Mémoire

Thiago Brevidelli Garcia

2024

Mapping Class Groups
&
their Representations

Thiago Brevidelli Garcia

Last updated July 9, 2024

This Master's Thesis is dedicated to my dear friend Lucas Schiezari.
May he rest in peace.

About

This is my M2 Mémoire (Master's Thesis), written in June 2024 under the supervision of Professor Maxime Wolff of the Institut of Mathématiques de Toulouse (IMT), France. The subject of the dissertation at hand is the notion of the mapping class group of a surface, their linear representations and some recent developments in the field. Namely, we discuss Korkmaz' proof of the triviality of low-dimensional representations.

Throughout these notes we will follow some guiding principles. First, lengthy proofs are favored over collections of smaller lemmas. This is a deliberate effort to emphasize the relevant results. That said, this dissertation is intended to be concise. Hence numerous results are left unproved. We refer the reader to proofs in other materials when appropriate.

Our main references are the beautiful book by Farb-Margalit [1], as well as the 2023 article by Korkmaz [2]. We assume the reader is already familiar with basic topology and group theory. Crucially, we *do not* assume any familiarity with representation theory. Because of tight deadlines, we have opted to hand-draw all the figures.

Acknowledgments

I would like to thank my friends and family for their tireless love and support. Special thanks to my girlfriend for her encouragement and understanding during this challenging year. I extend my gratitude to Professor Wolff for accepting me as his student and providing me with his invaluable guidance. I should also thank Matthieu Faitg for his help with the discussion on TQFT representations.

License

Contents

Chapter 1

Introduction

Ever since ancestral humans first stepped foot on the surface of Earth, Mankind has pondered the shape of the planet we inhabit. More recently, mathematicians have spent the past centuries trying to understand the topology of manifolds and, in particular, surfaces. Orientable compact surfaces were perhaps first classified by Gauss in the early 19th century. The proof of the following formulation of the classification, often attributed to Möbius, was completed in the 1920s with the work of Radò and others.

Theorem 1.1 (Classification of surfaces). *Any closed connected orientable surface is homeomorphic to the connected sum Σ_g of the sphere $\mathbb{S}^2$ with $g \geqslant 0$ copies of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Any compact connected orientable surface Σ is homeomorphic to the surface Σ_g^b obtained from Σ_g by removing $b \geqslant 0$ open disks with disjoint closures.*

The integer $g \geqslant 0$ in Theorem 1.1 is called *the genus of* Σ. We also have the noncompact surface $\Sigma_{g,r}^b = \Sigma_g^b \setminus \{x_1, \ldots, x_r\}$, where $x_1, \ldots, x_r$ lie in the interior of Σ_g^b. The points $x_1, \ldots, x_r$ are called the *punctures* of $\Sigma_{g,r}^b$. Throughout these notes, all surfaces considered will be of the form $\Sigma = \Sigma_{g,r}^b$. Any such Σ admits a natural compactification $\overline{\Sigma}$ obtained by filling its punctures. We denote $\Sigma_{g,r} = \Sigma_{g,r}^0$. All closed curves $\alpha, \beta \subseteq \Sigma$ we consider lie in the interior of Σ and intersect transversely. Unless explicitly stated otherwise, the curves α, β are assumed to be *unoriented* – i.e. we regard them as subsets of Σ.

Despite the apparent clarity of the picture painted by Theorem 1.1, there are still plenty of interesting, sometimes unanswered, questions about surfaces and their homeomorphisms. For instance, we can use the classification of surfaces to deduce information about how different curves in Σ are related by its homeomorphisms.

Observation 1.2 (Change of coordinates principle). Given oriented nonseparating simple closed curves $\alpha, \beta : \mathbb{S}^1 \longrightarrow \Sigma = \Sigma_{g,r}^b$, we can find an orientation-preserving homeomorphism $\phi : \Sigma \xrightarrow{\sim} \Sigma$ fixing $\partial\Sigma$ pointwise such that $\phi(\alpha) = \beta$ with orientation. To see this, we consider the surface Σ_α obtained by cutting Σ along α: we subtract the curve α from Σ and then add one additional boundary component δ_i in each side of α, as shown in Figure 1.1. By identifying δ_1 with δ_2 we can see Σ as a quotient of Σ_α.

Since α is nonseparating, Σ_α is a connected surface of genus $g-1$. In other words, $\Sigma_\alpha \cong \Sigma_{g-1,r}^{b+2}$. Similarly, $\Sigma_\beta \cong \Sigma_{g-1,r}^{b+2}$ also has two additional boundary components $\delta_1', \delta_2' \subseteq \partial\Sigma_\beta$. Now by the classification of surfaces we can find an orientation-preserving homeomorphism $\tilde{\phi} : \Sigma_\alpha \xrightarrow{\sim} \Sigma_\beta$. Even more so, we can choose $\tilde{\phi}$ taking δ_i to δ_i'. The homeomorphism $\tilde{\phi}$ then descends to a self-homeomorphism ϕ of the quotient surface $\Sigma \cong \Sigma_\alpha/\sim \;\cong \Sigma_\beta/\sim$ with $\phi(\alpha) = \beta$, as desired.

By cutting Σ along curves $\alpha, \alpha' \subseteq \Sigma$ crossing once, we can also show the following result.

Observation 1.3. Let $\alpha, \beta, \alpha', \beta' \subseteq \Sigma$ be nonseparating curves such that each pair $(\alpha, \alpha'), (\beta, \beta')$ crosses exactly once. Then we can find an orientation-preserving $\phi : \Sigma \xrightarrow{\sim} \Sigma$ fixing $\partial\Sigma$ pointwise such that $\phi(\alpha) = \beta$ and $\phi(\alpha') = \beta'$ – without orientation.

Figure 1.1: The surface $\Sigma_\alpha \cong \Sigma^{b+2}_{g-1,r}$ for a certain $\alpha \subseteq \Sigma$.

Given a surface Σ, the group $\mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ of orientation-preserving homeomorphisms of Σ fixing each point in $\partial\Sigma$ is a topological group[1] with a rich geometry, but its algebraic structure is often regarded as too complex to tackle. More importantly, all of this complexity is arguably unnecessary for most topological applications, in the sense that usually we are only really interested in considering *homeomorphisms up to isotopy*. For example,

(i) Isotopic $\phi \simeq \psi \in \mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ determine the same application $\phi_* = \psi_* : \pi_1(\Sigma, x) \longrightarrow \pi_1(\Sigma, x)$ and $\phi_* = \psi_* : H_1(\Sigma, \mathbb{Z}) \longrightarrow H_1(\Sigma, \mathbb{Z})$ at the levels of homotopy and homology.

(ii) The diffeomorphism class of the mapping torus $M_\phi = {}^{\Sigma \times [0,1]}\!\big/_{(x,0) \sim (\phi(x),1)}$ – a fundamental construction in low-dimensional topology – is invariant under isotopy.

It is thus more natural to consider the group of connected components of $\mathrm{Homeo}^+(\Sigma, \partial\Sigma)$, a countable discrete group known as *the mapping class group*. This will be the focus of the dissertation at hand.

Definition 1.4. The *mapping class group* $\mathrm{Mod}(\Sigma)$ *of an orientable surface* Σ is the group of isotopy classes of orientation-preserving homeomorphisms $\Sigma \xrightarrow{\sim} \Sigma$, where both the homeomorphisms and the isotopies are assumed to fix $\partial\Sigma$ pointwise.

$$\mathrm{Mod}(\Sigma) = {}^{\mathrm{Homeo}^+(\Sigma, \partial\Sigma)}\!\big/_{\simeq}$$

There are many variations of Definition 1.4.

Observation 1.5. Any $\phi \in \mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ extends uniquely to a homeomorphism $\tilde{\phi}$ of $\overline{\Sigma}$ that permutes the set $\{x_1, \ldots, x_r\} = \overline{\Sigma} \setminus \Sigma$ of punctures of Σ. We may thus define an action $\mathrm{Mod}(\Sigma) \circlearrowright \{x_1, \ldots, x_r\}$ via $f \cdot x_i = \tilde{\phi}(x_i)$ for $f = [\phi] \in \mathrm{Mod}(\Sigma)$ – which is independent of the choice of representative ϕ of f.

Definition 1.6. Given an orientable surface Σ and a puncture $x \in \overline{\Sigma}$ of Σ, denote by $\mathrm{Mod}(\Sigma, x) \subseteq \mathrm{Mod}(\Sigma)$ the subgroup of mapping classes that fix x. The *pure mapping class group* $\mathrm{PMod}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)$ *of* Σ is the subgroup of mapping classes that fix every puncture of Σ.

Observation 1.7. Given an oriented simple closed curve $\alpha : \mathbb{S}^1 \longrightarrow \Sigma$, denote by $\overrightarrow{[\alpha]}$ and $[\alpha]$ the isotopy classes of α with and without orientation, respectively – i.e $\overrightarrow{[\alpha]} = \overrightarrow{[\beta]}$ if $\alpha \simeq \beta$ as functions and $[\alpha] = [\beta]$ if $\overrightarrow{[\alpha]} = \overrightarrow{[\beta]}$ or $\overrightarrow{[\alpha]} = \overrightarrow{[\beta^{-1}]}$. There are natural actions $\mathrm{Mod}(\Sigma) \circlearrowright \{\overrightarrow{[\alpha]} \mid \alpha : \mathbb{S}^1 \longrightarrow \Sigma\}$ and $\mathrm{Mod}(\Sigma) \circlearrowright \{[\alpha] \mid \alpha \subseteq \Sigma\}$ given by

$$f \cdot \overrightarrow{[\alpha]} = \overrightarrow{[\phi(\alpha)]} \qquad\qquad f \cdot [\alpha] = [\phi(\alpha)]$$

for $f = [\phi] \in \mathrm{Mod}(\Sigma)$.

Definition 1.8. Given a simple closed curve $\alpha \subseteq \Sigma$, we denote by $\mathrm{Mod}(\Sigma)_{\overrightarrow{[\alpha]}}$ and $\mathrm{Mod}(\Sigma)_{[\alpha]}$ the subgroups of mapping classes that fix $\overrightarrow{[\alpha]}$ – for any given choice of orientation of α – and $[\alpha]$, respectively.

[1] Here we endow $\mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ with the compact-open topology.

While trying to understand the mapping class group of some surface Σ, it is interesting to consider how the geometric relationship between Σ and other surfaces affects $\mathrm{Mod}(\Sigma)$. Indeed, different embeddings $\Sigma' \hookrightarrow \Sigma$ translate to homomorphisms at the level of mapping class groups.

Example 1.9 (Inclusion homomorphism). Let $\Sigma' \subseteq \Sigma$ be a closed subsurface. Given $\phi \in \mathrm{Homeo}^+(\Sigma', \partial\Sigma')$, we may extend ϕ to $\tilde{\phi} \in \mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ by setting $\tilde{\phi}(x) = x$ for $x \in \Sigma$ outside of Σ' – which is well defined since ϕ fixes every point in $\partial\Sigma'$. This construction yields a group homomorphism

$$\begin{aligned} \mathrm{Mod}(\Sigma') &\longrightarrow \mathrm{Mod}(\Sigma) \\ [\phi] &\longmapsto [\tilde{\phi}], \end{aligned}$$

known as *the inclusion homomorphism*.

Example 1.10 (Capping homomorphism). Let $\delta \subseteq \partial\Sigma$ be a boundary component of Σ. We refer to the inclusion homomorphism $\mathrm{cap} : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Mod}(\Sigma \cup_\delta (\mathbb{D}^2 \setminus \{0\}))$ as *the capping homomorphism*.

Example 1.11 (Cutting homomorphism). Given a simple closed curve $\alpha \subseteq \Sigma$, any $f \in \mathrm{Mod}(\Sigma_{g+1})_{\overrightarrow{[\alpha]}}$ has a representative $\phi \in \mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ fixing α point-wise – so that ϕ restricts to a homeomorphism of $\Sigma \setminus \alpha$. Furthermore, if $\phi\restriction_{\Sigma\setminus\alpha} \simeq 1$ in $\Sigma \setminus \alpha$ then $\phi \simeq 1 \in \mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ – see [1, Proposition 3.20]. There is thus a group homomorphism

$$\begin{aligned} \mathrm{cut} : \mathrm{Mod}(\Sigma)_{\overrightarrow{[\alpha]}} &\longrightarrow \mathrm{Mod}(\Sigma \setminus \alpha) \\ [\phi] &\longmapsto [\phi\restriction_{\Sigma\setminus\alpha}], \end{aligned}$$

known as *the cutting homomorphism*.

As goes for most groups, another approach to understanding the mapping class group of a given surface Σ is to study its actions. We have already seen simple examples of such actions in Observation 1.5 and Observation 1.7. An important class of actions of $\mathrm{Mod}(\Sigma)$ are its *linear representations* – i.e. the group homomorphisms $\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{GL}_n(\mathbb{C})$. These may be seen as actions $\mathrm{Mod}(\Sigma) \circlearrowright \mathbb{C}^n$ where each $f \in \mathrm{Mod}(\Sigma)$ acts via some linear isomorphism $\mathbb{C}^n \xrightarrow{\sim} \mathbb{C}^n$.

1.1 Representations

Here we collect a few fundamental examples of linear representations of $\mathrm{Mod}(\Sigma)$.

Observation 1.12. Recall $H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, with standard basis given by $[\alpha_1], [\beta_1], \ldots, [\alpha_g], [\beta_g] \in H_1(\Sigma_g, \mathbb{Z})$ for $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ as in Figure 1.2. The Abelian group $H_1(\Sigma_g, \mathbb{Z})$ is endowed with a natural $\mathbb{Z}$-bilinear alternating form given by the *algebraic intersection number* $[\alpha] \cdot [\beta] = \sum_{x \in \alpha\cap\beta} \mathrm{ind}\, x$ – where the index $\mathrm{ind}\, x = \pm 1$ of an intersection point is given by Figure 1.3. In terms of the standard basis of $H_1(\Sigma_g, \mathbb{Z})$, this form is given by

$$[\alpha_i] \cdot [\beta_j] = \delta_{ij} \qquad [\alpha_i] \cdot [\alpha_j] = 0 \qquad [\beta_i] \cdot [\beta_j] = 0 \tag{1.1}$$

and thus coincides with the pullback of the standard $\mathbb{Z}$-bilinear symplectic form in $\mathbb{Z}^{2g}$.

Example 1.13 (Symplectic representation). Given $f = [\phi] \in \mathrm{Mod}(\Sigma_g)$, we may consider the map $\phi_* : H_1(\Sigma_g, \mathbb{Z}) \longrightarrow H_1(\Sigma_g, \mathbb{Z})$ induced at the level of singular homology. By homotopy invariance, the map ϕ_* is independent of the choice of representative ϕ of f. By the functoriality of homology groups we then get a $\mathbb{Z}$-linear action $\mathrm{Mod}(\Sigma_g) \circlearrowright H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ given by $f \cdot [\alpha] = \phi_*([\alpha]) = [\phi(\alpha)]$. Since pushforwards by orientation-preserving homeomorphisms preserve the indices of intersection points, $(f \cdot [\alpha]) \cdot (f \cdot [\beta]) = [\alpha] \cdot [\beta]$ for all $\alpha, \beta \subseteq \Sigma_g$ and $f \in \mathrm{Mod}(\Sigma_g)$. In light of (1.1), this implies $\mathrm{Mod}(\Sigma_g)$ acts on $\mathbb{Z}^{2g}$ via symplectomorphisms. We thus obtain a group homomorphism $\psi : \mathrm{Mod}(\Sigma_g) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \subseteq \mathrm{GL}_{2g}(\mathbb{C})$, known as *the symplectic representation of* $\mathrm{Mod}(\Sigma_g)$.

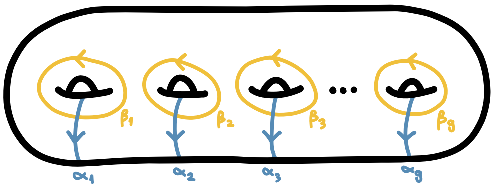

Figure 1.2: The curves $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \subseteq \Sigma_g$ that generate its first homology group.

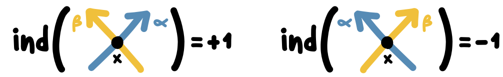

Figure 1.3: The index of an intersection point $x \in \alpha \cap \beta$.

The symplectic representation already allows us to compute some important examples of mapping class groups: namely, that of the torus $\mathbb{T}^2 = \Sigma_1$ and the once-punctured torus $\Sigma_{1,1}$.

Observation 1.14 (Alexander trick). The group $\mathrm{Homeo}^+(\mathbb{D}^2, \mathbb{S}^1)$ of homeomorphisms of the unit disk $\mathbb{D}^2 \subseteq \mathbb{C}$ is contractible. In particular, $\mathrm{Mod}(\mathbb{D}^2) = 1$. Indeed, for any $\phi \in \mathrm{Homeo}^+(\mathbb{D}^2, \mathbb{S}^1)$ the isotopy

$$\begin{aligned} \phi_t : \mathbb{D}^2 &\longrightarrow \mathbb{D}^2 \\ z &\longmapsto \begin{cases} (1-t)\phi(z/1-t) & \text{if } 0 \leqslant |z| \leqslant 1-t \\ z & \text{otherwise} \end{cases} \end{aligned}$$

that "fixes the band $\{z \in \mathbb{D}^2 : |z| \geqslant 1-t\}$ and does ϕ inside the sub-disk $\{z \in \mathbb{D}^2 : |z| \leqslant 1-t\}$" joins $\phi = \phi_0$ and $1 = \phi_1$.

Observation 1.15. By the same token, $\mathrm{Mod}(\mathbb{D}^2 \setminus \{0\}) = 1$.

Observation 1.16 (Linearity of $\mathrm{Mod}(\mathbb{T}^2)$). The symplectic representation $\psi : \mathrm{Mod}(\mathbb{T}^2) \longrightarrow \mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$ is a group isomorphism. In particular, $\mathrm{Mod}(\mathbb{T}^2) \cong \mathrm{SL}_2(\mathbb{Z})$. To see ψ is surjective, first observe $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant. Hence any matrix $g \in \mathrm{SL}_2(\mathbb{Z})$ descends to an orientation-preserving homeomorphism ϕ_g of the quotient $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, which satisfies $\psi([\phi_g]) = g$. To see ψ is injective we consider the curves α_1 and β_1 from Figure 1.2. Given $f = [\phi] \in \mathrm{Mod}(\mathbb{T}^2)$ with $\psi(f) = 1$, $f \cdot \overrightarrow{[\alpha_1]} = \overrightarrow{[\alpha_1]}$ and $f \cdot \overrightarrow{[\beta_1]} = \overrightarrow{[\beta_1]}$, so we may choose a representative ϕ of f fixing $\alpha_1 \cup \beta_1$ pointwise. Such ϕ determines a homeomorphism $\tilde{\phi}$ of the surface $\mathbb{T}^2_{\alpha_1\beta_1} \cong \mathbb{D}^2$ obtained by cutting $\mathbb{T}^2$ along α_1 and β_1, as in Figure 1.4. Now by the Alexander trick from Observation 1.14, $\tilde{\phi}$ must be isotopic to the identity. The isotopy $\tilde{\phi} \simeq 1 \in \mathrm{Homeo}^+(\mathbb{D}^2, \mathbb{S}^1)$ then descends to an isotopy $\phi \simeq 1 \in \mathrm{Homeo}^+(\mathbb{T}^2)$, so $f = 1 \in \mathrm{Mod}(\mathbb{T}^2)$ as desired.

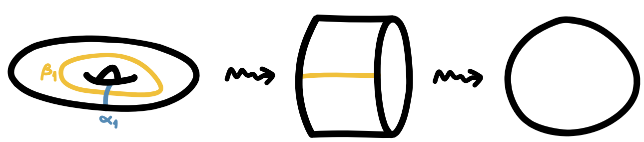

Figure 1.4: By cutting $\mathbb{T}^2$ along α_1 we obtain a cylinder, where β_1 determines a yellow arc joining the two boundary components. Now by cutting along this yellow arc we obtain a disk.

Observation 1.17. By the same token, $\mathrm{Mod}(\Sigma_{1,1}) \cong \mathrm{SL}_2(\mathbb{Z})$.

Remark. Despite the fact $\psi : \mathrm{Mod}(\mathbb{T}^2) \longrightarrow \mathrm{SL}_2(\mathbb{Z})$ is an isomorphism, the symplectic representation is ***not*** injective for surfaces of genus $g \geqslant 2$ – see [1, Section 6.5] for a description of its kernel. Korkmaz and Bigelow-Budney [3, 4] showed there exist injective linear representations of $\mathrm{Mod}(\Sigma_2)$, but the question of linearity of $\mathrm{Mod}(\Sigma_g)$ remains wide-open for $g \geqslant 3$. Recently, Korkmaz [2, Theorem 3] established the lower bound of $3g - 3$ for the dimension of an injective representation of $\mathrm{Mod}(\Sigma_g)$ in the $g \geqslant 3$ case – if one such representation exists.

Another fundamental class of examples of representations are the so-called *TQFT representations.*

Definition 1.18. A *cobordism* between closed oriented surfaces Σ and Σ' is a triple (W, ϕ_+, ϕ_-) where W is a smooth oriented compact 3-manifold with $\partial W = \partial_+ W \amalg \partial_- W$, $\phi_+ : \Sigma \xrightarrow{\sim} \partial_+ W$ is an orientation preserving diffeomorphism and $\phi_- : \Sigma' \xrightarrow{\sim} \partial_- W$ is an orientation-reversing diffeomorphism. We may abuse the notation and denote $W = (W, \phi_+, \phi_-)$.

Definition 1.19. We denote by $\mathbf{Cob}_3^+$ the category whose objects are (possibly disconnected) closed oriented surfaces and whose morphisms $\Sigma \longrightarrow \Sigma'$ are diffeomorphism classes[2] of cobordisms between Σ and Σ', with composition given by

$$[W, \phi_-, \phi_+] \circ [W', \psi_-, \psi_+] = [W \cup_{\psi_- \circ \phi_+^{-1}} W', \phi_-, \psi_+]$$

for $[W, \phi_-, \phi_+] : \Sigma \longrightarrow \Sigma'$ and $[W', \psi_-, \phi_+] : \Sigma' \longrightarrow \Sigma''$. We endow $\mathbf{Cob}_3^+$ with the monoidal structure given by

$$\Sigma \otimes \Sigma' = \Sigma \amalg \Sigma' \qquad [W, \phi_+, \phi_-] \otimes [W', \psi_+, \psi_-] = [W \amalg W', \phi_+ \amalg \psi_+, \phi_- \amalg \psi_-].$$

Definition 1.20 (TQFT). A *topological quantum field theory* (abbreviated by *TQFT*) is a functor $\mathcal{F} : \mathbf{Cob}_3^+ \longrightarrow \mathbf{Vect}(\mathbb{C})$ satisfying

$$\mathcal{F}(\emptyset) = \mathbb{C} \qquad \mathcal{F}(\Sigma \otimes \Sigma') = \mathcal{F}(\Sigma) \otimes \mathcal{F}(\Sigma') \qquad \mathcal{F}([W] \otimes [W']) = \mathcal{F}([W]) \otimes \mathcal{F}([W']),$$

where $\mathbf{Vect}(\mathbb{C})$ denotes the category of finite-dimensional complex vector spaces.

Observation 1.21. Given $\phi \in \mathrm{Homeo}^+(\Sigma_g)$, we may consider the so-called *mapping cylinder* $C_\phi = (\Sigma_g \times [0,1], \phi, 1)$, a cobordism between Σ_g and itself – where $\partial_+(\Sigma_g \times [0,1]) = \Sigma_g \times 0$ and $\partial_-(\Sigma_g \times [0,1]) = \Sigma_g \times 1$. The diffeomorphism class of C_ϕ is independent of the choice of representative of $f = [\phi] \in \mathrm{Mod}(\Sigma_g)$, so $C_f = [C_\phi] : \Sigma_g \longrightarrow \Sigma_g$ is a well defined morphism in $\mathbf{Cob}_3^+$.

Example 1.22 (TQFT representations). It is clear that C_1 is the identity morphism $\Sigma_g \longrightarrow \Sigma_g$ in $\mathbf{Cob}_3^+$. In addition, $C_{f \cdot g} = C_f \circ C_g$ for all $f, g \in \mathrm{Mod}(\Sigma_g)$ – see [5, Lemma 2.5]. Now given a TQFT $\mathcal{F} : \mathbf{Cob}_3^+ \longrightarrow \mathbf{Vect}(\mathbb{C})$, by functoriality we obtain a linear representation

$$\begin{aligned} \rho_{\mathcal{F}} : \mathrm{Mod}(\Sigma_g) &\longrightarrow \mathrm{GL}(\mathcal{F}(\Sigma_g)) \\ f &\longmapsto \mathcal{F}(C_f). \end{aligned}$$

As simple as the construction in Example 1.22 is, in practice it is not that easy to come across functors as the ones in Definition 1.20. This is because, in most interesting examples, we are required to attach some extra data to our surfaces to get a well defined association $\Sigma_g \longmapsto \mathcal{F}(\Sigma_g)$. Moreover, the condition $\mathcal{F}([W] \circ [W']) = \mathcal{F}([W]) \circ \mathcal{F}([W'])$ may only hold up to multiplication by scalars.

Hence constructing an actual functor typically requires *extending* $\mathbf{Cob}_3^+$ and *tweaking* $\mathbf{Vect}(\mathbb{C})$. Such functors give rise to linear and projective representations of the *extended mapping class groups* $\mathrm{Mod}(\Sigma_g) \times \mathbb{Z}$. We refer the reader to [5, 6] for constructions of one such TQFT and its corresponding representations: the so-called SU_2 *TQFT of level* r, first introduced by Witten and Reshetikhin-Tuarev [7, 8] in their foundational papers on quantum topology.

Besides Example 1.13 and Example 1.22, not a lot of other linear representations of $\mathrm{Mod}(\Sigma_g)$ are known. Indeed, the representation theory of mapping class groups remains a mystery at large. In Chapter 4 we provide a brief overview of the field, as well as some recent developments. More specifically, we highlight Korkmaz' [2] proof of the triviality of low-dimensional representations and comment on his classification of $2g$-dimensional representations. To that end, in Chapter 2 and Chapter 3 we survey the group structure of mapping class groups: its relations and known presentations.

[2] Here we only consider orientation-preserving diffeomorphisms $\varphi : W \xrightarrow{\sim} W'$ that are compatible with the boundary identifications in the sense that $\varphi(\partial_\pm W) = \partial_\pm W'$ and $\psi_\pm = \varphi \circ \phi_\pm$.

Chapter 2

Dehn Twists

With the goal of studying the linear representations of mapping class groups in mind, we now start investigating the group structure of $\mathrm{Mod}(\Sigma)$. We begin by computing some fundamental examples and then explore how we can use these examples to understand the structure of the mapping class groups of other surfaces. Namely, we compute $\mathrm{Mod}(\mathbb{S}^1 \times [0,1]) \cong \mathbb{Z}$, and discuss how its generator gives rise to a convenient generating set for $\mathrm{Mod}(\Sigma)$, known as the set of *Dehn twists*.

The idea here is to reproduce the proof of injectivity in Observation 1.16: by cutting along curves and arcs, we can always decompose a surface into copies of $\mathbb{D}^2$ and $\mathbb{D}^2 \setminus \{0\}$. Observation 1.14 and Observation 1.15 then imply the triviality of mapping classes fixing such arcs and curves. Formally, this translates to the following result.

Proposition 2.1 (Alexander method). *Let $\alpha_1, \ldots, \alpha_n \subseteq \Sigma$ be essential simple closed curves or proper arcs satisfying the following conditions.*

(i) $[\alpha_i] \neq [\alpha_j]$ *for* $i \neq j$.

(ii) *Each pair* (α_i, α_j) *crosses at most once.*

(iii) *Given distinct* i, j, k, *at least one of* $\alpha_i \cap \alpha_j, \alpha_i \cap \alpha_k, \alpha_j \cap \alpha_k$ *is empty.*

(iv) *The surface obtained by cutting* Σ *along the* α_i *is a disjoint union of disks and once-punctured disks.*

Suppose $f \in \mathrm{Mod}(\Sigma)$ *is such that* $f \cdot \overrightarrow{[\alpha_i]} = \overrightarrow{[\alpha_i]}$ *for all* i. *Then* $f = 1 \in \mathrm{Mod}(\Sigma)$.

See [1, Proposition 2.8] for a proof of Proposition 2.1. We now state some *fundamental* applications of the Alexander method.

Example 2.2. The mapping class group $\mathrm{Mod}(\mathbb{S}^1 \times [0,1])$ is freely generated by $f = [\phi]$, where

$$\begin{aligned} \phi : \mathbb{S}^1 \times [0,1] &\xrightarrow{\sim} \mathbb{S}^1 \times [0,1] \\ (e^{2\pi i t}, s) &\longmapsto (e^{2\pi i (t-s)}, s) \end{aligned}$$

is the map illustrated in Figure 2.1. In particular, $\mathrm{Mod}(\mathbb{S}^1 \times [0,1]) \cong \mathbb{Z}$.

Example 2.3. The mapping class group $\mathrm{Mod}(\mathbb{D}^2 \setminus \{-1/2, 1/2\})$ of the twice punctured unit disk in $\mathbb{C}$ is freely generated by $f = [\phi]$, where

$$\begin{aligned} \phi : \mathbb{D}^2 \setminus \{-1/2, 1/2\} &\xrightarrow{\sim} \mathbb{D}^2 \setminus \{-1/2, 1/2\} \\ z &\longmapsto -z \end{aligned}$$

is the map from Figure 2.2. In particular, $\mathrm{Mod}(\mathbb{D}^2 \setminus \{-1/2, 1/2\}) \cong \mathbb{Z}$.

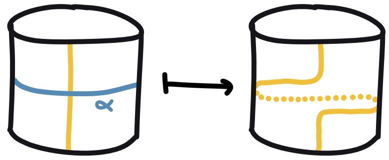

Figure 2.1: The generator f of $\mathrm{Mod}(\mathbb{S}^1 \times [0,1]) \cong \mathbb{Z}$ takes the yellow arc on the left-hand side to the arc on the right-hand side that winds about the curve α.

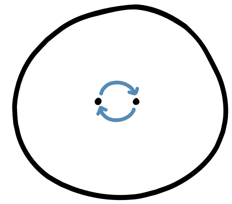

Figure 2.2: The generator f of $\mathrm{Mod}(\mathbb{D}^2 \setminus \{-1/2, 1/2\}) \cong \mathbb{Z}$ corresponds to the clockwise rotation by π about the origin.

Let Σ be an orientable surface, possibly with punctures and non-empty boundary. Given some closed $\alpha \subseteq \Sigma$, we may envision doing something similar to Example 2.2 in Σ by looking at annular neighborhoods of α. These are precisely the *Dehn twists*, illustrated in Figure 2.3 in the case of the bitorus Σ_2.

Definition 2.4. Given a simple closed curve $\alpha \subseteq \Sigma$, fix a closed annular neighborhood $A \subseteq \Sigma$ of α – i.e. $A \cong \mathbb{S}^1 \times [0,1]$. Let $f \in \mathrm{Mod}(A) \cong \mathrm{Mod}(\mathbb{S}^1 \times [0,1]) \cong \mathbb{Z}$ be the generator from Example 2.2. The *Dehn twist* $\tau_\alpha \in \mathrm{Mod}(\Sigma)$ *about* α is defined as the image of f under the inclusion homomorphism $\mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(\Sigma)$.

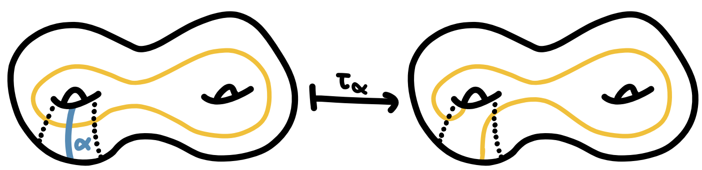

Figure 2.3: The Dehn twist about the curve α takes the peanut-shaped curve on the left-hand side to the yellow curve on the right-hand side.

Similarly, using the description of the mapping class group of the twice-puncture disk derived in Example 2.3, the generator of $\mathrm{Mod}(\mathbb{D}^2 \setminus \{-1/2, 1/2\})$ gives rise the so-called *half-twists*. These are examples of mapping classes that permute the punctures of Σ.

Definition 2.5. Given an arc $\alpha \subseteq \Sigma$ joining two punctures in the interior of Σ, fix a closed neighborhood $D \subseteq \Sigma$ of α with $D \cong \mathbb{D}^2 \setminus \{-1/2, 1/2\}$. Let $f \in \mathrm{Mod}(D) \cong \mathrm{Mod}(\mathbb{D}^2 \setminus \{-1/2, 1/2\}) \cong \mathbb{Z}$ be the generator from Example 2.3. The *half-twist* $h_\alpha \in \mathrm{Mod}(\Sigma)$ *about* α is defined as the image of f under the inclusion homomorphism $\mathrm{Mod}(D) \longrightarrow \mathrm{Mod}(\Sigma)$.

We can use the Alexander method to describe the kernel of capping and cutting morphisms in terms of Dehn twists.

Observation 2.6 (Capping exact sequence). Let $\delta \subseteq \partial\Sigma$ be a boundary component of Σ and $\mathrm{cap} : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Mod}(\Sigma \cup_\delta (\mathbb{D}^2 \setminus \{0\}))$ be the corresponding the capping homomorphism from Example 1.10. There is an exact sequence

$$1 \longrightarrow \langle \tau_\delta \rangle \longrightarrow \mathrm{Mod}(\Sigma) \xrightarrow{\mathrm{cap}} \mathrm{Mod}(\Sigma \cup_\delta (\mathbb{D}^2 \setminus \{0\}), 0) \longrightarrow 1,$$

known as *the capping exact sequence* – see [1, Proposition 3.19] for a proof.

Observation 2.7. Let $\alpha \subseteq \Sigma$ be a simple closed curve and $\mathrm{cut} : \mathrm{Mod}(\Sigma)_{\overrightarrow{[\alpha]}} \longrightarrow \mathrm{Mod}(\Sigma \setminus \alpha)$ be the cutting homomorphism from Example 1.11. Then $\ker \mathrm{cut} = \langle \tau_\alpha \rangle \cong \mathbb{Z}$.

It is also interesting to study how the geometry of two curves affects the relationship between their corresponding Dehn twists. For instance, by investigating the geometric intersection number

$$\#(\alpha \cap \beta) = \min\{|\alpha' \cap \beta'| : [\alpha'] = [\alpha] \text{ and } [\beta'] = [\beta]\}$$

we can distinguish between powers of Dehn twists [1, Proposition 3.2].

Proposition 2.8. *Let $\alpha \subseteq \Sigma$ be a simple closed curve and T_α be a representative of $\tau_\alpha \in \mathrm{Mod}(\Sigma)$. Then $\#(T_\alpha^k(\beta) \cap \beta) = |k| \cdot \#(\alpha \cap \beta)^2$ for any $k \in \mathbb{Z}$. In particular, if α is nontrivial then τ_α has infinite order.*

Observation 2.9. Given $\alpha, \beta \subseteq \Sigma$, $\tau_\alpha = \tau_\beta \iff [\alpha] = [\beta]$. Indeed, if α and β are non-isotopic, we can find γ with $\#(\gamma \cap \alpha) > 0$ and $\#(\gamma \cap \beta) = 0$. It thus follows from Proposition 2.8 that $\#(T_\alpha(\gamma) \cap \gamma) > \#(T_\beta(\gamma) \cap \gamma)$, so $\tau_\alpha \neq \tau_\beta$.

Many other relations between Dehn twists can derived be in a geometric fashion too.

Observation 2.10. Given $f = [\phi] \in \mathrm{Mod}(\Sigma)$, $\tau_{\phi(\alpha)} = f \tau_\alpha f^{-1}$.

Observation 2.11 (Disjointness relations). Given $f \in \mathrm{Mod}(\Sigma)$, $[f, \tau_\alpha] = 1 \iff f \cdot [\alpha] = [\alpha]$. In particular, $[\tau_\alpha, \tau_\beta] = 1$ for α and β disjoint, for we can choose a representative of τ_β whose support is disjoint from α.

Observation 2.12. If $\alpha, \beta \subseteq \Sigma$ are both nonseparating then $\tau_\alpha, \tau_\beta \in \mathrm{Mod}(\Sigma)$ are conjugate. Indeed, by the change of coordinates principle we can find $f \in \mathrm{Mod}(\Sigma)$ with $f \cdot [\alpha] = [\beta]$ and then apply Observation 2.10.

Observation 2.13 (Braid relations). Given $\alpha, \beta \subseteq \Sigma$ with $\#(\alpha \cap \beta) = 1$, it is not hard to check that $\tau_\beta \tau_\alpha \cdot [\beta] = [\alpha]$. From Observation 2.10 we then get $(\tau_\alpha \tau_\beta) \tau_\alpha (\tau_\alpha \tau_\beta)^{-1} = \tau_\beta$, from which follows the *braid relation*

$$\tau_\alpha \tau_\beta \tau_\alpha = \tau_\beta \tau_\alpha \tau_\beta.$$

A perhaps less obvious fact about Dehn twists is the following.

Theorem 2.14. *Let $\Sigma_{g,r}^b$ be the orientable surface of genus $g \geqslant 1$ with r punctures and b boundary components. Then the pure mapping class group $\mathrm{PMod}(\Sigma_{g,r}^b)$ is generated by finitely many Dehn twists about nonseparating curves or boundary components.*

The proof of Theorem 2.14 is simple in nature: we proceed by induction in g, b and r. On the other hand, the induction steps require two ingredients we have not encountered so far, namely the *Birman exact sequence* and the *modified graph of curves.* We now provide a concise account of these ingredients.

2.1 The Birman Exact Sequence

Having the proof of Theorem 2.14 in mind, it is interesting to consider the relationship between the mapping class group of $\Sigma_{g,r}^b$ and that of $\Sigma_{g,r+1}^b = \Sigma_{g,r}^b \setminus \{x\}$ for some x in the interior $(\Sigma_{g,r}^b)^\circ$ of $\Sigma_{g,r}^b$. Indeed, this will later allow us to establish the induction step on the number of punctures r.

Given an orientable surface Σ and $x_1, \ldots, x_n \in \Sigma^\circ$, denote by $\mathrm{Mod}(\Sigma \setminus \{x_1, \ldots, x_n\})_{\{x_1, \ldots, x_n\}} \subseteq \mathrm{Mod}(\Sigma \setminus \{x_1, \ldots, x_n\})$ the subgroup of mapping classes f that permute $x_1, \ldots, x_n$ – i.e. $f \cdot x_i = x_{\sigma(i)}$ for some permutation $\sigma \in S_n$. We certainly have a surjective homomorphism

$$\begin{aligned} \text{forget} : \mathrm{Mod}(\Sigma \setminus \{x_1, \ldots, x_n\})_{\{x_1, \ldots, x_n\}} &\longrightarrow \mathrm{Mod}(\Sigma) \\ [\phi] &\longmapsto [\tilde{\phi}] \end{aligned}$$

which "*forgets* the additional punctures $x_1, \ldots, x_n$ of $\Sigma \setminus \{x_1, \ldots, x_n\}$," but what is its kernel?

To answer this question, we consider the configuration space $C(\Sigma, n) = C^{\text{ord}}(\Sigma, n)/S_n$ of n (unordered) points in the interior of Σ – where $C^{\text{ord}}(\Sigma, n) = \{(x_1, \ldots, x_n) \in (\Sigma^\circ)^n : x_i \neq x_j \text{ for } i \neq j\}$. Denote $\text{Homeo}^+(\Sigma, \partial\Sigma)_{x_1,\ldots,x_n} = \{\phi \in \text{Homeo}^+(\Sigma, \partial\Sigma) : \phi(x_i) = x_i\}$. From the fibration[1]

$$\begin{aligned} \text{Homeo}^+(\Sigma, \partial\Sigma)_{x_1,\ldots,x_n} \hookrightarrow \text{Homeo}^+(\Sigma, \partial\Sigma) &\twoheadrightarrow C(\Sigma, n) \\ \phi &\longmapsto [\phi(x_1), \ldots, \phi(x_n)] \end{aligned}$$

and its long exact sequence in homotopy we then obtain the following fundamental result.

Theorem 2.15 (Birman exact sequence). *Suppose $\pi_1(\text{Homeo}^+(\Sigma, \partial\Sigma), 1) = 1$. Then there is an exact sequence*

$$1 \longrightarrow \pi_1(C(\Sigma, n), [x_1, \ldots, x_n]) \xrightarrow{\text{push}} \text{Mod}(\Sigma \smallsetminus \{x_1, \ldots, x_n\})_{\{x_1,\ldots,x_n\}} \xrightarrow{\text{forget}} \text{Mod}(\Sigma) \longrightarrow 1.$$

Remark. Notice that $C(\Sigma, 1) = \Sigma^\circ \simeq \Sigma$. Hence for $n = 1$ Theorem 2.15 gives us a sequence

$$1 \longrightarrow \pi_1(\Sigma, x) \xrightarrow{\text{push}} \text{Mod}(\Sigma \smallsetminus \{x\}, x) \xrightarrow{\text{forget}} \text{Mod}(\Sigma) \longrightarrow 1.$$

We may regard a simple loop $\alpha : \mathbb{S}^1 \longrightarrow C(\Sigma, n)$ based at $[x_1, \ldots, x_n]$ as n disjoint curves $\alpha_1, \ldots, \alpha_n : [0, 1] \longrightarrow \Sigma$ with $\alpha_i(0) = x_i$ and $\alpha_i(1) = x_{\sigma(i)}$ for some $\sigma \in S_n$. The element $\text{push}([\alpha]) \in \text{Mod}(\Sigma)$ can then be seen as the mapping class that "*pushes* a neighborhood of $x_{\sigma(i)}$ towards x_i along the curve α_i^{-1}," as shown in Figure 2.4 for the case $n = 1$. Indeed, this goes to show $\text{push}([\alpha])$ can be descrived as a product of Dehn twists.

Fundamental Observation 2.16. Using the notation of Figure 2.4, $\text{push}([\alpha]) = \tau_{\delta_1}\tau_{\delta_2}^{-1} \in \text{Mod}(\Sigma)$.

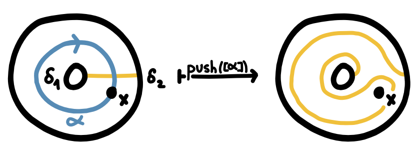

Figure 2.4: The inclusion $\text{push} : \pi_1(\Sigma, x) \longrightarrow \text{Mod}(\Sigma)$ maps a simple loop $\alpha : \mathbb{S}^1 \longrightarrow \Sigma$ to the mapping class supported at an annular neighborhood A of α. Inside this neighborhood, $\text{push}([\alpha])$ takes the arc joining the boundary components $\delta_i \subseteq \partial A$ on the left-hand side to the yellow arc on the right-hand side.

2.2 The Modified Graph of Curves

Having established Theorem 2.15, we now need to address the induction step in the genus g of $\Sigma_{g,r}^b$. Our strategy is to apply the following lemma from geometric group theory.

Lemma 2.17. *Let G be a group and Γ be a* connected *graph with $G \circlearrowright \Gamma$ via graph automorphisms. Suppose that G acts transitively on both $V(\Gamma)$ and $\{(v, w) \in V(\Gamma)^2 : v — w \text{ in } \Gamma\}$. If $v, w \in V(\Gamma)$ are connected by an edge and $g \in G$ is such that $g \cdot w = v$ then G is generated by g and the stabilizer G_v.*

We are interested, of course, in the group $G = \text{PMod}(\Sigma_{g,r}^b)$. As for the the role of Γ, we consider the following graph.

[1] See [9, Chapter 4] for a reference.

Definition 2.18. The *modified graph of nonseparating curves* $\hat{\mathcal{N}}(\Sigma)$ *of a surface* Σ is the graph whose vertices are (unoriented) isotopy classes of nonseparating simple closed curves in Σ and

$$[\alpha] — [\beta] \text{ in } \hat{\mathcal{N}}(\Sigma) \iff \#(\alpha \cap \beta) = 1,$$

where $\#(\alpha \cap \beta)$ is the geometric intersection number of α and β.

It is clear from the change of coordinates principle and Observation 1.3 that the actions of $\mathrm{Mod}(\Sigma^b_{g,r})$ on $V(\hat{\mathcal{N}}(\Sigma^b_{g,r}))$ and $\{([\alpha],[\beta]) \in V(\hat{\mathcal{N}}(\Sigma^b_{g,r}))^2 : \#(\alpha \cap \beta) = 1\}$ are both transitive. But why should $\hat{\mathcal{N}}(\Sigma^b_{g,r})$ be connected? Historically, the modified graph of nonseparating curves first arose as a *modified* version of another graph, known as *the graph of of curves*.

Definition 2.19. Given a surface Σ, the *graph of curves* $\mathscr{C}(\Sigma)$ *of* Σ is the graph whose vertices are (unoriented) isotopy classes of essential simple closed curves in Σ and

$$[\alpha] — [\beta] \text{ in } \mathscr{C}(\Sigma) \iff \#(\alpha \cap \beta) = 0.$$

The *graph of nonseparating curves* $\mathcal{N}(\Sigma)$ is the subgraph of $\mathscr{C}(\Sigma)$ whose vertices consist of non-separating curves.

Lickorish [10] essentially showed that, apart from a small number of sporadic cases, $\mathscr{C}(\Sigma_{g,r})$ is connected.

Theorem 2.20. *If* $\Sigma_{g,r}$ *is not one* $\Sigma_0 = \mathbb{S}^2, \Sigma_{0,1}, \ldots, \Sigma_{0,4}, \Sigma_1 = \mathbb{T}^2$ *and* $\Sigma_{1,1}$ *then* $\mathscr{C}(\Sigma_{g,r})$ *is connected.*

In other words, given simple closed curves $\alpha, \beta \subseteq \Sigma_{g,r}$, we can find closed $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ in $\Sigma_{g,r}$ with α_i disjoint from α_{i+1}. Now if α and β are nonseparating, by inductively adjusting this sequence of curves we obtain the following corollary.

Corollary 2.21. *If* $g \geqslant 2$ *then both* $\mathcal{N}(\Sigma_{g,r})$ *and* $\hat{\mathcal{N}}(\Sigma_{g,r})$ *are connected.*

See [1, Section 4.1] for a proof of Corollary 2.21. We are now ready to show Theorem 2.14.

Proof of Theorem 2.14. Let $\Sigma^b_{g,r}$ be the orientable surface of genus $g \geqslant 1$ with r punctures and b boundary components. We want to establish that $\mathrm{PMod}(\Sigma^b_{g,r})$ is generated by a finite number of Dehn twists about nonseparating simple closed curves or boundary components. As promised, we proceed by triple induction on r, g and b.

For the base case, it is clear from Observation 1.16 and Observation 1.17 that $\mathrm{Mod}(\mathbb{T}^2) \cong \mathrm{Mod}(\Sigma_{1,1}) \cong \mathrm{SL}_2(\mathbb{Z})$ are generated by the Dehn twists about the curves α and β from Figure 2.5, each corresponding to one of the standard generators

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

of $\mathrm{SL}_2(\mathbb{Z})$.

Figure 2.5: The curves α and β whose Dehn twists generate $\mathrm{Mod}(\mathbb{T}^2)$ and $\mathrm{Mod}(\Sigma_{1,1})$.

Now suppose $\mathrm{PMod}(\Sigma_{g,r})$ is finitely-generated by twists about nonseparating curves for $g \geqslant 2$ or $g = 1$ and $r > 1$. In both case, $\chi(\Sigma_{g,r}) = 2 - 2g - r < 0$ and thus $\pi_1(\mathrm{Homeo}^+(\Sigma_{g,r})) = 1$ – see [1, Theorem 1.14]. The Birman exact sequence from Theorem 2.15 then gives us

$$1 \longrightarrow \pi_1(\Sigma_{g,r}, x) \xrightarrow{\text{push}} \mathrm{PMod}(\Sigma_{g,r+1}) \xrightarrow{\text{forget}} \mathrm{PMod}(\Sigma_{g,r}) \longrightarrow 1,$$

where $\Sigma_{g,r+1} = \Sigma_{g,r} \setminus \{x\}$. Since $g \geqslant 1$, $\pi_1(\Sigma_{g,r}, x)$ is generated by finitely many nonseparating loops. We have seen in Observation 2.16 that push : $\pi_1(\Sigma_{g,r}, x) \longrightarrow \mathrm{Mod}(\Sigma_{g,r+1}, x)$ takes nonseparating simple loops to products of twists about nonseparating simple curves. Furthermore, we may lift the generators of $\mathrm{PMod}(\Sigma_{g,r})$ to Dehn twists about the corresponding curves in $\Sigma_{g,r+1}$. This goes to show that $\mathrm{PMod}(\Sigma_{g,r+1})$ is also generated by finitely many twists about simple curves, concluding the induction step on r.

As for the induction step on g, fix $g \geqslant 1$ and suppose that, for each $r \geqslant 0$, $\mathrm{PMod}(\Sigma_{g,r})$ is finitely generated by twists about nonseparating curves or boundary components. Let us show that the same holds for $\mathrm{Mod}(\Sigma_{g+1})$. To that end, we consider the action $\mathrm{Mod}(\Sigma_{g+1}) \circlearrowright \hat{\mathcal{N}}(\Sigma_{g+1})$. Since $g+1 \geqslant 2$, $\hat{\mathcal{N}}(\Sigma_{g+1})$ is connected and the conditions of Lemma 2.17 are met. Now recall from Observation 2.13 that, given nonseparating $\alpha, \beta \subseteq \Sigma_{g+1}$ crossing once, $\tau_\beta \tau_\alpha \cdot [\beta] = [\alpha]$. It thus follows from Lemma 2.17 that $\mathrm{Mod}(\Sigma_{g+1})$ is generated by $\tau_\beta\tau_\alpha$ and $\mathrm{Mod}(\Sigma_{g+1})_{[\alpha]} = \{f \in \mathrm{Mod}(\Sigma_{g+1}) : f \cdot [\alpha] = [\alpha]\}$.

In turn, $\mathrm{Mod}(\Sigma_{g+1})_{[\alpha]}$ has its index 2 subgroup

$$\mathrm{Mod}(\Sigma_{g+1})_{\overrightarrow{[\alpha]}} = \{f \in \mathrm{Mod}(\Sigma_{g+1}) : f \cdot \overrightarrow{[\alpha]} = \overrightarrow{[\alpha]}\}$$

of mapping classes fixing any given choice of orientation of α. One can check that $\tau_\beta \tau_\alpha^2 \tau_\beta \in \mathrm{Mod}(\Sigma_{g+1})_{[\alpha]}$ inverts the orientation of α and is thus a representative of the nontrivial $\mathrm{Mod}(\Sigma_{g+1})_{\overrightarrow{[\alpha]}}$-coset in $\mathrm{Mod}(\Sigma_{g+1})_{[\alpha]}$. In particular, $\mathrm{Mod}(\Sigma_{g+1})$ is generated by $\mathrm{Mod}(\Sigma_{g+1})_{\overrightarrow{[\alpha]}}$, $\tau_\beta\tau_\alpha$ and $\tau_\beta \tau_\alpha^2 \tau_\beta$.

We now claim $\mathrm{Mod}(\Sigma_{g+1})_{\overrightarrow{[\alpha]}}$ is generated by finitely many twists about nonseparating curves. First observe that $\Sigma_{g+1} \setminus \alpha \cong \Sigma_{g,2}$, as shown in Figure 2.6. Observation 2.7 then gives us an exact sequence

$$1 \longrightarrow \langle \tau_\alpha \rangle \longrightarrow \mathrm{Mod}(\Sigma_{g+1})_{\overrightarrow{[\alpha]}} \xrightarrow{\text{cut}} \mathrm{PMod}(\Sigma_{g,2}) \longrightarrow 1. \tag{2.1}$$

But by the induction hypothesis, $\mathrm{PMod}(\Sigma_{g,2})$ is finitely-generated by twists about nonseparating simple closed curves. As before, these generators may be lifted to appropriate twists in $\mathrm{Mod}(\Sigma_{g+1})_{\overrightarrow{[\alpha]}}$. Now by (2.1) we get that $\mathrm{Mod}(\Sigma_{g+1})_{\overrightarrow{[\alpha]}}$ is finitely generated by twists about nonseparating curves, as desired. This concludes the induction step in g.

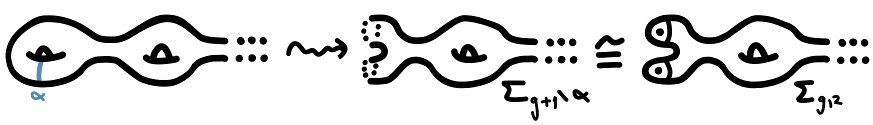

Figure 2.6: The homeomorphism $\Sigma_{g+1} \setminus \alpha \cong \Sigma_{g,2}$: removing the curve α has the same effect as cutting along α and then capping the two resulting boundary components with once-punctured disks, which gives us $\Sigma_{g,2}$.

Finally, we handle the induction in b. The boundaryless case $b = 0$ was already dealt with before. Now suppose $\mathrm{PMod}(\Sigma_{g,s}^b)$ is finitely generated by twists about simple closed curves or boundary components for all g and s. Fix some boundary component $\delta \subseteq \partial\Sigma_{g,r}^{b+1}$. From the homeomorphism $\Sigma_{g,r+1}^b \cong \Sigma_{g,r}^{b+1} \cup_\delta (\mathbb{D}^2 \setminus \{0\})$ and the capping exact sequence from Observation 2.6 we obtain a sequence

$$1 \longrightarrow \langle \tau_\delta \rangle \longrightarrow \mathrm{PMod}(\Sigma_{g,r}^{b+1}) \xrightarrow{\text{cap}} \mathrm{PMod}(\Sigma_{g,r+1}^b) \longrightarrow 1.$$

Now by induction hypothesis we may once again lift the generators of $\mathrm{PMod}(\Sigma_{g,r+1}^b)$ to Dehn twists about the corresponding curves in $\Sigma_{g,r}^{b+1}$ and add τ_δ to the generating set, concluding the induction in $b \geqslant 0$. We are done. ∎

There are many possible improvements to this last result. For instance, in [1, Section 4.4] Farb-Margalit exhibit an explicit set of generators of $\mathrm{Mod}(\Sigma_g^b)$ by adapting the induction steps in the proof of Theorem 2.14. These are known as the *Lickorish generators.*

Theorem 2.22 (Lickorish generators). *If $g \geqslant 1$ then $\mathrm{Mod}(\Sigma_g^b)$ is generated by the Dehn twists about the curves $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}, \eta_1, \dots, \eta_{b-1}$ as in Figure 2.7*

In the boundaryless case $b = 0$, we can write $\tau_{\alpha_3}, \dots, \tau_{\alpha_g} \in \mathrm{Mod}(\Sigma_g)$ as products of the twists about the remaining curves, from which we get the so-called *Humphreys generators.*

Corollary 2.23 (Humphreys generators). *If $g \geqslant 2$ then $\mathrm{Mod}(\Sigma_g)$ is generated by the Dehn twists about the curves $\alpha_0, \dots, \alpha_{2g}$ as in Figure 2.8.*

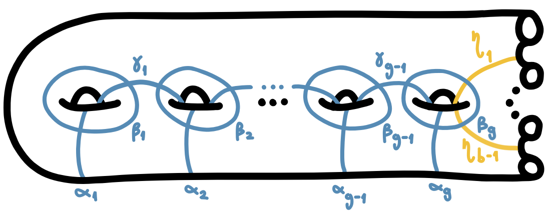

Figure 2.7: The curves from Lickorish generators of $\mathrm{Mod}(\Sigma_g^b)$.

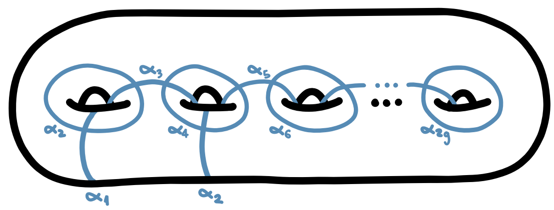

Figure 2.8: The curves from Humphreys generators of $\mathrm{Mod}(\Sigma_g)$.

Chapter 3

Relations Between Twists

Having found a convenient set of generators for $\mathrm{Mod}(\Sigma_g)$, it is now natural to ask what the relations between such generators are. In this chapter, we highlight some additional relations between Dehn twists and the geometric intuition behind them, culminating in the statement of a presentation for $\mathrm{Mod}(\Sigma_g)$ whose relations can be entirely explained in terms of the geometry of curves in Σ_g – see Theorem 3.15.

Fundamental Observation 3.1 (Lantern relation). Let Σ_0^4 be the surface of genus 0 with 4 boundary components and $\alpha, \beta, \gamma, \delta_1, \ldots, \delta_4 \subseteq \Sigma_0^4$ be as in Figure 3.1. Consider the surfaces $\Sigma_0^3 = \Sigma_0^4 \cup_{\delta_1} \mathbb{D}^2$ and $\Sigma_{0,1}^3 = \Sigma_0^4 \cup_{\delta_1} (\mathbb{D}^2 \setminus \{0\})$, as well as the map push : $\pi_1(\Sigma_0^3, 0) \longrightarrow \mathrm{Mod}(\Sigma_{0,1}^3)$. Let $\eta_1, \eta_2, \eta_3 : \mathbb{S}^1 \longrightarrow \Sigma_0^3$ be the loops from Figure 3.2, so that $[\eta_1] \cdot [\eta_2] = [\eta_3]$ in $\pi_1(\Sigma_0^3, 0)$. From Observation 2.16 we obtain

$$(\tau_{\delta_2}\tau_\alpha^{-1})(\tau_{\delta_3}\tau_\gamma^{-1}) = \mathrm{push}([\eta_1]) \cdot \mathrm{push}([\eta_2]) = \mathrm{push}([\eta_3]) = \tau_\beta\tau_{\delta_4}^{-1} \in \mathrm{Mod}(\Sigma_{0,1}^3).$$

Using the capping exact sequence from Observation 2.6, we can then see $\tau_{\delta_2}\tau_\alpha^{-1}\tau_{\delta_3}\tau_\gamma^{-1}, \tau_\beta\tau_{\delta_4}^{-1} \in \mathrm{Mod}(\Sigma_0^4)$ differ by a power of τ_{δ_1}. In fact, one can show $(\tau_{\delta_2}\tau_\alpha^{-1}\tau_{\delta_3}\tau_\gamma^{-1})(\tau_\beta\tau_{\delta_4}^{-1})^{-1} = \tau_{\delta_1}^{-1} \in \mathrm{Mod}(\Sigma_0^4)$. Now the disjointness relations $[\tau_{\delta_i}, \tau_\alpha] = [\tau_{\delta_i}, \tau_\beta] = [\tau_{\delta_i}, \tau_\gamma] = 1$ give us the *lantern relation* (3.1) in $\mathrm{Mod}(\Sigma_0^4)$.

$$\tau_\alpha\tau_\beta\tau_\gamma = \tau_{\delta_1}\tau_{\delta_2}\tau_{\delta_3}\tau_{\delta_4} \tag{3.1}$$

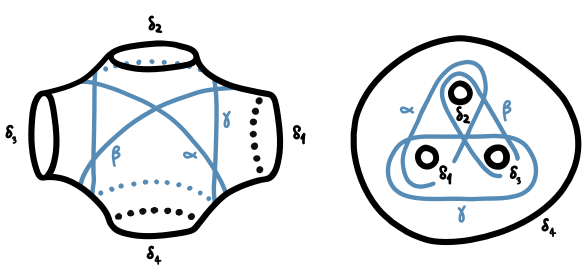

Figure 3.1: Two views of Σ_0^4: on the left-hand side we see the *lantern-like* surface we get by subtracting 4 disjoint open disks from $\mathbb{S}^2$, and on the right-hand side we see the disk with three open disks subtracted from its interior.

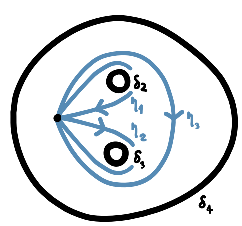

Figure 3.2: The curves $\eta_1, \eta_2, \eta_3 \subseteq \Sigma_0^3$ from the proof of the lantern relation.

We may exploit different embeddings $\Sigma_0^4 \hookrightarrow \Sigma$ and their corresponding inclusion homomorphisms $\mathrm{Mod}(\Sigma_0^4) \longrightarrow \mathrm{Mod}(\Sigma)$ to obtain interesting relations between the corresponding Dehn twists in $\mathrm{Mod}(\Sigma)$. For example, the lantern relation can be used to compute $\mathrm{Mod}(\Sigma_g^b)^{\mathrm{ab}}$ for $g \geqslant 3$.

Proposition 3.2. *The Abelianization* $\mathrm{Mod}(\Sigma_g^b)^{\mathrm{ab}} = \mathrm{Mod}(\Sigma_g^b)/[\mathrm{Mod}(\Sigma_g), \mathrm{Mod}(\Sigma_g)]$ *is cyclic. Moreover, if* $g \geqslant 3$ *then* $\mathrm{Mod}(\Sigma_g^b)^{\mathrm{ab}} = 0$. *In other words,* $\mathrm{Mod}(\Sigma_g)$ *is a perfect group for* $g \geqslant 3$.

Proof. By Theorem 2.22, $\mathrm{Mod}(\Sigma_g^b)^{\mathrm{ab}}$ is generated by the image of the Lickorish generators, which are all conjugate and thus represent the same class in the Abelianization. In fact, any nonseparating $\alpha \subseteq \Sigma_g^b$ is conjugate to the Lickorish generators too, so $\mathrm{Mod}(\Sigma_g^b)^{\mathrm{ab}} = \langle [\alpha] \rangle$.

Now for $g \geqslant 3$ we can embed Σ_0^4 in Σ_g^b in such a way that all the corresponding curves $\alpha, \beta, \gamma, \delta_1, \ldots, \delta_4 \subseteq \Sigma_g^b$ are nonseparating, as shown in Figure 3.3. The lantern relation (3.1) then becomes

$$3 \cdot [\tau_\alpha] = [\tau_\alpha] + [\tau_\beta] + [\tau_\gamma] = [\tau_{\delta_1}] + [\tau_{\delta_2}] + [\tau_{\delta_3}] + [\tau_{\delta_4}] = 4 \cdot [\tau_\alpha]$$

in $\mathrm{Mod}(\Sigma_g^b)^{\mathrm{ab}}$. In other words, $[\tau_\alpha] = 0$ and thus $\mathrm{Mod}(\Sigma_g^b)^{\mathrm{ab}} = 0$. ∎

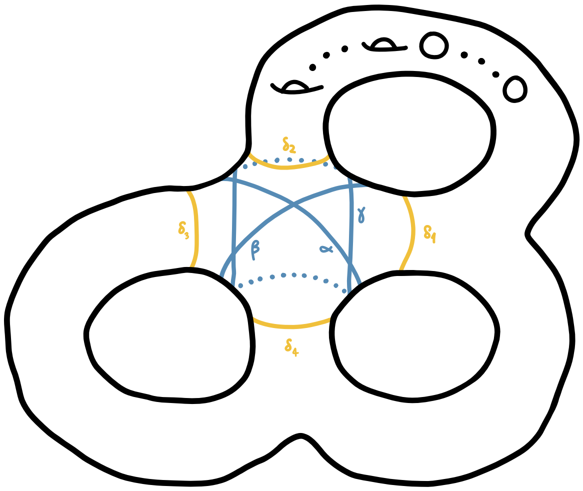

Figure 3.3: The embedding of Σ_0^4 in Σ_g^b for $g \geqslant 3$.

To get extra relations we need to investigate certain branched covers $\Sigma \longrightarrow \mathbb{D}^2 \smallsetminus \{x_1, \ldots, x_r\}$, as well as the relationship between $\mathrm{Mod}(\Sigma)$ and $\mathrm{Mod}(\mathbb{D}^2 \smallsetminus \{x_1, \ldots, x_r\})$. This is what is known as *the Birman-Hilden theorem.*

3.1 The Birman-Hilden Theorem

Let $\Sigma_{0,r}^1 = \mathbb{D}^2 \smallsetminus \{x_1, \ldots, x_r\}$ be the surface of genus 0 with r punctures and one boundary component. We begin our investigation by providing an alternative description of its mapping class group. Namely, we show that $\mathrm{Mod}(\Sigma_{0,r}^1)$ is the braid group on r strands.

Definition 3.3. The *braid group on n strands B_n* is the fundamental group $\pi_1(C(\mathbb{D}^2, n), *)$ of the configuration space $C(\mathbb{D}^2, n) = {C^{\mathrm{ord}}(\mathbb{D}^2, n)}/{S_n}$ of n points in the interior of the disk. The elements of B_n are referred to as *braids.*

Example 3.4. Given $i = 1, \ldots, n-1$, we define $\sigma_i \in B_n$ as in Figure 3.4.

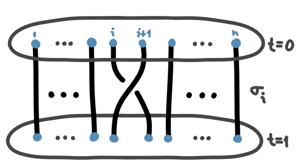

Figure 3.4: The braid σ_i.

The third Reidemeister move translates to the so-called *braid relations*

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_1 \sigma_i$$

in B_n, which motivates the name used in Observation 2.13. In his seminal paper on braid groups, Artin [11] gave the following finite presentation of B_n.

Theorem 3.5 (Artin).

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} : \begin{array}{cl} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for all } i, \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } j \neq i+1 \text{ and } j \neq i-1 \end{array} \right\rangle .$$

As promised, we now show that B_n coincides with $\mathrm{Mod}(\Sigma^1_{0,n})$. Recall from Theorem 2.15 that there is an exact sequence

$$1 \longrightarrow B_n \xrightarrow{\text{push}} \mathrm{Mod}(\Sigma^1_{0,n}) \longrightarrow \cancelto{1}{\mathrm{Mod}(\mathbb{D}^2)} \longrightarrow 1,$$

for $\mathrm{Homeo}^+(\mathbb{D}^2, \mathbb{S}^1)$ is contractible by Observation 1.14. We thus obtain the following result.

Proposition 3.6. *The map* push : $B_n \longrightarrow \mathrm{Mod}(\Sigma^1_{0,n})$ *is a group isomorphism.*

Observation 3.7. Using the capping exact sequence from Observation 2.6 and the Alexander method, one can check that the center $Z(\mathrm{Mod}(\Sigma^1_{0,n}))$ of $\mathrm{Mod}(\Sigma^1_{0,n})$ is freely generated by the Dehn twist τ_δ about the boundary $\delta = \partial\Sigma^1_{0,n}$. It is also not very difficult to see that push : $B_n \longrightarrow \mathrm{Mod}(\Sigma^1_{0,n})$ takes $\sigma_1 \cdots \sigma_{n-1}$ to the rotation by ${}^{2\pi}/_{n}$ as in Figure 3.5, which is an n-th root of τ_δ. Hence the center $Z(B_n)$ is freely generated by $z = (\sigma_1 \cdots \sigma_{n-1})^n$.

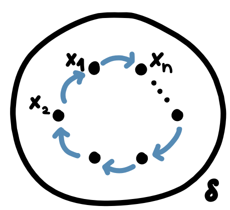

Figure 3.5: The clockwise rotation by ${}^{2\pi}/_{n}$ about an axis centered around the punctures $x_1, \dots, x_n$ of $\Sigma^1_{0,n}$.

To get from $\Sigma^1_{0,n}$ to surfaces of genus $g > 0$ we may consider the *hyperelliptic involution* $\iota : \Sigma_g \xrightarrow{\sim} \Sigma_g$, which rotates Σ_g by π around some axis as in Figure 3.6. Given $\ell < g$ and $b = 1, 2$, we can also embed Σ^b_ℓ in Σ_g in such way that ι restricts to an involution[1] $\Sigma^b_\ell \xrightarrow{\sim} \Sigma^b_\ell$.

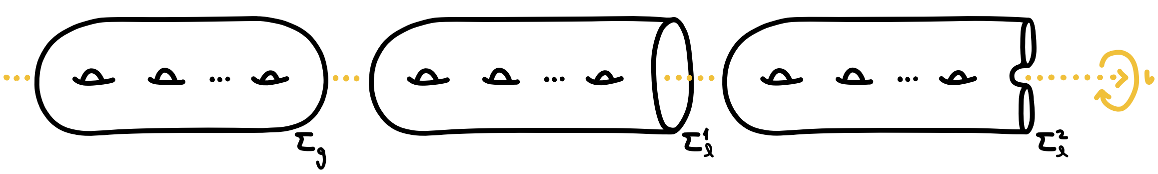

Figure 3.6: The hyperelliptic involution ι.

It is clear from Figure 3.6 that the quotients ${}^{\Sigma^1_\ell}/_{\iota}$ and ${}^{\Sigma^2_\ell}/_{\iota}$ are both disks, with boundary corresponding to the projection of the boundaries of Σ^1_ℓ and Σ^2_ℓ, respectively. Given $b = 1, 2$, the quotient map $\Sigma^b_\ell \longrightarrow {}^{\Sigma^b_\ell}/_{\iota} \cong \mathbb{D}^2$ is a double cover with $2\ell + b$ branch points corresponding to the fixed points of ι. We may thus regard ${}^{\Sigma^b_\ell}/_{\iota}$ as the disk $\Sigma^1_{0,2\ell+b}$ with $2\ell + b$ punctures in its interior, as shown in Figure 3.7. We also draw the curves $\alpha_1, \dots, \alpha_{2\ell} \subseteq \Sigma^b_\ell$ of the Humphreys generators of $\mathrm{Mod}(\Sigma_g)$. Since these curves are invariant under the action of ι, they descend to arcs $\bar{\alpha}_1, \dots, \bar{\alpha}_{2\ell+b} \subseteq \Sigma^1_{0,2\ell+b}$ joining the punctures of the quotient surface.

[1] This involution does not fix $\partial\Sigma^b_\ell$ point-wise.

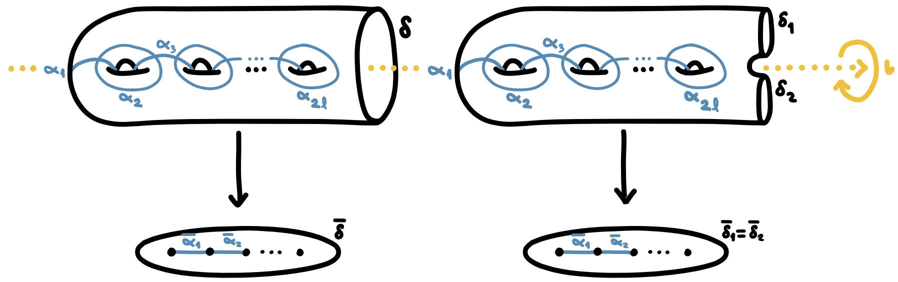

Figure 3.7: The double branched covers given by ι.

Observation 3.8. The map push : $B_{2\ell+b} \longrightarrow \mathrm{Mod}(\Sigma^1_{0,2\ell+b})$ takes σ_i to the half-twist $h_{\bar{\alpha}_i}$ about the arc $\bar{\alpha}_i \subseteq \Sigma^1_{0,2\ell+b}$.

We now study the homeomorphisms of Σ^1_ℓ and Σ^2_ℓ that descend to the quotient surfaces and their mapping classes, known as *the symmetric mapping classes.*

Definition 3.9. Let $\ell \geqslant 0$ and $b = 1, 2$. The *group of symmetric homeomorphisms of* Σ^b_ℓ is $\mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell) = \{\phi \in \mathrm{Homeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell) : [\phi, \iota] = 1\}$. The *symmetric mapping class group of* Σ^b_ℓ is the subgroup $\mathrm{SMod}(\Sigma^1_\ell) = \{[\phi] \in \mathrm{Mod}(\Sigma^b_\ell) : \phi \in \mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell)\}$.

Fix $b = 1$ or 2. It follows from the universal property of quotients that any $\phi \in \mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell)$ defines a homeomorphism $\bar{\phi} : \Sigma^1_{0,2\ell+b} \xrightarrow{\sim} \Sigma^1_{0,2\ell+b}$. This yields a homomorphism of topological groups

$$\begin{aligned} \mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell) &\longrightarrow \mathrm{Homeo}^+(\Sigma^1_{0,2\ell+b}, \partial\Sigma^1_{0,2\ell+b}) \\ \phi &\longmapsto \bar{\phi}, \end{aligned}$$

which is surjective because any $\psi \in \mathrm{Homeo}^+(\Sigma^1_{0,2\ell+b}, \partial\Sigma^1_{0,2\ell+b})$ lifts to Σ^b_ℓ.

It is also not difficult to see $\mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell) \longrightarrow \mathrm{Homeo}^+(\Sigma^1_{0,2\ell+b}, \partial\Sigma^1_{0,2\ell+b})$ is injective: the only candidates for elements of its kernel are 1 and ι, but ι is not an element of $\mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell)$ since it does not fix $\partial\Sigma^b_\ell$ point-wise. Now since we have a continuous bijective homomorphism we find

$$\begin{aligned} \pi_0(\mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell)) &\cong \pi_0(\mathrm{Homeo}^+(\Sigma^1_{0,2\ell+b}, \partial\Sigma^1_{0,2\ell+b})) \\ &= \mathrm{Homeo}^+(\Sigma^1_{0,2\ell+b}, \partial\Sigma^1_{0,2\ell+b})\big/ \simeq \\ &= \mathrm{Mod}(\Sigma^1_{0,2\ell+b}) \\ &\cong B_{2\ell+b}. \end{aligned}$$

We would like to say $\pi_0(\mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell)) = \mathrm{SMod}(\Sigma^b_\ell)$, but a priori the story looks a little more complicated: $\phi, \psi \in \mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell)$ define the same class in $\mathrm{SMod}(\Sigma^b_\ell)$ if they are isotopic, but they may not lie in same connected component of $\mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell)$ if they are not isotopic *through symmetric homeomorphisms.* Birman-Hilden [12] showed that this is never the case.

Theorem 3.10 (Birman-Hilden)**.** *If $\phi, \psi \in \mathrm{SHomeo}^+(\Sigma^b_\ell, \partial\Sigma^b_\ell)$ are isotopic then ϕ and ψ are isotopic through symmetric homeomorphisms. In particular, there is an isomorphism*

$$\begin{aligned} \mathrm{SMod}(\Sigma^b_\ell) &\xrightarrow{\sim} \mathrm{Mod}(\Sigma_{0,2\ell+b}) \\ [\phi] &\longmapsto [\bar{\phi}]. \end{aligned}$$

Observation 3.11. Using the notation of Figure 3.7, the Birman-Hilden isomorphism $\mathrm{SMod}(\Sigma^b_\ell) \xrightarrow{\sim} \mathrm{Mod}(\Sigma_{0,2g+b})$ takes τ_{α_i} to the half twist $h_{\bar{\alpha}_i} \in \mathrm{Mod}(\Sigma_{0,2g+b})$. This can be checked by looking at ι-invariant annular neighborhoods of the curves α_i – [1, Section 9.4].

Fundamental Observation 3.12 (k-chain relations). The Birman-Hilden isomorphism $\mathrm{SMod}(\Sigma_\ell^1) \xrightarrow{\sim} \mathrm{Mod}(\Sigma_{0,2\ell+1}^1)$ takes the twists $\tau_\delta \in \mathrm{SMod}(\Sigma_\ell^1)$ about the boundary $\delta = \partial\Sigma_\ell^1$ to $\tau_{\bar{\delta}}^2 \in \mathrm{Mod}(\Sigma_{0,2\ell+1}^1)$. Similarly, $\mathrm{SMod}(\Sigma_\ell^2) \xrightarrow{\sim} \mathrm{Mod}(\Sigma_{0,2\ell+2})$ takes $\tau_{\delta_1}\tau_{\delta_2} \in \mathrm{SMod}(\Sigma_\ell^2)$ to $\tau_{\bar{\delta}_1} = \tau_{\bar{\delta}_2}$. In light of Observation 3.8, Observation 3.7 translates into the so-called *k-chain relations* in $\mathrm{SMod}(\Sigma_\ell^b) \subseteq \mathrm{Mod}(\Sigma_g)$.

$$(\sigma_1 \cdots \sigma_k)^{2k+2} = z^2 \in B_{k+1} \rightsquigarrow (\tau_{\alpha_1} \cdots \tau_{\alpha_k})^{2k+2} = \tau_\delta \quad \text{for } k = 2\ell \text{ even}$$
$$(\sigma_1 \cdots \sigma_k)^{k+1} = z \in B_{k+1} \rightsquigarrow (\tau_{\alpha_1} \cdots \tau_{\alpha_k})^{k+1} = \tau_{\delta_1}\tau_{\delta_2} \quad \text{for } k = 2\ell+1 \text{ odd}$$

We may also exploit the quotient ${}^{\Sigma_g}/_{\iota} \cong \mathbb{S}^2$ to obtain other relations. Since ι has $2g+2$ fixed points in Σ_g, we get branched double cover $\Sigma_g \longrightarrow \Sigma_{0,2g+2}$.

Theorem 3.13 (Birman-Hilden without boundary). *If $g \geqslant 2$ then we have an exact sequence*

$$1 \longrightarrow \langle[\iota]\rangle \longrightarrow C_{\mathrm{Mod}(\Sigma_g)}([\iota]) \longrightarrow \mathrm{Mod}(\Sigma_{0,2g+2}) \longrightarrow 1,$$

where $C_{\mathrm{Mod}(\Sigma_g)}([\iota]) \subseteq \mathrm{Mod}(\Sigma_g)$ is the commutator subgroup of $[\iota]$ and the right map takes $[\phi] \in C_{\mathrm{Mod}(\Sigma_g)}([\iota])$ to $[\bar{\phi}] \in \mathrm{Mod}(\Sigma_{0,2g+2})$.

Fundamental Observation 3.14 (Hyperelliptic relations). Let $\alpha_1, \ldots, \alpha_{2g}, \delta \subseteq \Sigma_g$ be as in Figure 3.8. Then

$$[\iota] = \tau_\delta\tau_{\alpha_{2g}} \cdots \tau_{\alpha_1}\tau_{\alpha_1} \cdots \tau_{\alpha_{2g}}\tau_\delta. \tag{3.2}$$

Indeed, $C_{\mathrm{Mod}(\Sigma_g)}([\iota]) \longrightarrow \mathrm{Mod}(\Sigma_{0,2g+2})$ takes $\tau_\delta\tau_{\alpha_{2g}} \cdots \tau_{\alpha_1}$ to the rotation from Figure 3.9, while $\tau_{\alpha_1} \cdots \tau_{\alpha_{2g}}\tau_\delta$ is taken to its inverse. By Theorem 3.13,

$$\tau_\delta\tau_{\alpha_{2g}} \cdots \tau_{\alpha_1}\tau_{\alpha_1} \cdots \tau_{\alpha_{2g}}\tau_\delta \in \ker(C_{\mathrm{Mod}(\Sigma_g)}([\iota]) \longrightarrow \mathrm{Mod}(\Sigma_{0,2g+2})) = \langle[\iota]\rangle \cong \mathbb{Z}/2.$$

One can then show $\tau_\delta\tau_{\alpha_{2g}} \cdots \tau_{\alpha_1}\tau_{\alpha_1} \cdots \tau_{\alpha_{2g}}\tau_\delta$ inverts the orientation of α_1, so $\tau_\delta\tau_{\alpha_{2g}} \cdots \tau_{\alpha_1}\tau_{\alpha_1} \cdots \tau_{\alpha_{2g}}\tau_\delta \neq 1$ and (3.2) follows. In particular, we obtain the so-called *hyperelliptic relations* (3.3) and (3.4) in $\mathrm{Mod}(\Sigma_g)$.

$$(\tau_\delta\tau_{\alpha_{2g}} \cdots \tau_{\alpha_1}\tau_{\alpha_1} \cdots \tau_{\alpha_{2g}}\tau_\delta)^2 = 1 \tag{3.3}$$
$$[\tau_\delta\tau_{\alpha_{2g}} \cdots \tau_{\alpha_1}\tau_{\alpha_1} \cdots \tau_{\alpha_{2g}}\tau_\delta, \tau_\delta] = 1 \tag{3.4}$$

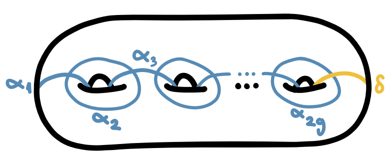

Figure 3.8: The curves from the Humphreys generators of $\mathrm{Mod}(\Sigma_g)$ and the curve δ from the hyperelliptic relations.

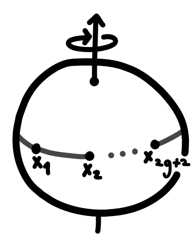

Figure 3.9: The clockwise rotation by ${}^{\pi}/_{g+1}$ about an axis centered around the punctures of $\Sigma_{0,2g+1}$.

3.2 Presentations of Mapping Class Groups

Having explored some of the relations in $\mathrm{Mod}(\Sigma)$, it is natural to ask if these relations are enough to completely describe the structure of $\mathrm{Mod}(\Sigma)$. Different presentations of mapping class groups are due to the work of Birman-Hilden [12], Gervais [13] and many others. Wajnryb [14] derived a presentation of $\mathrm{Mod}(\Sigma_g)$ only using the relations discussed in Chapter 2 and Section 3.1. This is quite a satisfactory result, for we have seen that all of these relations can be explained in terms of the topology of Σ_g.

Theorem 3.15 (Wajnryb). *Suppose $g \geqslant 3$. If $\alpha_0, \ldots, \alpha_g$ are as in Figure 2.8 and $a_i = \tau_{\alpha_i} \in \mathrm{Mod}(\Sigma_g)$ are the Humphreys generators, then there is a presentation of $\mathrm{Mod}(\Sigma_g)$ with generators $a_0, \ldots a_{2g}$ subject to the following relations.*

(i) *The* disjointness relations $[a_i, a_j] = 1$ *for α_i and α_j disjoint.*

(ii) *The* braid relations $a_i a_j a_i = a_j a_i a_j$ *for α_i and α_j crossing once.*

(iii) *The* 3-chain relation $(a_1 a_2 a_3)^4 = a_0 b_0$, *where*

$$b_0 = (a_4 a_3 a_2 a_1 a_1 a_2 a_3 a_4) a_0 (a_4 a_3 a_2 a_1 a_1 a_2 a_3 a_4)^{-1}.$$

(iv) *The* lantern relation $a_0 b_2 b_1 = a_1 a_3 a_5 b_3$, *where*

$$\begin{aligned} b_1 &= (a_4 a_5 a_3 a_4)^{-1} a_0 (a_4 a_5 a_3 a_4) \\ b_2 &= (a_2 a_3 a_1 a_2)^{-1} b_1 (a_2 a_3 a_1 a_2) \\ b_2 &= u b_1 u^{-1} \\ u &= (a_6 a_5)(a_4 a_3 a_2)(a_6 a_5)^{-1} b_1 (a_6 a_5) a_1^{-1} (a_4 a_3 a_2)^{-1}. \end{aligned}$$

(v) *The* hyperelliptic relation $[a_{2g} \cdots a_1 a_1 \cdots a_{2g}, d] = 1$, *where $d = n_g$ for $n_1 = a_1$, $n_2 = b_0$ and*

$$\begin{aligned} n_{i+2} &= w_i n_i w_i^{-1} \\ w_i &= (a_{2i+4} a_{2i+3} a_{2i+2} n_{i+1})(a_{2i+1} a_{2i}^2 a_{2i+1})(a_{2i+3} a_{2i+2} a_{2i+4} a_{2i+3})(n_1 a_{2i+2} a_{2i+1} a_{2i}). \end{aligned}$$

Remark. The mapping classes $b_0, \ldots, b_3, d$ in the statement of Theorem 3.15 correspond to the Dehn twists about the curves $\beta_0, \ldots, \beta_3, \delta \subseteq \Sigma_g$ highlighted in Figure 3.10, so Wajnryb's presentation is not as intractable as it might look at first glance.

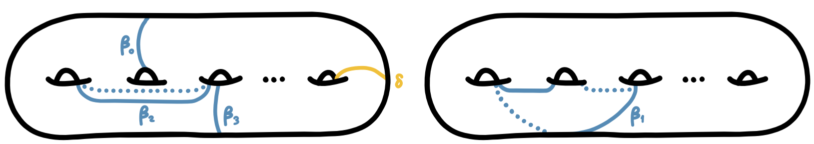

Figure 3.10: The curves from Wajnryb's presentation.

Different presentations can be used to compute the Abelianization of $\mathrm{Mod}(\Sigma_g)$ for $g \leqslant 2$. Indeed, if $G = \langle g_1, \ldots, g_n : R \rangle$ is a finitely-presented group, then $G^{\mathrm{ab}} = \langle g_1, \ldots, g_n : R, [g_i, g_j] \text{ for all } i, j \rangle$. Using this approach, Farb-Margalit [1, Section 5.1.3] show the Abelianization is given by

g	Σ_g	$\mathrm{Mod}(\Sigma_g)^{\mathrm{ab}}$
0	$\mathbb{S}^2$	0
1	$\mathbb{T}^2$	$\mathbb{Z}/12$
2	Σ_2	$\mathbb{Z}/10$

for closed surfaces of small genus. In [15] Korkmaz-McCarthy showed that even though $\mathrm{Mod}(\Sigma_2^b)$ is not perfect, its commutator subgroup is. In addition, they also show $[\mathrm{Mod}(\Sigma_g^b), \mathrm{Mod}(\Sigma_g^b)]$ is normally generated by a single mapping class.

Proposition 3.16. *The commutator subgroup $\mathrm{Mod}(\Sigma_2^b)' = [\mathrm{Mod}(\Sigma_2^b), \mathrm{Mod}(\Sigma_2^b)]$ is perfect – i.e. $\mathrm{Mod}(\Sigma_2^b)^{(2)} = [\mathrm{Mod}(\Sigma_2^b)', \mathrm{Mod}(\Sigma_2^b)']$ is the whole of $\mathrm{Mod}(\Sigma_2^b)'$.*

Proposition 3.17. *If* $g \geqslant 2$ *and* $\alpha, \beta \subseteq \Sigma_g$ *are simple closed crossing only once, then* $\mathrm{Mod}(\Sigma_g)'$ *is* normally generated *by* $\tau_\alpha \tau_\beta^{-1}$ *– i.e. if* $\tau_\alpha \tau_\beta^{-1} \in N \triangleleft \mathrm{Mod}(\Sigma_g)'$ *then* $\mathrm{Mod}(\Sigma_g)' \subseteq N$.

The different presentations of $\mathrm{Mod}(\Sigma_g)$ may also be used to study its representations. Indeed, in light of Theorem 3.15, a representation $\rho : \mathrm{Mod}(\Sigma_g) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ is nothing other than a choice of $2g+1$ matrices $\rho(\tau_{\alpha_0}), \ldots, \rho(\tau_{\alpha_{2g}}) \in \mathrm{GL}_n(\mathbb{C})$ satisfying the relations **(i)** to **(v)** as above. In the next chapter, we will discuss how these relations may be used to derive obstructions to the existence of nontrivial representations of certain dimensions.

Chapter 4

Low-Dimensional Representations

Having built a solid understanding of the combinatorics of Dehn twists, we are now ready to attack the problem of classifying the representations of $\mathrm{Mod}(\Sigma_g)$ of sufficiently small dimension. As promised, our strategy is to make use of the *geometrically-motivated* relations derived in Chapter 2 and Chapter 3.

Historically, these relations have been exploited by Funar [16], Franks-Handel [17] and others to establish the triviality of low-dimensional representations, culminating in Korkmaz' [2] recent classification of representations of dimension $n \leqslant 2g$ for $g \geqslant 3$. The goal of this chapter is to provide a concise account of Korkmaz' results.

Theorem 4.1 (Korkmaz). *Let Σ_g^b be the compact surface of genus $g \geqslant 1$ with b boundary components and $\rho : \mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be a linear representation with $n < 2g$. Then the image of ρ is Abelian. In particular, if $g \geqslant 3$ then ρ is trivial.*

Like some of the results we have encountered so far, the proof of Theorem 4.1 is elementary in nature: we proceed by induction on g and tedious case analysis. We begin by the base case $g = 2$.

Proposition 4.2. *Given $\rho : \mathrm{Mod}(\Sigma_2^b) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ with $n \leqslant 3$, the image of ρ is Abelian.*

Sketch of proof. Given $\alpha \subseteq \Sigma_2^b$, let $L_\alpha = \rho(\tau_\alpha)$ and denote by $E_{\alpha=\lambda} = \{v \in \mathbb{C}^n : L_\alpha v = \lambda v\}$ its eigenspaces. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \eta_1, \ldots, \eta_{b-1} \subseteq \Sigma_2^b$ be the curves of the Lickorish generators from Theorem 2.22, as shown in Figure 4.1.

If $n = 1$ then $\rho(\mathrm{Mod}(\Sigma_2^b)) \subseteq \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ is Abelian. Now if $n = 2$ or 3, by Proposition 3.17 it suffices to show $L_{\alpha_1} = L_{\beta_1}$, so that $\tau_{\alpha_1}\tau_{\beta_1}^{-1} \in \ker\rho$ and thus $\mathrm{Mod}(\Sigma_2^b)' \subseteq \ker\rho$ – i.e. $\rho(\mathrm{Mod}(\Sigma_2^b))$ is Abelian. Given the braid relation

$$L_{\alpha_1} L_{\beta_1} L_{\alpha_1} = L_{\beta_1} L_{\alpha_1} L_{\beta_1}, \tag{4.1}$$

this amounts to showing L_{α_1} and L_{β_1} commute.

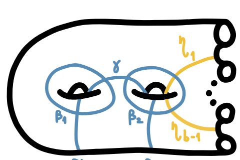

Figure 4.1: The Lickorish generators for $g = 2$.

To that end, we exhaustively analyze all of the possible Jordan forms

$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} \quad (1) \qquad \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} \quad (2) \qquad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} \quad (3)$$

$$\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix} \quad (4) \qquad \begin{pmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \nu\end{pmatrix} \quad (5) \qquad \begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix} \quad (6)$$

$$\begin{pmatrix}\lambda & 0 & 0\\ 0 & \mu & 1\\ 0 & 0 & \mu\end{pmatrix} \quad (7) \qquad \begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix} \quad (8) \qquad \begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{pmatrix} \quad (9)$$

of L_{α_2} – where $\lambda, \mu, \nu \in \mathbb{C}^\times$ are all distinct. By changing basis we may assume without loss of generality that the matrix L_{α_2} is exactly its Jordan form, so that $E_{\alpha_2=\lambda} = \bigoplus_{i \leqslant \dim E_{\alpha_2}} \mathbb{C}e_i$.

For cases (1) to (6), we use the change of coordinates principle and different relations to show L_{α_1} and L_{β_1} lie inside some Abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$.

(1) & (4) By the change of coordinates principle, both L_{α_1} and L_{β_1} are conjugate to $L_{\alpha_2} = \lambda$. But the only matrix conjugate to λ is λ itself. Hence $L_{\alpha_1} = L_{\beta_1} = \lambda \in \mathbb{C}^\times$.

(2) & (5) Since α_2 is disjoint from both α_1 and β_1, it follows from the disjointness relations $[\tau_{\alpha_1}, \tau_{\alpha_2}] = [\tau_{\beta_1}, \tau_{\alpha_2}] = 1$ that L_{α_1} and L_{β_1} preserve the eigenspaces of L_{α_2}, which are all 1-dimensional. Hence L_{α_1} and L_{β_1} lie inside the subgroup of diagonal matrices – an Abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$.

(3) & (6) As before, it follows from the disjointness relations that $E_{\alpha_2=\lambda} = \ker(L_{\alpha_2} - \lambda)$ and $\ker(L_{\alpha_2} - \lambda)^2$ are invariant under both L_{α_1} and L_{β_1}. This implies L_{α_1} and L_{β_1} are upper triangular matrices with λ along their diagonals. Any such pair of matrices satisfying the braid relation (4.1) commute.

Similarly, in case (7) we use the braid relation and the disjointness relations to show L_{α_1} and L_{β_1} commute – see [2, Proposition 5.1] for a full proof. Cases (8) and (9) require some extra thought. We consider the curve β_2. In these cases, the eigenspace $E_{\alpha_2=\lambda}$ is 2-dimensional. Since L_{α_2} and L_{β_2} are conjugate, $E_{\beta_2=\lambda}$ is also 2-dimensional – indeed, conjugate operators have the same Jordan form. Now either $E_{\alpha_2=\lambda} = E_{\beta_2=\lambda}$ or $E_{\alpha_2=\lambda} \neq E_{\beta_2=\lambda}$. We begin by the first case.

We claim that if $E_{\alpha_2=\lambda} = E_{\beta_2=\lambda}$ then $E_{\alpha_2=\lambda}$ is $\mathrm{Mod}(\Sigma_2^b)$-invariant. Indeed, by Observation 1.3 we can always find $f, g, h_i \in \mathrm{Mod}(\Sigma_2^b)$ with

$$\begin{aligned} f \cdot [\alpha_2] &= [\alpha_1] & g \cdot [\alpha_2] &= [\beta_1] & h_i \cdot [\alpha_2] &= [\alpha_2] \\ f \cdot [\beta_2] &= [\beta_1] & g \cdot [\beta_2] &= [\gamma] & h_i \cdot [\beta_2] &= [\eta_i]. \end{aligned}$$

In particular,

$$\begin{aligned} f\tau_{\alpha_2}f^{-1} &= \tau_{\alpha_1} & g\tau_{\alpha_2}g^{-1} &= \tau_{\beta_1} & h_i\tau_{\alpha_2}h_i^{-1} &= \tau_{\alpha_2} \\ f\tau_{\beta_2}f^{-1} &= \tau_{\beta_1} & g\tau_{\beta_2}g^{-1} &= \tau_{\gamma} & h_i\tau_{\beta_2}h_i^{-1} &= \tau_{\eta_i}. \end{aligned}$$

and thus

$$\begin{aligned} E_{\alpha_1=\lambda} &= \rho(f)E_{\alpha_2=\lambda} = \rho(f)E_{\beta_2=\lambda} = E_{\beta_1=\lambda} \\ E_{\beta_1=\lambda} &= \rho(g)E_{\alpha_2=\lambda} = \rho(g)E_{\beta_2=\lambda} = E_{\gamma=\lambda} \\ E_{\eta_i=\lambda} &= \rho(h_i)E_{\alpha_2=\lambda} = \rho(h_i)E_{\beta_2=\lambda} = E_{\beta_2=\lambda}. \end{aligned}$$

In other words, $E_{\alpha_1=\lambda} = E_{\alpha_2=\lambda} = E_{\beta_1=\lambda} = E_{\beta_2=\lambda} = E_{\gamma=\lambda} = E_{\eta_1=\lambda} = \cdots = E_{\eta_{b-1}=\lambda}$ is invariant under the action of all Lickorish generators.

Hence ρ restricts to a subrepresentation $\bar{\rho} : \mathrm{Mod}(\Sigma_2^b) \longrightarrow \mathrm{GL}(E_{\alpha_2=\lambda}) = \mathrm{GL}_2(\mathbb{C})$ – recall $E_{\alpha_2=\lambda} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. By case (2), $\bar{\rho}(f) = 1$ for all $f \in \mathrm{Mod}(\Sigma_2^b)'$, given that $\bar{\rho}(\mathrm{Mod}(\Sigma_2^b))$ is Abelian. Thus

$$\rho(\mathrm{Mod}(\Sigma_2^b)') \subseteq \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix}$$

lies inside the group of upper triangular matrices, a solvable subgroup of $\mathrm{GL}_3(\mathbb{C})$. Now by Proposition 3.16 we get $\rho(\mathrm{Mod}(\Sigma_2^b)') = 1$: any homomorphism from a perfect group to a solvable group is trivial.

Finally, if $E_{\alpha_2=\lambda} \neq E_{\beta_2=\lambda}$ and the Jordan form of L_{α_2} is given by (8) then the disjointness relations $[\tau_{\alpha_2}, \tau_{\alpha_1}] = [\tau_{\alpha_2}, \tau_{\beta_1}] = [\tau_{\beta_2}, \tau_{\alpha_1}] = [\tau_{\beta_2}, \tau_{\beta_1}] = 1$ implies that L_{α_1} and L_{β_1} preserve the eigenspaces of both L_{α_2} and L_{β_2}, so

$$0 \subsetneq E_{\alpha_2=\lambda} \cap E_{\beta_2=\lambda} \subsetneq E_{\alpha_2=\lambda} \subsetneq V$$

is a flag of subspaces invariant under L_{α_1} and L_{β_1}. In this case we can find a basis for $\mathbb{C}^3$ with respect to which the matrices of L_{α_1} and L_{β_1} are both upper triangular with λ along the diagonal: take $v_1, v_2, v_3 \in \mathbb{C}^3$ with $v_1 \in E_{\alpha_2=\lambda} \cap E_{\beta_2=\lambda}$ and $v_2 \in V_{L_{\alpha_2}}$. Any such pair of matrices satisfying the braid relation (4.1) commute.

Similarly, if L_{α_2} has Jordan form (9) and $E_{\alpha_2=\lambda} \neq E_{\beta_2=\lambda}$ we use (4.1) to conclude L_{α_1} and L_{β_1} commute – again, see [2, Proposition 5.1]. We are done. ∎

We are now ready to establish the triviality of low-dimensional representations.

Proof of Theorem 4.1. Let $g \geqslant 1$, $b \geqslant 0$, and fix $\rho : \mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ with $n < 2g$. We want to show $\rho(\mathrm{Mod}(\Sigma_g^b))$ is Abelian. As promised, we proceed by induction on g. The base case $g = 1$ is again clear from the fact $n = 1$ and $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. The case $g = 2$ was also established in Proposition 4.2.

Now suppose $g \geqslant 3$ and every m-dimensional representation of $\Sigma_{g-1}^{b'}$ has Abelian image for $m < 2(g-1)$. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g-1}, \eta_1, \ldots, \eta_{b-1} \subseteq \Sigma_g^b$ be the curves from the Lickorish generators of $\mathrm{Mod}(\Sigma_g^b)$, as in Figure 2.7. Once again, let $L_\alpha = \rho(\tau_\alpha)$ and denote by $E_{\alpha=\lambda}$ the eigenspace of L_α associated to $\lambda \in \mathbb{C}$. Let $\Sigma \cong \Sigma_{g-1}^1$ be the closed subsurface highlighted in Figure 4.2.

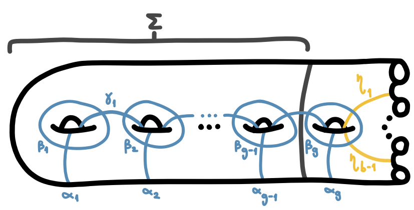

Figure 4.2: The subsurface $\Sigma \subseteq \Sigma_g^b$.

We claim that it suffices to find a m-dimensional $\mathrm{Mod}(\Sigma)$-invariant[1] subspace $W \subseteq \mathbb{C}^n$ with $2 \leqslant m \leqslant n-2$. Indeed, in this case $m < 2(g-1)$ and $\dim \mathbb{C}^n/W = n - m < 2(g-1)$. Thus both representations

$$\rho_1 : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{GL}(W) \cong \mathrm{GL}_m(\mathbb{C}) \qquad \rho_2 : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{GL}(\mathbb{C}^n/W) \cong \mathrm{GL}_{n-m}(\mathbb{C})$$

fall into the induction hypothesis – i.e. $\rho_i(\mathrm{Mod}(\Sigma))$ is Abelian. In particular, $\rho_i(\mathrm{Mod}(\Sigma)') = 1$ and we can find some basis for $\mathbb{C}^n$ with respect to which

$$\rho(f) = \left(\begin{array}{c|c} 1_m & * \\ \hline 0 & 1_{n-m} \end{array}\right)$$

for all $f \in \mathrm{Mod}(\Sigma)'$ – where 1_k denotes the $k \times k$ identity matrix. Since the group of upper triangular matrices is solvable, it follows from Proposition 3.16 that ρ annihilates all of $\mathrm{Mod}(\Sigma)'$ and, in particular,

[1] Here we view $\mathrm{Mod}(\Sigma)$ as a subgroup of $\mathrm{Mod}(\Sigma_g^b)$ via the inclusion homomorphism $\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Mod}(\Sigma_g^b)$ from Example 1.9, which can be shown to be injective in this particular case.

$\tau_{\alpha_1}\tau_{\beta_1}^{-1} \in \ker \rho$. Now recall from Proposition 3.17 that $\mathrm{Mod}(\Sigma_g^b)'$ is normally generated by $\tau_{\alpha_1}\tau_{\beta_1}^{-1}$, from which we conclude $\rho(\mathrm{Mod}(\Sigma_g^b)') = 1$, as desired.

As before, we exhaustively analyze all possible Jordan forms of L_{α_g}. First, consider the case where we can find eigenvalues $\lambda_1, \ldots, \lambda_k$ of L_{α_g} such that the sum $W = \bigoplus_i E_{\alpha_g = \lambda_i}$ of the corresponding eigenspaces has dimension m with $2 \leqslant m \leqslant n-2$. In this case, it suffices to observe that since α_g lies outside of Σ, each $E_{\alpha_g=\lambda_i}$ is $\mathrm{Mod}(\Sigma)$-invariant: the Lickorish generators $\tau_{\alpha_1}, \ldots, \tau_{\alpha_{g-1}}, \tau_{\beta_1}, \ldots, \tau_{\beta_{g-1}}, \tau_{\gamma_1}, \ldots, \tau_{\gamma_{g-2}}$ of $\Sigma \cong \Sigma_{g-1}^1$ all commute with τ_{α_g} and thus preserve the eigenspaces of its action on $\mathbb{C}^n$.

If no sum of the form $\bigoplus_i E_{\alpha_g=\lambda_i}$ has dimension lying between 2 and $n-2$, then there must be at most 2 distinct eigenvalues λ of L_{α_g}, and $\dim E_{\alpha_g=\lambda} = 1, n-1, n$ for all such λ. Hence the Jordan form of L_{α_g} has to be one of

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad (1) \qquad \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad (2)$$

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad (3) \qquad \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu \end{pmatrix} \quad (4)$$

for $\lambda \neq \mu$. We analyze the first two sporadic cases individually.

(1) Here we use the change of coordinates principle: each $L_{\alpha_i}, L_{\beta_i}, L_{\gamma_i}, L_{\eta_i}$ is conjugate to $L_{\alpha_g} = \lambda$, so all Lickorish generators of $\mathrm{Mod}(\Sigma_g^b)$ act on $\mathbb{C}^n$ as scalar multiplication by λ as well. Hence $\rho(\mathrm{Mod}(\Sigma_g^b)) = \langle \lambda \rangle$ is Abelian.

(2) In this case, $W = \ker(L_{\alpha_g} - \lambda)^2$ is a 2-dimensional $\mathrm{Mod}(\Sigma)$-invariant subspace.

For cases (3) and (4), we consider two situations: $E_{\alpha_g=\lambda} \neq E_{\beta_g=\lambda}$ or $E_{\alpha_g=\lambda} = E_{\beta_g=\lambda}$. If $E_{\alpha_2=\lambda} \neq E_{\beta_2=\lambda}$, then $W = E_{\alpha_g=\lambda} \cap E_{\beta_g=\lambda}$ is a $(n-2)$-dimensional $\mathrm{Mod}(\Sigma)$-invariant subspace: since β_g lies outside of Σ and $L_{\alpha_g}, L_{\beta_g}$ are conjugate, both $E_{\alpha_g=\lambda}$ and $E_{\beta_g=\lambda}$ are $\mathrm{Mod}(\Sigma)$-invariant $(n-1)$-dimensional subspaces.

Finally, we consider the case where $E_{\alpha_g=\lambda} = E_{\beta_g=\lambda}$. In this situation, as in the proof of Proposition 4.2, it follows from Observation 1.3 that there are $f_i, g_i, h_i \in \mathrm{Mod}(\Sigma_g^b)$ with

$$f_i \tau_{\alpha_g} f_i^{-1} = \tau_{\alpha_i} \qquad g_i \tau_{\alpha_g} g_i^{-1} = \tau_{\beta_i} \qquad h_i \tau_{\alpha_g} h_i^{-1} = \tau_{\alpha_g}$$
$$f_i \tau_{\beta_g} f_i^{-1} = \tau_{\beta_i} \qquad g_i \tau_{\beta_g} g_i^{-1} = \tau_{\gamma_i} \qquad h_i \tau_{\beta_g} h_i^{-1} = \tau_{\eta_i}$$

and thus $E_{\alpha_1=\lambda} = \cdots = E_{\alpha_g=\lambda} = E_{\beta_1=\lambda} = \cdots = E_{\beta_g=\lambda} = E_{\gamma_1=\lambda} = \cdots = E_{\gamma_{g-1}=\lambda} = E_{\eta_1=\lambda} = \cdots = E_{\eta_{b-1}=\lambda}$.

In particular, we can find a basis for $\mathbb{C}^n$ with respect to which the matrix of any Lickorish generator has the form

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 & * \\ 0 & \lambda & \ldots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & * \\ 0 & 0 & \cdots & 0 & * \end{pmatrix}.$$

Since the group of upper triangular matrices is solvable and $\mathrm{Mod}(\Sigma_g^b)$ is perfect, it follows that $\rho(\mathrm{Mod}(\Sigma_g^b))$ is trivial. This concludes the proof $\rho(\mathrm{Mod}(\Sigma_g^b))$ is Abelian.

To see that $\rho(\mathrm{Mod}(\Sigma_g^b)) = 1$ for $g \geqslant 3$ we note that, since ρ has Abelian image and thus factors though the Abelianization map $\mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{Mod}(\Sigma_g^b)^{\mathrm{ab}} = \mathrm{Mod}(\Sigma_g^b)/[\mathrm{Mod}(\Sigma_g^b), \mathrm{Mod}(\Sigma_g^b)]$. Now recall from Proposition 3.2 that $\mathrm{Mod}(\Sigma_g^b)^{\mathrm{ab}} = 0$ for $g \geqslant 3$. We are done. ∎

Having established the triviality of the low-dimensional representations $\rho : \mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{GL}_n(\mathbb{C})$, all that remains for us is to understand the $2g$-dimensional representations of $\mathrm{Mod}(\Sigma_g^b)$. We certainly know a nontrivial example of such, namely the symplectic representation $\psi : \mathrm{Mod}(\Sigma_g) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$ from Example 1.13. Surprisingly, this turns out to be *essentially* the only example of a nontrivial $2g$-dimensional representation in the compact case. More precisely,

Theorem 4.3 (Korkmaz). *Let $g \geqslant 3$ and $\rho : \mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{GL}_{2g}(\mathbb{C})$. Then ρ is either trivial or conjugate to the symplectic representation*[2] $\mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$ *of* $\mathrm{Mod}(\Sigma_g^b)$.

Unfortunately, the limited scope of these master's thesis does not allow us to dive into the proof of Theorem 4.3. The heart of this proof lies in a result about representations of the product $B_3^n = B_3 \times \cdots \times B_3$, which Korkmaz refers to as *the main lemma.*

Lemma 4.4 (Korkmaz' main lemma). *Given $i = 1, \ldots, n$, denote by*

$$a_i = (1, \ldots, 1, \sigma_1, 1, \ldots, 1) \qquad\qquad b_i = (1, \ldots, 1, \sigma_2, 1, \ldots, 1)$$

the n-tuples in B_3^n whose i-th coordinates are σ_1 and σ_2, respectively, and with all other coordinates equal to 1. Let $m \geqslant 2n$ and $\rho : B_3^n \longrightarrow \mathrm{GL}_m(\mathbb{C})$ be a representation satisfying:

(i) *The only eigenvalue of $\rho(a_i)$ is 1 and its eigenspace is $(m-1)$-dimensional.*

(ii) *The eigenspaces of $\rho(a_i)$ and $\rho(b_i)$ associated to the eigenvalue 1 do not coincide.*

Then ρ is conjugate to the representation

$$\begin{aligned} B_3^n &\longrightarrow \mathrm{GL}_m(\mathbb{C}) \\ a_i &\longmapsto \left(\begin{array}{c|cc|c} 1_{2(i-1)} & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1_{m-2i} \end{array}\right) \\ b_i &\longmapsto \left(\begin{array}{c|cc|c} 1_{2(i-1)} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1_{m-2i} \end{array}\right), \end{aligned}$$

where 1_k denotes the $k \times k$ identity matrix.

This is proved in [2, Lemma 7.6] using the braid relations. Notice that for $n = g$ and $m = 2g$ the matrices in Lemma 4.4 coincide with the action of the Lickorish generators $\tau_{\alpha_1}, \ldots, \tau_{\alpha_g}, \tau_{\beta_1}, \ldots, \tau_{\beta_g} \in \mathrm{Mod}(\Sigma_g^b)$ on $H_1(\Sigma_g, \mathbb{C}) \cong \mathbb{C}^{2g}$ – represented in the standard basis $[\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g]$ for $H_1(\Sigma_g, \mathbb{C})$.

$$(\tau_{\alpha_i})_* = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \qquad\qquad (\tau_{\beta_i})_* = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

[2]Here the map $\mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$ is given by the composition of the inclusion morphism $\mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{Mod}(\Sigma_g)$ with the usual symplectic representation $\psi : \mathrm{Mod}(\Sigma_g) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$.

Hence by embedding B_3^g in $\mathrm{Mod}(\Sigma_g^b)$ via

$$\begin{aligned} B_3^g &\longrightarrow \mathrm{Mod}(\Sigma_g^b) \\ a_i &\longmapsto \tau_{\alpha_i} \\ b_i &\longmapsto \tau_{\beta_i} \end{aligned}$$

we can see that any $\rho : \mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{GL}_{2g}(\mathbb{C})$ in a certain class of representation satisfying some technical conditions must be conjugate to the symplectic representation $\mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$ when restricted to B_3^g.

Korkmaz then goes on to show that such technical conditions are met for any nontrivial $\rho : \mathrm{Mod}(\Sigma_g^b) \longrightarrow \mathrm{GL}_{2g}(\mathbb{C})$. Furthermore, Korkmaz also argues that we can find a basis for $\mathbb{C}^{2g}$ with respect to which the matrices of $\rho(\tau_{\gamma_1}), \ldots, \rho(\tau_{\gamma_{g-1}}), \rho(\tau_{\eta_1}), \ldots, \rho(\tau_{\eta_{b-1}})$ also agree with the action of $\mathrm{Mod}(\Sigma_g^b)$ on $H_1(\Sigma_g, \mathbb{C})$, concluding the classification of $2g$-dimensional representations.

Recently, Kasahara [18] also classified the $(2g+1)$-dimensional representations of $\mathrm{Mod}(\Sigma_g^b)$ for $g \geqslant 7$ in terms of certain twisted 1-cohomology groups. On the other hand, the representations of dimension $n > 2g+1$ are still poorly understood, and fundamental questions remain unsweared. In the short and mid-terms, the works of Korkmaz and Kasahara lead to many follow-up questions. For example,

(i) In the $g \geqslant 3$ case, Korkmaz [2, Theorem 3] established the lower bound of $3g-3$ for the dimension of an injective linear representation of $\mathrm{Mod}(\Sigma_g)$ – if one such representation exists. Can we improve this lower bound?

(ii) What is the minimal dimension for a representation of $\mathrm{Mod}(\Sigma_g)$ which does not annihilate the entire kernel of the symplectic representation $\psi : \mathrm{Mod}(\Sigma_g) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$? In particular, do the $(2g+1)$-dimensional representations classified by Kasahara [18] annihilate all of $\ker \psi$?

These are some of the questions which I plan to work on during my upcoming PhD.

Bibliography

[1] B. Farb and D. Margalit, *A Primer on Mapping Class Groups* (Princeton Mathematical Series), English. Princeton University Press, 2011, vol. 49, ISBN: 978-0-691-14794-9.

[2] M. Korkmaz, "Low-dimensional linear representations of mapping class groups," English, *Journal of Topology*, vol. 16, no. 3, pp. 899–935, 2023, ISSN: 1753-8416. DOI: 10.1112/topo.12305.

[3] M. Korkmaz, "On the linearity of certain mapping class groups," *Turkish Journal of Mathematics*, vol. 24, pp. 367–371, 2000. [Online]. Available: https://journals.tubitak.gov.tr/math/vol24/iss4/5/.

[4] S. Bigelow and R. Budney, "The mapping class group of a genus two surface is linear," *Algebraic & Geometric Topology*, vol. 1, no. 2, pp. 699–708, Nov. 2001, ISSN: 1472-2747. DOI: 10.2140/agt.2001.1.699.

[5] F. Costantino, "Notes on topological quantum field theories," in *Winter Braids Lecture Notes*, vol. 2, Cellule MathDoc/CEDRAM, Feb. 2015, pp. 1–45. DOI: 10.5802/wbln.7.

[6] J. Marché, "Introduction to quantum representations of mapping class groups," English, in *Topology and geometry. A collection of essays dedicated to Vladimir G. Turaev*, European Mathematical Society, 2021, pp. 109–130, ISBN: 978-3-98547-001-3; 978-3-98547-501-8. DOI: 10.4171/IRMA/33-1/7.

[7] E. Witten, "Quantum field theory and the jones polynomial," *Communications in Mathematical Physics*, vol. 121, no. 3, pp. 351–399, Sep. 1989, ISSN: 1432-0916. DOI: 10.1007/bf01217730.

[8] N. Reshetikhin and V. G. Turaev, "Invariants of 3-manifolds via link polynomials and quantum groups," *Inventiones Mathematicae*, vol. 103, no. 1, pp. 547–597, Dec. 1991, ISSN: 1432-1297. DOI: 10.1007/bf01239527.

[9] A. Hatcher, *Algebraic Topology*, English, 1st ed. Cambridge University Press, 2001, ISBN: 9780521795401; 0521795400.

[10] W. B. R. Lickorish, "A finite set of generators for the homeotopy group of a 2-manifold," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 60, no. 4, pp. 769–778, Oct. 1964, ISSN: 1469-8064. DOI: 10.1017/s030500410003824x.

[11] E. Artin, "Theory of braids," *The Annals of Mathematics*, vol. 48, no. 1, pp. 101–126, Jan. 1947, ISSN: 0003-486X. DOI: 10.2307/1969218.

[12] J. Birman and H. Hilden, "On isotopies of homeomorphisms of riemann surfaces," *Annals of Mathematics*, vol. 97, no. 3, pp. 424–439, 1973, ISSN: 0003486X.

[13] S. Gervais, "Presentation and central extensions of mapping class groups," *Transactions of the American Mathematical Society*, vol. 348, no. 8, pp. 3097–3132, 1996, ISSN: 1088-6850. DOI: 10.1090/s0002-9947-96-01509-7.

[14] B. Wajnryb, "A simple presentation for the mapping class group of an orientable surface," *Israel Journal of Mathematics*, vol. 45, no. 2–3, pp. 157–174, Jun. 1983, ISSN: 1565-8511. DOI: 10.1007/bf02774014.

[15] M. Korkmaz and J. D. McCarthy, "Surface mapping class groups are ultrahopfian," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 129, no. 1, pp. 35–53, 2000. DOI: 10.1017/S0305004199004259.

[16] L. Funar, "Two questions on mapping class groups," English, *Proceedings of the American Mathematical Society*, vol. 139, no. 1, pp. 375–382, 2011. DOI: 10.1090/S0002-9939-2010-10555-5.

[17] J. Franks and M. Handel, "Triviality of some representations of MCG(S_g) in $GL(n, \mathbb{C})$, Diff(S^2) and Homeo($\mathbb{T}^2$)," *Proceedings of the American Mathematical Society*, vol. 141, no. 9, pp. 2951–2962, 2013, ISSN: 0002-9939. DOI: 10.1090/S0002-9939-2013-11556-X.

[18] Y. Kasahara, "Crossed homomorphisms and low dimensional representations of mapping class groups of surfaces," *Transactions of the American Mathematical Society*, vol. 377, no. 2, pp. 1183–1218, Oct. 2023, ISSN: 1088-6850. DOI: 10.1090/tran/9037.