REPRESENTATION THEORY

The Algebraic & the Geometric

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This notes are dedicated to my dear friend Lucas Schiezari, who somehow convinced me to apply for a bachelor's degree in pure mathematics. May he rest in peace.

About This Notes

Remark. Under construction!

These notes mostly amount to an amalgamation of thoughts and ideas I came across when studding the representation theory of groups. The primary focus of this notes is the beautiful interaction between algebra and geometry that occurs in representation theory. At first glance representation theory may seem like just another branch of abstract algebra. Historically, however, algebraic proofs in the representation theory of groups have been preceded by geometric proofs – sometimes by several decades. This is something Georgie Williamson discusses at length in the excellent *Representation theory and geometry* [Wil18], but perhaps the idea is better synthesized in the eloquent words of Élie Cartan:

> ... the difficulty, dare I not say the impossibility, of finding a proof which does not leave the strict infinitesimal domain shows the necessity of not sacrificing either point of view ...

The last quote is something Cartan wrote to Herman Weyl in 1925, after Weyl published his proof of complete reducibility of representations of complex semisimple Lie algebras – those being "the strict infinitesimal domain". His proof relied heavily on Weyl's previous work on smooth representations of compact Lie groups, and a purely algebraic proof would only surface after about a decade. This is a particular example of the common phenomena described by Williamson.

Throughout this notes we'll follow the following guiding principles:

(i) Lengthy proves are favored as opposed to collections of smaller lemmas. This is a deliberate effort to emphasize the relevant results.

(ii) Geometric proofs, as opposed to purely algebraic proofs, are generally preferred. This is again a deliberate effort to emphasize the connections between the geometric and the algebraic. We should clarify that when we say *geometric* we mean it in a very general sense – basically anything vaguely motivated by some notion of *space*. That is to say, when we say *geometry* we don't necessarily mean *differential geometry* or *algebraic geometry*.

(iii) We prefer, whenever possible, to outsource proofs. This is because I don't fancy reinventing the wheel: I'll write down proofs in here *only* when I fell like I have something to add to the proofs provided by other materials. Otherwise we refer the reader to the proofs in other books or articles – which will likely be indicated just after the statement of the theorem.

We'll assume basic knowledge of abstract algebra, group theory and differential geometry. Additional topics will be covered when needed.

Contents

Chapter 1

Groups & Actions

> A group is a groupoid with a single element
>
> *Paolo Aluffi*
> *Algebra: Chapter 0*

Group theory has a special place in abstract algebra: groups are simple and elegant, yet interesting enough to study – unlike monoids or magmas. The interesting thing about groups though is they transcend abstract algebra. What I mean by that is that in some sense groups are much more fundamental than what their simple algebraic structure may imply. Groups are algebraic incarnation of *actions* and *symmetries*.

Definition. If $\mathcal{C}$ is a category and A is an object of $\mathcal{C}$, an *action* of G in A is a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(A)$$

If ρ is injective ρ is called a faithful action.

Remark. When $\mathcal{C}$ is a subcategory of the category **Set** of sets and the map ρ is clear from the context we write $g(x)$ instead of $(\rho(g))(x)$.

In practice, when $\mathcal{C}$ is a subcategory of **Set** the fact $\rho : G \longrightarrow \mathrm{Aut}(A)$ is a group homomorphism means $(g \cdot h)(a) = g(h(a))$ for all $a \in A$. Just like men – and women – a group is known by its actions. Indeed, Caley's theorem establishes that every group is isomorphic to a subgroup of a permutation group. In other words, every group G is characterized by a set X and a faithful action of G in X, namely $X = G$ and

$$\begin{aligned} \rho : G &\longrightarrow S_X \\ g &\longmapsto \rho(g) : X \longrightarrow X \\ &\qquad\qquad\quad x \longmapsto g \cdot x \end{aligned}$$

This implies a group is essentially a group action in **Set**. In the abstract *categorical* terms of the last definition, Caley's theorem amounts to the epigraph if this chapter – which is a fancy way of saying *a group is a thing that acts on some other* arbitrary *thing*.

We are usually interested in actions of a group G with some extra structure that *respect the extra structure of* G in some sense or another, such as smooth isometric actions of a Lie group over Riemannian manifolds.

Example. If you squint your eyes enough, a discrete dynamical system may be thought of as an action of the additive group $\mathbb{Z}$ by continuous functions. Indeed, given a topological space X, an action $\mathbb{Z} \longrightarrow \mathrm{Homeo}(X)$ over X is determined by the action of 1: if 1 acts via $f : X \longrightarrow X$ then the action of n is given by the iteration f^n. Conversely, any homeomorphism $f : X \longrightarrow X$ defines an action of $\mathbb{Z}$ in X. Likewise, a continuous dynamical system can be though of as an action $\mathbb{R} \longrightarrow \mathrm{Homeo}(X)$ of the additive group $\mathbb{R}$ which is continuous in the sense that the map

$$\mathbb{R} \times X \longrightarrow X$$

is continuous – i.e. it preserves the topological structure of $\mathbb{R}$.

Example. If G is a Lie group acting on a Riemannian manifold M, we say G acts by isometries if for each $g \in G$ the operator $g : M \longrightarrow M$ is an isometry – i.e. a diffeomorphism that preserves the Riemannian metric. If the map

$$\begin{aligned} G \times M &\longrightarrow M \\ (g, m) &\longmapsto g(m) \end{aligned}$$

is smooth we say the action of G is smooth. Alternatively, by *the Myners-Steenrod theorem* if M is connected[1] then the group $\mathrm{Isom}(M)$ of isometries of M – i.e. the group of automorphisms of M in the category of Riemannian manifolds – has the canonical structure of a Lie group with respect to the compact-open topology, and we say the action of G is smooth if the group homomorphism $G \longrightarrow \mathrm{Isom}(M)$ is smooth. These two notions of smoothness of the action of G are compatible for every connected M.

We'll look into some other examples of this when we discuss continuous representations of compact groups in chapter 2 and smooth representations of Lie groups in chapter 3, but before we get into that let's review some basic concepts in representation theory.

1.1 Representations & Complete Reducibility

The single most well understood category in the entirety of mathematics is $\mathbb{C}$-**Vect**. Hence actions on $\mathbb{C}$-**Vect** are a natural starting point for understanding the behavior of a given group.

Definition 1.1.1. A complex vector space V equipped with an action $\rho : G \longrightarrow \mathrm{GL}(V)$ is called a *representation* of G – or alternatively a *linear action of G*. It's common to denote $\rho(g)$ by simply g when ρ can be inferred from the surrounding context.

We should note that there is nothing stopping us to consider representations over arbitrary fields. I other words, it's perfectly reasonable consider linear actions of G in a k-vector space: just replace $\mathbb{C}$ by k in the previous definition. We will see examples of representations over fields other than $\mathbb{C}$ in chapter 3 – namely $k = \mathbb{R}$. Nevertheless, throughout this notes we will be primarily concerned with representations over $\mathbb{C}$.

There are multiple reasons for doing so. The rich algebraic properties of $\mathbb{C}$, namely the fact that it is an algebraically closed field of characteristic zero, play an important role in the theory we will develop in this chapter. Furthermore, the geometry of $\mathbb{C}$, namely the fact that it is a complete field, will prove fundamental to the theory of later chapters. We are getting ahead of ourselves, however. For now, let's focus on some concrete examples.

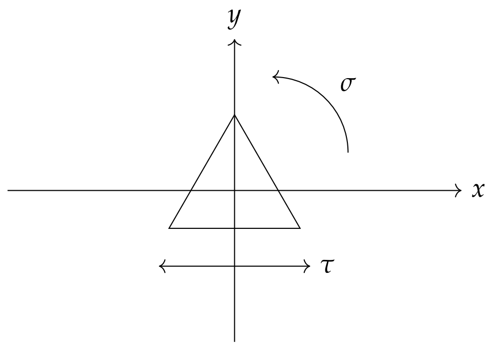
Example 1.1.1. Given any group G, the space $\mathbb{C}$ with

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}(\mathbb{C}) \\ g &\longmapsto \mathrm{Id} \end{aligned}$$

is a representation of G, called the *trivial representation G*.

Example 1.1.2. The dihedral group $D_3 = \langle \sigma, \tau : \sigma^3 = \tau^2 = (\tau\sigma)^2 = e \rangle$ acts linearly on the Cartesian plane via

[1]Clearly, every manifold is connected – otherwise we're really talking about multiple manifolds.



This is an example of a *faithful* representation of D_3 – i.e. a representation where ρ is injective.

Example 1.1.3. The permutation group S_n acts on $\mathbb{C}^n$ by permuting coordinates. In other words,

$$e_i \overset{\sigma}{\longmapsto} e_{\sigma(i)}$$

gives $\mathbb{C}^n$ the structure of a representation of S_n. This is known as *the permutation representation of* S_n.

Example 1.1.4. Given a group G, the group algebra $\mathbb{C}[G]$ – whose elements are formal sums of elements of G and the product is given by the product of G – equipped with the left multiplication

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}(\mathbb{C}[G]) \\ g &\longmapsto \rho(g) : \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \\ &\qquad\qquad\quad x \longmapsto g \cdot x \end{aligned}$$

is a representation of G.

For finite groups, $\mathbb{C}[G]$ is identified with the space $\mathbb{C}^G$ of functions $G \longrightarrow \mathbb{C}$, which is called *the regular representation of* G. Under this identification,

$$(g \cdot f)(h) = f(g^{-1}h)$$

The regular representation of a group G is quite important for a number of reasons. Perhaps the primary one is. . .

Theorem 1.1.1. *There is one-to-one correspondence between $\mathbb{C}[G]$-modules and representations of G. This correspondence takes a finitely-generated modules to a finite-dimensional representation.*

Proof. It suffices to note that given a $\mathbb{C}[G]$-module M the restriction of the map

$$\begin{aligned} \rho : \mathbb{C}[G] &\longrightarrow \mathrm{End}_{\mathbb{C}[G]}(M) \\ a &\longmapsto \rho(a) : M \longrightarrow M \\ &\qquad\qquad\quad m \longmapsto a \cdot m \end{aligned}$$

to $G \subseteq \mathbb{C}[G]$ gives M the structure of a representation of G. Conversely a group homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$ can be extended by linearity to an algebra homomorphism $\mathbb{C}[G] \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$, which gives the abelian group V the structure of a $\mathbb{C}[G]$-module. ∎

In general, given a field k there is a one-to-one correspondence between $k[G]$-modules and representations of G over k. This is fundamental to the theory of modular representations, which is the study of representations of a finite group G over $\mathbb{F}_{p^\ell}$ where $p \mid |G|$. We will, however, focus on representations over $\mathbb{C}$, in which case this correspondence isn't strictly necessary to the theory. Nevertheless, this last result is still useful to us: among other things, it tells us we can expect the realm of representation theory to be closed under the sort of constructions one studies in linear algebra. Indeed. . .

Example 1.1.5. Given two representations V and W of G, the spaces $V \oplus W$, $V \otimes W$, V^* and $\mathrm{Hom}(V, W)$ are all representations of G, where the action of G is given by

$$g(v + w) = gv + gw \qquad g(v \otimes w) = gv \otimes gw$$
$$(gf)(v) = f(g^{-1}v) \qquad (gT)(v) = g(T(g^{-1}v))$$

Definition 1.1.2. Given a group G and a representation V of G, a subspace $W \subseteq V$ such that $GW \subseteq W$ is called a *subrepresentation* of V.

Example 1.1.6. Given a representation V of a group G and a subrepresentation $W \subseteq V$, the space ${}^{V}/{}_{W}$ is a representation of G where

$$v + W \overset{g}{\longmapsto} gv + W$$

Example 1.1.7. Given a representation V of G and a representation W of H, the space $V \boxtimes W = V \otimes W$ with

$$(g, h)\ v \otimes w = gv \otimes hw$$

is a representation of $G \times H$.

As you might have guessed, *representation theory* (of groups) is the study of *representations* of groups. The general goal of representation theory is to extract as much information about a given group G as possible from its representations. It's also worth noting that understanding the relationship between representations is an integral part of representation theory. This brings us to the following definitions.

Definition 1.1.3. Given a group G and two representations V and W of G, a linear map $T : V \longrightarrow W$ such that

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ g\big\downarrow & & \big\downarrow g \\ V & \xrightarrow[T]{} & W \end{array}$$

for all $g \in G$ is called an *intertwing* operator or an *intertwiner*. Intertwing operators can be thought-of as *maps that preserve the action of* G.

Example 1.1.8. Given a representation V of G and a subrepresentation $W \subseteq V$, the projection $V \longrightarrow {}^{V}/{}_{W}$ and the inclusion $W \longrightarrow V$ are both intertwining operators.

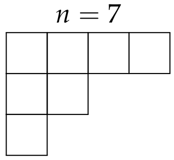
Definition 1.1.4. Let V be a representation of a group G. If V admits no proper subrepresentations – subrepresentations other than 0 and V – then V is called an *irreducible* representation. On the other hand, if V cannot be broken into the direct sum of two proper non-zero subrepresentations then V is called *indecomposable*.

Example 1.1.9. Example 1.1.1 is an example of an irreducible representations. In fact, every 1-dimensional representation V of a group G is irreducible: V has no non-zero proper subspaces, let alone G-invariant subspaces.

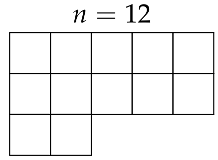
Example 1.1.10. Example 1.1.2 is an example of an irreducible representation: τ rotates a line through the origin by 60°, so that no such line is stable under the action of D_3.

Example 1.1.11. The irreducible representations of the symmetric groups S_n are classified by Young diagrams with n boxes, i.e. rectangular arrays of boxes where the number of boxes in each row decreases, and whose total number of boxes is precisely n, such as in the following examples.

$n = 7$



$n = 12$



These definitions naturally induce a category $\mathbf{Rep}(G)$, whose objects are representations of G and whose morphisms are intertwining operators. We'll abbreviate $\operatorname{Hom}_{\mathbf{Rep}(G)}(V, W)$ to $\operatorname{Hom}_G(V, W)$ and denote the full subcategory of finite-dimensional representations of G by $\mathbf{rep}(G)$. Theorem 1.1.1 then translates to an isomorphism of categories $\mathbb{C}[G]$-$\mathbf{Mod} \xrightarrow{\sim} \mathbf{Rep}(G)$ which restricts to an isomorphism $\mathbb{C}[G]$-$\mathbf{mod} \xrightarrow{\sim} \mathbf{rep}(G)$, while the assertion that "the realm of representation theory is closed under linear-algebraic constructions" translates to the fact that $\mathbf{Rep}(G)$ is an Abelian category.

It turns out much of the structure of a group G can be reconstructed from $\mathbf{rep}(G)$. In fact, a multitude of reconstruction theorems from the likes of Lev Pontryagin and Tadao Tannaka establish that, in certain contexts, the entirety of G can be reconstructed from $\mathbf{rep}(G)$. This hopefully establishes that *representation theory is useful*, but it also poses a problem. The reconstruction theorems require us to understand the whole of $\mathbf{Rep}(G)$ – or at least a large chunk of it. In other words, understanding individual representations won't get us anywhere, we need to study the collective behavior of *all* representations of group to be able to extract useful information from them.

Hence the classical problem in representation theory is classifying all representations of a given group G up to isomorphism. This turns out to be hard. The general strategy for classifying finite-dimensional representations of a group is to classify the indecomposable representations. This is because. . .

Theorem 1.1.2 (Krull-Schmidt)**.** *Every finite-dimensional representation of a group can be uniquely – up the isomorphisms and reordering of the summands – decomposed into a direct sum of indecomposable representations.*

Hence finding the indecomposable representations suffices to find *all* (finite-dimensional) representations: they are the direct sum of indecomposable representations. The existence of the decomposition should be clear from the definitions. Indeed, if V is representation of G a simple argument via induction in $\dim V$ suffices to prove the existence: if V is indecomposable then there is nothing to prove, and if V is not indecomposable then $V = W \oplus U$ for some $W, U \subsetneq V$ non-zero subrepresentations, so that their dimensions are both strictly smaller than $\dim V$ and the existence follows from the induction hypothesis. For a proof of uniqueness please refer to [Eti11].

Finding the indecomposable representations of an arbitrary group, however, turns out to be a bit of a circular problem: the indecomposable representations are the ones that cannot be decomposed, which is to say, those that are *not* decomposable. Ideally, we would like to find some other condition, equivalent to indecomposability, but which is easier to work with. It is clear from the definitions that every irreducible representation is indecomposable, but there is no reason to believe the converse is true. Indeed, this is not always the case. For instance. . .

Example 1.1.12. The space $V = \mathbb{C}^2$ endowed with the operators

$$\rho(n) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$$

is a representation of the additive group $\mathbb{Z}$. Indeed, $\rho(a+b) = \rho(a)\rho(b)$, which is to say, $\rho : \mathbb{Z} \longrightarrow \mathrm{GL}(V)$ is a group homomorphism. Notice V has a single non-zero proper subrepresentation, which is spanned by the vector $(1, 0)$. This is because if $(x + y, y) = \rho(1)\ (x, y) = \lambda(x, y)$ for some $\lambda \in \mathbb{C}$ then $\lambda = 1$ and $y = 0$. Hence V is indecomposable – it cannot be broken into a direct sum of 1-dimensional subrepresentations – but it is evidently not irreducible.

This counterexample poses an interesting question: are there conditions one can impose on a group G under which every indecomposable representation of G is irreducible? Alternatively, are there classes of representations – i.e. subcategories of $\mathbf{Rep}(G)$ – such that every indecomposable representations within this classes is irreducible? This is what is known in representation theory as *complete reducibility* or *semisimplicity*.

Definition 1.1.5. A representation V of a group G is called *completely reducible* if every subrepresentation $W \subseteq V$ admits a G-invariant complement – i.e. a subrepresentation $U \subseteq V$ such that $V = W \oplus U$.

Example 1.1.13. Example 1.1.12 shows that not every representation is completely reducible.

In case the relationship between complete reducibility and the irreducibility of indecomposable representations is unclear, the following results should clear things up.

Proposition 1.1.1. *Given a group G, the following conditions are equivalent.*

- **(i)** *Every finite-dimensional representation of G is completely reducible.*
- **(ii)** *Every exact sequence of finite-dimensional representations of G splits.*
- **(iii)** *Every finite-dimensional indecomposable representation of G is irreducible.*
- **(iv)** *Every finite-dimensional representation of G can be uniquely decomposed as a direct sum of irreducible representations.*

Proof. We begin by **(i)** $\Rightarrow$ **(ii)**. Let

$$0 \longrightarrow W \xrightarrow{\ f\ } V \xrightarrow{\ g\ } U \longrightarrow 0$$

be an exact sequence of representations of G. We can suppose without any loss of generality that $W \subseteq V$ is a subrepresentation and f is the usual inclusion, for if this is not the case there is an isomorphism of sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{\ f\ } & V & \xrightarrow{\ g\ } & U & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \| & & \| & & \\ 0 & \longrightarrow & f(W) & \longrightarrow & V & \xrightarrow[g]{} & U & \longrightarrow & 0 \end{array}$$

It then follows from **(i)** that there exists a subrepresentation $U' \subseteq V$ such that $V = W \oplus U'$. Finally, the projection $\pi : V \longrightarrow W$ is an intertwiner satisfying

$$0 \longrightarrow W \underset{\pi}{\overset{f}{\rightleftarrows}} V \xrightarrow{\ g\ } U \longrightarrow 0$$

Next is **(ii)** $\Rightarrow$ **(iii)**. If V is an indecomposable representation of G and $W \subseteq V$ is a subrepresentation, we have an exact sequence

$$0 \longrightarrow W \xrightarrow{\ i\ } V \xrightarrow{\ \pi\ } {}^{V}\!/_{W} \longrightarrow 0$$

of representations of G.

Since our sequence splits, we must have $V \cong W \oplus {}^{V}\!/_{W}$. But V is indecomposable, so that either $W = V$ or $W = 0$. Since this holds for all $W \subseteq V$, V is irreducible. For **(iii)** $\Rightarrow$ **(iv)** it suffices to apply theorem 1.1.2.

Finally, for **(iv)** $\Rightarrow$ **(i)**, if we assume **(iii)** and let V be a representation of G with decomposition into irreducible subrepresentations

$$V = \bigoplus_i V_i$$

and $W \subseteq V$ is a subrepresentation. Take some maximal set of indexes $\{i_1, \ldots, i_n\}$ so that $\left(\bigoplus_k V_{i_k}\right) \cap W = 0$ and let $U = \bigoplus_k V_{i_k}$. We want to establish $V = W \oplus U$.

Suppose without any loss in generality that $i_k = k$ for all k and let $j > n$. By the maximality of our set of indexes, there is some non-zero $w \in (V_j \oplus U) \cap W$. Say $w = v_j + v_1 + \cdots + v_n$ with each $v_i \in V_i$. Then $v_j = w - v_1 - \cdots - v_n \in V_j \cap (W \oplus U)$ is non-zero. Indeed, if this is not the case we find $0 \neq w = v_1 + \cdots + v_n \in (\bigoplus_{i=1}^n V_i) \cap W$, a contradiction. This implies $V_j \cap (W \oplus U)$ is a non-zero subrepresentation of V_j. Since V_j is irreducible, $V_j = V_j \cap (W \oplus U)$ and therefore $V_j \subseteq W \oplus U$. Given the arbitrary choice of j, it then follows $V = W \oplus U$. ∎

We should point out that this last proposition holds for modules over arbitrary finite-dimensional algebras. Indeed, nothing in our proof is specific to $\mathbb{C}[G]$. Furthermore, the assumption of finite-dimensionality can be dropped if we drop the last equivalence too. The advantage of working with irreducible representations as opposed to indecomposable ones is that they are generally much easier to find. The relationship between irreducible representations – i.e. the hom-sets between them – tends to be simple to understand too. In general, if we cannot afford the comfort of complete reducibility we end up either...

- Searching for an intermediate notion between irreducibility and indecomposability. In other words, we are looking for some property P such that

  $$V \text{ is irreducible im } pliesP(V) \text{ im } pliesV \text{ is indecomposable}$$

  We are interested, of course, in notions that turn out to be equivalent to indecomposability, but whose representations are easier to look for than the indecomposables per say. It is also very useful to classify specific classes of indecomposable representations – i.e. take $P(V) \iff V$ is indecomposable and $Q(V)$ for some other property $Q(V)$. This shows up pervasively in modular representation theory, such as in the analysis of projective indecomposable representations.

- Constructing each and every representation by hand, regardless of them being indecomposable or not. This is obviously very hard to do, but we can instead focus on constructing large classes of representations. The name of the game is *constructing representations as if your life depended on it*. The idea is that this may eventually lead to a classification of all of $\mathbf{Rep}(G)$ – if we stick in long enough to construct all of its objects – or at least to the insight necessary to solve the classification problem.

Both this strategies illustrate how complex the classification problem can get when complete reducibility does not hold, that is, when there are representations that are not completely reducible. In fact, we should note that the problem of classifying the representations of a group for which complete reducibility does not hold is, in a formal sense, *arbitrarily hard*. For instance, there are groups whose indecomposable representations are as hard to classify as simple $\mathbb{C}\langle x, y\rangle$-modules, known as *wild groups* – the meaning of "as hard as" can be made precise. The question now is: which restrictions can we impose to achieve complete reducibility?

The answer to this question turns out to be highly dependent on the structure of the group in question. In particular, if we're dealing with a group with some extra structure, such as a topological group or a Lie group, the answer to this question depends a lot on what this extra structure actually is. For instance, we'll see in chapter 2 that when working with topological groups it is only natural to impose restrictions regarding local compactness and continuity.

One such restriction if finiteness: it turns out that every finite-dimensional representation of a finite group is completely reducible. This result is known as Maschke's theorem and it, together with other structural results in the theory of representations of finite groups, will be the focus of our next section.

1.2 Finite-dimensional Representations of Finite Groups

The theory of finite-dimensional representations of finite groups usually serves as a first introduction to representation theory, while the simplicity and elegance of some of its core arguments serve as inspiration for the theory used in more complicated settings.

We should note that this a very rough outline of the theory, and by no means a complete account of anything near what is currently known about representations of finite groups – not even if we restrict ourselves exclusively to representations over $\mathbb{C}$. Nevertheless, the theory outlined in this section is an illustration of some of the general motifs of representation theory and we hope it can serve as a stepping stone to this vast field of study. Without further ado by restating...

Theorem 1.2.1 (Maschke)**.** *Every finite-dimensional representation of a finite group G is completely reducible.*

Proof. Let V be a finite-dimensional representation of G. If V is irreducible, there is nothing to show. If V is not irreducible, then by definition there is a proper subrepresentations $W \subsetneq V$. We want to establish that there exists a subrepresentation $U \subseteq W$ such that $V = W \oplus U$.

Given a Hermitian inner product H in V, consider

$$\langle v, u\rangle = \frac{1}{|G|}\sum_{g\in G} H(gv, gh)$$

Note $\langle\,,\rangle$ is a G-invariant inner product on V: given $v, w \in V$ we have

$$\langle gv, gw\rangle = \frac{1}{|G|}\sum_{h\in G} H(hgv, hgv) = \frac{1}{|G|}\sum_{h\in G} H(hv, hv) = \langle v, w\rangle$$

We claim that $U = W^{\perp}$ is invariant under the action of G. Indeed, if $v \in V$ is such that $\langle v, w\rangle = 0$ for all $w \in W$, it follows from the fact that W is a subrepresentation that $\langle gv, w\rangle = \langle v, g^{-1}w\rangle = 0$ for each $g \in G$. We are done. ■

Another fundamental observation is something known as *Schur's lemma*.

Lemma 1.2.1 (Schur)**.** *If V and W are irreducible representations of a group G and $T : V \longrightarrow W$ is an intertwining operator then T is either 0 or an isomorphism of representations. Furthermore, if $V = W$ then T is a scalar multiple of the identity.*

Proof. For the first statement it suffices to note that $\ker T \subseteq V$ and $\operatorname{im} T \subseteq W$ are both subrepresentations: either $\ker T = V$ and $\operatorname{im} T = 0$ or $\ker T = 0$ and $\operatorname{im} T = W$. For the second statement, if $T : V \longrightarrow V$ fix any eigenvalue $\lambda \in \mathbb{C}$ of T – whose existence follows from the fact V is finite dimensional. Then $\ker T - \lambda\,\mathrm{Id} \neq 0$ is an intertwining operator, so that $\ker T - \lambda\,\mathrm{Id} \neq 0$ is a subrepresentation of V. It then follows that $T = \lambda\,\mathrm{Id}$. ■

Corollary 1.2.1. *Every irreducible unitary representation of a finite Abelian group is 1-dimensional.*

Proof. It suffices to note that the condition that $gh = hg$ for every $h \in G$ implies $g : V \longrightarrow V$ is an intertwiner. ■

It should be noted that Schur's lemma holds for modules over any complex algebra A, or any algebra over a algebraically closed field. This lemma implies that by classifying the finite-dimensional irreducible representations of a group we get to know all objects $\mathbf{rep}(G)$, *as well as the relationship between them*. In fact, if $\{V_i\}_i$ are the irreducible representations of G, it follows from the universal property of the direct sum that

$$\mathrm{Hom}_G\left(\bigoplus_i n_i \cdot V_i, \bigoplus_j m_j \cdot V_j\right) \cong \bigoplus_{ij} \mathrm{Hom}_G(V_i, V_j)^{n_i m_j} \cong \bigoplus_i \mathbb{C}^{n_i^2}$$

The relationship between representations of different groups is also worth noting. This leads us to the following definitions.

Definition 1.2.1. Given a finite group G, a subgroup $H \subseteq G$ and a representation V of G, the space $\mathrm{Res}^G_H V = V$ is a representation of H, known as *the restriction of V to H* – here the action of H is given by restricting $\rho : G \longrightarrow \mathrm{GL}(V)$ to H.

Notice that every intertwining operator $T : V \longrightarrow W$ between representations of G is also an intertwiner with respect to the H-representation structure of $\mathrm{Res}^G_H V = V$ and $\mathrm{Res}^G_H W$. Hence by defining $\mathrm{Res}^G_H T = T$ we get a functor $\mathrm{Res}^G_H : \mathbf{Rep}(G) \longrightarrow \mathbf{Rep}(H)$.

Definition 1.2.2. Given a finite group G, a subgroup $H \subseteq G$ and a representation V of H, the vector space $\operatorname{Ind}_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$, where $\mathbb{C}[G]$ is regarded as a right $\mathbb{C}[H]$-module with $g \cdot h = gh$, has the natural structure of a representation of G given by

$$g(k \otimes v) = gk \otimes v,$$

This is known as *the representation induced by V*. This construction is once again functorial: to each pair of representations V and W of H and intertwiner $T : V \longrightarrow W$ there corresponds an intertwiner

$$\begin{aligned} \operatorname{Ind}_H^G T : \operatorname{Ind}_H^G V &\longrightarrow \operatorname{Ind}_H^G W \\ g \otimes v &\longmapsto g \otimes Tv \end{aligned}$$

These functors serve as an illustration of a very useful technique in representation theory: we may understand the representations of a given group G by studying the representations of a certain subgroup $H \subseteq G$ and weather of not this representations can somehow be extended to all of G. This technique shows up pervasively in solutions to the problem of classifying the irreducible representations of a particular group G, such as in the case of $G = S_n$ – where we take $H = A_n$. It is natural to expect some sort of compatibility between Res_H^G and Ind_H^G. Lo and behold...

Theorem 1.2.2 (Frobenius)**.** *Given a finite group G, a subgroups $H \subseteq G$, a representation V of H and a representation W of G, the map*

$$\begin{aligned} \Phi : \operatorname{Hom}_G(\operatorname{Ind}_H^G V, W) &\longrightarrow \operatorname{Hom}_H(V, \operatorname{Res}_H^G W) \\ T &\longmapsto \Phi(T) : V \longrightarrow \operatorname{Res}_H^G W \\ & \qquad\qquad\quad v \longmapsto T(e \otimes v) \end{aligned}$$

is a linear isomorphism. In other words, there is an adjunction $\operatorname{Res}_H^G \vdash \operatorname{Ind}_H^G$.

This last theorem is known as *Frobenius reciprocity*, and it is a particular case of the adjunction between restricting scalars and extending scalars: given an algebra A and a subalgebra $B \subseteq A$ there are analogous functors $\operatorname{Res}_B^A : A\text{-}\mathbf{Mod} \longrightarrow B\text{-}\mathbf{Mod}$ and $\operatorname{Ind}_B^A : B\text{-}\mathbf{Mod} \longrightarrow A\text{-}\mathbf{Mod}$ which are again left and right adjoints of one another. The proof of Frobenius reciprocity is precisely the same as that of the general case – just take $A = \mathbb{C}[G]$ and $B = \mathbb{C}[H]$ – and can be found in any introductory book on module theory.

An analogous notion to that of the induced representation $\operatorname{Ind}_H^G V$ is that of the *coinduced representation*.

Definition 1.2.3. Given a finite group G, a subgroup $H \subseteq G$ and a representation V of H, the space $\operatorname{Coind}_H^G V = \{f : G \longrightarrow V \mid f(hg) = hf(g)\}$ has the natural structure of a representation of G given by

$$(gf)(k) = f(kg)$$

Again, this is a functorial construction: given two representations V and W of H and an intertwiner $T : V \longrightarrow W$ there is an intertwiner

$$\begin{aligned} \operatorname{Coind}_H^G T : \operatorname{Coind}_H^G V &\longrightarrow \operatorname{Coind}_H^G W \\ f &\longmapsto T \circ f \end{aligned}$$

The coinduced representation is relevant to us since $\operatorname{Coind}_H^G \vdash \operatorname{Res}_H^G$, which is again a particular case of the adjunction between the coinduced module and restriction of scalar for modules over some arbitrary algebra. Furthermore...

Proposition 1.2.1. *Given a finite group G, a subgroup $H \subseteq G$ and a representation V of G, there is a natural isomorphism* $\operatorname{Ind}_H^G V \xrightarrow{\sim} \operatorname{Coind}_H^G V$.

Proof. The main ingredient of this proof is a canonical isomorphism of vector spaces

$$\operatorname{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], \mathbb{C}[H]) \otimes_{\mathbb{C}[H]} V \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], V)$$

Given an algebra A and two A-modules M and N, there is a natural linear map

$$\begin{aligned} \phi : \operatorname{Hom}_A(M, A) \otimes_A N &\longrightarrow \operatorname{Hom}_A(M, N) \\ T \otimes n &\longmapsto \phi(T \otimes n) : M \longrightarrow N \\ &\qquad\qquad m \longmapsto T(m)n \end{aligned}$$

where the right A-module structure of $\operatorname{Hom}_A(M, A)$ is given by $(T \cdot a)(m) = (Tm) \cdot a$.

This should look familiar to anyone who has ever attended a linear algebra course: if $A = k$ is its own ground field this is the canonical isomorphism $V^* \otimes W \xrightarrow{\sim} \operatorname{Hom}(V, W)$. In general Φ is far from being an isomorphism, but this is always the case if M is a free A-module of finite dimension. As it turns out, $\mathbb{C}[G]$ is a free $\mathbb{C}[H]$-module of finite dimension!

To see this, for each coset in G/H fix some representative, say g_i. We claim $\{g_i\}_i$ is a basis of the $\mathbb{C}[H]$-module $\mathbb{C}[G]$. The fact that $\mathbb{C}[G]$ is spanned by $\{g_i\}_i$ is clear from the definitions, for if $g \in G$ there exists precisely one $h \in H$ and $i = 1, 2, \ldots, n$ such that $g = hg_i$. The fact that the g_i's are linearly independence over $\mathbb{C}[H]$ follows from the fact that G is a basis of $\mathbb{C}[G]$.

It then follows that

$$\operatorname{Coind}_H^G V \cong \operatorname{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], V) \cong \operatorname{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], \mathbb{C}[H]) \otimes_{\mathbb{C}[H]} V$$

as vector spaces. Now by tensoring the isomorphism of right $\mathbb{C}[H]$-modules

$$\begin{aligned} \psi : \mathbb{C}[G] &\longrightarrow \operatorname{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], \mathbb{C}[H]) \\ g &\longmapsto \psi(g) : \mathbb{C}[G] \longrightarrow \mathbb{C}[H] \\ &\qquad\qquad k \longmapsto \begin{cases} kg & \text{if } kg \in H \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

with the identity $\operatorname{Id} : V \longrightarrow V$ we get a linear isomorphism

$$\operatorname{Ind}_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], \mathbb{C}[H]) \otimes_{\mathbb{C}[H]} V$$

If we compose our isomorphisms we arrive at a linear isomorphism $\operatorname{Ind}_H^G V \xrightarrow{\sim} \operatorname{Coind}_H^G V$ that takes $g \otimes v$ to the map

$$k \longmapsto \begin{cases} kgv & \text{if } kg \in H \\ 0 & \text{otherwise} \end{cases}$$

One can readily check this map is invariant under the action of G in $\operatorname{Ind}_H^G V$ and $\operatorname{Coind}_H^G V$, so that we have an isomorphism of representations $\operatorname{Ind}_H^G V \xrightarrow{\sim} \operatorname{Coind}_H^G V$. ■

We should note that while we had to fix a choice of representatives of the cosets in G/H to prove the map we later arrived at is an isomorphism, this map does not depend on the choice of representatives. In terms of the isomorphism $\operatorname{Ind}_H^G V \xrightarrow{\sim} \operatorname{Coind}_H^G V$, the inverse of the adjunction map Φ is given by the map

$$\begin{aligned} \Phi^{-1} : \operatorname{Hom}_H(V, \operatorname{Res}_H^G W) &\longrightarrow \operatorname{Hom}_G(\operatorname{Coind}_H^G V, W) \\ T &\longmapsto \Phi^{-1}(T) : \operatorname{Coind}_H^G V \longrightarrow W \\ &\qquad\qquad f \longmapsto \textstyle\sum_i g_i^{-1} T f(g_i) \end{aligned}$$

This last proposition is fundamental to later generalizations we will discuss in chapter 2. In fact, we'll see that Coind is really the *correct* formulation of induction if we want to talk about Frobenius reciprocity in the realm of compact groups.

We already know that every finite-dimensional representation of finite group G is a direct sum of irreducible representations, and we already know how two given representations are related in terms of their decomposition's as direct sums of irreducible representations. Furthermore, we already know how representations of G are related to representations of a subgroup $H \subseteq G$. The only missing peace of the puzzle are the irreducible representations themselves.

Our next goal is to provide a framework to classify the irreducible finite-dimensional representations of a given finite group G. To that end we introduce an invariant, known as *the character of a representation*. As you might expect, the study of the characters of the representations of a given group is known as…

1.2.1 Character Theory

Definition 1.2.4. Given a finite-dimensional representation V of a group G, consider the function

$$\begin{aligned} \chi_V : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \operatorname{Tr}(g\restriction_V) \end{aligned}$$

This is called the *character of* V. If V is irreducible, χ_V is called an irreducible character.

Example 1.2.1. The character of the trivial representation $\mathbb{C}$ of a group G is the constant function 1.

Example 1.2.2. Given a finite group G and $g \in G$, g acts on $\mathbb{C}[G]$ by permuting the elements of the basis G. This implies that all entries of the matrix of g in the basis G are either zeros or ones. Hence

$$\begin{aligned} \chi_{\mathbb{C}[G]}(g) &= \operatorname{Tr}(g\restriction_{\mathbb{C}[G]}) \\ &= \text{number of 1's in the diagonal of } [g]_G \\ &= \text{number of } h \in G \text{ fixed by } g \\ &= \begin{cases} |G| & \text{if } g = e \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Clearly, the characters of two isomorphic representations coincide, since the trace of a matrix is invariant under changes of basis. Hence characters are invariants of representations. The same argument can be used to show that characters of representations of a group G are constant in the conjugacy classes of G – i.e. characters are *class functions*. Moreover, it's easy to check…

Lemma 1.2.2. *Given a group G and finite-dimensional representations V and W of G,*

(i) $\chi_{V\oplus W} = \chi_V + \chi_W$

(ii) $\chi_{V\otimes W} = \chi_V \cdot \chi_W$

(iii) $\chi_{V^*} = \overline{\chi_V}$

(iv) $\chi_{\operatorname{Hom}(V,W)} = \overline{\chi_V} \cdot \chi_W$

Why are we doing all this again? Well, it turns out characters of finite groups are not only invariants, they are *perfect invariants*. In other words, a finite-dimensional representation of a finite group is completely determined by its character – i.e. if V and W are finite-dimensional representations of a finite group G and $\chi_V = \chi_W$ then $V \cong W$. Moreover, it is quite easy to determine whether or not a given character is irreducible. These are remarkably powerful results, which follow almost immediately from the fact that…

Theorem 1.2.3. *The characters of irreducible representations of a finite group G are* orthonormal *in $\mathbb{C}[G]$ – viewed as the space of complex valued functions on G – with the inner product given by*

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

For a proof of theorem 1.2.3 please see [Wil91]. It then follows...

Corollary 1.2.2. *A finite-dimensional representation of a finite group G is completely determined by its character.*

Proof. Given a finite-dimensional representation V of G, it follows from Maschke's theorem that

$$V \cong \bigoplus_{i=1}^{n} V_i$$

for some irreducible representations $V_1, V_2, \dots V_n$ of G. This implies $\chi_V = \chi_{V_1} + \cdots + \chi_{V_n}$ is a linear combination of the irreducible characters of G.

Since the irreducible characters of G are orthonormal, they form a basis for the subspace of functions $G \longrightarrow \mathbb{C}$ spanned by the irreducible characters themselves. This implies that χ_V can be uniquely expressed as a linear combination of irreducible characters of G.

Now let W be a finite-dimensional representation of G such that $\chi_W = \chi_V$. Suppose

$$W \cong \bigoplus_{i=1}^{m} W_i$$

for some irreducible representations $W_1, W_2, \dots, W_m$ of G. It then follow that

$$\chi_{V_1} + \cdots + \chi_{V_n} = \chi_V = \chi_W = \chi_{W_1} + \cdots + \chi_{W_m},$$

so $W_i = V_i$. This establishes that $V \cong W$. ■

As promised, characters come in handy for verifying that a given representation is irreducible, since...

Corollary 1.2.3. *A finite-dimensional representation V of G is irreducible if and only if*

$$\langle \chi_V, \chi_V \rangle = 1$$

In other words, the irreducible characters of a finite group are precisely the characters which are normal with respect to the norm induced by $\langle \, , \rangle$.

Proof. That the irreducible characters of a group are normal is part of the statement of theorem 1.2.3. To see that a representation V with $\|\chi_V\| = 1$ is irreducible, suppose

$$V \cong \bigoplus_i n_i \cdot V_i$$

where $\{V_i\}_i$ are non-isomorphic irreducible representations of G.

It then follows from lemma 1.2.2 that

$$1 = \|\chi_V\|^2 = \langle \chi_V, \chi_V \rangle = \sum_{ij} n_i n_j \langle \chi_{V_i}, \chi_{V_j} \rangle = \sum_i n_i^2$$

so that each n_i must be 0, expect for one of then, which must be 1. Hence $V \cong V_i$ for some i. ■

More generally...

Corollary 1.2.4. *If V and W are finite-dimensional representations of a finite group G with V irreducible, then $\langle \chi_V, \chi_W \rangle$ is the multiplicity of V in decomposition of W as a direct sum of irreducible representations.*

Proof. Suppose

$$W \cong \bigoplus_i n_i \cdot V_i$$

where V_i are non-isomorphic irreducible representations of G and $V_1 = V$. Then

$$\langle \chi_V, \chi_W \rangle = \sum_i n_i \langle \chi_V, \chi_{V_i} \rangle = n_1$$

■

Corollary 1.2.3 can be used to show that...

Theorem 1.2.4. *The irreducible characters of a finite group G form an orthonormal basis for the space $\mathcal{C}(G)$ of class functions $G \longrightarrow \mathbb{C}$. In particular the number of irreducible representations of G up to isomorphism coincides with the number of conjugacy classes of G.*

Proof. The fact that the irreducible characters of G are orthonormal and linearly independent in $\mathcal{C}(G)$ is clear from theorem 1.2.3. All it's left is to show that any class function is a linear combination of irreducible characters.

To do so, we show that a class function $f \in \mathcal{C}(G)$ such that $\langle f, \chi_V \rangle = 0$ for all irreducible representations V of G must be 0. Given such f and an irreducible representation V of G, consider the linear map

$$T = \sum_{g \in G} f(g) \cdot g : V \longrightarrow V$$

We claim this map is an intertwiner. Indeed,

$$\begin{aligned} T(gv) &= \sum_{h \in G} f(h) \cdot h(gv) \\ \text{(because conjugation by } g \text{ is a bijection } G \longrightarrow G) &= \sum_{h \in G} f(ghg^{-1}) \cdot ghg^{-1}(gv) \\ &= g\left(\sum_{h \in G} f(ghg^{-1}) \cdot hv\right) \\ \text{(because } f \text{ is a class function)} &= g\left(\sum_{h \in G} f(h) \cdot hv\right) \\ &= g(Tv) \end{aligned}$$

Hence by Schur's lemma $T = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{C}$. Now since $\|\chi_V^*\|^2 = \overline{\|\chi_V\|}^2 = 1$, V^* is irreducible and thus

$$\lambda = \frac{1}{\dim V} \operatorname{Tr}(T) = \frac{1}{\dim V} \sum_{g \in G} f(g) \chi_V(g) = \frac{|G|}{\dim V} \overline{\langle f, \chi_{V^*} \rangle} = 0,$$

This implies the operator

$$T = \sum_{g \in G} f(g) \cdot g : \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$$

is zero. It then follows from the fact that G is a basis of $\mathbb{C}[G]$ that $f(g) = 0$ for all $g \in G$. In other words, $f = 0$ in $\mathcal{C}(G)$. Hence every $f \in \mathcal{C}(G)$ such that $\langle f, \chi_V \rangle = 0$ for all irreducible V is zero. Another way of putting it is to say $\langle \chi_V : V \text{ is irreducible} \rangle^{\perp} = 0$ – i.e. $\mathcal{C}(G)$ is spanned by the irreducible characters of G. We are done. ∎

This last theorem may serve as a philosophical justification for example 1.1.11: the number of isomorphism classes of irreducible representations of S_n is the number of conjugacy classes of S_n, which is the number of partitions of n. Hence it is only natural to expect the irreducible representations of S_n to be parameterized by partitions of n – Young diagrams are just visual representations of partitions. A fundamental consequence of theorem 1.2.4, which is at the very heart of the origins of representation theory as a field of study, is...

Theorem 1.2.5. *If $\widehat{G}$ is the set of all irreducible representations of G up to isomorphism then*

$$\mathbb{C}[G] \cong \bigoplus_{V \in \widehat{G}} \dim V \cdot V$$

Proof. If $\{V_i\}_i = \widehat{G}$ and

$$\mathbb{C}[G] \cong \bigoplus_i n_i \cdot V_i$$

then it follows from corollary 1.2.4 and example 1.2.2 that

$$n_i = \langle \chi_{V_i}, \chi_{\mathbb{C}[G]} \rangle = \frac{1}{|G|} \chi_{V_i}(e)\,|G| = \mathrm{Tr}(\mathrm{Id} : V_i \longrightarrow V_i) = \dim V_i$$

■

Granted, we haven't explicitly constructed the irreducible representations of an arbitrary finite group G, but in practice these observations are enough to *completely annihilate* and *utterly destroy* our initial classification problem. This is because in practice it is reasonably easy to find the irreducible characters of a given finite group in an ad-hoc manner. For instance. . .

Example 1.2.3. The irreducible characters of $\mathbb{Z}/5\mathbb{Z}$ are indexed by 5-th roots of unity. To see this, first notice that $\mathbb{Z}/5\mathbb{Z}$ is Abelian, so that an irreducible representation of $\mathbb{Z}/5\mathbb{Z}$ is just a homomorphism $\mathbb{Z}/5\mathbb{Z} \longrightarrow \mathbb{C}^\times$. Since $\mathbb{Z}/5\mathbb{Z}$ is cyclic, such a homomorphism is determined by its value at 1, which must be a 5-th root of unity. If $\omega = e^{\frac{2\pi i}{5}}$, the irreducible characters of $\mathbb{Z}/5\mathbb{Z}$ are thus given by the following table.

$\mathbb{Z}/5\mathbb{Z}$	0	1	2	3	4
χ_1	1	1	1	1	1
χ_ω	1	ω	ω^2	ω^3	ω^4
χ_{ω^2}	1	ω^2	ω^4	ω	ω^3
χ_{ω^3}	1	ω^3	ω	ω^4	ω^2
χ_{ω^4}	1	ω^4	ω^3	ω^2	ω

The i-th row of this table represents the character χ_{ω^i}: the value of $\chi_{\omega^i}(n)$ is given by the value of the i-th row at the column indexed by n.

This last example is what's known as *the character table of* $\mathbb{Z}/5\mathbb{Z}$. It should be clear from our previous discussion that we can represent the characters of a given finite group in a table as in example 1.2.3. Notice, however, that this representation bares some redundancy, as columns indexed by elements in the same conjugacy class are equal to one another – because characters are class functions. For this reason, we usually index the columns in the character table of a group G by representatives of the conjugacy classes of G. The number of elements in each conjugacy class is usually marked bellow the representatives.

We can use the orthogonality relations of theorem 1.2.3 to compute the characters of a group. For example,

Example 1.2.4. The character table of S_4 is given by the following.

S_4	1	(12)	(123)	(1234)	(12)(34)
	1	6	8	6	3
$\mathbb{1}$	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
$\chi_{\mathrm{std.}}$	3	1	0	-1	-1
$\chi_{\mathrm{std.}} \cdot \chi_{\mathrm{sgn}}$	3	-1	0	1	-1
χ_V	2	0	-1	0	2

To see this, we first construct the first three rows of the table. The first two rows are given by the trivial representation and the so called *sign* representation of S_4, where each permutation $\sigma \in S_4$ acts on the plane via

$$\begin{aligned} \sigma : \mathbb{C} &\longrightarrow \mathbb{C} \\ \lambda &\longmapsto \mathrm{sgn}(\sigma) \cdot \lambda \end{aligned}$$

These are both irreducible representations of S_4, because they are 1-dimensional. The third row is given by the so called *standard* representation of S_4, which is the quotient of the permutation representation $\mathbb{C}^4$ – where S_4 acts by permuting coordinates – by the 1-dimensional subrepresentation spanned by $e_1 + e_2 + e_3 + e_4$ – which is a copy of the trivial representation inside $\mathbb{C}^4$.

The explicit description of the action of S_4 in this quotient provided by example 1.1.6 allows us to compute the character $\chi_{\text{sdt.}}$ of the standard representation, arriving at the values at the third row of our table. An easy calculation then shows $\|\chi_{\text{std.}}\| = 1$, which establishes that $\chi_{\text{std.}}$ is irreducible.

The forth row is obtained by tensoring the standard representation with the sign representation. In our case, we multiplying the characters χ_{sgn} and $\chi_{\text{std.}}$. Again, one can readily check this is an irreducible representation of S_4 by verifying $\|\chi_{\text{sgn}} \cdot \chi_{\text{std.}}\| = 1$. Finally, to get the last row we use the orthogonality relations: if we name the missing irreducible representation V and take $x_1 = \chi_V(1)$, $x_2 = \chi_V(12)$, etc, then we can solve

$$\begin{cases} 0 = x_1 + 6x_2 + 8x_3 + 6x_4 + 3x_5 \\ 0 = x_1 - 6x_2 + 8x_3 - 6x_4 + 3x_5 \\ 0 = 3x_1 + 6x_2 - 6x_4 - 3x_5 \\ 0 = 3x_1 - 6x_2 + 6x_4 - 3x_5 \end{cases}$$

for $x_1, x_2, x_3, x_4, x_5 \in \mathbb{C}$ – these are the equations we get if we expand $\langle \chi_V, \mathbb{1} \rangle = \cdots = \langle \chi_V, \chi_{\text{sgn}} \cdot \chi_{\text{std.}} \rangle = 0$. If we further impose $\|\chi_V\| = 1$ and $0 \neq x_1 \in \mathbb{N}$ – since $x_1 = \operatorname{Tr}(\operatorname{Id} : V \longrightarrow V) = \dim V$ – we find $x_1 = 2$, $x_2 = 0$, $x_3 = -1$, etc, as in our table.

Example 1.2.5. The following is an illustration of the character of the Monster simple group $\mathbb{M}$. Surprisingly, this table was computed even before Robert Griess first constructed $\mathbb{M}$.

Unfortunately, not every group is finite. Our next question is... what happens if G is infinite? We begin our inquire by investigating the case of compact groups, which provides us a welcoming introduction to the world of topological groups and their representations. Many of the details omitted in this section will be covered in the next chapter. Please refer to [Car18], [Wil91] and [Eti11] for further details.

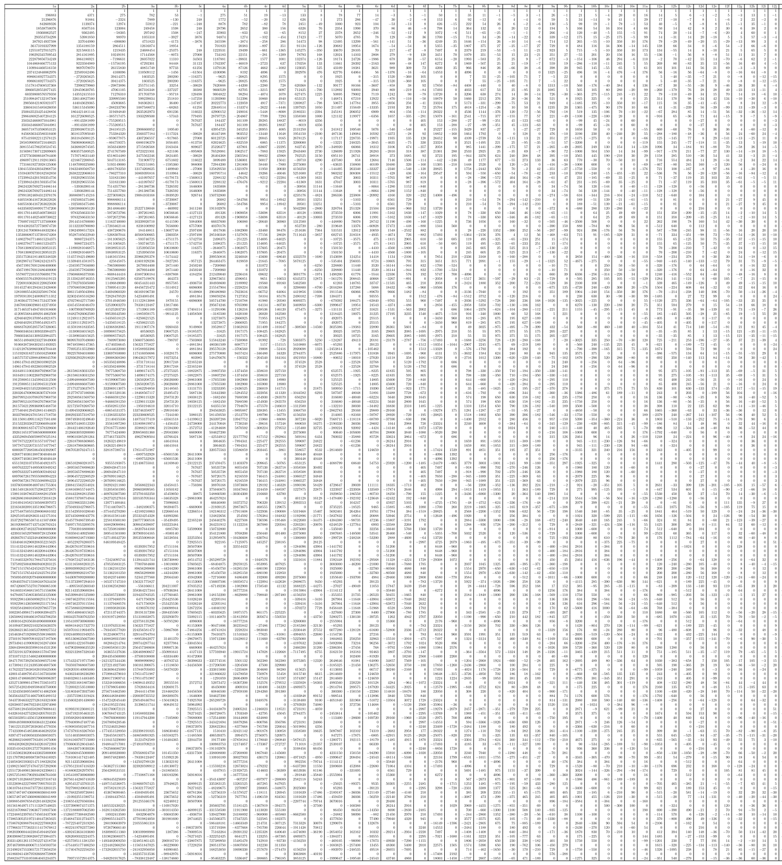


Figure 1.1: The Monster simple group character table

Chapter 2

Continuous Representations of Compact Groups

The following chapter is generally based on the third chapter of *A Journey Through Representation Theory: From Finite Groups to Quivers via Algebras* [Car18].

The algebraic theory of finite groups and their representations is quite rich, but its infinite counterpart is generally lacking in comparison. Infinite groups are complex beasts on their own, and one usually has to endow them with geometric structure to get interesting results. The simplest geometric structure in town is topology, so one naturally pays special attention to topological groups – i.e. group objects in the category **Top** of topological spaces.

$$\begin{array}{ccc} \mathbf{GrpTop} & \longrightarrow & \mathbf{Grp} \\ \downarrow & & \downarrow \\ \mathbf{Top} & \longrightarrow & \mathbf{Set} \end{array}$$

Just as one only considers continuous group homomorphisms when dealing with topological groups, given a topological group G it's usual practice to ignore representations that *do not respect the topological structure of G*. But what is that supposed to mean?

Definition. A representation V of a topological group G is called *continuous* if V is a (Hausdorff) topological vector space and the map

$$(g, v) \longmapsto gv$$

is continuous.

An alternative formulation I've seen around is something along the lines of *a finite-dimensional representation V of G over $\mathbb{R}$ or $\mathbb{C}$ is continuous if*

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

is continuous. Note that $\mathrm{GL}(V)$ is a topological group, since $\mathrm{GL}(V)$ is one of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. The former definition is, of course, much more general, but I still find the latter more intuitive.

Likewise, we only consider continuous intertwining operators between continuous representations of G, and closed subrepresentations of continuous representations of G. Note that in this context, the phrase *V is irreducible* means *the only* closed *subrepresentations of V are* 0 *and V*.

I suppose this makes sense from the *categorical* perspective, but endowing groups and representations with a topological structure just for the sake of it isn't productive at all. Why is any of this useful?

Well, the point of all this is that the topology of G and its representations allows us to reproduce many of the results of the theory of representations of finite groups, as we'll establish in the following. The parallels between representations of finite groups and representations of topological groups come mainly in the form of *tools derived from the Haar measure on compact groups* and the study of *unitary representations*. The closest thing in topology to finite groups are compact groups, so those will be the focus of the following chapter.

2.1 The Haar Measure on Compact Groups

As one would expect, the study of representations of finite groups is greatly facilitated by the fact that *finite groups are finite*. In practice this means things like

$$\sum_{g \in G} f(g)$$

are well defined for finite G. This is enormously helpful when building G-invariant construction on representations of G, since we can just take *any* construction on G and make it invariant by averaging over G.

The classic example of this is the existence of a G-invariant Hermitian inner product on a representation V of G we verified in our proof of Maschke's theorem. Ideally, we would like to extend some of this tools to the study of locally compact groups.

It turns out...

Theorem 2.1.1. *Every Hausdorff locally compact group G admits a non-trivial G-invariant – either by left or right translations – Borel measure. This measure is unique up to multiplication by a positive scalar, locally finite, regular and $\mu(U) > 0$ for every non-empty open subset $U \subseteq G$.*

The fact that G is required to be Hausdorff may seem like a huge limitation, but remember every T_0 topological group is actually Hausdorff. This is called the – either *left* or *right* – *Haar measure* of G, and it allows us to reproduce some of the averaging arguments used for finite groups by replacing sums with integrals

$$\frac{1}{|G|} \sum_{g \in G} f(g) \rightsquigarrow \frac{1}{\mu(G)} \int_G f(g) \, \mathrm{d}g$$

Remark. From now on we'll denote the Haar measure of a Hausdorff locally compact group G by μ.

Example 2.1.1. Let G be a discrete group and μ be its left Haar measure. Then $\{e\}$ is an open subset of G and therefore $\{e\}$ is measurable. Let $\lambda = \mu(\{e\})$.

Since $\{e\}$ is non-empty, $\lambda > 0$. Furthermore, since μ is left-invariant, $\mu(\{g\}) = \mu(\{e\}) = \lambda$ for each $g \in G$. Hence given a finite subset $A \subseteq G$

$$\mu(A) = \sum_{g \in A} \mu(\{g\}) = \sum_{g \in A} \lambda = \lambda \, |A|$$

If $A \subseteq G$ is a countably infinite subset then

$$\mu(A) = \sum_{g \in A} \mu(\{g\}) = \infty$$

If $B \subseteq G$ is uncountable, then there exists a countably infinite subset $A \subseteq B$. Hence

$$\mu(B) \geqslant \mu(A) = \infty$$

and therefore $\mu(B) = \infty$. In conclusion, μ is a scalar multiple of the counting measure.

Example 2.1.2. The Lebesgue measure $m : \mathfrak{B}(\mathbb{R}^n) \longrightarrow [0, \infty]$ is the Haar measure of the additive group $\mathbb{R}^n$. Indeed, the Lebesgue measure does check all of the boxes of theorem 2.1.1: it is a non-trivial Borel measure invariant under translations. In fact the proof of theorem 2.1.1 is very similar in spirit to that of the existence of the Lebesgue measure – see [Mar16] for further details.

Example 2.1.3. The Haar measure of the circle group $\mathbb{S}^1$ is the so called *angular measure*, which is given by

$$\mu(A) = m(\pi^{-1}(A) \cap [0, 2\pi]),$$

where $\pi : \mathbb{R} \longrightarrow \mathbb{S}^1$ is the canonical projection $t \longmapsto e^{2\pi i t}$.

Example 2.1.4. The Haar measure of $\mathrm{GL}_n(\mathbb{R})$ is given by

$$\mu(A) = \int_A \frac{1}{|\det X|^n} \, \mathrm{d}X,$$

where integration is taken under the Lebesgue measure of $\mathbb{R}^{n^2}$.

Example 2.1.5. The Haar measure of a profinite group $G = \varprojlim_i G_i$ is given by the net limit of the normalized counting measure of each G_i.

$$\mu(A) = \lim_i \frac{|\pi_i(A)|}{|G_i|}$$

The left Haar measure of a locally compact group G isn't necessarily proportional to the right Haar measure of G, but this is the case for compact groups. Furthermore, if G is compact then G has finite measure and therefore the Haar measure of G can be normalized so that $\mu(G) = 1$. We won't normalize the Haar measure in this notes though, since we want to emphasize the parallels with finite-dimensional representations of finite groups.

The proof that every (Hausdorff) locally compact group admits a Haar measure is somewhat involved. A special class of locally compact groups are the Lie groups [1], also known as *groups that are also smooth manifolds* – i.e. group objects in the category **Mnfd** of smooth manifolds. It's much easier to show that every Lie group admits a Haar measure, precisely because Lie groups are orientable manifolds and therefore they admit a (G-invariant) volume form, as we'll establish in the following.

2.1.1 Differential Forms

As previously mentioned, differential forms allow us to easily prove that every Lie group admits a Haar measure. We'll go over the definition of a differential form in a smooth manifold before proceeding to the proof mentioned above. The following comments are based on Marcos Alexandrino's notes on Riemannian Geometry [Ale19].

Definition 2.1.1. Let M be an n-dimensional smooth manifold. Given $p \in G$, the tangent T_pM space at p is an n-dimensional real vector space. Now consider the real vector space $\wedge^k(T_pM)^*$ of k-linear functionals $\omega_p : T_pM \times \cdots \times T_pM \longrightarrow \mathbb{R}$ such that

$$\omega_p(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma) \cdot \omega_p(v_1, \ldots, v_k)$$

for all $\sigma \in S_k$.

Since $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_i$ is a basis for T_pM,

$$\{\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} : i_1 < \cdots < i_n\}$$

is a is a basis for $\wedge^k(T_pM)^*$ – where $\{\mathrm{d}x_i\}_i$ is the dual basis for $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_i$. This implies

$$\dim \wedge^k(T_pM)^* = \binom{n}{k}$$

[1] As manifolds, Lie groups are locally Euclidean. Hence Lie groups are *locally locally compact*, i.e. locally compact.

This construction induces a vector bundle

$$\begin{array}{c} \coprod_{p\in M} \wedge^k (T_pM)^* \\ \big\downarrow \\ M \end{array}$$

commonly denoted by $\wedge^k T^*M$.

A (smooth) section of $\wedge^k T^*M$ – i.e. a smooth function $\omega : M \longrightarrow \wedge^k T^*M$ that takes each $p \in M$ to some $\omega_p \in \wedge^k (T_pM)^*$ – is called a *differential form of degree k*. Note that $\dim \wedge^n T_p^*M = 1$. Hence differential forms of degree n are called *differential forms of maximal degree*. Nowhere-vanishing – non-zero at every point – differential forms of maximal degree are called *volume forms*.

The set of all differential forms of degree k is usually denoted by $\Omega^k(M)$.

Notice $\Omega^k(M)$ is a $C^\infty(M)$-module where

$$(f \cdot \omega)_p = f(p) \cdot \omega_p$$

Moreover, an element $\omega \in \Omega^k(M)$ can be thought-of as a k-linear alternating functional

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

where $\mathfrak{X}(M)$ is the C^∞-module of (smooth) vector fields over M and

$$(\omega(V^1, \ldots, V^k))(p) = w_p(V_p^1, \ldots, V_p^k)$$

Why is this useful to us though? Well, besides various applications on differential geometry – which I'm completely unaware of – given a Lie group G, a differential form of maximal degree $\omega \in \Omega^n(G)$ can be integrated over of G. In other words,

$$\int_G \omega \in [-\infty, \infty]$$

is a thing that exists and we can use this to build a measure over G. I'll spear the reader from the boring details of the construction of the integral of a differential form, all we need to know at the moment is that integration of volume forms is something generally well behaved. Please refer to [Ale19] for more information on this topic.

Theorem 2.1.2. *Every Lie group G admits a left Haar measure.*

Proof. Suppose $\dim G = n$. Given $g \in G$, the left translations $\ell_{g^{-1}} : G \longrightarrow G$ by g^{-1} is a diffeomorphism that takes g to e. Consider the induced isomorphism

$$\ell_{g^{-1}}^* : \wedge^n T_e^*G \longrightarrow \wedge^n T_g^*G$$

given by $\wedge^n (d\ell_{g^{-1}})_g^*$ – where $(d\ell_{g^{-1}})_g^* : T_e^*G \longrightarrow T_g^*G$ is the dual of $(d\ell_{g^{-1}})_g : T_gG \longrightarrow T_eG$.

Let $\omega_e = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \wedge^n T_e^*G$. Then the map

$$\begin{aligned} \omega : G &\longrightarrow \wedge^n T^*G \\ g &\longmapsto \ell_{g^{-1}}^*(\omega_e) \end{aligned}$$

is a volume form. We claim ω is left-invariant too. Indeed,

$$\begin{aligned} (\ell_g^*\omega)_h &= \wedge^n (d\ell_g)_h^* \, \omega_{gh} \\ &= \wedge^n (d\ell_g)_h^* \wedge^n (d\ell_{h^{-1}g^{-1}})_{gh}^* \, \omega_e \\ &= \wedge^n ((d\ell_g)_h^* (d\ell_{h^{-1}g^{-1}})_{gh}^*) \, \omega_e \\ &= \wedge^n ((d\ell_{h^{-1}g^{-1}})_{gh} (d\ell_g)_h)^* \, \omega_e \\ \text{(chain rule to the rescue)} &= \wedge^n (d\ell_{h^{-1}})_h^* \, \omega_e \\ &= \ell_{h^{-1}}^*(\omega_e) \\ &= \omega_h \\ \therefore \ell_g^*\omega &= \omega \end{aligned}$$

Suppose without any loss of generality that

$$\int_G \omega \geqslant 0,$$

so that

$$\mu(K) = \int_K \omega \geqslant 0$$

for each compact subset $K \subseteq G$.

Clearly, $\mu(K) < \infty$, and it's easy to check that $\mu : \mathcal{K}(G) \longrightarrow [0, \infty)$ is a finitely additive monotonic function too. Moreover,

$$\mu(gK) = \int_{\ell_g(K)} \omega = \int_K \ell_g^* \omega = \int_K \omega = \mu(K)$$

and therefore $\mu : \mathcal{K}(G) \longrightarrow [0, \infty)$ is a left-invariant pre-measure.

By Caratheodory's extension theorem, μ can be extended to every Borel subset of G, yielding a left-invariant Borel measure $\mu : \mathfrak{B}(G) \longrightarrow [0, \infty]$ – see San Martin's construction of the Haar measure [Mar16] for further details. Furthermore, since ω is nowhere-vanishing, μ is non-trivial. We are done. ■

Remark. Notice we've also proved G is orientable – i.e. G admits a volume form.

The proof that every Lie group admits a right Haar measure is essentially the same – just replace ℓ with $\imath$ everywhere. Let's assume that every locally compact group admits a Haar measure and proceed to some applications of this fact, the most important of which being, of course, . . .

2.1.2 Maschke's Theorem

Theorem 2.1.3 (Maschke)**.** *Every finite-dimensional continuous representation of a compact group G is completely reducible.*

Proof. We proceed as in our proof of theorem 1.2.1: given a finite-dimensional continuous representation V of G we fix some Hermitian product $H : V \times V \longrightarrow \mathbb{C}$ and consider the bilinear form

$$\langle v, u \rangle = \frac{1}{\mu(G)} \int_G H(gv, gh) \, \mathrm{d}g$$

The form $\langle \, , \rangle$ is again a G-invariant Hermitian inner product, for if $v, w \in V$ then

$$\begin{aligned} \langle gv, gw \rangle &= \frac{1}{\mu(G)} \int_G H(hgv, hgv) \, \mathrm{d}h \\ \text{(due to the right-invariance of } \mu) &= \frac{1}{\mu(G)} \int_G H(hv, hv) \, \mathrm{d}h \\ &= \langle v, w \rangle \end{aligned}$$

Hence $U = W^\perp$ is a G-invariant closed subrepresentation of V such that $V = W \oplus U$. ■

Note that for finite G and discrete, μ is simply the counting measure and the integral is the standard summation, so this construction is compatible with the construction used for finite groups. We should point out that the essential ingredient of the last proof is the fact that every finite-dimensional representation admits a G-invariant inner product. This may seem like a trivial observation, but notice that it allows us to further generalize Maschke's theorem to arbitrary representations endowed with G-invariant inner products, regardless of their dimensions. This leads us to our next section.

2.2 Unitary Representations

Unitary representations play a special role in our theory, in the sense that they allow us to apply tools from functional analysis to understand the continuous representations of a compact group. As one would expect, a unitary representation of a topological group G is a representation in which every element $g \in G$ acts *unitarily* – i.e. g is a unitary operator. In other words. . .

Definition 2.2.1. A continuous representation V of a topological group G is called *unitary* if V is a Hilbert space and

$$\langle v, w \rangle = \langle gv, gw \rangle$$

for all $g \in G$. Alternatively, the group $\mathrm{U}(V)$ of unitary operators $V \longrightarrow V$ is a topological group under the compact-open topology and the continuity of the map $G \times V \longrightarrow V$ is equivalent to the continuity of the group homomorphism $G \longrightarrow \mathrm{U}(V)$ [Sch13].

Example 2.2.1. If G is a compact group then the space $L^2(G)$ of functions $f : G \longrightarrow \mathbb{C}$ such that

$$\int_G |f(g)|^2 \, \mathrm{d}g$$

exists is a unitary representation of G, where the action of $g \in G$ is given by

$$(g \cdot f)(h) = f(g^{-1}h)$$

and

$$\langle f_1, f_2 \rangle = \frac{1}{\mu(G)} \int_G f_1(g)\overline{f_2(g)} \, \mathrm{d}g$$

This is called the regular representation of G.

When G is finite and discrete the integral over G is just the regular summation and $L^2(G)$ is the space of all complex-valued functions on G – also known as $\mathbb{C}[G]$. In other words, this is a strict generalization of the regular representation of finite groups. In fact, we will soon see that $L^2(G)$ plays a similar role to that played by $\mathbb{C}[G]$ in the context of finite groups.

We already have everything we need to prove. . .

Theorem 2.2.1 (Maschke)**.** *Every unitary representation of a compact group G is completely reducible. In other words, given a unitary representation V of G and a closed subrepresentation $W \subseteq V$, there exists a (closed) subrepresentation $U \subseteq V$ such that*

$$V = W \oplus U$$

This proof is even simpler than that of theorem 2.1.3: it suffices to note that if W is closed then $V = W \oplus W^\perp$ and $W^\perp$ is, again, invariant under the action of G. In fact, our proof of theorem 2.1.3 amounts to the construction of a G-invariant inner product such that V is a unitary representation under this inner product. Theorem 2.2.1 proves to be a big milestone in the theory, but there is still a lot we don't know about the general structure of unitary representations. This we'll be the focus of this section, and we begin with a technical, yet fundamental, result.

Proposition 2.2.1. *Every irreducible unitary representation of a compact group G is finite-dimensional*

Remark. This proof is quite involved so buckle your seat-belts.

Proof. Let V be a unitary representation. We want to establish that V has a finite-dimensional non-zero closed subrepresentation.

Given $u \in V$ with $\|u\| = 1$, define

$$\begin{aligned} Q : V &\longrightarrow V \\ v &\longmapsto \int_G gTg^{-1}v \, \mathrm{d}g \end{aligned}$$

where

$$\begin{aligned} T : V &\longrightarrow V \\ v &\longmapsto \langle v, u \rangle u \end{aligned}$$

We want to establish that $\ker Q - \lambda \, \mathrm{Id}$ is a non-zero finite-dimensional subrepresentation of V for some $\lambda > 0$.

Notice Q is a self-adjoint intertwining operator. Furthermore

$$\begin{aligned} \langle v, Qv \rangle &= \left\langle v, \int_G \langle g^{-1}v, u \rangle gu \, \mathrm{d}g \right\rangle \\ &= \int_G \langle v, gu \rangle \overline{\langle v, gu \rangle} \, \mathrm{d}g \\ &= \int_G |\langle v, gu \rangle|^2 \, \mathrm{d}g \\ &\geqslant 0 \end{aligned}$$

so Q is semipositive. We claim Q is compact too.

Indeed, it follows from the Cauchy-Schwartz inequality that T is bounded. Moreover,

$$\dim \operatorname{im} T = \dim \langle u \rangle = 1$$

so gTg^{-1} is a bounded operator with 1-dimensional image for all $g \in G$. This implies

$$Q = \int_G gTg^{-1} \, \mathrm{d}g$$

is compact – since the ideal $\mathcal{C}(V) \subseteq \mathcal{B}(V)$ of all compact operators $V \longrightarrow V$ is the closure of the ideal $\mathcal{F}(V) \subseteq \mathcal{B}(V)$ of bounded operators $V \longrightarrow V$ with finite-dimensional image.

In other words, Q is a compact semipositive self-adjoint intertwining operator. Hence

$$\lambda = \sup_{\|v\|=1} \langle v, Qv \rangle$$

is either 0 or an eigenvalue of T. Notice

$$\langle u, Qu \rangle = \int_G |\langle u, gu \rangle|^2 \, \mathrm{d}g > 0$$

since

$$g \longmapsto |\langle u, gu \rangle|^2$$

is a continuous map that takes every element of G to a non-negative real number and

$$|\langle u, u \rangle|^2 = 1 > 0$$

This implies $\lambda > 0$, so λ is a positive eigenvalue of Q. Let $W = \ker Q - \lambda \, \mathrm{Id}$. Notice W is a (closed) subrepresentation of V, since it is the kernel of a continuous intertwining operator. Furthermore, W is non-zero since it is the eigenspace of T associated to $\lambda > 0$.

Since Q is compact and self-adjoint and $\lambda \neq 0$, it follows from the spectral theorem that W is finite-dimensional. Finally, W is a finite-dimensional (closed) subrepresentation of V! Hence if V is irreducible then W must be all of V, so that $\dim V = \dim W < \infty$. ■

An important consequence of this result is something known as *Schur's lemma*.

Lemma 2.2.1 (Schur)**.** *Let V and W be two irreducible unitary representations of a compact group G and $T : V \longrightarrow W$ be a continuous intertwining operator. Then either $T = 0$ or there exists $\lambda > 0$ such that λT is an isometry. Furthermore, if $V = W$ then T is scalar multiple of the identity.*

Proof. We begin by showing the last statement. Suppose $T : V \longrightarrow V$ is a continuous intertwiner. Since V is unitary, V is finite-dimensional. It then follows that T has at least one eigenvalue, so $\ker T - \lambda \operatorname{Id} \neq 0$ for some $\lambda \in \mathbb{C}$.

Notice $T - \lambda \operatorname{Id}$ is again a continuous intertwiner. This implies that $\ker T - \lambda \operatorname{Id}$ is a closed subrepresentation of V. Since $\ker T - \lambda \operatorname{Id} \neq 0$, $\ker T - \lambda \operatorname{Id} = V$ and therefore $T = \lambda \operatorname{Id}$. Now to the first statement: let $T : V \longrightarrow W$ be a continuous intertwiner and suppose $T \neq 0$. It follows that $T^*T = \mu \operatorname{Id}$ for some $\mu \in \mathbb{C}$. We want to establish that μ is a positive real number.

Since μ is an eigenvalue of the bounded semipositive self-adjoint operator T^*T, μ is a non-negative real number. But μ cannot be 0, for if this was the case we could conclude $T = 0$. Hence $\mu \neq 0$ and therefore $\mu > 0$. Let $\lambda = \frac{1}{\sqrt{\mu}}$ and $U = \lambda T$. Then

$$\begin{aligned} U^*U &= \frac{1}{\sqrt{\mu}} T^* \frac{1}{\sqrt{\mu}} T \\ &= \frac{1}{|\mu|}\, \mu \operatorname{Id} \\ &= \operatorname{Id} \end{aligned}$$

In other words, $U = \lambda T$ is an intertwining isometry. ■

Corollary 2.2.1. *Every irreducible unitary representation of a compact Abelian group is* 1*-dimensional.*

It should be noted that Schur's lemma – and by extension corollary 2.2.1 – also holds for *arbitrary* irreducible (continuous) representations of any *locally* compact group – see [Car18] for a proof. This is essential for establishing the connection between Pontryagin's duality and representation theory. For more information on the topic please refer to the excellent *Fourier Analysis on Number Fields* [Din98]. This generalization of Schur's lemma will also come in handy in the following proof.

Theorem 2.2.2. *Every irreducible continuous representation of a compact group G is (isomorphic to) a finite-dimensional closed subrepresentation of the regular representation $L^2(G)$.*

Proof. Let V be an irreducible representation of G. We begin by showing that V is isomorphic to a subrepresentation of $L^2(G)$. Fix some non-zero $f \in V'$ and define

$$\begin{aligned} T : V &\longrightarrow L^2(G) \\ v &\longmapsto Tv : G \longrightarrow \mathbb{C} \\ &\qquad\qquad g \longmapsto f(g^{-1}v) \end{aligned}$$

The fact that f is continuous implies Tv is continuous for all $v \in V$, so that T is well-defined. By the same token, T must itself be continuous. We claim T is an intertwiner. Indeed,

$$(T(gv))(h) = f(h^{-1}gv) = f((g^{-1}h)^{-1}v) = (Tv)(g^{-1}h) = (g(Tv))(h)$$

for each $g, h \in G$.

Since V is irreducible, $\ker T = 0$, so that T is injective. Now, it follows from Schur's lemma that the image of T must be contained in a single irreducible component of $L^2(G)$, for if this was not the case we could find two non-zero intertwiners $T_1 : V \longrightarrow V_1 \subseteq L^2(G)$, $T_2 : V \longrightarrow V_2 \subseteq L^2(G)$ with $V_1 \ncong V_2$. Hence V is isomorphic to this irreducible component of $L^2(G)$.

In particular, the image of T is closed in $L^2(G)$, so that $V \cong \operatorname{im} T$ is a Hilbert space and a unitary representation. In conclusion, V is a irreducible unitary subrepresentation of V, so it must be finite-dimensional. ■

Another interesting parallel between the representation theory of finite groups and that of unitary representations of compact groups is that of Frobenius reciprocity. Specifically. . .

Definition 2.2.2. Given a compact group G and a closed subgroup $H \subseteq G$, the space ${}_H\backslash^G$ of right cosets has a natural Borel measure

$$\begin{aligned} \nu : \mathfrak{B}\left({}_H\backslash^G\right) &\longrightarrow [0, \infty] \\ A &\longmapsto \mu(\pi^{-1}(A)) \end{aligned}$$

which is invariant under left and right translations by elements of G.

Definition 2.2.3. Given a compact group G, a closed subgroup $H \subseteq G$ and a unitary representation V of H, the (Banach) space $\operatorname{Coind}_H^G V$ of weakly measurable functions $G \longrightarrow V$ such that $f(hg) = hf(g)$ for all $h \in H$ and

$$\int_{{}_H\backslash^G} \|f(g)\| \, \mathrm{d}g < \infty \tag{2.1}$$

with

$$(g \cdot f)(k) = f(kg)$$

is a continuous representation of G, where the integral in (2.1) is taken under the measure ν[2].

This construction is functorial: given two representations V and W of H and a continuous intertwiner $T : V \longrightarrow W$ there is a continuous intertwiner

$$\begin{aligned} \operatorname{Coind}_H^G T : \operatorname{Coind}_H^G V &\longrightarrow \operatorname{Coind}_H^G W \\ f &\longmapsto T \circ f \end{aligned}$$

Theorem 2.2.3 (Moore)**.** *Given a compact group G, a closed subgroups $H \subseteq G$, a unitary representation V of H and a unitary representation W of G, the map*

$$\begin{aligned} \Phi : \operatorname{Hom}_H(V, \operatorname{Res}_H^G W) &\longrightarrow \operatorname{Hom}_G(\operatorname{Coind}_H^G V, W) \\ T &\longmapsto \Phi(T) : \operatorname{Coind}_H^G V \longrightarrow W \\ & \qquad\qquad f \longmapsto \int_{{}_H\backslash^G} g^{-1} T f(g) \, \mathrm{d}g \end{aligned}$$

is an isometry[3].

It should be obvious that theorem 2.2.3 is a strict generalization of the Frobenius reciprocity theorem for finite groups. However, we should note that unlike the case of finite-dimensional representations of finite groups – where $\operatorname{Coind}_H^G V$ is always guaranteed to be finite-dimensional – the representation $\operatorname{Coind}_H^G V$ coinduced by some unitary V needs not to be unitary! Hence the codomain of Coind_H^G is actually the category of *continuous* representations of G, and therefore there is no adjunction as in theorem 1.2.2 – because the domains and codomains of Coind_H^G and Res_H^G don't match.

The proof of theorem 2.2.3 is unfortunately too long to fit in this notes, but a proof of a slightly more general statement can be found in the original paper where Calvin Moore first introduced this formulation of Frobenius reciprocity [Moo62]. Having just generalized the Frobenius reciprocity theorem, the only thing from chapter 1 left to generalize is character theory. This brings us to. . .

[2] The integral in (2.1) is well defined because $\|f(hg)\| = \|hf(g)\| = \|f(g)\|$ for all $h \in H$

[3] The integral in the definition of $\Phi(T)$ is well defined because $(hg)^{-1}Tf(hg) = g^{-1}Tf(g)$ for all $h \in H$

2.3 The Peter-Weyl Theorem & Character Theory

The goal of this section is to generalize most of the results from section 1.2.1 to compact groups, in particular theorem 1.2.3 and theorem 1.2.5. To that end, we will introduce a technical yet useful tool, known as *matrix coefficients*.

Definition 2.3.1. Given a unitary representation V of a compact group G and $v, w \in V$. Consider

$$\begin{aligned} f_{v,w} : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \langle gv, w \rangle \end{aligned}$$

This is called a *matrix coefficient* of V.

This is particularly useful for computing the character of finite-dimensional representations, since if $g = (a_{ij})_{ij}$ in an orthonormal basis $\{e_1, \dots, e_n\}$ then $f_{e_i,e_j}(g) = a_{ij}$. Hence

$$\chi_V(g) = \mathrm{Tr}(g \restriction_V) = \sum_{i=1}^{n} f_{e_i,e_i}(g)$$

Using matrix coefficients, it's easy enough to show that theorem 1.2.3, corollary 1.2.2 and corollary 1.2.3 all hold for compact groups [Car18], but ideally we would also like to show that theorem 1.2.5 holds. Moreover, we would like to show that the irreducible characters of a compact G span the subspace $\mathcal{C}(G) \subseteq L^2(G)$ of class functions – the word *span* is doing some legwork here, we'll get to the precise formulation of this. Proving this requires *the Peter-Weyl theorem*. This theorem may not seem particularly interesting on its own, but it is essential for generalizing theorem 1.2.5 and theorem 1.2.4.

Theorem 2.3.1 (Peter-Weyl)**.** *The matrix coefficients of all irreducible representations of compact group G span a dense subspace in $L^2(G)$.*

Lemma 2.3.1. *The map*

$$\begin{aligned} \Phi : \overline{\bigoplus_{V \in \widehat{G}} V^* \boxtimes V} &\longrightarrow L^2(G) \\ e_i^* \otimes e_j &\longmapsto f_{e_j,e_i} \end{aligned}$$

is an isomorphism of $G \times G$-representations where the action of $G \times G$ in $L^2(G)$ is given by

$$((g,h)f)(k) = f(g^{-1}kh)$$

Proof. It's clear that Φ is linear, and it follows from the Peter-Weyl theorem that Φ is also surjective. To see that Φ is an intertwiner, it suffices to observe that

$$\begin{aligned} ((g,h)\phi(e_i^* \otimes e_j))(k) &= ((g,h)f_{e_j,e_i})(k) \\ &= f_{e_j,e_i}(g^{-1}kh) \\ &= \langle g^{-1}khe_j, e_i \rangle \\ &= \langle khe_j, ge_i \rangle \\ &= f_{he_j,ge_i}(k) \\ &= (\phi(ge_i^* \otimes he_j))(k) \\ &= (\phi((g,h)e_i^* \otimes e_j))(k) \\ &= (g,h)\phi(v \otimes w) = \phi((g,h)v \otimes w) \end{aligned}$$

Now given $V \in \widehat{G}$, notice $\|\chi_{V^* \boxtimes V}\| = \|\chi_{V^*}\| \cdot \|\chi_V\| = 1$. It then follow from corollary 1.2.3 that $V^* \boxtimes V$ is irreducible. Hence $\ker \Phi \restriction_V$ is either 0 or V. Since $\Phi(e_1^* \otimes e_1) = f_{e_1,e_1} \neq 0$, $\ker \Phi \restriction_V = 0$. This implies Φ is injective.

In conclusion, Φ is an isomorphism of representations. ■

Again, this results aren't that appealing on their own. What we're really interested in is...

Corollary 2.3.1 (Peter-Weyl)**.**

$$L^2(G) \cong \overline{\bigoplus_{V \in \widehat{G}} \dim V \cdot V}$$

Proof. Notice $G \times \{e\} \cong G$ as topological groups. On the one hand, $\operatorname{Res}^{G \times G}_{G \times \{e\}} L^2(G)$ is precisely the regular representation $L^2(G)$. On the other hand, given $V \in \widehat{G}$ it's easy to check that

$$\begin{aligned}
\operatorname{Res}^{G \times G}_{G \times \{e\}} V^* \boxtimes V &\cong V^* \otimes \bigoplus_{i=1}^{\dim V} \mathbb{C} \\
&\cong \bigoplus_{i=1}^{\dim V} V^* \otimes \mathbb{C} \\
&\cong \bigoplus_{i=1}^{\dim V^*} V^* \\
&= \dim V^* \cdot V^*
\end{aligned}$$

Moreover, since V is irreducible, V^* is irreducible. This implies

$$\operatorname{Res}^{G \times G}_{G \times \{e\}} \overline{\bigoplus_{V \in \widehat{G}} V^* \boxtimes V} \cong \overline{\bigoplus_{V \in \widehat{G}} \dim V \cdot V}$$

Hence Φ can be thought of as an isomorphism of G-representations that takes $\overline{\bigoplus_{V \in \widehat{G}} \dim V \cdot V}$ to $L^2(G)$. ∎

Corollary 2.3.2 (Peter-Weyl)**.** *The irreducible characters of G form an orthonormal basis for the subspace $\mathcal{C}(G)$ of class-functions – in the sense that they span a dense subspace in $\mathcal{C}(G)$.*

Remark. Yet another lengthy proof lies ahead.

Proof. Given an irreducible representation V of G consider the algebra $\operatorname{End}(V)$ with

$$T \cdot S = \frac{1}{\dim V} TS$$

and

$$\begin{aligned}
\Psi : \overline{\bigoplus_{V \in \widehat{G}} \operatorname{End}(V)} &\longrightarrow L^2(G) \\
E_{ij} &\longmapsto f_{e_j, e_i}
\end{aligned}$$

We want to establish that Ψ is an isomorphism of algebras between $\overline{\bigoplus_{V \in \widehat{G}} \operatorname{End}(V)}$ and $L^2(G)$ equipped with the convolution product

$$(f_1 * f_2)(g) = \frac{1}{\mu(G)} \int_G f_1(h) f_2(h^{-1} g) \, \mathrm{d}h,$$

This may all seem *extremely arbitrary* – and indeed it is! The point of all this is we'll show that the center of $L^2(G)$ is $\mathcal{C}(G)$ and the center of $\overline{\bigoplus_{V \in \widehat{G}} \operatorname{End}(V)}$ is precisely what we want it to be. First of all, note that Ψ factors through Φ.

$$\begin{array}{ccc}
\overline{\bigoplus_{V \in \widehat{G}} V^* \boxtimes V} & \xrightarrow{\sim} & \overline{\bigoplus_{V \in \widehat{G}} \operatorname{End}(V)} \\
& \searrow^{\Phi} & \downarrow \Psi \\
& & L^2(G)
\end{array}$$

This implies Ψ is a linear isomorphism. To see that Φ is a homomorphism of algebras, it suffices to observe that

$$\begin{aligned}
(\Psi(E_{ij} \cdot E_{jk}))(g) &= \frac{1}{\dim V}(\Psi(E_{ik}))(g) \\
&= \frac{1}{\dim V} f_{e_k,e_i}(g) \\
&= \frac{1}{\dim V}\langle e_j, e_j\rangle\langle ge_k, e_i\rangle \\
\text{(see theorem 2.1 of [Car18])} &= \langle f_{e_j,e_i}, f_{e_j,ge_k}\rangle \\
&= \frac{1}{\mu(G)}\int_G \langle he_j, e_i\rangle\overline{\langle he_j, ge_k\rangle}\ \mathrm{d}h \\
&= \frac{1}{\mu(G)}\int_G \langle he_j, e_i\rangle\langle h^{-1}ge_k, e_j\rangle\ \mathrm{d}h \\
&= \frac{1}{\mu(G)}\int_G f_{e_j,e_i}(h) f_{e_k,e_j}(h^{-1}g)\ \mathrm{d}h \\
&= (f_{e_j,e_i} * f_{e_k,e_j})(g) \\
&= (\Psi(E_{ij}) * \Psi(E_{jk}))(g) \\
&\therefore \Psi(T \cdot S) = \Psi(T) * \Psi(S)
\end{aligned}$$

Notice that $\mathcal{C}(G)$ is the center of $L^2(G)$ under the convolution product. Moreover, the center of $\operatorname{End}(V)$ is clearly $\mathbb{C}\,\mathrm{Id}$. We claim that the image of $\mathbb{C}\,\mathrm{Id}$ under Ψ is $\mathbb{C}\chi_V$. Indeed,

$$(\Psi(\mathrm{Id}))(g) = \sum_{i=1}^{n}(\Psi(E_{ii}))(g) = \sum_{i=1}^{n} f_{e_i,e_i}(g) = \chi_V(g)$$

Hence

$$\begin{aligned}
\mathcal{C}(G) &= Z(L^2(G)) \\
&= \Psi\left(Z\left(\overline{\bigoplus_{V\in\widehat{G}} \operatorname{End}(V)}\right)\right) \\
&= \overline{\bigoplus_{V\in\widehat{G}} Z(\Psi(\operatorname{End}(V)))} \\
&= \overline{\bigoplus_{V\in\widehat{G}} \mathbb{C}\chi_V}
\end{aligned}$$

We are done. ■

This entire chapter hopefully establishes that the representation theory of compact groups is *precisely the same* as the theory of representations of finite groups. Next we will turn our attention to Lie groups and their representations, which substantially deviates from the case of finite groups.

Chapter 3

Smooth Representations of Lie Groups

This chapter is generally based on the second part of *Representation theory: A first course* [Wil91]. We've already discussed Lie groups, but for the uninitiated among you: a Lie group is *a group that is also a smooth manifold* – i.e. a group object in the category **Mnfd** of smooth manifolds.

$$\begin{array}{ccc} \textbf{LieGrp} & \longrightarrow & \textbf{Grp} \\ \downarrow & & \downarrow \\ \textbf{Mnfd} & \longrightarrow & \textbf{Set} \end{array}$$

Example. The *classical* examples of Lie groups are the *classical groups*, i.e. . .

(i) The groups of invertible $n \times n$ matrices $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$

(ii) $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) : \det M = 1\}$ and $\mathrm{SL}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : \det M = 1\}$

(iii) The group SO_n of orientation preserving isometries of $\mathbb{R}^n$– i.e. orthogonal invertible matrices with real coefficients

(iv) The group SU_n of orientation preserving isometries of $\mathbb{C}^n$– i.e. orthogonal invertible matrices with complex coefficients

(v) The symplectic group

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \left\{ M \in \mathrm{SL}_{2n}(\mathbb{R}) : M \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} M^{-1} = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \right\}$$

All of the groups above are endowed with the Euclidean topology. Since $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous, given $M \in \mathrm{GL}_n(\mathbb{R})$ there should exist some open neighborhood of M completely contained in $\mathrm{GL}_n(\mathbb{R})$. Hence $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$ – in particular, it is a closed submanifold of $M_n(\mathbb{R})$. The other real groups can be realized as the inverse image of regular values of smooth maps $\mathrm{GL}_n(\mathbb{R}) \longrightarrow M$ – for appropriate M's.

It's easy to see that the product in such groups is smooth. Moreover, it follows from Cramer's formula that the inverse map is also smooth. This establishes that the real linear groups are Lie groups. The exact same goes for the complex linear groups.

This examples are very general, in the sense that every connected Lie group is locally isomorphic – i.e. there's a morphism of Lie groups whose restriction to some neighborhood of the identity is invertible with smooth inverse – to a subgroup of $\mathrm{GL}_n(\mathbb{R})$ for large enough n, but we're getting ahead of ourselves. We should point-out, however, that not every Lie group is a matrix group – the simplest counterexample is the quotient of the Heisenberg group

$$\mathbb{H} = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$$

by the normal subgroup

$$\begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is not isomorphic to any subgroup of $\mathrm{GL}_n(\mathbb{R})$ – for all n.

A morphism of Lie groups $G \longrightarrow H$ is a smooth group homomorphism $G \longrightarrow H$. Again, it doesn't make any sense to forget the geometric structure of given Lie group G when studying its representations – how else are we supposed to leverage the fact that G is a smooth manifold? This translates to...

Definition. A finite-dimensional representation V of a Lie group G over $\mathbb{R}$ or $\mathbb{C}$ is called *smooth* if

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

is a smooth map.

Remark. While fairly intuitive, this definition is entirely unnecessary! This is because every continuous group homomorphism between Lie groups is actually smooth. Hence smooth representations are simply finite-dimensional continuous representations. This implies we have already classified the smooth representations of compact Lie groups.

Example. Consider the action of G over G given by conjugation

$$\begin{aligned} \Psi : G &\longrightarrow \mathrm{Inn}(G) \\ g &\longmapsto \Psi(g) : \begin{aligned}[t] G &\longrightarrow G \\ h &\longmapsto ghg^{-1} \end{aligned} \end{aligned}$$

This action induces a smooth (real) representation $V = T_e G$ where

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}(V) \\ g &\longmapsto (d\Psi(g))_e \end{aligned}$$

known as *the adjoint representation of G*.

Notice in this case that there is *no* concept of *infinite-dimensional representations of a Lie group*. This is because smooth manifolds are, by definition, finite-dimensional. Hence in general $\mathrm{GL}(V)$ isn't a Lie group for infinite-dimensional V. This is quite convenient for us, since *finite-dimensional stuff* is easier to understand. This implies there's no need to define *smooth intertwining operators*, since every linear map $\mathbb{C}^n \longrightarrow \mathbb{C}^m$ is smooth.

Remark. Besides the last paragraph, there is something called *geometric analysis*, which allows us to talk about *infinite-dimensional manifolds* and *infinite-dimensional Lie groups*. We won't discuss that in here though.

Our goal is, once again, to classify the (finite-dimensional smooth) representations of a given Lie group up to isomorphism. Just like the *topology* of *topological* groups allows us to effectively study them and their representations, the geometric structure of Lie groups play a vital – and frankly essential – role in this chapter. Surprisingly however, the representation theory of Lie groups is much more *algebraic* than its topological counter-part.

It's important to note that we are primarily interested in *complex* representations of *real* Lie groups – i.e. Lie group morphisms $G \longrightarrow \mathrm{GL}_n(\mathbb{C})$. Hence from now on when we say *representation* we really mean *complex representation*. I personally find the distinction between the ground fields a bit eye-rolling, but this is a necessary evil. As it stands, this is an extremely inconsequential decision: everything said in this chapter works for real representations too. Nevertheless, the distinction between $\mathbb{R}$ and $\mathbb{C}$ will play a vital role in the next chapter

One could also study *analytic representations of complex Lie groups*, on which case we'd be interested in *holomorphic* group homomorphisms $G \longrightarrow \mathrm{GL}_n(\mathbb{C})$. Even though holomorphic functions are subject to much greater constraints than smooth maps – for instance, every locally constant holomorphic map is constant – the entirety of what we'll discuss in this chapter can be translated to the context of complex Lie groups. We'll initially focus on real Lie groups though, mainly because arguments regarding ordinary differential equations can be expressed in clearer terms in the real case. Back to real Lie groups then...

The instrumental peace of the puzzle here is the notion of *Lie algebras*, which can be treated as purely-algebraic structures. We regard Lie algebras primarily as tools for the study of Lie groups, they are not our focus in here. Nevertheless, despite its title this chapter is mostly dedicated to the study of Lie algebras and their relationship with Lie groups. This is because, as we'll see, the study of representations of Lie algebras is the key to solving our initial classification problem – the one about smooth representations of Lie groups.

3.1 Lie Algebras

In the spirit of dedicating a chapter entitled *Smooth Representations of Lie Groups* to Lie algebras, we'll start a section entitled *Lie Algebras* by talking about Lie groups. It turns out Lie groups are *embarrassingly symmetric objects*. Lie *groups* are *groups*, of course. But even more so, the interplay between their geometry and their group structure makes the former tremendously regular and homogeneous – i.e. *Lie groups are also "symmetric" as smooth manifolds* in some vague sense, even though not all Lie groups are *symmetric spaces*. In particular, if G and H are Lie groups it's easy to check that

(i) Given $g \in G$, the translation $\ell_g : G \longrightarrow G$ by g is a diffeomorphism that takes e to g – i.e. Lie groups are homogeneous spaces

(ii) If G is connected and U is a neighborhood of the identity then $G = \langle U \rangle$

(iii) If G and H are simply connected and locally isomorphic at the identity then $G \cong H$

(iv) Given a surjective morphism $\Phi : G \longrightarrow H$ with $(d\Phi)_e$ invertible, Φ is a covering map

This implies that an invariant vector field $X \in \mathfrak{X}(G)$ – that is, X such that the pushforward $(\ell_g)_* X = (d\ell_g)X = X$ – is uniquely determined by X_e. Hence the space $\mathfrak{g}$ of invariant vector fields over G can be identified with $T_e G$. Now consider the bilinear map

$$\begin{aligned} [\,,] : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (X, Y) &\longmapsto XY - YX \end{aligned}$$

where the composition of fields is given point-wise. The map $[\,,]$ is usually called *the Lie bracket on* $\mathfrak{g}$. It's not hard to check that $[\,,]$ is skew-symmetric and satisfies the Jacobian identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{3.1}$$

This brings us to the following definition...

Definition 3.1.1. A Lie algebra $\mathfrak{g}$ over a field k is k-vector space endowed with a skew-symmetric bilinear product $[\,,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the Jacobian identity (3.1). A homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ of Lie algebras is a linear operator such that

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$$

Clearly, the Lie algebra of invariant vector fields over a Lie group G *is an actual Lie algebra* – which is usually called *the Lie algebra associated with* G. In particular,

Example 3.1.1. The Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ associated with $\mathrm{GL}_n(\mathbb{C})$ can be identified with $T_e \mathrm{GL}_n(\mathbb{C}) = M_n(\mathbb{C})$. Under this identification, the Lie bracket $[\,,] : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}_n(\mathbb{C})$ is given by

$$[X, Y] = XY - YX,$$

where the product is the usual matrix product, and the Lie algebras of $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ correspond to the subalgebras

$$\mathfrak{sl}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_n(\mathbb{C}) : \mathrm{Tr}\, X = 0\}$$
$$\mathfrak{sp}_{2n}(\mathbb{C}) = \left\{X \in \mathfrak{gl}_{2n}(\mathbb{C}) : X^* \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \overline{X}\right\}$$

respectively.

Example 3.1.2. If G is an Abelian Lie group, its Lie algebra $\mathfrak{g}$ satisfies

$$[X, Y] = 0 \tag{3.2}$$

for all $X, Y \in \mathfrak{g}$. In fact, the Lie algebra of a connected Lie group satisfies (3.2) if and only if such group is Abelian – see the section on Abelian Lie groups of the second chapter of [Gor21]. For this reason, the Lie algebras satisfying (3.2) are known as *Abelian Lie algebras*.

Example 3.1.3. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, their direct sum $\mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra under the brackets given by

$$[X, Y] = [\pi_1(X), \pi_1(Y)] + [\pi_2(X), \pi_2(Y)]$$

If G and H are Lie groups, the Lie algebra of $G \times H$ is $\mathfrak{g} \oplus \mathfrak{h}$.

Definition 3.1.2. A representation of a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$ is a vector space V endowed with a homomorphism of Lie algebras

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

Example 3.1.4. Given a subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{C})$, the space $V = \mathbb{C}^n$ has the *natural* structure of a complex representation of $\mathfrak{g}$, where the action of $X \in \mathfrak{g}$ is given by the operator $V \longrightarrow V$ whose matrix in the standard basis is X. Of course, the same holds to $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{R})$ and $V = \mathbb{R}^n$. This is known as *the natural representation of* $\mathfrak{g}$.

Notice we have defined Lie algebras over *arbitrary* fields. This will come in handy later, when we discuss the *complexification* of a real Lie algebra. For now, however, we're really only interested in the (real) Lie algebra associated with a given Lie group.

This definitions are simple enough, but why are we doing this again? Well, the point is that Lie algebras are invariants: given two Lie groups G and H, a local isomorphism $G \longrightarrow H$ induces an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{h}$. Even more so, if G and H are connected, an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{h}$ induces a local isomorphism $G \longrightarrow H$. This allows us to given substance to our claim that *every connected Lie group is locally isomorphic to a subgroup of* $\mathrm{GL}_n(\mathbb{R})$, whose proof follows directly from something known as *Ado's theorem*.

Theorem 3.1.1 (Ado)**.** *Every finite-dimensional real Lie algebra is isomorphic to some Lie subalgebra of* $\mathfrak{gl}_n(\mathbb{R})$ *for large enough* n*. Equivalently, every finite-dimensional real Lie algebra has a finite-dimensional faithful representation.*

What we're particularly interested in is that, *in certain contexts*, there is a correspond between morphisms of Lie groups $G \longrightarrow H$ and homomorphism of Lie algebras $\mathfrak{g} \longrightarrow \mathfrak{h}$. In particular, there is a strong correspondence between representations of G and representations of $\mathfrak{g}$ – we'll get to a precise formulation of this soon.

First of all, why should this be true? It may come as a surprise, but given $X, Y \in \mathfrak{X}(G)$ in general

$$[X, Y] = XY - YX \neq 0$$

In other words, invariant fields over arbitrary Lie groups dot not commute – indeed the same holds for arbitrary fields over a manifold. In fact, the Lie bracket $[\,,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ measures *the degree to which the commutativity of invariant fields over G fails*, which is very much part of the geometric structure of G. For instance, the Lie bracket is closely related with the curvature of any given connection in a vector bundle over G. This is the whole point of Lie algebras in this context: $\mathfrak{g}$ encapsulates much of the geometric information of G in an algebraic, easily-manipulatable structure.

I hope that this botched attempt at motivating the definition of a Lie algebra was sufficiently convincing, but that's not really what we're here for – for an alternative, in-depth motivation see [Wil91]. We're here because we want to use representations of Lie algebras to study representations of Lie groups and apparently there's a so called *strong correspondence* between them. What's the correspondence then?

Theorem 3.1.2. *If G and H are Lie groups with G simply connected then there's a one-to-one correspondence between* $\mathrm{Hom}(G, H)$ *and* $\mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$. *Explicitly, given a morphism $\Phi : G \longrightarrow H$, $\Phi_* : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a homomorphism of Lie algebras and given a homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ there is a unique morphism $\Phi : G \longrightarrow H$ such that $\varphi = \Phi_*$.*

In particular,

Corollary 3.1.1. *If G is a simply connected Lie group then there is a one-to-one correspondence between representations of G and representations of $\mathfrak{g}$.*

Proof. It suffices to note that given a vector space V there is a one-to-one correspondence between $\mathrm{Hom}(G, \mathrm{GL}(V))$ and $\mathrm{Hom}(\mathfrak{g}, \mathfrak{gl}(V))$. ∎

Example 3.1.5. The trivial representation of G induces a representation $V = \mathbb{C}$ of $\mathfrak{g}$ where $Xv = 0$ for each $X \in \mathfrak{g}$ and $v \in V$. Indeed, the derivative of the constant map $g \longmapsto \mathrm{Id}$ is zero.

Example 3.1.6. The adjoint representation of G induces a real representation $V = \mathfrak{g}$ of $\mathfrak{g}$. Surprisingly, this is precisely the "regular" representation

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow \mathfrak{gl}(V) \\ X &\longmapsto \rho(X) : V \longrightarrow V \\ &\qquad\qquad\quad Y \longmapsto [X, Y] \end{aligned}$$

of $\mathfrak{g}$ – see section 2.2 of [Gor21].

Example 3.1.7. Given representations V and W of G, $\mathfrak{g}$ acts on the representations associated to V^*, $V \otimes W$ and $\mathrm{Sym}^n V$ via

$$Xv^* = -\overline{X}^* v \qquad X(v \otimes w) = Xv \otimes w + v \otimes Xw \qquad X(v^n) = (Xv)^n$$

respectively.

In categorical terms, theorem 3.1.2 amounts to saying the functor $\mathrm{Lie} : \mathbf{LieGrp}_{\mathrm{simpl}} \longrightarrow \mathbf{LieAlg}$ – between the category of simply connected Lie groups and the category of finite-dimensional real Lie algebras – that takes G to $\mathfrak{g}$ and a map $G \longrightarrow H$ to its derivative at the identity is fully-faithful. What's perhaps more surprising is that this functor is essentially surjective – i.e. every finite-dimensional real Lie algebra is the Lie algebra associated with some simply connected Lie group [Wan13], which is known as *Lie's third fundamental theorem*.

Hence our functor is an equivalence of categories $\mathbf{LieGrp}_{\mathrm{simpl}} \xrightarrow{\sim} \mathbf{LieAlg}$. Given that every connected Lie group is locally isomorphic to its universal cover[1], this implies that Lie algebras are

[1] Given a connected Lie group G, its universal cover $\widetilde{G}$ has a unique group structure under which $\widetilde{G}$ is a Lie group and the covering map $\widetilde{G} \longrightarrow G$ is a Lie group morphism and a submersion. The covering map $\widetilde{G} \longrightarrow G$ can then be shown to be a local isomorphism. [Yua13]

perfect invariants of "connected Lie groups up to universal cover", which is to say, two connected Lie groups share the same Lie algebra if and only if their universal covers are the same. As interesting as it may sound, however, this functorial formulation of the correspondence is lacking – in the sense that that only G is required to be simply connected for $\mathrm{Hom}(G, H) \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ to hold.

Nevertheless, this correspondence extends to an equivalence between the category of smooth representations of a given (simply connected) Lie group G and that of its Lie algebra – which takes a representation V of G to the corresponding representation of $\mathfrak{g}$ and an intertwining operator $T : V \longrightarrow W$ to itself. This not only establishes a close connection between representations of Lie groups and algebras, as much as it shows they are essentially the same thing – for simply connected G. Our next goal is to prove theorem 3.1.2, whose essential ingredient is something known as...

3.2 The Exponential Map

Consider the map

$$\begin{aligned} \exp : M_n(\mathbb{C}) &\longrightarrow M_n(\mathbb{C}) \\ X &\longmapsto \mathrm{Id} + X + \frac{X^2}{2} + \frac{X^3}{3!} + \cdots \end{aligned}$$

This is called *the exponential map*.

First of all, notice that

$$\exp(X + Y) = \exp(X)\exp(Y) \tag{3.3}$$

for all $X, Y \in M_n(\mathbb{C})$ with $XY = YX$. In other words, the name exp wasn't chosen at random – the exponential map turns *addition* into *multiplication*, at least in the case where the factors commute. Since $\exp(0) = \mathrm{Id}$, (3.3) implies $\exp(-X) = \exp(X)^{-1}$. Hence exp is really a smooth map $\mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$.

Moreover, exp is natural in the sense that

$$\begin{array}{ccc} \mathfrak{gl}_n(\mathbb{C}) & \xrightarrow{\Phi_*} & \mathfrak{gl}_m(\mathbb{C}) \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ \mathrm{GL}_n(\mathbb{C}) & \xrightarrow[\Phi]{} & \mathrm{GL}_m(\mathbb{C}) \end{array}$$

for all morphisms $\Phi : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C})$. Now notice that the derivative of the exponential map at the origin is the identity – in particular, it is a linear isomorphism. Hence the restriction $\exp\restriction_U : U \longrightarrow \exp(U)$ is a diffeomorphism for some neighborhood of U of the origin. It's also important to note that $\mathrm{Id} = \exp(0) \in \exp(U)$.

This is particularly useful for us since it implies that, given that $\mathrm{GL}_n(\mathbb{C})$ is connected, every map $\mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C})$ is determined by its derivative at the identity. We may now ask if there is any analogue of the exponential map for some arbitrary connected G. This is the next step we'll take in the direction of establishing theorem 3.1.2: somehow generalize the exponential map. Given $X \in \mathfrak{gl}_n(\mathbb{C})$, a simple calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX) = X \cdot \exp(tX) \tag{3.4}$$

Now since $\exp(0) = \mathrm{Id}$, this implies that

$$\begin{aligned} \gamma_X : \mathbb{R} &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ t &\longmapsto \exp(tX) \end{aligned}$$

is the integral curve of X that passes trough Id in $t = 0$. In other words, γ_X is the unique curve over $\mathrm{GL}_n(\mathbb{C})$ such that $\dot{\gamma}_X(t) = X \cdot \gamma_X(t)$ and $\gamma_X(0) = \mathrm{Id}$. Hence $\exp(X) = \gamma_X(1)$, where γ_X is the integral curve described above. That's quite a useful alternative to us, since nothing about this is specific to $\mathrm{GL}_n(\mathbb{C})$.

Given a Lie group G and $X \in \mathfrak{g}$, it's not hard to show that there exists a single smooth curve $\gamma : I \longrightarrow G$ such that $\dot{\gamma}(t) = X_{\gamma(t)}$ and $\gamma(0) = e$ – where $I \subseteq \mathbb{R}$ is some (maximal) open interval containing 0. We claim that γ is *local group homomorphism* too – i.e. $\gamma(s+t) = \gamma(s) \cdot \gamma(t)$ for all $s, t \in I$ with $s + t \in I$.

Indeed, if we fix $s \in I$ and define

$$\begin{aligned} \gamma_1(t) &= \gamma(s+t) \\ \gamma_2(t) &= \gamma(s) \cdot \gamma(t) \end{aligned}$$

it's easy to see that

$$\dot{\gamma_1}(t) = X_{\gamma_1(t)} \tag{3.5}$$

$$\dot{\gamma_2}(t) = X_{\gamma_2(t)} \tag{3.6}$$

for all t.

Moreover, $\gamma_1(0) = \gamma_2(0) = \gamma(s)$. It then follows from the uniqueness of the integral curve of a vector field over a manifold that $\gamma_1 = \gamma_2$. This implies $I = \mathbb{R}$. In other words, γ_X is a smooth group homomorphism $\mathbb{R} \longrightarrow G$ – which is usually referred to as a *one-parameter subgroup of G* – such that $\dot{\gamma_X}(t) = X_{\gamma_X(t)}$. Lo and behold, we arrive at our general definition...

Definition 3.2.1. The map exponential map on G is given by

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow G \\ X &\longmapsto \gamma_X(1) \end{aligned}$$

This extended definition is useful to us not only because it generalizes the exponential map of $\mathrm{GL}_n(\mathbb{C})$ – in the sense that both definitions coincide in this case – but primarily because it preserves most of its key features. Specifically...

Proposition 3.2.1. *The exponential map is the unique map from $\mathfrak{g}$ to G that takes 0 to e whose derivative at the origin is the identity and whose restriction to the lines through the origin in $\mathfrak{g}$ are one-parameter subgroups of G.*

Proposition 3.2.2. *The exponential map is such that*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Phi_*} & \mathfrak{h} \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\Phi]{} & H \end{array}$$

for all $\Phi : G \longrightarrow H$.

Proof. First of all, notice that γ_X is uniquely characterized by the fact that it is a homomorphism and $\dot{\gamma_X}(0) = X_e$. Indeed, given a smooth homomorphism $\gamma : \mathbb{R} \longrightarrow G$ with $\dot{\gamma}(0) = X_e$,

$$\begin{aligned} X_{\gamma(t)} &= (\ell_{\gamma(t)})_* \, X_e \\ &= (d\ell_{\gamma(t)})_e \, \dot{\gamma}(0) \\ \text{(chain rule)} &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \gamma(t)\gamma(s) \\ \text{(because } \gamma \text{ is a homomorphism)} &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \gamma(t+s) \\ \text{(chain rule)} &= \dot{\gamma}(t+0) \\ &= \dot{\gamma}(t) \end{aligned}$$

In other words, γ is the integral curve γ_X. This implies $\gamma_{\Phi_* X} = \Phi \circ \gamma_X$ – since both of these curves satisfy the conditions above. In particular,

$$\begin{aligned}\Phi(\exp(X)) &= \Phi(\gamma_X(1)) \\ &= \gamma_{\Phi_* X}(1) \\ &= \exp(\Phi_* X)\end{aligned}$$

for all $X \in \mathfrak{g}$. ∎

As previously mentioned, this last proposition implies. . .

Corollary 3.2.1. *For connected G, a Lie group morphism $\Phi : G \longrightarrow H$ is determined by its derivative at the identity Φ_*.*

Proof. Let $\Psi : G \longrightarrow H$ be a morphism such that $\Psi_* = \Phi_*$. We want to establish that $\Psi = \Phi$.

Since the derivative of exp at the origin is the identity, $\exp\restriction_U : U \longrightarrow \exp(U)$ is a diffeomorphism for some open $U \subseteq \mathfrak{g}$ containing the origin – i.e it is a local diffeomorphism at the origin. It then follows from proposition 3.2.2 that $\Phi\restriction_{\exp(U)}$ and $\Psi\restriction_{\exp(U)}$ coincide.

Notice $e \in \exp(U)$ – because the exponential map takes 0 to the identity in G. In particular, $\exp(U)$ is an open neighborhood of e. Now since G is connected, $G = \langle \exp(U) \rangle$, from which we conclude that Φ and Ψ coincide in all of G – given that Φ and Ψ are both group homomorphisms. We are done. ∎

An important consequence of proposition 3.2.1 is the fact that (3.4) holds for all G, in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(tX) = X_{\exp(tX)}$$

for all $X \in \mathfrak{g}$. Moreover, given a representation V of G,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho(\exp(tX)) = \rho_* \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX) \right) = \rho_*(X)$$

Hence. . .

Corollary 3.2.2. *Let V and W be representations of G and $T : V \longrightarrow W$ be an intertwining operator . Then T is also an intertwiner with respect to the corresponding representations of $\mathfrak{g}$. In other words, $\mathrm{Hom}_G(V, W) \subseteq \mathrm{Hom}_{\mathfrak{g}}(V, W)$.*

Proof. Given $X \in \mathfrak{g}$, it suffices to note that (3.4) implies

$$\begin{aligned} X\,T &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX)\,T \\ \text{(since } T \text{ is an intertwiner)} &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T\,\exp(tX) \\ &= T\,X \end{aligned}$$

where $\exp(tX)$ and X stand for their respective actions in V and W. ∎

We are now one step closer to establishing our correspondence. Essentially, what we've just proved in the last results is that our map $\mathrm{Hom}(G, H) \longrightarrow \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ is injective for connected G – and in particular for simply connected G – and that our map $\mathrm{Hom}_G(V, W) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(V, W)$ is well defined. All that's left is to show that these maps are also surjective for simply connected G, which is remarkably simple.

Theorem 3.2.1. *For simply connected G, every homomorphism $\mathfrak{g} \longrightarrow \mathfrak{h}$ is the derivative of some morphism $G \longrightarrow H$ at the identity.*

Proof. Let $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a homomorphism. Now consider the group $G \times H$, whose Lie algebra is $\mathfrak{g} \oplus \mathfrak{h}$. The condition that φ is a homomorphism is equivalent to the condition that $\mathfrak{j} = \{g \oplus \varphi(g) : g \in \mathfrak{g}\}$ is Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$. We claim that there exists some (immersed) subgroup $J \subseteq G \times H$ whose Lie algebra is precisely $\mathfrak{j}$.

If we allow ourselves some leeway and suppose this is the case in a gesture of faith, it's easy to see the derivative of the projection map $\pi_1 : J \longrightarrow G$ at the identity $(\pi_1)_* : \mathfrak{j} \longrightarrow \mathfrak{g}$ is a linear isomorphism: indeed, $(\pi_1)_*$ is just the projection to the first coordinate in $\mathfrak{j}$. Hence π_1 is a local diffeomorphism. Let $U \subseteq J$ be an open neighborhood of (e, e) in J so that $\pi_1 : U \longrightarrow \pi_1(U)$ is a diffeomorphism.

The first conclusion we arrive at is that π_1 is surjective. This is because since G is connected $G = \langle \pi_1(U) \rangle \subseteq \operatorname{im} \pi_1$. Furthermore, since $\dim \ker \pi_1 = \dim \ker(\pi_1)_* = 0$, $\ker \pi_1$ is a discrete subgroup of J, so that we can suppose without any loss in generality that $\ker \pi_1 \cap U = \{e\}$. This implies $\pi_1^{-1}(\pi_1(U)) = \coprod_{k \in \ker \pi_1} kU$, so that $\pi_1(U)$ is a uniformly covered (open) neighborhood of the identity e in G. Now by translating $\pi_1(U)$ by g we find a uniformly covered neighborhood of g for any $g \in G$.

Hence π_1 is a covering map. Now since G is simply connected, π_1 is an isomorphism. It then follows that the composition $\Phi = \pi_2 \circ \pi_1^{-1} : G \longrightarrow H$ is a morphism such that $\Phi_* = \varphi$. ■

Theorem 3.2.2. *Let V and W be representations of G for some connected G and $T : V \longrightarrow W$ be an intertwining operator between the corresponding representations of $\mathfrak{g}$. Then T is also an intertwiner with respect to G. In other words, $\operatorname{Hom}_{\mathfrak{g}}(V, W) \subseteq \operatorname{Hom}_G(V, W)$.*

Proof. Let $X \in \mathfrak{g}$ and define

$$\begin{aligned} \phi_1 : \mathbb{R} &\longrightarrow \operatorname{Hom}(V, W) \\ t &\longmapsto T \exp(tX) \end{aligned}$$

$$\begin{aligned} \phi_2 : \mathbb{R} &\longrightarrow \operatorname{Hom}(V, W) \\ t &\longmapsto \exp(tX)\, T \end{aligned}$$

It then follows from (3.4) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi_1(t) &= T\, X \exp(tX) \\ (T \text{ is an intertwiner}) &= X\, T \exp(tX) \\ &= X\, \phi_1(t) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi_2(t) &= X \exp(tX)\, T \\ &= X\, \phi_2(t) \end{aligned}$$

Hence ϕ_1 and ϕ_2 are both solutions to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = X\phi(t)$$

with $\phi(0) = T$. It then follows from the existence and uniqueness of solutions of ordinary differential equations that $\phi_1 = \phi_2$. In particular, $\phi_1(1) = \phi_2(1)$, which means $T \exp(X) = \exp(X)\, T$ for all $X \in \mathfrak{g}$. Now since G is connected, $G = \langle \exp(\mathfrak{g}) \rangle$ and therefore

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[T]{} & W \end{array}$$

for all $g \in G$. ■

Problem solved! Here's our correspondence, proved once and for all. Representations of simply connected Lie groups and Lie algebras are, indeed, *the same exact thing*. We are done. Well... not really, are we? We still need to show that the subgroup $J \subseteq G \times H$ of theorem 3.2.2 does, in fact, exist. We're almost there, but once again we'll need the help of some additional tools to finish our proofs, in particular...

3.3 The Campbell-Hausdorff Formula

Our new goal is to prove that...

Proposition 3.3.1. *Let G be a Lie group and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra of its Lie algebra. Then there exists an immersed subgroup $H \subseteq G$ such that $\mathfrak{h}$ is the Lie algebra of H.*

We expect, perhaps naively, that the exponential map $\exp : \mathfrak{h} \longrightarrow H$ is the restriction of $\exp : \mathfrak{g} \longrightarrow G$ to $\mathfrak{h}$. So according to proposition 3.2.1 the one-parameter subgroups of H should be the restriction of $\exp : \mathfrak{g} \longrightarrow G$ to the lines through the origin in $\mathfrak{h}$. Hence a natural candidate is

$$H = \bigcup_{\substack{X \in \mathfrak{h} \\ t}} \exp(tX) = \exp(\mathfrak{h})$$

In other words, H should be the image of the curves over G tangent to the elements of $\mathfrak{h}$. The question now is: is $\exp(\mathfrak{h})$ a subgroup of G? Well, if it is a subgroup, its Lie algebra is clearly $\mathfrak{h}$. Also, regardless of whether or not it is a subgroup, $\langle \exp(\mathfrak{h}) \rangle$ is. The issue with setting $H = \langle \exp(\mathfrak{h}) \rangle$, however, is that we lose control of its Lie algebra. So to all intents and purposes it would be quite convenient for us if $\exp(\mathfrak{h}) = \langle \exp(\mathfrak{h}) \rangle$. Fortunately, this is the case, but how do we prove it?

Clearly, $\exp(\mathfrak{h}) \subseteq \langle \exp(\mathfrak{h}) \rangle$. Now given $X, Y \in \mathfrak{h}$, we need to find some $Z \in \mathfrak{h}$ such that $\exp(Z) = \exp(X)\exp(Y)$. Recall that $\exp : \mathfrak{g} \longrightarrow G$ is a local diffeomorphism at the origin. This means that the restriction of the exponential map to some neighborhood U of the origin in $\mathfrak{g}$ is invertible. Let $\log : \exp(U) \longrightarrow U$ be the inverse of exp. Then for $X, Y \in \mathfrak{h}$ close enough of the origin,

$$X * Y = \log(\exp(X)\exp(Y))$$

is such that $\exp(X * Y) = \exp(X)\exp(Y)$.

Given $X \in \mathfrak{h}$, it's easy to check that the integral curve $\gamma_X : \mathbb{R} \longrightarrow G$ passes through $\exp(U)$ at $t = 0$. Hence there should exist some $n > 0$ large enough such that $\gamma_X\left(\frac{1}{n}\right) \in \exp(U)$. By taking $t = \frac{1}{n}$ we arrive at

$$\exp(X) = \gamma_X(1) = \gamma_X(n \cdot t) = \gamma_X(t)^n$$

This implies $\langle \exp(U \cap \mathfrak{h}) \rangle = \langle \exp(\mathfrak{h}) \rangle$, so it suffices to show that $\exp(\mathfrak{h})$ is locally closed under multiplication – i.e. $X * Y \in \mathfrak{h}$ for all $X, Y \in U \cap \mathfrak{h}$. We'll, again, begin by studying a simpler case: the case of $\mathfrak{gl}_n(\mathbb{R})$.

In this case, a simple calculation shows

$$\log(M) = (M - \mathrm{Id}) - \frac{(M - \mathrm{Id})^2}{2} + \frac{(M - \mathrm{Id})^3}{3} - \dots,$$

which converges only for M sufficiently close to Id.

Now by expanding

$$\begin{aligned} \exp(X)\exp(Y) &= \left(\mathrm{Id} + X + \frac{X^2}{2} + \cdots\right)\left(\mathrm{Id} + Y + \frac{Y^2}{2} + \cdots\right) \\ &= \mathrm{Id} + (X + Y) + \left(\frac{X^2}{2} XY + \frac{Y^2}{2}\right) + \cdots \end{aligned}$$

we arrive at

$$\begin{aligned} X * Y &= (X+Y) + \left(-\frac{(X+Y)^2}{2} + \left(\frac{X^2}{2} + XY + \frac{Y^2}{2}\right)\right) + \cdots \\ &= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \ldots \end{aligned} \tag{3.7}$$

These calculations are, of course, quite trivial, as long as you simply copy them from a book without even bothering to check the second term like I did. Anyway, the point of (3.7) isn't its proof or the precise formula of the series in the right site: the point is that each term in the right side of the last equality is a scalar multiple of a Lie bracket. This implies that $X * Y \in \mathfrak{h}$ for $X, Y \in \mathfrak{h} \subseteq \mathfrak{gl}_n(\mathbb{R})$ sufficiently close to the origin.

Equation (3.7) is called *the Campbell-Hausdorff formula*, and its proof is nothing short of a painful exercise in real analysis as far as I can tell – we won't include it in here and neither does the book I copied it from [Wil91]. Now given an arbitrary Lie group G, remember that G is locally isomorphic to some subgroup of $\mathrm{GL}_n(\mathbb{R})$ at the identity for n large enough. Hence the Campbell-Hausdorff formula holds for *all* Lie groups. We are essentially done.

Proof of Proposition 3.3.1. Let $H = \exp(\mathfrak{h})$. Then given $X, Y \in \mathfrak{h}$ sufficiently close to the origin,

$$X * Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots \in \mathfrak{h}$$

since $\mathfrak{h}$ is closed under the Lie bracket and the limit of sequences – every subspace of a finite-dimensional space is closed. So there exists $Z \in \mathfrak{h}$ such that $\exp(Z) = \exp(X)\exp(Y)$ – given by $Z = X * Y$. This implies $\langle H \rangle \subseteq H$, which is to say, H is a subgroup of G. ■

Notice that Lie's third fundamental theorem also follows from proposition 3.3.1. Indeed, given a finite-dimensional real Lie algebra $\mathfrak{g}$, Ado's theorem and proposition 3.3.1 imply that there exists some connected subgroup $G \subseteq \mathrm{GL}_n(\mathbb{R})$ such that $\mathfrak{g}$ is the Lie algebra of G – explicitly, $G = \exp(\mathfrak{g})$. By taking the universal cover $\widetilde{G}$ of G we arrive at the desired conclusion – see [Wan13] for further details.

Remark. We call $\widetilde{G}$ the *simply connected form of* $\mathfrak{g}$.

With our correspondence in hands we can finally give a precise meaning to our claim that "everything works the same for complex Lie groups". Namely, we can finally the following theorem.

Theorem 3.3.1. *If G and H are complex Lie groups with G simply connected then the complex Lie functor* $\mathbf{LieGrp}(\mathbb{C}) \longrightarrow \mathbf{LieAlg}(\mathbb{C})$ *induces a bijection*

$$\mathrm{Hom}_{\mathbb{C}}(G,H) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g},\mathfrak{h})$$

between the set $\mathrm{Hom}_{\mathbb{C}}(G,H)$ *holomorphic group homomorphisms* $G \longrightarrow H$ *and the set* $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g},\mathfrak{h})$ *of homomorphisms of complex Lie algebras* $\mathfrak{g} \longrightarrow \mathfrak{h}$.

Proof. If we regard G as a real Lie group, notice that the real algebra of G can be obtained by restricting scalars in the complex Lie algebra $\mathfrak{g}$ of G. This is because every left invariant vector field is holomorphic – the proof is the same as in the real case. Of course, the same holds for H. Hence if we denote the set real Lie algebra homomorphisms $\mathfrak{g} \longrightarrow \mathfrak{h}$ by $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{h})$ there is a natural inclusion $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g},\mathfrak{g}) \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{h})$. Namely, $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g},\mathfrak{g})$ is the set of $\mathbb{C}$-linear real Lie algebra homomorphisms $\mathfrak{g} \longrightarrow \mathfrak{h}$.

On the other hand, since every holomorphic map is smooth there is a natural inclusion $\mathrm{Hom}_{\mathbb{C}}(G,H) \subseteq \mathrm{Hom}_{\mathbb{R}}(G,H)$ – where $\mathrm{Hom}_{\mathbb{R}}(G,H)$ denotes the set of smooth group homomorphisms $G \longrightarrow H$. We claim that the image of $\mathrm{Hom}_{\mathbb{C}}(G,H)$ under the map

$$\mathrm{Hom}_{\mathbb{R}}(G,H) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{h})$$

given by the Lie functor is precisely $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathfrak{h}$. Indeed, if $\varphi : G \longrightarrow H$ is a holomorphic group homomorphism then $\varphi_* = d\varphi_e$ is a $\mathbb{C}$-linear map. Likewise, if $\varphi : G \longrightarrow H$ is only required to be smooth but φ_* is $\mathbb{C}$-linear, then so is $d\varphi_g = (d\ell_{\varphi(g)})_e \varphi_* (d\ell_g)_g^{-1}$ for all $g \in G$ – i.e. the total derivative $d\varphi : TG \longrightarrow TH$ is $\mathbb{C}$-linear. This is equivalent to the condition that φ is holomorphic, so $\varphi \in \mathrm{Hom}_{\mathbb{C}}(G, H)$. ■

Corollary 3.3.1. *The complex Lie functor* Lie : $\mathbf{LieGrp}(\mathbb{C})_{\mathrm{simpl}} \longrightarrow \mathbf{LieAlg}(\mathbb{C})$ *between the category of simply connected complex Lie groups and the category of finite-dimensional complex Lie algebras is an equivalence of categories.*

Remark. One more lengthy proof lies ahead.

Proof. We've just shown that the Lie functor is fully faithful. All it's left is to show that Lie is essentially surjective. In other words, we want the establish that every finite-dimensional complex Lie algebra is (isomorphic to) the complex Lie algebra of a simply connected complex Lie group.

Given a finite-dimensional complex Lie algebra $\mathfrak{g}$, if we regard $\mathfrak{g}$ as a real Lie algebra and apply Lie's third theorem we find some simply connected real Lie group G whose Lie algebra is $\mathfrak{g}$. The group G has a natural G-invariant almost complex structure J given by

$$(JX)_g = (d\ell_g)_e i (d\ell_{g^{-1}})_g X_g$$

We claim J is integrable. Indeed, if we regard the elements of $\mathfrak{g}$ as left invariants smooth vector fields over G a simple calculation shows that $JX = iX$ for all $X \in \mathfrak{g}$. It follows from the fact that $[,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is $\mathbb{C}$-bilinear,

$$\begin{aligned} N_J(X, Y) &= [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \\ &= [X, Y] + i[iX, Y] + i[X, iY] - [iX, iY] \\ &= 0 \end{aligned}$$

for all $X, Y \in \mathfrak{g}$, where N_J denotes the Nijenhuis tensor of J. Since N_J is a tensor, we find $N = 0$ – see proposition 1.3 of [Sal19] for further details. Hence by the NewlanderNirenberg theorem J is integrable. In other words, G is a complex manifold where the multiplication by i in each $T_g G$ is given by $J_g : T_g G \longrightarrow T_g G$. All its left is to show that the group operations in G are holomorphic, which is the same as showing their total derivatives are $\mathbb{C}$-linear – in the sense that they commute with J.

By construction, J is left invariant, so that the map $\ell_g : G \longrightarrow G$ is holomorphic for all $g \in G$. We claim $\imath_g : G \longrightarrow G$ is also holomorphic. To see this, first notice that since $\mathfrak{g}$ is a complex Lie algebra, $\mathrm{ad}(X)$ is $\mathbb{C}$-linear for all $X \in \mathfrak{g}$, so that

$$\begin{aligned} \mathrm{Ad}(\exp(X)) iY = \exp(\mathrm{ad}(X)) iY &= \sum_{n=1}^{\infty} \frac{\mathrm{ad}(X)^n iY}{n!} \\ &= i \sum_{n=1}^{\infty} \frac{\mathrm{ad}(X)^n Y}{n!} = i \exp(\mathrm{ad}(X)) Y = i \,\mathrm{Ad}(\exp(X)) Y \end{aligned}$$

for all $Y \in \mathfrak{g}$ – i.e. $\mathrm{Ad}(\exp(X))$ is $\mathbb{C}$-linear. Now since $G = \langle \exp(\mathfrak{g}) \rangle$, this implies $\mathrm{Ad}(g)$ is $\mathbb{C}$-linear for all g and therefore

$$\begin{aligned} J_g (d\imath_g)_e X &= (d\ell_g)_e i (d\ell_{g^{-1}})_g (d\imath_g)_e X \\ \text{(chain rule)} &= (d\ell_g)_e i \,\mathrm{Ad}(g^{-1}) X \\ &= (d\ell_g)_e \,\mathrm{Ad}(g^{-1}) iX \\ &= (d\ell_g)_e (d\ell_{g^{-1}})_g (d\imath_g)_e iX \\ &= (d\imath_g)_e J_e X \end{aligned}$$

Finally, since $(d\mathfrak{r}_g)_h = (d\mathfrak{r}_{hg})_e (d\mathfrak{r}_h)_e^{-1}$ is $\mathbb{C}$-linear for all $h \in G$, the total derivative $d\mathfrak{r}_g$ is $\mathbb{C}$-linear. This implies that the multiplication map $G \times G \longrightarrow G$ is holomorphic. To see that the inversion map $\iota : G \longrightarrow G$ is holomorphic first notice that given $X \in \mathfrak{g}$

$$\begin{aligned} d\iota_e X &= d\iota_e \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX)^{-1} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(-tX) \\ &= -X \end{aligned}$$

It then follows from the identity $\iota \circ \ell_g = \mathfrak{r}_{g^{-1}} \circ \iota$ that

$$d\iota_g = (d\mathfrak{r}_{g^{-1}})_e d\iota_e (d\ell_g)_e^{-1} = -(d\mathfrak{r}_{g^{-1}})_e (d\ell_g)_e^{-1}$$

is $\mathbb{C}$-linear for all $g \in G$, so that the total derivative $d\iota$ is again $\mathbb{C}$-linear. Hence G is a complex Lie group, and by our previous assertion that every invariant vector field is holomorphic its complex Lie algebra must be $\mathfrak{g}$. ■

Corollary 3.3.2. *Given a simply connected complex Lie group G, the category $\mathbf{Rep}(G)$ of analytic representations of G is equivalent to the category $\mathbf{Rep}(\mathfrak{g})$ of finite-dimensional representations of its complex Lie algebra $\mathfrak{g}$.*

If we return to the real case for a moment, we've now reduced the problem of classifying the complex representations of a simply connected real Lie group to that of classifying the representations of its Lie algebra. Before proceeding to the next chapter, however, we would like to go a step further: we claim we can further reduce the problem to that of classifying the representations of complex Lie algebras. Indeed, we will prove that for each a simply connected G,

$$\mathbf{Rep}(G) \cong \mathbf{Rep}(\mathfrak{g}) \cong \mathbf{Rep}(\mathfrak{g}_{\mathbb{C}})$$

for some *complex* Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

The obvious advantage of complex Lie algebras over real ones is the fact that $\mathbb{C}$ is an algebraically closed field, which tends to be a big feature when we're working with algebraic objects such as Lie algebras. However, as we shall see, this is not the only reason why we prefer to work with complex algebras as opposed to real algebras. Needless to say, this step is entirely superfluous in the case where G is a complex Lie group. The essential ingredient of this proof is the notion of...

3.4 The Complexification of a Lie Algebra

Given a real Lie algebra $\mathfrak{g}$, the space $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is, of course, a complex vector space, and it has the natural structure of a *complex* Lie algebra where the brackets are given by

$$[X \otimes z_1, Y \otimes z_2] = z_1 z_2 [X, Y]$$

The algebra $\mathfrak{g}_{\mathbb{C}}$ is usually referred to as *the complexification of* $\mathfrak{g}$. This construction induces a functor $-_{\mathbb{C}} : \mathbf{LieAlg} \longrightarrow \mathbf{LieAlg}(\mathbb{C})$ between the category of finite-dimensional real Lie algebra and that of complex Lie algebras, which takes a homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ to $\varphi \otimes \mathrm{Id} : \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathfrak{h}_{\mathbb{C}}$. This is neither an equivalence of categories nor a fully-faithful functor – for instance $\mathfrak{su}_2 \otimes \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{R})_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$ while $\mathfrak{su}_2 \not\cong \mathfrak{sl}_2(\mathbb{R})$ – but what's interesting about it is it enjoys the universal property that

$$\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{h}) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$$

for every complex Lie algebra $\mathfrak{h}$. In other words, there is an adjunction $-_{\mathbb{C}} \dashv R$, where $R : \mathbf{LieAlg}(\mathbb{C}) \longrightarrow \mathbf{LieAlg}$ is the "restriction of scalars" functor.

Indeed, by considering the maps $\alpha_{\mathfrak{g}} : \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{h}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ and $\beta_{\mathfrak{g}} : \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{h})$ given by

$$\begin{aligned} \alpha_{\mathfrak{g}}(\varphi) : \mathfrak{g}_{\mathbb{C}} &\longrightarrow \mathfrak{h} \\ X \otimes z &\longmapsto z\varphi(X) \end{aligned} \qquad \begin{aligned} \beta_{\mathfrak{g}}(\psi) : \mathfrak{g} &\longrightarrow \mathfrak{h} \\ X &\longmapsto \psi(X \otimes 1) \end{aligned}$$

we can very quickly check that $\beta_{\mathfrak{g}} = \alpha_{\mathfrak{g}}^{-1}$, establishing a natural bijection.

In particular, $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{gl}(V)) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{gl}(V))$ for each complex vector space V – i.e. $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ have the same *complex* representation theory. Indeed, this natural bijection does correspond to an equivalence of categories $\mathbf{Rep}(\mathfrak{g}) \cong \mathbf{Rep}(\mathfrak{g}_{\mathbb{C}})$. This allows us to give substance to our claim that "studying complex representations as opposed to real representations is a necessary evil in the realm of Lie groups": there is no clear correspondence between the real representations of a Lie group and the complex representations of the complex form of its Lie algebra.

Remark. Actually, there is a correspondence, but it is much more involved than the one we've just highlighted. Essentially, if G is a Lie group, every complex representation of $\mathfrak{g}$ is the torsion of a real representation of $\mathfrak{g}$.

This goes to show

$$\mathbf{Rep}(G) \cong \mathbf{Rep}(\mathfrak{g}) \cong \mathbf{Rep}(\mathfrak{g}_{\mathbb{C}}),$$

but it also implies something interesting: since SU_2 and $\mathrm{SL}_2(\mathbb{C})$ are both simply connected,

$$\mathbf{Rep}(\mathrm{SU}_2) \cong \mathbf{Rep}(\mathfrak{su}_2) \cong \mathbf{Rep}(\mathfrak{su}_2 \otimes \mathbb{C}) = \mathbf{Rep}(\mathfrak{sl}_2(\mathbb{C})) \cong \mathbf{Rep}(\mathrm{SL}_2(\mathbb{C}))$$

while $\mathrm{SU}_2 \ncong \mathrm{SL}_2(\mathbb{C})$. In other words, there is no reconstruction theorem for Lie groups, not even for the simply connected ones.

No matter how much representation-theoretic information we have, we will never be able to reconstruct a group and its differential structure solely out of its representations the way we could do for compact groups. This may disappoint Pontryagin and Tanaka, but it is actually very convenient for us: because there are multiple (non-isomorphic) simply connected forms of a complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, we may choose a form whose representations are well-behaved, lift this behavior to $\mathfrak{g}_{\mathbb{C}}$ and then lift this behavior for all real forms of $\mathfrak{g}_{\mathbb{C}}$. This is the primary feature of the complexification functor $-_{\mathbb{C}} : \mathbf{LieAlg} \longrightarrow \mathbf{LieAlg}(\mathbb{C})$.

We've spoken very little about representations so for. Now that we've established our correspondence, we'll carry out our initial goal of *classifying the smooth representations of simply connected Lie groups* by passing to the algebraic side of the force and studying the representations of the complexification of their Lie algebras. This is not to say that we will completely abandon the geometric point of view, quite to the contrary. From now on, however, we will primarily focus on the algebraic problem of classifying representations of Lie algebras.

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