Mapping Class Groups & their Linear Representations

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- Σ an orientable compact surface of genus $g \geqslant 0$
- $\mathrm{Diff}^+(\Sigma, \partial\Sigma) = \{\phi \in \mathrm{Diff}^+(\Sigma) : \phi(x) = x \text{ for all } x \in \partial\Sigma\}$
- $\phi \simeq \psi \in \mathrm{Diff}^+(\Sigma, \partial\Sigma) \rightsquigarrow \phi_* = \psi_* : H_1(\Sigma, \mathbb{C}) \longrightarrow H_1(\Sigma, \mathbb{C})$
- $M_\phi = {}^{\Sigma \times [0,1]}\big/_{(\phi(x),0) \sim (x,1)}$ is invariant under isotopy

Definition

The *mapping class group* of Σ is

$$\mathrm{Mod}(\Sigma) := {}^{\mathrm{Diff}^+(\Sigma, \partial\Sigma)}\big/_{\text{isotopy}}.$$

Linear Representations

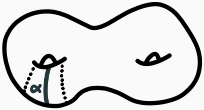
- $\rho : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{GL}_n(\mathbb{C})$
- First examples
  - Symplectic representation $\Psi : \mathrm{Mod}(\Sigma_g) \longrightarrow \mathrm{GL}(H_1(\Sigma_g, \mathbb{C}))$
  - TQFT representations
- The full picture is known for $n \leqslant 2g + 1$

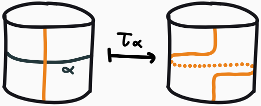
Theorem (Korkmaz, '23)
Let $g \geqslant 2$, $n < 2g$ and $\rho : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{GL}_n(\mathbb{C})$.

1. $\rho(\mathrm{Mod}(\Sigma))$ is Abelian.
2. If $g \geqslant 3$ then ρ is trivial.

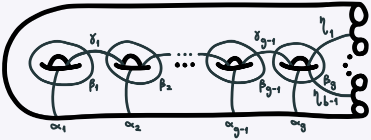
Dehn Twists

- $\alpha \subset \Sigma$ simple closed curve $\rightsquigarrow \tau_\alpha \in \mathrm{Mod}(\Sigma)$



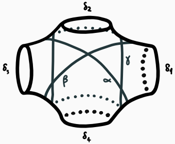


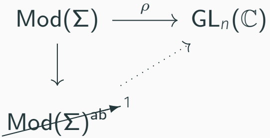
Theorem (Lickorish generators)
$\mathrm{Mod}(\Sigma)$ is generated by the twists about the followings curves.



- Lantern relation:

  $\tau_\alpha \tau_\beta \tau_\gamma = \tau_{\delta_1} \tau_{\delta_2} \tau_{\delta_3} \tau_{\delta_4}$



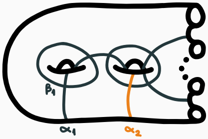


Theorem
$\mathrm{Mod}(\Sigma)^{\mathrm{ab}} = {}^{\mathrm{Mod}(\Sigma)}/_{[\mathrm{Mod}(\Sigma),\,\mathrm{Mod}(\Sigma)]}$ is trivial for $g \geqslant 3$.

Theorem (Korkmaz, '00)
$[\mathrm{Mod}(\Sigma), \mathrm{Mod}(\Sigma)]$ is *normally generated* by $\tau_\alpha \tau_\beta^{-1}$ with $\#(\alpha \cap \beta) = 1$ for $g \geqslant 2$.

$\rho(\mathrm{Mod}(\Sigma))$ is Abelian

- $\rho : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{GL}_n(\mathbb{C})$, $n < 2g$
- $g = 2$



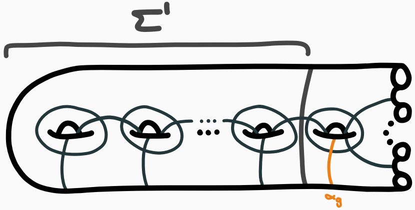
$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

$\rho(\mathrm{Mod}(\Sigma))$ is Abelian

- Induction in $g \geqslant 3$



- The generators of $\mathrm{Mod}(\Sigma')$ commute with τ_{α_g}!
- The eigenspaces of $\rho(\tau_{\alpha_g})$ are $\mathrm{Mod}(\Sigma')$-invariant ∎

Thank You!