}(S) \text{ or simply } V_{|L|,\delta}$$

parameterizing irreducible curves in $|L|$ having only δ nodes as singularities; this is called the *Severi variety* of δ-nodal curves in $|L|$. We will let $g := p - \delta$, the geometric genus of the curves in $V_{|L|,\delta}$.

It is well-known that, if $V_{|L|,\delta}$ is non-empty, then all of its irreducible components V have dimension $\dim(V) \geq \dim |L| - \delta$. More precisely, the Zariski tangent space to $V_{|L|,\delta}$ at the point corresponding to C is

$$T_{[C]}V_{|L|,\delta} \simeq H^0(L \otimes \mathcal{J}_N)/ < C >, \tag{1}$$

where $\mathcal{J}_N = \mathcal{J}_{N|S}$ is the ideal sheaf of subscheme N of S consisting of the δ nodes of C (see, e.g., [4, §1]). Thus, $V_{|L|,\delta}$ is *smooth of dimension* $\dim |L| - \delta$ at $[C]$ if and only if the set of nodes N imposes independent conditions on $|L|$. In this case, $V_{|L|,\delta}$ is said to be *regular* at $[C]$. An irreducible component V of $V_{|L|,\delta}$ will be said to be *regular* if the condition of regularity is satisfied at any of its points, equivalently, if it is smooth of dimension $\dim |L| - \delta$.

The *existence and regularity problems of* $V_{|L|,\delta}(S)$ have been studied in many cases and are the most basic problems one may ask on Severi varieties. We only mention some of known results. In the case $S \simeq \mathbb{P}^2$, Severi proved the existence and regularity of $V_{|L|,\delta}(S)$ in [14]. The description of the tangent space is due to Severi and later to Zariski [15]. The existence and regularity of $V_{|L|,\delta}(S)$ when S is of general type has been studied in [4] and [3]. Further regularity results are provided in [10]. More recently Severi varieties on K3 surfaces have received a lot of attention for many reasons. In this case Severi varieties are known to be regular (cf. [13]) and are nonempty on general K3 surfaces by Mumford and Chen (cf. [12], [2]).

As far as we know, Severi varieties on Enriques surfaces have not been studied yet, apart from [8, Thm. 4.12] which limits the singularities of a general member of the Severi variety $V^g_{|L|}$ of irreducible genus g curves in $|L|$, and gives a sufficient condition for the density of the latter in the Severi variety $V_{|L|,p-g}$ of $(p-g)$-nodal curves. In particular, the existence problem is mainly open and we intend to treat it in a forthcoming article. The result of this paper is Proposition 1, which answers the regularity question for Severi varieties of nodal curves on Enriques surfaces.

2 Regularity of Severi varieties on Enriques surfaces

Let S be a smooth Enriques surface, i.e. a smooth complex surface with nontrivial canonical bundle $\omega_S \not\simeq \mathcal{O}_S$, such that $\omega_S^{\otimes 2} \simeq \mathcal{O}_S$ and $H^1(\mathcal{O}_S) = 0$. We denote linear (resp. numerical) equivalence by $\sim$ (resp. $\equiv$).

Let L be a line bundle on S such that $L^2 > 0$. It is well-known that $|L|$ contains smooth, irreducible curves if and only if it contains irreducible curves (see [5, Thm.

4.1 and Prop. 8.2]); in other words, *on Enriques surfaces the Bertini linear systems are the linear systems that contain irreducible curves*. Moreover, by [6, Prop. 2.4], this is equivalent to L being nef and not of the form $L \sim P + R$, with $|P|$ an elliptic pencil and R a smooth rational curve such that $P \cdot R = 2$ (in which case $p = 2$). If $|L|$ is a Bertini linear system, the adjunction formula, the Riemann–Roch theorem, and Mumford vanishing yield that

$$L^2 = 2(p-1) \quad \text{and} \quad \dim |L| = p - 1$$

(see, e.g., [5, 7]).

Let K_S be the canonical divisor. It defines an étale double cover

$$\pi : X \longrightarrow S \tag{2}$$

where X is a smooth, projective $K3$ surface (that is, $\omega_X \simeq \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$), endowed with a fixed-point-free involution ι, which is the universal covering of S. Conversely, the quotient of any $K3$ surface by a fixed-point-free involution is an Enriques surface.

Let $C \subset S$ be a reduced and irreducible curve of genus $g \geq 2$. We will henceforth denote by $\nu_C : \widetilde{C} \to C$ the normalization of C and define $\eta_C := \mathcal{O}_C(K_S) = \mathcal{O}_C(-K_S)$, a nontrivial 2-torsion element in $\mathrm{Pic}^0 C$, and $\eta_{\widetilde{C}} := \nu_C^* \eta_C$. The fact that η_C is nontrivial follows from the cohomology of the restriction sequence

$$0 \longrightarrow \mathcal{O}_S(K_S - C) \longrightarrow \mathcal{O}_S(K_S) \longrightarrow \eta_C \longrightarrow 0,$$

which yields $h^0(\eta_C) = h^1(K_S - C) = h^1(C) = 0$, the latter vanishing as C is big and nef. One has the fiber product

$$\begin{array}{ccc} (\pi^{-1}C) \times_C \widetilde{C} & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow \nu_C \\ (\pi^{-1}C) & \xrightarrow[\pi|_{\pi^{-1}(C)}]{} & C, \end{array}$$

where $\pi|_{\pi^{-1}(C)}$ and the upper horizontal map are the double coverings induced respectively by η_C and $\eta_{\widetilde{C}}$. By standard results on coverings of complex manifolds (cf. [1, Sect. I.17]), two cases may happen:

- $\eta_{\widetilde{C}} \not\simeq \mathcal{O}_{\widetilde{C}}$ and $\pi^{-1}C$ is irreducible, as in Fig. 1;
- $\eta_{\widetilde{C}} \simeq \mathcal{O}_{\widetilde{C}}$ and $\pi^{-1}C$ consists of two irreducible components conjugated by the involution ι. These two components are *not* isomorphic to C, as η_C is nontrivial, as in Fig. 2 (each component of $\tilde{C}$ is a partial normalization of C).

As mentioned in the Introduction, it is well-known that *any* irreducible component of a Severi variety on a $K3$ surface is regular when nonempty (see, e.g., [4, Ex. 1.3]; see also [8, §4.2]). The corresponding result on Enriques surfaces is the following.

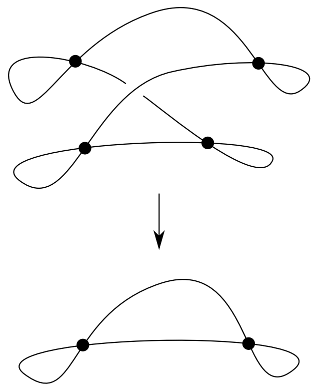

Fig. 1 $\eta_{\tilde{C}} = \nu_C^*(\eta_C) \neq 0$

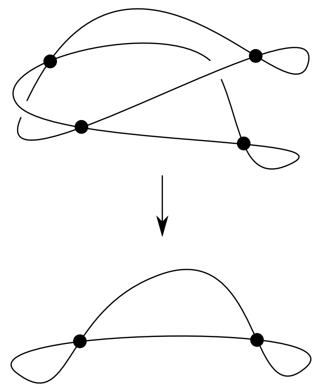

Fig. 2 $\eta_{\tilde{C}} = \nu_C^*(\eta_C) = 0$

First note that, in the above notation, the dimension of the Severi variety of genus $g = p_g(C)$ curves in $|L| = |C|$ at the point $[C]$ satisfies the inequalities

$$g - 1 \leq \dim_{[C]}(V^g_{|L|}) \leq h^0(\omega_{\tilde{C}} \otimes \eta_{\tilde{C}}) = \begin{cases} g-1 & \text{if } \eta_{\tilde{C}} \not\simeq \mathcal{O}_{\tilde{C}} \\ g & \text{if } \eta_{\tilde{C}} \simeq \mathcal{O}_{\tilde{C}} \end{cases} \tag{3}$$

(see [8, ineq. (2.6) and Lem. 2.3]). Our result implies that the second inequality in (3) is in fact an equality when C is nodal, and gives a concrete geometric description of the situation in both cases.

Proposition 1 *Let L be a Bertini linear system, with $L^2 > 0$, on a smooth Enriques surface S. Then the Severi variety $V_{|L|,\delta}(S)$ is smooth and every irreducible component $V \subseteq V_{|L|,\delta}(S)$ has either dimension $g-1$ or g; in the former case the component is regular. Furthermore, with the notation introduced above,*

1. *for any curve C in a $(g-1)$-dimensional irreducible component V, $\pi^{-1}C$ is irreducible (whence an element in $V_{|\pi^*L|,2\delta}(X)$);*
2. *for any g-dimensional component V, there is a line bundle L' on X with $(L')^2 = 2(p-d)-2$ and $L' \cdot \iota^*L' = 2d$ for some integer d satisfying*

$$\frac{p-1}{2} \leq d \leq \delta,$$

such that $\pi^*L \simeq L' \otimes \iota^*L'$, and the curves parametrized by $V \subseteq V_{|L|,\delta}(S)$ are the birational images by π of the curves in $V_{|L'|,\delta-d}(X)$ intersecting their conjugates by ι transversely (in $2d$ points). In other words, for any $[C] \in V$, we have $\pi^{-1}C = Y + \iota(Y)$, with $[Y] \in V_{|L'|,\delta-d}(X)$ and $[\iota(Y)] \in V_{|\iota^*L'|,\delta-d}(X)$ intersecting transversely.
Furthermore, if $L' \simeq \iota^*L'$, which is the case if S is general in moduli, then $d = \frac{p-1}{2}$ and $L \sim 2M$, for some $M \in \operatorname{Pic} S$ such that $M^2 = d$.

We will henceforth refer to components of dimension $g-1$ as *regular* and the ones of dimension g as *nonregular*. Note however that from a parametric perspective the Severi variety has the expected dimension and is smooth in both cases, as the fact that (3) is an equality indicates; we do not dwell on this here, and refer to [8] for a discussion of the differences between the parametric and Cartesian points of view (the latter is the one we adopted in this text).

Note that Proposition 1 does not assert that the Severi variety $V_{|L|,\delta}$ is necessarily non-empty: in such a situation, $V_{|L|,\delta}$ does not have any irreducible component and the statement is empty.

Proof Pick any curve C in an irreducible component V of $V_{|L|,\delta}(S)$. Let $f : \widetilde{S} \to S$ be the blow-up of S at N, the scheme of the δ nodes of C, denote by $\mathfrak{e}$ the (total) exceptional divisor and by $\widetilde{C}$ the strict transform of C. Thus $f_{|\widetilde{C}} = \nu_C$ and we have

$$K_{\widetilde{S}} \sim f^*K_S + \mathfrak{e} \quad \text{and} \quad \widetilde{C} \sim f^*C - 2\mathfrak{e}.$$

>From the restriction sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{S}}(\mathfrak{e}) \longrightarrow \mathcal{O}_{\widetilde{S}}(\widetilde{C} + \mathfrak{e}) \longrightarrow \omega_{\widetilde{C}}(\eta_{\widetilde{C}}) \longrightarrow 0$$

we find

$$\begin{aligned} \dim T_{[C]}V_{|L|,\delta}(S) = \dim |L \otimes \mathcal{J}_N| &= h^0(L \otimes \mathcal{J}_N) - 1 = h^0(f^*L - \mathfrak{e}) - 1 \\ &= h^0(\mathcal{O}_{\widetilde{S}}(\widetilde{C} + \mathfrak{e})) - 1 = h^0(\omega_{\widetilde{C}}(\eta_{\widetilde{C}})) \\ &= \begin{cases} g-1, & \text{if } \eta_{\widetilde{C}} \not\simeq \mathcal{O}_{\widetilde{C}}, \\ g, & \text{if } \eta_{\widetilde{C}} \simeq \mathcal{O}_{\widetilde{C}}. \end{cases} \end{aligned} \tag{4}$$

In the upper case, by (1), we have that $V_{|L|,\delta}$ is smooth at $[C]$ of dimension $g - 1 = p - \delta - 1 = \dim |L \otimes \mathcal{J}_N|$.

Assume next that we are in the lower case. Then, by the discussion prior to the proposition, we have $\pi^{-1}C = Y + \iota(Y)$ for an irreducible curve Y on X, such that π maps both Y and $\iota(Y)$ birationally, but not isomorphically, to C. In particular, Y and $\iota(Y)$ have geometric genus $p_g(Y) = p_g(\iota(Y)) = p_g(C) = p-\delta = g$. Set $L' := \mathcal{O}_X(Y)$ and $2d := Y \cdot \iota(Y)$. Note that d is an integer because, if $y = \iota(x) \in Y \cap \iota(Y)$, then $\iota(y) = x \in Y \cap \iota(Y)$. Since $Y \simeq \iota(Y)$ and π is étale, both Y and $\iota(Y)$ are nodal with $\delta - d$ nodes and they intersect transversely at $2d$ points, which are pairwise conjugate by ι, and therefore map to d nodes of C. Hence $d \leq \delta$. We have

$$p_a(Y) = p_a(\iota(Y)) = g + \delta - d = p - \delta + \delta - d = p - d. \tag{5}$$

whence

$$(L')^2 = 2(p - 1 - d).$$

By the Hodge index theorem, we have

$$4(p-1-d)^2 = \left((L')^2\right)^2 = (L')^2(\iota^*L')^2 \leq \left(L' \cdot \iota^*L'\right)^2 = 4d^2,$$

whence $p-1 \leq 2d$.

By the regularity of Severi varieties on $K3$ surfaces, any irreducible component of $V_{|L'|,\delta-d}(X)$ has dimension $\dim|L'| - (\delta-d) = p_g(Y) = g$. Hence, V is g-dimensional; more precisely, the curves parameterized by V are the (birational) images by π of the curves in an irreducible component of $V_{|L'|,\delta-d}(X)$ intersecting their conjugates by ι transversely (in $2d$ points). By (4), it also follows that $\dim V = \dim T_{[C]}V_{|L|,\delta}(S)$, so that $[C]$ is a smooth point of $V_{|L|,\delta}(S)$.

To prove the final assertion of the proposition, observe that, by the regularity of Severi varieties on $K3$ surfaces, we may deform Y and $\iota(Y)$ on X to irreducible curves Y' and $\iota(Y')$ with any number of nodes $\leq \delta - d$ and intersecting transversally in $2d$ points; in particular, we may deform Y and $\iota(Y)$ to *smooth* curves Y' and $\iota(Y')$. Thus, $C' := \pi(Y')$ is a member of $V_{|L|,d}$, whence of geometric genus $p-d$. Since $\dim|Y'| = p_a(Y') = p_g(C') = p_a(C') - d = p-d$, the component of $V_{|L|,d}$ containing $[C']$ has dimension $\dim|L| - d + 1 = p-d$. We thus have $\dim|L \otimes \mathcal{J}_{N'}| = \dim|L| - d + 1$, where N' is the set of d nodes of C', hence N' does not impose independent conditions on $|L|$.

Assume now that $L' \simeq \iota^*L'$, which — as is well-known (see, e.g., [9, §11]) — is the case occurring for generic S, as then $\operatorname{Pic} X$ is precisely the invariant part under ι of $H_2(X,\mathbb{Z})$. Then $2d = L' \cdot \iota^*L' = (L')^2 = 2(p-1-d)$, so that $p-1 = 2d$. Since $L^2 = 2(p-1) = 4d$ and N' does not impose independent conditions on $|L|$, by [11, Prop. 3.7] there is an effective divisor $D \subset S$ containing N' satisfying $L - 2D \geq 0$ and

$$L \cdot D - d \leq D^2 \overset{(i)}{\leq} \frac{1}{2} L \cdot D \overset{(ii)}{\leq} d, \tag{6}$$

with equality in (i) or (ii) only if $L \equiv 2D$; moreover, since $L - 2D \geq 0$, the numerical equivalence $L \equiv 2D$ implies the linear equivalence $L \sim 2D$. Now since $N' \subset D$, we must have $L \cdot D = C' \cdot D \geq 2d$, hence the inequalities in (6) are all equalities, and thus $D^2 = d$ and $L \sim 2D$. □

The following corollary is a straightforward consequence of Prop. 1 and the fact that the nodes on curves in a regular component in a Severi variety (on any surface and in particular on a K3 surface) can be independently smoothened.

Corollary 1 *If a Severi variety $V_{|L|,\delta}$ on an Enriques surface has a regular (resp., nonregular) component, then for any $0 \leq \delta' \leq \delta$ (resp., $d \leq \delta' \leq \delta$, with d as in Prop. 1), also $V_{|L|,\delta'}$ contains a regular (resp., nonregular) component.*

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