SEVERI VARIETIES ON BLOW–UPS OF THE SYMMETRIC SQUARE OF AN ELLIPTIC CURVE

CIRO CILIBERTO, THOMAS DEDIEU, CONCETTINA GALATI, AND ANDREAS LEOPOLD KNUTSEN

Abstract. We prove that certain Severi varieties of nodal curves of positive genus on general blow–ups of the twofold symmetric product of a general elliptic curve are non–empty and smooth of the expected dimension. This result, besides its intrinsic value, is an important preliminary step for the proof of nonemptiness of Severi varieties on general Enriques surfaces in [10].

1. Introduction

Let S be a smooth, projective complex surface and $\xi \in \mathrm{Num}(S)$. Let

$$p_a(\xi) = \frac{1}{2}\xi \cdot (\xi + K_S) + 1$$

be the *arithmetic genus* of ξ. If L is a line bundle or divisor on S with class ξ in $\mathrm{Num}(S)$ we set $p_a(L) = p_a(\xi)$. We denote by $V^\xi(S)$ the locus in the Hilbert scheme of S parametrizing the curves C on S such that the class of $\mathcal{O}_S(C)$ in $\mathrm{Num}(S)$ coincides with ξ. Assume that L is a line bundle or divisor on S with class ξ in $\mathrm{Num}(S)$ and that $p_a(L) \geqslant 0$. For any integer δ satisfying $0 \leqslant \delta \leqslant p_a(\xi)$ we denote by $V^\xi_\delta(S)$ the *Severi variety* parametrizing irreducible δ–nodal curves contained in $V^\xi(S)$. This is a possibly empty locally closed set in $V^\xi(S)$.

Let V be an irreducible component of $V^\xi(S)$ and, for any δ such that $0 \leqslant \delta \leqslant p_a(\xi)$, let V_δ be an irreducible component of $V \cap V^\xi_\delta(S)$. It is well known that

$$\dim(V_\delta) \geqslant \dim(V) - \delta, \tag{1}$$

where the right hand side is called the *expected dimension* of V_δ. Moreover if the equality holds in (1), then, for all $0 \leqslant \delta' \leqslant \delta$, the closure of the intersection of $V^\xi_{\delta'}(S)$ with V contains V_δ, and any of its components whose closure contains V_δ has the expected dimension $\dim(V) - \delta'$ (see [25, Thm. 6.3]).

Severi varieties were introduced by Severi in Anhang F of [26], where he proved that all Severi varieties of irreducible δ-nodal curves of degree d in $\mathbb{P}^2$ are nonempty and smooth of the expected dimension. Severi also claimed irreducibility of such varieties, but his proof contains a gap. The irreducibility was proved by Harris [17] more than 60 years later.

Severi varieties on other surfaces have received much attention in recent years, especially in connection with enumerative formulas computing their degrees.

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Nonemptiness is known to hold for all Severi varieties associated to big and nef classes on Del Pezzo surfaces (as well as rational surfaces under certain assumptions) by [16, Thms. 3-4] and for Hirzebruch surfaces, a result implicitly contained in [27, §3]. In both cases of Del Pezzo and Hirzebruch surfaces, all Severi varieties are smooth of the expected dimensions, cf., e.g. [27, Lemma 2.9] or [7, p. 45]. Moreover, all Severi varieties of Hirzebruch surfaces are irreducible [31], and Severi varieties parametrizing rational curves on Del Pezzo surfaces of degrees $\geqslant 2$ are irreducible as well [30]; the same holds true for *general* Del Pezzo surfaces of degree one, except for the Severi variety parametrizing rational curves in the anticanonical class [29, Cor. 6.4].

On a general primitively polarized $K3$ surface (S, ξ), all Severi varieties $V_\delta^{n\xi}(S)$, where $0 \leqslant \delta \leqslant p_a(n\xi)$, are nonempty by a result of Mumford [23] if $n = 1$ and of Chen [5] for all n; moreover, all components are always smooth of the expected dimension $p_a(n\xi) - \delta$ [28, 7]. The irreduciblity question (for $\delta < p_a(n\xi)$) has been the object of much attention, see [18, 19, 8, 1, 13], and was recently solved in the case $n = 1$ for all $\delta \leqslant p_a(\xi) - 4$ in the preprint [2].

Similarly, on a general primitively polarized abelian surface (S, ξ), all Severi varieties $V_\delta^{n\xi}(S)$, where $0 \leqslant \delta \leqslant p_a(n\xi) - 2$, are nonempty (by [21] if $n = 1$ and [20] in general) and smooth of the expected dimension $p_a(n\xi) - \delta$ [22]. Irreducibility does not hold: the various irreducible components in the case $n = 1$ have been described by Zahariuc [32].

Very little is known on other surfaces, where problems such as nonemptiness, smoothness, dimension and irreducibility are regarded as very hard. In particular, Severi varieties may have unexpected behaviour: examples are given in [6] of surfaces of general type with reducible Severi varieties, and also with components of dimension different from the expected one.

In this paper we consider the case of blow–ups of a particular type of ruled surface over an elliptic curve.

Let E be a general smooth irreducible projective curve of genus one and set $R := \mathrm{Sym}^2(E)$. Let $\widetilde{R}$ be the blow–up of R at any finite set of general points. Our main result in this paper shows that Severi varieties of a large class of line bundles on $\widetilde{R}$ are well-behaved:

Theorem 1.1. *In the above setting, let L be a line bundle on $\widetilde{R}$ verifying condition $(\star)$ (cf. Definition 2.1) and let ξ be the class of L in $\mathrm{Num}(\widetilde{R})$. Let δ be an integer satisfying $0 \leqslant \delta < p_a(L)$. Then $V_\delta^\xi(\widetilde{R})$ is nonempty and smooth with all components of the expected dimension $-L \cdot K_{\widetilde{R}} + p_a(L) - \delta - 1$.*

The statement about smoothness and dimension follows from standard arguments of deformation theory, once non–emptiness has been proved, cf. Proposition 2.2 below. Moreover we remark that, by what we said above, it suffices to prove Theorem 1.1 for the maximal number of nodes, i.e., $\delta = p_a(L) - 1$. This will follow from Proposition 2.3 below, which treats the special case in which the blown–up points are in a special position. In [10] we will make use of Proposition 2.3 in order to prove nonemptiness of Severi varieties on Enriques surfaces. The question of smoothness and dimension of Severi varieties on Enriques surfaces has been treated in [9].

The irreducibility question for $V_\delta^\xi(\widetilde{R})$ is not treated in this paper; thus, we pose:

Question 1.2. Are the varieties $V_\delta^\xi(\widetilde{R})$ from Theorem 1.1 irreducible?

The paper is organised as follows. In §2 we recall some preliminaries concerning twofold symmetric products of elliptic curves. Section 3 is devoted to recalling a degeneration of the symmetric product of a general elliptic curve studied in [11]. In §4 we construct certain families of curves on some blow–ups of the projective plane that turn out to be useful in the proof of Proposition 2.3, which is proved by degeneration in §5.

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2. The twofold symmetric product of an elliptic curve

Let E be a smooth irreducible projective elliptic curve. Denote by $\oplus$ (and $\ominus$) the group operation on E and by e_0 the neutral element. Let $R := \mathrm{Sym}^2(E)$ and $\pi : R \to E$ be the (Albanese) morphism sending $x+y$ to $x \oplus y$. We denote the fiber of π over a point $e \in E$ by

$$\mathfrak{f}_e := \pi^{-1}(e) = \{x + y \in \mathrm{Sym}^2(E) \mid x \oplus y = e \text{ (equivalently, } x + y \sim e + e_0)\},$$

(where $\sim$ denotes linear equivalence), which is the $\mathbb{P}^1$ defined by the linear system $|e+e_0|$. We denote the algebraic equivalence class of the fibers by $\mathfrak{f}$.

For each $e \in E$ we define the curve $\mathfrak{s}_e$ (called D_e in [4]) as the image of the section $E \to R$ of the Albanese morphism mapping x to $e+(x\ominus e)$. We let $\mathfrak{s}$ denote the algebraic equivalence class of these sections, which are the ones with minimal self-intersection, namely 1, cf. [4]. One has

$$K_R \sim -2\mathfrak{s}_{e_0} + \mathfrak{f}_{e_0}.$$

Let $y_1, \ldots, y_n \in R$ be distinct points and let $\widetilde{R} := \mathrm{Bl}_{y_1,\ldots,y_n}(R) \to R$ denote the blow–up of R at $y_1, \ldots, y_n$, with exceptional divisors $\mathfrak{e}_i$ over y_i. We denote the strict transforms of $\mathfrak{s}$ and $\mathfrak{f}$ on $\widetilde{R}$ by the same symbols.

Definition 2.1. A line bundle or Cartier divisor L on $\widetilde{R}$ is said to verify condition $(\star)$ if it is of the form $L \equiv \alpha\mathfrak{s} + \beta\mathfrak{f} - \sum_{i=1}^n \gamma_i\mathfrak{e}_i$ (where $\equiv$ denotes numerical or, equivalently, algebraic equivalence), with α, β, $\gamma_1, \ldots, \gamma_n$ are integers such that:
(i) $\alpha \geqslant 1$, $\beta \geqslant 0$;
(ii) $\alpha \geqslant \gamma_i$ for $i = 1, \ldots, n$;
(iii) $\alpha + \beta \geqslant \sum_{i=1}^n \gamma_i$;
(iv) $\alpha + 2\beta \geqslant \sum_{i=1}^n \gamma_i + 4$.

Condition (ii) is satisfied if L is nef. Condition (iv) is equivalent to $-L \cdot K_{\widetilde{R}} \geqslant 4$.

The statement about smoothness and dimension in Theorem 1.1 follows from the following more general result, well–known to experts:

Proposition 2.2. *Let S be a smooth projective complex surface and $\xi \in \mathrm{Num}(S)$ such that $-\xi \cdot K_S > 0$. Let δ be an integer satisfying $0 \leqslant \delta \leqslant p_a(\xi)$.*

If $V^{\xi}_{\delta}(S)$ is nonempty, it is smooth and every component has the expected dimension $-\xi \cdot K_S + p_a(\xi) - \delta - 1$.

Proof. Let X be any curve in $V^\xi_\delta(S)$ and let $V^\xi(S)$ be the Hilbert scheme defined in the introduction. Since

$$\deg(\mathcal{N}_{X/S}) = \xi^2 = \xi\cdot(\xi+K_S) - \xi\cdot K_S = 2p_a(\xi) - 2 - \xi\cdot K_S > 2p_a(\xi)-2,$$

the normal bundle $\mathcal{N}_{X/S}$ is nonspecial, whence $V^\xi(S)$ is smooth at $[X]$ of dimension $h^0(\mathcal{N}_{X/S}) = -\xi\cdot K_S + p_a(\xi) - 1$ (cf., e.g., [24, §4.3]).

Let $\varphi : \widetilde{X} \to S$ be the composition of the normalization $\widetilde{X}\to X$ with the inclusion $X\subset S$ and consider the *normal sheaf* $\mathcal{N}_\varphi$ defined by the short exact sequence

$$0 \longrightarrow \mathcal{T}_{\widetilde{X}} \longrightarrow \varphi^*\mathcal{T}_S \longrightarrow \mathcal{N}_\varphi \longrightarrow 0.$$

The tangent space to $V^\xi_\delta(S)$ at $[X]$ is isomorphic to $H^0(\widetilde{X},\mathcal{N}_\varphi)$, and $\mathcal{N}_\varphi$ is a line bundle, as X is nodal (cf., e.g., [24, §3.4.3] or [14]). Let g be the geometric genus of X. Since $\deg\mathcal{N}_\varphi = -X\cdot K_S + 2g-2 > 2g-2$ by the above sequence, the line bundle $\mathcal{N}_\varphi$ is nonspecial, and

$$h^0(\mathcal{N}_\varphi) = -\xi\cdot K_S + g - 1 = -\xi\cdot K_S + p_a(\xi) - \delta - 1 = \dim(V^\xi(S)) - \delta,$$

which is the expected dimension of $V^\xi_\delta(S)$. Thus, $V^\xi_\delta(S)$ is smooth at $[X]$ and of the expected dimension. □

By what we said in the introduction, it suffices to prove Theorem 1.1 for the highest possible δ, that is, for $\delta = p_a(L)-1$, in which case the Severi variety in question parametrizes nodal curves of geometric genus one. We will prove the theorem by specializing the points $y_1,\dots,y_n$ as we now explain.

Let η be any of the three nonzero 2-torsion points of E. The map $E\to R$ defined by mapping e to $e+(e\oplus\eta)$ realizes E as an unramified double cover of its image curve

$$T := \{e + (e\oplus\eta) \mid e\in E\}.$$

We have

$$T \sim -K_R + \mathfrak{f}_\eta - \mathfrak{f}_{e_0} \sim 2\mathfrak{s}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta, \tag{2}$$

by [4, (2.10)]. In particular,

$$T \not\sim -K_R \quad\text{and}\quad 2T \sim -2K_R. \tag{3}$$

We denote the strict transform of T on $\widetilde{R}$ by the same symbol. Suppose that $y_1,\dots,y_n\in T$ are general points. Then by (2)–(3) we have

$$T \sim 2\mathfrak{s}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta - \mathfrak{e}_1 - \cdots - \mathfrak{e}_n \not\sim -K_{\widetilde{R}}, \quad 2T\sim -2K_{\widetilde{R}}$$

on $\widetilde{R}$. In particular,

$$T \equiv -K_{\widetilde{R}} \equiv 2\mathfrak{s} - \mathfrak{f} - \mathfrak{e}_1 - \cdots - \mathfrak{e}_n.$$

As remarked in the introduction, Theorem 1.1 is a consequence of the following result, which we will prove in §5.

Proposition 2.3. *Let E be a general irreducible smooth projective curve of genus one. Let $y_1,\dots,y_n\in T$ be general points, with T on $R=\mathrm{Sym}^2(E)$ as defined above. Let L be a line bundle on $\widetilde{R}=\mathrm{Bl}_{y_1,\dots,y_n}(R)$ verifying condition $(\star)$ with class ξ in $\mathrm{Num}(\widetilde{R})$. Then $V^\xi_{p_a(L)-1}(\widetilde{R})$ is nonempty and smooth, of the expected dimension $L\cdot T = -L\cdot K_{\widetilde{R}}$.*

3. A degeneration of the twofold symmetric product of a general elliptic curve

Let E be a smooth irreducible projective elliptic curve. We recall a degeneration of $R = \mathrm{Sym}^2(E)$ from [11], to which we refer the reader for details.

Let $\mathcal{X} \to \mathbb{D}$ be a flat projective family of curves over the unit disk $\mathbb{D}$, with $\mathcal{X}$ smooth, such that the fiber X_0 over $0 \in \mathbb{D}$ is an irreducible rational 1-nodal curve and all other fibers X_t, $t \in \mathbb{D} \setminus \{0\}$, are smooth irreducible elliptic curves. Let $p : \mathcal{Y} \to \mathbb{D}$ be the relative 2-symmetric product. Then, for $t \neq 0$, the fiber $Y_t = p^{-1}(t) \simeq \mathrm{Sym}^2(X_t)$ is smooth, whereas the special fiber $Y_0 = p^{-1}(0) = \mathrm{Sym}^2(X_0)$ is irreducible, but singular. The singular locus of Y_0 consists of the curve

$$X_P := \{x + P \mid x \in X_0\},$$

where P is the node of X_0.

Let $\nu : \mathbb{P}^1 \simeq \widetilde{X}_0 \to X_0$ be the normalization, with P_1 and P_2 the preimages of P (with the notation of [11, p. 328], this is the case $g = 1$ with $P = P_1$). Then ν induces a birational morphism

$$\mathrm{Sym}^2(\nu) : \mathbb{P}^2 \simeq \mathrm{Sym}^2(\widetilde{X}_0) \longrightarrow \mathrm{Sym}^2(X_0) = Y_0. \tag{4}$$

Under the isomorphism on the left the diagonal in $\mathrm{Sym}^2(\widetilde{X}_0)$ corresponds to a smooth conic Γ in $\mathbb{P}^2$ that is mapped by $\mathrm{Sym}^2(\nu)$ to the diagonal Δ_0 of Y_0 and, for each $x \in \widetilde{X}_0$, the curve

$$\{x + Q \mid Q \in \widetilde{X}_0\} \subset \mathrm{Sym}^2(\widetilde{X}_0)$$

corresponds to the line in $\mathbb{P}^2$ tangent to Γ at the point corresponding to $2x$.

The threefold $\mathcal{Y}$ has local equations at the point $2P \in Y_0$ given by

$$z_5^2 - z_3 z_4 = 0, \quad z_1 z_5 + z_2 z_3 = 0, \quad z_1 z_4 + z_2 z_5 = 0, \quad z_1 z_2 + z_5 + t = 0,$$

with $(z_1, ..., z_5, t) \in \mathbb{C}^5 \times \mathbb{D}$ (cf. [11, p. 329]). In particular, $\mathcal{Y}$ is singular only at the point $2P$, corresponding to the origin $\underline{0}$. Its special fibre Y_0 is locally reducible at $\underline{0} = 2P$, where it consists of three irreducible components $S^1 \cup S^2 \cup S^3$ (named S_1^i in [11]), where S^1 is the $z_2 z_4$-plane, S^2 is the $z_1 z_3$-plane and S^3 has equations $z_3 = z_1^2$, $z_4 = z_2^2$, $z_5 = -z_1 z_2$, meeting as in [11, Fig. 1]. The singular locus $X_P = \mathrm{Sing}(Y_0)$ of Y_0 has a node at the origin, Y_0 has double normal crossing singularities along $X_P \setminus 2P$ and the intersection curves $C^1 = S^3 \cap S^1$ and $C^2 = S^3 \cap S^2$ (named C_1^i in [11]) are the two branches of the curve X_P at $\underline{0} = 2P$. Finally, in these local coordinates, the diagonal Δ_0 of Y_0 ($\Delta_0 = \Delta_{0,1}^1 \cup \Delta_{0,2}^1$ in [11, Fig. 1]) consists of the z_2, z_1-axes and it has a node at the point $2P$.

Let $\mu : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be the blow–up at the point $2P \in \mathrm{Sym}^2(X_0) = Y_0 \subset \mathcal{Y}$ and denote the exceptional divisor by $\mathcal{E}$ (called E_1 in [11]). Then $\mathcal{E} \simeq \mathbb{F}_1$ and $\widetilde{\mathcal{Y}}$ is smooth (see [11, p. 330]). All fibers over $t \neq 0$ are unchanged. The special fiber $\widetilde{Y}_0$ of $\widetilde{\mathcal{Y}} \to \mathbb{D}$ is the union of $\mathcal{E}$ and of an irreducible surface $\widetilde{S}$, which is the strict transform of Y_0. We have

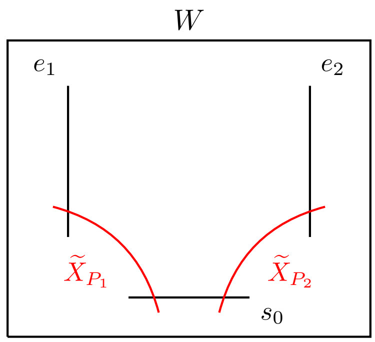
$$\widetilde{S} \cap \mathcal{E} = s_0 + e_1 + e_2,$$

where e_1 and e_2 are two fibers of $\mathcal{E} \simeq \mathbb{F}_1$ and s_0 (called η_1 in [11, Fig. 2]) is the section satisfying $s_0^2 = -1$. The surface $\widetilde{S}$ is singular, with double normal crossings singularities along the proper transform $\widetilde{X}_P$ of the curve X_P. The proper transform on $\widetilde{\mathcal{Y}}$ of the diagonal of $\mathcal{Y}$ intersects $\widetilde{Y}_0$ along

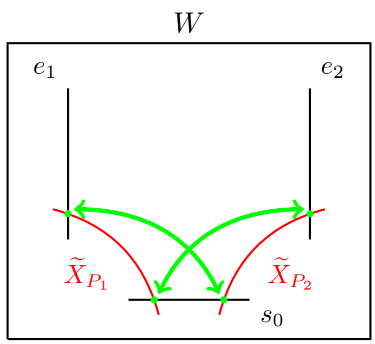
$$\widetilde{\Delta}_0 + s_0,$$

where $\widetilde{\Delta}_0$ is the proper transform of the diagonal Δ_0 on Y_0.

To normalize $\widetilde{S}$ one unfolds along $\widetilde{X}_P$. The resulting surface W is smooth. Denote the normalization map by $\sigma : W \to \widetilde{S}$. The preimage of $\widetilde{X}_P$ is a pair of curves, which we call $\widetilde{X}_{P_1}$ and $\widetilde{X}_{P_2}$. Denoting the inverse images on W of the curves e_1, e_2, s_0 on $\widetilde{S}$ by the same symbols, the intersection configuration between the curves $e_1, e_2, s_0, \widetilde{X}_{P_1}, \widetilde{X}_{P_2}$ on W looks as follows:



Under the map σ the two curves $\widetilde{X}_{P_1}$ and $\widetilde{X}_{P_2}$ are identified: we denote the identification morphism by $\omega : \widetilde{X}_{P_1} \simeq \widetilde{X}_{P_2}$. Under this morphism, the intersection points of the above configuration are mapped as follows:



Definition 3.1. We say that a curve $C \subset W$ is *ω-compatible* if C contains neither $\widetilde{X}_{P_1}$ nor $\widetilde{X}_{P_2}$ and ω maps the 0–dimensional intersection scheme of C with $\widetilde{X}_{P_1}$ to the intersection scheme of C with $\widetilde{X}_{P_2}$.

If the curve C is ω–compatible then $\sigma(C)$ is a Cartier divisor on $\widetilde{S}$. Conversely any curve on $\widetilde{S}$ that is a Cartier divisor and does not contain the singular curve of $\widetilde{S}$ is the image by σ of an ω–compatible curve on W.

One sees that the curves s_0, e_1, e_2 are (-1)–curves on W (see [11, p. 331–332]). Contracting them we obtain a morphism $\phi : W \to \mathbb{P}^2 \simeq \mathrm{Sym}^2(\widetilde{X}_0)$ such that

$$\phi(s_0) = P_1 + P_2, \quad \phi(e_1) = 2P_1, \quad \phi(e_2) = 2P_2$$

and

$$\phi(\widetilde{X}_{P_i}) = X_{P_i} := \{P_i + Q \mid Q \in \widetilde{X}_0\},$$

fitting in a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "\sigma"] \arrow[d, "\phi"'] & \widetilde{S} \subset \widetilde{S} \cup \mathcal{E} = & \widetilde{Y}_0 \arrow[d, "\mu|_{\widetilde{Y}_0}"] & \subset & \widetilde{\mathcal{Y}} \arrow[d, "\mu"] \\
\mathbb{P}^2 \arrow[rr, "\mathrm{Sym}^2(\nu)"'] & & Y_0 \arrow[d, "p|_{Y_0}"] & \subset & \mathcal{Y} \arrow[d, "p"] \\
 & & \{0\} & \subset & \mathbb{D}
\end{tikzcd}$$

(see [11, p. 332]). This is shown in the next picture:

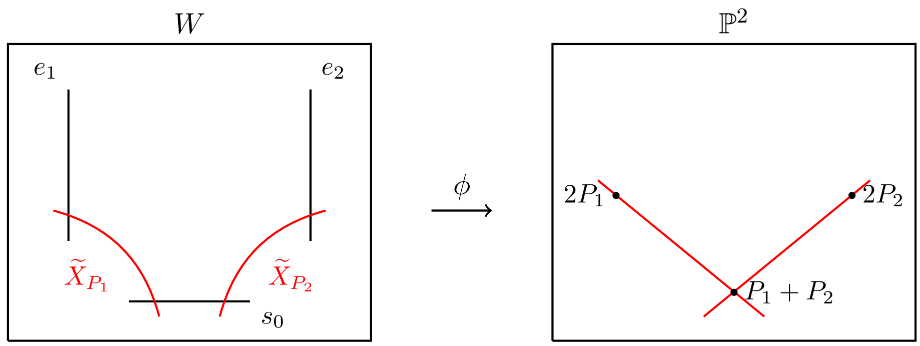


Figure A

Remark 3.2. The morphism $\omega : \widetilde{X}_{P_1} \to \widetilde{X}_{P_2}$ is geometrically interpreted in the following way (see [11]). Via ϕ the curves $\widetilde{X}_{P_1}$ and $\widetilde{X}_{P_2}$ map isomorphically to the two lines on the plane $\mathbb{P}^2$ in red in Figure A joining the point $P_1 + P_2$ with the points $2P_1$ and $2P_2$ respectively. In $\mathbb{P}^2$ we have the conic Γ (mapped by $\mathrm{Sym}^2(\nu)$ to the diagonal Δ_0 of Y_0), which is tangent to these lines at the points $2P_1$ and $2P_2$. The map ω associates two points if and only if their images in the plane lie on a tangent line to Γ. (The two points $P_1 + Q$ and $P_2 + Q$ of $\mathbb{P}^2$ lie on the tangent line to Γ at the point $2Q$ and are the intersection points of this tangent line with the two lines joining $2P_1$ with $P_1 + P_2$ and $2P_2$ with $P_1 + P_2$, namely $\phi(\widetilde{X}_{P_1})$ and $\phi(\widetilde{X}_{P_2})$.)

The Picard group of W is generated by h, e_1, e_2, s_0, where h is the pullback by ϕ of a line. In particular,

$$\widetilde{X}_{P_i} \sim h - s_0 - e_i, \quad i = 1, 2. \tag{5}$$

One has

$$-K_W = 3h - e_1 - e_2 - s_0.$$

Let us look at what happens to the classes of $\mathfrak{s}$ and $\mathfrak{f}$ under the degeneration of R to $\widetilde{Y}_0$. This is described in [11, §2] together with the more general description of the degeneration of line bundles on R under the degeneration of R to $\widetilde{Y}_0$, which we are now going to recall.

Let h' be an ω-compatible member of $|h|$ on W (cf. Definition 3.1), not containing any of e_1, e_2 or s_0. There is a one-dimensional irreducible family of such curves whose general member is the strict transform on W of a general tangent line to the conic Γ of $\mathbb{P}^2$ mapped to the diagonal of Y_0 by $\mathrm{Sym}^2(\nu)$ (cf. Remark 3.2). Since $h \cdot e_1 = h \cdot e_2 = h \cdot s_0 = 0$, we have $\sigma(h') \cap \mathcal{E} = \emptyset$, so that $\sigma(h') \subset \widetilde{S}$ determines a Cartier divisor on $\widetilde{Y}_0$. The class of $\mathfrak{s}$ on R degenerates to the class of $\sigma(h')$.

The class $h - s_0$ on W satisfies $(h - s_0) \cdot s_0 = 1$ and $(h - s_0) \cdot e_i = (h - s_0) \cdot \widetilde{X}_{P_i} = 0$, $i = 1, 2$. Thus the general member F of the pencil $|h - s_0|$ is ω-compatible and $\sigma(F)$ intersects $\mathcal{E}$ in one point along s_0. Therefore, the union of $\sigma(F)$ with the fiber of $\mathcal{E}$ over the intersection point on s_0 is a Cartier divisor on $\widetilde{Y}_0$, which turns out to be the limit of $\mathfrak{f}$.

Let $C \equiv a\mathfrak{s} + b\mathfrak{f}$ on R, with $a, b \geqslant 0$, and let C_0 be its limit on $\widetilde{Y}_0$. Assume that it neither contains any of e_1, e_2, s_0 nor the double curve of $\widetilde{S}$. We may write $C_0 = C_{\widetilde{S}} \cup C_{\mathcal{E}}$ with $C_{\widetilde{S}} \subset \widetilde{S}$ and $C_{\mathcal{E}} \subset \mathcal{E}$. Then $C_{\widetilde{S}} \cap C_{\mathcal{E}} \subset s_0$ and $C_{\mathcal{E}}$ is a union of fibers of $\mathcal{E}$. We have $C_{\widetilde{S}} = \sigma(C_W)$, with C_W a ω-compatible curve satisfying

$$C_W \sim ah + b(h - s_0) = (a + b)h - bs_0.$$

This is because the transform of the limit of $\mathfrak{s}$ is numerically equivalent to h on W and the transform of the limit of $\mathfrak{f}$ is equivalent to $(h - s_0)$. This means that $\phi(C_W) \subset \mathbb{P}^2$ is a plane curve of degree $a + b$ with a point of multiplicity b at $P_1 + P_2$, with intersection points with X_{P_1} and X_{P_2} satisfying the suitable "gluing conditions" given by ω.

Conversely, we have:

Lemma 3.3. *Let $a, b \geqslant 0$ and $C_W \in |(a + b)h - bs_0|$ be an ω-compatible curve not containing any of e_1, e_2, s_0 and intersecting s_0 in distinct points. Let $C_{\mathcal{E}}$ denote the union of fibers on $\mathcal{E} \simeq \mathbb{F}_1$ such that $C_{\mathcal{E}} \cap s_0 = C_W \cap s_0$. Then $\sigma(C_W) \cup C_{\mathcal{E}}$ is the flat limit of a curve algebraically equivalent to $a\mathfrak{s} + b\mathfrak{f}$.*

Proof. Since $C_W \cdot \widetilde{X}_{P_i} = a$, the locus of ω-compatible curves in $|C_W|$ has dimension

$$\dim |C_W| - a = \frac{1}{2}\left(a^2 + 3a + 2ab + 2b\right) - a = \frac{1}{2}\left(a^2 + a + 2ab + 2b\right),$$

which equals the dimension of the Hilbert scheme of curves algebraically equivalent to $a\mathfrak{s} + b\mathfrak{f}$. The result follows from the discussion prior to the lemma. □

Let us now go back to the degeneration $\mathcal{X} \to \mathbb{D}$ of a general elliptic curve E to a rational nodal curve X_0 we considered at the beginning of this section. This can be viewed as a degeneration of the group E to $\mathbb{C}^*$, where (keeping the notation introduced at the beginning of this section) $\mathbb{C}^* = \mathbb{P}^1 \setminus \{P_1, P_2\}$. Since $\mathbb{C}^*$ has a unique non–trivial point of order 2, i.e., -1, we see that in the degeneration $\mathcal{X} \to \mathbb{D}$ one of the three non–trivial points of order 2 of the general fibre degenerates to -1, so it is fixed by the monodromy of the family $\mathcal{X} \to \mathbb{D}$. (The other two non–trivial points of order 2 must degenerate to the node of X_0.) This implies that we have a divisor $\mathcal{T}$ on $\widetilde{\mathcal{Y}}$ such that the fiber T_t for $t \neq 0$ is a curve $T \subset R$ like the one we considered in §2. Since $T \equiv 2\mathfrak{s} - \mathfrak{f}$, the proper transform T_W on W of the limit of the curve T is such that

$$T_W \sim 2h - (h - s_0) \sim h + s_0$$

(remember that the pull–back on W of the limit of $\mathfrak{s}$ and $\mathfrak{f}$ are h and $h - s_0$ respectively). More precisely, since T has zero intersection with the diagonal of R, the image of T_W in $\mathbb{P}^2$ via $\phi : W \to \mathbb{P}^2$ must intersect the conic Γ, which is mapped by $\mathrm{Sym}^2(\nu)$ to the diagonal of Y_0 (see (4)), only in points of Γ that are blown up by ϕ, i.e., in the points corresponding to $2P_1$ and $2P_2$ (see Figure A). This implies that

$$T_W = h_0 + e_1 + e_2 + s_0,$$

where h_0 is the strict transform by ϕ of the line in $\mathbb{P}^2$ through $2P_1$ and $2P_2$.

Since T_W contains s_0, the divisor $\mathcal{T}$ contains $\mathcal{E}$. By subtracting $\mathcal{E}$ from $\mathcal{T}$, the resulting irreducible effective divisor $\mathcal{T} - \mathcal{E}$ intersects the central fibre in a curve that consists of two components: one component on $\widetilde{S}$, which is $\sigma(h_0)$, and another component sitting on $\mathcal{E}$ that is the pull–back on $\mathcal{E} \simeq \mathbb{F}_1$ of the unique line of $\mathbb{P}^2$ passing through the two points in which h_0 intersects e_1 and e_2. However, what will be important for us in what follows is that $\sigma(h_0)$ is in the limit of T.

4. A useful family of rational curves on some blow–ups of the plane

In this section we prove some results on certain line bundles on some blow–ups of the surface W introduced in the previous section. They will be useful in the proof of Proposition 2.3 in §5. We go on keeping the notation and convention we introduced in the previous section.

Let $y_1, \ldots, y_n \in h_0$ be general points. Choose sections of $p : \widetilde{\mathcal{Y}} \to \mathbb{D}$ passing through $\sigma(y_1), \ldots, \sigma(y_n) \in \sigma(h_0)$ and through general points $y_1^t, \ldots, y_n^t \in T_t$ on a general fiber Y_t. Blowing up $\widetilde{\mathcal{Y}}$ along these sections, we obtain a smooth threefold $\mathcal{Y}'$ with a morphism $p' : \mathcal{Y}' \to \mathbb{D}$ with general fiber the blow–up of $Y_t = \mathrm{Sym}^2(X_t)$ at n general points of T_t and special fiber $Y' := S' \cup \mathcal{E}$, where $S' = \mathrm{Bl}_{\sigma(y_1),\ldots,\sigma(y_n)}(\widetilde{S})$, and there is a normalization morphism $\sigma' : W' \to S'$, where $W' = \mathrm{Bl}_{y_1,\ldots,y_n}(W)$. Let e_{y_i} denote the exceptional divisor in W' over y_i, for $i = 1, \ldots, n$. We denote the strict transforms of $e_1, e_2, s_0, \widetilde{X}_{P_1}, \widetilde{X}_{P_2}, h_0$ on W' by the same symbols. Note that (5) still holds; furthermore

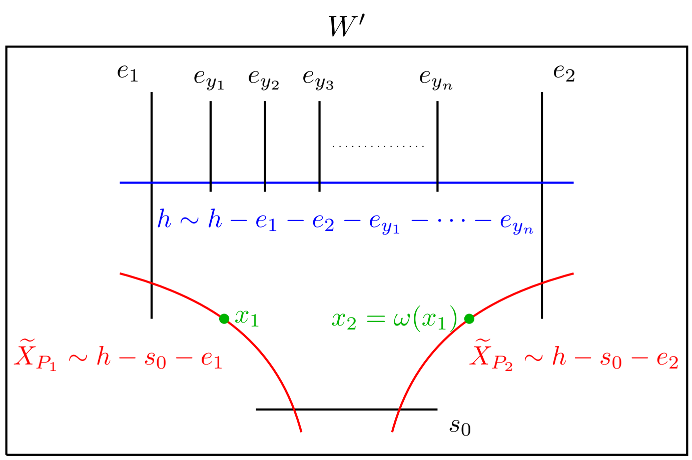
$$h_0 \sim h - e_1 - e_2 - e_{y_1} - \cdots - e_{y_n}$$

and

$$-K_{W'} = 3h - e_1 - e_2 - s_0 - e_{y_1} - \cdots - e_{y_n} \sim h_0 + e_1 + e_2 + \widetilde{X}_{P_1} + \widetilde{X}_{P_2} + s_0. \tag{6}$$

Moreover, the pull–back on W' of the limit of T on S' contains h_0.

We next fix a general point $x_1 \in \widetilde{X}_{P_1}$ and set $x_2 = \omega(x_2) \in \widetilde{X}_{P_2}$. The following picture summarizes the situation:



We introduce the following notation. For a line bundle M on W', we denote by V_M the locus of curves C in $|M|$ on W' such that

- C is irreducible and rational,
- C intersects $\widetilde{X}_{P_i}$ only at x_i, $i = 1, 2$, and it is unibranch there.

We denote by V_M^* the open sublocus of V_M of curves C with the further property that

- C intersects s_0 transversely,
- C is smooth at x_i, $i = 1, 2$,
- C is nodal.

Lemma 4.1. *Assume $V_M \neq \emptyset$.*

(i) If $M \cdot (h + s_0 - e_{y_1} - \cdots - e_{y_n}) \geqslant 1$, then for each component V of V_M one has

$$\dim(V) = -K_{W'} \cdot M - 1 - M \cdot \widetilde{X}_{P_1} - M \cdot \widetilde{X}_{P_2} = M \cdot (h + s_0 - e_{y_1} - \cdots - e_{y_n}) - 1.$$

(ii) If $M \cdot (h + s_0 - e_{y_1} - \cdots - e_{y_n}) \geqslant 4$, then $V_M^* \neq \emptyset$.

Proof. The result follows from [3, §2], as outlined in [12, Thm. (1.4)]. □

Proposition 4.2. *Let $\alpha, \beta, \gamma_1, \ldots, \gamma_n$ be non–negative integers verifying conditions (i)–(iv) in Definition 2.1. Set $M = (\alpha+\beta)h - \beta s_0 - \sum \gamma_i e_{y_i}$. Then V_M^* is non–empty with all components of dimension $\alpha + 2\beta - \sum \gamma_i - 1$.*

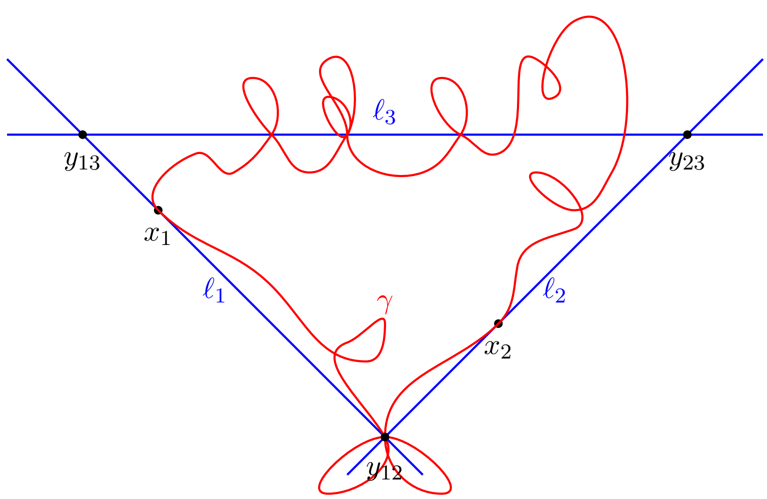
In the proof of Proposition 4.2 we will need the following:

Lemma 4.3. *Given three lines ℓ_1, ℓ_2, ℓ_3 in the plane $\mathbb{P}$ not passing through the same point, we set $y_{ij} = \ell_i \cap \ell_j$ for $1 \leqslant i < j \leqslant 3$. Fix integers $d > m \geqslant 0$, $n \geqslant 0$, $m_1, \ldots, m_n \geqslant 1$, such that*

$$d \geqslant \sum_{i=1}^{n} m_i \quad \text{and} \quad d \geqslant m + m_i, \quad i = 1, \ldots, n.$$

Then there is a reduced and irreducible rational curve γ in $\mathbb{P}$ of degree d with the following properties:

- *γ has a point of multiplicity m at y_{12},*
- *the pull–back on the normalization of γ of the g^1_{d-m} cut out by the lines through y_{12} has two total ramification points mapping to generic points $x_1 \in \ell_1$ and $x_2 \in \ell_2$ respectively (in particular different from y_{12}, y_{13}, y_{23}),*
- *γ has n points of multiplicities $m_1, \ldots, m_n$ that are pairwise distinct points on ℓ_3 (in particular different from y_{13} and y_{23}).*



Proof. Set $\delta = d - m$. The assertion is trivial if $\delta = 1$. So we assume $\delta \geqslant 2$. Consider a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree δ with two points of total ramification, that is a g^1_δ with no base points. Fix a general effective divisor D of degree m on $\mathbb{P}^1$, so that $g^1_\delta + D$ is a g^1_d. Fix n general points $P_1, \ldots, P_n$ of $\mathbb{P}^1$, and consider the fibres

$$f^{-1}(P_i) = P_{i1} + \cdots + P_{i\delta}, \quad i = 1, \ldots, n.$$

Consider then the divisor

$$E = \sum_{i=1}^{n} \sum_{j=1}^{m_i} P_{ij} + F$$

where F is a general effective divisor of degree $d - \sum_{i=1}^{n} m_i$ on $\mathbb{P}^1$. The divisor E has no common point with the general divisor of $g^1_\delta + D$. Hence E and $g^1_\delta + D$ span a g^2_d with no base points. Moreover, this g^2_d is birational. Indeed, if g^2_d were composed with a g^1_ν, then, by the generality of the divisor D, the g^1_δ would have base points, a contradiction.

Let γ be the image of $\mathbb{P}^1$ via the g^2_d. This is a rational plane curve of degree d, with a point of multiplicity m at a point y_{12} of $\mathbb{P}^2$. Moreover there are two lines ℓ_1, ℓ_2 passing through y_{12}, that each intersects γ at one point apart from y_{12}, call it x_1 and x_2 respectively, where the g^1_δ has total ramification. Finally there is a third line ℓ_3 that pulls back to $\mathbb{P}^1$ to the divisor E. By the choices we made, this line does not pass through x_1 and x_2 and the divisors $\sum_{j=1}^{m_i} P_{ij}$, for $i = 1, \ldots, n$, are contracted by the g^2_d to n distinct points on ℓ_3 that have multiplicities $m_1, \ldots, m_n$. The genericity of x_1, x_2 can be achieved by acting with projective transformations of the plane fixing the lines ℓ_1, ℓ_2, ℓ_3, which keep the points of multiplicities $m_1, \ldots, m_n$ pairwise distinct. □

Proof of Proposition 4.2. Let $d = \alpha + \beta$, $m_i = \gamma_i$ and $m = \beta$. Consider the plane $\mathbb{P}$ containing the curve γ constructed in Lemma 4.3. Let us blow up the points y_{12}, y_{13}, y_{23} and the n points of multiplicities $m_1, \ldots, m_n$ on γ along ℓ_3.

We will consider the family $\mathcal{W}$ of the surfaces W' as above where the points $y_1, \ldots, y_n$ are no longer general but simply pairwise distinct. We call $b > 0$ the dimension of the parameter space of this family. There is a line bundle $\mathcal{M}$ on $\mathcal{W}$ that restricts on each member of $\mathcal{W}$ to the line bundle M as in the statement of the proposition. Accordingly we can consider the families $\mathcal{V}_{\mathcal{M}}$ and $\mathcal{V}^*_{\mathcal{M}}$ of all varieties V_M and V^*_M as before.

The blow–up at the beginning of the proof can be interpreted as a member W'_0 of $\mathcal{W}$ with $\omega(x_1) = x_2$, since there is a unique irreducible conic Γ tangent to the lines ℓ_1, ℓ_2 at y_{13} and y_{23} respectively, and tangent also to the line joining the two points x_1 and x_2 (cf. Remark 3.2). We denote by M_0 the restriction of $\mathcal{M}$ to W'_0.

Lemma 4.3 implies that V_{M_0} is non–empty, which by Lemma 4.1 implies in turn that $V^*_{M_0}$ is non–empty, with all components of the expected dimension $\alpha + 2\beta - \sum \gamma_i - 1$. This yields that $\mathcal{V}^*_{\mathcal{M}}$ is non–empty. Then its dimension is at least the expected dimension which is $\alpha + 2\beta - \sum \gamma_i - 1 + b$ that is strictly larger than the dimension of $V^*_{M_0}$. This implies that for W' general in $\mathcal{W}$ the variety V^*_M is non–empty of the expected dimension $\alpha + 2\beta - \sum \gamma_i - 1$, as wanted. □

5. Proof of Proposition 2.3

Let us go back to R and $\widetilde{R} = \mathrm{Bl}_{y_1,\ldots,y_n}(R)$, where $y_1, \ldots, y_n$ are general points on T, with exceptional divisors $\mathfrak{e}_i$ over y_i, for $i = 1, \ldots, n$.

Proof of Proposition 2.3. As we already noted, condition (iv) in Definition 2.1 of $(\star)$ is equivalent to $-K_{\widetilde{R}} \cdot L \geqslant 4$. Hence, as remarked for Theorem 1.1 in the introduction, the statements about dimension and smoothness follow from Proposition 2.2 once nonemptiness is proved. So it remains to prove nonemptiness.

We prove the result by degeneration of $\widetilde{R}$ to Y', as indicated in §4, from which we keep the notation.

On the surface W' consider

$$L_0 \sim \alpha h + \beta(h - s_0) - \sum \gamma_i e_{y_i}.$$

Denote by $\{L_0\}_{W'} \subset |L_0|$ the sublocus of ω–compatible curves.

Claim 5.0.1. $\dim(\{L_0\}_{W'}) = \dim(|L_0|) - \alpha = \frac{1}{2}\left(\alpha^2 + \alpha - \sum \gamma_i - \sum \gamma_i^2\right) + \beta(\alpha + 1)$.

Proof of claim. By Proposition 4.2, the linear system $|L_0|$ contains an irreducible curve. Since $h^1(\mathcal{O}_{W'}) = 0$, it therefore follows that

$$h^1(-L_0) = h^1(L_0 + K_{W'}) = 0. \tag{7}$$

Set $A := h_0 + e_1 + e_2 + \widetilde{X}_{P_1} + \widetilde{X}_{P_2} + s_0$. Then A is a reduced cycle of rational curves, thus of arithmetic genus one, and it is anticanonical by (6). Since $L_0 \cdot A \geqslant 4$ by condition (iv) of $(\star)$, it follows in particular that $|L_0 + A|$ contains a reduced, connected member. It therefore follows that

$$h^1(-L_0 - A) = h^1(L_0) = 0 \text{ and } h^0(-L_0 - A) = h^2(L_0) = 0. \tag{8}$$

In particular, using Riemann-Roch on W', one computes that

$$\dim(|L_0|) = \frac{1}{2}\left(\alpha^2 + 3\alpha - \sum \gamma_i - \sum \gamma_i^2\right) + \beta(\alpha + 1),$$

thus proving the right hand equality of the claim.

We have left to prove that $\{L_0\}_{W'}$ has codimension α in $|L_0|$.

To this end, let $Z_1 \in \mathrm{Sym}^\alpha(\widetilde{X}_{P_1})$ be general, and set $Z_2 := \omega(Z_1) \in \mathrm{Sym}^\alpha(\widetilde{X}_{P_2})$. From the two restriction sequences

$$0 \longrightarrow L_0 + K_{W'} \longrightarrow L_0 \longrightarrow L_0|_A \longrightarrow 0$$

and

$$0 \longrightarrow L_0 + K_{W'} \longrightarrow L_0 \otimes \mathcal{J}_{Z_1 \cup Z_2} \longrightarrow L_0|_A(-Z_1 - Z_2) \longrightarrow 0\ ,$$

together with (7), we see that

$$\operatorname{codim}\Big(\,|L_0 \otimes \mathcal{J}_{Z_1\cup Z_2}|\,,|L_0|\,\Big) = \operatorname{codim}\Big(\,|L_0|_A(-Z_1 - Z_2)|\,,|L_0|\,\Big).$$

A standard computation involving restriction sequences to the various components of A shows that the latter codimension is 2α. Therefore,

$$\dim\left(|L_0 \otimes \mathcal{J}_{Z_1\cup Z_2}|\right) = \dim(|L_0|) - 2\alpha.$$

(This equality can also be obtained applying [12, Thm. (1.4.0)].) Varying $Z_1 \in \mathrm{Sym}^\alpha(\widetilde{X}_{P_1})$, we obtain the whole of $\{L_0\}_{W'}$. Thus,

$$\dim(\{L_0\}_{W'}) = \dim\left(|L_0 \otimes \mathcal{J}_{Z_1\cup Z_2}|\right) + \dim(\mathrm{Sym}^\alpha(\widetilde{X}_{P_1})) = \dim(|L_0|) - \alpha,$$

finishing the proof of the claim. □

Denote by $\{L_0\}$ the locus of curves on $Y' = S' \cup \mathcal{E}$ of the form $\sigma'(C) \cup C_{\mathcal{E}}$, where C is an element of $\{L_0\}_{W'}$ and $C_{\mathcal{E}}$ is the union of fibers on $\mathcal{E} \simeq \mathbb{F}_1$ such that $C_{\mathcal{E}} \cap S' = \sigma'(C) \cap s_0$. Then there is a one-to-one correspondence between $\{L_0\}_{W'}$ and $\{L_0\}$. Thus, by the last claim,

$$\dim(\{L_0\}) = \frac{1}{2}\left(\alpha^2 + \alpha - \sum \gamma_i - \sum \gamma_i^2\right) + \beta(\alpha+1). \tag{9}$$

Note that all members of $\{L_0\}$ are Cartier divisors on Y'. Moreover, by Lemma 3.3, the closure of the locus $\{L_0\}$ is (a component of) the limit of the algebraic system $\{L\}$ on $\widetilde{R}$ of curves of class $\alpha\mathfrak{s} + \beta\mathfrak{f} - \sum \gamma_i \mathfrak{e}_i$. Since the anticanonical divisor on $\widetilde{R}$ is effective, we have $h^2(L) = h^0(K_{\widetilde{R}} - L) = 0$, whence Riemann-Roch yields

$$\begin{aligned}
\dim\{L\} &= \dim|L| + 1 = \chi(L) + h^1(L) \\
&= \frac{1}{2} L \cdot (L - K_{\widetilde{R}}) + h^1(L) \\
&= \frac{1}{2}\left(\alpha^2 + \alpha - \sum \gamma_i - \sum \gamma_i^2\right) + \beta(\alpha+1) + h^1(L).
\end{aligned}$$

By (9) and semicontinuity, we must have $h^1(L) = 0$ and

$$\dim(\{L_0\}) = \dim(\{L\}). \tag{10}$$

Let x_1 and x_2 be as in §4 and pick a general C in a component of $V^*_{L_0}$ in W' (which is non–empty by Proposition 4.2). Then C intersects s_0 transversely at $L_0 \cdot s_0 = \beta$ distinct points. Denote as above by $C_{\mathcal{E}}$ the union of the β fibers on $\mathcal{E}$ such that $C_{\mathcal{E}} \cap s_0 = \sigma'(C) \cap s_0$. Then $\sigma'(C) \cup C_{\mathcal{E}}$ is a member of $\{L_0\}$, with an α-tacnode at $\sigma'(x_1) = \sigma'(x_2)$, and nodal otherwise, stably equivalent to $\sigma'(C)$. Varying x_1, we obtain by Proposition 4.2 a family $\mathcal{C}$ of dimension $\alpha + 2\beta - \sum \gamma_i = -L \cdot K_{\widetilde{R}}$ of such curves, and this is the expected dimension of $V^{\xi}_{p_a(L)-1}(\widetilde{R})$.

Let δ_0 be the number of nodes of C, which equals the number of singular points of $\sigma'(C)$ on the smooth locus of Y'. Then

$$\delta_0 = p_a(L_0) = \frac{1}{2}\left(\alpha^2 - 3\alpha + \sum \gamma_i - \sum \gamma_i^2\right) + \beta(\alpha - 1) + 1.$$

Grant for the moment the following[1]:

Claim 5.0.2. *The family of curves in $\{L_0\}$ passing through the δ_0 nodes of $\sigma'(C)$ and having an $(\alpha-1)$–tacnode at $\sigma'(x_1) = \sigma'(x_2)$ has codimension $\delta_0 + \alpha - 1$ in $\{L_0\}$, that is, it has dimension $\alpha + 2\beta - \sum \gamma_i = \dim \mathcal{C}$.*

Arguing as in [15, Thm. 3.3, Cor. 3.12 and proof of Thm. 1.1][2], we may deform Y' to $\widetilde{R}$ deforming the α-tacnode of $\sigma'(C)$ to $\alpha - 1$ nodes, while preserving the δ_0 nodes

[1]From a deformation-theoretic point of view the claim asserts the smoothness of the *equisingular deformation locus* of $\sigma'(C) \cup C_{\mathcal{E}}$, which is a dense open subset of $\mathcal{C}$, cf. [15, Lemma 3.4].

[2]The setting in [15] is slightly different, as the central fibre in the degeneration is a transversal union of two smooth surfaces, whereas S' in the present setting is singular. Moreover, the central fiber in the degeneration in [15] is regular, whence linear and algebraic equivalence coincide, which is not the case on Y'. However, the reasoning in [15] is local, so the proof goes through in the present setting as well. The two hypotheses (1)-(2) in [15, Thm. 3.3] correspond, respectively, to (10) and the statement in Claim 5.0.2.

and smoothing the nodes $\sigma'(C) \cap C_{\mathcal{E}}$. Thus $\sigma'(C) \cup C_{\mathcal{E}}$ deforms to a curve algebraically equivalent to L with δ nodes, where

$$\delta = \delta_0 + \alpha - 1 = \frac{1}{2}\left(\alpha^2 - \alpha + \sum \gamma_i - \sum \gamma_i^2\right) + \beta(\alpha - 1).$$

One computes

$$p_a(L) = \frac{1}{2}\left(L^2 + L \cdot K_{\widetilde{R}}\right) + 1 = \delta + 1.$$

This shows that C has geometric genus one, as wanted.

We have left to prove the claim.

Proof of Claim 5.0.2. Let $\mathcal{F}$ be the family of curves in $\{L_0\}_{W'}$ passing through the δ_0 nodes of C and being tangent to $\widetilde{X}_{P_i}$ at x_i with order $\alpha - 1$, for $i = 1, 2$. The statement of the claim is equivalent to $\dim \mathcal{F} = \alpha + 2\beta - \sum \gamma_i$.

Denoting by N the scheme of the δ_0 nodes of C and by $Z_i = (\alpha-1)x_i$ the subscheme on $\widetilde{X}_i$, whence on W', we have that $\mathcal{F}$ is the locus of ω-compatible curves in $|L_0 \otimes \mathcal{J}_{N \cup Z_1 \cup Z_2}|$, which has codimension one, as $L_0 \cdot \widetilde{X}_i = \alpha$. Thus,

$$\dim(\mathcal{F}) = \dim(|L_0 \otimes \mathcal{J}_{N \cup Z_1 \cup Z_2}|) - 1.$$

To compute this, let $q : W'' \to W'$ denote the blow–up of W' at N and denote the total exceptional divisor by E. Denote the inverse images of Z_i by the same names. Then

$$\dim(\mathcal{F}) = \dim\left(|(q^*L_0 - E) \otimes \mathcal{J}_{Z_1 \cup Z_2}|\right) - 1. \tag{11}$$

Let $\widetilde{C}$ be the strict transform of C on W'', which is a smooth rational curve. Then $\widetilde{C} \sim q^*L_0 - 2E$. We therefore have a short exact sequence

$$0 \longrightarrow \mathcal{O}_{W''}(E) \longrightarrow (q^*L_0 - E) \otimes \mathcal{J}_{Z_1 \cup Z_2} \longrightarrow \mathcal{O}_{\widetilde{C}}(q^*L_0 - E)(-(\alpha-1)(x_1 + x_2)) \longrightarrow 0,$$

whence, from (11) we have

$$\begin{aligned}
\dim(\mathcal{F}) &= h^0((q^*L_0 - E) \otimes \mathcal{J}_{Z_1 \cup Z_2}) - 2 \\
&= h^0(\mathcal{O}_{\widetilde{C}}(q^*L_0 - E)(-(\alpha-1)(x_1 + x_2))) - 1 \\
&= \deg(\mathcal{O}_{\widetilde{C}}(q^*L_0 - E)(-(\alpha-1)(x_1 + x_2))) \\
&= (q^*L_0 - 2E)(q^*L_0 - E) - 2(\alpha - 1) \\
&= L_0^2 - 2\delta_0 - 2\alpha + 2 \\
&= \alpha + 2\beta - \sum \gamma_i,
\end{aligned}$$

as desired. □

The proof of Proposition 2.3 is now complete. □

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