Enumerative geometry of $K3$ surfaces

Thomas Dedieu

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Introduction

The scope of these notes is to explain various enumerative results about $K3$ surfaces without assuming familiarity with Gromov–Witten theory; in fact, they represent an attempt on my part to understand what these results mean in classical terms.

The enumerative results in question are due to Beauville, Bryan and Leung, Pandharipande, Maulik, Thomas, and others, and confirm conjectures made by Yau–Zaslow, Göttsche, and Katz–Klemm–Vafa. They are listed in (0.1) below.

They fall in three categories : (i) some don't really need Gromov–Witten theory at all either to be formulated or to be proved; (ii) others may be formulated without Gromov–Witten theory but their proofs we know so far heavily rely on techniques from this theory; (iii) the remaining ones require an understanding of Gromov–Witten theory to be fully apreciated. It was therefore unavoidable to assume that the reader nevertheless has a minimal idea of what Gromov–Witten invariants are; it should be more than enough to know the relevant facts listed in (0.2) below.

(0.1) Contents description. In Section 1, I state a formula giving the number of rational curves in a primitive linear system on a $K3$ surface, and give its proof by Beauville using the universal compactified Jacobian, following the strategy suggested by Yau–Zaslow; this falls in category (i). I also give two geometric interpretations, due to Fantechi–Göttsche–van Straten of the multiplicity with which a given rational curve is counted, namely the topological Euler number of its compactified Jacobian.

This is generalized in Section 2 to a formula giving the number of genus g curves in a primitive linear system passing through g general points, which had been conjectured by Göttsche. I give an outline of its proof by degeneration to an elliptic $K3$ surface due to Bryan–Leung, as detailed as the scope of these notes and the ability of the author permit ; it requires the formulation of the result in terms of twisted Gromov–Witten invariants specifically designed for algebraic $K3$ surfaces (see subsection 2.1), and relies among other things on a multiple cover formula for nodal rational curves.

Essentially all remaining results fall in category (iii). The goal of Section 3 is to explain the extension of the Yau–Zaslow formula to non-primitive linear systems, which has been proven by Klemm–Maulik–Pandharipande–Scheidegger (this proof is streamlined in subsection 5.3). This features the Aspinwall–Morrison multiple cover formula, and its application to define corrected Gromov–Witten invariants known as BPS states numbers. I also discuss other degenerate contributions, striving to sort out the relation between the number given by the formula and the actual number of integral rational curves.

Section 4 is devoted to various generalizations. Special care is accorded to the close connection between Gromov–Witten integrals on $K3$ surfaces and curve counts on threefolds. For instance I discuss the Katz–Klemm–Vafa formula, proved by Pandharipande–Thomas, which has to be seen as computing, in any genus, the excess contribution of a $K3$ surface to the Gromov–Witten invariants of any fibered threefold in which it appears as a fibre.

Section 5 introduces Noether–Lefschetz numbers for families of lattice-polarized $K3$ surfaces, and states a result due to Maulik–Pandharipande which shows, on a threefold fibered in lattice-polarized $K3$ surfaces, how these Noether–Lefschetz numbers give an explicit relation between Gromov–Witten invariants of the threefold and of the $K3$ fibres. Eventually, I discuss the application of this formula to the proof of the Yau–Zaslow formula for non-primitive linear systems. It involves a mirror symmetry theorem that enables the computation of Gromov–Witten invariants of anticanonical sections of toric 4-manifolds, as well as modularity results for Noether–Lefschetz numbers following from the work of Borcherds and Kudla–Millson ; the latter enable the computation of all Noether–Lefschetz numbers of the family of lattice-polarized $K3$ surfaces considered in the proof.

(0.2) Gromov–Witten theory. Let X be a projective manifold, say. The starting idea of Gromov–Witten theory is to view genus g curves on X as *stable maps*, i.e. morphisms $f : C \to X$ where C is a connected nodal curve of arithmetic genus g such that there are only finitely many automorphisms ϕ of C satisfying the identity $f \circ \phi = f$. The latter condition is called the *stability condition*, and amounts to the requirement that each irreducible component of arithmetic genus 0 (resp. 1) of C which is contracted by f carries at least 3 (resp. 1) special points, i.e. either intersection points with other irreducible components of C or, if relevant, marked points. An integral embedded curve $C \subset X$ is then encoded as the map $f : \bar{C} \to X$ obtained by composing the normalization of C with its embedding in X.

The point in choosing this point of view is to compactify the space of curves on X, which is a prerequisite to the definition of well-formed invariants counting curves on X. There are of course other possible ways to do so ; they all come with some specific drawbacks, but this is inevitable. See the enlightening survey [28] for more on this question.

This being said, Gromov–Witten invariants are integrals (or intersection products if one prefers)

$$\int_{[\overline{M}_{g,k}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cup \ldots \cup \mathrm{ev}_k^*(\gamma_k), \tag{0.2.1}$$

where β is a homology class in $H_2(X, \mathbf{Z})$, $\overline{M}_{g,k}(X,\beta)$ is the moduli space of genus g stable maps $f : C \to X$ such that $[f_*(C)] = \beta$ with k marked points $x_1, \ldots, x_k \in C$, $\mathrm{ev}_i : \overline{M}_{g,k}(X,\beta) \to X$ is the evaluation at the i-th marked point sending $(f : C \to X, x_1, \ldots, x_k)$ to $f(x_i) \in X$ for $i = 1, \ldots, k$, and $\gamma_1, \ldots, \gamma_k$ are cohomology classes in $H^*(X, \mathbf{Z})$; the *virtual fundamental class* $[\overline{M}_{g,k}(X,\beta)]^{\mathrm{vir}}$ is a rational homology class in $H_{2\,\mathrm{vdim}}\big(\overline{M}_{g,k}(X,\beta), \mathbf{Q}\big)$ where vdim is the virtual (or expected if one prefers) dimension of $\overline{M}_{g,k}(X,\beta)$

$$\mathrm{vdim}\, \overline{M}_{g,k}(X,\beta) = (\dim X - 3)(1-g) - K_X \cdot \beta + k\,^{1}, \tag{0.2.2}$$

and the integral (0.2.1) is defined to be 0 if the degree of the integrand does not match the dimension of the virtual class. The virtual class is the usual fundamental class when the moduli space $\overline{M}_{g,k}(X,\beta)$ has the expected dimension ; otherwise it is given by an excess formula (it is the top Chern class of the obstruction bundle when $\overline{M}_{g,k}(X,\beta)$ is non-singular). Typically the cohomology classes $\gamma_1, \ldots, \gamma_k$ are the Poincaré duals to algebraic cycles $\Gamma_1, \ldots, \Gamma_k$ on X ; in this case, the condition that the degrees of $\gamma_1, \ldots, \gamma_k$ sum up to $2\,\mathrm{vdim}\, \overline{M}_{g,k}(X,\beta)$ is equivalent to the equality

$$\sum\nolimits_{i=1}^{k} \big(\mathrm{codim}_X(\Gamma_i) - 1\big) = \mathrm{vdim}\, \overline{M}_{g,0}(X,\beta),$$

which means that the incidence conditions imposed by $\Gamma_1, \ldots, \Gamma_k$ to genus g curves in the class β are expected to define a finite number of curves. Therefore, under suitable transversality assumptions, and provided the moduli space $\overline{M}_{g,k}(X,\beta)$ (or equivalently $\overline{M}_{g,0}(X,\beta)$) has the expected dimension, the Gromov–Witten invariant (0.2.1) gives the number of genus g curves in the class β (interpreted as stable maps, and counted with multiplicities) which pass through the cycles $\Gamma_1, \ldots, \Gamma_k$. We will be mainly concerned with the case when all Γ_i's are points, which is the only relevant case when X is a surface.

I refer to [43] for a short introduction to Gromov–Witten theory at the same level as the present set of notes.

(0.3) Terminology and conventions. We always work over the field of complex numbers.

Let C be a curve. Its arithmetic genus, denoted by $p_a(C)$, is the integer $1 - \chi(\mathcal{O}_C)$. If C is reduced, its geometric genus is the arithmetic genus of its normalization, and is denoted by $p_g(C)$. When I write 'genus', this means 'geometric genus'.

A reduced curve C is *immersed* when the differential of its normalization map is everywhere non-degenerate. Concretely this means that C has no cuspidal points ; it may have however points of any multiplicity, and non-ordinary singularities (e.g., a tacnode, i.e. a point at which there are two smooth local branches tangent one to another). A *node* is an ordinary double point.

A $K3$ surface S is a smooth surface with trivial canonical bundle and vanishing irregularity ; we may occasionally qualify as $K3$ a surface with canonical singularities, the minimal smooth model of which is a smooth $K3$ surface. Let p be a positive integer. A $K3$ surface of genus p is a pair (S, L), where S is a $K3$ surface and L an effective line bundle on S, such that $L^2 = 2p - 2$ (in particular, the $K3$ surface S is algebraic). Under these assumptions, the complete linear system $|L|$ has dimension p, and its general member is a smooth curve of genus p. The pair (S, L) is *primitive* if the line bundle L is indivisible, i.e. there is no line bundle L' on S such that $L \cong (L')^{\otimes m}$ for some integer $m > 1$.

In the notation of (0.2), we write $\overline{M}_g(X,\beta)$ for $\overline{M}_{g,0}(X,\beta)$. If S is a surface equipped with an effective line bundle $L \to S$, we write $\overline{M}_{g,k}(X, L)$ for $\overline{M}_{g,k}(X,\beta)$ where β is the homology class of the members of $|L|$.

1. if one can find a stable $f : C \to X$ corresponding to a point of $\overline{M}_{g,k}(X,\beta)$ such that f is unramified on a dense open subset of X, this may be computed as $\chi(N_f) + k$ where N_f is the normal sheaf of f, i.e. the cokernel of the injective map $T_C \to f^*T_X$; see [38, § 3.4.2] or [28, § $1\frac{1}{2}$] for how to do this in general.

(0.4) Let (S, L) be a $K3$ surface of genus p. Members of $|L|$ with exactly δ nodes as singularities have geometric genus $p - \delta$, and are expected to fill up a locus of codimension δ in $|L|$. For this reason (and because $|L|$ has dimension p), the locus of genus g curves in $|L|$ has expected dimension g; note that this does *not* match with the virtual dimension (0.2.2) of $\overline{M}_g(S, L)$ (see subsection 2.1). One can actually prove that this is indeed the correct dimension, and that the locus of genus g curves is equidimensional (see [11, § 4.2]). This implies that for a general set of g points $x_1, \ldots, x_g \in S$, there is a finite number of genus g curves in $|L|$ passing through all points $x_1, \ldots, x_g$.

(0.5) Acknowledgments. I thank Jim Bryan and Rahul Pandharipande for patiently answering my naive questions.

1 – Rational curves in a primitive class

In this Section we discuss the following result proved by Beauville [3], following a strategy proposed by Yau and Zaslow [45].

(1.1) Theorem. (Yau–Zaslow, Beauville) *Let (S, L) be a smooth primitive $K3$ surface of genus p_0, and assume that* Pic $S \cong \mathbf{Z} \cdot L$. *Then there is a finite number N^{p_0} of rational curves in the complete linear system $|L|$, and it is determined by the formula*

(1.1.1)
$$\begin{aligned} 1 + \sum_{p=1}^{+\infty} N^p q^p &= \prod_{n=1}^{+\infty} \frac{1}{(1-q^n)^{24}} \\ &= 1 + 24q + 324q^2 + 3200q^3 + \cdots \end{aligned}$$

Of course, N^p has to be understood as the number of rational curves counted with multiplicities for formula (1.1.1) to hold without any further genericity assumption. As we shall see, the multiplicity with which a given integral rational curve counts is the topological Euler number of its compactified Jacobian $e(\bar{J}C)$, which depends only on its singularities, and may be explicitly computed; it is 1 whenever the curve is immersed. For a very general (S, L), all rational curves in $|L|$ are actually nodal by [7], hence (1.1.1) holds without multiplicities.

The assumption about (S, L) that is really used in the proof is that *all* members of $|L|$ are integral curves. Although it may possible to drop the assumption that all members of $|L|$ are irreducible, it seems unavoidable to require that they are all reduced (see however Section 3 for some hints on how to handle this situation).

Theorem (1.1) is a particular case of the more general result that we treat in Section 2.2. We will recall there the relevant facts from the theory of modular forms needed to explore the modular aspects of formula (1.1.1), and give more values of N^p for small p.

The strategy of Yau–Zaslow was inspired by physics; it is an elaboration of the elementary argumentation using Euler numbers presented in Subsec. 1.1. The BPS state counts for Calabi-Yau 3-folds introduced by Gopakumar and Vafa are conjecturally computable in a similar way, see [28, Sec. $2\frac{1}{2}$] for an introduction. The corresponding invariants are considered in Section 3.

1.1 – An elementary topological counting formula

Let X be a complex variety, and $\mathfrak{f}$ a 1-dimensional family of divisors of X, the general member of which is smooth. It is possible to count the number of singular members of $\mathfrak{f}$ using the standard topological Lemma (1.3).

(1.2) Euler number. Let X be a topological space. Recall that the (topological) Euler number of X is

$$e(X) := \sum_i (-1)^i \dim \mathrm{H}^i(X, \mathbf{Z}),$$

where it is understood that the cohomology groups $\mathrm{H}^i(X, \mathbf{Z})$ should be replaced by the cohomology groups with compact support $\mathrm{H}^i_{\mathrm{c}}(X, \mathbf{Z})$ whenever X is not compact.

If $F \subset X$ is a closed subset, there is a long exact sequence

$$\cdots \to \mathrm{H}^i_{\mathrm{c}}(X - F, \mathbf{Z}) \to \mathrm{H}^i(X, \mathbf{Z}) \to \mathrm{H}^i(F, \mathbf{Z}) \to \mathrm{H}^{i+1}_{\mathrm{c}}(X - F, \mathbf{Z}) \to \cdots$$

which implies the additivity formula

$$e(X) = e(X - F) + e(F).$$

(1.3) Lemma. *Let $f : X \to C$ be a surjective morphism from a projective manifold onto a smooth curve. One has*

$$e(X) = e(F_{\mathrm{gen}})\, e(C) + \sum_{y \in \mathrm{Disc}\, f} \big(e(F_y) - e(F_{\mathrm{gen}}) \big), \tag{1.3.1}$$

where F_{gen} and F_y respectively denote the fibres of f over the generic point of B and a closed point $y \in C$, and $\mathrm{Disc}\, f$ is the set of points above which f is not smooth.

This may be applied to the situation described in the introduction of this subsection by replacing X by its blow-up at the base points of the family $\mathfrak{f}$.

Proof. Set $U := X - \bigcup_{y \in \mathrm{Disc}\, f} F_y$. The map $f : U \to C - \mathrm{Disc}\, f$ is a topological fibre bundle, hence

$$e(U) = e(C - \mathrm{Disc}\, f)\, e(F_{\mathrm{gen}}).$$

The formula then follows by additivity of the Euler number. □

(1.4) When X is a surface and the schematic fibre over y is reduced, the difference $e(F_y) - e(F_{\mathrm{gen}})$ is determined by the singularities of F_y.

Let D be a reduced projective curve, Σ its singular locus, $\nu : \bar{D} \to D$ its normalization, and $\bar{\Sigma} = \nu^{-1}(\Sigma)$. By additivity of the Euler number, one has

$$\begin{aligned} e(D) &= e(D - \Sigma) + e(\Sigma) \\ &= e(\bar{D} - \bar{\Sigma}) + e(\bar{\Sigma}) + e(\Sigma) - e(\bar{\Sigma}) \\ &= e(\bar{D}) - \big(\mathrm{Card}(\bar{\Sigma}) - \mathrm{Card}(\Sigma) \big). \end{aligned}$$

Let $f : S \to C$ be a surjective morphism from a smooth projective surface to a smooth curve, and consider a point $y \in C$ such that the schematic fibre F_y is reduced. Then the curve F_y has the same arithmetic genus as the general fibre F_{gen}, hence

$$e(\bar{F}_y) = e(F_{\mathrm{gen}}) + 2\delta,$$

where $\delta = p_a(F_y) - p_g(F_y)$ is the sum of the δ-invariants of all singularities of F_y, and

$$e(F_y) - e(F_{\text{gen}}) = 2\delta - \big(\operatorname{Card}(\bar{\Sigma}_y) - \operatorname{Card}(\Sigma_y)\big).$$

This proves the following.

(1.4.1) Lemma. *In the above notation, the multiplicity with which the fibre F_y is counted in formula* (1.3.1) *is a sum of local multiplicities computed at the singular points of F_y, namely*

$$e(F_y) - e(F_{\text{gen}}) = \sum_{x \in \operatorname{Sing} F_y} \left(\delta(F_y, x) - \#\binom{\textit{local branches}}{\textit{of } F_y \textit{ at } x} + 1 \right).$$

This gives for example the local multiplicity 1 for a node, 2 for an ordinary cusp, 3 for a tacnode, and 4 for an ordinary triple point.

(1.5) Application to elliptically fibred $K3$ surfaces. Let (S, L) be a primitive $K3$ surface of genus 1. Then $|L|$ is a base-point-free pencil of elliptic curves, and all its members are reduced since L is not divisible. Since S is a $K3$ surface, one has $e(S) = 24$; the Euler number of a smooth elliptic curve being 0, it follows from formula (1.3.1) that $|L|$ has 24 singular members, counted with the multiplicity given in Lemma (1.4.1). This agrees with Theorem (1.1).

1.2 – Proof of the Beauville–Yau–Zaslow formula

Let p be a positive integer, (S, L) a smooth primitive $K3$ surface of genus p, and call $\mathfrak{L}$ the complete linear system $|L|$. We assume that all members of $\mathfrak{L}$ are integral.

The relevant feature of the map $S \to \mathbf{P}^1$ considered in (1.5) is that its generic fibre is a complex torus, hence the only fibres with non-vanishing Euler number are those corresponding to a rational curve in the pencil. We let $\mathcal{C}$ be the universal curve over $\mathfrak{L}$, and consider

$$\pi : \bar{\mathcal{J}}^p\mathcal{C} \to \mathfrak{L}$$

the component of the compactified Picard scheme of the family $\mathcal{C} \to \mathfrak{L}$ parametrizing pairs (C, M) where C is any member of $\mathfrak{L}$ and M is a rank 1, torsion-free coherent sheaf of degree p on C. The total space $\bar{\mathcal{J}}^p\mathcal{C}$ is a projective variety of dimension $2p$.

(1.6) Beauville proves that the Euler number of a fibre $\pi^{-1}([C]) = \bar{J}^pC$ is zero if C is not rational, and positive if C is rational (see Propositions (1.8) and (1.11) below). Let us now explain how this shows that the Euler number $e(\bar{\mathcal{J}}^p\mathcal{C})$ is the number of rational curves in $\mathfrak{L}$ counted with multiplicities.

This is basically an elaboration of the proof of Lemma (1.3). There exists a stratification $\mathfrak{L} = \coprod_\alpha \Sigma_\alpha$ by locally closed subsets such that π is locally trivial above each stratum Σ_α [42]. For each α one has

$$e\big(\pi^{-1}(\Sigma_\alpha)\big) = e(\Sigma_\alpha) \times e(J_\alpha)$$

where J_α stands for the fibre of π over any point in the stratum Σ_α. Applying repeatedly the additivity of the Euler number, one then gets

$$e(\bar{\mathcal{J}}^p\mathcal{C}) = \sum_\alpha e\big(\pi^{-1}(\Sigma_\alpha)\big).$$

Eventually, the fact that $e(\bar{J}^pC) = 0$ if the curve C is not rational implies that

(1.6.1) $$e(\bar{\mathcal{J}}^p\mathcal{C}) = \sum_{[C] \in \mathfrak{L}_{\text{rat}}} e(\bar{J}^pC),$$

where $\mathfrak{L}_{\mathrm{rat}}$ is the union of those strata Σ_α, the points of which correspond to rational curves; it is necessarily a finite set (see, e.g., [11, Prop. (4.7)]).

Equation (1.6.1) says that $e(\bar{\mathcal{J}}^p\mathcal{C})$ is the number of rational curves in $\mathfrak{L}$, each rational curve C being counted with the multipicity $e(\bar{J}^pC)$ which is a positive integer; this proves the claim made at the beginning of the paragraph.

(1.7) On the other hand it is possible to identify the Euler number of $\bar{\mathcal{J}}^p\mathcal{C}$ with the knowledge at our disposal, and this together with (1.6) ends the proof of Theorem (1.1).

First, as noted in [27, Example 0.5] $\bar{\mathcal{J}}^p\mathcal{C}$ is a connected component of the moduli space of simple sheaves on the $K3$ surface S, and this shows that it is actually smooth and Hyperkähler.

Next, one proves as follows that $\bar{\mathcal{J}}^p\mathcal{C}$ is birational to $S^{[p]}$, the component of the Hilbert scheme of S parametrizing 0-dimensional subschemes of length p, which as well is a smooth Hyperkähler variety. There is an open subset $U \subseteq \bar{\mathcal{J}}^p\mathcal{C}$ whose points are pairs (C, M) with C a smooth curve and M a non-special line bundle. For such a pair one has $h^0(C, M) = 1$, and this associates to (S, M) the unique divisor D in the complete linear system $|M|$, which has degree p hence may be seen as a point of $S^{[p]}$. On the other hand, the fact that $h^0(C, \mathcal{O}_C(D)) = 1$ implies that D imposes p independent linear conditions to $\mathfrak{L}$, or in other words that C is the unique member of $\mathfrak{L}$ that contains D : this shows that the mapping $(C, M) \mapsto D$ is $1:1$, and ends the proof.

One concludes that $\bar{\mathcal{J}}^p$ and $S^{[p]}$ are actually deformation equivalent by a theorem of Huybrechts [19, p. 65], hence share the same Betti numbers, so that $e(\bar{\mathcal{J}}^p) = e(S^{[p]})$. This finally proves as required that $e(\bar{\mathcal{J}}^p)$ is the coefficient of q^p in the Fourier expansion of $\prod_{n=1}^{\infty}(1-q^n)^{-24}$ as in (1.1.1), thanks to the computation by Göttsche of the Betti and Euler numbers of $S^{[p]}$ for any complex smooth projective surface [17]. The latter computation is based on the by now rather widespread yet wonderful idea of using the Weil conjectures (proved by Deligne) to translate this into the problem of counting the points of $S^{[p]}$ over finite fields.

1.3 – Compactified Jacobian of an integral curve.

Let C be an integral curve. We now turn to the study of the Euler number of the compactified Jacobian $\bar{J}^dC$ of rank one torsion free coherent sheaves of degree d on C. This is required for Beauville's proof of the Yau–Zaslow formula, displayed in Subsection 1.2 above; in particular, we shall justify the assertions at the beginning of (1.6).

As is well-known, the choice of an invertible sheaf of degree d on C induces an isomorphism between $\bar{J}^dC$ and $\bar{J}^0C =: \bar{J}C$, so we will restrict our attention to the latter variety.

(1.8) Proposition. *If C is an integral curve of positive geometric genus, then $e(\bar{J}C) = 0$.*

Proof. There is an exact sequence

$$0 \to H \to JC \to J\tilde{C} \to 0$$

where $\tilde{C}$ is the normalization of C [1], J denotes the Jacobian Pic^0, and H is a product of copies of $(\mathbf{C}, +)$ and $(\mathbf{C}^*, \times)$ (this is standard; see, e.g., [22, Thm. 7.5.19]). It splits as an exact sequence of Abelian groups since H is divisible, so we may find for every positive integer n a subgroup $G_n < JC$ of order n that injects in $J\tilde{C}$ (here we use the fact that $J\tilde{C}$ is not trivial, given by the assumption on the geometric genus of C).

Then [3, Lem. 2.1] tells us that for all $M \in G_n < JC$ and $\mathcal{F} \in \bar{J}C$ the two sheaves $\mathcal{F}$ and $\mathcal{F} \otimes M$ are *not* isomorphic. Thus G_n acts freely on $\bar{J}C$, which implies that n divides $e(\bar{J}C)$.

1. exceptionally, I do not use the notation $\bar{C}$ in order to avoid unpleasant confusions between $\bar{J}C$ and $J\bar{C}$.

This being true for any n, we conclude that $e(\bar{J}C) = 0$. □

We need the following local construction (cf. [3, §3.6] and the references therein) in order to explicit $e(\bar{J}C)$ for a rational curve C.

(1.9) Let (C, x) be a germ of curve which we assume to be unibranch (i.e. C is analitically locally irreducible at x), and $\tilde{C}$ the normalization of C ; there is only one point in the preimage of x, which we also call x. Set $\delta_x := \dim \mathcal{O}_{\tilde{C},x}/\mathcal{O}_{C,x}$ (this is the number by which a singularity equivalent to (C, x) makes the geometric genus drop), and $\mathbf{c}'_x$ the ideal $\mathcal{O}_{\tilde{C}}(-2\delta \cdot x)$ [2]. We then consider the two finite-dimensional algebras $A_x := \mathcal{O}_{C,x}/\mathbf{c}'$ and $\tilde{A}_x := \mathcal{O}_{\tilde{C},x}/\mathbf{c}'$.

Eventually, let $\mathbf{G}_x$ be the closed subvariety of the Grassmannian $\mathbf{G}(\delta_x, \tilde{A}_x)$ parametrizing codimension δ_x subspaces of $\tilde{A}_x$ with the additional property of being sub-A_x-modules of $\tilde{A}_x$; it may also be seen as the variety parametrizing codimension δ_x sub-$\mathcal{O}_{C,x}$-modules of $\mathcal{O}_{\tilde{C},x}$. It only depends on the completion $\hat{\mathcal{O}}_{C,x}$, hence only on the analytic type of the singularity (C, x).

(1.10) Let C be a curve. It is *unibranch* if its normalization is a homeomorphism, or equivalently if it is everywhere analytically locally irreducible. Any curve C has a “unibranchization” $\breve{\nu} : \breve{C} \to C$, i.e. there is a unique such partial normalization such that any other partial normalization $\nu' : C' \to C$ with C' unibranch factors through $\breve{\nu}$.

If C is a unibranch curve with singular locus $\Sigma \subset C$, the product $\prod_{x\in\Sigma} \mathbf{G}_x$ parametrizes sub-$\mathcal{O}_C$-modules $\mathcal{F} \subset \mathcal{O}_{\tilde{C}}$ such that $\dim \mathcal{O}_{\tilde{C},x}/\mathcal{F}_x = \delta_x$ for all x. Such an $\mathcal{F}$ enjoys the property that $\chi(\mathcal{F}) = \chi(\mathcal{O}_C)$, which implies $\mathcal{F} \in \bar{J}C$. This defines a morphism

$$\varepsilon : \textstyle\prod_{x\in\Sigma} \mathbf{G}_x \to \bar{J}C.$$

(1.11) Proposition. (i) [3, Prop. 3.3] *If C is an integral curve, then $e(\bar{J}C) = e(\bar{J}\breve{C})$.*
(ii) [3, Prop. 3.8] *If C is a unibranch rational curve with singular locus Σ, then $e(\bar{J}C) = \prod_{x\in\Sigma} e(\mathbf{G}_x)$.*

If C is not integral, it is certainly not true that $e(\bar{J}C) = e(\bar{J}\breve{C})$. Part (ii) in the above statement is proved by showing that the morphism $\varepsilon : \prod_{x\in\Sigma} \mathbf{G}_x \to \bar{J}C$ is a homeomorphism if C is rational, though in general not an isomorphism. Note that since $\mathbf{G}_x$ is a point when (C, x) is a smooth curve germ, one has $\prod_{x\in\Sigma} e(\mathbf{G}_x) = \prod_{x\in C} e(\mathbf{G}_x)$.

As a consequence of (i), one sees that $e(\bar{J}C) = 1$ for an immersed rational curve C. Part (ii) on the other hand shows that, for any rational curve C (unibranch or not, thanks to (i)), $e(\bar{J}C)$ only depends on the singularities of C. The fact that $e(\bar{J}C) > 0$ for any rational curve C is best seen as an immediate consequence of (1.15). Note moreover that whenever C has only planar singularities (a condition which obviously holds when C is contained in a surface), the satisfactory fact that $e(\bar{J}C)$ actually only depends on the topological type of the singularities of C has been proven by Maulik [24] (see (1.16) below).

(1.12) Examples. [3, § 4] If (C, x) is the germ of curve given by the equation $u^p + v^q = 0$ at the origin in the affine plane, with p and q relatively prime, then

$$e(\mathbf{G}_x) = \frac{1}{p+q}\binom{p+q}{p}. \qquad (1.12.1)$$

2. we reserve the notation $\mathbf{c}_x$ for the conductor ideal, which contains $\mathbf{c}'_x$.

This is particularly meaningful if one takes into account the constancy of $e(\mathbf{G}_x)$ in topological equivalence classes of planar singularities. As a particular case, one gets $e(\mathbf{G}_x) = \ell + 1$ for (C, x) the cuspidal singularity defined by the equation $u^2 + v^{2\ell+1} = 0$.

Using the fact (Proposition (1.11), (ii)) that the local contribution $e(\mathbf{G}_x)$ of a germ (C, x) is the product of the local contributions of all local irreducible branches of (C, x), (1.12.1) is enough to determine the local contribution of any simple curve singularity, see [3, Prop. 4.5].

(1.13) Remark. The fact that any immersed rational curve counts with multiplicity 1 seems to disagree with the results of subsection. 1.1, see in particular Lemma (1.4.1). However if $|F|$ is a complete pencil of elliptic curves, the assumption that all curves in $|F|$ are integral readily implies that all rational curves in $|F|$ are curves of arithmetic genus 1 with either a node or an ordinary cusp as their unique singular point, in which cases the two multiplicities agree.

On the other hand, if $|F|$ has non-integral members, then Proposition (1.11) does not hold for them. Assume for instance there is a member C of $|F|$ that splits as a cycle of two rational curves, i.e. C is a degenerate fibre of Kodaira type I_2. Then there are two distinct partial normalizations of C with arithmetic genus 0, so from the point of view of stable maps — which seems to be the appropriate one, see (1.17.1) and Lemma (2.4) —, the curve C should count with multiplicity 2, in agreement with Lemma (1.4.1).

(1.14) Remark. Returning to the case of a p-dimensional linear system $\mathfrak{L}$ with all members integral, Beauville makes a remark similar to (1.13), deeming “rather surprising” the fact that “some highly singular [immersed] curves count with multiplicity one”, and considers the case $p = 2$ to provide a confirming example. I shall add some more details about this example in (1.14.2) below.

A good conceptual explanation of this fact is, as we have already mentioned, that the numbers N^p should be seen as counting stable maps rather than embedded curves, and stable maps don’t make any difference between nodal and arbitrary immersed curves. Yet this does not give a satisfactory “embedded” explanation. I propose a particular instance of such an explanation in (1.14.1) below; ultimately, it relies on the smoothness of the equigeneric deformation space of an immersed singularity (see (1.18) in the next subsection).

(1.14.1) Let S be a non-degenerate surface in $\mathbf{P}^3$. The linear system $|L| := |\mathcal{O}_S(1)|$ identifies with the dual projective space $\check{\mathbf{P}}^3$, the locus of singular curves in $|L|$ with the dual surface $\check{S} \subset \check{\mathbf{P}}^3$ (which by definition parametrizes hyperplanes in $\mathbf{P}^3$ tangent to S), and the closure of the locus of 2-nodal (resp. 1-cuspidal) curves with the ordinary double curve D_b (resp. cuspidal double curve D_c) of $\check{S}$.

Of course, the $K3$ surfaces in $\mathbf{P}^3$ are quartic hypersurfaces, and their hyperplane sections have arithmetic genus 3, so that rational curves among them are expected to be 3-nodal (at any rate, they have δ-invariant 3). Still, I shall discuss the geometry of the locus of 2-nodal curves, as it gives in my opinion a clearer picture of what is going on.

It is classically known [35, § 612], see [33, 34] for more up-to-date treatments[3], that the locus of tacnodal curves in $|L|$ consists of those intersection points of D_b and D_c at which D_b is smooth and D_c has a cuspidal point. This implies that 1-tacnodal curves, as they correspond to simple points of D_b, count for one co-genus 2 curve as do ordinary 2-nodal curves; they count however for two cuspidal curves.

The local description of the dual $\check{S}$ at a tacnodal curve reflects the geometry of various strata in the semi-universal deformation space of a tacnode. One may obtain with the same ingredients a local description of $\check{S}$ around a point corresponding to an immersed rational curve, e.g., a curve

3. beware that in [34, p. 391] the geometries of $T_xS \cap S$ in cases d) and e) have been mistakenly exchanged.

with one tacnode and one node, or one oscnode, and it would confirm that it counts for one rational curve only. I will not undertake this here.

(1.14.2) Let (S, L) be a general $K3$ surface of genus $p = 2$; then S is a double covering of the plane ramified over a smooth sextic curve B, and the members of $|L|$ are the pull-back of lines. Rational, 2-nodal, curves correspond to bitangent lines of B.

When B is Plücker general, i.e. when its dual curve $\check{B}$ has only nodes and cusps as singularities, the number of bitangents to B may be computed using the Plücker formulæ. It will be useful to unfold this explicitely, in order to handle more special cases later on. The dual curve $\check{B}$ as degree $6 \times (6-1) = 30$, and its cusps correspond to the inflection points of B; the latter are the intersection points of B with its Hessian hypersurface, which has degree $3 \times (6-2) = 12$; it follows that $\check{B}$ has $\check{\kappa} = 72$ cusps. The number $\check{\delta}$ of nodes of $\check{B}$ may then be derived arguing that the geometric genus of $\check{B}$ equals that of B, which is 10. This gives $\check{\delta} = p_a(\check{B}) - 10 - 72 = 324$, in accord with (1.1.1).

Now assume that B has a hyperflex o of order 4, i.e. the tangent line $\mathbf{T}_{B,o}$ has contact of order 4 with B at o; the pull-back of this line to S is an immersed rational curve, with one ordinary tacnode as only singularity. I shall now explain why it counts as one ordinary rational curve only. A local computation shows that the hyperflex o corresponds to a singularity on $\check{B}$ of the kind $y^4 = x^3$ at the point $\check{o} := (\mathbf{T}_{B,o})^{\perp}$. Such a singularity has δ-invariant 3, i.e. it makes the genus of $\check{B}$ drop by 3 with respect to the arithmetic genus $p_a(\check{B})$. On the other hand, B has a contact of order 2 with its Hessian at o, so it amounts for two ordinary flexes, and correspondingly $\check{o}$ amounts for two cusps of $\check{B}$. The fact that the δ-invariant of $(\check{B}, \check{o})$ be 3 then implies that $\check{o}$ amounts for one node of $\check{B}$, and correspondingly the line $\mathbf{T}_{B,o}$ amounts for one bitangent only, hence the pull-back of $\mathbf{T}_{B,o}$ amounts for one rational curve only. To sum up, the tangent line $\mathbf{T}_{B,o}$ amounts at the same time for one bitangent and two flex tangents, similar to what happened in (1.14.1).

In the next subsection we will see two results of Fantechi, Göttsche, and van Straten which extend and confirm the considerations of Remark (1.14) above.

1.4 – Two fundamental interpretations of the multiplicity

In this last subsection, I state two enlightening geometric interpretations of the local multiplicities $e(\mathbf{G}_x)$ defined in the previous subsection 1.3, and their global counterpart the product $\prod_{x \in C} e(\mathbf{G}_x)$. They have been obtained by Fantechi, Göttsche and van Straten [12].

(1.15) Theorem. [12, Thm. 1] *Let (C, x) be a reduced plane curve singularity, and $\mathbf{G}_x$ be as in (1.9). Then the topological Euler number $e(\mathbf{G}_x)$ equals the multiplicity at the point $[(C, x)]$ of the equigeneric locus $\mathrm{EG}(C, x)$ in the semi-universal deformation space of the singularity (C, x).*

Recall that the *equigeneric locus* $\mathrm{EG}(C, x)$ is defined as the reduced subscheme of the semi-universal deformation space of the singularity (C, x) supported on those points corresponding to singularities with the same δ-invariant as (C, x); see, e.g., [37] for more details.

(1.16) Together with Proposition (1.11), this implies that for C an integral rational curve with only planar singularities, the topological Euler number $e(\bar{J}C)$ of the compactified Jacobian of C equals the multiplicity at the point $[C]$ of the equigeneric stratum $\mathrm{EG}(C)$ in a semi-universal deformation space of C. The latter result has been subsequently generalized by Shende [40] to (the closures of) all δ-constant strata in the semi-universal deformation space of C.

Given a reduced plane curve singularity (C, x), there exists a rational curve $\tilde{C}$ with (C, x) as its only singularity (this follows for instance from [26]), and one then has $\mathbf{G}_x \cong \bar{J}\tilde{C}$. This, in conjunction with Maulik's main theorem in [24] gives the aforementioned constancy of the invariant $e(\mathbf{G}_x)$ on topological equivalence classes of plane curve singularities. Similarly, Shende and Maulik results together give the constancy on topological equivalence classes of the multiplicities at $[(C, x)]$ of (the closures of) all δ-constant strata in the semi-universal deformation space of (C, x) (see [24, § 6.5]).

Let C be an integral curve of geometric genus g. Recall that $\overline{M}_g(C, [C])$ is the space of genus g stable maps with target C and realizing the class $[C] \in \mathrm{H}_2(C, \mathbf{Z})$. This is a 0-dimensional scheme, which contains a single closed point, corresponding to the normalization $[\nu : \bar{C} \to C]$ of C.

(1.17) Theorem. [12, Thm. 2] *In the above notation, assume the curve C has only planar singularities. Then the length of the 0-dimensional scheme $\overline{M}_g(C, [C])$ equals the multiplicity at $[C]$ of the semi-universal deformation space of the curve C.*

Together with Theorem (1.15) above, this implies that the length of $\overline{M}_g(C, [C])$ equals $\prod_{x \in C} e(\mathbf{G}_x)$.

When C is rational, this is precisely $e(\bar{J}C)$. It is thus tempting to interpret Theorem (1.17) as telling us that what the Yau–Zaslow formula (1.1.1) really computes are the numbers of genus 0 stable maps realizing primitive classes on $K3$ surfaces.

Actually, if C is an isolated genus g curve in a smooth manifold X, then $\overline{M}_g(C, [C])$ is a subscheme of $\overline{M}(X, [C])$ and the length of the former scheme is a lower bound for the length of the latter at the normalization of C. For rational curves on $K3$ surfaces, Fantechi, Göttsche and van Straten show that this is in fact an equality.

(1.17.1) [12, Thm. 2] *Let C be an integral rational curve contained in a smooth $K3$ surface S. Then the topological Euler number $e(\bar{J}C)$ equals the length of the space of stable maps $\overline{M}_0(S, [C])$ at the closed point corresponding to the normalization of C.*

Lemma (2.4) in the next Section somehow deals with the same question for curves of any genus on a $K3$ surface. We refer to [11, § 2.2] for a general analysis, given an integral curve C on a smooth surface S, of the local relationship between $\overline{M}_g(S, [C])$ and the Severi variety of equigeneric deformations of C in S.

(1.18) As a corollary of Theorem (1.15) and Proposition (1.11), one obtains that if (C, x) is an immersed planar curve singularity, then the equigeneric locus $\mathrm{EG}(C, x)$ in the space of semi-universal deformations is smooth at the point $[(C, x)]$.

Certainly, this is merely a baroque way to prove a result otherwise accessible by a more straightforward argument. Still, I don't know wether the converse holds.

(1.18.1) Question. *Let (C, x) be a unibranch non-immersed planar curve singularity. Is it true that the equigeneric locus $\mathrm{EG}(C, x)$ is singular at the point $[(C, x)]$? equivalently, is it true that $e(\mathbf{G}_x) > 1$?*

I believe this is related to the question asked in [11, (3.16)] : let (C, x) be a non-immersed planar singularity ; is it true that the respective pull-back of the adjoint and equisingular ideals to the normalization $\bar{C}$ are different ?

2 – Curves of any genus in a primitive class

2.1 – Reduced Gromov–Witten theories for $K3$ surfaces

(2.1) A vanishing phenomenon. It happens that all Gromov–Witten invariants of $K3$ surfaces are trivial. The fundamental reason for this is that Gromov–Witten invariants are deformation invariant (and this is indeed a desirable feature of any well-behaved counting invariants), and there exist non-algebraic $K3$ surfaces, which in general do not contain any curve at all.

Somewhat more concretely, the explanation is that the virtual and the actual dimensions of the moduli spaces of stable maps on $K3$ surfaces do not match, as we already pointed out in (0.4). Let S be a $K3$ surface, and $C \subset S$ an integral curve of geometric genus g. Consider the stable map $f : \bar{C} \to S$ obtained by composing the normalization $\bar{C} \to C$ with the inclusion $C \subset S$. The normal sheaf N_f of f is isomorphic to the canonical bundle $\omega_{\bar{C}}$, and therefore $h^0(N_f) = g$ and $h^1(N_f) = 1$. It follows that the virtual dimension of $\overline{M}_g(S, [C])$ is $g - 1$, whereas the curve C actually moves in a g-dimensional family of curves of genus g on S (see [11, § 4.2] for more details). This implies that the Gromov–Witten invariants counting genus g curves on S passing through the appropriate number of points (namely g) vanishes for mere degree reasons.

The following two paths have been successfully followed to circumvent this phenomenon, and define modified invariants for algebraic $K3$ surfaces which capture the relevant enumerative information.

(2.2) Invariants of families of symplectic structures. [5, § 2–3] This has been chronologically the first workaround to be proposed, and enabled the counting of curves of any genus in a primitive class on a $K3$ surface reported on in subsection 2.2 below.

Let S be a polarized $K3$ surface. The idea here is really to take into account the existence of non-algebraic deformations of S. To this effect, instead of counting curves directly on S, one counts curves in a family of Kähler surfaces defined over the 2-sphere $\mathbf{S}^2$, canonically attached to S, and in which roughly speaking S is the only one to be algebraic, so that all the curves we count are actually concentrated on S.

This family of Kähler surfaces is the twistor family of S (cf. [36, p. 124]) : the polarization on S determines a Kähler class α, and Yau's celebrated theorem asserts that there is a unique Kählerian metric g in α with vanishing Ricci curvature. Then the holonomy defines an action of $\mathbf{H}$ (the field of quaternion numbers) on the holomorphic tangent bundle TS by parallel endomorphisms. The quaternions of square -1 define those complex structures on the differentiable manifold S for which the metric g remains Kählerian. There is a 2-sphere worth of such quaternions, and it parametrizes the family we are interested in.

(2.3) Reduced Gromov–Witten theory. [25, 2.2] (see also [23]). In this second approach, the idea is to plug in the fact that, for a stable map f as in (2.1), the space $\mathrm{H}^1(\bar{C}, N_f)$ although non-trivial does not contain any actual obstruction to deform f as a map with target S, following for instance Ran's results on deformation theory and the semiregularity map. To this end, Maulik and Pandharipande define a suitable perfect obstruction theory which they dub *reduced*, and which provides, following the construction pioneered by Behrend and Fantechi, a reduced virtual fundamental class $[\overline{M}_g(S, \beta)]^{\mathrm{red}}$ for all integers $g \geqslant 0$ and algebraic class $\beta \in \mathrm{H}_2(S, \mathbf{Z})$, which has the appropriate (real) dimension $2g$.

This in turn gives reduced Gromov–Witten invariants, by replacing the virtual fundamental class by its reduced version in the integral (0.2.1). They are invariant under *algebraic* deformations of $K3$ surfaces.

2.2 – The Göttsche–Bryan–Leung formula

In this subsection, I discuss a result of Bryan and Leung [5] giving the number of curves in a primitive linear system on a $K3$ surface that have a given genus and pass through the appropriate number of base points. The formula was conjectured by Göttsche [18] as a particular case of a more general framework.

Let

$$N_g^p := \int_{[\overline{M}_{g,g}(S,L)]^{\mathrm{red}}} \mathrm{ev}_1^*(\mathrm{pt}) \cup \cdots \cup \mathrm{ev}_g^*(\mathrm{pt})$$

be the reduced Gromov–Witten invariant counting curves of genus g in the linear system $|L|$ on a primitive $K3$ surface (S, L) of genus p ($\mathrm{pt} \in \mathrm{H}^4(S, \mathbf{Z})$ is the point class, and the $\mathrm{ev}_1, \ldots, \mathrm{ev}_g$ are the evaluation maps $\overline{M}_{g,g}(S, L) \to S$). The following result tells us that these invariants do indeed count curves.

(2.4) Lemma. *The invariants N_g^p are strongly enumerative, in the following sense : let (S, L) be a very general primitive $K3$ surface of genus p ; then N_g^p is the actual number of genus g curves in $|L|$ passing through a general set of g points, all counted with multiplicity* 1.

Proof. The key fact is that genus g curves on a $K3$ surface move in g-dimensional families, see e.g., [11, Prop. (4.7)] ; for $g > 0$, this implies by a deformation argument that the general member of such a family is an immersed curve [11, Prop. (4.8)]. The same holds for $g = 0$ by the more difficult result of Chen [7], which requires the generality assumption and asserts that all rational curves in $|L|$ are nodal.

Let $\mathbf{x} = (x_1, \ldots, x_g)$ be a general (ordered) set of g points on S. The invariant N_g^p may be computed by integration against a virtual class on the cut-down moduli space $\overline{M}(S, \mathbf{x}) \subset \overline{M}_{g,g}(S, L)$ consisting of those genus g stable maps sending the i-th marked point to $x_i \in S$ for $i = 1, \ldots, g$ [5, Appendix A].

Let $[f : C \to S] \in \overline{M}(S, \mathbf{x})$. Thanks to the generality assumption on (S, L), we may and will assume that L generates the Picard group of S. The condition $f_*C \in |L|$ thus imposes that f_*C is an integral cycle, hence that f contracts all irreducible components of C but one, and restricts to a birational map on the latter component, which we will call C_1. Now the points $x_1, \ldots, x_g$, being general, impose g general independent linear conditions on $|L|$, which implies that the curve $f(C) = f(C_1)$ must have geometric genus at least g by the key fact mentioned at the beginning of the proof. Therefore C_1 as well must have geometric genus at least g, and because of the inequality of arithmetic genera

$$p_a(C_1) \leqslant p_a(C) = g,$$

this implies that C_1 is smooth of genus g ; moreover, the stability conditions then imply that $C = C_1$, hence f is the normalization of the integral genus g curve $f(C_1)$.

This already tells us that the space $\overline{M}(S, \mathbf{x})$ is 0-dimensional, and isomorphic as a set to the space of genus g curves in $|L|$ passing through $x_1, \ldots, x_g$. Since $\overline{M}(S, \mathbf{x})$ has the expected dimension, the virtual class on it simply encodes its schematic structure. It follows that the number N_g^p is weakly enumerative, i.e. it gives the number of genus g curves passing through $x_1, \ldots, x_g$ counted with multiplicities.

The fact that these multiplicities all equal 1 follows from the fact that all the curves we consider are immersed, as follows. Let $[f : C \to S] \in \overline{M}(S, \mathbf{x})$ as above. Since f is an immersion, it has normal bundle

$$N_f = f^*\omega_S \otimes \omega_C = \omega_C$$

by [11, (2.3)], hence $\mathrm{h}^0(N_f) = g$ and the full moduli space $\overline{M}_{g,0}(S, L)$ is smooth of dimension g at the point $[f]$. By [11, Lemma (2.5)], there is a surjective map e from a neighbourhood of

$[f]$ in $\overline{M}_{g,0}(S,L)$ onto a neighbourhood of $f(C)$ in the locally closed subset of $|L|$ parametrizing genus g curves. By generality of the points $x_1,\ldots,x_g$, the latter space is smooth of dimension g at the point $[f(C)]$. Therefore e is a local isomorphism at $[f]$, and this implies that the scheme $\overline{M}(S,\mathbf{x})$ is reduced at $[f]$, which shows that the stable map f counts with multiplicity 1. □

For all integers $g \geqslant 0$, set

$$F_g(q) := \sum_{p=g}^{+\infty} N_g^p\, q^p \tag{2.5.1}$$

as a formal power series in the variable q, where we set N_0^0 by convention so that F_0 equals the power series of (1.1.1). Beware the shift in degree between the definition (2.5.1) of the series F_g and that given in [5]. Note that $N_g^p = 0$ whenever $p < g$.

(2.5) Theorem. (Bryan–Leung) *The power series F_g is the Fourier expansion of*

$$\left(\sum_{n=1}^{+\infty} n\sigma_1(n)\, q^n\right)^g \prod_{m=1}^{+\infty} \frac{1}{(1-q^m)^{24}}, \tag{2.5.2}$$

where $\sigma_1(n) := \sum_{d|n} d$ is the sum of all positive integer divisors of n.

This of course gives the possibility to explicitly compute as many numbers N_g^p as we want. Table 1 (p. 15) gives sample values for small p and g. Note that since columns are indexed by $\delta := p-g$, the Fourier coefficients of a given F_g are read along a diagonal; this gives for instance F_0 as in (1.1.1),

$$F_1(q) = q + 30q^2 + 480q^3 + 5460q^4 + \cdots$$

and so on.

We discuss the proof of Theorem (2.5) in subsection 2.3 below.

(2.6) Modularity. There is a modular form theoretic aspect to formula (2.5.2), which I explicitly state in subparagraph (2.6.5) below. There is somehow a meaning to this, but I will not try to discuss it here. I will however make a couple of points, at least to set things right and introduce notation for further use (I follow [39]).

(2.6.1) For every integer $k > 1$, define the k-th *Eisenstein series* to be

$$G_k(z) := \sum_{\substack{(m,n)\in\mathbf{Z}^2:\\ (m,n)\neq(0,0)}} \frac{1}{(m+nz)^{2k}};$$

it is a modular form of weight $2k$ [39, Prop. VII.4], which means that it is holomorphic and $G_k(z)dz^k$ is invariant under the action of $\mathrm{PSL}_2(\mathbf{Z})$. Its Fourier expansion at infinity is

$$G_k(z) = \frac{2^{2k}}{(2k)!} B_k \pi^{2k} + 2\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{+\infty} \sigma_{2k-1}(n) q^n,$$

where $q = e^{2\pi i z}$, $\sigma_k(n) = \sum_{d|n} d^k$, and B_k is the k-th Bernoulli number, defined by the formula

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{k=1}^{+\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}$$

p \ δ	1	2	3	4	5	6	7	8	9
1	24								
2	30	324							
3	36	480	3200						
4	42	672	5460	25650					
5	48	900	8728	49440	176256				
6	54	1164	13220	88830	378420	1073720			
7	60	1464	19152	150300	754992	2540160	5930496		
8	66	1800	26740	241626	1412676	5573456	15326880	30178575	
9	72	2172	36200	371880	2499648	11436560	36693360	84602400	143184000
10	78	2580	47748	551430	4213332	22116456	81993600	219548277	432841110
11		3024	61600	791940	6808176	40588544	172237344	531065070	1210781880
12			77972	1106370	10603428	71127680	342358560	1205336715	3154067950
13				1508976	15990912	119665872	647773200	2582847180	7698660544
14					23442804	194196632	1172896512	5255204625	17710394230
15						305225984	2041899840	10205262330	38607114200
16							3431986848	19002853575	80149394030
17								34070137272	159184435520
18									303705014550

Table 1 – First values of N_g^p ($\delta = p - g$)

[39, Prop. VII.8]. We set

$$E_k(z) := \frac{(2k)!}{2^{2k}B_k\pi^{2k}}G_k(z) = 1 + (-1)^k\frac{4k}{B_k}\sum_{n=1}^{+\infty}\sigma_{2k-1}(n)q^n$$
$$= 1 + (-1)^k\frac{4k}{B_k}\sum_{n=1}^{+\infty}n^{2k-1}\frac{q^n}{1-q^n}.$$

(2.6.2) Define

$$\Delta(z) := \big(60G_2(z)\big)^3 - 27\big(140G_3(z)\big)^2,$$

the discriminant of the cubic polynomial $4X^3 - 60G_2X - 140G_3$ divided by 16. It is a modular form of weight 12 vanishing at infinity, and it is a theorem of Jacobi [39, Thm. VII.6] that

$$\Delta(z) = (2\pi)^{12}q\prod_{n+1}^{+\infty}(1-q^n)^{24}.$$

(2.6.3) In the $k = 1$ case, we set

$$G_1(z) := \sum_{n\in\mathbf{Z}}\sum_{\substack{m\in\mathbf{Z}:\\(m,n)\neq(0,0)}}\frac{1}{m+nz^2}$$

(note that the order of summation is significant). It has the Fourier expansion at infinity

$$G_1(z) = \frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{+\infty}\sigma_1(n)q^n$$

[39, § VII.4.4], and we set

$$E_1(z) := \frac{3}{\pi^2}G_1(z) = 1 - 24\sum_{n=1}^{+\infty}\sigma_1(n)q^n.$$

One has the identity [39, § VII.4.4]

$$\frac{d\Delta}{\Delta} = 2\pi i E_1(z)dz.$$

(2.6.4) The function G_1 is *not* a modular form, but still it does satisfy a functional equation close to that equivalent to the invariance of $G_1(z)dz$ under the action of $\mathrm{PSL}_2(\mathbf{Z})$ [4, Prop. 6 p. 19]. For this reason, it is called a *quasi-modular* form.

One may then define the ring of quasi-modular forms as the $\mathbf{C}$-algebra generated by G_1 and the algebra of modular forms (see [4] for a more intrinsic definition). Since the ring of modular forms is generated by G_2 and G_3 [39, Cor. VII.2], the ring of quasi-modular forms may be concretely described as $\mathbf{C}[G_1, G_2, G_3]$. The ring of quasi-modular forms is closed under differentiation by the operator

$$D := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$$

[4, Prop. 15 p. 49].

(2.6.5) *Quasi-modularity of* $qF_g(q)$. Taking into account the various stunning formulæ above, (2.5.2) may be rewritten as

$$qF_g(q) = \left(-\frac{1}{24}q\,\frac{dE_1}{dq}\right)^g \left(\frac{\Delta}{(2\pi)^{12}}\right)^{-1},$$

from which it follows that $qF_g(q)$ is a quasi-modular form, with a simple pole at infinity (i.e. at $q = 0$) if $g = 0$.

2.3 – Proof of the Göttsche–Bryan–Leung formula

As Theorem (2.5) really is about counting actual curves, as Lemma (2.4) attests, one may prefer in the first place to avoid the complications of Gromov–Witten theory to prove it. It will yet be clear in a moment that this is not really possible as far as the proof proposed by Bryan and Leung goes, as the latter fundamentally relies on the agile possibilities featured by Gromov–Witten theory, precisely as a reward to the aforementioned complications.

(2.7) Degeneration to an elliptic $K3$**.** Let $g \geqslant 0$, $p > 0$ be integers. By deformation invariance, we are free to compute the number N_g^p on our favourite primitively polarized $K3$ surface (S, L) of genus p. We let S be an elliptic $K3$ surface with a section E, and denote by F the class of the ellitic fibres ; the intersection form (or its restriction to the subspace $\langle E, F\rangle$) is given in the basis (E, F) by the matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

We set $L := \mathcal{O}_S(E + pF)$. Then $L^2 = 2p - 2$, and L is a primitive polarization of genus p.

We shall compute the numbers N_g^p on the pair (S, L). Note that while in the proof of the Yau–Zaslow formula above we considered a construction generalizing the structure of Jacobian fibration of elliptic $K3$ surfaces, this time we really degenerate to an actual elliptic $K3$.

(2.8) The linear system $|E + pF|$ has dimension p and consists solely of reducible curves $E + F_1 + \cdots + F_p$, where the F_i's are (not necessarily distinct) in the class F. From this it readily follows that if we fix a general set of g points $\mathbf{y} = (y_1, \ldots, y_g)$ on S, then the moduli space

$$\overline{M}_{g,\mathbf{y}}(S, E + pF) := \overline{M}_{g,g}(S, E + pF) \cap \mathrm{ev}_1^*(y_1) \cap \ldots \cap \mathrm{ev}_g^*(y_g)$$

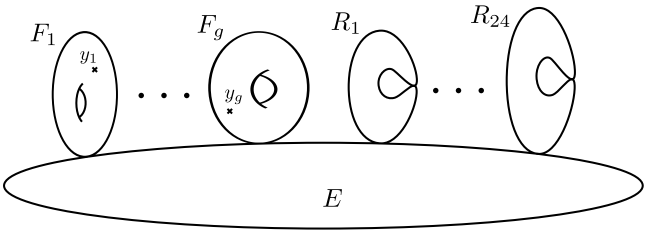
of genus g stable maps passing through the points $y_1, \ldots, y_g$ decomposes as the disjoint union

$$\coprod_{\mathbf{a},\mathbf{b}} \overline{M}_{\mathbf{a},\mathbf{b}},$$

where $\mathbf{a} = (a_1, \ldots, a_{24})$ and $\mathbf{b} = (b_1, \ldots, b_g)$ range through $\mathbf{Z}^{24}_{\geqslant 0}$ and $\mathbf{Z}^{g}_{>0}$ respectively subject to the condition that $\sum_i a_i + \sum_j b_j = p$, and $\overline{M}_{\mathbf{a},\mathbf{b}}$ is the moduli space of genus g stable maps $(f : C \to S, x_1, \ldots, x_g)$ such that $f(x_j) = y_j$ for $j = 1, \ldots, g$ and

$$f_*C = E + \sum_{i=1}^{24} a_i R_i + \sum_{j=1}^{g} b_j F_j,$$

$R_1, \ldots, R_{24}$ being the 24 rational members of the pencil $|F|$, and F_j being the unique member of $|F|$ containing the point y_j for $j = 1, \ldots, g$ (see figure below). We may and do assume that all members of $|F|$ are irreducible, and the R_i's are 1-nodal.



(2.9) Partition function. For all positive integers n, we let $p(n)$ be the *number of partitions* of n, i.e. the number of ways to write $n = \lambda_1 + \cdots + \lambda_k$, $\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 1$ (k is not fixed). The numbers $p(n)$ may be computed using the generating series

$$
\begin{aligned}
1 + \sum_{n=1}^{+\infty} p(n)\, t^n &= (1 + t^1 + t^{1+1} + t^{1+1+1} + \cdots) \times (1 + t^2 + t^{2+2} + t^{2+2+2} + \cdots) \\
&\qquad \times (1 + t^3 + t^{3+3} + t^{3+3+3} + \cdots) \times \cdots \\
&= \prod_{n=1}^{+\infty} \frac{1}{1 - t^n}. \qquad (2.9.1)
\end{aligned}
$$

See [13, Chap. 4] for much more about this.

The following result is the key to formula (2.5.2) for (S, L) an elliptic $K3$ surface as set-up in § (2.7)–(2.8).

(2.10) Proposition. *The contribution of* $\overline{M}_{\mathbf{a},\mathbf{b}}$ *to* N_g^p *is*

$$
\left(\prod_{i=1}^{24} p(a_i) \right) \left(\prod_{j=1}^{g} b_j \sigma_1(b_j) \right). \qquad (2.10.1)
$$

The latter results yields Formula (2.5.2) after a series of elementary manipulations which I don't reproduce here (see [5, p. 383] for details). Note that the identity (2.9.1) comes into play. The rest of this subsection is dedicated to the proof of Proposition (2.10).

(2.11) Enumeration of elliptic multiple covers. We first explain the factors $b_j\sigma_1(b_j)$ in (2.10.1). They are simple to understand, as they are of a combinatorial nature.

Let $f : C \to S$ be a member of $\overline{M}_{\mathbf{a},\mathbf{b}}$. It is necessarily shaped as follows : the curve C consists of (i) a smooth rational curve mapped isomorphically to the section $E \subset S$, which we will abusively call E as well, (ii) g smooth elliptic curves $G_1, \ldots, G_g$, pairwise disjoint and each attached at one point to E, and (iii) 24 trees of smooth rational curves $T_1, \ldots, T_{24}$, pairwise disjoint, each disjoint from the G_j's and attached to E at one point ; for $j = 1, \ldots, g$, f maps G_j to the elliptic fibre F_j with degree b_j and there is a marked point $x_j \in G_j$ mapped to y_j, and for $i = 1, \ldots, 24$ one has $f_* T_i = a_i R_i$.

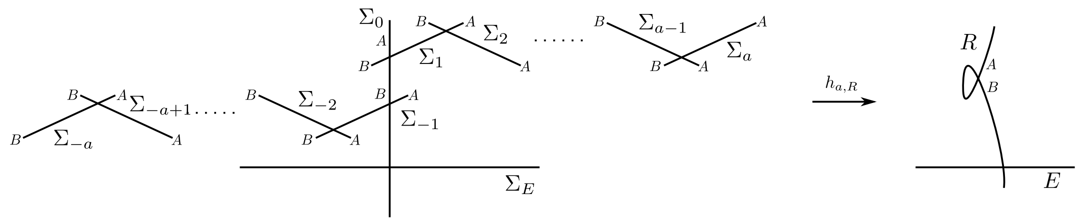
For all j, we may fix the intersection point with E as the origin of G_j and F_j respectively, which makes $f|_{G_j} : G_j \to F_j$ a degree b_j homomorphism of elliptic curves. Such homomorphisms are in $1:1$ correspondence with index b_j sublattices of the lattice defining F_j as a complex torus, and the number of such sublattices is $\sigma_1(b_j)$ [39, § VII.5.2]. Next, there are b_j possibilities to choose the marked point x_j in the preimage of y_j in G_j.

Once the data of the homomorphisms $G_j \to F_j$ and the marked points $x_j \in G_f$ is fixed, the corresponding sub-moduli space of $\overline{M}_{\mathbf{a},\mathbf{b}}$ decomposes as a product

$$\prod_{i=1}^{24} \overline{M}_{a_i \mathbf{e}_i, 0},$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_{24} = (0, \ldots, 0, 1)$ denotes the canonical basis of $\mathbf{Z}^{24}$. The moduli space $\overline{M}_{\mathbf{a},\mathbf{b}}$ thus consists of $\prod_{j=1}^{g} b_j \sigma_1(b_j)$ disjoint copies of the space $\prod_{i=1}^{24} \overline{M}_{a_i \mathbf{e}_i, 0}$, and the rest of the proof of Proposition (2.10) consists in showing that each of those contributes by $\prod_{i=1}^{24} p(a_i)$ to N_g^p. Before we can proceed, we need the following.

(2.12) Description of $\overline{M}_{a_i \mathbf{e}_i, 0}$. Let us start by defining a model stable map $h_{a,R} : \mathbf{\Sigma}_a \to S$ for all positive integers a and 1-nodal rational curves $R \in \{R_1, \ldots, R_{24}\}$. The curve $\mathbf{\Sigma}_a$ is a tree of $2a+2$ smooth rational curves $\Sigma_E, \Sigma_{-a}, \ldots, \Sigma_0, \ldots \Sigma_{+a}$ as depicted on the figure below.



The map $h_{a,R}$ is chosen so that it restricts to an isomorphism $\Sigma_E \cong E$ (hence from now on we will denote Σ_E by E) and to $2a+1$ copies $\Sigma_i \to R$ of the normalization of the 1-nodal rational curve R, in such a way that it is everywhere a local isomorphism between $\mathbf{\Sigma}_a$ and $E \cup R$. Concretely, the latter requirement is that locally at every node $\Sigma_i \cap \Sigma_{i+1}$, the map $h_{a,R}$ should send Σ_i to one of the two local branches of R at its node and Σ_{i+1} to the other.

There are basically two possible (indifferent) choices. We indicate one of them on the above figure by decorating each local branch at a node of $\Sigma_{-a} \cup \ldots \cup \Sigma_{+a}$ with a letter A or B, with the convention that A and B label the two local branches of R at its node.

The following lemma provides a basic description of the objects in $\overline{M}_{a_i \mathbf{e}_i, 0}$.

(2.12.1) Lemma. *For every $(f : C \to S) \in \overline{M}_{a\mathbf{e}_i, 0}$, there is a unique lift of f to a stable map $\tilde{f} : C \to \mathbf{\Sigma}_a$, meaning that $f = h_{a,R_i} \circ \tilde{f}$.*

Proof. The curve C is necessarily made of a smooth rational component mapped isomorphically to E, which we denote by E as well, and a tree of smooth rational curves T attached at one point to E, as in (2.11). For each irreducible component C_s of T, one has $f_* C_s = k_s R_i$ for some non-negative integer k_s. We determine the lift $\tilde{f}$ by exploring the dual graph of T along all possible paths from its root to one of its leaves, as follows.

There is a unique irreducible component C_0 of T intersecting E ; we call the corresponding vertex of the dual graph of T the *root* of the dual graph. The *leaves* are those vertices corresponding to irreducible components of T intersecting only one other irreducible component. Now the lift $\tilde{f}$, should it exist, necessarily maps C_0 to Σ_0, and there is a unique suitable map $C_0 \to \Sigma_0$ (possibly contracting C_0 to the point $\Sigma_0 \cap E$) by the universal property of normalization.

Suppose a putative lift $\tilde{f}$ is determined on an irreducible component C_s of T, and consider an arbitrary component C_{s+1} of T intersecting C_s at one point z_{s+1}. I claim that the behaviour of $\tilde{f}$ on C_{s+1} is uniquely determined by the already constructed piece $\tilde{f}\big|_{C_s}$. If C_{s+1} is contracted by f, this is clear ; otherwise, it is enough by the universal property of normalization to determine which component of $\mathbf{\Sigma}_a$ the lift $\tilde{f}$ should map C_{s+1} to. If $\tilde{f}(z_{s+1})$ is a smooth point of $\mathbf{\Sigma}_a$, then

$\tilde{f}$ has to map C_{s+1} to the same component it maps C_s to ; if not, $\tilde{f}(z_{s+1})$ is a node $\Sigma_{t_s} \cap \Sigma_{t_s+\varepsilon}$, $\varepsilon \in \{\pm 1\}$, and $\tilde{f}$ has to map C_{s+1} to Σ_{t_s} or $\Sigma_{t_s+\varepsilon}$, depending on which of the local branches A and B of R_i at its node the local branch of C_{s+1} at z_{s+1} is mapped to by f.

This discussion shows that one may algorithmically construct an $\tilde{f}$ such that $h_{a,R_i} \circ \tilde{f} = f$, and that there is a unique such lift. (Note that the chain of rational curves $\Sigma_{-a} \cup \ldots \cup \Sigma_a$ is long enough for the construction to go through without any trouble : since $f_* C = E + aR_i$, if at some point during the algorithm we hit an irreducible component C_s of C that has to be mapped to Σ_{-a+1} or Σ_{a-1}, then the push-forward by f of the sum of all components already visited by the algorithm fills out the class aR_i, hence all components of C not yet touched by the algorithm are contracted by f, and we don't have to go beyond Σ_{-a+1} or Σ_{a-1} in $\mathbf{\Sigma}_a$). □

Using Lemma (2.12.1), one can associate to every stable map $(f : C \to S) \in \overline{M}_{a\mathbf{e}_i,0}$ a combinatorial datum called an *admissible sequence* of weight a : this is a sequence of $2a+1$ non-negative integers

$$\mathbf{k} = (0, \ldots, 0, k_{-m}, \ldots, k_0, \ldots, k_n, 0, \ldots, 0)$$

with $m, n \geqslant 0$, $k_{-m}, \ldots, k_n > 0$, and $k_{-m} + \cdots + k_n = a$.

The association goes as follows. Let $h_{a,R_i} \circ \tilde{f}$ be the factorization of f, and write

$$\tilde{f}_* C = E + \sum_{s=-a}^{a} k_s \Sigma_s.$$

It follows from the construction of $\tilde{f}$ in the proof of Lemma (2.12.1) that $(k_{-a}, \ldots, k_a)$ is an admissible sequence of weight a.

(2.12.2) The moduli space $\overline{M}_{a\mathbf{e}_i,0}$ thus decomposes as the disjoint union

$$\overline{M}_{a\mathbf{e}_i,0} = \coprod_{\mathbf{k}} \overline{M}_{\mathbf{k}},$$

where $\mathbf{k}$ ranges through all weight a admissible sequences, and $\overline{M}_{\mathbf{k}}$ is the sub-moduli space of $\overline{M}_{a\mathbf{e}_i,0}$ parametrizing those f with associated admissible sequence $\mathbf{k}$.

(2.13) Identification of the virtual class. Recall that in (2.11) we saw the moduli space $\overline{M}_{\mathbf{a},\mathbf{b}}$ decomposes in a disjoint union of copies of the product $\prod_1^{24} \overline{M}_{a_i\mathbf{e}_i,0}$ (each corresponding to a given behaviour over the elliptic fibres $F_1, \ldots, F_g$) ; each $\overline{M}_{a_i\mathbf{e}_i,0}$ in turn decomposes as a disjoint union of moduli spaces $\overline{M}_{\mathbf{k}_i}$ of stable maps with target the curve $\mathbf{\Sigma}_{a_i}$, with $\mathbf{k}_i$ an admissible sequence of weight a_i, as we have described in (2.12). Eventually, $\overline{M}_{\mathbf{a},\mathbf{b}}$ is thus a disjoint union of various products $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$.

The heart of the proof of Bryan and Leung is the explicit identification of the restriction to the product $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$ of the virtual class giving rise to the invariant N_g^p [5, § 5.2]. This is by far the most demanding part of their article, and I will not attempt to give any idea of the proof.

The result is that (i) the virtual class on $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$ is a product of virtual classes on the various factors, and (ii) the virtual class on the factor $\overline{M}_{\mathbf{k}_i}$ is computed by means of a "virtual tangent bundle" T on the target curve $\mathbf{\Sigma}_{a_i}$. This virtual tangent bundle is the vector bundle T on $\mathbf{\Sigma}_{a_i}$ defined by the conditions that it is isomorphic to $h^*_{a_i,R_i} T_S$ on $\Sigma_{-a} \cup \ldots \cup \Sigma_a$ and to $T_E \oplus \mathcal{O}_E(-1)$ on E.

Note that $h^*_{a_i,R_i} T_S$ restricts to $T_E \oplus \mathcal{O}_E(-2)$ on E ; the correction made to define T corresponds to the fact that we want to kill the obstruction space $\mathrm{H}^1(C, N_f)$ as we know the actual obstruction space is trivial although $\mathrm{H}^1(C, N_f)$ is not (see (2.3)).

(2.14) Planar model. Let a be a non-negative integer. Thanks to the result of (2.13), it is possible to construct a model for $\mathbf{\Sigma}_a$ embedded in a familiar surface where its actual deformation theory is isomorphic to the virtual theory leading to the invariants N_g^p. This will eventually let us compute the contribution of the $\overline{M}_{\mathbf{k}_i}$'s (and hence of the $\prod_1^{24} \overline{M}_{\mathbf{k}_i}$'s) to the invariant N_g^p.

Consider three distinct points p, p_{-1}, p_1 lying on a line in the projective plane $\mathbf{P}^2$. Let $P_1 \to \mathbf{P}^2$ be the blow-up at these three points, and call E the exceptional divisor over p, Σ_0 the proper transform of the line through p, p_{-1}, p_1, and Σ_{-1}, Σ_1 the exceptional divisors over p_{-1}, p_1 respectively. Next, recursively for all $s = 1, \dots, a$, we perform the blow-up $P_{s+1} \to P_s$ at two general points of Σ_{-s} and Σ_s respectively, and let Σ_{-s-1} and Σ_{s+1} be the two corresponding exceptional divisors; we call $E, \Sigma_{-s}, \dots, \Sigma_s$ respectively the proper transforms in P_{s+1} of the curves with the same name in P_s.

The curve $\Sigma_{-a} \cup \ldots \cup \Sigma_0 \cup \Sigma_a$ (note that this excludes the last two exceptional curves Σ_{-a-1} and Σ_{a+1}) is isomorphic as an abstract curve to $\mathbf{\Sigma}_a$. Moreover, the tangent bundle of P_{a+1} restricts to $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ on E and to $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$ on $\Sigma_{-a}, \dots, \Sigma_a$, and is therefore isomorphic to the "virtual tangent bundle" T introduced in (2.13). As a consequence, Bryan–Leung prove the following.

(2.14.1) Lemma. *For all admissible sequences $\mathbf{k} = (k_{-a}, \dots, k_0, \dots, k_a)$, the "local" contribution $\int_{[\overline{M}_{\mathbf{k}}]^{\mathrm{vir}}} 1$ of $\overline{M}_{\mathbf{k}}$ to the invariant N_g^p equals the ordinary genus 0 Gromov–Witten integral $\int_{[\overline{M}_0(P_{a+1},\beta)]^{\mathrm{vir}}} 1$ for the class $\beta = E + \sum_{-a}^{a} k_s \Sigma_s$.*

This follows from (2.13) and the isomorphism between the restriction of $T_{P_{a+1}}$ and T, provided the two moduli spaces $\overline{M}_{\mathbf{k}}$ and the ordinary $\overline{M}_0(P_{a+1}, E + \sum_{-a}^{a} k_s \Sigma_s)$ are isomorphic as sets. Bryan and Leung are able to prove this by elementary arguments using a slightly more evolved set-up : they start with a linear $\mathbf{C}^*$ action (not $(\mathbf{C}^*)^2$) on $\mathbf{P}^2$ leaving the line $\langle p, p_{-1}, p_1 \rangle$ and a point q fixed, and this $\mathbf{C}^*$ action survives in P_{a+1}. We refer to [5, Lem. 5.7] for the proof.

Eventually, by deformation invariance of Gromov–Witten invariants we may transport the computation on the projective plane blown-up at $2a+3$ general points. This gives the following.

(2.14.2) Lemma. *Let $\tilde{\mathbf{P}}^2$ be the blow-up of $\mathbf{P}^2$ at a general set of $2a+3$ points, with exceptional divisors $E, E_{-a-1}, \dots, E_{-1}, E_1, \dots, E_{a+1}$ (all (-1)-curves of course). We call H the pull-back of the line class. Then for all admissible sequences $\mathbf{k} = (k_{-a}, \dots, k_0, \dots, k_a)$, the "local" contribution $\int_{[\overline{M}_{\mathbf{k}}]^{\mathrm{vir}}} 1$ of $\overline{M}_{\mathbf{k}}$ to the invariant N_g^p equals the ordinary genus 0 Gromov–Witten integral $\int_{[\overline{M}_0(\tilde{\mathbf{P}}^2,\beta_{\mathbf{k}})]^{\mathrm{vir}}} 1$ for the class*

$$\begin{aligned}\beta_{\mathbf{k}} &= E + \sum_{s=1}^{a} k_{-s}(E_{-s} - E_{-s-1}) + k_0(H - E - E_1 - E_{-1}) + \sum_{s=1}^{a} k_s(E_s - E_{s+1}) \\ &= k_0 H + (1-k_0)E + \sum_{s=1}^{a} (k_s - k_{s-1})E_s - k_a E_{a+1} + \sum_{s=1}^{a} (k_{-s} - k_{-s+1})E_{-s} - k_{-a}E_{-a-1}.\end{aligned}$$

Note that

$$(k_0 - 1) + \sum_{s=1}^{a} (k_{s-1} - k_s) + k_a + \sum_{s=1}^{a} (k_{-s+1} - k_{-s}) + k_{-a} = 3k_0 - 1,$$

so the virtual class $[\overline{M}_0(\tilde{\mathbf{P}}^2, \beta_{\mathbf{k}})]^{\mathrm{vir}}$ has dimension 0 (see (2.15) below).

(2.15) The computation on the blown-up plane. The Gromov–Witten invariants gotten in Lemma (2.14.2) are computable in practice thanks to the analysis of genus 0 Gromov–Witten invariants of blow-ups of $\mathbf{P}^2$ carried out by Göttsche and Pandharipande [16].

Let n be a non-negative integer, $d, \alpha_1, \ldots, \alpha_n$ integers. We call $N(d; \alpha_1, \ldots, \alpha_n)$ the genus 0 Gromov–Witten invariant for $\tilde{\mathbf{P}}^2$ and the class $dH - \sum_i \alpha_i E_i$ (mind the minus sign, introduced for obvious geometric reasons), where $\tilde{\mathbf{P}}^2$ is the projective plane blown-up at a general set of n points, H the pull-back of the line class, and $E_1, \ldots, E_n$ the exceptional (-1)-curves. The corresponding moduli space of stable maps has virtual dimension $3d - 1 - \sum_i \alpha_i$; if this is positive, then we impose the appropriate number of point constraints, and if this is negative, then we set the invariant to 0.

These invariants enjoy the following properties (see [16]) :

(i) $N(1) = 1$;

(ii) $N(d; \alpha_1, \ldots, \alpha_{n-1}, 1) = N(d; \alpha_1, \ldots, \alpha_{n-1}, 0) = N(d; \alpha_1, \ldots, \alpha_{n-1})$;

(iii) $N(d; \alpha_1, \ldots, \alpha_n) = N(d; \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$ for any permutation $\sigma \in \mathfrak{S}_n$;

(iv) $N(d; \alpha_1, \ldots, \alpha_n) = 0$ if there is an index i for which $\alpha_i < 0$, unless $dH - \sum_i \alpha_i E_i = E_{i_0}$ for some i_0 in which case the invariant is 1;

(v) The invariant $N(d; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n)$ is invariant under the isomorphism given by the quadratic Cremona transformation corresponding to the linear system $|2H - E_1 - E_2 - E_3|$, i.e.

$$N(d; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n) = N(2d - \alpha_1 - \alpha_2 - \alpha_3, d - \alpha_2 - \alpha_3, d - \alpha_2 - \alpha_3, d - \alpha_2 - \alpha_3, \alpha_4, \ldots, \alpha_n).$$

We need the following definition to state the result. An admissible sequence $(k_{-a}, \ldots, k_0, \ldots, k_a)$ is *1-pyramidal* if

$$k_s - 1 \leqslant k_{s+1} \leqslant k_s \qquad \text{and} \qquad k_{-s} - 1 \leqslant k_{-s-1} \leqslant k_{-s}$$

for $s = 0, \ldots, a-1$.

(2.15.1) Lemma. *Let $\mathbf{k} = (k_{-a}, \ldots, k_0, \ldots, k_a)$ be an admissible sequence of weight a. Then the genus 0 Gromov–Witten invariant*

$$N(\mathbf{k}) := N(k_0;\, k_0 - 1,\, k_{-a},\, k_{-a+1} - k_{-a},\, \ldots,\, k_0 - k_{-1},\, k_0 - k_1,\, \ldots,\, k_{a-1} - k_a,\, k_a)$$

equals 1 if $\mathbf{k}$ is 1-pyramidal, and 0 otherwise.

Proof. [1] We first show that $N(\mathbf{k}) = 0$ if $\mathbf{k}$ is not 1-pyramidal. Since $k_0 > 0$ by definition of an admissible sequence, it follows from Property (iv) above that $N(\mathbf{k}) \neq 0$ implies

$$k_{-s} \leqslant k_{-s+1} \qquad \text{and} \qquad k_{s-1} \geqslant k_s \tag{2.15.2}$$

for all $s \in \{1, \ldots, a\}$. Next, we apply the Cremona transformation defined by $|2H - E - E_1 - E_{s+1}|$ $(2 \leqslant s \leqslant a-1)$ in the notation of Lemma (2.14.2) and get

$$N(\mathbf{k}) = N(1 + k_1 + k_{s+1} - k_s;\, k_1 - k_s + k_{s+1},\, 1 - k_s + k_{s+1},\, 1 - k_0 + k_1,\, \ldots)$$

by Property (v) above. If $N(\mathbf{k}) \neq 0$, we have $k_s \leqslant k_1$ by (2.15.2), hence $1 + k_1 + k_{s+1} - k_s > 0$, and Property (iv) then implies that $k_1 \geqslant k_0 - 1$ and $k_{s+1} \geqslant k_s - 1$. An analogous move shows that $k_{-s-1} \geqslant k_s - 1$ for $s = 0, \ldots, a-1$ if $N(\mathbf{k}) \neq 0$, so that eventually we see that the non-vanishing of $N(\mathbf{k})$ implies that $\mathbf{k}$ is 1-pyramidal.

Conversely, let's assume that $\mathbf{k}$ is 1-pyramidal and of weight a. Then $k_a = k_{-a} = 0$ (otherwise the weight exceeds a; we have somehow already made this observation in the course of the proof of Lemma (2.12.1)), and all coefficients $k_s - k_{s-1}$ and $k_{-s} - k_{-s+1}$, $1 \leqslant s \leqslant a$,

1. there is a transcription mistake in [5, p. 399] for the class $\beta_{\mathbf{k}}$ of our Lemma (2.14.2) ; this leads to a minor correction in the present proof.

equal 0 or 1. It thus follows from Property (ii) that $N(\mathbf{k}) = N(k_0; k_0 - 1)$, which is readily seen to equal 1 : the moduli space of genus 0 stable maps $\overline{M}_0\big(\tilde{\mathbf{P}}^2, k_0 H - (k_0 - 1)E\big)$ has only one enumeratively meaningful irreducible component, isomorphic to the family of degree k_0 plane curves with multiplicity $k_0 - 1$ at a fixed point $x_E \in \mathbf{P}^2$, and this is a linear system. □

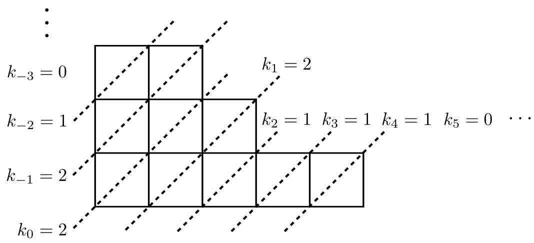
(2.16) Conclusion. The proof of Theorem (2.5) will be completed once we show the following clever combinatorial result.

(2.16.1) Lemma. *The number of* 1*-pyramidal admissible sequences* $(k_{-a}, \ldots, k_0, \ldots, k_a)$ *of weight* a *equals the partition number* $p(a)$ *(cf. (2.9)).*

Indeed, together with (2.12.2) and Lemmata (2.14.2) and (2.15.1), this shows that the "local" contribution of $\overline{M}_{a_i \mathbf{e}_i, 0}$ equals $p(a_i)$, hence the contribution of each copy of the product $\prod_1^{24} \overline{M}_{a_i \mathbf{e}_i, 0}$ equals $\prod_1^{24} p(a_i)$, which proves Proposition (2.10) thanks to the enumeration of elliptic multiple covers performed in (2.11); as we have seen in (2.8), the latter Proposition implies Theorem (2.5).

Proof of Lemma (2.16.1). Partitions of an integer a are in bijective correspondence with Young diagrams of size a [13, Chap. 4]; we exhibit a bijective correspondence between Young diagrams of size a and 1-pyramidal admissible sequences $(k_{-a}, \ldots, k_0, \ldots, k_a)$ of weight a as follows.

We see Young diagrams as embedded in the upper-right quadrant of a Cartesian plane, leaning on both the x and y axes, and with blocks squares of size 1. Given such a Young diagram, we let k_s be the number of blocks on the line $y - x = s$ for $s = -a, \ldots, 0, \ldots, a$. We give an example of the procedure in the figure below.



We leave it to the reader to check that this is indeed a bijection. □

3 – BPS state counts

In this Section, I discuss why and how curve counting in non-primitive classes imply the use of multiple covers formulæ. This features the generalization of the Yau–Zaslow formula of Theorem (1.1) to non-primitive classes.

3.1 – Rational curves on the quintic threefold

To describe the picture in its simplest form, let me first discuss a question slightly at the margin of the scope of these notes, that of counting rational curves on a general quintic hypersurface V of $\mathbf{P}^4$.

(3.1) The Clemens conjecture (see (3.5) below) predicts there are finitely many such curves of any given degree d; this is in keeping with the virtual dimension

$$\operatorname{vdim} \overline{M}_g(V,\beta) = (\dim V - 3)(g-1) - K_V \cdot \beta \tag{3.1.1}$$

being 0 for any homology class on the Calabi–Yau threefold V. This suggests that the numbers

$$N_d^V := \int_{[\overline{M}_0(V,d\ell)]^{\mathrm{vir}}} 1$$

(where ℓ denotes the homology class of a line) may indeed give the actual number of rational curves of degree d in V. This would be particularly appealing, since the numbers N_d^V may in theory be rigorously computed using the predictions of mirror symmetry, and they are for small values of d (see [9] for a thorough discussion).

A first objection to such an ideal statement to hold is that there may be rational curves with non-trivial infinitesimal deformations, but the Clemens conjecture predicts as well that this does not happen.

(3.2) Multiple cover formula. A somehow more serious grain of salt comes from the existence of components of $\overline{M}_0(V,d\ell)$ of dimension larger than expected : suppose we are given a smooth degree d rational curve $C \subset V$, and let $f : \mathbf{P}^1 \to V$ be the stable map induced by its normalization; then for any positive integer k, the degree k covers

$$\mathbf{P}^1 \xrightarrow{k:1} \mathbf{P}^1 \xrightarrow{f} V \tag{3.2.1}$$

constitute an irreducible variety M_{kC} of dimension $2k-2$.

(3.2.2) *The Aspinwall-Morrison formula asserts that the corresponding irreducible component* $\overline{M}_{kC}$ *of* $\overline{M}_0(V,kd\ell)$ *contributes by* $\frac{1}{k^3}$ *to the integral* N_{kd}^V. *This has been mathematically proved by Kontsevich and Manin, and Voisin* (see [44, § 5.6], [9, Thm. 7.4.4]).

To explain where the factor $\frac{1}{k^3}$ comes from, it is convenient to replace the integrals N_d^V by their close cousins

$$\langle I_{0,3,d\ell}^V\rangle(\omega_1,\omega_2,\omega_3) := \int_{[\overline{M}_{0,3}(V,d\ell)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\omega_1)\wedge \mathrm{ev}_2^*(\omega_2)\wedge \mathrm{ev}_3^*(\omega_3),$$

where $\overline{M}_{0,3}(V,d\ell)$ is the space of genus 0 stable maps with 3 marked points (which has the advantage of identifying locally with the Hilbert scheme $\mathrm{Hom}(\mathbf{P}^1,V)$ at stable maps with source $\mathbf{P}^1$), and the ω_i are Kähler forms on V. It follows from the divisorial axiom of Gromov–Witten theory that

$$\langle I_{0,3,d\ell}^V\rangle(\omega_1,\omega_2,\omega_3) = \int_{d\ell}\omega_1 \times \int_{d\ell}\omega_2 \times \int_{d\ell}\omega_3 \times N_d^V.$$

On the other hand each physical rational curve $C \subset V$ contributes through its normalization $f : \mathbf{P}^1 \to V$ by

$$\int_{\mathbf{P}^1} f^*\omega_1 \times \int_{\mathbf{P}^1} f^*\omega_2 \times \int_{\mathbf{P}^1} f^*\omega_3.$$

Assume we knew what the compactification $\overline{M}_{kC}$ of M_{kC} in $\overline{M}_{0,3}(V,kd\ell)$ looks like, and we had a vector bundle E on it with fibre over $g \in M_{kC}$ (as in (3.2.1)) the obstruction space $E_g = \mathrm{H}^1(\mathbf{P}^1, g^*T_V)$. Then we could compute the contribution of $\overline{M}_{kC}$ by the excess formula

$$\int_{\overline{M}_{kC}} c_{2k-2}(E)\wedge \mathrm{ev}_1^*(\omega_1)\wedge \mathrm{ev}_2^*(\omega_2)\wedge \mathrm{ev}_3^*(\omega_3). \tag{3.2.3}$$

The heuristic computation of [44, p. 115–116] shows that a convenient model for $\overline{M}_{kC}$ leads to the excess contribution (3.2.3) being

$$\int_{\mathbf{P}^1} f^*\omega_1 \times \int_{\mathbf{P}^1} f^*\omega_2 \times \int_{\mathbf{P}^1} f^*\omega_3. = \left(\frac{1}{k}\int_{k[C]}\omega_1\right) \times \left(\frac{1}{k}\int_{k[C]}\omega_2\right) \times \left(\frac{1}{k}\int_{k[C]}\omega_3\right),$$

which would justify the contribution by $\frac{1}{k^3}$ of $\overline{M}_{kC}$ to the integral $\langle I^V_{0,3,kd\ell}\rangle(\omega_1,\omega_2,\omega_3)$.

(3.3) Instanton numbers. We now wish to define new invariants n^V_d from the Gromov–Witten integrals N^V_d that better reflect the enumerative geometry of the Calabi–Yau threefold V, taking into account the multiple cover phenomenon described in paragraph (3.2) above. The relations that these numbers should satisfy are

(3.3.1) $$N^V_d = \sum_{k|d} \frac{1}{k^3} n^V_{\frac{d}{k}}$$

for all positive integers d, where the sum runs over all positive integral divisors of d. This is an invertible triangular set of relations, and it follows that the number n^V_d is uniquely determined by the invariants $N^V_{d'}$ for all positive divisors d' of d.

These new numbers are traditionally called instanton numbers. The name obviously bears some physical meaning; I shall not discuss this here.

(3.4) Reducible covers of singular curves. There is yet another phenomenon, first observed by Pandharipande, that prevents the instanton numbers n^V_d defined in (3.3) above to be the actual numbers of integral degree d rational curves on V. It is linked to the existence of singular integral rational curves.

To describe the simplest instance of this phenomenon, let $C \subset V$ be an integral rational curve with normalization $f : \mathbf{P}^1 \to V$, and assume it has an ordinary double point at $x \in C$. Let x_1 and x_2 be the two preimages of x in the normalization, and consider the nodal rational curve $\mathbf{P}^1 \cup_{x_1,x_2} \mathbf{P}^1$ obtained by glueing tranversaly two copies of $\mathbf{P}^1$ in such a way that x_1 in the first copy is identified with x_2 in the second copy. Then the map

$$f_x : \mathbf{P}^1 \cup_{x_1,x_2} \mathbf{P}^1 \to V,$$

the restriction of which to both components equals f, is a stable map of genus 0 realizing the homology class $2[C]$ on V, hence contributes to the Gromov–Witten invariant $N^V_{2\deg C}$.

The latter contribution is by 1 if C is infinitesimally rigid (cf. [9, § 9.2.3]). It follows that a δ-nodal rational curve of degree d (i.e. a curve with exactly δ ordinary double points as singularities) contributes in the above described fashion by δ to the Gromov–Witten integral N^V_{2d}.

Fortunately, the Clemens conjecture below predicts that the complications don't go beyond this in this particular situation. Note in particular that part (iii) of the conjecture implies that the numbers N^V_d (or n^V_d) count *irreducible* physical rational curves $C \subset V$, since by definition the source of a stable map is connected.

(3.5) Conjecture. *Let $V \subset \mathbf{P}^4$ be a general quintic hypersurface.*
(i) For each integer $d \geqslant 1$, there are only finitely many irreducible rational curves $C \subset V$ of degree d.
(ii) For every integral rational curve $C \subset V$ with normalisation $f : \mathbf{P}^1 \to V$, the normal bundle

N_f *of the map* f *is isomorphic to* $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ *(i.e.* C *is infinitesimally rigid).*
(iii) All *the integral rational curves on* V *(of any degree) are pairwise disjoint.*

It is completely proved in degree $\leqslant 11$ (cf. [8, 10] for the latest steps), and part (i) is known in degree 12 [1].

We end this prologue about quintic threefolds by an explicit example displaying all these phenomena together.

(3.6) Rational curves of degree 10 **on a quintic threefold.** (cf. [9, § 9.2.3] for a thorough analysis). First note that by definition of the instanton numbers, the Gromov–Witten integral N_{10}^V decomposes as

$$N_{10} = \frac{1}{10^3} n_1 + \frac{1}{5^3} n_2 + \frac{1}{2^3} n_5 + n_{10}$$

(I dropped the superscript V to lighten notations), and one has

$$\begin{aligned} n_1 &= 2,875 \\ n_2 &= 609,250 \\ n_5 &= 229,305,888,887,625 \\ n_{10} &= 704,288,164,978,454,686,113,488,249,750, \end{aligned}$$

cf. [6]. While n_1 and n_2 are simply the numbers of lines and conics respectively on a general quintic threefold, n_5 counts two kinds of rational curves of degree 5. Indeed, the planes in $\mathbf{P}^4$ are parametrized by a 6-dimensional Grassmannian, and for a general quintic $V \subset \mathbf{P}^4$, finitely many of them are 6-tangent to V ; the corresponding plane sections of V are 6-nodal plane quintic curves, and in particular they are rational curves. Vainsencher [41] has been able to compute their number

$$n_5' = 17,601,000.$$

Each such curve contributes to n_5 (or N_5) by 1 [9, Lem. 9.2.4], and

$$n_5'' := n_5 - n_5'$$

is indeed the number of *smooth* rational curves of degree 5 on V.

Whereas n_5 still is the number of rational curves of degree 5 on V, this is no longer true for n_{10} as the discussion in (3.4) indicates. The actual number of degree 10 integral rational curves on V is

$$n_{10}^\circ := n_{10} - 6n_5'$$

[9, Thm. 9.2.6] [1].

3.2 – Degree 8 rational curves on a sextic double plane

In this subsection, I give the enumerative interpretation due to Gathmann [15] of the reduced Gromov–Witten invariant

$$N_{0,2}^2 := \int_{[\overline{M}_0(S,2L)]^{\mathrm{red}}} 1$$

of a general primitively polarized $K3$ surface (S, L) of genus 2 (i.e. S is a double covering of the plane $\pi : S \to \mathbf{P}^2$ branched over a general sextic curve B, and L is the pull-back of the line class).

1. there is a misprint there : $6\frac{1}{8}$ should be replaced by $6 + \frac{1}{8}$.

(3.7) The analysis carried out in subsection 3.1 indicates that the integral $N^2_{0,2}$ is a sum of contributions corresponding to the following types of curves.
(i) 5-nodal integral curves in $|2L|$; these are the preimages of the conics in $\mathbf{P}^2$ tangent to the branch sextic B at 5 distinct points. There are $70,956$ of those, as Gathmann was able to compute using his theory of relative Gromov–Witten invariants for hypersurfaces.
(ii) Reducible rational curves made of two distinct rational curves in $|L|$; let C_1, C_2 be two such curves (each of these is the pull-backs of line bitangent to B), with normalizations $f_i : \mathbf{P}^1 \to S$; they intersect in two points x, x' where both of them are smooth. There are correspondingly two distinct stable maps

$$f : \mathbf{P}^1 \cup_{f_1^{-1}(x), f_2^{-1}(x)} \mathbf{P}^1 \to S \quad \text{and} \quad f' : \mathbf{P}^1 \cup_{f_1^{-1}(x'), f_2^{-1}(x')} \mathbf{P}^1 \to S$$

with source the union of two $\mathbf{P}^1$'s meeting transversely at one point, which realize the physical curve $C_1 + C_2 \in |2L|$.

Each of these contributes by 1 to $N^2_{0,2}$, so each of the $\binom{324}{2}$ pairs of distinct rational curves in $|L|$ contributes by 2, thus giving a total contribution of $104,652$ to $N^2_{0,2}$ (recall that there are 324 bitangent lines to B, as can be classicaly computed, or extracted from Thm. (1.1)).
(iii) Reducible double coverings of rational curves in $|L|$, as in (3.4). Since all rational curves in $|L|$ are 2-nodal, all 324 of them give 2 stable maps with reducible source contributing by 1 each to $N^2_{0,2}$, for a total contribution of 648.
(iv) Double covers of rational curves in $|L|$, as in (3.2). One may expect that each of the 324 corresponding irreducible components of $\overline{M}_0(S, 2L)$ gives a contribution to $N^2_{0,2}$ similar to that prescribed by the Aspinwall–Morrison formula, although there is at first sight no obvious reason for this to be the case. Gathmann [15, Lem. 4.1] proves that indeed each irreducible component contributes by $\frac{1}{8}$.

(3.8) Remark. An important difference with the case of the quintic threefold, although rather innocent-looking, is that in the present situation integral rational curves do intersect each other, contrary to the prediction of part (iii) of Clemens' Conjecture. In the above discussion (3.7), this amounts to case (ii) needing to be added with respect to the discussion for quintic threefolds, which is still manageable. When looking at linear systems $|mL|$ with $m \geqslant 3$ however, this soon gets much more complicated, see (3.15).

(3.9) From the enumeration of (3.7), one may deduce the value of $N^2_{0,2}$, which was not known before. But the striking observation of [15] is that the sum of the contributions (i)–(iii) above, which would be the instanton number $n^2_{0,2}$ in the language of (3.3), actually equals

$$N^5_{0,1} := N^5_0 = 176,256,$$

the number of degree 8 rational curves in a primitively polarized $K3$ surface of genus 5, computed by formula (1.1.1). This suggests the amazing possibility that the number of rational curves of degree d on a $K3$ surface only depends on d, and not on the algebraic geometry of the $K3$ surface! We will come back to this in detail in subsection 3.4 below.

3.3 – Elliptic curves in a 2-divisible class on a $K3$ surface

Here, I report on the computation by Lee and Leung [21] of the reduced Gromov–Witten invariant

$$N^p_{1,2} := \int_{[\overline{M}_{1,1}(S,2L)]^{\mathrm{red}}} \mathrm{ev}_1^*(\mathrm{pt})$$

of a general primitively polarized $K3$ surface (S, L) of genus p, which "counts" genus 1 curves in $|2L|$ passing through 1 general point on S. Using a suitable version of topological recursion, they prove the following formula.

(3.10) Theorem. [21] *One has*

$$N_{1,2}^{p} = N_{1,1}^{4p-3} + 2N_{1,1}^{p}.$$

The numbers $N_{1,1}^{q} := N_{1}^{q}$ are those giving the number of elliptic curves through a general point in the primitive class of a $K3$ surface of genus q, as in formula (2.5). Note that $p' := 4p-3$ is the integer such that $(2L)^2 = 2p' - 2$.

(3.11) Lee and Leung propose the following interpretation of their formula (3.10), to put it in tune with (3.9) and more generally with the results of subsection 3.4 below.

Given a smooth elliptic curve E, there are $\sigma_1(2) = 1 + 2 = 3$ morphisms of elliptic curves $E' \to E$ of degree 2 (note that we require that the origin is respected), as we have already seen in (2.11). This implies that each of the $N_{1,1}^{p}$ elliptic curves C in the primitive linear system $|L|$ passing through a general point $x_1 \in S$ gives via double covers of its normalization 3 genus 1 stable maps realizing the homology class $2[C]$, each contributing by 1 to the number $N_{1,2}^{p}$.

Lee and Leung deduce from this that N_{1}^{4p-3} is the actual number of physical elliptic curves in $|2L|$, meaning that it counts each integral elliptic curve $C \in |2L|$ for 1 and each $2C$ where $C \in |L|$ is an integral elliptic curve for 1 as well. This is indeed a striking interpretation, although arguably debatable.

There is at any rate a phenomenon that prevents this interpretation to be anything more than philosophical, namely that reducible curves $C_0 + C_1$, where C_0 (resp. C_1) is a rational (resp. elliptic) integral curve in $|L|$, also contribute to the invariant $N_{1,2}^{p}$. The two curves C_0 and C_1 intersect (transversely, say) at $2p-2$ points $y_1, \ldots, y_{2p-2}$, and this gives $2p-2$ genus 1 stable maps

$$\mathbf{P}^1 \cup_{y_i} \bar{C}_1 \to S$$

realizing the class $2L$ and passing through the appropriate fixed point whenever C_1 does (the source is the transverse union of $\mathbf{P}^1$ and the normalization of C_1 attached at one point, the preimages of y_i in the normalizations of C_0 and C_1 respectively).

3.4 – The Yau–Zaslow formula for non-primitive classes

We now come to the general statement confirming (3.9) and recently proved by Klemm, Maulik, Pandharipande, and Scheidegger.

(3.12) Let S be a $K3$ surface, and $L \in \operatorname{Pic} S$. It follows from deformation invariance of reduced Gromov–Witten integrals and the global Torelli theorem for $K3$ surfaces that the integral $\int_{[\overline{M}_0(S,L)]^{\mathrm{red}}} 1$ only depends on the self-intersection L^2 and the divisibility index of L in $\operatorname{Pic} S$, i.e. the largest integer m for which there exists $L' \in \operatorname{Pic} S$ such that $L = mL'$. We may thus make the following definition.

For integers $p > 0$ and $m \geqslant 1$, we let

$$N_{0,m}^{p} := \int_{[\overline{M}_0(S,mL)]^{\mathrm{red}}} 1$$

where (S, L) is any primitively polarized $K3$ surface of genus p.

(3.13) BPS states. Similar to what has been done in (3.3), and following the insight of (3.9), we now define new invariants from the $N^p_{0,m}$ of (3.12) above by applying the corrections indicated by the Aspinwall–Morrison formula.

Let us formulate this in terms of generating series as follows. Given a positive integer p, set

$$F^p(v) := \sum_{m \geqslant 1} N^p_{0,m}\, v^m \tag{3.13.1}$$

as a formal power series in the variable v. Then the new set of invariants $n^p_{0,m}$ is uniquely determined by the rewriting of the generating series as

$$F^p(v) := \sum_{m \geqslant 1} n^p_{0,m} \Big(\sum_{d>0} \frac{1}{d^3}\, v^{dm} \Big) \tag{3.13.2}$$

(note that this is exactly the same modification as that of (3.3.1)).

Note that this is not mere makeshift reformulation. The invariants $n^p_{0,m}$ are believed to count objects named BPS states by the physicists, after Bogomol'nyi, Prasad, and Sommerfield; the mathematical nature of these objects is however not clear yet. In particular, it should be possible to define the $n^p_{0,m}$ intrinsically, not relying on the $N^p_{0,m}$; the relation (3.13.2) would then tie together these two sets of indipendently defined invariants. See the enlightening survey [28, § $2\frac{1}{2}$] for more about this. There is moreover a physical meaning to the introduction of generating series, that I will not discuss.

(3.14) Theorem. [20] *The invariants $n^p_{0,m}$ do not depend upon the divisibility index, i.e. one has for all integers $m, p \geqslant 1$*

$$n^p_{0,m} = n^{m^2p-m^2+1}_{0,1} = N^{m^2p-m^2+1}_0 \tag{3.14.1}$$

(the integer $p' = m^2p - m^2 + 1$ is designed such that $(mL)^2 = 2p' - 2$ if $L^2 = 2p - 2$).

Recall that N^p_0 was defined in section 1; the second equality in (3.14.1) is by definition of $n^p_{0,1}$ and N^p_0. This statement was part of the Yau–Zaslow conjecture [45]. Together with Theorem (1.1), which was also part of the Yau–Zaslow conjecture, it implies that all the $n^p_{0,m}$'s may be computed by means of formula (1.1.1). The set of relations (3.13.1) being triangular invertible, this also gives all genus 0 reduced Gromov–Witten invariants of $K3$ surfaces. Section 5 below contains an overview of the proof given by Klemm, Maulik, Pandharipande, and Scheidegger [20] of Theorem (3.14).

As we already noted in (3.9), the truly remarkable feature of the invariants $n^p_{0,m}$ displayed by this statement is that the number of rational curves of prescribed degree in an algebraic $K3$ surface does not depend on the algebraic geometry of the surface.

(3.15) In spite of formula (3.13.2) taking into account the Aspinwall-Morrison multiple cover correction, the invariants $n^p_{0,m}$ do not in general count the actual number of rational curves in $|mL|$.

One reason for this is the existence of more non-reduced curves with rational support than those taken in consideration in the correction (3.13.2), namely curves with reducible support. For instance let $m = 3$ and consider two integral rational curves C_1, C_2. Then $2C_1 + C_2 \in |3L|$, and there are correspondingly finitely many positive-dimensional components of $\overline{M}_0(S, 3L)$, the general points of which correspond to stable maps

$$\mathbf{P}^1 \cup_x \mathbf{P}^1 \to S$$

with source a transverse union of two $\mathbf{P}^1$'s, consisting of a double cover of C_1 on the first component and the normalization of C_2 on the other. These certainly give an excess contribution to the invariant $N^p_{0,3}$, which is not taken into account in the definition of $n^p_{0,3}$.

Such problematic phenomena do not occur for $m \leqslant 2$, so that $n^p_{0,1}$ is directly enumerative as was already noted in Section 1, and $n^p_{0,2}$ counts reduced rational curves in the way described in subsection 3.2. It would be very interesting to relate $n^p_{0,m}$ to the number of integral rational curves in $|mL|$ for $m \geqslant 3$.

There were, at least conjecturally, no such phenomena at work in the case of the quintic threefold discussed in subsection 3.1, as part (iii) of the Clemens conjecture (3.5) asserts that two integral rational curves in a general quintic threefold never intersect. In a surface, there is of course not enough space for two curves to avoid each other, so we inevitably have to deal with the aforementoined degenerate contributions.

The philosophy, as R. Pandharipande communicated to me, is that what the BPS numbers for $K3$ surfaces are virtually counting, are rational curves in some perturbation of the twistor family of the $K3$ surface (a threefold, cf. (2.2)). We shall consider in more detail the close interplay between counting invariants for $K3$ surfaces and Calabi–Yau threefolds in the next Section 4.

4 – Relations with threefold invariants

It is already visible in the very foundation of the theory of Gromov–Witten invariants for algebraic $K3$ surfaces developped by Bryan–Leung [5], see (2.2), that these invariants are fundamentally attached to a threefold (even though the approach by Maulik–Pandharipande [25], see (2.3), enables one to bypass this). Another revealing evidence of the 3-dimensional nature of these invariants is the meaningful role played by the Aspinwall–Morrison in the Yau–Zaslow statement discussed in subsection 3.4 above, a tool specifically designed for Calabi–Yau threefolds.

In this section we will try to describe this relation in a more conceptual way. It it wise to keep in mind the symplectic nature of Gromov–Witten invariants throughout.

4.1 – Two obstruction theories

(4.1) A threefold degenerate contribution. Let V be a Calabi–Yau threefold. It follows from formula (3.1.1) that the virtual dimension of any space of stable maps of any genus is always 0 (it is actually true even if the canonical class K_V is not trivial that the dimension only depends on the homology class β). This makes the following phenomenon happen.

Let $C_0 \subset V$ be a rational curve (smooth and infinitesimally rigid, say). Its normalization $f : \mathbf{P}^1 \to V$ contributes regularly by 1 to the integral $\int_{[\overline{M}_0(V,[C_0])]^{\mathrm{vir}}} 1$. But for any stable curve C' of genus $g \geqslant 1$ we may obtain a genus g stable map realizing the class $[C_0]$ by attaching C' to the normalization of C_0 over a smooth point x, and letting

$$f_{C',x} : \mathbf{P}^1 \cup_x C_0 \to V$$

equal to f along $\mathbf{P}^1$ and collapsing C' to x. This produces a positive dimensional moduli space of genus g stable maps all having the same image $C_0 \subset V$; its contribution to the Gromov–Witten invariant $\int_{[\overline{M}_g(V,[C_0])]^{\mathrm{vir}}} 1$ must be computed via Hodge integrals over the moduli space of stable curves of genus g. This has been studied by Faber and Pandharipande, see [28, § $1\frac{1}{2}$] and the references therein.

(4.2) Curves in the twistor space of a $K3$. Let S be a $K3$ surface, together with an algebraic class $\beta \in \mathrm{H}_2(S, \mathbf{Z})$. We consider its twistor space $T \to \mathbf{S}^2$ described in (2.2) above (we emphasize that this is a real 6-dimensional variety), and let $\iota : S \hookrightarrow T$ be the canonical inclusion of S. Since curves in T can only appear in the fibre S, we have the equality of moduli spaces of stable maps

$$\overline{M}_g(T, \iota_*\beta) = \overline{M}_g(S, \beta),$$

a priori only as sets but in fact as Deligne–Mumford stacks. They come however with two different obstructions theories, hence also with two different virtual classes. Gromov–Witten invariants on T are related to those on S (within the reduced theory for $K3$ surfaces, cf. (2.3)) by the formula

$$\int_{[\overline{M}_g(T,\iota_*\beta)]^{\mathrm{vir}}} 1 \;=\; \int_{[\overline{M}_g(S,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g, \tag{4.2.1}$$

where λ_g stands for the top Chern class $c_g(\mathbf{E}_g)$ of the Hodge bundle $\mathbf{E}_g \to \overline{M}_g(S, \beta)$, whose fibre over the stable map $f : C \to S$ is $\mathrm{H}^0(C, \omega_C)$.

(4.3) Hodge integrals. It follows from the invariance of reduced Gromov–Witten invariants under algebraic deformation and the global Torelli theorem for $K3$ surfaces that the right-hand side of (4.2.1) depends only on the self-intersection β^2 and the divisibility index of β as an algebraic class. We may thus formulate the following definition.

For integers $g \geqslant 0$ and $p, m \geqslant 1$, let

$$R^p_{g,m} := \int_{[\overline{M}_g(S,mL)]^{\mathrm{red}}} (-1)^g \lambda_g \tag{4.3.1}$$

where (S, L) is any primitively polarized $K3$ surface of genus p, and λ_g is the top Chern class of the Hodge bundle as in (4.2) above.

This extends the definition of the invariants $N^p_{0,m}$ in (3.12) above, in the sense that $N^p_{0,m} = R^p_{0,m}$ (note however that the invariants $N^p_{1,2}$ used in subsection 3.3 do not coincide with the $R^p_{1,2}$). For $g > 0$, these invariants are certainly not counting curves on S ; rather, formula (4.2.1) tells us that they virtually give the excess contribution of S to the vertical Gromov–Witten theory of any $K3$-fibred threefold in which it appears as a fibre. This philosophy is put into concrete form by Theorem (5.9) below.

4.2 – The Katz–Klemm–Vafa formula

This is an extension of the Yau–Zaslow conjecture discussed in subsection 3.4 above to the invariants $R^p_{g,m}$. It has been proved by Pandharipande and Thomas [32], see also [31].

(4.4) BPS invariants. It is admittedly better to organize the invariants $R^p_{g,m}$ in BPS form as in (3.13). We have now a clear justification for this, as we have seen in subsection 4.1 above that these invariants really count objects on threefolds.

We first let

$$F^p(u, v) := \sum_{g \geqslant 0} \sum_{m > 0} R^p_{g,m}\, u^{2g-2} v^m$$

as formal power series in the two variables u, v for all positive integers p. One then defines new invariants $r^p_{g,m}$ for all integers $p, m > 0$, $g \geqslant 0$, by setting

$$F^p(u, v) = \sum_{m>0} \sum_{g \geqslant 0} r^p_{g,m} u^{2g-2} \left(\sum_{d>0} \frac{1}{d} \left(\frac{\sin d\frac{u}{2}}{\frac{u}{2}} \right)^{2g-2} v^{dm} \right)$$

$$= \sum_{m>0} \Bigg(r^p_{0,m} u^{-2} \sum_{d>0} \Big(\frac{1}{d^3} + \frac{1}{12d} u^2 + \frac{d}{240} u^4 + \frac{d^3}{6048} u^6 + \frac{d^5}{172800} u^8 + \cdots \Big) v^{dm}$$

$$+ n^p_{1,m} \sum_{d>0} \frac{1}{d} v^{dm}$$

$$+ n^p_{2,m} u^2 \sum_{d>0} \Big(d - \frac{d^3}{12} u^2 + \frac{d^5}{360} u^4 - \frac{d^7}{20160} u^6 + \frac{d^9}{1814400} u^8 + \cdots \Big) v^{dm}$$

$$+ n^p_{3,m} u^4 \sum_{d>0} \Big(d^3 - \frac{d^5}{6} u^2 + \frac{d^7}{80} u^4 - \frac{17 d^9}{30240} u^6 + \frac{31 d^{11}}{1814400} u^8 + \cdots \Big) v^{dm}$$

$$+ \cdots \Bigg).$$

The modifications for genus $g > 0$ objects did not appear earlier in this text. Note that every object counted by $r^p_{g,m}$ contributes to the invariants $R^p_{g',m}$ for all $g' \geqslant g$ (except when $g = 0$), with alternated sign if $g \geqslant 2$. This is in accord with what the phenomenon described in (4.1) suggests.

(4.5) Theorem. (Katz–Klemm–Vafa formula, [32]) *The invariants $r^p_{g,m}$ do not depend on the divisibility index, meaning that one has*

$$r^p_{g,m} = r^{m^2p-m^2+1}_{g,1} = R^{m^2p-m^2+1}_g$$

for all integers $p, m > 0$, $g \geqslant 0$.

They are all determined by the formula

$$\sum_{p\geqslant 0} \sum_{g\geqslant 0} (-1)^g r^p_g (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{2g} q^p = \prod_{n\geqslant 1} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}, \tag{4.5.1}$$

where we set $r^p_g := r^p_{g,1}$ (and $r^0_0 = 1$, $r^0_g = 0$ if $g > 0$, for convenience).

Setting $y = 1$ in the formula restricts to the invariants r^p_0, and recovers the Yau–Zaslow formula of Theorem (1.1). As a first corollary, one gets that $r^p_g = 0$ if $g > p$, and $r^p_p = (-1)^p(p+1)$. The first values of r^p_g are tabulated below.

g \ p	0	1	2	3	4
0	1	24	324	3200	25650
1		−2	−54	−800	−8550
2			3	88	1401
3				−4	−126
4					5

Table 2 – First values of r^p_g

4.3 – Further Gromov–Witten integrals

I close this section with a short discussion of further results about Gromov–Witten integrals on $K3$ surfaces. They all come from [23].

(4.6) Hodge integrals with point insertions. As a direct generalization of the invariants (4.3.1), one may consider the integrals

$$R^p_{g,k,m} := \int_{[\overline{M}_{g,k}(S,mL)]^{\mathrm{red}}} (-1)^{g-k} \lambda_{g-k} \cup \mathrm{ev}_1^*(\mathrm{pt}) \cup \cdots \cup \mathrm{ev}_k^*(\mathrm{pt}),$$

where (S, L) is a primitive $K3$ surface of genus p, $\overline{M}_{g,k}(S, mL)$ is the moduli space of genus g stable maps with k marked points, and λ_i is the i-th Chern class of the Hodge bundle $\mathbf{E}_{g,k} \to \overline{M}_{g,k}(S, mL)$ as in (4.2).

For *primitive* classes on $K3$ surfaces, the following formula is proved by Maulik, Pandharipande and Thomas [23, Thm. 3] :

(4.6.1)
$$\sum_{g=0}^{+\infty}\sum_{p=0}^{+\infty} R^p_{g,k,1}\, u^{2g-2} q^p \; = \; q\, \frac{(2\pi)^{12}}{u^2 \Delta(q)} \cdot \exp\left(\sum_{g=1}^{+\infty} u^{2g} \frac{B_{2g}}{g(2g)!} E_g(q) \right) \cdot \left(\sum_{m=1}^{+\infty} q^m \sum_{d|m} \frac{m}{d} \big(2 \sin d\tfrac{u}{2}\big)^2 \right)^k$$

(notation is as in (2.6)).

Note that in the $k = 0$ case, this contains nothing new with respect to the formula given in Theorem (4.5), as the expression (4.6.1) may be deduced from (4.5.1) using known identities, see [23, § 5.4].

(4.7) Descendent Gromov–Witten invariants. So far, we have been essentially concerned with the reduced Gromov–Witten invariants (see (2.3))

$$N_g(S, \beta) = \int_{[\overline{M}_{g,g}(S,\beta)]^{\mathrm{red}}} \mathrm{ev}_1^*(\mathrm{pt}) \cup \cdots \cup \mathrm{ev}_g^*(\mathrm{pt})$$

counting curves in the algebraic class $\beta \in \mathrm{H}_2(S, \mathbf{Z})$ on the $K3$ surface S and passing through g general fixed points ($\mathrm{pt} \in \mathrm{H}^4(S, \mathbf{Z})$ denotes the (co)homology class of a point). It is of course possible to pull-back more general cohomology classes $\gamma_i \in \mathrm{H}^*(S, \mathbf{Z})$ by the evaluation maps, thus encoding more general incidence conditions than the passing through a given point (although this is not of crucial interest for surfaces due to the divisor axiom of Gromov–Witten theory). Beware that when doing so one gets integrals that do depend on the class β itself, and not only on its self-intersection and divisibility index, as classes in $\mathrm{H}^2(S, \mathbf{Z})$ are not monodromy invariant.

A more sensible generalization is to integrate *descendent classes.* Let $\overline{M}_{g,k}(S, \beta)$ be the moduli space of genus g stable maps with k marked points realizing the class β, and $\mathrm{ev}_1, \ldots, \mathrm{ev}_k$ the corresponding evaluation maps $\overline{M}_{g,k}(S, \beta) \to S$. For all $i = 1, \ldots, k$, define the *i-th cotangent line bundle* L_i to be the line bundle over $\overline{M}_{g,k}(S, \beta)$ the fibre of which over the point $(f : C \to S, p_1, \ldots, p_k)$ is the $\mathbf{C}$-line Ω^1_{C,p_i}. The descendent classes on $\overline{M}_{g,k}(S, \beta)$ are those gotten from the Chern classes of these line bundles.

Let $\psi_i := c_1(L_i) \in \mathrm{H}^2\big(\overline{M}_{g,k}(S, \beta), \mathbf{Q}\big)$. For all cohomology classes $\gamma_1, \ldots, \gamma_k \in \mathrm{H}^*(S, \mathbf{Z})$ and non-negative integers $n_1, \ldots, n_k$ we define the reduced descendent Gromov–Witten invariants

(4.7.1)
$$\big\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \big\rangle_g^{S,\beta} := \int_{[\overline{M}_{g,k}(S,\beta)]^{\mathrm{red}}} \psi_1^{k_1} \cup \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \psi_k^{k_k} \cup \mathrm{ev}_k^*(\gamma_k)$$

whenever the degree of the integrand equals the (real) dimension $2g + 2k$ of the reduced virtual class, and $\big\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \big\rangle_g^{S,\beta} := 0$ otherwise.

How to geometrically interpret the insertion of the classes ψ_i is not straightforward; I refer to [30] and [14] for some discussions about this. See however [14, Thm. 2.2.6], where descendent classes are used to define Gromov–Witten invariants of a projective manifold X *relative* to a smooth very ample hypersurface Y, i.e. invariants virtually counting curves in X with prescribed tangency conditions along Y.

(4.8) Quasi-modularity. The integrals (4.7.1) for fixed integrand and fixed g and divisibility index of β are expected to fit together as the Fourier coefficients of a quasi-modular form, as in Theorem (2.5). Due to their dependency on the class β and not only on its numerical characters, this is formulated as follows.

Let S be an arbitrarily fixed $K3$ surface possessing an elliptic fibration $\pi : S \to \mathbf{P}^1$ and a section E of π. Call $\mathsf{e}, \mathsf{f} \in \mathrm{H}_2(S, \mathbf{Z})$ the classes of E and the fibres of π respectively. It follows from deformation invariance and the same standard degeneration argument as in the proof of Theorem (2.5) that any integral of the form (4.7.1) on any algebraic $K3$ surface equals an integral of the same kind on S with $\beta = a\mathsf{e} + b\mathsf{f}$, a, b non-negative integers.

For all integers $g \geqslant 0$ and $m > 0$, we set

$$F^S_{g,m}\big(\tau_{n_1}(\gamma_1)\cdots\tau_{n_k}(\gamma_k)\big) := \sum_{n\geqslant 0} \big\langle \tau_{n_1}(\gamma_1)\cdots\tau_{n_k}(\gamma_k)\big\rangle_g^{S, m\mathsf{e}+n\mathsf{f}} q^{m(n-m)}$$

as a formal power series in the variable q. Maulik and Pandharipande conjecture the following.

(4.8.1) Conjecture. ([29, Conj. 3] and [23, § 7.5]) *The power series $F^S_{g,m}\big(\tau_{n_1}(\gamma_1)\cdots\tau_{n_k}(\gamma_k)\big)$ is the Fourier expansion in q of a quasi-modular form of level m^2 with pole at $q = 0$ of order at most m^2.*

(A quasi-modular form of level N with possible pole at $q = 0$ is by definition an element of the $\mathbf{C}$-algebra generated by the Eisenstein series G_1 (see (2.6) and modular forms of level N; recall in addition that a modular form of level N is a form satisfying the modular equation for transformations in the congruence subgroup $\Gamma_0(N)$ consisting of elements of $\mathrm{PSL}_2(\mathbf{Z})$ congruent to the identity matrix modulo N).

For $m = 1$, i.e. for primitive classes, this has been proved by Maulik, Pandharipande and Thomas [23, Thm. 4]. Note however that, even in the primitive case, there is as far as I know no general explicit formula for the modular form in question. Theorem (2.5) provides particular instances of such a formula. At any rate, modularity strongly constrains the invariants and in favorable cases enables one to compute them all (see (5.11) for an example in a different context).

Although I will say nothing about the proofs of the results presented in this section, I would like to point out that one fundamental ingredient for them is the use of other counting invariants than those coming from Gromov–Witten theory, together with correspondence theorems between the two. They are more algebraic in nature than Gromov–Witten invariants, and more agile to study the problems we have been discussing. These invariants virtually count *stable pairs*; they were defined by Pandharipande and Thomas, specifically for threefolds up to now. See [28] for a presentation.

5 – Noether–Lefschetz theory and applications

5.1 – Lattice polarized $K3$ surfaces and Noether–Lefschetz theory

In this subsection we define Noether–Lefschetz divisors in the moduli spaces of lattice polarized $K3$ surfaces. While the version we will use is the refined one of (5.4), the elementary version of (5.3) is needed to give a proper definition.

Let $\mathbf{L}_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ be the $K3$ lattice (see, e.g., [2]) and consider throughout this subsection a fixed lattice Λ of rank r and signature $(1, r-1)$ together with a primitive embedding $\iota : \Lambda \hookrightarrow \mathbf{L}_{K3}$ (an embedding is primitive if the corresponding quotient $\mathbf{L}_{K3}/i(\Lambda)$ is torsion-free).

(5.1) Definition. *Let S be a $K3$ surface. A Λ-polarization on S is a primitive embedding $j : \Lambda \hookrightarrow \operatorname{Pic} S$ such that*
(i) there is a nef and big class in $j(\Lambda) \subset \operatorname{Pic} S$;
(ii) there exists an isometry $\phi : \mathrm{H}^2(S, \mathbf{Z}) \to \mathbf{L}_{K3}$ such that $\phi \circ j = \iota$.

A Λ-polarized $K3$ surface is a pair (S, j) where S is a $K3$ surface and j is a Λ-polarization on S.

There exists a moduli space $\mathcal{K}_\Lambda$ of Λ-polarized $K3$ surfaces, which may be constructed relying on the global Torelli theorem by adapting the method of [36, Exp. XIII, §3].

(5.2) Define the discriminant of a rank s lattice L to be the signed determinant

$$\operatorname{Disc} L := (-1)^{s-1} \det\big(\langle v_i, v_j\rangle\big)_{1 \leqslant i,j \leqslant s}$$

where $(v_1, \ldots, v_s)$ is an integral basis of L (the sign has been added to the usual definition so that $\operatorname{Disc}\Lambda > 0$) ; this does not depend on the choice of the basis.

Let $\mathbf{L}$ be a rank $r+1$ lattice with an even symmetric bilinear form, together with a primitive embedding $i : \Lambda \hookrightarrow \mathbf{L}$. There is an invariant of the pair $(\mathbf{L}, i)$ called the *coset*, which is defined as follows. Consider any vector $v \in \mathbf{L}$ such that $\mathbf{L} = i(\Lambda) \oplus v$; the pairing with v determines an element $\ell_v \in \Lambda^\vee$ in the lattice dual to Λ. On the other hand let $G_\Lambda := \Lambda^\vee/\Lambda$ be the quotient of the injection defined by the pairing on Λ ; it is an abelian group of order $\operatorname{Disc}\Lambda$. Now the coset δ of $(\mathbf{L}, i)$ is the class of ℓ_v in $G/\pm$; it does not depend on the choice of v.

Two pairs $(\mathbf{L}, i)$ and $(\mathbf{L}', i')$ as above are isomorphic (i.e. there exists an isometry $\phi : \mathbf{L} \to \mathbf{L}'$ such that $\phi \circ i = i'$) if and only if the two following conditions both hold : (i) $\operatorname{Disc}(\mathbf{L}) = \operatorname{Disc}(\mathbf{L}')$, and (ii) $\delta(\mathbf{L}, i) = \delta(\mathbf{L}', i')$.

(5.3) Elementary Noether–Lefschetz divisors. The Noether–Lefschetz divisor $P^\Lambda_{\Delta,\delta} \subset \mathcal{K}_\Lambda$ is defined as the closure of the locus of Λ-polarized $K3$ surfaces (S, j) such that $\operatorname{Pic} S$ has rank $r+1$ and discriminant Δ, and the coset $\delta(\operatorname{Pic} S, j)$ equals δ.

It follows from the Hodge index theorem that the divisor $P^\Lambda_{\Delta,\delta}$ is empty when $\Delta \leqslant 0$.

(5.4) Refined Noether–Lefschetz divisors. We now fix an integral basis $\mathbf{v}_\Lambda = (v_1, \ldots, v_r)$ for Λ, and let $m \in \mathbf{Z}_{>0}$, $(p, \mathbf{d}) = (p, d_1, \ldots, d_r) \in \mathbf{Z}^{r+1}$. We want to define a Noether–Lefschetz divisor $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} \subset \mathcal{K}_\Lambda$ corresponding to Λ-polarized $K3$ surfaces (S, j) with an extra class $\beta \in \operatorname{Pic} S$ of divisibility index m, and such that $\langle \beta, \beta\rangle = 2p-2$ and $\langle \beta, v_i\rangle = d_i$ for $i = 1, \ldots, r$.

This goes as follows : let

$$\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} := (-1)^r \begin{vmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_r\rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r, v_1\rangle & \cdots & \langle v_r, v_r\rangle & d_r \\ d_1 & \cdots & d_r & 2p-2 \end{vmatrix} ;$$

– if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} > 0$, set

$$D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} := \sum_{\Delta,\delta} \mu^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}(\Delta, \delta) \cdot P^\Lambda_{\Delta,\delta}$$

where the sum runs over all Δ, δ such that there exists a pair $(\mathbf{L}, i)$ as in (5.2) with $\operatorname{Disc} \mathbf{L} = \Delta$ and $\delta(\mathbf{L}, i) = \delta$ (the pair $(\mathbf{L}, i)$ is then unique up to isomorphism), and $\mu^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}(\Delta, \delta)$ is the number of elements $\beta \in \mathbf{L}$ having divisibility index m and satisfying $\langle \beta, \beta \rangle = 2p-2$ and $\langle \beta, v_i \rangle = d_i$ for $i = 1, \ldots, r$. Note that $\mu^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}(\Delta, \delta)$ may be 0 ; in particular its non-vanishing implies that Δ divides $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}}$, so the above sum has only finitely many terms. The condition $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} > 0$ implies that any β such that $\langle \beta, \beta \rangle = 2p-2$ and $\langle \beta, v_i \rangle = d_i$ for all i does not belong to $i(\Lambda)$;

– if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} < 0$, set $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} := 0$;

– if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} = 0$ and $m = \gcd(d_1, \ldots, g_r)$, let $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}$ be the divisor associated to the dual of the Hodge line bundle $\mathcal{E} \to \mathcal{K}_\Lambda$ (the fibre of $\mathcal{E}$ over the point (S, i) is $\mathrm{H}^{2,0}(S)$) ;

– if $\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} = 0$ and $m \neq \gcd(d_1, \ldots, g_r)$, set $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}} := 0$.

5.2 – Invariants of families of lattice polarized $K3$ surfaces

(5.5) Families of lattice polarized $K3$ surfaces. Let $\iota : \Lambda \hookrightarrow \mathbf{L}_{K3}$ be a primitive embedding of a lattice Λ of rank r and signature $(1, r-1)$. A 1-parameter family of Λ-polarized $K3$ surfaces is a smooth family $\pi : X \to C$ of $K3$ surfaces equipped with line bundles $L_1, \ldots, L_r$ on X such that :
(i) X is a compact 3-dimensional complex manifold (not necessarily algebraic), C is a complete smooth complex curve, and π is a holomorphic submersion ;
(ii) for each $t \in C$, the fibre X_t of π over t is a (smooth) $K3$ surface ;
(iii) there exists a linear combination L^π of the holomorphic line bundles L_i on X, the restriction of which to every fibre of π is nef and big ;
(iv) there exists an integral basis $(v_1, \ldots, v_r)$ of Λ such that for each $t \in C$, the map $j_t : \Lambda \to \operatorname{Pic} X_t$ defined by $v_i \mapsto L_{i,t}$ (the restriction of L_i to X_t) is a Λ-polarization of X_t.

For the remainder of this subsection, we consider $(\pi : X \to C, L_1, \ldots, L_r)$ a 1-parameter family of Λ-polarized $K3$ surfaces as in Definition (5.5) above.

(5.6) Noether–Lefschetz numbers. Let $m \in \mathbf{Z}_{>0}$ and $(p, \mathbf{d}) = (p, d_1, \ldots, d_r) \in \mathbf{Z}^{r+1}$. The Noether–Lefschetz number $\mathrm{NL}^\pi_{m,p,\mathbf{d}}$ is defined as

$$\mathrm{NL}^\pi_{m,p,\mathbf{d}} := \int_C f_\pi^*(D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}),$$

where $f : C \to \mathcal{K}_\Lambda$ is the morphism induced from $(\pi : X \to C, L_1, \ldots, L_r)$ by the universal property of $\mathcal{K}_\Lambda$, and $\mathbf{v}_\Lambda$ is the integral basis of Λ defined by $(L_1, \ldots, L_r)$ through point (iv) of Definition (5.5).

Note that this is a classical intersection product (i.e. there is no need to define a virtual class), although it may be given by an excess formula in case the image $f_\pi(C)$ is fully contained in the divisor $D^{\mathbf{v}_\Lambda}_{m,p,\mathbf{d}}$.

(5.7) Gromov–Witten invariants for vertical curve classes. Although it may not be a projective variety, the total space X carries a $(1,1)$-form ω_π which is Kähler on the fibres of π ; this is sufficient to define Gromov–Witten theory for non-zero *vertical* classes $\gamma \in \mathrm{H}_2(X, \mathbf{Z})^\pi$, i.e. classes $\gamma \in \mathrm{H}_2(X, \mathbf{Z})$ such that $\pi_*(\gamma) = 0$ (see [25, §2.1] for details).

We thus have a set of invariants

$$N^X_{g,\gamma} := \int_{[\overline{M}_g(X,\gamma)]^{\mathrm{vir}}} 1$$

for non-zero *vertical* classes γ, where the moduli spaces of genus g stable maps $\overline{M}_g(X,\gamma)$ all have virtual dimension 0. We consider the invariants $n^X_{g,\gamma}$ obtained from the $N^X_{g,\gamma}$ by applying the BPS corrections packaged in the formula of (4.4) : we let

$$F^X(u,v) := \sum_{g\geqslant 0} \sum_{0\neq\gamma\in \mathrm{H}_2(Z,\mathbf{Z})^\pi} N^X_{g,\gamma}\, u^{2g-2} v^\gamma$$

as a formal power series in the variables u, v, where the powers of v are indexed by $\mathrm{H}_2(Z,\mathbf{Z})^\pi$, and set

$$F^X(u,v) := \sum_{g\geqslant 0} \sum_{0\neq\gamma\in \mathrm{H}_2(Z,\mathbf{Z})^\pi} n^X_{g,\gamma}\, u^{2g-2} \left(\sum_{d>0} \frac{1}{d} \left(\frac{\sin d\frac{u}{2}}{\frac{u}{2}} \right)^{2g-2} v^{d\gamma} \right).$$

Eventually, for a non-zero multidegree $\mathbf{d} = (d_1,\dots,d_r) \in \mathbf{Z}^r$, we let $n^X_{g,\mathbf{d}}$ be the invariant counting genus g stable maps in vertical classes of degree $d_1,\dots,d_r$ with respect to $L_1,\dots,L_r$ respectively, i.e.

$$n^X_{g,\mathbf{d}} := \sum_{\gamma\in \mathrm{H}_2(X,\mathbf{Z})^\pi:\ \int_\gamma L_i = d_i} n^X_{g,\gamma}. \tag{5.7.1}$$

(5.8) Reduced Gromov–Witten invariants of $K3$ fibres. We also consider the invariants $r^p_{g,m}$ for $K3$ surfaces which have been defined in (4.4) ; recall they are the reduced Hodge integrals (4.3.1) put under BPS form.

We need to maintain the dependency on the divisibility index m, because Theorem (5.9) below is needed for the proof of the independence on m conjectured by Yau–Zaslow.

A multidegree $\mathbf{d} = (d_1,\dots,d_r) \in \mathbf{Z}^r$ is positive with respect to L^π if for any line bundle M on some fibre X_t of π, $(M, L_{i,t}) = d_i$ for all i implies $(M, L^\pi) > 0$; since L^π is a linear combination of the L_i this is an elementary linear algebraic condition.

(5.9) Theorem. [25] *Let $\mathbf{d} = (d_1,\dots,d_r) \in \mathbf{Z}^r$ be a multidegree positive with respect to L^π. Then*

$$n^X_{g,\mathbf{d}} = \sum_{p=0}^{+\infty} \sum_{m=1}^{+\infty} r^p_{g,m} \cdot \mathrm{NL}^\pi_{m,p,\mathbf{d}}. \tag{5.9.1}$$

(This is stated in [25] in the $r = 1$ case (*i.e.* $\mathbf{d} \in \mathbf{Z}$), but as noted in [20] the same proof goes through in general).

The philosophy behind this relation is rather natural, and ought to be compared to the discussion of subsection 4.1 above. Consider the genus $g = 0$ case for simplicity ; then the invariant $n^X_{0,\mathbf{d}}$ counts vertical rational curves in X of prescribed degrees with respect to $L_1,\dots,L_r$, and these are virtually in finite number. There are on the other hand finitely many members of the family π with algebraic divisor classes of the prescribed degrees with respect $L_1,\dots,L_r$, and each of these provides a finite number of rational curves. The theorem morally says that the number of rational curves in X is the sum of these isolated contributions from the fibres.

Of course the actual story is more complicated than this, if only because of the existence of 1-dimensional families of rational curves on X, coming from finitely many rational curves in *all* $K3$ members of the family (which all have algebraic divisor classes, as they are Λ-polarized), in spite of the virtual dimension being 0. In other words, a Calabi-Yau threefold X as in Theorem (5.9) above is far from satisfying the same properties than the perturbations of the twistor families of algebraic $K3$ surfaces on which BPS numbers are supposed to count curves.

5.3 – Application to the Yau–Zaslow conjecture

In this subsection we give an outline of the proof by Klemm, Maulik, Pandharipande, and Scheidegger of the Yau–Zaslow conjecture (Theorem (3.14) above). Recall that the invariants $r^p_{g,m}$ being invariant under algebraic deformations of the $K3$ surface, it is enough to prove the result for our favourite $K3$ surface. These invariants for certain elliptic $K3$ surfaces are approached by means of the relation (5.9) for a particular family.

(5.10) The ***STU*** **model.** The central character of the proof is a smooth projective Calabi–Yau 3-fold X, known as the STU model and coming from physics (quoting [20], the letter S stands for the dilaton and T and U label the torus moduli in the heterotic string). It is contructed as an anticanonical section of a smooth projective toric 4-fold Y defined by an explicit fan in $\mathbf{Z}^4$.

The variety X has the structure of a fibration $\pi : X \to \mathbf{P}^1$, the general fibre of which is a smooth $K3$ surface, itself with an elliptic fibration. It comes with two line bundles $L_1, L_2 \to X$, defining a Λ-polarization on the family $\pi : X \to \mathbf{P}^1$ (leaving aside the fact that there are inevitably singular members), where Λ is the lattice with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The family π has the shape of a Lefschetz pencil, in particular each of its singular members has a unique ordinary double point as its only singularity. One may thus build an actual family $\tilde{\pi} : \tilde{X} \to C$ of Λ-polarized $K3$ surfaces from π as follows. One first performs a base change by $t \mapsto t^2$ around each singular member ; to do so, one considers the $2:1$ covering $\varepsilon : C \to \mathbf{P}^1$ with branch divisor $\mathrm{Disc}(\pi)$, the set of points above which π fails to be smooth, and let $\pi^\flat : X^\flat \to C$ be the family obtained from π by applying the base change $\varepsilon : C \to \mathbf{P}^1$. The new total space $X^\flat$ is singular, precisely it has an ordinary 3-fold double point at each singular point of a fibre (analytically locally around such a point, X is defined by the equation $x^2 + y^2 + z^2 = t$ in a 4-dimensional complex ball, hence $X^\flat$ is defined by $x^2 + y^2 + z^2 = t^2$). One then chooses for $\tilde{X}$ any small resolution of all these singularities : this may be understood as first blowing-up once all singular points, and then contracting one ruling of each exceptional divisors (they are all smooth quadric surfaces). This has the effect of replacing each fibre of $X^\flat$ by its minimal model.

One may determine the number of singular members of π by the same topological Euler characteristic computation as in subsection 1.1. The Euler number $e(X)$ is found to be -480 by toric intersection in the 4-fold Y, and then the number of singular fibres equals

$$e(K3) \cdot e(\mathbf{P}^1) - e(X) = 528.$$

(5.11) Modularity for Noether–Lefschetz numbers. It is a stunning application of a theory developed by Borcherds and Kudla–Millson (see [25, 20] and the references therein) that the Noether–Lefschetz numbers of a family of Λ-polarized $K3$ surfaces fit into a vector valued modular form.

Let notation be as in (5.6) for a moment, in order to state this precisely (see [25, § 4] for a complete treatment). One may define divisors $D^{\mathbf{v}_\Lambda}_{p,\mathbf{d}}$ and subsequently numbers $\mathrm{NL}^\pi_{p,\mathbf{d}}$ by dropping the requirement on the divisibility index m in (5.4). It is an elementary result [20, Lemma 1] that the full set of the numbers $\mathrm{NL}^\pi_{p,\mathbf{d}}$ determine the refined Noether–Lefschetz numbers $\mathrm{NL}^\pi_{m,p,\mathbf{d}}$. Let $\mathrm{Mp}_2(\mathbf{Z})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbf{Z})$. There is a canonical representation

$$\rho^*_\Lambda : \mathrm{Mp}_2(\mathbf{Z}) \to \mathrm{End}\big(\mathbf{C}[G_\Lambda]\big)$$

associated to Λ (recall that $G_\Lambda = \Lambda^\vee / \Lambda$).

(5.11.1) Theorem. (Borcherds, Kudla–Millson, Maulik–Pandharipande) *There exists a vector-valued modular form*

$$\Phi^\pi(q) = \sum_{\gamma\in G} \Phi^\pi_\gamma(q)\, u^\gamma \in \mathbf{C}[[q^{\frac{1}{2\,\mathrm{Disc}\,\Lambda}}]] \otimes \mathbf{C}[G]$$

of weight $\frac{22-r}{2}$ *and type* ρ^*_Λ*, such that the Noether–Lefschetz number* $\mathrm{NL}^\pi_{p,\mathbf{d}}$ *is the coefficient of* Φ^π_γ *in* q *to the power* $\frac{\Delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}}}{2\,\mathrm{Disc}\,\Lambda}$*, where* $\gamma \in G$ *is any of the two liftings of the coset* $\delta^{\mathbf{v}_\Lambda}_{p,\mathbf{d}} \in G/\pm$ *represented by the linear functional* $v_i \mapsto d_i$*.*

Taking advantage of the strong structure results for modular forms, Maulik and Pandharipande are able to use this theorem to derive explicitly the Noether–Lefchetz numbers of classical families of $K3$ surfaces of genus $2 \leqslant p \leqslant 5$ (i.e. double planes and complete intersection $K3$'s).

A similar calculation is carried out in [20] for the STU family, as one of the key steps in the proof of the Yau–Zaslow conjecture. We now return to the notation of (5.10). Theorem (5.11.1) tells that the Noether–Lefschetz numbers of the family $\tilde{\pi} : \tilde{X} \to C$ are the Fourier coefficients of a scalar modular form of weight 10. The vector space of such forms has dimension 1 and is generated by the Eisenstein series

$$E_5(q) = E_2(q)E_3(q) = 1 - 264 \sum_{n=1}^{+\infty} \sigma_9(n) q^n$$

[39, § VII.3.2] (notation as in (2.6)). It follows that it is enough to know one Noether–Lefschetz number to determine the full modular form, and since we do know of them, given by the number 528 of singular members of the STU family, one obtains that the number $\mathrm{NL}^{\tilde{\pi}}_{p,d_1,d_2}$ is the coefficient in q to the power $\frac{1}{2}\Delta(p, d_1, d_2)$ of the modular form $-4E_2(q)E_3(q)$, where

$$\Delta(p, d_1, d_2) = \begin{vmatrix} 0 & 1 & d_1 \\ 1 & 0 & d_2 \\ d_1 & d_2 & 2p-2 \end{vmatrix}.$$

(5.12) Mirror symmetry. The STU model X being an anticanonical section of a smooth semi-positive toric variety, its genus 0 Gromov–Witten invariants are known by mathematically proven mirror symmetry results. This gives the corresponding invariants of $\tilde{X}$, the latter being twice those of X [25, § 5.2].

Precisely, Givental has proven the relation of the genus 0 Gromov–Witten invariants of X by mirror transformation to hypergeometric solutions of the Picard–Fuchs equations of the Batyrev–Borisov mirror, see [25, 20] and the references therein. This gives the following formula of Klemm–Mayr–Lerche [20, Prop. 5]

$$\sum_{(d_1,d_2)\in\mathcal{P}} (d_2)^3 N^X_{0,(d_1,d_2)}\, q_1^{d_1} q_2^{d_2} = -2 + 2\,\frac{E_2(q_1)E_3(q_1)}{(2\pi)^{-12}\Delta(q_1)}\,\frac{E_2(q_2)}{j(q_1) - j(q_2)}, \qquad (5.12.1)$$

where

$$j(q) := 1728\,\frac{\big(60G_2(q)\big)^3}{\Delta(q)} = (2\pi)^{12}\frac{E_2(q)^3}{\Delta(q)} = \frac{1}{q} + 744 + 196884q + \cdots$$

(notation as in (2.6)) is the normalized j function, $\mathcal{P} = \big\{(d_1, d_2) \neq (0,0) : d_1 \geqslant 0, d_1 \geqslant -d_2\big\}$, and $N^X_{0,(d_1,d_2)}$ is defined by formula (5.7.1) from the various $N^X_{0,\gamma}$, $\gamma \in \mathrm{H}_2(X, \mathbf{Z})^\pi$.

(5.13) Conclusion : the Harvey–Moore identity. Using the fact that the lattice Λ has rank 2, Klemm–Maulik–Pandharipande–Scheidegger then show that the invariants $r^p_{0,m}$ are uniquely determined by the relations (5.9.1) for the family $\tilde{\pi} : \tilde{X} \to C$ and the numbers $n^{\tilde{X}}_{0,(d_1,d_2)}$ and $\mathrm{NL}^{\tilde{\pi}}_{m,p,(d_1,d_2)}$ [20, Prop. 3]. The latter two sets of numbers being known by the results of (5.11) and (5.12), it is therefore enough, in order to end the proof of Theorem (3.14), to show that the numbers $r^p_{0,m}$ predicted by the Yau–Zaslow conjecture (i.e. $r^p_{0,m} = r^{m^2p-m^2+1}_{0,1}$ together with the formula of (1.1) giving the $r^p_{0,1}$'s) indeed fit in the relations (5.9.1).

This takes the form of an identity between modular forms : let

$$f(z) := \frac{E_2(z)E_3(z)}{(2\pi)^{-12}\Delta(z)} = \sum_{n=-1}^{+\infty} c(n)q^n, \quad q = e^{2\pi i z};$$

what has to be proven is

$$\frac{f(z_1)E_2(z_2)}{j(z_1)-j(z_2)} = \frac{q_1}{q_1-q_2} + E_2(z_2) - \sum_{d,k,l>0} l^3 c(kl)\, q_1^{kd} q_2^{ld}.$$

This is the Harvey–Moore identity, which has been proven by Zagier, see [20, § 4.2].

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