SEMISIMPLE LIE ALGEBRAS & THEIR REPRESENTATIONS

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To Violeta

About These Notes

Remark. Under construction!

These notes are perhaps better understood as a coming-of-age tale. They were originally part of some notes of mine on representations of Lie groups, written in early 2021 as part of my second scientific internship project with professor Iryna Kashuba of the department of mathematics of the Institute of Mathematics and Statistics of the University of São Paulo (IME-USP), Brazil. These were later adapted and expanded into my undergraduate dissertation, produced in late 2022 under the supervision of professor Kashuba. In mid 2023, after the publication of my undergraduate dissertation, the notes were once again expanded with the addition of their final chapter. All in all, I have been working on the prose that follows for the better part of my early higher education.

As they currently stand, the subject of these notes is a select topic in the representation theory of semisimple Lie algebras: Olivier Mathieu's classification of simple weight modules. Its first four chapters consist of a pretty standard account of the basic theory of semisimple Lie algebras and their finite-dimensional representations, providing a concise exposition of the background required for understanding the classification. On the other hand, the last two chapter of the notes should be understood as a reading guide for Mathieu's original paper [Mat00], with an emphasis on the intuition behind its major results.

Throughout these notes we will follow some guiding principles. First, lengthy proofs are favored as opposed to collections of smaller lemmas. This is a deliberate effort to emphasize the relevant results. Secondly, and this is more important, we are primarily interested in the broad strokes of the theory highlighted in the following chapters. This is because the topic of the dissertation at hand is a profoundly technical one. In particular, certain proofs can sometimes feel like an unmotivated pile of technical arguments. Instead, we prefer to focus on the intuition behind the relevant results.

Hence some results are left unproved. Nevertheless, we include numerous references throughout the text to other materials where the reader can find complete proofs. We will assume basic knowledge of abstract algebra. In particular, we assume that the reader is familiarized with multilinear algebra, the theory of modules over an algebra and exact sequences. We also assume familiarity with the language of categories, functors and adjunctions. Understanding some examples in the introductory chapter requires basic knowledge of differential and algebraic geometry, as well as rings of differential operators, but these examples are not necessary to the comprehension of the notes as a whole. Additional topics will be covered in the notes as needed.

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Finally, I would like to thank my dear friend Lucas Dias Schiezari for somehow convincing me to apply for a bachelors degree in pure mathematics, as well as the moments of joy we shared. May he rest in peace.

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Chapter 1

Introduction

Associative algebras have proven themselves remarkably useful throughout mathematics. There is no lack of natural and interesting examples coming from a diverse spectrum of different fields: topology, number theory, analysis, you name it. Associative algebras have thus been studied at length, specially the commutative ones. On the other hand, non-associative algebras have never sustained the same degree of scrutiny. To this day, non-associative algebras remain remarkably mysterious. Many have given up on attempting a systematic investigation and focus instead on understanding particular classes of non-associative algebras – i.e. algebras satisfying *pseudo-associativity* conditions.

Perhaps the most fascinating class of non-associative algebras are the so called *Lie algebras*, and these will be the focus of these notes.

Definition 1.1. Given a field K, a Lie algebra over K is a K-vector space $\mathfrak{g}$ endowed with an antisymmetric bilinear map $[\,,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ – which we call its *Lie bracket* – satisfying the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Definition 1.2. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ over K, a homomorphism of Lie algebras $\mathfrak{g} \longrightarrow \mathfrak{h}$ is a K-linear map $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ which *preserves bracket* in the sense that

$$f([X, Y]) = [f(X), f(Y)]$$

for all $X, Y \in \mathfrak{g}$. The dimension $\dim \mathfrak{g}$ of $\mathfrak{g}$ is its dimension as a K-vector space.

The collection of Lie algebras over a fixed field K thus form a category, which we call K-**LieAlg**. We are primarily interested in finite-dimensional Lie algebras over algebraically closed fields of characteristic 0. Hence from now on we assume K is algebraically closed and $\operatorname{char} K = 0$ unless explicitly stated otherwise. Ironically, perhaps the most basic examples of Lie algebras are derived from associative algebras.

Example 1.3. Given an associative K-algebra A, we can view A as a Lie algebra over K with the Lie bracket given by the commutator $[a, b] = ab - ba$. In particular, given a K-vector space V we may view the K-algebra $\operatorname{End}(V)$ as a Lie algebra, which we call $\mathfrak{gl}(V)$. We may also regard the Lie algebra $\mathfrak{gl}_n(K) = \mathfrak{gl}(K^n)$ as the space of $n \times n$ matrices with coefficients in K.

Example 1.4. Let $n \leqslant m$. Then the map

$$\begin{aligned} \mathfrak{gl}_n(K) &\longrightarrow \mathfrak{gl}_m(K) \\ X &\longmapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

is a homomorphism of Lie algebras.

While straightforward enough, I always found the definition of a Lie algebra unconvincing on its own. Specifically, the Jacobi identity can look very alien to someone who has never ventured outside of the realms of associativity. Traditional abstract algebra courses offer little in the way of a motivation for studying non-associative algebras in general. Why should we drop the assumption of associativity if every example of an algebraic structure we have ever seen is an associative one? Instead, the most natural examples of Lie algebras often come from an entirely different field: geometry.

Here the meaning of *geometry* is somewhat vague. Topics such as differential and algebraic geometry are prominently featured, but examples from fields such as the theory of differential operators and D-modules also show up a lot in the theory of representations – which we will soon discuss. Perhaps one of the most fundamental themes of the study of Lie algebras is their relationship with groups, specially in geometric contexts. We will now provide a brief description of this relationship through a series of examples.

Example 1.5. Let A be an associative K-algebra and $\operatorname{Der}(A)$ be the space of all derivations on A – i.e. all linear maps $D : A \longrightarrow A$ satisfying the Leibniz rule $D(a \cdot b) = a \cdot Db + (Da) \cdot b$. The commutator $[D, D']$ of two derivations $D, D' \in \operatorname{Der}(A)$ in the ring $\operatorname{End}(A)$ of K-linear endomorphisms of A is a derivation. Hence $\operatorname{Der}(A)$ is a Lie algebra.

One specific instance of this last example is...

Example 1.6. Given a smooth manifold M, the space $\mathfrak{X}(M)$ of all smooth vector fields is canonically identified with $\operatorname{Der}(M) = \operatorname{Der}(C^\infty(M))$ – where a field $X \in \mathfrak{X}(M)$ is identified with the map $C^\infty(M) \longrightarrow C^\infty(M)$ which takes a function $f \in C^\infty(M)$ to its derivative in the direction of X. This gives $\mathfrak{X}(M)$ the structure of a Lie algebra over $\mathbb{R}$.

Example 1.7. Given a Lie group G – i.e. a smooth manifold endowed with smooth group operations – we call $X \in \mathfrak{X}(G)$ left invariant if $(d\ell_g)_1 X_1 = X_g$ for all $g \in G$, where $\ell_g : G \longrightarrow G$ denotes the left translation by g. The commutator of invariant fields is invariant, so the space $\mathfrak{g} = \operatorname{Lie}(G)$ of all invariant vector fields has the structure of a Lie algebra over $\mathbb{R}$ with bracket given by the usual commutator of fields. Notice that an invariant field X is completely determined by $X_1 \in T_1 G$. Hence there is a linear isomorphism $\mathfrak{g} \xrightarrow{\sim} T_1 G$. In particular, $\mathfrak{g}$ is finite-dimensional.

We should point out that the Lie algebra $\mathfrak{g}$ of a complex Lie group G – i.e. a complex manifold endowed with holomorphic group operations – has the natural structure of a complex Lie algebra. Indeed, every left invariant field $X \in \mathfrak{X}(G)$ is holomorphic, so $\mathfrak{g}$ is a (complex) subspace of the complex vector space of holomorphic vector fields over G. There is also an algebraic analogue of this last construction.

Example 1.8. Let G be an affine algebraic K-group – i.e. an affine variety over K with rational group operations – and $K[G]$ denote the ring of regular functions $G \longrightarrow K$. We call a derivation $D : K[G] \longrightarrow K[G]$ left invariant if $D(g \cdot f) = g \cdot Df$ for all $g \in G$ and $f \in K[G]$ – where the action of G on $K[G]$ is given by $(g \cdot f)(h) = f(g^{-1}h)$. The commutator of left invariant derivations is invariant too, so the space $\operatorname{Lie}(G) = \operatorname{Der}(G)^G$ of invariant derivations in $K[G]$ has the structure of a Lie algebra over K with bracket given by the commutator of derivations. Again, $\operatorname{Lie}(G)$ is isomorphic to the Zariski tangent space $T_1 G$, which is finite-dimensional.

Example 1.9. The Lie algebra $\operatorname{Lie}(\mathrm{GL}_n(K))$ is canonically isomorphic to the Lie algebra $\mathfrak{gl}_n(K)$. Likewise, the Lie algebra $\operatorname{Lie}(\mathrm{SL}_n(K))$ is canonically isomorphic to the Lie algebra $\mathfrak{sl}_n(K)$ of traceless $n \times n$ matrices.

$$\mathfrak{sl}_n(K) = \{X \in \mathfrak{gl}_n(K) : \operatorname{Tr} X = 0\}$$

Example 1.10. The elements

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis for $\mathfrak{sl}_2(K)$ and are subject to the following relations.

$$[e,f] = h \qquad\qquad [h,f] = -2f \qquad\qquad [h,e] = 2e$$

Example 1.11. The Lie algebra of the affine algebraic group

$$\mathrm{Sp}_{2n}(K) = \{g \in \mathrm{GL}_{2n}(K) : \omega(g \cdot v, g \cdot w) = \omega(v,w)\,\forall v,w \in K^{2n}\}$$

is canonically isomorphic to the Lie algebra

$$\mathfrak{sp}_{2n}(K) = \left\{ \begin{pmatrix} X & Y \\ Z & -X^\top \end{pmatrix} : X,Y,Z \in \mathfrak{gl}_n(K), Y = Y^\top, Z = Z^\top \right\},$$

with bracket given by the usual commutator of matrices – where

$$\omega((v_1,\ldots,v_n,\dot{v}_1,\ldots,\dot{v}_n),(w_1,\ldots,w_n,\dot{w}_1,\ldots,\dot{w}_n)) = v_1\dot{w}_1 + \cdots + v_n\dot{w}_n - \dot{v}_1 w_1 - \cdots - \dot{v}_n w_n$$

is, of course, the standard symplectic form of K^{2n}.

It is important to point out that the construction of the Lie algebra $\mathfrak{g}$ of a Lie group G in Example 1.7 is functorial. Specifically, one can show the derivative $df_1 : \mathfrak{g} \cong T_1 G \longrightarrow T_1 H \cong \mathfrak{h}$ of a smooth group homomorphism $f : G \longrightarrow H$ is a homomorphism of Lie algebras, and the chain rule implies $d(f \circ g)_1 = df_1 \circ dg_1$. This is known as the *the Lie functor* $\mathrm{Lie} : \mathbf{LieGrp} \longrightarrow \mathbb{R}\text{-}\mathbf{LieAlg}$ between the category of Lie groups and smooth group homomorphisms and the category of Lie algebras.

This goes to show Lie algebras are invariants of Lie groups. What is perhaps more surprising is the fact that, in certain contexts, Lie algebras are perfect invariants. Even more so…

Theorem 1.12 (Lie). *The restriction* $\mathrm{Lie} : \mathbf{LieGrp}_{\mathrm{simpl}} \longrightarrow \mathbb{R}\text{-}\mathbf{LieAlg}$ *of the Lie functor to the full subcategory of simply connected Lie groups is an equivalence of categories onto the full subcategory of finite-dimensional real Lie algebras.*

This last theorem is a direct corollary of the so called *first and third fundamental Lie Theorems*. Lie's first Theorem establishes that if G is a simply connected Lie group and H is a connected Lie group then the induced map $\mathrm{Hom}(G,H) \longrightarrow \mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ is bijective, which implies the Lie functor is fully faithful. On the other hand, Lie's third Theorem states that every finite-dimensional real Lie algebra is the Lie algebra of a simply connected Lie group – i.e. the Lie functor is essentially surjective.

This goes to show that the relationship between Lie groups and Lie algebras is deeper than the fact they share a name: in a very strong sense, studying simply connected Lie groups is *precisely* the same as studying finite-dimensional Lie algebras. Such a vital connection between apparently distant subjects is bound to produce interesting results. Indeed, the passage from the geometric setting to its algebraic counterpart and vice-versa has proven itself a fruitful one.

This correspondence can be extended to the complex case too. In other words, the Lie functor $\mathbf{CLieGrp}_{\mathrm{simpl}} \longrightarrow \mathbb{C}\text{-}\mathbf{LieAlg}$ is also an equivalence of categories between the category of simply connected complex Lie groups and the full subcategory of finite-dimensional complex Lie algebras. The situation is more delicate in the algebraic case. For instance, consider the complex Lie algebra homomorphism

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathfrak{sl}_2(\mathbb{C}) \\ \lambda &\longmapsto \lambda h = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \end{aligned}$$

Since $\mathfrak{sl}_2(\mathbb{C}) = \mathrm{Lie}(\mathrm{SL}_2(\mathbb{C}))$ and $\mathrm{SL}_2(\mathbb{C})$ is simply connected, we know there exists a unique holomorphic group homomorphism $g : \mathbb{C} \longrightarrow \mathrm{SL}_2(\mathbb{C})$ between the affine line $\mathbb{C}$ and the complex *algebraic* group $\mathrm{SL}_2(\mathbb{C})$ such that $f = dg_1$. Indeed, this homomorphism is

$$\begin{aligned} g : \mathbb{C} &\longrightarrow \mathrm{SL}_2(\mathbb{C}) \\ \lambda &\longmapsto \exp(\lambda h) = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}, \end{aligned}$$

which is not a rational map. It then follows from the uniqueness of g that there is no rational group homomorphism $\mathbb{C} \longrightarrow \mathrm{SL}_2(\mathbb{C})$ whose derivative at the identity is f.

In particular, the Lie functor $\mathbb{C}$-$\mathbf{Grp}_{\mathrm{simpl}} \longrightarrow \mathbb{C}$-$\mathbf{LieAlg}$ – between the category $\mathbb{C}$-$\mathbf{Grp}_{\mathrm{simpl}}$ of simply connected complex algebraic groups and the category of complex Lie algebras – fails to be full. Similarly, the functor $\mathbb{C}$-$\mathbf{Grp}_{\mathrm{simpl}} \longrightarrow \mathbb{C}$-$\mathbf{LieAlg}$ is *not* essentially surjective onto the subcategory of finite-dimensional algebras: every finite-dimensional complex Lie algebra is isomorphic to the Lie algebra of a unique simply connected complex Lie group, but there are simply connected complex Lie groups which are not algebraic groups. Nevertheless, Lie algebras are still powerful invariants of algebraic groups. An interesting discussion of some of these delicacies can be found in sixth section of [DG70, ch. II].

All in all, there is a profound connection between groups and finite-dimensional Lie algebras throughout multiple fields. While perhaps unintuitive at first, the advantages of working with Lie algebras over their group-theoretic counterparts are numerous. First, Lie algebras allow us to avoid much of the delicacies of geometric objects such as real and complex Lie groups. Even when working without additional geometric considerations, groups can be complicated beasts themselves. They are, after all, nonlinear objects. On the other hand, Lie algebras are linear by nature, which makes them much more flexible than groups.

Having thus hopefully established that Lie algebras are interesting, we are now ready to dive deeper into them. We begin by analyzing some of their most basic properties.

1.1 Lie Algebras

However bizarre Lie algebras may seem at a first glance, they actually share a lot a structural features with their associative counterparts. For instance, it is only natural to define...

Definition 1.13. Given a Lie algebra $\mathfrak{g}$, a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called *a subalgebra of* $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$. A subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is called *an ideal of* $\mathfrak{g}$ if $[X, Y] \in \mathfrak{a}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$, in which case we write $\mathfrak{a} \triangleleft \mathfrak{g}$.

Remark. In the context of associative algebras, it is usual practice to distinguish between *left ideals* and *right ideals*. This is not necessary when dealing with Lie algebras, however, since any "left ideal" of a Lie algebra is also a "right ideal": given $\mathfrak{a} \triangleleft \mathfrak{g}$, $[Y, X] = -[X, Y] \in \mathfrak{a}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$.

Example 1.14. Let $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a homomorphism between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then $\ker f \subseteq \mathfrak{g}$ and $\operatorname{im} f \subseteq \mathfrak{h}$ are subalgebras. Furthermore, $\ker f \triangleleft \mathfrak{g}$.

Example 1.15. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be a Lie algebras over K. Then the space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Lie algebra with bracket

$$[X_1 + X_2, Y_1 + Y_2] = [X_1, Y_1] + [X_2, Y_2],$$

and $\mathfrak{g}_1, \mathfrak{g}_2 \triangleleft \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Example 1.16. Let G be an affine algebraic K-group and $H \subseteq G$ be a connected closed subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. The inclusion $H \longrightarrow G$ induces an injective homomorphism $\mathfrak{h} \longrightarrow \mathfrak{g}$. We may thus regard $\mathfrak{h}$ as a subalgebra of $\mathfrak{g}$. In addition, $\mathfrak{h} \triangleleft \mathfrak{g}$ if, and only if $H \triangleleft G$.

There is also a natural analogue of quotients.

Definition 1.17. Given a Lie algebra $\mathfrak{g}$ and $\mathfrak{a} \lhd \mathfrak{g}$, the space ${}^{\mathfrak{g}}/_{\mathfrak{a}}$ has the natural structure of a Lie algebra over K, where
$$[X + \mathfrak{a}, Y + \mathfrak{a}] = [X, Y] + \mathfrak{a}$$

Proposition 1.18. *Given a Lie algebra $\mathfrak{g}$ and $\mathfrak{a} \lhd \mathfrak{g}$, every homomorphism of Lie algebras $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ such that $\mathfrak{a} \subseteq \ker f$ uniquely factors through the projection $\mathfrak{g} \longrightarrow {}^{\mathfrak{g}}/_{\mathfrak{a}}$.*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d] & \mathfrak{h} \\ {}^{\mathfrak{g}}/_{\mathfrak{a}} \arrow[ur, dotted] & \end{tikzcd}$$

Definition 1.19. A Lie algebra $\mathfrak{g}$ is called *Abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.

Example 1.20. Let G be a connected algebraic K-group and $\mathfrak{g}$ be its Lie algebra. Then G is Abelian if, and only if $\mathfrak{g}$ is Abelian.

Remark. Notice that an Abelian Lie algebra is determined by its dimension. Indeed, any linear map $\mathfrak{g} \longrightarrow \mathfrak{h}$ between Abelian Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a homomorphism of Lie algebras. In particular, any linear isomorphism $\mathfrak{g} \xrightarrow{\sim} K^n$ – where K^n is endowed with the trivial bracket $[v, w] = 0$, $v, w \in K^n$ – is an isomorphism of Lie algebras for Abelian $\mathfrak{g}$.

Example 1.21. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{z} = \{X \in \mathfrak{g} : [X, Y] = 0, Y \in \mathfrak{g}\}$. Then $\mathfrak{z}$ is an Abelian ideal of $\mathfrak{g}$, known as *the center of* $\mathfrak{z}$.

Due to their relationship with Lie groups and algebraic groups, Lie algebras also share structural features with groups. For example…

Definition 1.22. A Lie algebra $\mathfrak{g}$ is called *solvable* if its derived series
$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq [[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]]] \supseteq \cdots$$
converges to 0 in finite time.

Example 1.23. Let G be a connected affine algebraic K-group and $\mathfrak{g}$ be its Lie algebra. Then G is solvable if, and only if $\mathfrak{g}$ is.

Definition 1.24. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if its lower central series
$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]] \supseteq \cdots$$
converges to 0 in finite time.

Example 1.25. Let G be a connected affine algebraic K-group and $\mathfrak{g}$ be its Lie algebra. Then G is nilpotent if, and only if $\mathfrak{g}$ is.

Other interesting classes of Lie algebras are the so called *simple* and *semisimple* Lie algebras.

Definition 1.26. A non-Abelian Lie algebra $\mathfrak{s}$ over K is called *simple* if its only ideals are 0 and $\mathfrak{s}$.

Example 1.27. The Lie algebra $\mathfrak{sl}_2(K)$ is simple. To see this, notice that any ideal $\mathfrak{a} \lhd \mathfrak{sl}_2(K)$ must be stable under the operator $\mathrm{ad}(h) : \mathfrak{sl}_2(K) \longrightarrow \mathfrak{sl}_2(K)$ given by $\mathrm{ad}(h)X = [h, X]$. But Example 1.10 implies $\mathrm{ad}(h)$ is diagonalizable, with eigenvalues 0 and ± 2. Hence $\mathfrak{a}$ must be spanned by some of the eigenvectors e, f, h of $\mathrm{ad}(h)$. If $h \in \mathfrak{a}$, then $[e, h] = -2e \in \mathfrak{a}$ and $[f, h] = 2f \in \mathfrak{a}$, so $\mathfrak{a} = \mathfrak{sl}_2(K)$. If $e \in \mathfrak{a}$ then $[f, e] = -h \in \mathfrak{a}$, so again $\mathfrak{a} = \mathfrak{sl}_2(K)$. Similarly, if $f \in \mathfrak{a}$ then $[e, f] = h \in \mathfrak{a}$ and $\mathfrak{a} = \mathfrak{sl}_2(K)$. More generally, the Lie algebra $\mathfrak{sl}_n(K)$ is simple for each $n > 1$ – see the section of [Kir08, ch. 6] on invariant bilinear forms and the semisimplicity of classical Lie algebras.

Example 1.28. The Lie algebras $\mathfrak{sp}_{2n}(K)$ are simple for all $n \geqslant 1$ – agina, see [Kir08, ch. 6].

Definition 1.29. A Lie algebra $\mathfrak{g}$ is called *semisimple* if it is the direct sum of simple Lie algebras. Equivalently, a Lie algebra $\mathfrak{g}$ is called *semisimple* if it has no nonzero solvable ideals.

Example 1.30. Let G be a connected affine algebraic K-group. Then G is semisimple if, and only if $\mathfrak{g}$ semisimple.

A slight generalization is...

Definition 1.31. A Lie algebra $\mathfrak{g}$ is called *reductive* if $\mathfrak{g}$ is the direct sum of a semisimple Lie algebra and an Abelian Lie algebra.

Example 1.32. The Lie algebra $\mathfrak{gl}_n(K)$ is reductive. Indeed,

$$X = \begin{pmatrix} a_{11} - \frac{\mathrm{Tr}(X)}{n} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} - \frac{\mathrm{Tr}(X)}{n} \end{pmatrix} + \begin{pmatrix} \frac{\mathrm{Tr}(X)}{n} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\mathrm{Tr}(X)}{n} \end{pmatrix}$$

for each matrix $X = (a_{ij})_{ij}$. In other words, $\mathfrak{gl}_n(K) = \mathfrak{sl}_n(K) \oplus K\,\mathrm{Id} \cong \mathfrak{sl}_n(K) \oplus K$.

As suggested by their names, simple and semisimple algebras are quite well behaved when compared with the general case. To a lesser degree, reductive algebras are also unusually well behaved. In the next chapter we will explore the question of why this is the case, but for now we note that we can get semisimple and reductive algebras by modding out by certain ideals, known as *radicals*.

Definition 1.33. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The sum $\mathfrak{a} + \mathfrak{b}$ of solvable ideals $\mathfrak{a}, \mathfrak{b} \lhd \mathfrak{g}$ is again a solvable ideal. Hence the sum of all solvable ideals of $\mathfrak{g}$ is a maximal solvable ideal, known as *the radical* $\mathfrak{rad}(\mathfrak{g})$ *of* $\mathfrak{g}$.

$$\mathfrak{rad}(\mathfrak{g}) = \sum_{\substack{\mathfrak{a} \lhd \mathfrak{g} \\ \text{solvable}}} \mathfrak{a}$$

Definition 1.34. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The sum of nilpotent ideals is a nilpotent ideal. Hence the sum of all nilpotent ideals of $\mathfrak{g}$ is a maximal nilpotent ideal, known as *the nilradical* $\mathfrak{nil}(\mathfrak{g})$ *of* $\mathfrak{g}$.

$$\mathfrak{nil}(\mathfrak{g}) = \sum_{\substack{\mathfrak{a} \lhd \mathfrak{g} \\ \text{nilpotent}}} \mathfrak{a}$$

Proposition 1.35. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$ is semisimple and $\mathfrak{g}/\mathfrak{nil}(\mathfrak{g})$ is reductive.*

We have seen in Example 1.3 that we can pass from an associative algebra A to a Lie algebra by taking its bracket as the commutator $[a, b] = ab - ba$. We should also not that any homomorphism of K-algebras $f : A \longrightarrow B$ preserves commutators, so that f is also a homomorphism of Lie algebras. Hence we have a functor $\mathrm{Lie} : K\text{-}\mathbf{Alg} \longrightarrow K\text{-}\mathbf{LieAlg}$. We can also go the other direction by embedding a Lie algebra $\mathfrak{g}$ in an associative algebra, known as *the universal enveloping algebra of* $\mathfrak{g}$.

Definition 1.36. Let $\mathfrak{g}$ be a Lie algebra and $T\mathfrak{g} = \bigoplus_n \mathfrak{g}^{\otimes n}$ be its tensor algebra – i.e. the free K-algebra generated by the elements of $\mathfrak{g}$. We call the K-algebra $\mathcal{U}(\mathfrak{g}) = {}^{T\mathfrak{g}}/_{I}$ *the universal enveloping algebra of* $\mathfrak{g}$, where I is the left ideal of $T\mathfrak{g}$ generated by the elements $[X, Y] - (X \otimes Y - Y \otimes X)$.

Notice there is a canonical homomorphism $\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ given by the composition

$$\mathfrak{g} \longrightarrow T\mathfrak{g} \longrightarrow {}^{T\mathfrak{g}}/_{I} = \mathcal{U}(\mathfrak{g})$$

Given $X_1, \ldots, X_n \in \mathfrak{g}$, we identify X_i with its image under the inclusion $\mathfrak{g} \longrightarrow T\mathfrak{g}$ and we write $X_1 \cdots X_n$ for $(X_1 \otimes \cdots \otimes X_n) + I$. This notation suggests the map $\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is injective, but at this point this is not at all clear – given that the projection $T\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is not injective. However, we will soon see this is the case. Intuitively, $\mathcal{U}(\mathfrak{g})$ is the smallest associative K-algebra containing $\mathfrak{g}$ as a Lie subalgebra. In practice this means. . .

Proposition 1.37. *Let $\mathfrak{g}$ be a Lie algebra and A be an associative K-algebra. Then every homomorphism of Lie algebras $f : \mathfrak{g} \longrightarrow A$ – where A is endowed with the structure of a Lie algebra as in Example 1.3 – can be uniquely extended to a homomorphism of algebras $\mathcal{U}(\mathfrak{g}) \longrightarrow A$.*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d] & A \\ \mathcal{U}(\mathfrak{g}) \arrow[ur, dotted] & \end{tikzcd}$$

Proof. Let $f : \mathfrak{g} \longrightarrow A$ be a homomorphism of Lie algebras. By the universal property of free algebras, there is a homomorphism of algebras $\tilde{f} : T\mathfrak{g} \longrightarrow A$ such that

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d] & A \\ T\mathfrak{g} \arrow[ur, dotted, "\tilde{f}"'] & \end{tikzcd}$$

Since f is a homomorphism of Lie algebras,

$$\tilde{f}([X, Y]) = f([X, Y]) = [f(X), f(Y)] = [\tilde{f}(X), \tilde{f}(Y)] = \tilde{f}(X \otimes Y - Y \otimes X)$$

for all $X, Y \in \mathfrak{g}$. Hence $I = ([X, Y] - (X \otimes Y - Y \otimes X) : X, Y \in \mathfrak{g}) \subseteq \ker \tilde{f}$ and therefore $\tilde{f}$ factors through the quotient $\mathcal{U}(\mathfrak{g}) = {}^{T\mathfrak{g}}/_{I}$.

$$\begin{tikzcd} T\mathfrak{g} \arrow[r, "\tilde{f}"] \arrow[d] & A \\ \mathcal{U}(\mathfrak{g}) \arrow[ur, dotted, "\bar{\tilde{f}}"'] & \end{tikzcd}$$

Combining the two previous diagrams, we can see that $\bar{\tilde{f}}$ is indeed an extension of f. The uniqueness of the extension then follows from the uniqueness of $\tilde{f}$ and $\bar{\tilde{f}}$. ∎

We should point out this construction is functorial. Indeed, if $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a homomorphism of Lie algebras then Proposition 1.37 implies there is a homomorphism of algebras $\mathscr{U}(f) : \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{h})$ satisfying

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d] & \mathfrak{h} \arrow[r] & \mathscr{U}(\mathfrak{h}) \\ \mathscr{U}(\mathfrak{g}) \arrow[urr, dotted, "\mathscr{U}(f)"'] & & \end{tikzcd}$$

It is important to note, however, that $\mathscr{U} : K\text{-}\mathbf{LieAlg} \longrightarrow K\text{-}\mathbf{Alg}$ is not the "inverse" of our functor $K\text{-}\mathbf{Alg} \longrightarrow K\text{-}\mathbf{LieAlg}$. For instance, if $\mathfrak{g} = K$ is the 1-dimensional Abelian Lie algebra then $\mathscr{U}(\mathfrak{g}) \cong K[x]$, which is infinite-dimensional. Nevertheless, Proposition 1.37 may be restated using the language of adjoint functors – as described in [Mac71] for instance.

Corollary 1.38. *If* $\mathrm{Lie} : K\text{-}\mathbf{Alg} \longrightarrow K\text{-}\mathbf{LieAlg}$ *is the functor described in Example 1.3, there is an adjunction* $\mathrm{Lie} \vdash \mathscr{U}$*.*

The structure of $\mathscr{U}(\mathfrak{g})$ can often be described in terms of the structure of $\mathfrak{g}$. For instance, $\mathfrak{g}$ is Abelian if, and only if $\mathscr{U}(\mathfrak{g})$ is commutative, in which case any basis $\{X_i\}_i$ for $\mathfrak{g}$ induces an isomorphism $\mathscr{U}(\mathfrak{g}) \cong K[x_1, x_2, \ldots, x_i, \ldots]$. More generally, we find...

Theorem 1.39 (Poincaré-Birkoff-Witt)**.** *Let* $\mathfrak{g}$ *be a Lie algebra over* K *and* $\{X_i\}_i \subseteq \mathfrak{g}$ *be an ordered basis for* $\mathfrak{g}$ *– i.e. a basis indexed by an ordered set. Then* $\{X_{i_1} \cdot X_{i_2} \cdots X_{i_n} : n \geqslant 0, i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n\}$ *is a basis for* $\mathscr{U}(\mathfrak{g})$*.*

This last result is known as *the PBW Theorem*. It is hugely important and will come up again and again throughout these notes. Among other things, it implies...

Corollary 1.40. *Let* $\mathfrak{g}$ *be a Lie algebra over* K*. Then* $\mathscr{U}(\mathfrak{g})$ *is a domain and the inclusion* $\mathfrak{g} \longrightarrow \mathscr{U}(\mathfrak{g})$ *is injective.*

The PBW Theorem can also be used to compute a series of examples.

Example 1.41. Consider the Lie algebra $\mathfrak{gl}_n(K)$ and its canonical basis $\{E_{ij}\}_{ij}$. Even though $E_{ij}E_{jk} = E_{ik}$ in the associative algebra $\mathrm{End}(K^n)$, the PBW Theorem implies $E_{ij}E_{jk} \neq E_{ik}$ in $\mathscr{U}(\mathfrak{gl}_n(K))$. In general, if A is an associative K-algebra then the elements in the image of the inclusion $A \longrightarrow \mathscr{U}(A)$ do not satisfy the same relations as the elements of A.

Example 1.42. Let $\mathfrak{g}$ be an Abelian Lie algebra. As previously stated, any choice of basis $\{X_i\}_i \subseteq \mathfrak{g}$ induces an isomorphism of algebras $\mathscr{U}(\mathfrak{g}) \xrightarrow{\sim} K[x_1, x_2, \ldots, x_i, \ldots]$ which takes $X_i \in \mathfrak{g}$ to the variable $x_i \in K[x_1, x_2, \ldots, x_i, \ldots]$.

Example 1.43. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over K. We claim that the natural map

$$\begin{aligned} f : \mathscr{U}(\mathfrak{g}_1) \otimes_K \mathscr{U}(\mathfrak{g}_2) &\longrightarrow \mathscr{U}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \\ u \otimes v &\longmapsto u \cdot v \end{aligned}$$

is an isomorphism of algebras. Since the elements of $\mathfrak{g}_1$ commute with the elements of $\mathfrak{g}_2$ in $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, a simple calculation shows that f is indeed a homomorphism of algebras. In addition, the PBW Theorem implies that f is a linear isomorphism.

The construction of $\mathscr{U}(\mathfrak{g})$ may seem like a purely algebraic affair, but the universal enveloping algebra of the Lie algebra of a Lie group G is in fact intimately related with the algebra $\mathrm{Diff}(G)$ of differential operators $C^\infty(G) \longrightarrow C^\infty(G)$ – i.e. $\mathbb{R}$-linear endomorphisms $C^\infty(G) \longrightarrow C^\infty(G)$ of finite order, as defined in [C C95, ch. 3] for example. Algebras of differential operators and their modules are the subject of the theory of D-modules, which has seen remarkable progress in the past century. Specifically, we find...

Proposition 1.44. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Denote by $\mathrm{Diff}(G)^G$ the algebra of G-invariant differential operators in G – i.e. the algebra of all differential operators $P : C^\infty(G) \longrightarrow C^\infty(G)$ such that $g \cdot Pf = P(g \cdot f)$ for all $f \in C^\infty(G)$ and $g \in G$. There is a canonical isomorphism of algebras $\mathcal{U}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Diff}(G)^G$.*

Proof. An order 0 G-invariant differential operator in G is simply multiplication by a constant in $\mathbb{R}$. A homogeneous order 1 G-invariant differential operator in G is simply a left invariant derivation $C^\infty(G) \longrightarrow C^\infty(G)$. All other G-invariant differential operators are generated by invariant operators of order 0 and 1. Hence $\mathrm{Diff}(G)^G$ is generated by $\mathrm{Der}(G)^G + \mathbb{R}$ – here $\mathrm{Der}(G)^G \subseteq \mathrm{Der}(G)$ denotes the Lie subalgebra of invariant derivations.

Now recall that there is a canonical isomorphism of Lie algebras $\mathfrak{X}(G) \xrightarrow{\sim} \mathrm{Der}(G)$. This isomorphism takes left invariant fields to left invariant derivations, so it restricts to an isomorphism $f : \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}(G)^G \subseteq \mathrm{Diff}(G)^G$. Since f is a homomorphism of Lie algebras, it can be extended to an algebra homomorphism $\tilde{f} : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{Diff}(G)^G$. We claim $\tilde{f}$ is an isomorphism.

To see that $\tilde{f}$ is injective, it suffices to notice

$$\tilde{f}(X_1 \cdots X_n) = \tilde{f}(X_1) \cdots \tilde{f}(X_n) = f(X_1) \cdots f(X_n) \neq 0$$

for all nonzero $X_1, \ldots, X_n \in \mathfrak{g}$ – the product of operators of positive order has positive order and is therefore nonzero. Since $\mathcal{U}(\mathfrak{g})$ is generated by the image of the inclusion $\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$, this implies $\ker \tilde{f} = 0$. Given that $\mathrm{Diff}(G)^G$ is generated by $\mathrm{Der}(G)^G + \mathbb{R}$, this also goes to show $\tilde{f}$ is surjective. ∎

As one would expect, the same holds for complex Lie groups and algebraic groups too – if we replace $C^\infty(G)$ by $\mathcal{O}(G)$ and $K[G]$, respectively. This last proposition has profound implications. For example, it affords us an analytic proof of certain particular cases of the PBW Theorem. Most surprising of all, Proposition 1.44 implies $\mathcal{U}(\mathfrak{g})$-modules are *precisely* the same as modules over the ring of G-invariant differential operators – i.e. $\mathrm{Diff}(G)^G$-modules. We can thus use $\mathcal{U}(\mathfrak{g})$ and its modules to understand the geometry of G.

Proposition 1.44 is in fact only the beginning of a profound connection between the theory of D-modules and *representation theory*, the latter of which we now explore in the following section.

1.2 Representation Theory

First introduced in 1896 by Georg Frobenius in his paper “Über Gruppencharakteren” [Fro96] in the context of group theory, representation theory is now one of the cornerstones of modern mathematics. In this section we provide a brief overview of basic concepts of the representation theory of Lie algebras. We should stress, however, that the representation theory of Lie algebras is only a small fragment of what is today known as “representation theory”, which is in general concerned with a diverse spectrum of algebraic and combinatorial structures – such as groups, quivers and associative algebras. An introductory exploration of some of these structures can be found in [Eti+11].

We begin by noting that any $\mathcal{U}(\mathfrak{g})$-module M may be regarded as a K-vector space endowed with a “linear action” of $\mathfrak{g}$. Indeed, by restricting the action map $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{End}(M)$ to $\mathfrak{g} \subseteq \mathcal{U}(\mathfrak{g})$ yields a homomorphism of Lie algebras $\mathfrak{g} \longrightarrow \mathfrak{gl}(M) = \mathrm{End}(M)$. In fact Proposition 1.37 implies that given a vector space M there is a one-to-one correspondence between $\mathcal{U}(\mathfrak{g})$-module structures for M and homomorphisms $\mathfrak{g} \longrightarrow \mathfrak{gl}(M)$. This leads us to the following definition.

Definition 1.45. Given a Lie algebra $\mathfrak{g}$ over K, *a representation V of $\mathfrak{g}$* is a K-vector space endowed with a homomorphism of Lie algebras $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

Hence there is a one-to-one correspondence between representations of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$-modules.

Example 1.46. Given a Lie algebra $\mathfrak{g}$, the zero map $0 : \mathfrak{g} \longrightarrow K$ gives K the structure of a representation of $\mathfrak{g}$, known as *the trivial representation*.

Example 1.47. Given a Lie algebra $\mathfrak{g}$, consider the homomorphism $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ given by $\mathrm{ad}(X)Y = [X, Y]$. This gives $\mathfrak{g}$ the structure of a representation of $\mathfrak{g}$, known as *the adjoint representation*.

Example 1.48. Given a Lie algebra $\mathfrak{g}$, the map $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathcal{U}(\mathfrak{g}))$ given by left multiplication endows $\mathcal{U}(\mathfrak{g})$ with the structure of a representation of $\mathfrak{g}$, known as *the regular representation of* $\mathfrak{g}$.

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathcal{U}(\mathfrak{g})) \\ X &\longmapsto \rho(X) : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \\ &\qquad\qquad\qquad u \longmapsto X \cdot u \end{aligned}$$

Example 1.49. Given a subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}_n(K)$, the inclusion $\mathfrak{g} \longrightarrow \mathfrak{gl}_n(K)$ endows K^n with the structure of a representation of $\mathfrak{g}$, known as *the natural representation of* $\mathfrak{g}$.

When the map $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is clear from context it is usual practice to denote the K-endomorphism $\rho(X) : V \longrightarrow V$, $X \in \mathfrak{g}$, simply by $X\restriction_V$. This leads us to the natural notion of *transformations* between representations.

Definition 1.50. Given a Lie algebra $\mathfrak{g}$ and two representations V and W of $\mathfrak{g}$, we call a K-linear map $f : V \longrightarrow W$ *an intertwining operator*, or *an intertwiner*, if it commutes with the action of $\mathfrak{g}$ on V and W, in the sense that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "X\restriction_V"'] & W \arrow[d, "X\restriction_W"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

commutes for all $X \in \mathfrak{g}$. We denote the space of all intertwiners $V \longrightarrow W$ by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ – as opposed the space $\mathrm{Hom}(V, W)$ of all K-linear maps $V \longrightarrow W$.

The collection of representations of a fixed Lie algebra $\mathfrak{g}$ thus forms a category, which we call $\mathbf{Rep}(\mathfrak{g})$. This allow us formulate the correspondence between representations of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$-modules in more precise terms.

Proposition 1.51. *There is a natural isomorphism of categories* $\mathbf{Rep}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})$-$\mathbf{Mod}$.

Proof. We have seen that given a K-vector space M there is a one-to-one correspondence between $\mathfrak{g}$-representation structures for M – i.e. homomorphisms $\mathfrak{g} \longrightarrow \mathfrak{gl}(M)$ – and $\mathcal{U}(\mathfrak{g})$-module structures for M – i.e. homomorphisms $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{End}(M)$. This gives us a surjective map that takes objects in $\mathbf{Rep}(\mathfrak{g})$ to objects in $\mathcal{U}(\mathfrak{g})$-$\mathbf{Mod}$.

As for the corresponding maps $\mathrm{Hom}_{\mathfrak{g}}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M, N)$, it suffices to note that a K-linear map between representations M and N is an intertwiner if, and only if it is a homomorphism of $\mathcal{U}(\mathfrak{g})$-modules. Our functor thus takes an intertwiner $M \longrightarrow N$ to itself. It should then be clear that our functor $\mathbf{Rep}(\mathfrak{g}) \longrightarrow \mathfrak{g}$-$\mathbf{Mod}$ is invertible. ∎

The language of representation is thus equivalent to that of $\mathcal{U}(\mathfrak{g})$-modules, which we call $\mathfrak{g}$-*modules*. Correspondingly, we refer to the category $\mathcal{U}(\mathfrak{g})$-**Mod** as $\mathfrak{g}$-**Mod**. The terms $\mathfrak{g}$-*submodule* and $\mathfrak{g}$-*homomorphism* should also be self-explanatory. To avoid any confusion, we will, for the most part, exclusively use the language of $\mathfrak{g}$-modules. It should be noted, however, that both points of view are profitable.

For starters, the notation for $\mathfrak{g}$-modules is much cleaner than that of representations: it is much easier to write "$X \cdot m$" than "$(\rho(X))(m)$" or even "$X \restriction_M (m)$". By using the language of $\mathfrak{g}$-modules we can also rely on the general theory of modules over associative algebras – which we assume the reader is already familiarized with. On the other hand, it is usually easier to express geometric considerations in terms of the language representations, particularly in group representation theory.

Often times it is easier to define a $\mathfrak{g}$-module M in terms of the corresponding map $\mathfrak{g} \longrightarrow \mathfrak{gl}(M)$ – this is technique we will use throughout the text. In general, the equivalence between both languages makes it clear that to understand the action of $\mathcal{U}(\mathfrak{g})$ on M it suffices to understand the action of $\mathfrak{g} \subseteq \mathcal{U}(\mathfrak{g})$. For instance, for defining a $\mathfrak{g}$-module M it suffices to define the action of each $X \in \mathfrak{g}$ and verify this action respects the commutator relations of $\mathfrak{g}$ – indeed, $\mathfrak{g}$ generates $\mathcal{U}(\mathfrak{g})$ as an algebra, and the only relations between elements of $\mathfrak{g}$ are the ones derived from the commutator relations.

Example 1.52. The space $K[x, y]$ is a $\mathfrak{sl}_2(K)$-module with

$$e \cdot p = x\frac{\mathrm{d}}{\mathrm{d}y}p \qquad f \cdot p = y\frac{\mathrm{d}}{\mathrm{d}x}p \qquad h \cdot p = \left(x\frac{\mathrm{d}}{\mathrm{d}x} - y\frac{\mathrm{d}}{\mathrm{d}y}\right)p$$

Example 1.53. Given a Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-modules M and N, the space $\operatorname{Hom}(M, N)$ of K-linear maps $M \longrightarrow N$ is a $\mathfrak{g}$-module where $(X \cdot f)(m) = X \cdot f(m) - f(X \cdot m)$ for all $X \in \mathfrak{g}$ and $f \in \operatorname{Hom}(M, N)$. In particular, if we take $N = K$ the trivial $\mathfrak{g}$-module, we can view M^* – the dual of M in the category of K-vector spaces – as a $\mathfrak{g}$-module where $(X \cdot f)(m) = -f(X \cdot m)$ for all $f : M \longrightarrow K$.

The fundamental problem of representation theory is a simple one: classifying all representations of a given Lie algebra up to isomorphism. However, understanding the relationship between representations is also of huge importance. In other words, to understand the whole of $\mathfrak{g}$-**Mod** we need to study the collective behavior of representations – as opposed to individual examples. For instance, we may consider $\mathfrak{g}$-submodules, quotients and tensor products.

Example 1.54. Let $K[x, y]$ be the $\mathfrak{sl}_2(K)$-module as in Example 1.52. Since e, f and h all preserve the degree of monomials, the space $K[x, y]^{(d)} = \bigoplus_{k+\ell=d} K x^k y^\ell$ of homogeneous polynomials of degree d is a finite-dimensional $\mathfrak{sl}_2(K)$-submodule of $K[x, y]$.

Example 1.55. Given a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-module M and $m \in M$, the subspace $\mathcal{U}(\mathfrak{g}) \cdot m = \{u \cdot m : u \in \mathcal{U}(\mathfrak{g})\}$ is a $\mathfrak{g}$-submodule of M, which we call *the submodule generated by m*.

Example 1.56. Given a Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-modules M and N, the space $M \otimes N = M \otimes_K N$ is a $\mathfrak{g}$-module where $X \cdot (m \otimes n) = X \cdot m \otimes n + m \otimes X \cdot n$. The exterior and symmetric products $M \wedge N$ and $M \odot N$ are both quotients of $M \otimes N$ by $\mathfrak{g}$-submodules. In particular, the exterior and symmetric powers $\wedge^r M$ and $\operatorname{Sym}^r M$ are $\mathfrak{g}$-modules.

Remark. We would like to stress that the monoidal structure of $\mathfrak{g}$-**Mod** we've just described is *not* given by the usual tensor product of modules. In other words, $M \otimes N$ is not the same as $M \otimes_{\mathcal{U}(\mathfrak{g})} N$.

It is also interesting to consider the relationship between representations of separate algebras. In particular, we may define…

Example 1.57. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be a subalgebra. Given a $\mathfrak{g}$-module M, denote by $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{h}} M = M$ the $\mathfrak{h}$-module where the action of $\mathfrak{h}$ is given by restricting the map $\mathfrak{g} \longrightarrow \mathfrak{gl}(M)$ to

$\mathfrak{h}$. Any homomorphism of $\mathfrak{g}$-modules $M \longrightarrow N$ is also a homomorphism of $\mathfrak{h}$-modules and this construction is clearly functorial.

$$\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{h}} : \mathfrak{g}\text{-}\mathbf{Mod} \longrightarrow \mathfrak{h}\text{-}\mathbf{Mod}$$

Example 1.58. Given a Lie algebra $\mathfrak{g}$, the adjoint $\mathfrak{g}$-module is a submodule of the restriction of the adjoint $\mathcal{U}(\mathfrak{g})$-module – where we consider $\mathcal{U}(\mathfrak{g})$ a Lie algebra as in Example 1.3, not as an associative algebra – to $\mathfrak{g}$.

Surprisingly, this functor has a right adjoint.

Example 1.59. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be a subalgebra. Given a $\mathfrak{h}$-module M, let $\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}} M = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} M$ – where the right $\mathfrak{h}$-module structure of $\mathcal{U}(\mathfrak{g})$ is given by right multiplication. Any $\mathfrak{h}$-homomorphism $f : M \longrightarrow N$ induces a $\mathfrak{g}$-homomorphism $\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}} f = \operatorname{id} \otimes f : \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}} M \longrightarrow \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}} N$ and this construction is clearly functorial.

$$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}} : \mathfrak{h}\text{-}\mathbf{Mod} \longrightarrow \mathfrak{g}\text{-}\mathbf{Mod}$$

Proposition 1.60. *Given a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, a $\mathfrak{h}$-module M and a $\mathfrak{g}$-module N, the map*

$$\begin{aligned} \alpha : \operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}} M, N) &\longrightarrow \operatorname{Hom}_{\mathfrak{h}}(M, \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{h}} N) \\ f &\longmapsto \alpha(f) : M \longrightarrow \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{h}} N \\ &\qquad\qquad m \longmapsto f(1 \otimes m) \end{aligned}$$

is a K-linear isomorphism. In other words, there is an adjunction $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{h}} \vdash \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}}$.

Proof. It suffices to note that the map

$$\begin{aligned} \beta : \operatorname{Hom}_{\mathfrak{h}}(M, \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{h}} N) &\longrightarrow \operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}} M, N) \\ f &\longmapsto \beta(f) : \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}} M \longrightarrow N \\ &\qquad\qquad u \otimes m \longmapsto u \cdot f(m) \end{aligned}$$

is the inverse of α. ■

This last proposition is known as *Frobenius reciprocity*, and was first proved by Frobenius himself in the context of finite groups. Another interesting construction is...

Example 1.61. Given two K-algebras A and B, an A-module M and a B-module N, $M \otimes N = M \otimes_K N$ has the natural structure of an $A \otimes_K B$-module. In light of Example 1.43, this implies that given Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, a $\mathfrak{g}_1$-module M_1 and a $\mathfrak{g}_2$-module M_2, the space $M_1 \otimes M_2$ has the natural structure of a $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module, where the action of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is given by

$$(X_1 + X_2) \cdot (m \otimes n) = X_1 \cdot m \otimes n + m \otimes X_2 \cdot n$$

Example 1.61 thus provides a way to describe representations of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in terms of the representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. We will soon see that in many cases *all* (simple) $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-modules can be constructed in such a manner. This concludes our initial remarks on $\mathfrak{g}$-modules. In the following chapters we will explore the fundamental problem of representation theory: that of classifying all representations of a given algebra up to isomorphism.

Chapter 2

Semisimplicity & Complete Reducibility

Having hopefully established in the previous chapter that Lie algebras and their representations are indeed useful, we are now faced with the Herculean task of trying to understand them. We have seen that representations can be used to derive geometric information about groups, but the question remains: how do we go about classifying the representations of a given Lie algebra? This question has sparked an entire field of research, and we cannot hope to provide a comprehensive answer in the 65 pages we have left. Nevertheless, we can work on particular cases.

For instance, one can readily check that a K^n-module M – here K^n denotes the n-dimensional Abelian Lie algebra – is nothing more than a choice of n commuting operators $M \longrightarrow M$ – corresponding to the action of the canonical basis elements $e_1, \ldots, e_n \in K^n$. In particular, a 1-dimensional K^n-module is just a choice of n scalars $\lambda_1, \ldots, \lambda_n$. Different choices of scalars yield non-isomorphic modules, so that the 1-dimensional K^n-modules are parameterized by points in K^n.

This goes to show that classifying the representations of Abelian algebras is not that interesting of a problem. Instead, we focus on a less trivial, yet reasonably well behaved case: the finite-dimensional modules of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field K of characteristic 0. But why are the modules of a semisimple Lie algebras simpler – or perhaps *semisimpler* – to understand than those of any old Lie algebra? We will get back to this question in a moment, but for now we simply note that, when solving a classification problem, it is often profitable to break down our structure is smaller pieces. This leads us to the following definitions.

Definition 2.1. A $\mathfrak{g}$-module is called *indecomposable* if it is not isomorphic to the direct sum of two nonzero $\mathfrak{g}$-modules.

Definition 2.2. A $\mathfrak{g}$-module is called *simple* if it has no nonzero proper $\mathfrak{g}$-modules.

Example 2.3. The trivial $\mathfrak{g}$-module K is an example of a simple $\mathfrak{g}$-module. In fact, every 1-dimensional $\mathfrak{g}$-module M is simple: M has no nonzero proper K-subspaces, let alone $\mathfrak{g}$-submodules.

Example 2.4. Given a finite-dimensional simple $\mathfrak{g}_1$-module M_1 and a finite-dimensional simple $\mathfrak{g}_2$-module M_2, the tensor product $M_1 \otimes M_2$ is a simple $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module. All finite-dimensional simple $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-modules have the form $M_1 \otimes M_2$ for unique (up to isomorphism) M_1 and M_2. In light of Example 1.43, this is a particular case of the fact that, given K-algebras A and B, all finite-dimensional simple $A \otimes_K B$-modules are given tensor products of simple A-modules with simple B-modules – see [Eti+11, ch. 3].

The general strategy for classifying finite-dimensional modules over an algebra is to classify the indecomposable modules. This is because...

Theorem 2.5 (Krull-Schmidt)**.** *Let $\mathfrak{g}$ be a Lie algebra. Then every finite-dimensional $\mathfrak{g}$-module can be uniquely – up to isomorphisms and reordering of the summands – decomposed into a direct sum of indecomposable $\mathfrak{g}$-modules.*

Hence finding the indecomposable $\mathfrak{g}$-modules suffices to find *all* finite-dimensional $\mathfrak{g}$-modules: they are the direct sum of indecomposable $\mathfrak{g}$-modules. The existence of the decomposition should be clear from the definitions. Indeed, if M is a finite-dimensional $\mathfrak{g}$-modules a simple argument via induction in $\dim M$ suffices to prove the existence: if M is indecomposable then there is nothing to prove, and if M is not indecomposable then $M = N \oplus L$ for some nonzero submodules $N, L \subsetneq M$, so that their dimensions are both strictly smaller than $\dim M$ and the existence follows from the induction hypothesis. For a proof of uniqueness please refer to [Eti+11].

Finding the indecomposable modules of an arbitrary Lie algebra, however, turns out to be a bit of a circular problem: the indecomposable $\mathfrak{g}$-modules are the ones that cannot be decomposed, which is to say, those that are *not* decomposable. Ideally, we would like to find some other condition, equivalent to indecomposability, but which is easier to work with. It is clear from the definitions that every simple $\mathfrak{g}$-module is indecomposable, but there is no reason to believe the converse is true. Indeed, this is not always the case. For instance...

Example 2.6. The space $M = K^2$ endowed with the action

$$x \cdot e_1 = e_1 \qquad\qquad x \cdot e_2 = e_1 + e_2$$

of the Lie algebra $K[x]$ is a $K[x]$-module. Notice M has a single nonzero proper submodule, which is spanned by the vector e_1. This is because if $(a+b)e_1 + be_2 = x \cdot (ae_1 + be_2) = \lambda \cdot (ae_1 + be_2)$ for some $\lambda \in K$ then $\lambda = 1$ and $b = 0$. Hence M is indecomposable – it cannot be broken into a direct sum of 1-dimensional submodules – but it is evidently not simple.

This counterexample poses an interesting question: are there conditions one can impose on an algebra $\mathfrak{g}$ under which every indecomposable $\mathfrak{g}$-module is simple? This is what is known in representation theory as *complete reducibility*.

Definition 2.7. A $\mathfrak{g}$-module M is called *completely reducible* if every $\mathfrak{g}$-submodule of M has a $\mathfrak{g}$-invariant complement – i.e. given $N \subseteq M$, there is a submodule $L \subseteq M$ such that $M = N \oplus L$.

Definition 2.8. A $\mathfrak{g}$-module M is called *semisimple* if it is the direct sum of simple $\mathfrak{g}$-modules.

In case the relationship between complete reducibility, semisimplicity of $\mathfrak{g}$-modules and the simplicity of indecomposable modules is unclear, the following results should clear things up.

Proposition 2.9. *The following conditions are equivalent.*

(i) *Every submodule of a finite-dimensional $\mathfrak{g}$-module is completely reducible.*

(ii) *Every exact sequence of finite-dimensional $\mathfrak{g}$-modules splits.*

(iii) *Every indecomposable finite-dimensional $\mathfrak{g}$-module is simple.*

(iv) *Every finite-dimensional $\mathfrak{g}$-module is semisimple.*

Proof. We begin by **(i)** $\implies$ **(ii)**. Let

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} L \longrightarrow 0$$

be an exact sequence of $\mathfrak{g}$-modules. We can suppose without loss of generality that $N \subseteq M$ is a submodule and f is its inclusion in M, for if this is not the case there is an isomorphism of sequences

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "f"'] & M \arrow[r, "g"] \arrow[d, equal] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & f(N) \arrow[r] & M \arrow[r, "g"'] & L \arrow[r] & 0 \end{tikzcd}$$

It then follows from **(i)** that there exists a $\mathfrak{g}$-submodule $L' \subseteq M$ such that $M = N \oplus L'$. Finally, the projection $s : M \longrightarrow N$ is $\mathfrak{g}$-homomorphism satisfying

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] & M \arrow[l, bend left, "s"] \arrow[r, "g"] & L \arrow[r] & 0 \end{tikzcd}$$

Next is **(ii)** $\implies$ **(iii)**. If M is an indecomposable $\mathfrak{g}$-module and $N \subseteq M$ is a submodule, we have an exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

of $\mathfrak{g}$-modules.

Since our sequence splits, we must have $M \cong N \oplus M/N$. But M is indecomposable, so that either $M = N$ or $M \cong M/N$, in which case $N = 0$. Since this holds for all $N \subseteq M$, M is simple. For **(iii)** $\implies$ **(iv)** it suffices to apply Theorem 2.5.

Finally, for **(iv)** $\implies$ **(i)**, if we assume **(iv)** and let M be a $\mathfrak{g}$-module with decomposition into simple submodules

$$M = \bigoplus_i M_i$$

and $N \subseteq M$ is a submodule. Take some maximal set of indexes $\{i_1, \ldots, i_r\}$ so that $\left(\bigoplus_k M_{i_k}\right) \cap M = 0$ and let $L = \bigoplus_k M_{i_k}$. We want to establish $M = N \oplus L$.

Suppose without any loss in generality that $i_k = k$ for all k and let $j > r$. By the maximality of our set of indexes, there is some nonzero $n \in (M_j \oplus L) \cap N$. Say $n = m_j + m_1 + \cdots + m_r$ with each $m_i \in M_i$. Then $m_j = n - m_1 - \cdots - m_r \in M_j \cap (N \oplus L)$ is nonzero. Indeed, if this is not the case we find $0 \neq n = m_1 + \cdots + m_r \in (\bigoplus_{i=1}^r M_i) \cap N$, a contradiction. This implies $M_j \cap (N \oplus L)$ is a nonzero submodule of M_j. Since M_j is simple, $M_j = M_j \cap (N \oplus L)$ and therefore $M_j \subseteq N \oplus L$. Given the arbitrary choice of j, it then follows $M = N \oplus L$. ■

While we are primarily interested in indecomposable $\mathfrak{g}$-modules – which is usually a strictly larger class of representations than that of simple $\mathfrak{g}$-modules – it is important to note that simple $\mathfrak{g}$-modules are generally much easier to find. The relationship between simple $\mathfrak{g}$-modules is also well understood. This is because of the following result, known as *Schur's Lemma*.

Lemma 2.10 (Schur). *Let M and N be simple $\mathfrak{g}$-modules and $f : M \longrightarrow N$ be a $\mathfrak{g}$-homomorphism. Then f is either 0 or an isomorphism. Furthermore, if $M = N$ is finite-dimensional then f is a scalar operator.*

Proof. For the first statement, it suffices to notice that $\ker f$ and $\operatorname{im} f$ are both submodules. In particular, either $\ker f = 0$ and $\operatorname{im} f = N$ or $\ker f = M$ and $\operatorname{im} f = 0$. Now suppose $M = N$ is finite-dimensional. Let $\lambda \in K$ be an eigenvalue of f – which exists because K is algebraically closed – and M_λ be its corresponding eigenspace. Given $m \in M_\lambda$, $f(X \cdot m) = X \cdot f(m) = \lambda X \cdot m$. In other words, M_λ is a $\mathfrak{g}$-submodule. It then follows $M_\lambda = M$, given that $M_\lambda \neq 0$. ■

We are now ready to answer our first question: the special thing about semisimple algebras is that the relationship between their indecomposable modules and their simple modules is much clearer. Namely...

Proposition 2.11. *Given a finite-dimensional Lie algebra $\mathfrak{g}$ over K, $\mathfrak{g}$ is semisimple if, and only if every finite-dimensional $\mathfrak{g}$-module is completely reducible.*

The proof of the fact that a finite-dimensional Lie algebra $\mathfrak{g}$ whose finite-dimensional modules are completely reducible is semisimple is actually pretty simple. Namely, it suffices to note that the adjoint $\mathfrak{g}$-module is the direct sum of simple submodules, which are all simple ideals of $\mathfrak{g}$ – so $\mathfrak{g}$ is the direct sum of simple Lie algebras. The proof of the converse is more nuanced, and this will be our next milestone.

Before proceeding to the proof of complete reducibility, however, we would like to introduce some basic tools which will come in handy later on, known as...

2.1 Invariant Bilinear Forms

Definition 2.12. A symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow K$ is called $\mathfrak{g}$-*invariant* if the operator $\operatorname{ad}(X) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is antisymmetric with respect to B for all $X \in \mathfrak{g}$.

$$B(\operatorname{ad}(X)Y, Z) + B(Y, \operatorname{ad}(X)Z) = 0$$

Remark. The etymology of the term *invariant form* comes from group representation theory. Namely, given a linear action of a group G on a vector space V equipped with a bilinear form B, B is called G-invariant if all $g \in G$ act via B-orthogonal operators. The condition of $\mathfrak{g}$-invariance can thus be though-of as an *infinitesimal approximation* of the notion of a G-invariant form. Indeed $\operatorname{Lie}(\mathrm{O}(B))$ is precisely the Lie subalgebra of $\mathfrak{gl}(V)$ consisting of antisymmetric operators $V \longrightarrow V$.

An interesting example of an invariant bilinear form is the so called *Killing form*.

Definition 2.13. Given a finite-dimensional Lie algebra $\mathfrak{g}$, the symmetric bilinear form

$$\begin{aligned} \kappa : \mathfrak{g} \times \mathfrak{g} &\longrightarrow K \\ (X, Y) &\longmapsto \operatorname{Tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) \end{aligned}$$

is called *the Killing form of* $\mathfrak{g}$.

The fact that the Killing form is an invariant form follows directly from the identity $\operatorname{Tr}([X, Y]Z) = \operatorname{Tr}(X[Y, Z])$, $X, Y, Z \in \mathfrak{gl}_n(K)$. In fact this same identity show...

Lemma 2.14. *Given a finite-dimensional $\mathfrak{g}$-module M, the symmetric bilinear form*

$$\begin{aligned} \kappa_M : \mathfrak{g} \times \mathfrak{g} &\longrightarrow K \\ (X, Y) &\longmapsto \operatorname{Tr}(X\restriction_M Y\restriction_M) \end{aligned}$$

is $\mathfrak{g}$-invariant.

The reason why we are discussing invariant bilinear forms is the following characterization of finite-dimensional semisimple Lie algebras, known as *Cartan's criterion for semisimplicity*.

Proposition 2.15. *Let $\mathfrak{g}$ be a Lie algebra. The following conditions are equivalent.*

(i) *$\mathfrak{g}$ is semisimple.*

(ii) *For each non-trivial finite-dimensional $\mathfrak{g}$-module M, the $\mathfrak{g}$-invariant bilinear form*

$$\begin{aligned} \kappa_M : \mathfrak{g} \times \mathfrak{g} &\longrightarrow K \\ (X, Y) &\longmapsto \mathrm{Tr}(X\restriction_M \, Y\restriction_M) \end{aligned}$$

is non-degenerate[1].

(iii) *The Killing form κ is non-degenerate.*

This proof is somewhat technical, but the idea behind it is simple. First, for **(i)** $\implies$ **(ii)** we show that $\mathfrak{a} = \{X \in \mathfrak{g} : \kappa_M(X, Y) = 0 \, \forall Y \in \mathfrak{g}\}$ is a solvable ideal of $\mathfrak{g}$. Hence $\mathfrak{a} = 0$. For **(ii)** $\implies$ **(iii)** it suffices to take $M = \mathfrak{g}$ the adjoint $\mathfrak{g}$-module. Finally, for **(iii)** $\implies$ **(i)** we note that the orthogonal complement of any $\mathfrak{a} \lhd \mathfrak{g}$ with respect to the Killing form κ is an ideal $\mathfrak{b}$ of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. Furthermore, the Killing form of $\mathfrak{a}$ is the restriction $\kappa\restriction_{\mathfrak{a}}$ of the Killing form of $\mathfrak{g}$ to $\mathfrak{a} \times \mathfrak{a}$, which is non-degenerate. It then follows from induction in $\dim \mathfrak{a}$ that $\mathfrak{g}$ is the sum of simple ideals.

We refer the reader to [E H73, ch. 5] for a complete proof. Without further ado, we may proceed to our. . .

2.2 Proof of Complete Reducibility

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over K. We want to establish that if $\mathfrak{g}$ is semisimple then all finite-dimensional $\mathfrak{g}$-modules are semisimple. Historically, this was first proved by Herman Weyl for $K = \mathbb{C}$, using his knowledge of smooth representations of compact Lie groups. Namely, Weyl showed that any finite-dimensional semisimple complex Lie algebra is (isomorphic to) the complexification of the Lie algebra of a unique simply connected compact Lie group, known as its *compact form*. Hence the category of the finite-dimensional modules of a given complex semisimple algebra is equivalent to that of the finite-dimensional smooth representations of its compact form, whose representations are known to be completely reducible because of Maschke's Theorem – see [GS18, ch. 3] for instance.

This proof, however, is heavily reliant on the geometric structure of $\mathbb{C}$. In other words, there is no hope for generalizing this for some arbitrary K. Fortunately for us, there is a much simpler, completely algebraic proof of complete reducibility, which works for algebras over any algebraically closed field of characteristic zero. The algebraic proof included in here is mainly based on that of [Kir08, ch. 6], and uses some basic homological algebra. Admittedly, much of the homological algebra used in here could be concealed from the reader, which would make the exposition more accessible – see [E H73] for instance.

However, this does not change the fact the arguments used in this proof are essentially homological in nature. Hence we consider it more productive to use the full force of the language of homological algebra, instead of burring the reader in a pile of unmotivated, yet entirely elementary arguments. Furthermore, the homological algebra used in here is actually *very basic*. In fact, all we need to know is. . .

[1] A symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow K$ is called non-degenerate if $B(X, Y) = 0$ for all $Y \in \mathfrak{g}$ implies $X = 0$.

Theorem 2.16. *There is a sequence of bifunctors* $\mathrm{Ext}^i : \mathfrak{g}\text{-}\mathbf{Mod}^{\mathrm{op}} \times \mathfrak{g}\text{-}\mathbf{Mod} \longrightarrow K\text{-}\mathbf{Vect}$, $i \geqslant 0$ *such that, given a* $\mathfrak{g}$*-module* L'*, every exact sequence of* $\mathfrak{g}$*-modules*

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} L \longrightarrow 0$$

induces long exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathfrak{g}}(L', N) \arrow[r, "f\circ -"] & \mathrm{Hom}_{\mathfrak{g}}(L', M) \arrow[r, "g\circ -"] & \mathrm{Hom}_{\mathfrak{g}}(L', L) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^1(L', N) \arrow[r] & \mathrm{Ext}^1(L', M) \arrow[r] & \mathrm{Ext}^1(L', L) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^2(L', N) \arrow[r] & \mathrm{Ext}^2(L', M) \arrow[r] & \mathrm{Ext}^2(L', L) \arrow[r, dashed] & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathfrak{g}}(L, L') \arrow[r, "-\circ g"] & \mathrm{Hom}_{\mathfrak{g}}(M, L') \arrow[r, "-\circ f"] & \mathrm{Hom}_{\mathfrak{g}}(N, L') \arrow[dll, rounded corners] \\
& \mathrm{Ext}^1(L, L') \arrow[r] & \mathrm{Ext}^1(M, L') \arrow[r] & \mathrm{Ext}^1(N, L') \arrow[dll, rounded corners] \\
& \mathrm{Ext}^2(L, L') \arrow[r] & \mathrm{Ext}^2(M, L') \arrow[r] & \mathrm{Ext}^2(N, L') \arrow[r, dashed] & \cdots
\end{tikzcd}$$

Theorem 2.17. *Given* $\mathfrak{g}$*-modules* N *and* L*, there is a one-to-one correspondence between elements of* $\mathrm{Ext}^1(L, N)$ *and isomorphism classes of short exact sequences*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

In particular, $\mathrm{Ext}^1(L, N) = 0$ *if, and only if every short exact sequence of* $\mathfrak{g}$*-modules with* N *and* L *in the extremes splits.*

We should point out that, although we have not provided an explicit definition of the bifunctors Ext^i, they are uniquely determined by the conditions of Theorem 2.16 and some additional minimality constraints. This is, of course, *far* from a comprehensive account of homological algebra. Nevertheless, this is all we need. We refer the reader to [Har08] for a complete exposition, or to part II of [Rib22] for a more modern account using derived categories.

We are particularly interested in the case where $L' = K$ is the trivial $\mathfrak{g}$-module. Namely, we may define...

Definition 2.18. Given a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module M, we refer to the Abelian group $H^i(\mathfrak{g}, M) = \mathrm{Ext}^i(K, M)$ as *the* i*-th Lie algebra cohomology group of* $\mathfrak{g}$ *with coefficients in* M.

Definition 2.19. Given a $\mathfrak{g}$-module M, we call the vector space $M^{\mathfrak{g}} = \{m \in M : X \cdot m = 0\ \forall X \in \mathfrak{g}\}$ *the space of invariants of* M. A simple calculations shows that a $\mathfrak{g}$-homomorphism $f : M \longrightarrow N$ takes invariants to invariants, so that f restricts to a map $M^{\mathfrak{g}} \longrightarrow N^{\mathfrak{g}}$. This construction thus yields a functor $-^{\mathfrak{g}} : \mathfrak{g}\text{-}\mathbf{Mod} \longrightarrow K\text{-}\mathbf{Vect}$.

Example 2.20. Let M be a $\mathfrak{g}$-module. Then M is a direct sum of copies of the trivial $\mathfrak{g}$-module if, and only if $M = M^{\mathfrak{g}}$.

Example 2.21. Let M and N be $\mathfrak{g}$-modules. Then $\mathrm{Hom}(M,N)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(M,N)$. Indeed, given a K-linear map $f : M \longrightarrow N$ we find

$$\begin{aligned} f \in \mathrm{Hom}(M,N)^{\mathfrak{g}} &\iff X \cdot f(m) - f(X \cdot m) = (X \cdot f)(m) = 0 \;\forall X \in \mathfrak{g}, m \in M \\ &\iff X \cdot f(m) = f(X \cdot m) \;\forall X \in \mathfrak{g}, m \in M \\ &\iff f \in \mathrm{Hom}_{\mathfrak{g}}(M,N) \end{aligned}$$

The Lie algebra cohomology groups are very much related to invariants of $\mathfrak{g}$-modules. Namely, constructing a $\mathfrak{g}$-homomorphism $f : K \longrightarrow M$ is precisely the same as fixing an invariant of M – corresponding to $f(1)$, which must be an invariant for f to be a $\mathfrak{g}$-homomorphism. Formally, this translates to the existence of a canonical isomorphism of functors $\mathrm{Hom}_{\mathfrak{g}}(K,-) \xrightarrow{\sim} -^{\mathfrak{g}}$ given by

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(K,M) &\xrightarrow{\sim} M^{\mathfrak{g}} \\ f &\longmapsto f(1) \end{aligned}$$

This implies. . .

Corollary 2.22. *Every short exact sequence of $\mathfrak{g}$-modules*

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} L \longrightarrow 0$$

induces a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & N^{\mathfrak{g}} \arrow[r, "f"] & M^{\mathfrak{g}} \arrow[r, "g"] & L^{\mathfrak{g}} \arrow[dll, rounded corners] \\ & H^1(\mathfrak{g},N) \arrow[r] & H^1(\mathfrak{g},M) \arrow[r] & H^1(\mathfrak{g},L) \arrow[dll, rounded corners] \\ & H^2(\mathfrak{g},N) \arrow[r] & H^2(\mathfrak{g},M) \arrow[r] & H^2(\mathfrak{g},L) \arrow[r, dashed] & \cdots \end{tikzcd}$$

Proof. We have an isomorphism of sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathfrak{g}}(K,N) \arrow[r, "f\circ-"] \arrow[d] & \mathrm{Hom}_{\mathfrak{g}}(K,M) \arrow[r, "g\circ-"] \arrow[d] & \mathrm{Hom}_{\mathfrak{g}}(K,L) \arrow[r] \arrow[d] & H^1(\mathfrak{g},N) \arrow[r, dashed] \arrow[d, equal] & \cdots \\ 0 \arrow[r] & N^{\mathfrak{g}} \arrow[r, "f"'] & M^{\mathfrak{g}} \arrow[r, "g"'] & L^{\mathfrak{g}} \arrow[r] & H^1(\mathfrak{g},N) \arrow[r, dashed] & \cdots \end{tikzcd}$$

By Theorem 2.16 the sequence on the top is exact. Hence so is the sequence on the bottom. ■

This is all well and good, but what does any of this have to do with complete reducibility? Well, in general cohomology theories really shine when one is trying to control obstructions of some kind. In our case, the bifunctor $H^1(\mathfrak{g}, \mathrm{Hom}(-,-)) : \mathfrak{g}\text{-}\mathbf{Mod}^{\mathrm{op}} \times \mathfrak{g}\text{-}\mathbf{Mod} \longrightarrow K\text{-}\mathbf{Vect}$ classifies obstructions to complete reducibility. Explicitly. . .

Theorem 2.23. *There is a natural isomorphism* $\mathrm{Ext}^1 \xrightarrow{\sim} H^1(\mathfrak{g}, \mathrm{Hom}(-,-))$. *In particular, given $\mathfrak{g}$-modules N and L, there is a one-to-one correspondence between elements of $H^1(\mathfrak{g}, \mathrm{Hom}(L,N))$ and isomorphism classes of short exact sequences*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

This is essentially a consequence of Example 2.21 and Theorem 2.17, as well as the minimality conditions that characterize Ext^1. For the readers already familiar with homological algebra: the correspondence between $H^1(\mathfrak{g}, \mathrm{Hom}(L,N))$ and short exact sequences of $\mathfrak{g}$-modules can be described in very concrete terms by considering a canonical free resolution

$$\cdots \dashrightarrow \mathcal{U}(\mathfrak{g}) \otimes (\wedge^2 \mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g}) \longrightarrow K \longrightarrow 0$$

of the trivial $\mathfrak{g}$-module K, known as *the Chevalley-Eilenberg resolution*, which provides an explicit construction of the cohomology groups – see [LB00, sec. 1.3C] or [GS84, sec. 24] for further details.

We will use the previous result implicitly in our proof, but we will not prove it in its full force. Namely, we will show that if $\mathfrak{g}$ is semisimple then $H^1(\mathfrak{g}, M) = 0$ for all finite-dimensional M, and that the fact that $H^1(\mathfrak{g}, \mathrm{Hom}(L,N)) = 0$ for all finite-dimensional N and L implies complete reducibility. To that end, we introduce a distinguished element of $\mathcal{U}(\mathfrak{g})$, known as *the Casimir element of a $\mathfrak{g}$-module*.

Definition 2.24. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and M be a finite-dimensional $\mathfrak{g}$-module. Let $\{X_i\}_i$ be a basis for $\mathfrak{g}$ and denote by $\{X^i\}_i \subseteq \mathfrak{g}$ its dual basis with respect to the form κ_M – i.e. the unique basis for $\mathfrak{g}$ satisfying $\kappa_M(X_i, X^j) = \delta_{ij}$, whose existence is a consequence of the non-degeneracy of κ_M. We call

$$\Omega_M = X_1 X^1 + \cdots + X_r X^r \in \mathcal{U}(\mathfrak{g})$$

the *Casimir element of M*.

Lemma 2.25. *The definition of Ω_M is independent of the choice of basis $\{X_i\}_i$.*

Proof. Whatever basis $\{X_i\}_i$ we choose, the image of Ω_M under the canonical isomorphism $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\sim} \mathrm{End}(\mathfrak{g})$ is the identity operator[2]. ■

Proposition 2.26. *The Casimir element $\Omega_M \in \mathcal{U}(\mathfrak{g})$ is central, so that $\Omega_M\restriction_N \colon N \longrightarrow N$ is a $\mathfrak{g}$-homomorphism for any $\mathfrak{g}$-module N. Furthermore, Ω_M acts on M as a nonzero scalar operator whenever M is a non-trivial finite-dimensional simple $\mathfrak{g}$-module.*

Proof. To see that Ω_M is central fix a basis $\{X_i\}_i$ for $\mathfrak{g}$ and denote by $\{X^i\}_i$ its dual basis with respect to κ_M, as in Definition 2.24. Given any $X \in \mathfrak{g}$, it follows from definition of the X^i that $X = \kappa_M(X, X^1)X_1 + \cdots + \kappa_M(X, X^r)X_r = \kappa_M(X, X_1)X^1 + \cdots + \kappa_M(X, X_r)X^r$.

In particular, it follows from the invariance of κ_M that

$$\begin{aligned}
[X, \Omega_M] &= \sum_i [X, X_i X^i] \\
&= \sum_i [X, X_i] X^i + \sum_i X_i [X, X^i] \\
&= \sum_{ij} \kappa_M([X, X_i], X^j) X_j X^i + \sum_{ij} \kappa_M([X, X^i], X_j) X_i X^j, \\
&= \sum_{ij} (\kappa_M([X, X_j], X^i) + \kappa_M(X_j, [X, X^i])) X_i X^j \\
&= 0
\end{aligned}$$

[2]Here the isomorphism $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{g}^*$ is given by tensoring the identity $\mathfrak{g} \longrightarrow \mathfrak{g}$ with the isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ induced by the form κ_M.

and Ω_M is central. This implies that $\Omega_M\restriction_N \colon N \longrightarrow N$ is a $\mathfrak{g}$-homomorphism for all $\mathfrak{g}$-modules N: its action commutes with the action of any other element of $\mathfrak{g}$.

In particular, it follows from Schur's Lemma that if M is finite-dimensional and simple then Ω_M acts on M as a scalar operator. To see that this scalar is nonzero we compute

$$\operatorname{Tr}(\Omega_M\restriction_M) = \operatorname{Tr}(X_1\restriction_M X^1\restriction_M) + \cdots + \operatorname{Tr}(X_r\restriction_M X^r\restriction_M) = \dim\mathfrak{g},$$

so that $\Omega_M\restriction_M = \lambda\,\mathrm{Id}$ for $\lambda = \frac{\dim\mathfrak{g}}{\dim M} \neq 0$. ■

As promised, the Casimir element of a $\mathfrak{g}$-module can be used to establish...

Proposition 2.27. *Suppose $\mathfrak{g}$ is semisimple and let M be a finite-dimensional $\mathfrak{g}$-module. Then $H^1(\mathfrak{g}, M) = 0$.*

Proof. We begin by the case where M is simple. Due to Theorem 2.17, it suffices to show that any exact sequence of the form

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} K \longrightarrow 0 \tag{2.1}$$

splits.

If $M = K$ is the trivial $\mathfrak{g}$-module then the exactness of

$$0 \longrightarrow K \xrightarrow{f} N \xrightarrow{g} K \longrightarrow 0 \tag{2.2}$$

implies N is 2-dimensional. Take any nonzero $n \in N$ outside of the image of f.

Since $\dim N = 2$, the simple component $\mathcal{U}(\mathfrak{g})\cdot n$ of n in N is either Kn or N itself. But this component cannot be N, since the image of f is a 1-dimensional $\mathfrak{g}$-module – i.e. a proper nonzero submodule. Hence Kn is invariant under the action of $\mathfrak{g}$. In particular, $X \cdot n = 0$ for all $X \in \mathfrak{g}$. Since n lies outside the image of f, $g(n) \neq 0$ – which is to say, $n \notin \ker g = \operatorname{im} f$. This implies the map $K \longrightarrow N$ that takes 1 to ${}^{n}/_{g(n)}$ is a splitting of (2.2).

Now suppose that M is non-trivial, so that Ω_M acts on M as λ for some $\lambda \neq 0$. Denote by N^μ the generalized eigenspace of $\Omega_M\restriction_N \colon N \longrightarrow N$ associated with $\mu \in K$. If we identify M with $f(M)$, it is clear that $M \subseteq N^\lambda$. The exactness of (2.1) implies $\dim N = \dim M + 1$, so that either $N^\lambda = M$ or $N^\lambda = N$. But if $N^\lambda = N$ then there is some nonzero $n \in N^\lambda$ with $n \notin M = \ker g$ such that

$$0 = (\Omega_M - \lambda)^r \cdot n = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \lambda^k \Omega_M^{r-k} \cdot n$$

for some $r \geqslant 1$.

In particular,

$$(-\lambda)^{r-1} g(n) = \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} \lambda^k g(\Omega_M^{r-k}\cdot n) = \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} \lambda^k \underbrace{\Omega_M^{r-k}\cdot g(n)}_{=0} = 0,$$

which is a contradiction – given that neither $(-\lambda)^{r-1}$ nor $g(n)$ are nil. Hence $M = N^\lambda$ and there must be some other eigenvalue μ of $\Omega_M\restriction_N$. For any such μ and any eigenvector $n \in N_\mu$,

$$\mu g(n) = g(\mu n) = g(\Omega_M \cdot n) = \Omega_M \cdot g(n) = 0$$

implies $\mu = 0$, so that the eigenvalues of the action of Ω_M on N are precisely λ and 0.

Now notice that N^0 is in fact a submodule of N. Indeed, given $n \in N^0$ and $X \in \mathfrak{g}$, it follows from the fact that Ω_M is central that

$$\Omega_M^r \cdot (X \cdot n) = X \cdot (\Omega_M^r \cdot n) = X \cdot 0 = 0$$

for some r. Hence $N = M \oplus N^0$ as $\mathfrak{g}$-modules. The homomorphism g thus induces an isomorphism $N^0 \cong {}^{N}\!/\!{}_{M} \xrightarrow{\sim} K$, which translates to a splitting of (2.1).

Finally, we consider the case where M is not simple. Suppose $H^1(\mathfrak{g}, N) = 0$ for all $\mathfrak{g}$-modules with $\dim N < \dim M$ and let $N \subseteq M$ be a proper nonzero submodule. Then the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

induces a long exact sequence of the form

$$\cdots \dashrightarrow H^1(\mathfrak{g}, N) \longrightarrow H^1(\mathfrak{g}, M) \longrightarrow H^1(\mathfrak{g}, M/N) \dashrightarrow \cdots \tag{2.3}$$

Since $\dim N < \dim M$, it follows $H^1(\mathfrak{g}, N) = 0$. In addition, since $\dim N > 0$, we find $\dim M/N < \dim M$ and thus $H^1(\mathfrak{g}, M/N) = 0$. The exactness of (2.3) then implies $H^1(\mathfrak{g}, M) = 0$. Hence by induction in $\dim M$ we find $H^1(\mathfrak{g}, M) = 0$ for all finite-dimensional M. We are done. ■

We are now finally ready to prove...

Theorem 2.28 (Weyl)**.** *Given a semisimple Lie algebra $\mathfrak{g}$, every finite-dimensional $\mathfrak{g}$-module is semisimple.*

Proof. Let

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} L \longrightarrow 0 \tag{2.4}$$

be a short exact sequence of finite-dimensional $\mathfrak{g}$-modules. We want to establish that (2.4) splits.

We have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(L, N) \xrightarrow{f \circ -} \mathrm{Hom}(L, M) \xrightarrow{g \circ -} \mathrm{Hom}(L, L) \longrightarrow 0$$

of vector spaces. Since all maps involved are $\mathfrak{g}$-homomorphisms, this is an exact sequence of $\mathfrak{g}$-modules. This then induces a long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}(L, N)^{\mathfrak{g}} \xrightarrow{f \circ -} \mathrm{Hom}(L, M)^{\mathfrak{g}} \xrightarrow{g \circ -} \mathrm{Hom}(L, L)^{\mathfrak{g}} \longrightarrow \\ \longrightarrow H^1(\mathfrak{g}, \mathrm{Hom}(L, N)) \longrightarrow H^1(\mathfrak{g}, \mathrm{Hom}(L, M)) \longrightarrow H^1(\mathfrak{g}, \mathrm{Hom}(L, L)) \dashrightarrow \cdots \end{array}$$

of vector spaces.

But $H^1(\mathfrak{g}, \mathrm{Hom}(L, N))$ vanishes because of Proposition 2.27. In addition, recall from Example 2.21 that $\mathrm{Hom}(L, L')^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(L, L')$. We thus have a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(L, N) \xrightarrow{f \circ -} \mathrm{Hom}_{\mathfrak{g}}(L, M) \xrightarrow{g \circ -} \mathrm{Hom}_{\mathfrak{g}}(L, L) \longrightarrow 0$$

In particular, there is some $\mathfrak{g}$-homomorphism $s : L \longrightarrow M$ such that $g \circ s : L \longrightarrow L$ is the identity operator. In other words

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] & M \arrow[r, "g"] & L \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd}$$

is a splitting of (2.4). ■

Theorem 2.28 typically fails in the infinite-dimensional setting. For instance, consider...

Example 2.29. The regular $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g})$ is an indecomposable module which is not simple. In particular, $\mathcal{U}(\mathfrak{g})$ is not semisimple. To see this, notice that the submodules of $\mathcal{U}(\mathfrak{g})$ are precisely its left ideals. If we suppose that $I, J \lhd \mathcal{U}(\mathfrak{g})$ are such that $\mathcal{U}(\mathfrak{g}) = I \oplus J$ as $\mathfrak{g}$-modules, we can find $u \in I$ and $v \in J$ such that $1 = u + v$. The PBW Theorem then implies that u and v commute, so that $uv = vu \in I \cap J = 0$. Since $\mathcal{U}(\mathfrak{g})$ is a domain, either $u = 0$ or $v = 0$. Given that $1 = u + v$, $u = 1$ or $v = 1$. Hence either $I = \mathcal{U}(\mathfrak{g})$ and $J = 0$ or $I = 0$ and $J = \mathcal{U}(\mathfrak{g})$, as required.

We should point out that these last results are just the beginning of a well developed cohomology theory. For example, a similar argument involving the Casimir elements can be used to show that $H^i(\mathfrak{g}, M) = 0$ for all semisimple $\mathfrak{g}$ and all non-trivial finite-dimensional simple M, $i > 0$. For $K = \mathbb{C}$, the Lie algebra cohomology groups of the algebra $\mathfrak{g} = \mathbb{C} \otimes \mathrm{Lie}(G)$ are intimately related with the topological cohomologies – i.e. singular cohomology, de Rham cohomology, etc. – of G with coefficients in $\mathbb{C}$. We refer the reader to [LB00] and [GS84, sec. 24] for further details.

Complete reducibility can be generalized for arbitrary – not necessarily semisimple – $\mathfrak{g}$, to a certain extent, by considering the exact sequence

$$0 \longrightarrow \mathfrak{rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{rad}(\mathfrak{g}) \longrightarrow 0$$

This sequence always splits for finite-dimensional $\mathfrak{g}$, which in light of Example 2.4 implies we can deduce information about $\mathfrak{g}$-modules by studying the modules of its "semisimple part" $\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$ – see Proposition 1.35. In practice this translates to. . .

Proposition 2.30 (Lie)**.** *Let $\mathfrak{g}$ be a solvable Lie algebra. Every finite-dimensional simple $\mathfrak{g}$-module is 1-dimensional.*

Corollary 2.31. *Let $\mathfrak{g}$ be a Lie algebra. Every finite-dimensional simple $\mathfrak{g}$-module is the tensor product of a simple $\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$-module and a 1-dimensional $\mathfrak{rad}(\mathfrak{g})$-module.*

Proof. This follows at once from Proposition 2.30 and Example 2.4. ■

Having finally reduced our initial classification problem to that of classifying the finite-dimensional simple $\mathfrak{g}$-modules, we can now focus exclusively in this particular class of $\mathfrak{g}$-modules. However, there is so far no indication on how we could go about understanding them. In the next chapter we will explore some concrete examples in the hopes of finding a solution to our general problem.

Chapter 3

Representations of $\mathfrak{sl}_2(K)$ & $\mathfrak{sl}_3(K)$

We are, once again, faced with the daunting task of classifying the finite-dimensional modules of a given (semisimple) algebra $\mathfrak{g}$. Having reduced the problem a great deal, all its left is classifying the simple $\mathfrak{g}$-modules. We have encountered numerous examples of simple $\mathfrak{g}$-modules over the previous chapter, but we have yet to subject them to any serious scrutiny. In this chapter we begin a systematic investigation of simple modules by looking at concrete examples. Specifically, we will classify the simple finite-dimensional modules of certain low-dimensional semisimple Lie algebras: $\mathfrak{sl}_2(K)$ and $\mathfrak{sl}_3(K)$.

The reason why we chose $\mathfrak{sl}_2(K)$ is a simple one: throughout the previous chapters $\mathfrak{sl}_2(K)$ has afforded us surprisingly illuminating examples. We begin our analysis by recalling that the elements

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis for $\mathfrak{sl}_2(K)$ and satisfy

$$[e, f] = h \qquad [h, f] = -2f \qquad [h, e] = 2e$$

Let M be a finite-dimensional simple $\mathfrak{sl}_2(K)$-module. We now turn our attention to the action of h on M, in particular, we investigate the subspace $\bigoplus_\lambda M_\lambda \subseteq M$ – where λ ranges over the eigenvalues of $h\restriction_M$ and M_λ is the corresponding eigenspace.

At this point, this is nothing short of a gamble: why look at the eigenvalues of h? The short answer is that, as we shall see, this will pay off. We will postpone the discussion about the real reason of why we chose h, but for now we may notice that, perhaps surprisingly, the action $h\restriction_M$ of h on a finite-dimensional simple $\mathfrak{sl}_2(K)$-module M is always a diagonalizable operator.

Let λ be any eigenvalue of $h\restriction_M$. Notice M_λ is in general not a $\mathfrak{sl}_2(K)$-submodule of M. Indeed, if $m \in M_\lambda$ then the identities

$$\begin{aligned} h \cdot (e \cdot m) &= 2e \cdot m + eh \cdot m = (\lambda + 2) e \cdot m \\ h \cdot (f \cdot m) &= -2f \cdot m + fh \cdot m = (\lambda - 2) f \cdot m \end{aligned}$$

follow. In other words, e sends an element of M_λ to an element of $M_{\lambda+2}$, while f sends it to an element of $M_{\lambda-2}$. Visually, we may draw

$$\begin{tikzcd} \cdots \arrow[r, bend left] & M_{\lambda-2} \arrow[r, bend left, "e"] \arrow[l, bend left] & M_\lambda \arrow[r, bend left, "e"] \arrow[l, bend left, "f"] & M_{\lambda+2} \arrow[r, bend left] \arrow[l, bend left, "f"] & \cdots \arrow[l, bend left] \end{tikzcd}$$

This implies $\bigoplus_\lambda M_\lambda$ is a $\mathfrak{sl}_2(K)$-submodule, so that $\bigoplus_\lambda M_\lambda$ is either 0 or the entirety of M – recall that M is simple. Since M is finite dimensional, $h\restriction_M$ has at least one eigenvalue and therefore

$$M = \bigoplus_\lambda M_\lambda$$

Even more so, we have seen that for any eigenvalue $\lambda \in K$ of $h\restriction_M$, $\bigoplus_{k\in\mathbb{Z}} M_{\lambda-2k}$ is a $\mathfrak{sl}_2(K)$-invariant subspace, which goes to show

$$M = \bigoplus_{k\in\mathbb{Z}} M_{\lambda-2k},$$

and the eigenvalues of h all have the form $\lambda - 2k$ for some k. By the same token, if a is the greatest $k \in \mathbb{Z}$ such that $V_{\lambda-2k} \neq 0$ and, likewise, b is the smallest $k \in \mathbb{Z}$ such that $V_{\lambda-2k} \neq 0$ then

$$M = \bigoplus_{\substack{k\in\mathbb{Z} \\ a\leqslant k\leqslant b}} M_{\lambda-2k}$$

The eigenvalues of h thus form an unbroken string

$$\ldots, \lambda-4, \lambda-2, \lambda, \lambda+2, \lambda+4, \ldots$$

around λ. Our main objective is to show M is determined by this string of eigenvalues. To do so, we suppose without any loss in generality that λ is the right-most eigenvalue of h, fix some nonzero $m \in M_\lambda$ and consider the set $\{m, f\cdot m, f^2\cdot m, \ldots\}$.

Proposition 3.1. *The set $\{m, f\cdot m, f^2\cdot m, \ldots\}$ is a basis for M. In addition, the action of $\mathfrak{sl}_2(K)$ on M is given by the formulas*

$$f^k\cdot m \xmapsto{e} k(\lambda+1-k)f^{k-1}\cdot m \quad f^k\cdot m \xmapsto{f} f^{k+1}\cdot m \quad f^k\cdot m \xmapsto{h} (\lambda-2k)f^k\cdot m \tag{3.1}$$

Proof. First of all, notice $f^k\cdot m$ lies in $M_{\lambda-2k}$, so that $\{m, f\cdot m, f^2\cdot m, \ldots\}$ is a set of linearly independent vectors. Hence it suffices to show $M = K\langle m, f\cdot m, f^2\cdot m, \ldots\rangle$, which in light of the fact that M is simple is the same as showing $K\langle m, f\cdot m, f^2\cdot m, \ldots\rangle$ is invariant under the action of $\mathfrak{sl}_2(K)$.

The fact that $h\cdot(f^k\cdot m) \in K\langle m, f\cdot m, f^2\cdot m, \ldots\rangle$ follows immediately from our previous assertion that $f^k\cdot m \in M_{\lambda-2k}$ – indeed, $h\cdot(f^k\cdot m) = (\lambda-2k)f^k\cdot m \in K\langle m, f\cdot m, f^2\cdot m, \ldots\rangle$, which also goes to show one of the formulas in (3.1). Seeing $e\cdot(f^k\cdot m) \in K\langle m, f\cdot m, f^2\cdot m, \ldots\rangle$ is a bit more complex. Clearly,

$$\begin{aligned} e\cdot(f\cdot m) &= h\cdot m + f\cdot(e\cdot m) \\ \text{(since λ is the right-most eigenvalue)} &= h\cdot m + f\cdot 0 \\ &= \lambda m \end{aligned}$$

Next we compute

$$\begin{aligned} e\cdot(f^2\cdot m) &= (h+fe)\cdot(f\cdot m) \\ &= h\cdot(f\cdot m) + f\cdot(\lambda m) \\ &= 2(\lambda-1)f\cdot m \end{aligned}$$

The pattern is starting to become clear: e sends $f^k\cdot m$ to a multiple of $f^{k-1}\cdot m$. Explicitly, it is not hard to check by induction that

$$e\cdot(f^k\cdot m) = k(\lambda+1-k)\cdot f^{k-1}m,$$

which which is the first formula of (3.1). ∎

The significance of Proposition 3.1 should be self-evident: we have just provided a complete description of the action of $\mathfrak{sl}_2(K)$ on M. In particular, this goes to show...

Corollary 3.2. *Every eigenspace of the action of h on M is 1-dimensional.*

Proof. It suffices to note $\{m, f \cdot m, f^2 \cdot m, \ldots\}$ is a basis for M consisting of eigenvalues of h and whose only element in $M_{\lambda-2k}$ is $f^k \cdot m$. ∎

Corollary 3.3. *The eigenvalues of h in M form a symmetric, unbroken string of integers separated by intervals of length 2 whose right-most value is* $\dim M - 1$.

Proof. If f^r is the lowest power of f that annihilates m, it follows from the formulas in (3.1) that

$$0 = e \cdot 0 = e \cdot (f^r \cdot m) = r(\lambda + 1 - r) f^{r-1} \cdot m$$

This implies $\lambda + 1 - r = 0$ – i.e. $\lambda = r - 1 \in \mathbb{Z}$. Now since $\{m, f \cdot m, f^2 \cdot m, \ldots, f^{r-1} \cdot m\}$ is a basis for M, $r = \dim V$. Hence if $\lambda = \dim V - 1$ then the eigenvalues of h are

$$\ldots, \lambda - 6, \lambda - 4, \lambda - 2, \lambda$$

To see that this string is symmetric around 0, simply note that the left-most eigenvalue of h is precisely $\lambda - 2(r-1) = -\lambda$. ∎

Visually, the situation it thus

$$\begin{tikzcd}
M_{-\lambda} \arrow[r, bend left, "e"] & M_{-\lambda+2} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & M_{-\lambda+4} \arrow[l, bend left, "f"] \arrow[r, bend left] & \cdots \arrow[l, bend left] \arrow[r, bend left] & M_{\lambda-4} \arrow[l, bend left] \arrow[r, bend left, "e"] & M_{\lambda-2} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & M_{\lambda} \arrow[l, bend left, "f"]
\end{tikzcd}$$

Corollary 3.3 can be used to find the eigenvalues of the action of h on an arbitrary finite-dimensional $\mathfrak{sl}_2(K)$-module. Namely, if M and N are $\mathfrak{sl}_2(K)$-modules, $m \in M_\mu$ and $n \in N_\mu$ then by computing

$$h \cdot (m + n) = h \cdot m + h \cdot n = \mu(m + n)$$

we can see that $(M \oplus N)_\mu = M_\mu + N_\mu$. Hence the set of eigenvalues of h in a $\mathfrak{sl}_2(K)$-module M is the union of the sets of eigenvalues in its simple components, and the corresponding eigenspaces are the direct sums of the eigenspaces of such simple components.

In particular, if the eigenvalues of M all have the same parity – i.e. they are either all even integers or all odd integers – and the dimension of each eigenspace is no greater than 1 then M must be simple, for if $N, L \subseteq M$ are submodules with $M = N \oplus L$ then either $N_\lambda = 0$ for all λ or $L_\lambda = 0$ for all $\lambda \in \mathfrak{h}^*$. To conclude our analysis all it is left is to show that for each $\lambda \in \mathbb{Z}$ with $\lambda \geqslant 0$ there is some finite-dimensional simple M whose highest weight is λ. Surprisingly, we have already encountered such a M.

Theorem 3.4. *For each $\lambda \geqslant 0$, $\lambda \in \mathbb{Z}$, there exists a unique simple $\mathfrak{sl}_2(K)$-module whose left-most eigenvalue of h is λ.*

Proof. Let $M = K[x, y]^{(\lambda)}$ be the $\mathfrak{sl}_2(K)$-module of homogeneous polynomials of degree λ in two variables, as in Example 1.54. A simple calculation shows $M_{n-2k} = Kx^{\lambda-k}y^k$ for $k = 0, \ldots, \lambda$ and $M_\mu = 0$ otherwise. In particular, the right-most eigenvalue of M is λ. Alternatively, one can readily check that if K^2 is the natural $\mathfrak{sl}_2(K)$-module, then $M = \operatorname{Sym}^\lambda K^2$ satisfies the relations of (3.1). Indeed, the map

$$\begin{aligned} K[x,y]^{(\lambda)} &\longrightarrow \operatorname{Sym}^\lambda K^2 \\ x^k y^\ell &\longmapsto e_1^k \cdot e_2^\ell \end{aligned}$$

is an isomorphism.

Either way, by the previous observation that a finite-dimensional $\mathfrak{sl}_2(K)$-module whose eigenvalues all have the same parity and whose corresponding eigenspace are all 1-dimensional must be simple, M is simple. As for the uniqueness of M, it suffices to notice that if N is a finite-dimensional simple $\mathfrak{sl}_2(K)$-module with right-most eigenvalue λ and $n \in N_\lambda$ is nonzero then relations (3.1) imply the map

$$\begin{aligned} M &\longrightarrow N \\ f^k \cdot m &\longmapsto f^k \cdot n \end{aligned}$$

is an isomorphism – this is, in effect, precisely how the isomorphism $K[x,y]^{(\lambda)} \xrightarrow{\sim} \operatorname{Sym}^\lambda K^2$ was constructed. ■

Our initial gamble of studying the eigenvalues of h may have seemed arbitrary at first, but it payed off: we have *completely* described *all* simple $\mathfrak{sl}_2(K)$-modules. It is not yet clear, however, if any of this can be adapted to a general setting. In the following section we shall double down on our gamble by trying to reproduce some of these results for $\mathfrak{sl}_3(K)$, hoping this will somehow lead us to a general solution. In the process of doing so we will find some important clues on why h was a sure bet and the race was fixed all along.

3.1 Representations of $\mathfrak{sl}_{2+1}(K)$

The study of representations of $\mathfrak{sl}_2(K)$ reminds me of the difference between the derivative of a function $\mathbb{R} \longrightarrow \mathbb{R}$ and that of a smooth map between manifolds: it is a simpler case of something greater, but in some sense it is too simple of a case, and the intuition we acquire from it can be a bit misleading in regards to the general setting. For instance, I distinctly remember my Calculus I teacher telling the class “the derivative of the composition of two functions is not the composition of their derivatives” – which is, of course, the *correct* formulation of the chain rule in the context of smooth manifolds.

The same applies to $\mathfrak{sl}_2(K)$. It is a simple and beautiful example, but unfortunately the general picture, modules of arbitrary semisimple algebras, lacks its simplicity. The general purpose of this section is to investigate to which extent the framework we developed for $\mathfrak{sl}_2(K)$ can be generalized to other semisimple Lie algebras. Of course, the algebra $\mathfrak{sl}_3(K)$ stands as a natural candidate for potential generalizations: $\mathfrak{sl}_3(K) = \mathfrak{sl}_{2+1}(K)$ after all.

Our approach is very straightforward: we will fix some simple $\mathfrak{sl}_3(K)$-module M and proceed step by step, at each point asking ourselves how we could possibly adapt the framework we laid out for $\mathfrak{sl}_2(K)$. The first obvious question is one we have already asked ourselves: why h? More specifically, why did we choose to study its eigenvalues and is there an analogue of h in $\mathfrak{sl}_3(K)$?

The answer to the former question is one we will discuss at length in the next chapter, but for now we note that perhaps the most fundamental property of h is that *there exists an eigenvector m of h that is annihilated by e* – that being the generator of the right-most eigenspace of h. This was instrumental to our explicit description of the simple $\mathfrak{sl}_2(K)$-modules culminating in Theorem 3.4.

Our first task is to find some analogue of h in $\mathfrak{sl}_3(K)$, but it is still unclear what exactly we are looking for. We could say we are looking for an element of M that is annihilated by some analogue of e, but the meaning of *some analogue of e* is again unclear. In fact, as we shall see, no such analogue exists and neither does such element. Instead, the actual way to proceed is to consider the subalgebra

$$\mathfrak{h} = \left\{ X \in \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix} : \mathrm{Tr}(X) = 0 \right\}$$

The choice of $\mathfrak{h}$ may seem like an odd choice at the moment, but the point is we will later show that there exists some $m \in M$ that is simultaneously an eigenvector of each $H \in \mathfrak{h}$ and annihilated by half of the remaining elements of $\mathfrak{sl}_3(K)$. This is exactly analogous to the situation we found in $\mathfrak{sl}_2(K)$: h corresponds to the subalgebra $\mathfrak{h}$, and the eigenvalues of h in turn correspond to linear functions $\lambda : \mathfrak{h} \longrightarrow k$ such that $H \cdot m = \lambda(H)m$ for each $H \in \mathfrak{h}$ and some nonzero $m \in M$. We call such functionals λ *eigenvalues of* $\mathfrak{h}$, and we say *m is an eigenvector of* $\mathfrak{h}$.

Once again, we will pay special attention to the eigenvalue decomposition

$$M = \bigoplus_{\lambda} M_\lambda \tag{3.2}$$

where λ ranges over all eigenvalues of $\mathfrak{h}$ and $M_\lambda = \{m \in M : H \cdot m = \lambda(H)m, \forall H \in \mathfrak{h}\}$. We should note that the fact that (3.2) holds is not at all obvious. This is because in general M_λ is not the eigenspace associated with an eigenvalue of any particular operator $H \in \mathfrak{h}$, but instead the eigenspace of the action of the entire algebra $\mathfrak{h}$. Fortunately for us, (3.2) always holds, but we will postpone its proof to the next chapter.

Next we turn our attention to the remaining elements of $\mathfrak{sl}_3(K)$. In our analysis of $\mathfrak{sl}_2(K)$ we saw that the eigenvalues of h differed from one another by multiples of 2. A possible way to interpret this is to say *the eigenvalues of h differ from one another by integral linear combinations of the eigenvalues of the adjoint action of h*. In English, since

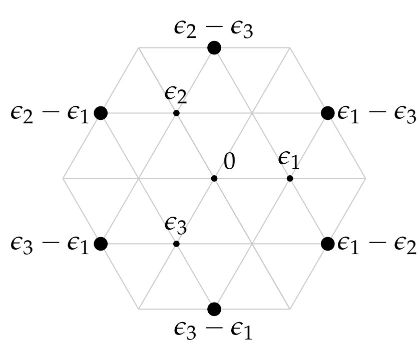
$$\mathrm{ad}(h)e = 2e \qquad \mathrm{ad}(h)f = -2f \qquad \mathrm{ad}(h)h = 0,$$

the eigenvalues of the adjoint actions of h are 0 and ± 2, and the eigenvalues of the action of h on a simple $\mathfrak{sl}_2(K)$-module differ from one another by integral multiples of 2.

In the case of $\mathfrak{sl}_3(K)$, a simple calculation shows that if $[H, X]$ is scalar multiple of X for all $H \in \mathfrak{h}$ then all but one entry of X are zero. Hence the eigenvectors of the adjoint action of $\mathfrak{h}$ are E_{ij} and its eigenvalues are $\epsilon_i - \epsilon_j$, where

$$\epsilon_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i$$

Visually we may draw

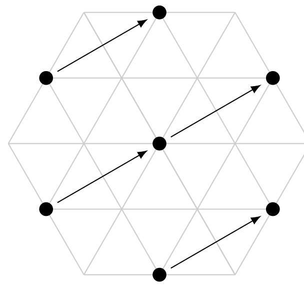


If we denote the eigenspace of the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3(K)$ associated to α by $\mathfrak{sl}_3(K)_\alpha$ and fix some $X \in \mathfrak{sl}_3(K)_\alpha$, $H \in \mathfrak{h}$ and $m \in M_\lambda$ then

$$\begin{aligned} H \cdot (X \cdot m) &= X \cdot (H \cdot m) + [H, X] \cdot m \\ &= X \cdot (\lambda(H)m) + \alpha(H) X \cdot m \\ &= (\lambda + \alpha)(H) X \cdot m \end{aligned}$$

so that X carries m to $M_{\lambda+\alpha}$. In other words, *$\mathfrak{sl}_3(k)_\alpha$ acts on M by translating vectors between eigenspaces*.

For instance $\mathfrak{sl}_3(K)_{\epsilon_1-\epsilon_3}$ will act on the adjoint $\mathfrak{sl}_3(K)$-modules via



This is again entirely analogous to the situation we observed in $\mathfrak{sl}_2(K)$. In fact, we may once more conclude...

Theorem 3.5. *The eigenvalues of the action of $\mathfrak{h}$ on a simple $\mathfrak{sl}_3(K)$-module M differ from one another by integral linear combinations of the eigenvalues $\epsilon_i - \epsilon_j$ of the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3(K)$.*

Proof. This proof goes exactly as that of the analogous statement for $\mathfrak{sl}_2(K)$: it suffices to note that if we fix some eigenvalue λ of $\mathfrak{h}$ and let i and j vary then

$$\bigoplus_{ij} M_{\lambda+\epsilon_i-\epsilon_j}$$

is an invariant subspace of M. ■

To avoid confusion we better introduce some notation to differentiate between eigenvalues of the action of $\mathfrak{h}$ on M and eigenvalues of the adjoint action of $\mathfrak{h}$.

Definition 3.6. Given a $\mathfrak{sl}_3(K)$-module M, we will call the *nonzero* eigenvalues of the action of $\mathfrak{h}$ on M *weights of M*. As you might have guessed, we will correspondingly refer to eigenvectors and eigenspaces of a given weight by *weight vectors* and *weight spaces*.

It is clear from our previous discussion that the weights of the adjoint $\mathfrak{sl}_3(K)$-module deserve some special attention.

Definition 3.7. The weights of the adjoint $\mathfrak{sl}_3(K)$-module are called *roots of $\mathfrak{sl}_3(K)$*. Once again, the expressions *root vector* and *root space* are self-explanatory.

Theorem 3.5 can thus be restated as...

Definition 3.8. The lattice $Q = \mathbb{Z}\langle \epsilon_i - \epsilon_j : i, j = 1, 2, 3\rangle$ is called *the root lattice of $\mathfrak{sl}_3(K)$*.

Corollary 3.9. *The weights of a simple $\mathfrak{sl}_3(K)$-module M are all congruent modulo the root lattice Q. In other words, the weights of M all lie in a single Q-coset $\xi \in \mathfrak{h}^*/Q$.*

At this point we could keep playing the tedious game of reproducing the arguments from the previous section in the context of $\mathfrak{sl}_3(K)$. However, it is more profitable to use our knowledge of $\mathfrak{sl}_2(K)$-modules instead. Notice that the canonical inclusion $\mathfrak{gl}_2(K) \longrightarrow \mathfrak{gl}_3(K)$ – as described in

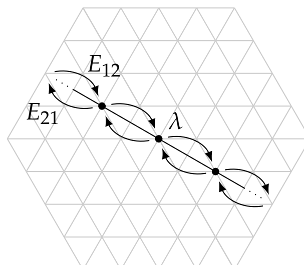
Example 1.4 – restricts to an injective homomorphism $\mathfrak{sl}_2(K) \longrightarrow \mathfrak{sl}_3(K)$. In other words, $\mathfrak{sl}_2(K)$ is isomorphic to the image $\mathfrak{s}_{12} = K\langle E_{12}, E_{21}, [E_{12}, E_{21}]\rangle \subseteq \mathfrak{sl}_3(K)$ of the inclusion $\mathfrak{sl}_2(K) \longrightarrow \mathfrak{sl}_3(K)$. We may thus regard M as a $\mathfrak{sl}_2(K)$-module by restricting to $\mathfrak{s}_{12}$.

Our first observation is that, since the root spaces act by translation, the subspace

$$\bigoplus_{k\in\mathbb{Z}} M_{\lambda-k(\epsilon_1-\epsilon_2)},$$

must be invariant under the action of E_{12} and E_{21} for all $\lambda \in \mathfrak{h}^*$. This goes to show $\bigoplus_k M_{\lambda-k(\epsilon_1-\epsilon_2)}$ is a $\mathfrak{sl}_2(K)$-submodule of M for all weights λ of M. Furthermore, one can easily see that the eigenspace of the action of h on $\bigoplus_{k\in\mathbb{Z}} M_{\lambda-k(\epsilon_1-\epsilon_2)}$ associated with the eigenvalue $\lambda(H) - 2k$ is precisely the weight space $M_{\lambda-k(\epsilon_2-\epsilon_1)}$.

Visually,



In general, we find...

Proposition 3.10. *Given $i < j$, the subalgebra $\mathfrak{s}_{ij} = K\langle E_{ij}, E_{ji}, [E_{ij}, E_{ji}]\rangle$ is isomorphic to $\mathfrak{sl}_2(K)$. In addition, given a weight $\lambda \in \mathfrak{h}^*$ of M, the space*

$$N = \bigoplus_{k\in\mathbb{Z}} M_{\lambda-k(\epsilon_i-\epsilon_j)}$$

is invariant under the action of $\mathfrak{s}_{ij}$ and

$$M_{\lambda-k(\epsilon_i-\epsilon_j)} = N_{\lambda([E_{ij},E_{ji}])-2k}$$

Proof. In effect, if $i \neq k \neq j$ then $\mathfrak{s}_{ij}$ is the subalgebra of matrices whose k-th row and k-th column are nil. For instance, if $i = 1$ and $j = 3$ then

$$\mathfrak{s}_{13} = \begin{pmatrix} K & 0 & K \\ 0 & 0 & 0 \\ K & 0 & K \end{pmatrix} \cap \mathfrak{sl}_3(K)$$

In this case, the map

$$\begin{aligned} \mathfrak{s}_{13} &\longrightarrow \mathfrak{sl}_2(K) \\ \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & -a \end{pmatrix} &\longmapsto \begin{pmatrix} a & \cancel{0} & b \\ \cancel{0} & \cancel{0} & \cancel{0} \\ c & \cancel{0} & -a \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \end{aligned}$$

is an isomorphism of Lie algebras. In general, the map

$$\begin{aligned} \mathfrak{s}_{ij} &\longrightarrow \mathfrak{sl}_2(K) \\ E_{ij} &\longmapsto e \\ E_{ji} &\longmapsto f \\ [E_{ij}, E_{ji}] &\longmapsto h \end{aligned}$$

which "erases the k-th row and the k-th column" of a matrix is an isomorphism.

To see that N is invariant under the action of $\mathfrak{s}_{ij}$, it suffices to notice E_{ij} and E_{ji} map $m \in M_{\lambda - k(\epsilon_i - \epsilon_j)}$ to $E_{ij} \cdot m \in M_{\lambda-(k-1)(\epsilon_i-\epsilon_j)}$ and $E_{ji} \cdot m \in M_{\lambda-(k+1)(\epsilon_i-\epsilon_j)}$, respectively. Moreover,

$$(\lambda - k(\epsilon_i - \epsilon_j))([E_{ij}, E_{ji}]) = \lambda([E_{ij}, E_{ji}]) - k(1-(-1)) = \lambda([E_{ij}, E_{ji}]) - 2k,$$

which goes to show $M_{\lambda-k(\epsilon_i-\epsilon_j)} \subseteq N_{\lambda([E_{ij},E_{ji}])-2k}$. On the other hand, if we suppose $0 < \dim M_{\lambda-k(\epsilon_i-\epsilon_j)} < \dim N_{\lambda([E_{ij},E_{ji}])-2k}$ for some k we arrive at

$$\dim N = \sum_k \dim M_{\lambda-k(\epsilon_i-\epsilon_j)} < \sum_k \dim N_{\lambda([E_{ij},E_{ji}])-2k} = \dim N,$$

a contradiction. ■

As a first consequence of this, we show...

Definition 3.11. The lattice $P = \mathbb{Z}\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle$ is called *the weight lattice of* $\mathfrak{sl}_3(K)$.

Corollary 3.12. *Every weight λ of M lies in the weight lattice P.*

Proof. It suffices to note $\lambda([E_{ij}, E_{ji}])$ is an eigenvalue of h in a finite-dimensional $\mathfrak{sl}_2(K)$-module, so it must be an integer. Now since

$$\lambda \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} = \lambda \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -b \end{pmatrix} = a\lambda([E_{13}, E_{31}]) + b\lambda([E_{23}, E_{32}]),$$

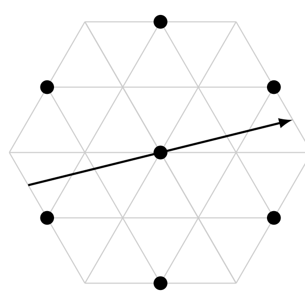
which is to say $\lambda = \lambda([E_{13}, E_{31}])\epsilon_1 + \lambda([E_{23}, E_{32}])\epsilon_2 \in P$. ■

There is a clear parallel between the case of $\mathfrak{sl}_3(K)$ and that of $\mathfrak{sl}_2(K)$, where we observed that the eigenvalues of the action of h all lied in the lattice $P = \mathbb{Z}$ and were congruent modulo the sublattice $Q = 2\mathbb{Z}$.

Among other things, this last result goes to show that the diagrams we have been drawing are in fact consistent with the theory we have developed. Namely, since all weights lie in the rational span of $\{\epsilon_1, \epsilon_2, \epsilon_3\}$, we may as well draw them in the Cartesian plane. In fact, the attentive reader may notice that $\kappa(E_{12}, E_{23}) = -1/2$, so that the angle – with respect to the Killing form κ – between the root vectors E_{12} and E_{23} is precisely the same as the angle between the points representing their roots $\epsilon_1 - \epsilon_2$ and $\epsilon_2 - \epsilon_3$ in the Cartesian plane. Since $\epsilon_1 - \epsilon_2$ and $\epsilon_2 - \epsilon_3$ span $\mathfrak{h}^*$, this implies the diagrams we've been drawing are given by an isometry $\mathbb{Q}P \xrightarrow{\sim} \mathbb{Q}^2$, where $\mathbb{Q}P$ is endowed with the bilinear form defined by $(\epsilon_i - \epsilon_j, \epsilon_k - \alpha_\ell) \longmapsto \kappa(E_{ij}, E_{k\ell})$ – which we denote by κ as well.

To proceed we once more refer to the previously established framework: next we saw that the eigenvalues of h form an unbroken string of integers symmetric around 0. To prove this we analyzed the right-most eigenvalues of h and their eigenvectors, providing an explicit description of the simple $\mathfrak{sl}_2(K)$-modules in terms of these vectors. We may reproduce these steps in the context of $\mathfrak{sl}_3(K)$ by fixing a direction in the plane an considering the weight lying the furthest in that direction.

For instance, let's say we fix the direction

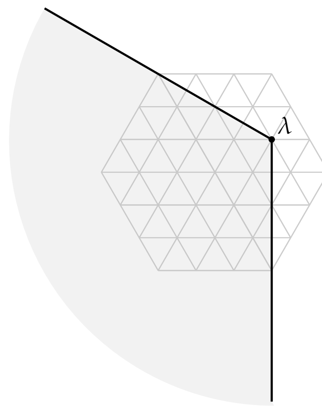


and let λ be the weight lying the furthest in this direction.

Its easy to see what we mean intuitively by looking at the previous picture, but its precise meaning is still allusive. Formally this means we will choose a linear functional $f : \mathbb{Q}P \longrightarrow \mathbb{Q}$ and pick the weight that maximizes f. To avoid any ambiguity we should choose the direction of a line irrational with respect to the root lattice Q – for if f is not irrational there may be multiple choices the "weight lying the furthest" along this direction.

Definition 3.13. We say that a root α is positive if $f(\alpha) > 0$ – i.e. if it lies to the right of the direction we chose. Otherwise we say α is negative. Notice that $f(\alpha) \neq 0$ since by definition $\alpha \neq 0$ and f is irrational with respect to the lattice Q.

The next observation we make is that all others weights of M must lie in a sort of $\frac{1}{3}$-cone with apex at λ, as shown in



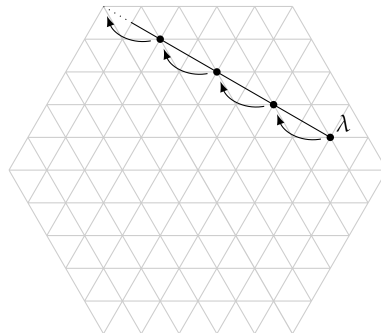
Indeed, if this is not the case then, by definition, λ is not the weight placed the furthest in the direction we chose. Given our previous assertion that the root spaces of $\mathfrak{sl}_3(K)$ act on the weight spaces of M via translation, this implies that E_{12}, E_{13} and E_{23} all annihilate M_λ, or otherwise one of $M_{\lambda+\epsilon_1-\epsilon_2}$, $M_{\lambda+\epsilon_1-\epsilon_3}$ and $M_{\lambda+\epsilon_2-\epsilon_3}$ would be nonzero – which contradicts the hypothesis that λ lies the furthest in the direction we chose. In other words...

Proposition 3.14. *There is a weight vector $m \in M$ that is annihilated by all positive root spaces of $\mathfrak{sl}_3(K)$.*

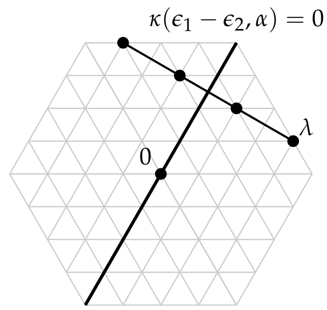
Proof. It suffices to note that the positive roots of $\mathfrak{sl}_3(K)$ are precisely $\epsilon_1 - \epsilon_2$, $\epsilon_1 - \epsilon_3$ and $\epsilon_2 - \epsilon_3$, with root vectors E_{12}, E_{13} and E_{23}, respectively. ■

We call λ *the highest weight of M*, and we call any nonzero $m \in M_\lambda$ *a highest weight vector*. Going back to the case of $\mathfrak{sl}_2(K)$, we then constructed an explicit basis for our simple module in terms of a highest weight vector, which allowed us to provide an explicit description of the action of $\mathfrak{sl}_2(K)$ in terms of its standard basis, and finally we concluded that the eigenvalues of h must be symmetrical around 0. An analogous procedure could be implemented for $\mathfrak{sl}_3(K)$ – and indeed that's what we will do later down the line – but instead we would like to focus on the problem of finding the weights of M in the first place.

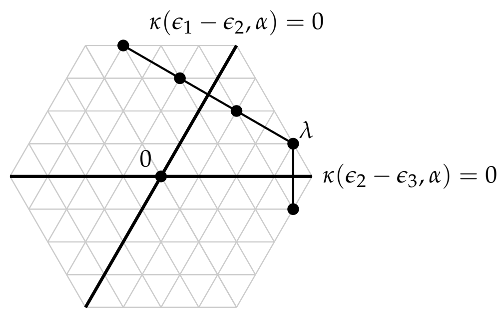
We will start out by trying to understand the weights in the boundary of previously drawn cone. As we have just seen, we can get to other weight spaces from M_λ by successively applying E_{21}.



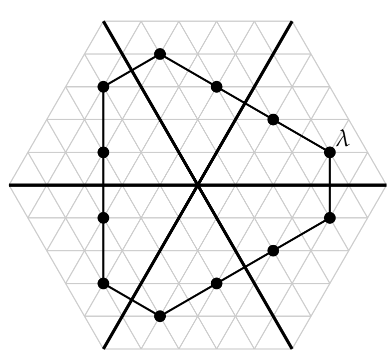
Notice that $\lambda([E_{12}, E_{21}]) \in \mathbb{Z}$ is the right-most eigenvalue of the $\mathfrak{sl}_2(K)$-module $\bigoplus_{k \in \mathbb{Z}} M_{\lambda - k(\epsilon_1 - \epsilon_2)}$. In particular, $\lambda([E_{12}, E_{21}])$ must be positive. In addition, since the eigenspace of the eigenvalue $\lambda([E_{12}, E_{21}]) - 2k$ of the action of h on $\bigoplus_{k \in \mathbb{N}} M_{\lambda - k(\epsilon_1 - \epsilon_2)}$ is $M_{\lambda - k(\epsilon_1 - \epsilon_2)}$, the weights of M appearing the string $\lambda, \lambda + (\epsilon_1 - \epsilon_2), \ldots, \lambda + k(\epsilon_1 - \epsilon_2), \ldots$ must be symmetric with respect to the line $\kappa(\epsilon_1 - \epsilon_2, \alpha) = 0$. The picture is thus



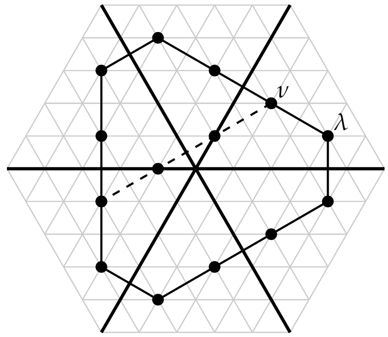
We could apply this same argument to the subspace $\bigoplus_k M_{\lambda - k(\epsilon_2 - \epsilon_3)}$, so that the weights in this subspace must be symmetric with respect to the line $\kappa(\epsilon_2 - \epsilon_3, \alpha) = 0$. The picture is now



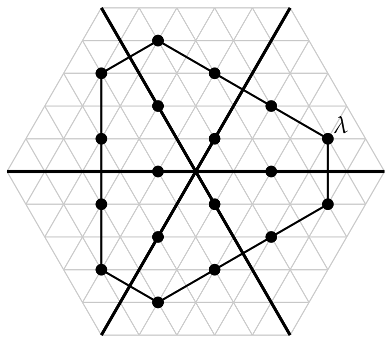
Needless to say, we could keep applying this method to the weights at the ends of our string, arriving at



We claim all dots μ lying inside the hexagon we have drawn must also be weights – i.e. $M_\mu \neq 0$. Indeed, by applying the same argument to an arbitrary weight ν in the boundary of the hexagon we get a $\mathfrak{sl}_2(K)$-module whose weights correspond to weights of M lying in a string inside the hexagon, and whose right-most weight is precisely the weight of M we started with.



By construction, ν corresponds to the right-most weight of a $\mathfrak{sl}_2(K)$-module, so that all dots lying on the dashed string must occur in $\mathfrak{sl}_2(K)$-module. Hence they must also be weights of M. The final picture is thus



This final picture is known as *the weight diagram of M*. Finally. . .

Theorem 3.15. *The weights of M are precisely the elements of the weight lattice P congruent to λ module the sublattice Q and lying inside hexagon with vertices the images of λ under the group generated by reflections across the lines $\kappa(\epsilon_i - \epsilon_j, \alpha) = 0$.*

Having found all of the weights of M, the only thing we are missing is an existence and uniqueness theorem analogous to Theorem 3.4. It is clear from the symmetries of the locus of weights found in Theorem 3.15 that if $\lambda \in P$ is the highest weight of some finite-dimensional simple $\mathfrak{sl}_3(K)$-module M then λ lies in the cone $\mathbb{N}\langle\epsilon_1, -\epsilon_3\rangle$. What's perhaps more surprising is the fact that this condition is sufficient for the existence of such a M. In other words, our next goal is establishing. . .

Definition 3.16. An element $\lambda \in P$ is called *dominant* if it lies in the cone $\mathbb{N}\langle\epsilon_1, -\epsilon_3\rangle$.

Theorem 3.17. *For each dominant $\lambda \in P$, there exists precisely one finite-dimensional simple $\mathfrak{sl}_3(K)$-module M whose highest weight is λ.*

To proceed further we once again refer to the approach we employed in the case of $\mathfrak{sl}_2(K)$: next we showed in Proposition 3.1 that any simple $\mathfrak{sl}_2(K)$-module is spanned by the images of its highest weight vector under f. A more abstract way of putting it is to say that a simple $\mathfrak{sl}_2(K)$-module M of is spanned by the images of its highest weight vector under successive applications of the action of half of the root spaces of $\mathfrak{sl}_2(K)$. The advantage of this alternative formulation is, of course, that the same holds for $\mathfrak{sl}_3(K)$. Specifically…

Proposition 3.18. *Given a simple $\mathfrak{sl}_3(K)$-module M and a highest weight vector $m \in M$, M is spanned by the images of m under successive applications of E_{21}, E_{31} and E_{32}.*

Proof. Given the fact M is simple, it suffices to show that the subspace N spanned by successive applications of E_{21}, E_{31} and E_{32} to m is stable under the action of $\mathfrak{sl}_3(K)$. In addition, since $[E_{21}, E_{31}] = [E_{31}, E_{32}] = 0$ and $[E_{21}, E_{32}] = -E_{31}$, all successive product of E_{21}, E_{31} and E_{32} in $\mathcal{U}(\mathfrak{sl}_3(K))$ can be written as $E_{21}^a E_{31}^b E_{31}^c$ for some a, b and c, so that N is spanned by the elements $E_{21}^a E_{31}^b E_{31}^c \cdot m$.

Recall that E_{ij} maps M_μ to $M_{\mu+\epsilon_i-\epsilon_j}$. In particular, $E_{21}^a E_{31}^b E_{31}^c \cdot m \in M_{\lambda-a(\epsilon_1-\epsilon_2)-b(\epsilon_1-\epsilon_3)-c(\epsilon_2-\epsilon_3)}$. In other words,

$$HE_{21}^a E_{31}^b E_{31}^c \cdot m = (\lambda - a(\epsilon_1 - \epsilon_2) - b(\epsilon_1 - \epsilon_3) - c(\epsilon_2 - \epsilon_3))(H) E_{21}^a E_{31}^b E_{31}^c \cdot m \in N$$

for all $H \in \mathfrak{h}$ and N is stable under the action of $\mathfrak{h}$. On the other hand, N is clearly stable under the action of E_{21}, E_{31} and E_{32}. All it is left is to show N is stable under the action of E_{12}, E_{13} and E_{23}.

We begin by analyzing the case of E_{12}. We have

$$\begin{aligned}
E_{12}E_{21}^a E_{31}^b E_{32}^c \cdot m &= ([E_{12}, E_{21}] + E_{21}E_{12})E_{21}^{a-1} E_{31}^b E_{32}^c \cdot m \\
&= E_{21}([E_{12}, E_{21}] + E_{21}E_{12})E_{21}^{a-2} E_{31}^b E_{32}^c \cdot m \\
&\quad + (\lambda - (a-1)(\epsilon_1 - \epsilon_2) - b(\epsilon_1 - \epsilon_3) - c(\epsilon_2 - \epsilon_3))([E_{12}, E_{21}])E_{21}^{a-1} E_{31}^b E_{32}^c \cdot m \\
&= E_{21}^2([E_{12}, E_{21}] + E_{21}E_{12})E_{21}^{a-3} E_{31}^b E_{32}^c \cdot m \\
&\quad + (\lambda - (a-1)(\epsilon_1 - \epsilon_2) - b(\epsilon_1 - \epsilon_3) - c(\epsilon_2 - \epsilon_3))([E_{12}, E_{21}])E_{21}^{a-1} E_{31}^b E_{32}^c \cdot m \\
&\quad + (\lambda - (a-2)(\epsilon_1 - \epsilon_2) - b(\epsilon_1 - \epsilon_3) - c(\epsilon_2 - \epsilon_3))([E_{12}, E_{21}])E_{21}^{a-2} E_{31}^b E_{32}^c \cdot m \\
&\vdots \\
&= E_{21}^a E_{12} E_{31}^b E_{32}^c \cdot m \\
&\quad + (\lambda - (a-1)(\epsilon_1 - \epsilon_2) - b(\epsilon_1 - \epsilon_3) - c(\epsilon_2 - \epsilon_3))([E_{12}, E_{21}])E_{21}^{a-1} E_{31}^b E_{32}^c \cdot m \\
&\quad + (\lambda - (a-2)(\epsilon_1 - \epsilon_2) - b(\epsilon_1 - \epsilon_3) - c(\epsilon_2 - \epsilon_3))([E_{12}, E_{21}])E_{21}^{a-2} E_{31}^b E_{32}^c \cdot m \\
&\quad \vdots \\
&\quad + (\lambda - (a-a)(\epsilon_1 - \epsilon_2) - b(\epsilon_1 - \epsilon_3) - c(\epsilon_2 - \epsilon_3))([E_{12}, E_{21}])E_{21}^{a-a} E_{31}^b E_{32}^c \cdot m
\end{aligned}$$

Since $(\lambda - (a-k)(\epsilon_1 - \epsilon_2) - b(\epsilon_1 - \epsilon_3) - c(\epsilon_2 - \epsilon_3))([E_{12}, E_{21}])E_{21}^{a-k} E_{31}^b E_{32}^c \cdot m \in N$ for all k, it suffices to show $E_{21}^a E_{12} E_{31}^b E_{32}^c \cdot m \in N$. But

$$\begin{aligned}
E_{12}E_{31}^b &= (E_{31}E_{12} - E_{32})E_{31}^{b-1} \\
&= E_{31}E_{12}E_{31}^{b-1} - E_{31}E_{32}E_{31}^{b-1} \\
&= E_{31}(E_{31}E_{12} - E_{32})E_{31}^{b-2} - E_{32}E_{31}^b, \\
&\vdots \\
&= E_{31}^b E_{12} - bE_{32}E_{31}^b
\end{aligned}$$

given $[E_{12}, E_{31}] = -E_{32}$ and $[E_{32}, E_{31}] = 0$. It then follows from the fact $E_{12} \cdot m = 0$ that

$$E_{21}^a E_{12} E_{31}^b E_{32}^c \cdot m = E_{21}^a E_{31}^b E_{32}^c E_{12} \cdot m - bE_{21}^a E_{31}^b E_{32}^{c+1} \cdot m = -bE_{21}^a E_{31}^b E_{32}^{c+1} \cdot m \in N,$$

given that E_{12} and E_{32} commute. Hence $E_{12} \cdot (E_{21}^a E_{31}^b E_{32}^c \cdot m) \in N$. Similarly,

$$E_{13} \cdot (E_{21}^a E_{31}^b E_{32}^c \cdot m), E_{23} \cdot (E_{21}^a E_{31}^b E_{32}^c \cdot m) \in N$$

■

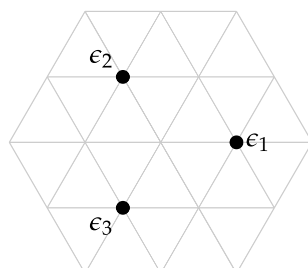
The same argument also goes to show...

Corollary 3.19. *Given a finite-dimensional $\mathfrak{sl}_3(K)$-module M with highest weight λ and $m \in M_\lambda$, the subspace spanned by successive applications of E_{21}, E_{31} and E_{32} to m is a simple submodule whose highest weight is λ.*

This is very interesting to us since it implies that finding *any* finite-dimensional module whose highest weight is λ is enough for establishing the "existence" part of Theorem 3.17. Moreover, constructing such a module turns out to be quite simple.

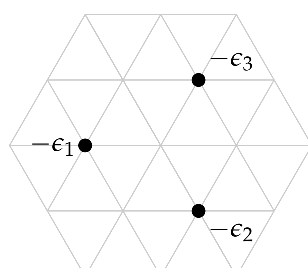
Proof of existence. Take $\lambda = k\epsilon_1 - \ell\epsilon_3 \in P$ with $k, \ell \geqslant 0$, so that λ is dominant. Consider the natural $\mathfrak{sl}_3(K)$-module K^3. We claim that the highest weight of $\operatorname{Sym}^k K^3 \otimes \operatorname{Sym}^\ell (K^3)^*$ is λ.

First of all, notice that the weight vector of K^3 are the canonical basis elements e_1, e_2 and e_3, whose corresponding weights are ϵ_1, ϵ_2 and ϵ_3 respectively. Hence the weight diagram of K^3 is



and ϵ_1 is the highest weight of K^3.

On the one hand, if $\{f_1, f_2, f_3\}$ is the dual basis for $\{e_1, e_2, e_3\}$ then $H \cdot f_i = -\epsilon_i(H) f_i$ for each $H \in \mathfrak{h}$, so that the weights of $(K^3)^*$ are precisely the opposites of the weights of K^3. In other words,



is the weight diagram of $(K^3)^*$ and ϵ_3 is the highest weight of $(K^3)^*$.

On the other hand if we fix two $\mathfrak{sl}_3(K)$-modules N and L, by computing

$$\begin{aligned} H \cdot (n \otimes l) &= H \cdot n \otimes l + n \otimes H \cdot l \\ &= \lambda(H) n \otimes l + n \otimes \mu(H) l \\ &= (\lambda + \mu)(H)\,(n \otimes l) \end{aligned}$$

for each $H \in \mathfrak{h}$, $n \in N_\lambda$ and $l \in L_\mu$ we can see that the weights of $N \otimes L$ are precisely the sums of the weights of N with the weights of L.

This implies that the highest weights of $\operatorname{Sym}^k K^3$ and $\operatorname{Sym}^\ell (K^3)^*$ are $k\epsilon_1$ and $-\ell\epsilon_3$ respectively – with highest weight vectors e_1^k and f_3^ℓ. Furthermore, by the same token the highest weight of $\operatorname{Sym}^k K^3 \otimes \operatorname{Sym}^\ell (K^3)^*$ must be $\lambda = ke_1 - \ell e_3$ – with highest weight vector $e_1^k \otimes f_3^\ell$. ■

The "uniqueness" part of Theorem 3.17 is even simpler than that.

Proof of uniqueness. Let M and N be two simple $\mathfrak{sl}_3(K)$-modules with highest weight λ. By Theorem 3.15, the weights of M are precisely the same as those of N.

Now by computing

$$H \cdot (m+n) = H \cdot m + H \cdot n = \mu(H)m + \mu(H)n = \mu(H)(m+n)$$

for each $H \in \mathfrak{h}$, $m \in M_\mu$ and $n \in N_\mu$, we can see that the weights of $M \oplus N$ are same as those of M and N. Hence the highest weight of $M \oplus N$ is λ – with highest weight vectors given by the sum of highest weight vectors of M and N.

Fix some $m \in M_\lambda$ and $n \in N_\lambda$ and consider the submodule $L = \mathcal{U}(\mathfrak{sl}_3(K)) \cdot m + n \subseteq M \oplus N$ generated by $m+n$. Since $m+n$ is a highest weight of $M \oplus N$, it follows from corollary 3.19 that L is simple. The projection maps $\pi_1 : L \longrightarrow M$, $\pi_2 : L \longrightarrow N$, being nonzero homomorphism between simple $\mathfrak{sl}_3(K)$-modules, must be isomorphism. Finally,

$$M \cong L \cong N$$

■

We have been very successful in our pursue for a classification of the simple modules of $\mathfrak{sl}_2(K)$ and $\mathfrak{sl}_3(K)$, but so far we have mostly postponed the discussion on the motivation behind our methods. In particular, we did not explain why we chose h and $\mathfrak{h}$, and neither why we chose to look at their eigenvalues. Apart from the obvious fact we already knew it would work a priory, why did we do all that? In the following chapter we will attempt to answer this question by looking at what we did in the last chapter through more abstract lenses and studying the representations of an arbitrary finite-dimensional semisimple Lie algebra $\mathfrak{g}$.

Chapter 4

Finite-Dimensional Simple Modules

In this chapter we classify the finite-dimensional simple $\mathfrak{g}$-modules for a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over K. At the heart of our analysis of $\mathfrak{sl}_2(K)$ and $\mathfrak{sl}_3(K)$ was the decision to consider the eigenspace decomposition

$$M = \bigoplus_{\lambda} M_\lambda \tag{4.1}$$

This was simple enough to do in the case of $\mathfrak{sl}_2(K)$, but the rational behind it and the reason why equation (4.1) holds are harder to explain in the case of $\mathfrak{sl}_3(K)$. The eigenspace decomposition associated with an operator $M \longrightarrow M$ is a very well-known tool, and readers familiarized with basic concepts of linear algebra should be used to this type of argument. On the other hand, the eigenspace decomposition of M with respect to the action of an arbitrary subalgebra $\mathfrak{h} \subseteq \mathfrak{gl}(M)$ is neither well-known nor does it hold in general: as indicated in the previous chapter, it may very well be that

$$\bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \subsetneq M$$

We should note, however, that these two cases are not as different as they may sound at first glance. Specifically, we can regard the eigenspace decomposition of a $\mathfrak{sl}_2(K)$-module M with respect to the eigenvalues of the action of h as the eigenvalue decomposition of M with respect to the action of the subalgebra $\mathfrak{h} = Kh \subseteq \mathfrak{sl}_2(K)$. Furthermore, in both cases $\mathfrak{h} \subseteq \mathfrak{sl}_n(K)$ is the subalgebra of diagonal matrices, which is Abelian. The fundamental difference between these two cases is thus the fact that $\dim \mathfrak{h} = 1$ for $\mathfrak{h} \subseteq \mathfrak{sl}_2(K)$ while $\dim \mathfrak{h} > 1$ for $\mathfrak{h} \subseteq \mathfrak{sl}_3(K)$. The question then is: why did we choose $\mathfrak{h}$ with $\dim \mathfrak{h} > 1$ for $\mathfrak{sl}_3(K)$?

The rational behind fixing an Abelian subalgebra $\mathfrak{h}$ is a simple one: we have seen in the previous chapter that representations of Abelian algebras are generally much simpler to understand than the general case. Thus it make sense to decompose a given $\mathfrak{g}$-module M of into subspaces invariant under the action of $\mathfrak{h}$, and then analyze how the remaining elements of $\mathfrak{g}$ act on these subspaces. The bigger $\mathfrak{h}$ is, the simpler our problem gets, because there are fewer elements outside of $\mathfrak{h}$ left to analyze.

Hence we are generally interested in maximal Abelian subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$, which leads us to the following definition.

Definition 4.1. A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called *a Cartan subalgebra of* $\mathfrak{g}$ if is self-normalizing – i.e. $[X, H] \in \mathfrak{h}$ for all $H \in \mathfrak{h}$ if, and only if $X \in \mathfrak{h}$ – and nilpotent. Equivalently for reductive $\mathfrak{g}$, $\mathfrak{h}$ is called *a Cartan subalgebra of* $\mathfrak{g}$ if it is Abelian, $\operatorname{ad}(H)$ is diagonalizable for each $H \in \mathfrak{h}$ and if $\mathfrak{h}$ is maximal with respect to the former two properties.

Proposition 4.2. *There exists a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.*

Proof. Notice that $0 \subseteq \mathfrak{g}$ is an Abelian subalgebra whose elements act as diagonal operators via the adjoint $\mathfrak{g}$-module. Indeed, 0, the only element of $0 \subseteq \mathfrak{g}$, is such that $\operatorname{ad}(0) = 0$ is a diagonalizable operator. Furthermore, given a chain of Abelian subalgebras

$$0 \subseteq \mathfrak{h}_1 \subseteq \mathfrak{h}_2 \subseteq \cdots$$

such that $\operatorname{ad}(H)$ is a diagonal operator for each $H \in \mathfrak{h}_i$, the subalgebra $\bigcup_i \mathfrak{h}_i \subseteq \mathfrak{g}$ is Abelian, and its elements also act diagonally in $\mathfrak{g}$. It then follows from Zorn's Lemma that there exists a subalgebra $\mathfrak{h}$ which is maximal with respect to both these properties, also known as a Cartan subalgebra. ■

We have already seen some concrete examples. Namely. . .

Example 4.3. The Lie subalgebra

$$\mathfrak{h} = \begin{pmatrix} K & 0 & \cdots & 0 \\ 0 & K & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K \end{pmatrix} \subseteq \mathfrak{gl}_n(K)$$

of diagonal matrices is a Cartan subalgebra. Indeed, every pair of diagonal matrices commutes, so that $\mathfrak{h}$ is an Abelian – and hence nilpotent – subalgebra. A simple calculation also shows that if $i \neq j$ then the coefficient of E_{ij} in $[E_{ii}, X]$ is the same as the coefficient of E_{ij} in X, for all $X \in \mathfrak{gl}_n(K)$. In particular, if $[E_{ii}, X]$ is diagonal for all i, then so is X – i.e. $\mathfrak{h}$ is self-normalizing.

Example 4.4. Let $\mathfrak{h}$ be as in Example 4.3. Then the subalgebra $\mathfrak{h} \cap \mathfrak{sl}_n(K)$ of traceless diagonal matrices is a Cartan subalgebra of $\mathfrak{sl}_n(K)$.

Example 4.5. It is easy to see from Example 1.11 that $\mathfrak{h} = \{X \in \mathfrak{sp}_{2n}(K) : X \text{ is diagonal}\}$ is a Cartan subalgebra.

Example 4.6. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras and $\mathfrak{h}_i \subseteq \mathfrak{g}_i$ be Cartan subalgebras. Then $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a Cartan subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.

The intersection of such subalgebra with $\mathfrak{sl}_n(K)$ – i.e. the subalgebra of traceless diagonal matrices – is a Cartan subalgebra of $\mathfrak{sl}_n(K)$. In particular, if $n = 2$ or $n = 3$ we get to the subalgebras described the previous chapter. The remaining question then is: if $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra and M is a $\mathfrak{g}$-module, does the eigenspace decomposition

$$M = \bigoplus_{\lambda} M_{\lambda}$$

of M hold? The answer to this question turns out to be yes. This is a consequence of something known as *simultaneous diagonalization*, which is the primary tool we will use to generalize the results of the previous section. What is simultaneous diagonalization all about then?

Definition 4.7. Given a K-vector space V, a set of operators $\{T_j : V \longrightarrow V\}_j$ is called *simultaneously diagonalizable* if there is a basis $\{v_1, \ldots, v_n\}$ for V such that $T_j v_i$ is a scalar multiple of v_i, for all i, j.

Proposition 4.8. *Given a* finite-dimensional *vector space V, a set of diagonalizable operators $V \longrightarrow V$ is simultaneously diagonalizable if, and only if all of its elements commute with one another.*

We should point out that simultaneous diagonalization *only works in the finite-dimensional setting*. In fact, simultaneous diagonalization is usually framed as an equivalent statement about diagonalizable $n \times n$ matrices. Simultaneous diagonalization implies that to show $M = \bigoplus_{\lambda} M_{\lambda}$ it suffices to show that $H\restriction_M : M \longrightarrow M$ is a diagonalizable operator for each $H \in \mathfrak{h}$. To that end, we introduce *the Jordan decomposition of an operator* and *the abstract Jordan decomposition of a semisimple Lie algebra*.

Proposition 4.9 (Jordan). *Given a finite-dimensional vector space V and an operator $T : V \longrightarrow V$, there are unique commuting operators $T_{\mathrm{ss}}, T_{\mathrm{nil}} : V \longrightarrow V$, with T_{ss} diagonalizable and T_{nil} nilpotent, such that $T = T_{\mathrm{ss}} + T_{\mathrm{nil}}$. The pair $(T_{\mathrm{ss}}, T_{\mathrm{nil}})$ is known as* the Jordan decomposition of T.

Proposition 4.10. *Given $\mathfrak{g}$ semisimple and $X \in \mathfrak{g}$, there are $X_{\mathrm{ss}}, X_{\mathrm{nil}} \in \mathfrak{g}$ such that $X = X_{\mathrm{ss}} + X_{\mathrm{nil}}$, $[X_{\mathrm{ss}}, X_{\mathrm{nil}}] = 0$, $\mathrm{ad}(X_{\mathrm{ss}})$ is a diagonalizable operator and $\mathrm{ad}(X_{\mathrm{nil}})$ is a nilpotent operator. The pair $(X_{\mathrm{ss}}, X_{\mathrm{nil}})$ is known as* the Jordan decomposition of X.

It should be clear from the uniqueness of $\mathrm{ad}(X)_{\mathrm{ss}}$ and $\mathrm{ad}(X)_{\mathrm{nil}}$ that the Jordan decomposition of $\mathrm{ad}(X)$ is $\mathrm{ad}(X) = \mathrm{ad}(X_{\mathrm{ss}}) + \mathrm{ad}(X_{\mathrm{nil}})$. What is perhaps more remarkable is the fact this holds for *any* finite-dimensional $\mathfrak{g}$-module. In other words. . .

Proposition 4.11. *Let M be a finite-dimensional $\mathfrak{g}$-module and $X \in \mathfrak{g}$. Denote by $X\restriction_M$ the action of X on M. Then $X_{\mathrm{ss}}\restriction_M = (X\restriction_M)_{\mathrm{ss}}$ and $X_{\mathrm{nil}}\restriction_M = (X\restriction_M)_{\mathrm{nil}}$.*

This last result is known as *the preservation of the Jordan form*, and a proof can be found in appendix C of [FH91]. As promised this implies. . .

Corollary 4.12. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra and M be any finite-dimensional $\mathfrak{g}$-module. Then there is a basis $\{m_1, \ldots, m_r\}$ of M so that each m_i is simultaneously an eigenvector of all elements of $\mathfrak{h}$ – i.e. each element of $\mathfrak{h}$ acts as a diagonal matrix in this basis. In other words, there are linear functionals $\lambda_i \in \mathfrak{h}^*$ so that $H \cdot m_i = \lambda_i(H) m_i$ for all $H \in \mathfrak{h}$. In particular,*

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$$

Proof. Fix some $H \in \mathfrak{h}$. It suffices to show that $H\restriction_M : M \longrightarrow M$ is a diagonalizable operator.

If we write $H = H_{\mathrm{ss}} + H_{\mathrm{nil}}$ for the abstract Jordan decomposition of H, we know $\mathrm{ad}(H_{\mathrm{ss}}) = \mathrm{ad}(H)_{\mathrm{ss}}$. But $\mathrm{ad}(H)$ is a diagonalizable operator, so that $\mathrm{ad}(H)_{\mathrm{ss}} = \mathrm{ad}(H)$. This implies $\mathrm{ad}(H_{\mathrm{nil}}) = \mathrm{ad}(H)_{\mathrm{nil}} = 0$, so that H_{nil} is a central element of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $H_{\mathrm{nil}} = 0$. Proposition 4.11 then implies $(H\restriction_M)_{\mathrm{nil}} = H_{\mathrm{nil}}\restriction_M = 0$, so $H\restriction_M = (H\restriction_M)_{\mathrm{ss}}$ is a diagonalizable operator. ■

We should point out that this last proof only works for semisimple Lie algebras. This is because we rely heavily on Proposition 4.11, as well in the fact that semisimple Lie algebras are centerless. In fact, Corollary 4.12 fails even for reductive Lie algebras. For a counterexample, consider the algebra $\mathfrak{g} = K$: the Cartan subalgebra of $\mathfrak{g}$ is $\mathfrak{g}$ itself, and a $\mathfrak{g}$-module is simply a vector space M endowed with an operator $M \longrightarrow M$ – which corresponds to the action of $1 \in \mathfrak{g}$ on M. In particular, if we choose an operator $M \longrightarrow M$ which is *not* diagonalizable we find $M \neq \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.

However, Corollary 4.12 does work for reductive $\mathfrak{g}$ if we assume that the $\mathfrak{g}$-module M in question is simple, since central elements of $\mathfrak{g}$ act on simple $\mathfrak{g}$-modules as scalar operators. The hypothesis of finite-dimensionality is also of huge importance. For instance, consider. . .

Example 4.13. Let $\mathcal{U}(\mathfrak{g})$ denote the regular $\mathfrak{g}$-module. Notice that $\mathcal{U}(\mathfrak{g})_\lambda = 0$ for all $\lambda \in \mathfrak{h}^*$. Indeed, since $\mathcal{U}(\mathfrak{g})$ is a domain, if $(H - \lambda(H))u = 0$ for some nonzero $H \in \mathfrak{h}$ then $u = 0$. In particular,

$$\bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{U}(\mathfrak{g})_\lambda = 0 \neq \mathcal{U}(\mathfrak{g})$$

As a first consequence of Corollary 4.12 we show. . .

Corollary 4.14. *The restriction of the Killing form κ to $\mathfrak{h}$ is non-degenerate.*

Proof. Consider the root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$ of the adjoint $\mathfrak{g}$-module, where α ranges over all nonzero eigenvalues of the adjoint action of $\mathfrak{h}$. We claim $\mathfrak{g}_0 = \mathfrak{h}$.

Indeed, since $\mathfrak{h}$ is Abelian, $\operatorname{ad}(\mathfrak{h})\mathfrak{h} = 0$ – i.e. $\mathfrak{h} \subseteq \mathfrak{g}_0$. On the other hand, since $\mathfrak{h}$ is self-normalizing, if $[X, H] = 0 \in \mathfrak{h}$ for all $H \in \mathfrak{h}$ then $X \in \mathfrak{h}$ – i.e. $\mathfrak{g}_0 \subseteq \mathfrak{h}$. So the eigenspace decomposition becomes

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$$

We furthermore claim that $\mathfrak{h} = \mathfrak{g}_0$ is orthogonal to $\mathfrak{g}_\alpha$ with respect to κ for any $\alpha \neq 0$. Indeed, given $X \in \mathfrak{g}_\alpha$ and $H_1, H_2 \in \mathfrak{h}$ with $\alpha(H_1) \neq 0$ we have

$$\alpha(H_1) \cdot \kappa(X, H_2) = \kappa([H_1, X], H_2) = -\kappa([X, H_1], H_2) = -\kappa(X, [H_1, H_2]) = 0$$

Hence the non-degeneracy of κ implies the non-degeneracy of its restriction. ■

We should point out that the restriction of κ to $\mathfrak{h}$ is *not* the Killing form of $\mathfrak{h}$. In fact, since $\mathfrak{h}$ is Abelian, its Killing form is identically zero – which is hardly ever a non-degenerate form.

Remark. Since κ induces an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$, it induces a bilinear form $(\kappa(X, \cdot), \kappa(Y, \cdot)) \longmapsto \kappa(X, Y)$ in $\mathfrak{h}^*$. As in section 3.1, we denote this form by κ as well.

We now have most of the necessary tools to reproduce the results of the previous chapter in a general setting. Let $\mathfrak{g}$ be a finite-dimensional semisimple algebra with a Cartan subalgebra $\mathfrak{h}$ and let M be a finite-dimensional simple $\mathfrak{g}$-module. We will proceed, as we did before, by generalizing the results of the previous two sections in order. By now the pattern should be starting to become clear, so we will mostly omit technical details and proofs analogous to the ones on the previous sections. Further details can be found in appendix D of [FH91] and in [E H73].

4.1 The Geometry of Roots and Weights

We begin our analysis, as we did for $\mathfrak{sl}_2(K)$ and $\mathfrak{sl}_3(K)$, by investigating the locus of roots of and weights of $\mathfrak{g}$. Throughout chapter 3 we have seen that the weights of any given finite-dimensional module of $\mathfrak{sl}_2(K)$ or $\mathfrak{sl}_3(K)$ can only assume very rigid configurations. For instance, we have seen that the roots of $\mathfrak{sl}_2(K)$ and $\mathfrak{sl}_3(K)$ are symmetric with respect to the origin. In this chapter we will generalize most results from chapter 3 regarding the rigidity of the geometry of the set of weights of a given module.

As for the aforementioned result on the symmetry of roots, this turns out to be a general fact, which is a consequence of the non-degeneracy of the restriction of the Killing form to the Cartan subalgebra.

Proposition 4.15. *The roots α of $\mathfrak{g}$ are symmetrical about the origin – i.e. $-\alpha$ is also a root – and they span all of $\mathfrak{h}^*$.*

Proof. We will start with the first claim. Let α and β be two roots. Notice $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Indeed, if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ then

$$[H, [X, Y]] = [X, [H, Y]] - [Y, [H, X]] = (\alpha + \beta)(H) \cdot [X, Y]$$

for all $H \in \mathfrak{h}$.

This implies that if $\alpha + \beta \neq 0$ then $\operatorname{ad}(X)\operatorname{ad}(Y)$ is nilpotent: if $Z \in \mathfrak{g}_\gamma$ then

$$(\operatorname{ad}(X)\operatorname{ad}(Y))^r Z = [X, [Y, [\ldots, [X, [Y, Z]]] \ldots] \in \mathfrak{g}_{r\alpha + r\beta + \gamma} = 0$$

for r large enough. In particular, $\kappa(X, Y) = \operatorname{Tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) = 0$. Now if $-\alpha$ is not an eigenvalue we find $\kappa(X, \mathfrak{g}_\beta) = 0$ for all roots β, which contradicts the non-degeneracy of κ. Hence $-\alpha$ must be an eigenvalue of the adjoint action of $\mathfrak{h}$.

For the second statement, note that if the roots of $\mathfrak{g}$ do not span all of $\mathfrak{h}^*$ then there is some nonzero $H \in \mathfrak{h}$ such that $\alpha(H) = 0$ for all roots α, which is to say, $\operatorname{ad}(H)X = [H, X] = 0$ for all $X \in \mathfrak{g}$. Another way of putting it is to say H is an element of the center $\mathfrak{z} = 0$ of $\mathfrak{g}$, a contradiction. ■

Furthermore, as in the case of $\mathfrak{sl}_2(K)$ and $\mathfrak{sl}_3(K)$ one can show...

Proposition 4.16. *The root spaces $\mathfrak{g}_\alpha$ are all 1-dimensional.*

The proof of the first statement of Proposition 4.15 highlights something interesting: if we fix some eigenvalue α of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ and a eigenvector $X \in \mathfrak{g}_\alpha$, then for each $H \in \mathfrak{h}$ and $m \in M_\lambda$ we find

$$H \cdot (X \cdot m) = XH \cdot m + [H, X] \cdot m = (\lambda + \alpha)(H) X \cdot m$$

Thus X sends m to $M_{\lambda+\alpha}$. We have encountered this formula twice in these notes: again, we find $\mathfrak{g}_\alpha$ *acts on M by translating vectors between eigenspaces*. In particular, if we denote by Δ the set of all roots of $\mathfrak{g}$ then...

Theorem 4.17. *The weights of a finite-dimensional simple $\mathfrak{g}$-module M are all congruent modulo the root lattice $Q = \mathbb{Z}\Delta$ of $\mathfrak{g}$. In other words, all weights of M lie in the same Q-coset $\xi \in \mathfrak{h}^*/Q$.*

Again, we may leverage our knowledge of $\mathfrak{sl}_2(K)$ to obtain further restrictions on the geometry of the locus of weights of M. Namely, as in the case of $\mathfrak{sl}_3(K)$ we show...

Proposition 4.18. *Given a root α of $\mathfrak{g}$ the subspace $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a subalgebra isomorphic to $\mathfrak{sl}_2(K)$.*

Corollary 4.19. *For all weights μ, the subspace*

$$\bigoplus_k M_{\mu - k\alpha}$$

is invariant under the action of the subalgebra $\mathfrak{s}_\alpha$ and the weight spaces in this string match the eigenspaces of h.

The proof of Proposition 4.18 is very technical in nature and we won't include it here, but the idea behind it is simple: recall that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are both 1-dimensional, so that $\dim[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is at most 1. We check that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$ and that no generator of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is annihilated by α, so that by adjusting scalars we can find $E_\alpha \in \mathfrak{g}_\alpha$ and $F_\alpha \in \mathfrak{g}_{-\alpha}$ such that $H_\alpha = [E_\alpha, F_\alpha]$ satisfies

$$[H_\alpha, F_\alpha] = -2F_\alpha \qquad\qquad [H_\alpha, E_\alpha] = 2E_\alpha$$

The elements $E_\alpha, F_\alpha \in \mathfrak{g}$ are not uniquely determined by this condition, but H_α is. As promised, the second statement of Corollary 4.19 imposes strong restrictions on the weights of M. Namely, if λ is a weight, $\lambda(H_\alpha)$ is an eigenvalue of h on some $\mathfrak{sl}_2(K)$-module, so it must be an integer. In other words...

Definition 4.20. The lattice $P = \{\lambda \in \mathfrak{h}^* : \lambda(H_\alpha) \in \mathbb{Z}\, \forall \alpha \in \Delta\} \subseteq \mathfrak{h}^*$ is called *the weight lattice of* $\mathfrak{g}$. We call the elements of P *integral*.

Proposition 4.21. *The weights of a finite-dimensional simple $\mathfrak{g}$-module M of all lie in the weight lattice P.*

Proposition 4.21 is clearly analogous to Corollary 3.12. In fact, the weight lattice of $\mathfrak{sl}_3(K)$ – as in Definition 4.20 – is precisely $\mathbb{Z}\langle\epsilon_1, \epsilon_2, \epsilon_3\rangle$. To proceed further, we would like to take *the highest weight of* M as in section 3.1, but the meaning of *highest* is again unclear in this situation. We could simply fix a linear function $\mathbb{Q}P \longrightarrow \mathbb{Q}$ – as we did in section 3.1 – and choose a weight λ of M that maximizes this functional, but at this point it is convenient to introduce some additional tools to our arsenal. These tools are called *basis*.

Definition 4.22. A subset $\Sigma = \{\beta_1, \ldots, \beta_r\} \subseteq \Delta$ of linearly independent roots is called *a basis for* Δ if, given $\alpha \in \Delta$, there are unique $k_1, \ldots, k_r \in \mathbb{N}$ such that $\alpha = \pm(k_1\beta_1 + \cdots + k_r\beta_r)$.

Example 4.23. Suppose $\mathfrak{g} = \mathfrak{sl}_n(K)$ and $\mathfrak{h} \subseteq \mathfrak{g}$ is the subalgebra of diagonal matrices, as in Example 4.4. Consider the linear functionals $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{h}^*$ such that $\epsilon_i(H)$ is the i-th entry of the diagonal of H. As observed in section 3.1 for $n = 3$, the roots of $\mathfrak{sl}_n(K)$ are $\epsilon_i - \epsilon_j$ for $i \neq j$ – with root vectors given by E_{ij} – and we may take the basis $\Sigma = \{\beta_1, \ldots, \beta_{n-1}\}$ with $\beta_i = \epsilon_i - \epsilon_{i+1}$.

Example 4.24. Suppose $\mathfrak{g} = \mathfrak{sp}_{2n}(K)$ and $\mathfrak{h} \subseteq \mathfrak{g}$ is the subalgebra of diagonal matrices, as in Example 4.5. Consider the linear functionals $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{h}^*$ such that $\epsilon_i(H)$ is the i-th entry of the diagonal of H. Then the roots of $\mathfrak{sp}_{2n}(K)$ are $\pm\epsilon_i \pm \epsilon_j$ for $i \neq j$ and $\pm 2\epsilon_i$ – see [FH91, ch. 16]. In this case, we may take the basis $\Sigma = \{\beta_1, \ldots, \beta_n\}$ with $\beta_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$ and $\beta_n = 2\epsilon_n$.

The interesting thing about basis for Δ is that they allow us to compare weights of a given $\mathfrak{g}$-module. At this point the reader should be asking himself: how? Definition 4.22 isn't exactly all that intuitive. Well, the thing is that any choice of basis Σ induces an order in Q, where elements are ordered by their Σ-*coordinates*.

Definition 4.25. Let $\Sigma = \{\beta_1, \ldots, \beta_r\}$ be a basis for Δ. Given $\alpha = k_1\beta_1 + \cdots + k_r\beta_r \in Q$ with $k_1, \ldots, k_r \in \mathbb{Z}$, we call the vector $\alpha_\Sigma = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ *the Σ-coordinate of* α. We say that $\alpha \preccurlyeq \beta$ if $\alpha_\Sigma \leqslant \beta_\Sigma$ in the lexicographical order.

Definition 4.26. Given a basis Σ for Δ, there is a canonical partition[1] $\Delta^+ \cup \Delta^- = \Delta$, where $\Delta^+ = \{\alpha \in \Delta : \alpha \succ 0\}$ and $\Delta^- = \{\alpha \in \Delta : \alpha \prec 0\}$. The elements of Δ^+ and Δ^- are called *positive* and *negative roots*, respectively.

Example 4.27. If $\mathfrak{g} = \mathfrak{sl}_3(K)$ and Σ is as in Example 4.23 then the partition $\Delta^+ \cup \Delta^-$ induced by Σ is the same as the one described in section 3.1.

Definition 4.28. Let Σ be a basis for Δ. The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ is called *the Borel subalgebra associated with* $\mathfrak{h}$ *and* Σ. A subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is called *parabolic* if $\mathfrak{p} \supseteq \mathfrak{b}$.

It should be obvious that the binary relation $\preccurlyeq$ in Q is a total order. In addition, we may compare the elements of a given Q-coset $\lambda + Q$ by comparing their difference with $0 \in Q$. In other

[1]Notice that $\mathrm{ht}(\alpha) = 0$ if, and only if $\alpha = 0$. Since 0 is, by definition, not a root, the sets Δ^+ and Δ^- account for all roots.

words, given $\lambda \in \mu + Q$, we say $\lambda \preccurlyeq \mu$ if $\lambda - \mu \preccurlyeq 0$. In particular, since the weights of M all lie in a single Q-coset, we may compare them in this way. Given a basis Σ for Δ we may take "the highest weight of M" as a maximal weight λ of M. The obvious question then is: can we always find a basis for Δ?

Proposition 4.29. *There is a basis Σ for Δ.*

The intuition behind the proof of this proposition is similar to our original idea of fixing a direction in $\mathfrak{h}^*$ in the case of $\mathfrak{sl}_3(K)$. Namely, one can show that $\kappa(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$, so that the Killing form κ restricts to a nondegenerate $\mathbb{Q}$-bilinear form $\mathbb{Q}\Delta \times \mathbb{Q}\Delta \longrightarrow \mathbb{Q}$. We can then fix a nonzero vector $\gamma \in \mathbb{Q}\Delta$ and consider the orthogonal projection $f : \mathbb{Q}\Delta \longrightarrow \mathbb{Q}\gamma \cong \mathbb{Q}$. We say a root $\alpha \in \Delta$ is *positive* if $f(\alpha) > 0$, and we call a positive root α *simple* if it cannot be written as the sum two other positive roots. The subset $\Sigma \subseteq \Delta$ of all simple roots is a basis for Δ, and all other basis can be shown to arise in this way.

Fix some basis Σ for Δ, with corresponding decomposition $\Delta^+ \cup \Delta^- = \Delta$. Let λ be a maximal weight of M. We call λ *the highest weight of* M, and we call any nonzero $m \in M_\lambda$ *a highest weight vector*. The strategy then is to describe all weight spaces of M in terms of λ and m, as in Theorem 3.15. Unsurprisingly we do so by reproducing the proof of the case of $\mathfrak{sl}_3(K)$.

First, we note that any highest weight vector $m \in M_\lambda$ is annihilated by all positive root spaces, for if $\alpha \in \Delta^+$ then $E_\alpha \cdot m \in M_{\lambda+\alpha}$ must be zero – or otherwise we would have that $\lambda + \alpha$ is a weight with $\lambda \prec \lambda + \alpha$. In particular,

$$\bigoplus_{k \in \mathbb{Z}} M_{\lambda - k\alpha} = \bigoplus_{k \in \mathbb{N}} M_{\lambda - k\alpha}$$

and $\lambda(H_\alpha)$ is the right-most eigenvalue of the action of h on the $\mathfrak{sl}_2(K)$-module $\bigoplus_k M_{\lambda - k\alpha}$.

This has a number of important consequences. For instance...

Corollary 4.30. *If $\alpha \in \Delta^+$ and $\sigma_\alpha : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ is the reflection in the hyperplane perpendicular to α with respect to the Killing form, the weights of M occurring in the line joining λ and σ_α are precisely the $\mu \in P$ lying between λ and $\sigma_\alpha(\lambda)$.*

Proof. Notice that any $\mu \in P$ in the line joining λ and $\sigma_\alpha(\lambda)$ has the form $\mu = \lambda - k\alpha$ for some k, so that M_μ corresponds the eigenspace associated with the eigenvalue $\lambda(H_\alpha) - 2k$ of the action of h on $\bigoplus_k M_{\lambda - k\alpha}$. If μ lies between λ and $\sigma_\alpha(\lambda)$ then k lies between 0 and $\lambda(H_\alpha)$, in which case $M_\mu \neq 0$ and therefore μ is a weight.

On the other hand, if μ does not lie between λ and $\sigma_\alpha(\lambda)$ then either $k < 0$ or $k > \lambda(H_\alpha)$. Suppose μ is a weight. In the first case $\mu \succ \lambda$, a contradiction. On the second case the fact that $M_\mu \neq 0$ implies $M_{\lambda + (k - \lambda(H_\alpha))\alpha} = M_{\sigma_\alpha(\mu)} \neq 0$, which contradicts the fact that $M_{\lambda + \ell\alpha} = 0$ for all $\ell \geqslant 0$. ■

This is entirely analogous to the situation of $\mathfrak{sl}_3(K)$, where we found that the weights of the simple $\mathfrak{sl}_3(K)$-modules formed continuous strings symmetric with respect to the lines $K\alpha$ with $\kappa(\epsilon_i - \epsilon_j, \alpha) = 0$. As in the case of $\mathfrak{sl}_3(K)$, the same sort of arguments leads us to the conclusion...

Definition 4.31. We refer to the (finite) group $W = \langle \sigma_\alpha : \alpha \in \Delta \rangle = \langle \sigma_\beta : \beta \in \Sigma \rangle \subseteq \mathrm{O}(\mathfrak{h}^*)$ as *the Weyl group of* $\mathfrak{g}$.

Theorem 4.32. *The weights of a simple $\mathfrak{g}$-module M with highest weight λ are precisely the elements of the weight lattice P congruent to λ modulo the root lattice Q lying inside the convex hull of the orbit of λ under the action of the Weyl group W.*

At this point we are basically done with results regarding the geometry of the weights of M, but it is convenient to introduce some further notation. Aside from showing up in the previous theorem, the Weyl group will also play an important role in chapter 5 by virtue of the existence of a canonical action of W on $\mathfrak{h}$.

Definition 4.33. The canonical action of W on $\mathfrak{h}^*$ given by $\sigma \cdot \lambda = \sigma(\lambda)$ is called *the natural action of* W. We also consider the equivalent "shifted" action $\sigma \bullet \lambda = \sigma(\lambda + \rho) - \rho$ of W on $\mathfrak{h}^*$, known as *the dot action of* W – here $\rho = {}^1/_2\beta_1 + \cdots {}^1/_2\beta_r$.

This already allow us to compute some examples of Weyl groups.

Example 4.34. Suppose $\mathfrak{g} = \mathfrak{sl}_n(K)$ and $\mathfrak{h} \subseteq \mathfrak{g}$ is as in Example 4.4. Let $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{h}^*$ be as in Example 4.23 and take the associated basis $\Sigma = \{\beta_1, \ldots, \beta_{n-1}\}$ for Δ, $\beta_i = \epsilon_i - \epsilon_{i+1}$. Then a simple calculation shows that σ_{β_i} permutes ϵ_i and ϵ_{i+1} and fixes the other ϵ_j. This translates to a canonical isomorphism

$$\begin{aligned} W &\xrightarrow{\sim} S_n \\ \sigma_{\beta_i} &\longmapsto \sigma_i = (i\ i{+}1) \end{aligned}$$

Example 4.35. Suppose $\mathfrak{g} = \mathfrak{sp}_{2n}(K)$ and $\mathfrak{h} \subseteq \mathfrak{g}$ is as in Example 4.5. Let $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{h}^*$ be as in Example 4.24 and take the associated basis $\Sigma = \{\beta_1, \ldots, \beta_n\}$ for Δ. Then a simple calculation shows that σ_{β_i} permutes ϵ_i and ϵ_{i+1} for $i < n$ and σ_{β_n} switches the sign of ϵ_n. This translates to a canonical isomorphism

$$\begin{aligned} W &\xrightarrow{\sim} S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n \\ \sigma_{\beta_i} &\longmapsto (\sigma_i, (\bar{0}, \ldots, \bar{0})) \\ \sigma_{\beta_n} &\longmapsto (1, (\bar{0}, \ldots, \bar{0}, \bar{1})), \end{aligned}$$

where $\sigma_i = (i\ i{+}1)$ are the canonical transpositions.

If we conjugate some $\sigma \in W$ by the isomorphism $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ afforded by the restriction of the Killing for to $\mathfrak{h}$ we get a linear action of W on $\mathfrak{h}$, which is given by $\kappa(\sigma \cdot H, \cdot) = \sigma \cdot \kappa(H, \cdot)$. As it turns out, this action can be extended to an action of W on $\mathfrak{g}$ by automorphisms of Lie algebras. This translates into the following results, which we do not prove – but see [E H73, sec. 14.3].

Proposition 4.36. *Given $\alpha \in \Delta^+$, there is an automorphism of Lie algebras $f_\alpha : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ such that $f_\alpha(H) = \sigma_\alpha \cdot H$ for all $H \in \mathfrak{h}$. In addition, these automorphisms can be chosen in such a way that the family $\{f_\alpha\}_{\alpha \in \Delta^+}$ defines an action of W on $\mathfrak{g}$ – which is obviously compatible with the natural action of W on $\mathfrak{h}$.*

Remark. We should notice the action of W on $\mathfrak{g}$ from Proposition 4.36 is not canonical, since it depends on the choice of E_α and F_α. Nevertheless, different choices of E_α and F_α yield isomorphic actions and the restriction of these actions to $\mathfrak{h}$ is independent of any choices.

We should point out that the results in this section regarding the geometry roots and weights are only the beginning of a well develop axiomatic theory of the so called *root systems*, which was used by Cartan in the early 20th century to classify all finite-dimensional simple complex Lie algebras in terms of Dynking diagrams. This and much more can be found in [E H73, p. III] and [FH91, ch. 21]. Having found all of the weights of M, the only thing we are missing for a complete classification is an existence and uniqueness theorem analogous to Theorem 3.4 and Theorem 3.17. This will be the focus of the next section.

4.2 Highest Weight Modules & the Highest Weight Theorem

It is already clear from the previous discussion that if λ is the highest weight of M then $\lambda(H_\alpha) \geqslant 0$ for all positive roots α. Indeed, as in the $\mathfrak{sl}_3(K)$, for each $\alpha \in \Delta^+$ we know $\lambda(H_\alpha)$ is the highest eigenvalue of the action of h in the $\mathfrak{sl}_2(K)$-module $\bigoplus_k M_{\lambda - k\alpha}$ – which must be a non-negative integer. This fact may be summarized in the following proposition.

Definition 4.37. An element λ of P such that $\lambda(H_\alpha) \geqslant 0$ for all $\alpha \in \Delta^+$ is referred to as an *dominant integral weight of* $\mathfrak{g}$. The set of all dominant integral weights is denotes by P^+.

Proposition 4.38. *Suppose M is a finite-dimensional simple $\mathfrak{g}$-module and λ is its highest weight. Then λ is a dominant integral weight of $\mathfrak{g}$.*

The condition that $\lambda \in P^+$ is thus necessary for the existence of a simple $\mathfrak{g}$-module with highest weight given by λ. Given our previous experience with $\mathfrak{sl}_2(K)$ and $\mathfrak{sl}_3(K)$, it is perhaps unsurprising that this condition is also sufficient.

Theorem 4.39. *For each dominant integral $\lambda \in P^+$ there exists precisely one finite-dimensional simple $\mathfrak{g}$-module M whose highest weight is λ.*

This is known as *the Highest Weight Theorem*, and its proof is the focus of this section. The "uniqueness" part of the theorem follows at once from the arguments used for $\mathfrak{sl}_3(K)$. However, the "existence" part of the theorem is more nuanced. Our first instinct is, of course, to try to generalize the proof used for $\mathfrak{sl}_3(K)$. Indeed, as in Proposition 3.14, one is able to show...

Proposition 4.40. *Let M be a finite-dimensional simple $\mathfrak{g}$-module. Then there exists a nonzero weight vector $m \in M$ which is annihilated by all positive root spaces of $\mathfrak{g}$ – i.e. $X \cdot m = 0$ for all $X \in \mathfrak{g}_\alpha$, $\alpha \in \Delta^+$.*

Proof. If λ is the highest weight of M, it suffices to take any $m \in M_\lambda$. Indeed, given $X \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta^+$, $X \cdot m \in M_{\lambda+\alpha} = 0$ because $\lambda + \alpha \succ \lambda$. ∎

Unfortunately for us, this is where the parallels with Proposition 3.14 end. The issue is that our proof relied heavily on our knowledge of the roots of $\mathfrak{sl}_3(K)$. It is thus clear that we need a more systematic approach for the general setting. We begin by asking a simpler question: how can we construct *any* $\mathfrak{g}$-module M whose highest weight is λ? In the process of answering this question we will come across a surprisingly elegant solution to our problem.

If M is a finite-dimensional simple module with highest weight λ and $m \in M_\lambda$, we already know that $X \cdot m = 0$ for any $m \in M_\lambda$ and $X \in \mathfrak{g}_\alpha$, $\alpha \in \Delta^+$. Since $M = \mathcal{U}(\mathfrak{g}) \cdot m$, the restriction of M to the Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ has a prescribed action. On the other hand, we have essentially no information about the action of the rest of $\mathfrak{g}$ on M. Nevertheless, given a $\mathfrak{b}$-module we may obtain a $\mathfrak{g}$-module by *freely* extending the action of $\mathfrak{b}$ via induction. This leads us to the following definition.

Definition 4.41. Given $\lambda \in \mathfrak{h}^*$, consider the $\mathfrak{b}$-module Km^+ where $H \cdot m^+ = \lambda(H)m^+$ for all $H \in \mathfrak{h}$ and $X \cdot m^+ = 0$ for $X \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta^+$. The $\mathfrak{g}$-module $M(\lambda) = \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} Km^+$ is called *the Verma module of weight* λ.

Example 4.42. If $\mathfrak{g} = \mathfrak{sl}_2(K)$, then we can take $\mathfrak{h} = Kh$ and $\mathfrak{b} = Ke \oplus Kh$. In this setting, the linear map $g : \mathfrak{h}^* \longrightarrow K$ defined by $g(h) = 1$ affords us a canonical identification $\mathfrak{h}^* = Kg \cong K$, so that given $\lambda \in K$ we may denote $M(\lambda g)$ simply by $M(\lambda)$. Using this notation $M(\lambda) = \bigoplus_{k \geqslant 0} K f^k \cdot m^+$, and the action of $\mathfrak{sl}_2(K)$ on $M(\lambda)$ is given by formula (4.2).

$$f^k \cdot m^+ \xmapsto{e} k(\lambda + 1 - k) f^{k-1} \cdot m^+ \quad f^k \cdot m^+ \xmapsto{f} f^{k+1} \cdot m^+ \quad f^k \cdot m^+ \xmapsto{h} (\lambda - 2k) f^k \cdot m^+ \tag{4.2}$$

Example 4.43. Consider the $\mathfrak{sl}_2(K)$-module $M(2)$ as described in Example 4.42. It follows from formula (4.2) that the action of $\mathfrak{sl}_2(K)$ on $M(2)$ is given by

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "-10"] & M(2)_{-6} \arrow[l, bend left, "1"] \arrow[r, bend left, "-4"] & M(2)_{-4} \arrow[l, bend left, "1"] \arrow[r, bend left, "0"] & M(2)_{-2} \arrow[l, bend left, "1"] \arrow[r, bend left, "2"] & M(2)_{0} \arrow[l, bend left, "1"] \arrow[r, bend left, "2"] & M(2)_{2} \arrow[l, bend left, "1"]
\end{tikzcd}$$

where $M(2)_{2-2k} = K f^k \cdot m^+$. Here the top arrows represent the action of e and the bottom arrows represent the action of f. The scalars labeling each arrow indicate to which multiple of $f^{k \pm 1} \cdot m^+$ the elements e and f send $f^k \cdot m^+$. The string of weight spaces to the left of the diagram is infinite. Since $e \cdot (f^3 \cdot m^+) = 0$, it is easy to see that subspace $\bigoplus_{k \geqslant 3} K f^k \cdot m^+$ is a (maximal) $\mathfrak{sl}_2(K)$-submodule, which is isomorphic to $M(-4)$.

These last examples show that, unlike most modules we have so far encountered, Verma modules are *highly infinite-dimensional*. Indeed, it follows from the PBW Theorem that the regular module $\mathcal{U}(\mathfrak{g})$ is a free $\mathfrak{b}$-module of infinite rank – equal to the codimension of $\mathcal{U}(\mathfrak{b})$ in $\mathcal{U}(\mathfrak{g})$. Hence $\dim M(\lambda)$, which is the same as the rank of $\mathcal{U}(\mathfrak{g})$ as a $\mathfrak{b}$-module, is also infinite. Nevertheless, it turns out that finite-dimensional modules and Verma module may both be seen as particular cases of a more general pattern. This leads us to the following definitions.

Definition 4.44. Let M be a $\mathfrak{g}$-module. A vector $m \in M$ is called *singular* if it is annihilated by all positive weight spaces of $\mathfrak{g}$ – i.e. $X \cdot m = 0$ for all $X \in \mathfrak{g}_\alpha$, $\alpha \in \Delta^+$.

Definition 4.45. A $\mathfrak{g}$-module M is called *a highest weight module* if there exists some singular weight vector $m^+ \in M_\lambda$ such that $M = \mathcal{U}(\mathfrak{g}) \cdot m^+$. Any such m^+ is called *a highest weight vector*, while λ is called *the highest weight of M*.

Example 4.46. Proposition 4.40 is equivalent to the fact that every finite-dimensional simple $\mathfrak{g}$-module is a highest weight module.

Example 4.47. It should be obvious from the definitions that $M(\lambda)$ is a highest weight module of highest weight λ and highest weight vector $m^+ = 1 \otimes m^+$ as in Definition 4.41. Indeed, $u \otimes m^+ = u \cdot m^+$ for all $u \in \mathcal{U}(\mathfrak{g})$, which already shows $M(\lambda)$ is generated by m^+. In particular,

$$\begin{aligned}
H \cdot m^+ &= H \otimes m^+ = 1 \otimes H \cdot m^+ = \lambda(H) m^+ \\
X \cdot m^+ &= X \otimes m^+ = 1 \otimes X \cdot m^+ = 0
\end{aligned}$$

for all $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$, $\alpha \in \Delta^+$.

While Verma modules show that a highest weight module needs not to be finite-dimensional, it turns out that highest weight modules enjoy many of the features we've grown used to in the past chapters. Explicitly, we may establish the properties described in the following proposition, whose statement should also explain the nomenclature of Definition 4.45.

Proposition 4.48. *Let M be a highest weight $\mathfrak{g}$-module with highest weight vector $m \in M_\lambda$. The weight spaces decomposition*

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$$

holds. Furthermore, $\dim M_\mu < \infty$ *for all* $\mu \in \mathfrak{h}^*$ *and* $\dim M_\lambda = 1$ *– i.e.* $M_\lambda = Km$*. Finally, given a weight μ of M, $\lambda \succcurlyeq \mu$ – so that the highest weight λ of M is unique and coincides with the largest of the weights of M.*

Proof. Since $M = \mathcal{U}(\mathfrak{g}) \cdot m$, the PBW Theorem implies that M is spanned by the vectors $F_{\alpha_{i_1}} F_{\alpha_{i_2}} \cdots F_{\alpha_{i_s}} \cdot m$ for $\Delta^+ = \{\alpha_1, \ldots, \alpha_r\}$ and $F_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ as in the proof of Proposition 4.18. But

$$\begin{aligned} H \cdot (F_{\alpha_{i_1}} F_{\alpha_{i_2}} \cdots F_{\alpha_{i_s}} \cdot m) &= ([H, F_{\alpha_{i_1}}] + F_{\alpha_{i_1}} H) F_{\alpha_{i_2}} \cdots F_{\alpha_{i_s}} \cdot m \\ &= -\alpha_{i_1}(H) F_{\alpha_{i_1}} \cdots F_{\alpha_{i_s}} \cdot m + F_{\alpha_{i_1}} ([H, F_{\alpha_{i_2}}] + F_{\alpha_{i_2}} H) F_{\alpha_{i_2}} \cdots F_{\alpha_{i_s}} \cdot m \\ &\ \ \vdots \\ &= (-\alpha_{i_1} - \cdots - \alpha_{i_s})(H) F_{\alpha_{i_1}} \cdots F_{\alpha_{i_s}} \cdot m + F_{\alpha_{i_1}} \cdots F_{\alpha_{i_s}} H \cdot m \\ &= (\lambda - \alpha_{i_1} - \cdots - \alpha_{i_s})(H) F_{\alpha_{i_1}} \cdots F_{\alpha_{i_s}} \cdot m \\ &\therefore F_{\alpha_{i_1}} \cdots F_{\alpha_{i_s}} \cdot m \in M_{\lambda - \alpha_{i_1} - \cdots - \alpha_{i_s}} \end{aligned}$$

Hence $M \subseteq \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$, as desired. In fact we have established

$$M \subseteq \bigoplus_{k_i \in \mathbb{N}} M_{\lambda - k_1 \cdot \alpha_1 - \cdots - k_r \cdot \alpha_r}$$

where $\{\alpha_1, \ldots, \alpha_r\} = \Delta^+$, so that all weights of M have the form $\mu = \lambda - k_1 \cdot \alpha_1 - \cdots - k_r \cdot \alpha_r$. This already gives us that the weights of M are bounded by λ.

To see that $\dim M_\mu < \infty$, simply note that there are only finitely many monomials $F_{\alpha_1}^{k_1} F_{\alpha_2}^{k_2} \cdots F_{\alpha_s}^{k_s}$ such that $\mu = \lambda + k_1 \cdot \alpha_1 + \cdots + k_s \cdot \alpha_s$. Since M_μ is spanned by the images of m under such monomials, we conclude $\dim M_\mu < \infty$. In particular, there is a single monomial $F_{\alpha_1}^{k_1} F_{\alpha_2}^{k_2} \cdots F_{\alpha_s}^{k_s}$ such that $\lambda = \lambda + k_1 \cdot \alpha_1 + \cdots + k_s \cdot \alpha_s$ – which is, of course, the monomial where $k_1 = \cdots = k_n = 0$. Hence $\dim M_\lambda = 1$. ∎

At this point it is important to note that, far from a "misbehaved" class of examples, Verma modules hold a very special place in the theory of highest weight modules. Intuitively speaking, the Verma module $M(\lambda)$ should really be though-of as "the freest highest weight $\mathfrak{g}$-module of highest weight λ". In practice, this translates to the following universal property.

Proposition 4.49. *Let M be a $\mathfrak{g}$-module and $m \in M_\lambda$ be a singular vector. Then there exists a unique $\mathfrak{g}$-homomorphism $f : M(\lambda) \longrightarrow M$ such that $f(m^+) = m$. Furthermore, all homomorphisms $M(\lambda) \longrightarrow M$ are given in this fashion.*

$$\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M) \cong \{m \in M_\lambda : m \text{ is singular}\}$$

Proof. The result follows directly from Proposition 1.60. Indeed, by the Frobenius Reciprocity Theorem, a $\mathfrak{g}$-homomorphism $f : M(\lambda) \longrightarrow M$ is the same as a $\mathfrak{b}$-homomorphism $g : Km^+ \longrightarrow M = \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M$. More specifically, given a $\mathfrak{b}$-homomorphism $g : Km^+ \longrightarrow M$, there exists a unique $\mathfrak{g}$-homomorphism $f : M(\lambda) \longrightarrow M$ such that $f(u \otimes m^+) = u \cdot g(m^+)$ for all $u \in \mathcal{U}(\mathfrak{g})$, and all $\mathfrak{g}$-homomorphism $M(\lambda) \longrightarrow M$ arise in this fashion.

Any K-linear map $g : Km^+ \longrightarrow M$ is determined by $m = g(m^+)$. Finally, notice that g is a $\mathfrak{b}$-homomorphism if, and only if m is a singular vector lying in M_λ. ∎

Why is any of this interesting to us, however? After all, Verma modules are not specially well suited candidates for a proof of the Highest Weight Theorem. Indeed, we have seen in Example 4.43 that in general $M(\lambda)$ is not simple, nor is it ever finite-dimensional. Nevertheless, we may use $M(\lambda)$ to establish Theorem 4.39 as follows.

Suppose M is a highest weight $\mathfrak{g}$-module of highest weight λ with highest weight vector m. By the last proposition, there is a $\mathfrak{g}$-homomorphism $f : M(\lambda) \longrightarrow M$ such that $f(m^+) = m$. Since $M = \mathcal{U}(\mathfrak{g}) \cdot m$, f is surjective and therefore $M \cong {}^{M(\lambda)}/_{\ker f}$. Hence...

Proposition 4.50. *Let M be a highest weight $\mathfrak{g}$-module of highest weight λ. Then M is quotient of $M(\lambda)$. If M is simple then M is the quotient of $M(\lambda)$ by a maximal $\mathfrak{g}$-submodule.*

Maximal submodules of Verma modules are thus of primary interest to us. As it turns out, these can be easily classified.

Proposition 4.51. *Every submodule $N \subseteq M(\lambda)$ is the direct sum of its weight spaces. In particular, $M(\lambda)$ has a unique maximal submodule $N(\lambda)$ and a unique simple quotient $L(\lambda) = {}^{M(\lambda)}/_{N(\lambda)}$. Any simple highest weight $\mathfrak{g}$-module has the form $L(\lambda)$ for some unique $\lambda \in \mathfrak{h}^*$.*

Proof. Let $N \subseteq M(\lambda)$ be a submodule and take any nonzero $n \in N$. Because of Proposition 4.48, we know there are $\mu_1, \ldots, \mu_r \in \mathfrak{h}^*$ and nonzero $m_i \in M(\lambda)_{\mu_i}$ such that $n = m_1 + \cdots + m_r$. We want to show $m_i \in N$ for all i.

Fix some $H_2 \in \mathfrak{h}$ such that $\mu_1(H_2) \neq \mu_2(H_2)$. Then

$$m_1 - \frac{(\mu_3 - \mu_1)(H_2)}{(\mu_2 - \mu_1)(H_2)} \cdot m_3 - \cdots - \frac{(\mu_r - \mu_1)(H_2)}{(\mu_2 - \mu_1)(H_2)} \cdot m_r = \left(1 - \frac{H_2 - \mu_1(H_2)}{(\mu_2 - \mu_1)(H_2)}\right) \cdot n \in N$$

Now take $H_3 \in \mathfrak{h}$ such that $\mu_1(H_3) \neq \mu_3(H_3)$. By applying the same procedure again we get

$$\begin{aligned} m_1 - \frac{(\mu_4 - \mu_3)(H_3) \cdot (\mu_4 - \mu_1)(H_2)}{(\mu_3 - \mu_1)(H_3) \cdot (\mu_2 - \mu_1)(H_2)} \cdot m_4 - \cdots - \frac{(\mu_r - \mu_3)(H_3) \cdot (\mu_r - \mu_1)(H_2)}{(\mu_3 - \mu_1)(H_3) \cdot (\mu_2 - \mu_1)(H_2)} \cdot m_r \\ = \left(1 - \frac{H_3 - \mu_1(H_3)}{(\mu_3 - \mu_1)(H_3)}\right)\left(1 - \frac{H_2 - \mu_1(H_2)}{(\mu_2 - \mu_1)(H_2)}\right) \cdot n \in N \end{aligned}$$

By applying the same procedure over and over again we can see that $m_1 = u \cdot n \in N$ for some $u \in \mathcal{U}(\mathfrak{g})$. Furthermore, if we reproduce all this for $m_2 + \cdots + m_r = n - m_1 \in N$ we get that $m_2 \in N$. All in all we find $m_1, \ldots, m_r \in N$. Hence

$$N = \bigoplus_\mu N_\mu = \bigoplus_\mu M(\lambda)_\mu \cap N$$

Since $M(\lambda) = \mathcal{U}(\mathfrak{g}) \cdot m^+$, if N is a proper submodule then $m^+ \notin N$. Hence any proper submodule lies in the sum of weight spaces other than $M(\lambda)_\lambda$, so the sum $N(\lambda)$ of all such submodules is still proper. This implies $N(\lambda)$ is the unique maximal submodule of $M(\lambda)$ and $L(\lambda) = {}^{M(\lambda)}/_{N(\lambda)}$ is its unique simple quotient. ∎

Corollary 4.52. *Let M be a simple weight $\mathfrak{g}$-module of weight λ. Then $M \cong L(\lambda)$.*

We thus know that $L(\lambda)$ is the only possible candidate for the $\mathfrak{g}$-module M in the statement of Theorem 4.39. We should also note that our past examples indicate that $L(\lambda)$ does fulfill its required role. Indeed...

Example 4.53. Consider the $\mathfrak{sl}_2(K)$ module $M(2)$ as described in Example 4.42. We can see from Example 4.43 that $N(2) = \bigoplus_{k \geqslant 3} K f^k \cdot m^+$, so that $L(2)$ is the 3-dimensional simple $\mathfrak{sl}_2(K)$-module – i.e. the finite-dimensional simple module with highest weight 2 constructed in chapter 3.

All its left to prove the Highest Weight Theorem is verifying that the situation encountered in Example 4.53 holds for any dominant integral $\lambda \in P^+$. In other words, we need to show...

Proposition 4.54. *If $\mathfrak{g}$ is semisimple and λ is dominant integral then the unique simple quotient $L(\lambda)$ of $M(\lambda)$ is finite-dimensional.*

The proof of Proposition 4.54 is very technical and we won't include it here, but the idea behind it is to show that the set of weights of $L(\lambda)$ is stable under the natural action of the Weyl group W on $\mathfrak{h}^*$. One can then show that the every weight of $L(\lambda)$ is conjugate to a single dominant integral weight of $L(\lambda)$, and that the set of dominant integral weights of $L(\lambda)$ is finite. Since W is finitely generated, this implies the set of weights of the unique simple quotient of $M(\lambda)$ is finite. But each weight space is finite-dimensional. Hence so is the simple quotient $L(\lambda)$.

We refer the reader to [E H73, ch. 21] for further details. We are now ready to prove the Highest Weight Theorem.

Proof of Theorem 4.39. We begin by the "existence" part of the theorem. Let λ be a dominant integral weight of $\mathfrak{g}$. Since $\dim L(\lambda) < \infty$, all its left is to show that $M = L(\lambda)$ is indeed a highest weight module of highest weight λ. It is clear from the definitions that $m^+ + N(\lambda) \in L(\lambda)_\lambda$ is singular and generates all of $L(\lambda)$. Hence it suffices to show that $m^+ + N(\lambda)$ is nonzero. But this is the same as checking that $m^+ \notin N(\lambda)$, which is also clear from the previous definitions. As for the uniqueness of M, it suffices to apply Corollary 4.52. ■

We would now like to conclude this chapter by describing the situation where $\lambda \notin P^+$. We begin by pointing out that Proposition 4.54 fails in the general setting. For instance, consider...

Example 4.55. The action of $\mathfrak{sl}_2(K)$ on $M(-4)$ is given by the following diagram. In general, it is possible to check using formula (4.2) that e always maps $f^{k+1} \cdot m^+$ to a nonzero multiple of $f^k \cdot m^+$, so we can see that $M(-4)$ has no proper submodules, $N(-4) = 0$ and thus $L(-4) \cong M(-4)$.

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "-28"] & M(-4)_{-10} \arrow[l, bend left, "1"] \arrow[r, bend left, "-18"] & M(-4)_{-8} \arrow[l, bend left, "1"] \arrow[r, bend left, "-10"] & M(-4)_{-6} \arrow[l, bend left, "1"] \arrow[r, bend left, "-4"] & M(-4)_{-4} \arrow[l, bend left, "1"]
\end{tikzcd},$$

While $L(\lambda)$ is always a highest weight module of highest weight λ, we can easily see that if $\lambda \notin P^+$ then $L(\lambda)$ is infinite-dimensional. Indeed, this is precisely the counterpositive of Proposition 4.38! If $\lambda = k_1\beta_1 + \cdots + k_r\beta_r \in P$ is integral and $k_i < 0$ for all i, then one is additionally able to show that $M(\lambda) \cong L(\lambda)$ as in Example 4.55. Verma modules can thus serve as examples of infinite-dimensional simple modules.

In the next chapter we expand our previous results by exploring the question: what are *all* the infinite-dimensional simple $\mathfrak{g}$-modules?

Chapter 5

Simple Weight Modules

In this chapter we will expand our results on finite-dimensional simple modules of semisimple Lie algebras by considering *infinite-dimensional* $\mathfrak{g}$-modules, which introduces numerous complications to our analysis.

For instance, in the infinite-dimensional setting we can no longer take complete-reducibility for granted. Indeed, we have seen that even if $\mathfrak{g}$ is a semisimple Lie algebra, there are infinite-dimensional $\mathfrak{g}$-modules which are not semisimple. For a counterexample look no further than Example 2.29: the regular $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g})$ is never semisimple. Nevertheless, for simplicity – or shall we say *semisimplicity* – we will focus exclusively on *semisimple* $\mathfrak{g}$-modules. Our strategy is, once again, that of classifying simple modules. The regular $\mathfrak{g}$-module hides further unpleasant surprises, however: recall from Example 4.13 that

$$\bigoplus_{\lambda} \mathcal{U}(\mathfrak{g})_\lambda = 0 \subsetneq \mathcal{U}(\mathfrak{g})$$

and the weight space decomposition fails for $\mathcal{U}(\mathfrak{g})$.

Indeed, our proof of the weight space decomposition in the finite-dimensional case relied heavily in the simultaneous diagonalization of commuting operators in a finite-dimensional space. Even if we restrict ourselves to simple modules, there is still a diverse spectrum of counterexamples to Corollary 4.12 in the infinite-dimensional setting. For instance, any $\mathfrak{g}$-module M whose restriction to $\mathfrak{h}$ is a free module satisfies $M_\lambda = 0$ for all λ as in Example 4.13. These are called $\mathfrak{h}$-*free* $\mathfrak{g}$-*modules*, and rank 1 simple $\mathfrak{h}$-free $\mathfrak{sp}_{2n}(K)$-modules where first classified by Nilsson in [Nil16]. Dimitar's construction of the so called *exponential tensor* $\mathfrak{sl}_n(K)$-*modules* in [GN20] is also an interesting source of counterexamples.

Since the weight space decomposition was perhaps the single most instrumental ingredient of our previous analysis, it is only natural to restrict ourselves to the case it holds. This brings us to the following definition.

Definition 5.1. A $\mathfrak{g}$-module M is called a *weight* $\mathfrak{g}$-*module* if $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ and $\dim M_\lambda < \infty$ for all $\lambda \in \mathfrak{h}^*$. The *support of* M is the set $\operatorname{supp} M = \{\lambda \in \mathfrak{h}^* : M_\lambda \neq 0\}$.

Example 5.2. Corollary 4.12 is equivalent to the fact that every finite-dimensional module of a semisimple Lie algebra is a weight module. More generally, every finite-dimensional simple module of a reductive Lie algebra is a weight module.

Example 5.3. We have seen that every finite-dimensional $\mathfrak{g}$-module is a weight module for semisimple $\mathfrak{g}$. In particular, if $\mathfrak{g}$ is finite-dimensional then the adjoint $\mathfrak{g}$-module $\mathfrak{g}$ is a weight module. More generally, a finite-dimensional Lie algebra $\mathfrak{g}$ is reductive if, and only if the adjoint $\mathfrak{g}$-module $\mathfrak{g}$ is a weight module, in which case its weight spaces are given by the root spaces of $\mathfrak{g}$

Example 5.4. Proposition 4.48 is equivalent to the fact that any highest weight $\mathfrak{g}$-module M of highest weight λ is a weight module whose support is contained in $\lambda + \mathbb{N}\Delta^- = \{\lambda - k_n\alpha_1 - \cdots - k_n\alpha_n : \alpha_i \in \Delta^+, k_i \in \mathbb{Z}, k_i \geqslant 0\}$. In particular, Verma modules are weight modules.

Example 5.5. Proposition 4.51 implies that the unique maximal submodule $N(\lambda)$ of $M(\lambda)$ is a weight module. In fact, the proof of Proposition 4.51 can be generalized to show that every submodule $N \subseteq M$ of a weight module M is a weight module, and $N_\lambda = M_\lambda \cap N$ for all $\lambda \in \mathfrak{h}^*$.

Example 5.6. Given a weight module M, a submodule $N \subseteq M$ and $\lambda \in \mathfrak{h}^*$, it is clear that ${}^{M_\lambda}/_{N} \subseteq ({}^{M}/_{N})_\lambda$. In addition, ${}^{M}/_{N} = \bigoplus_{\lambda\in\mathfrak{h}^*} {}^{M_\lambda}/_{N}$. Hence ${}^{M}/_{N}$ is weight $\mathfrak{g}$-module with $({}^{M}/_{N})_\lambda = {}^{M_\lambda}/_{N} \cong {}^{M_\lambda}/_{N_\lambda}$.

Example 5.7. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras, M_1 be a weight $\mathfrak{g}_1$-module and M_2 a weight $\mathfrak{g}_2$-module. Recall from Example 4.6 that if $\mathfrak{h}_i \subseteq \mathfrak{g}_i$ are Cartan subalgebras then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{h}^* = \mathfrak{h}_1^* \oplus \mathfrak{h}_2^*$. In this setting, one can readily check that $M_1 \otimes M_2$ is a weight $\mathfrak{g}$-module with

$$(M_1 \otimes M_2)_{\lambda_1+\lambda_2} = (M_1)_{\lambda_1} \otimes (M_2)_{\lambda_2}$$

for all $\lambda_i \in \mathfrak{h}_i^*$ and $\operatorname{supp}(M_1 \otimes M_2) = \operatorname{supp} M_1 \oplus \operatorname{supp} M_2 = \{\lambda_1 + \lambda_2 : \lambda_i \in \operatorname{supp} M_i \subseteq \mathfrak{h}_i^*\}$.

Example 5.8. Let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_r$ be a reductive Lie algebra, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{s}_1, \ldots, \mathfrak{s}_r$ are its simple components. As in Example 2.4, any simple weight $\mathfrak{g}$-module M can be decomposed as

$$M \cong Z \otimes M_1 \otimes \cdots \otimes M_r$$

where Z is a 1-dimensional representation of $\mathfrak{z}$ and M_i is a simple weight $\mathfrak{s}_i$-module. The modules Z and M_i are uniquely determined up to isomorphism.

Example 5.9. We would like to show that the requirement of finite-dimensionality in Definition 5.1 is not redundant. Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra and consider the adjoint $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g})$ – where $X \in \mathfrak{g}$ acts by taking commutators. Given $\alpha \in Q$, a simple computation shows $K\langle X_1 \cdots X_n H_1 \cdots H_m : X_i \in \mathfrak{g}_{\alpha_i}, H_i \in \mathfrak{h}, \alpha_i \in \Delta, \alpha = \alpha_1 + \cdots + \alpha_n\rangle \subseteq \mathcal{U}(\mathfrak{g})_\alpha$. The PBW Theorem and Example 5.3 thus imply that $\mathcal{U}(\mathfrak{g}) = \bigoplus_{\alpha\in Q} \mathcal{U}(\mathfrak{g})_\alpha$ where $\mathcal{U}(\mathfrak{g})_\alpha = K\langle X_1 \cdots X_n H_1 \cdots H_m : X_i \in \mathfrak{g}_{\alpha_i}, H_i \in \mathfrak{h}, \alpha_i \in \Delta, \alpha = \alpha_1 + \cdots + \alpha_n\rangle$. However, $\dim \mathcal{U}(\mathfrak{g})_\alpha = \infty$. For instance, $\mathcal{U}(\mathfrak{g})_0$ is *precisely* the commutator of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$, which contains $\mathcal{U}(\mathfrak{h})$ and is therefore infinite-dimensional.

Remark. We should stress that the weight spaces $M_\lambda \subseteq M$ of a given weight $\mathfrak{g}$-module M are *not* $\mathfrak{g}$-submodules. Nevertheless, M_λ is a $\mathfrak{h}$-submodule. More generally, M_λ is a $\mathcal{U}(\mathfrak{g})_0$-submodule, where $\mathcal{U}(\mathfrak{g})_0$ is the centralizer of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$ – which coincides with the weight space of $0 \in \mathfrak{h}^*$ in the adjoint $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g})$, as seen in Example 5.9.

A particularly well behaved class of examples are the so called *bounded* modules.

Definition 5.10. A weight $\mathfrak{g}$-module M is called *bounded* if $\dim M_\lambda$ is bounded. The lowest upper bound $\deg M$ for $\dim M_\lambda$ is called *the degree of* M. The *essential support* of M is the set $\operatorname{supp}_{\text{ess}} M = \{\lambda \in \mathfrak{h}^* : \dim M_\lambda = \deg M\}$.

Example 5.11. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras with Cartan subalgebras $\mathfrak{h}_i \subseteq \mathfrak{g}_i$ and take $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Given bounded $\mathfrak{g}_i$-modules M_i, it follows from Example 5.7 that $M_1 \otimes M_2$ is a bounded $\mathfrak{g}$-module with $\deg M_1 \otimes M_2 = \deg M_1 \cdot \deg M_2$ and

$$\operatorname{supp}_{\text{ess}}(M_1 \otimes M_2) = \operatorname{supp}_{\text{ess}} M_1 \oplus \operatorname{supp}_{\text{ess}} M_2 = \{\lambda_1 + \lambda_2 : \lambda_i \in \operatorname{supp}_{\text{ess}} M_i \subseteq \mathfrak{h}_i^*\}$$

Example 5.12. There is a natural action of $\mathfrak{sl}_2(K)$ on the space $K[x, x^{-1}]$ of Laurent polynomials, given by the formulas in (5.1). One can quickly verify $K[x, x^{-1}]_{2k} = Kx^k$ and $K[x, x^{-1}]_\lambda = 0$ for any $\lambda \notin 2\mathbb{Z}$, so that $K[x, x^{-1}] = \bigoplus_{k\in\mathbb{Z}} Kx^k$ is a degree 1 bounded weight $\mathfrak{sl}_2(K)$-module. It follows from the remark at the end of Example 5.5 that any nonzero submodule $N \subseteq K[x, x^{-1}]$ must contain a monomial x^k. But since the operators $-\frac{d}{dx} + \frac{x^{-1}}{2}, x^2\frac{d}{dx} + \frac{x}{2} : K[x, x^{-1}] \longrightarrow K[x, x^{-1}]$ are both injective, this implies all other monomials can be found in N by successively applying f and e. Hence $N = K[x, x^{-1}]$ and $K[x, x^{-1}]$ is a simple module.

$$e \cdot p = \left(x^2 \frac{d}{dx} + \frac{x}{2}\right) p \qquad f \cdot p = \left(-\frac{d}{dx} + \frac{x^{-1}}{2}\right) p \qquad h \cdot p = 2x\frac{d}{dx} p \tag{5.1}$$

Notice that the support of $K[x, x^{-1}]$ is the trivial $2\mathbb{Z}$-coset $0 + 2\mathbb{Z}$. This is representative of the general behavior in the following sense: if M is a simple weight $\mathfrak{g}$-module, since $M[\lambda] = \bigoplus_{\alpha \in Q} M_{\lambda+\alpha}$ is stable under the action of $\mathfrak{g}$ for all $\lambda \in \mathfrak{h}^*$, $\bigoplus_{\alpha \in Q} M_{\lambda+\alpha}$ is either 0 or all of M. In other words, the support of a simple weight module is always contained in a single Q-coset.

However, the behavior of $K[x, x^{-1}]$ deviates from that of an arbitrary bounded $\mathfrak{g}$-module in the sense its essential support is precisely the entire Q-coset it inhabits – i.e. $\operatorname{supp}_{\text{ess}} K[x, x^{-1}] = 2\mathbb{Z}$. This isn't always the case. Nevertheless, in general we find…

Proposition 5.13. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and M be a simple infinite-dimensional bounded $\mathfrak{g}$-module. The essential support* $\operatorname{supp}_{\text{ess}} M$ *is Zariski-dense*[1] *in $\mathfrak{h}^*$.*

This proof was deemed too technical to be included in here, but see Proposition 3.5 of [Mat00] for the case where $\mathfrak{g} = \mathfrak{s}$ is a simple Lie algebra. The general case then follows from Example 5.8, Example 5.11 and the asserting that the product of Zariski-dense subsets in K^n and K^m is Zariski-dense in $K^{n+m} = K^n \times K^m$.

We now begin a systematic investigation of the problem of classifying the infinite-dimensional simple weight modules of a given Lie algebra $\mathfrak{g}$. As in the previous chapter, let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. As a first approximation of a solution to our problem, we consider the Verma modules $M(\lambda)$ for $\lambda \in \mathfrak{h}^*$ which is not dominant integral. After all, the simple quotients of Verma modules form a remarkably large class of infinite-dimensional simple weight modules – at least as large as $\mathfrak{h}^* \setminus P^+$! More generally, the induction functor $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} : \mathfrak{b}\text{-}\mathbf{Mod} \longrightarrow \mathfrak{g}\text{-}\mathbf{Mod}$ has proven itself a powerful tool for constructing modules.

We claim this is not an unmotivated guess. Specifically, there are very good reasons behind the choice to consider induction over the Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$. First, the fact that $\mathfrak{h} \subseteq \mathfrak{g}$ affords us great control over the weight spaces of $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} M$: by assigning a prescribed action of $\mathfrak{h}$ to M we can ensure that $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} M = \bigoplus_\lambda (\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} M)_\lambda$. In addition, we have seen in the proof of Proposition 4.48 that by requiring that the positive part of $\mathfrak{b}$ acts on M by zero we can ensure that $\dim(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} M)_\lambda < \infty$. All in all, the nature of $\mathfrak{b}$ affords us just enough control to guarantee that $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} M$ is a weight module for sufficiently well behaved M.

Unfortunately for us, this is still too little control: there are simple weight modules which are not of the form $L(\lambda)$. More generally, we may consider induction over some parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ – i.e. some subalgebra such that $\mathfrak{p} \supseteq \mathfrak{b}$. This leads us to the following definition.

Definition 5.14. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra and M be a simple $\mathfrak{p}/\mathfrak{nil}(\mathfrak{p})$-module. We can view M as a $\mathfrak{p}$-module where $\mathfrak{nil}(\mathfrak{p})$ acts by zero by setting $X \cdot m = (X + \mathfrak{nil}(\mathfrak{p})) \cdot m$ for all $m \in M$ and $X \in \mathfrak{p}$ – which is the same as the $\mathfrak{p}$-module given by composing the action map $\mathfrak{p}/\mathfrak{nil}(\mathfrak{p}) \longrightarrow \mathfrak{gl}(M)$ with the projection $\mathfrak{p} \longrightarrow \mathfrak{p}/\mathfrak{nil}(\mathfrak{p})$. The module $M_{\mathfrak{p}}(M) = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M$ is called *generalized Verma module associated with M*.

Example 5.15. It is not hard to see that $\mathfrak{b}/\mathfrak{nil}(\mathfrak{b}) = \mathfrak{h}$. If we take $\lambda \in \mathfrak{h}^*$ and let Km^+ be the 1-dimensional $\mathfrak{h}$-module where $\mathfrak{h}$ acts by λ then $M(\lambda) = M_{\mathfrak{b}}(Km^+)$.

[1] Any choice of basis for $\mathfrak{h}^*$ induces a K-linear isomorphism $\mathfrak{h}^* \xrightarrow{\sim} K^n$. In particular, a choice of basis induces a unique topology in $\mathfrak{h}^*$ such that the map $\mathfrak{h}^* \longrightarrow K^n$ is a homeomorphism onto K^n with the Zariski topology. Any two basis induce the same topology in $\mathfrak{h}^*$, which we call *the Zariski topology of $\mathfrak{h}^*$*.

As promised, $M_{\mathfrak{p}}(M)$ is generally well behaved for well behaved M. In particular, if M is highest weight ${}^{\mathfrak{p}}/_{\mathfrak{nil}(\mathfrak{p})}$-module then $M_{\mathfrak{p}}(M)$ is also a highest weight $\mathfrak{g}$-module, and if M is a weight ${}^{\mathfrak{p}}/_{\mathfrak{nil}(\mathfrak{p})}$-module then $M_{\mathfrak{p}}(M)$ is a weight module with $M_{\mathfrak{p}}(M)_\lambda = \sum_{\alpha+\mu=\lambda} \mathcal{U}(\mathfrak{g})_\alpha \otimes_{\mathcal{U}(\mathfrak{p})} M_\mu$, – see Lemma 1.1 of [Mat00] for a full proof. However, $M_{\mathfrak{p}}(M)$ is not simple in general. Indeed, regular Verma modules not necessarily simple. This issue may be dealt with by passing to the simple quotients of $M_{\mathfrak{p}}(M)$.

Let M be a simple weight ${}^{\mathfrak{p}}/_{\mathfrak{nil}(\mathfrak{p})}$-module. As it turns out, the situation encountered in Proposition 4.51 is also verified in the general setting. Namely, since $M_{\mathfrak{p}}(M)$ is generated by $K \otimes_{\mathcal{U}(\mathfrak{p})} M = \bigoplus_{\lambda \in Q + \operatorname{supp} M} M_{\mathfrak{p}}(M)_\lambda$, it follows that any proper submodule of $M_{\mathfrak{p}}(M)$ is contained in $\bigoplus_{\lambda \notin Q + \operatorname{supp} M} M_{\mathfrak{p}}(M)_\lambda$. The sum $N_{\mathfrak{p}}(M)$ of all such submodules is thus the unique maximal submodule of $M_{\mathfrak{p}}(M)$ and $L_{\mathfrak{p}}(M) = {}^{M_{\mathfrak{p}}(M)}/_{N_{\mathfrak{p}}(M)}$ is its unique simple quotient – again, we refer the reader to [Mat00] for a complete proof. This leads us to the following definition.

Definition 5.16. A simple weight $\mathfrak{g}$-module is called *parabolic induced* if it is isomorphic to $L_{\mathfrak{p}}(M)$ for some proper parabolic subalgebra $\mathfrak{p} \subsetneq \mathfrak{g}$ and some simple weight ${}^{\mathfrak{p}}/_{\mathfrak{nil}(\mathfrak{p})}$-module M. A *cuspidal* $\mathfrak{g}$*-module* is a simple weight $\mathfrak{g}$-module which is *not* parabolic induced.

The first breakthrough regarding our classification problem was given by Fernando in his now infamous paper "Lie algebra modules with finite-dimensional weight spaces. I" [Fer90], where he proved that every simple weight $\mathfrak{g}$-module is parabolic induced by a cuspidal module.

Theorem 5.17 (Fernando)**.** *Any simple weight $\mathfrak{g}$-module is isomorphic to $L_{\mathfrak{p}}(M)$ for some parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ and some cuspidal ${}^{\mathfrak{p}}/_{\mathfrak{nil}(\mathfrak{p})}$-module M.*

We should point out that the relationship between simple weight $\mathfrak{g}$-modules and pairs $(\mathfrak{p}, M)$ is not one-to-one. Nevertheless, this relationship is well understood. Namely, Fernando himself established...

Proposition 5.18 (Fernando)**.** *Given a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, there exists a basis Σ for Δ such that $\Sigma \subseteq \Delta_{\mathfrak{p}} \subseteq \Delta$, where $\Delta_{\mathfrak{p}}$ denotes the set of roots of $\mathfrak{p}$. Furthermore, if $\mathfrak{p}' \subseteq \mathfrak{g}$ is another parabolic subalgebra, M is a cuspidal ${}^{\mathfrak{p}}/_{\mathfrak{nil}(\mathfrak{p})}$-module and N is a cuspidal ${}^{\mathfrak{p}'}/_{\mathfrak{nil}(\mathfrak{p}')}$-module then $L_{\mathfrak{p}}(M) \cong L_{\mathfrak{p}'}(N)$ if, and only if $\mathfrak{p}' = {}^\sigma\mathfrak{p}$ and $M \cong {}^\sigma N$ as $\mathfrak{p}$-modules for some*[2] *$\sigma \in W_M$, where*

$$W_M = \langle \sigma_\beta : \beta \in \Sigma, H_\beta + \mathfrak{nil}(\mathfrak{p}) \text{ is central in } {}^{\mathfrak{p}}/_{\mathfrak{nil}(\mathfrak{p})} \text{ and } H_\beta \text{ acts on } M \text{ as a positive integer} \rangle \subseteq W$$

Remark. The definition of the subgroup $W_M \subseteq W$ is independent of the choice of basis Σ.

As a first consequence of Fernando's Theorem, we provide two alternative characterizations of cuspidal modules.

Corollary 5.19 (Fernando)**.** *Let M be a simple weight $\mathfrak{g}$-module. The following conditions are equivalent.*

(i) *M is cuspidal.*

(ii) *F_α acts injectively on M for all $\alpha \in \Delta$.*

(iii) *The support of M is precisely one Q-coset.*

Example 5.20. As noted in Example 5.12, the element $f \in \mathfrak{sl}_2(K)$ acts injectively on the space of Laurent polynomials. Hence $K[x, x^{-1}]$ is a cuspidal $\mathfrak{sl}_2(K)$-module.

[2]Here ${}^\sigma\mathfrak{p}$ denotes the image of $\mathfrak{p}$ under the automorphism of $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ given by the canonical action of W on $\mathfrak{g}$ and ${}^\sigma N$ is the $\mathfrak{p}$-module given by composing the map $\mathfrak{p}' \longrightarrow \mathfrak{gl}(N)$ with the restriction $\sigma\restriction_{\mathfrak{p}} : \mathfrak{p} \longrightarrow \mathfrak{p}'$.

Having reduced our classification problem to that of classifying cuspidal modules, we are now faced the daunting task of actually classifying them. Historically, this was first achieved by Olivier Mathieu in the early 2000's in his paper "Classification of irreducible weight modules" [Mat00]. To do so, Mathieu introduced new tools which have since proved themselves remarkably useful throughout the field, known as...

5.1 Coherent Families

We begin our analysis with a simple question: how to do we go about constructing cuspidal modules? Specifically, given a cuspidal $\mathfrak{g}$-module, how can we use it to produce new cuspidal modules? To answer this question, we look back at the single example of a cuspidal module we have encountered so far: the $\mathfrak{sl}_2(K)$-module $K[x,x^{-1}]$ of Laurent polynomials – i.e. Example 5.12.

Our first observation is that $\mathfrak{sl}_2(K)$ acts on $K[x,x^{-1}]$ via differential operators. In other words, the action map $\mathcal{U}(\mathfrak{sl}_2(K)) \longrightarrow \mathrm{End}(K[x,x^{-1}])$ factors through the inclusion of the algebra $\mathrm{Diff}(K[x,x^{-1}]) = K\left[x,x^{-1},\frac{\mathrm{d}}{\mathrm{d}x}\right]$ of differential operators in $K[x,x^{-1}]$.

$$\mathcal{U}(\mathfrak{sl}_2(K)) \longrightarrow \mathrm{Diff}(K[x,x^{-1}]) \longrightarrow \mathrm{End}(K[x,x^{-1}])$$

The space $K[x,x^{-1}]$ can be regarded as a $\mathrm{Diff}(K[x,x^{-1}])$-module in the natural way, and we can produce new $\mathrm{Diff}(K[x,x^{-1}])$-modules by twisting $K[x,x^{-1}]$ by automorphisms of $\mathrm{Diff}(K[x,x^{-1}])$. For example, given $\lambda \in K$ we may take the automorphism

$$\begin{aligned}\varphi_\lambda : \mathrm{Diff}(K[x,x^{-1}]) &\longrightarrow \mathrm{Diff}(K[x,x^{-1}])\\ x &\longmapsto x\\ x^{-1} &\longmapsto x^{-1}\\ \frac{\mathrm{d}}{\mathrm{d}x} &\longmapsto \frac{\mathrm{d}}{\mathrm{d}x} + \frac{\lambda}{2}x^{-1}\end{aligned}$$

and consider the twisted module ${}^{\varphi_\lambda}K[x,x^{-1}] = K[x,x^{-1}]$, where some operator $P \in \mathrm{Diff}(K[x,x^{-1}])$ acts as $\varphi_\lambda(P)$.

By composing the action map $\mathrm{Diff}(K[x,x^{-1}]) \longrightarrow \mathrm{End}({}^{\varphi_\lambda}K[x,x^{-1}])$ with the homomorphism of algebras $\mathcal{U}(\mathfrak{sl}_2(K)) \longrightarrow \mathrm{Diff}(K[x,x^{-1}])$ we can give ${}^{\varphi_\lambda}K[x,x^{-1}]$ the structure of an $\mathfrak{sl}_2(K)$-module. Diagrammatically, we have

$$\mathcal{U}(\mathfrak{sl}_2(K)) \longrightarrow \mathrm{Diff}(K[x,x^{-1}]) \xrightarrow{\varphi_\lambda} \mathrm{Diff}(K[x,x^{-1}]) \longrightarrow \mathrm{End}(K[x,x^{-1}]) \ ,$$

where the maps $\mathcal{U}(\mathfrak{sl}_2(K)) \longrightarrow \mathrm{Diff}(K[x,x^{-1}])$ and $\mathrm{Diff}(K[x,x^{1}]) \longrightarrow \mathrm{End}(K[x,x^{-1}])$ are the ones from the previous diagram.

Explicitly, we find that the action of $\mathfrak{sl}_2(K)$ on ${}^{\varphi_\lambda}K[x,x^{-1}]$ is given by

$$p \overset{e}{\longmapsto} \left(x^2\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1+\lambda}{2}x\right)p \qquad p \overset{f}{\longmapsto} \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1-\lambda}{2}x^{-1}\right)p \qquad p \overset{h}{\longmapsto} \left(2x\frac{\mathrm{d}}{\mathrm{d}x} + \lambda\right)p,$$

so we can see ${}^{\varphi_\lambda}K[x,x^{-1}]_{2k+\frac{\lambda}{2}} = Kx^k$ for all $k \in \mathbb{Z}$ and ${}^{\varphi_\lambda}K[x,x^{-1}]_\mu = 0$ for all other $\mu \in \mathfrak{h}^*$.

Hence ${}^{\varphi_\lambda}K[x,x^{-1}]$ is a degree 1 bounded $\mathfrak{sl}_2(K)$-module with $\mathrm{supp}\,{}^{\varphi_\lambda}K[x,x^{-1}] = \frac{\lambda}{2} + 2\mathbb{Z}$. One can also quickly check that if $\lambda \notin 1+2\mathbb{Z}$ then e and f act injectively in ${}^{\varphi_\lambda}K[x,x^{-1}]$, so that ${}^{\varphi_\lambda}K[x,x^{-1}]$ is simple. In particular, if $\lambda,\mu \notin 1+2\mathbb{Z}$ with $\lambda \notin \mu+2\mathbb{Z}$ then ${}^{\varphi_\lambda}K[x,x^{-1}]$ and ${}^{\varphi_\mu}K[x,x^{-1}]$ are non-isomorphic cuspidal $\mathfrak{sl}_2(K)$-modules, since their supports differ. These cuspidal modules can be "glued together" in a *monstrous concoction* by summing over $\lambda \in K$, as in

$$\mathscr{M} = \bigoplus_{\lambda+2\mathbb{Z}\in K/2\mathbb{Z}} {}^{\varphi_\lambda}K[x,x^{-1}],$$

To a distracted spectator, $\mathscr{M}$ may look like just another, innocent, $\mathfrak{sl}_2(K)$-module. However, the attentive reader may have already noticed some of the its bizarre features, most noticeable of which is the fact that $\mathscr{M}$ is very big. In fact, $\mathscr{M}$ is as big a degree 1 bounded module gets: $\operatorname{supp} \mathscr{M} = \operatorname{supp}_{\mathrm{ess}} \mathscr{M}$ is the entirety of $\mathfrak{h}^*$. This may look very alien the reader familiarized with the finite-dimensional setting, where the configuration of weights is very rigid. For this reason, $\mathscr{M}$ deserves to be called "a monstrous concoction".

On a perhaps less derogatory note, $\mathscr{M}$ also deserves to be called *a family*. This is because $\mathscr{M}$ consists of lots of smaller cuspidal modules which fit together inside of it in a *coherent* fashion. Mathieu's ingenious breakthrough was the realization that $\mathscr{M}$ is a particular example of a more general pattern, which he named *coherent families*.

Definition 5.21. A *coherent family $\mathscr{M}$ of degree d* is a weight $\mathfrak{g}$-module $\mathscr{M}$ such that

(i) $\dim \mathscr{M}_\lambda = d$ for *all* $\lambda \in \mathfrak{h}^*$ – i.e. $\operatorname{supp}_{\mathrm{ess}} \mathscr{M} = \mathfrak{h}^*$.

(ii) For any $u \in \mathscr{U}(\mathfrak{g})$ in the centralizer $\mathscr{U}(\mathfrak{g})_0$ of $\mathfrak{h}$ in $\mathscr{U}(\mathfrak{g})$, the map

$$\begin{aligned} \mathfrak{h}^* &\longrightarrow K \\ \lambda &\longmapsto \operatorname{Tr}(u \restriction_{\mathscr{M}_\lambda}) \end{aligned}$$

is polynomial in λ.

Example 5.22. The module $\mathscr{M} = \bigoplus_{\lambda + 2\mathbb{Z} \in K/2\mathbb{Z}} {}^{\varphi_\lambda}K[x, x^{-1}]$ is a degree 1 coherent $\mathfrak{sl}_2(K)$-family.

Example 5.23. Given $\lambda \in K$, $\mathscr{M}(\lambda) = \bigoplus_{\mu \in K} K x^\mu$ with

$$p \xmapsto{e} \left(x^2 \frac{\mathrm{d}}{\mathrm{d}x} + \lambda x\right) p \qquad p \xmapsto{f} \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \lambda x^{-1}\right) p \qquad p \xmapsto{h} 2x \frac{\mathrm{d}}{\mathrm{d}x} p,$$

is a degree 1 coherent $\mathfrak{sl}_2(K)$-family – where $x^{\pm 1}, {}^{\mathrm{d}}/_{\mathrm{d}x} : \mathscr{M}(\lambda) \longrightarrow \mathscr{M}(\lambda)$ are given by $x^{\pm 1} x^\mu = x^{\mu \pm 1}$ and ${}^{\mathrm{d}}/_{\mathrm{d}x} x^\mu = \mu x^{\mu - 1}$. It is easy to check $\mathscr{M}$ from Example 5.22 is isomorphic to $\mathscr{M}(1/2)$ and $(\mathscr{M}(1/2))[0] \cong K[x, x^{-1}]$.

Remark. We would like to stress that coherent families have proven themselves useful for problems other than the classification of cuspidal $\mathfrak{g}$-modules. For instance, Nilsson's classification of rank 1 $\mathfrak{h}$-free $\mathfrak{sp}_{2n}(K)$-modules is based on the notion of coherent families and the so called *weighting functor*.

Our hope is that given a cuspidal module M, we can somehow fit M inside of a coherent $\mathfrak{g}$-family, such as in the case of $K[x, x^{-1}]$ and $\mathscr{M}$ from Example 5.22. In addition, we hope that such coherent families are somehow *uniquely determined* by M. This leads us to the following definition.

Definition 5.24. Given a bounded $\mathfrak{g}$-module M of degree d, a *coherent extension $\mathscr{M}$ of M* is a coherent family $\mathscr{M}$ of degree d that contains M as a subquotient.

Our goal is now showing that every simple bounded module has a coherent extension. The idea then is to classify coherent families, and classify which submodules of a given coherent family are actually cuspidal modules. If every simple bounded $\mathfrak{g}$-module fits inside a coherent extension, this would lead to classification of all cuspidal $\mathfrak{g}$-modules, which we now know is the key for the solution of our classification problem. However, there are some complications to this scheme.

Leaving aside the question of existence for a second, we should point out that coherent families turn out to be rather complicated on their own. In fact they are too complicated to classify in general. Ideally, we would like to find *nice* coherent extensions – ones we can actually classify. For instance, we may search for *irreducible* coherent extensions, which are defined as follows.

Definition 5.25. A coherent family $\mathscr{M}$ is called *irreducible* if it contains no proper coherent subfamilies – i.e. $\mathscr{M}$ is a simple object in the full subcategory of $\mathfrak{g}$-**Mod** consisting of coherent families. Equivalently, we call $\mathscr{M}$ irreducible if $\mathscr{M}_\lambda$ is a simple $\mathscr{U}(\mathfrak{g})_0$-module for some $\lambda \in \mathfrak{h}^*$.

Another natural candidate for the role of "nice extensions" are the semisimple coherent families – i.e. families which are semisimple as $\mathfrak{g}$-modules. These turn out to be very easy to produce. Namely, there is a construction, known as *the semisimplification of a coherent family*, which takes a coherent extension of M to a semisimple coherent extension of M.

Lemma 5.26. *Given a coherent family $\mathscr{M}$ and $\lambda \in \mathfrak{h}^*$, $\mathscr{M}[\lambda]$ has finite length as a $\mathfrak{g}$-module.*

Proposition 5.27. *Let $\mathscr{M}$ be a coherent family of degree d. There exists a unique semisimple coherent family $\mathscr{M}^{\text{ss}}$ of degree d such that the composition series of $\mathscr{M}^{\text{ss}}[\lambda]$ is the same as that of $\mathscr{M}[\lambda]$ for all $\lambda \in \mathfrak{h}^*$, called* the semisimplification of $\mathscr{M}$.
Namely, if $\lambda \in \mathfrak{h}^*$ and $0 = \mathscr{M}_{\lambda 0} \subseteq \mathscr{M}_{\lambda 1} \subseteq \cdots \subseteq \mathscr{M}_{\lambda r_\lambda} = \mathscr{M}[\lambda]$ is a composition series[3],

$$\mathscr{M}^{\text{ss}} \cong \bigoplus_{\substack{\lambda+Q \in \mathfrak{h}^*/Q \\ i}} \mathscr{M}_{\lambda i+1}/\mathscr{M}_{\lambda i}$$

Proof. The uniqueness of $\mathscr{M}^{\text{ss}}$ should be clear: since $\mathscr{M}^{\text{ss}}$ is semisimple, so is $\mathscr{M}^{\text{ss}}[\lambda]$. Hence by the Jordan-Hölder Theorem

$$\mathscr{M}^{\text{ss}}[\lambda] \cong \bigoplus_i \mathscr{M}_{\lambda i+1}/\mathscr{M}_{\lambda i}$$

As for the existence of the semisimplification, it suffices to show

$$\mathscr{M}^{\text{ss}} = \bigoplus_{\substack{\lambda+Q \in \mathfrak{h}^*/Q \\ i}} \mathscr{M}_{\lambda i+1}/\mathscr{M}_{\lambda i}$$

is indeed a semisimple coherent family of degree d.

We know from Examples 5.5 and 5.6 that each quotient $\mathscr{M}_{\lambda i+1}/\mathscr{M}_{\lambda i}$ is a weight module. Hence $\mathscr{M}^{\text{ss}}$ is a weight module. Furthermore, given $\mu \in \mathfrak{h}^*$

$$\mathscr{M}^{\text{ss}}_\mu = \bigoplus_{\substack{\lambda+Q \in \mathfrak{h}^*/Q \\ i}} \left(\mathscr{M}_{\lambda i+1}/\mathscr{M}_{\lambda i}\right)_\mu = \bigoplus_i \left(\mathscr{M}_{\mu i+1}/\mathscr{M}_{\mu i}\right)_\mu \cong \bigoplus_i (\mathscr{M}_{\mu i+1})_\mu/(\mathscr{M}_{\mu i})_\mu$$

In particular,

$$\dim \mathscr{M}^{\text{ss}}_\mu = \sum_i \dim(\mathscr{M}_{\mu i+1})_\mu - \dim(\mathscr{M}_{\mu i})_\mu = \dim \mathscr{M}[\mu]_\mu = \dim \mathscr{M}_\mu = d$$

Likewise, given $u \in \mathscr{U}(\mathfrak{g})_0$ the value

$$\operatorname{Tr}(u\restriction_{\mathscr{M}^{\text{ss}}_\mu}) = \sum_i \operatorname{Tr}(u\restriction_{(\mathscr{M}_{\mu i+1})_\mu}) - \operatorname{Tr}(u\restriction_{(\mathscr{M}_{\mu i})_\mu}) = \operatorname{Tr}(u\restriction_{\mathscr{M}[\mu]_\mu}) = \operatorname{Tr}(u\restriction_{\mathscr{M}_\mu})$$

is polynomial in $\mu \in \mathfrak{h}^*$. ■

[3]Notice that $\mathscr{M}[\lambda] = \mathscr{M}[\mu]$ for any $\mu \in \lambda + Q$. Hence the sum $\bigoplus_{\lambda+Q\in\mathfrak{h}^*/Q} \bigoplus_i \mathscr{M}_{\lambda i+1}/\mathscr{M}_{\lambda i}$ is independent of the choice of representative for $\lambda + Q$ – provided we choose $\mathscr{M}_{\mu i} = \mathscr{M}_{\lambda i}$ for all $\mu \in \lambda + Q$ and i.

Remark. Although we have provided an explicit construction of $\mathscr{M}^{\mathrm{ss}}$ in terms of $\mathscr{M}$, we should point out this construction is not functorial. First, given a $\mathfrak{g}$-homomorphism $f : \mathscr{M} \longrightarrow \mathscr{N}$ between coherent families, it is unclear what $f^{\mathrm{ss}} : \mathscr{M}^{\mathrm{ss}} \longrightarrow \mathscr{N}^{\mathrm{ss}}$ is supposed to be. Secondly, and this is more relevant, our construction depends on the choice of composition series $0 = \mathscr{M}_{\lambda 0} \subseteq \cdots \subseteq \mathscr{M}_{\lambda r_\lambda} = \mathscr{M}[\lambda]$. While different choices of composition series yield isomorphic results, there is no canonical isomorphism. In addition, there is no canonical choice of composition series.

The proof of Lemma 5.26 is extremely technical and will not be included in here. It suffices to note that, as in Proposition 5.13, the general case follows from the case where $\mathfrak{g}$ is simple, which may be found in [Mat00] – see Lemma 3.3. As promised, if $\mathscr{M}$ is a coherent extension of M then so is $\mathscr{M}^{\mathrm{ss}}$.

Proposition 5.28. *Let M be a simple bounded $\mathfrak{g}$-module and $\mathscr{M}$ be a coherent extension of M. Then $\mathscr{M}^{\mathrm{ss}}$ is a coherent extension of M, and M is in fact a submodule of $\mathscr{M}^{\mathrm{ss}}$.*

Proof. Since M is simple, its support is contained in a single Q-coset. This implies that M is a subquotient of $\mathscr{M}[\lambda]$ for any $\lambda \in \operatorname{supp} M$. If we fix some composition series $0 = \mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \cdots \subseteq \mathscr{M}_r = \mathscr{M}[\lambda]$ of $\mathscr{M}[\lambda]$ with $M \cong \mathscr{M}_{i+1}/\mathscr{M}_i$, there is a natural inclusion

$$M \xrightarrow{\sim} \mathscr{M}_{i+1}/\mathscr{M}_i \longrightarrow \bigoplus_j \mathscr{M}_{j+1}/\mathscr{M}_j \cong \mathscr{M}^{\mathrm{ss}}[\lambda]$$

∎

Given the uniqueness of the semisimplification, the semisimplification of any semisimple coherent extension $\mathscr{M}$ is $\mathscr{M}$ itself and therefore...

Corollary 5.29. *Let M be a simple bounded $\mathfrak{g}$-module and $\mathscr{M}$ be a semisimple coherent extension of M. Then M is contained in $\mathscr{M}$.*

These last results provide a partial answer to the question of existence of well behaved coherent extensions. As for the uniqueness $\mathscr{M}$ in Corollary 5.29, it suffices to show that the multiplicities of the simple weight $\mathfrak{g}$-modules in $\mathscr{M}$ are uniquely determined by M. These multiplicities may be computed via the following lemma.

Lemma 5.30. *Let M be a semisimple weight $\mathfrak{g}$-module. Then M_λ is a semisimple $\mathscr{U}(\mathfrak{g})_0$-module for any $\lambda \in \operatorname{supp} M$. Moreover, if L is a simple weight $\mathfrak{g}$-module such that $\lambda \in \operatorname{supp} L$ then L_λ is a simple $\mathscr{U}(\mathfrak{g})_0$-module and the multiplicity L in M coincides with the multiplicity of L_λ in M_λ as a $\mathscr{U}(\mathfrak{g})_0$-module.*

Proof. We begin by showing that L_λ is simple. Let $N \subseteq L_\lambda$ be a nontrivial $\mathscr{U}(\mathfrak{g})_0$-submodule. We want to establish that $N = L_\lambda$.

If $\mathscr{U}(\mathfrak{g})_\alpha$ denotes the root space of α in $\mathscr{U}(\mathfrak{g})$ under the adjoint action of $\mathfrak{g}$ as in Example 5.9, $\alpha \in Q$, a simple calculation shows $\mathscr{U}(\mathfrak{g})_\alpha \cdot N \subseteq L_{\lambda+\alpha}$. Since L is simple and N is nonzero, it follows from Example 5.9 that

$$L = \mathscr{U}(\mathfrak{g}) \cdot N = \bigoplus_{\alpha \in Q} \mathscr{U}(\mathfrak{g})_\alpha \cdot N$$

and thus $L_{\lambda+\alpha} = \mathscr{U}(\mathfrak{g})_\alpha \cdot N$. In particular, $L_\lambda = \mathscr{U}(\mathfrak{g})_0 \cdot N \subseteq N$ and $N = L_\lambda$.

Now given a semisimple weight $\mathfrak{g}$-module $M = \bigoplus_i M_i$ with M_i simple, it is clear $M_\lambda = \bigoplus_i (M_i)_\lambda$. Each $(M_i)_\lambda$ is either 0 or a simple $\mathscr{U}(\mathfrak{g})_0$-module, so that M_λ is a semisimple $\mathscr{U}(\mathfrak{g})_0$-module. In addition, to see that the multiplicity of L in M coincides with the multiplicity of L_λ in M_λ it suffices to show that if $(M_i)_\lambda \cong (M_j)_\lambda$ are both nonzero then $M_i \cong M_j$.

If $I(M_i) = \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{g})_0} (M_i)_\lambda$, the inclusion of $\mathscr{U}(\mathfrak{g})_0$-modules $(M_i)_\lambda \longrightarrow M_i$ induces a $\mathfrak{g}$-homomorphism

$$\begin{aligned} I(M_i) &\longrightarrow M_i \\ u \otimes m &\longmapsto u \cdot m \end{aligned}$$

Since M_i is simple and $\lambda \in \operatorname{supp} M_i$, $M_i = \mathscr{U}(\mathfrak{g}) \cdot (M_i)_\lambda$. The homomorphism $I(M_i) \longrightarrow M_i$ is thus surjective. Similarly, if $I(M_j) = \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{g})_0} (M_j)_\lambda$ then there is a natural surjective $\mathfrak{g}$-homomorphism $I(M_j) \longrightarrow M_j$. Now suppose there is an isomorphism of $\mathscr{U}(\mathfrak{g})_0$-modules $f : (M_i)_\lambda \xrightarrow{\sim} (M_j)_\lambda$. Such an isomorphism induces an isomorphism of $\mathfrak{g}$-modules

$$\begin{aligned} \tilde{f} : I(M_i) &\xrightarrow{\sim} I(M_j) \\ u \otimes m &\longmapsto u \otimes f(m) \end{aligned}$$

By composing $\tilde{f}$ with the projection $I(M_j) \longrightarrow M_j$ we get a surjective homomorphism $I(M_i) \longrightarrow M_j$. We claim $\ker(I(M_i) \longrightarrow M_i) = \ker(I(M_i) \longrightarrow M_j)$. To see this, notice that $\ker(I(M_i) \longrightarrow M_i)$ coincides with the largest submodule $Z(M_i) \subseteq I(M_i)$ contained in $\bigoplus_{\alpha \neq 0} \mathscr{U}(\mathfrak{g})_\alpha \otimes_{\mathscr{U}(\mathfrak{g})_0} (M_i)_\lambda$. Indeed, a simple computation shows $\ker(I(M_i) \longrightarrow M_i) \cap (\mathscr{U}(\mathfrak{g})_0 \otimes_{\mathscr{U}(\mathfrak{g})_0} (M_i)_\lambda) = 0$, which implies $\ker(I(M_i) \longrightarrow M_i) \subseteq Z(M_i)$. Since M_i is simple, $\ker(I(M_i) \longrightarrow M_i)$ is maximal and thus $\ker(I(M_i) \longrightarrow M_i) = Z(M_i)$. By the same token, $\ker(I(M_j) \longrightarrow M_j)$ is the largest submodule of $I(M_j)$ contained in $\bigoplus_{\alpha \neq 0} \mathscr{U}(\mathfrak{g})_\alpha \otimes_{\mathscr{U}(\mathfrak{g})_0} (M_j)_\lambda$ and therefore $\ker(I(M_i) \longrightarrow M_i) = \tilde{f}^{-1}(\ker(I(M_j) \longrightarrow M_j)) = \ker(I(M_i) \longrightarrow M_j)$.

Hence there is an isomorphism ${}^{I(M_i)}\!/\!{}_{\ker(I(M_i) \longrightarrow M_i)} \xrightarrow{\sim} M_j$ satisfying

$$\begin{tikzcd} I(M_i) \arrow[r, "\tilde{f}"] \arrow[d] & I(M_j) \arrow[d] \\ {}^{I(M_i)}\!/\!{}_{\ker(I(M_i) \longrightarrow M_i)} \arrow[r, "\sim"] & M_j \end{tikzcd}$$

and finally $M_i \cong {}^{I(M_i)}\!/\!{}_{\ker(I(M_i) \longrightarrow M_i)} \cong M_j$. ■

A complementary question now is: which submodules of a *nice* coherent family are cuspidal?

Proposition 5.31 (Mathieu). *Let $\mathscr{M}$ be an irreducible coherent family of degree d and $\lambda \in \mathfrak{h}^*$. The following conditions are equivalent.*

(i) *$\mathscr{M}[\lambda]$ is simple.*

(ii) *$F_\alpha \restriction_{\mathscr{M}[\lambda]}$ is injective for all $\alpha \in \Delta$.*

(iii) *$\mathscr{M}[\lambda]$ is cuspidal.*

Proof. The fact that **(i)** and **(iii)** are equivalent follows directly from Corollary 5.19. Likewise, it is clear from the corollary that **(iii)** implies **(ii)**. All it is left is to show **(ii)** implies **(iii)**. This isn't already clear from Corollary 5.19 because, at first glance, $\mathscr{M}[\lambda]$ may not be simple for some λ satisfying **(ii)**. We will show this is never the case.

Suppose F_α acts injectively on the submodule $\mathscr{M}[\lambda]$, for all $\alpha \in \Delta$. Since $\mathscr{M}[\lambda]$ has finite length, $\mathscr{M}[\lambda]$ contains an infinite-dimensional simple $\mathfrak{g}$-submodule M. Moreover, again by Corollary 5.19 we conclude M is a cuspidal module, and its degree is bounded by d. We want to show $\mathscr{M}[\lambda] = M$.

We claim the set $U = \{\mu \in \mathfrak{h}^* : \mathscr{M}_\mu \text{ is a simple } \mathscr{U}(\mathfrak{g})_0\text{-module}\}$ is Zariski-open. If we suppose this is the case for a moment or two, it follows from the fact that M is simple and $\operatorname{supp}_{\text{ess}} M$ is Zariski-dense that $U \cap \operatorname{supp}_{\text{ess}} M$ is non-empty. In other words, there is some $\mu \in \mathfrak{h}^*$ such that $\mathscr{M}_\mu$ is a simple $\mathscr{U}(\mathfrak{g})_0$-module and $\dim M_\mu = \deg M$.

In particular, $M_\mu \neq 0$, so $M_\mu = \mathcal{M}_\mu$. Now given any simple $\mathfrak{g}$-module L, it follows from Lemma 5.30 that the multiplicity of L in $\mathcal{M}[\lambda]$ is the same as the multiplicity L_μ in $\mathcal{M}_\mu$ as a $\mathcal{U}(\mathfrak{g})_0$-module – which is, of course, 1 if $L \cong M$ and 0 otherwise. Hence $\mathcal{M}[\lambda] = M$ and $\mathcal{M}[\lambda]$ is cuspidal. ■

To finish the proof, we now show...

Lemma 5.32. *Let $\mathcal{M}$ be a coherent family. The set $U = \{\lambda \in \mathfrak{h}^* : \mathcal{M}_\lambda \text{ is a simple } \mathcal{U}(\mathfrak{g})_0\text{-module}\}$ is Zariski-open.*

Proof. For each $\lambda \in \mathfrak{h}^*$ we introduce the bilinear form

$$\begin{aligned} B_\lambda : \mathcal{U}(\mathfrak{g})_0 \times \mathcal{U}(\mathfrak{g})_0 &\longrightarrow K \\ (u,v) &\longmapsto \operatorname{Tr}(uv\restriction_{\mathcal{M}_\lambda}) \end{aligned}$$

and consider its rank – i.e. the dimension of the image of the induced operator

$$\begin{aligned} \mathcal{U}(\mathfrak{g})_0 &\longrightarrow \mathcal{U}(\mathfrak{g})_0^* \\ u &\longmapsto B_\lambda(u,\cdot) \end{aligned}$$

Our first observation is that $\operatorname{rank} B_\lambda \leqslant d^2$. This follows from the commutativity of

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g})_0 \arrow[r] \arrow[d] & \mathcal{U}(\mathfrak{g})_0^* \\ \operatorname{End}(\mathcal{M}_\lambda) \arrow[r, "\sim"] & \operatorname{End}(\mathcal{M}_\lambda)^* \arrow[u] \end{tikzcd},$$

where the map $\mathcal{U}(\mathfrak{g})_0 \longrightarrow \operatorname{End}(\mathcal{M}_\lambda)$ is given by the action of $\mathcal{U}(\mathfrak{g})_0$, the map $\operatorname{End}(\mathcal{M}_\lambda)^* \longrightarrow \mathcal{U}(\mathfrak{g})_0^*$ is its dual, and the isomorphism $\operatorname{End}(\mathcal{M}_\lambda) \xrightarrow{\sim} \operatorname{End}(\mathcal{M}_\lambda)^*$ is induced by the trace form

$$\begin{aligned} \operatorname{End}(\mathcal{M}_\lambda) \times \operatorname{End}(\mathcal{M}_\lambda) &\longrightarrow K \\ (T,S) &\longmapsto \operatorname{Tr}(TS) \end{aligned}$$

Indeed, $\operatorname{rank} B_\lambda \leqslant \operatorname{rank}(\mathcal{U}(\mathfrak{g})_0 \longrightarrow \operatorname{End}(\mathcal{M}_\lambda)) \leqslant \dim \operatorname{End}(\mathcal{M}_\lambda) = d^2$. Furthermore, if $\operatorname{rank} B_\lambda = d^2$ then we must have $\operatorname{rank}(\mathcal{U}(\mathfrak{g})_0 \longrightarrow \operatorname{End}(\mathcal{M}_\lambda)) = d^2$ – i.e. the map $\mathcal{U}(\mathfrak{g})_0 \longrightarrow \operatorname{End}(\mathcal{M}_\lambda)$ is surjective. In particular, if $\operatorname{rank} B_\lambda = d^2$ then $\mathcal{M}_\lambda$ is a simple $\mathcal{U}(\mathfrak{g})_0$-module, for if $M \subseteq \mathcal{M}_\lambda$ is invariant under the action of $\mathcal{U}(\mathfrak{g})_0$ then M is invariant under any K-linear operator $\mathcal{M}_\lambda \longrightarrow \mathcal{M}_\lambda$, so that $M = 0$ or $M = \mathcal{M}_\lambda$.

On the other hand, if $\mathcal{M}_\lambda$ is simple then by Burnside's Theorem on matrix algebras the map $\mathcal{U}(\mathfrak{g})_0 \longrightarrow \operatorname{End}(\mathcal{M}_\lambda)$ is surjective. Hence the commutativity of the previously drawn diagram, as well as the fact that $\operatorname{rank}(\mathcal{U}(\mathfrak{g})_0 \longrightarrow \operatorname{End}(\mathcal{M}_\lambda)) = \operatorname{rank}(\operatorname{End}(\mathcal{M}_\lambda)^* \longrightarrow \mathcal{U}(\mathfrak{g})_0^*)$, imply that $\operatorname{rank} B_\lambda = d^2$. This goes to show that U is precisely the set of all λ such that B_λ has maximal rank d^2. We now show that U is Zariski-open. First, notice that

$$U = \bigcup_{\substack{V \subseteq \mathcal{U}(\mathfrak{g})_0 \\ \dim V = d}} U_V,$$

where $U_V = \{\lambda \in \mathfrak{h}^* : \operatorname{rank} B_\lambda\restriction_V = d^2\}$. Here V ranges over all d-dimensional subspaces of $\mathcal{U}(\mathfrak{g})_0$ – V is not necessarily a $\mathcal{U}(\mathfrak{g})_0$-submodule.

Indeed, if $\operatorname{rank} B_\lambda = d^2$ it follows from the subjectivity of the map $\mathcal{U}(\mathfrak{g})_0 \longrightarrow \operatorname{End}(\mathcal{M}_\lambda)$ that there is some $V \subseteq \mathcal{U}(\mathfrak{g})_0$ with $\dim V = d$ such that the restriction $V \longrightarrow \operatorname{End}(\mathcal{M}_\lambda)$ is surjective. The commutativity of

$$
\begin{tikzcd}
V \arrow[r] \arrow[d] & V^* \\
\operatorname{End}(\mathscr{M}_\lambda) \arrow[r, "\sim"] & \operatorname{End}(\mathscr{M}_\lambda)^* \arrow[u]
\end{tikzcd}
$$

then implies rank $B_\lambda \restriction_V = d^2$. In other words, $U \subseteq \bigcup_V U_V$.

Likewise, if rank $B_\lambda \restriction_V = d^2$ for some V, then the commutativity of

$$
\begin{tikzcd}
V \arrow[r] \arrow[d] & V^* \\
\mathcal{U}(\mathfrak{g})_0 \arrow[r] & \mathcal{U}(\mathfrak{g})_0^* \arrow[u]
\end{tikzcd}
$$

implies rank $B_\lambda \geqslant d^2$, which goes to show $\bigcup_V U_V \subseteq U$.

Given $\lambda \in U_V$, the surjectivity of $V \longrightarrow \operatorname{End}(\mathscr{M}_\lambda)$ and the fact that $\dim V < \infty$ imply $V \longrightarrow V^*$ is invertible. Since $\mathscr{M}$ is a coherent family, B_λ depends polynomially in λ. Hence so does the induced maps $V \longrightarrow V^*$. In particular, there is some Zariski neighborhood U' of λ such that the map $V \longrightarrow V^*$ induced by $B_\mu \restriction_V$ is invertible for all $\mu \in U'$.

But the surjectivity of the map induced by $B_\mu \restriction_V$ implies rank $B_\mu = d^2$, so $\mu \in U_V$ and therefore $U' \subseteq U_V$. This implies U_V is open for all V. Finally, U is the union of Zariski-open subsets and is therefore open. We are done. ■

The major remaining question for us to tackle is that of the existence of coherent extensions, which will be the focus of our next section.

5.2 Localizations & the Existence of Coherent Extensions

Let M be a simple bounded $\mathfrak{g}$-module of degree d. Our goal is to prove that M has a (unique) irreducible semisimple coherent extension $\mathscr{M}$. Since M is simple, we know $M \subseteq \mathscr{M}[\lambda]$ for any $\lambda \in \operatorname{supp} M$. Our first task is constructing $\mathscr{M}[\lambda]$. The issue here is that $\operatorname{supp}_{\text{ess}} M$ may not be all of $\lambda + Q = \operatorname{supp}_{\text{ess}} \mathscr{M}[\lambda]$, so we may find $M \subsetneq \mathscr{M}[\lambda]$. In fact, we may find $\operatorname{supp} M \subsetneq \lambda + Q$.

This wasn't an issue an Example 5.12 because we verified that the action of $f \in \mathfrak{sl}_2(K)$ on $K[x, x^{-1}]$ is injective. Since all weight spaces of $K[x, x^{-1}]$ are 1-dimensional, this implies the action of f is actually bijective, so we can obtain a nonzero vector in $K[x, x^{-1}]_{2k} = Kx^k$ for any $k \in \mathbb{Z}$ by translating between weight spaced using f and f^{-1} – here f^{-1} denotes the K-linear operator $(-{}^{\mathrm{d}}/_{\mathrm{d}x} + {}^{x^{-1}}/_2)^{-1}$, which is the inverse of the action of f on $K[x, x^{-1}]$.

$$
\begin{tikzcd}
\cdots \arrow[r, bend left, "f^{-1}"] & Kx^{-2} \arrow[l, bend left, "f"] \arrow[r, bend left, "f^{-1}"] & Kx^{-1} \arrow[l, bend left, "f"] \arrow[r, bend left, "f^{-1}"] & K \arrow[l, bend left, "f"] \arrow[r, bend left, "f^{-1}"] & Kx \arrow[l, bend left, "f"] \arrow[r, bend left, "f^{-1}"] & Kx^2 \arrow[l, bend left, "f"] \arrow[r, bend left, "f^{-1}"] & \cdots \arrow[l, bend left, "f"]
\end{tikzcd}
$$

In the general case, the action of some $F_\alpha \in \mathfrak{g}$ with $\alpha \in \Delta$ in M may not be injective. In fact, we have seen that the action of F_α is injective for all $\alpha \in \Delta^+$ if, and only if M is cuspidal. Nevertheless, we could intuitively *make it injective* by formally inverting the elements $F_\alpha \in \mathcal{U}(\mathfrak{g})$. This would allow us to obtain nonzero vectors in M_μ for all $\mu \in \lambda + Q$ by successively applying elements of $\{F_\alpha^{\pm 1}\}_{\alpha \in \Delta}$ to a nonzero weight vector $m \in M_\lambda$. Moreover, if the actions of the F_α were to be invertible, we would find that all M_μ are d-dimensional for $\mu \in \lambda + Q$.

In a commutative domain, this can be achieved by tensoring our module by the field of fractions. However, $\mathcal{U}(\mathfrak{g})$ is hardly ever commutative – $\mathcal{U}(\mathfrak{g})$ is commutative if, and only if $\mathfrak{g}$ is Abelian – and the situation is more delicate in the non-commutative case. For starters, a non-commutative K-algebra A may not even have a "field of fractions" – i.e. an over-ring where all elements of A

have inverses. Nevertheless, it is possible to formally invert elements of certain subsets of A via a process known as *localization*, which we now describe.

Definition 5.33. Let A be a K-algebra. A subset $S \subseteq A$ is called *multiplicative* if $s \cdot t \in S$ for all $s, t \in S$ and $0 \notin S$. A multiplicative subset S is said to satisfy *Ore's localization condition* if for each $a \in A$ and $s \in S$ there exists $b, c \in A$ and $t, t' \in S$ such that $sa = bt$ and $as = t'c$.

Theorem 5.34 (Ore-Asano). *Let $S \subseteq A$ be a multiplicative subset satisfying Ore's localization condition. Then there exists a (unique) K-algebra $S^{-1}A$, with a canonical algebra homomorphism $A \longrightarrow S^{-1}A$, enjoying the universal property that each algebra homomorphism $f : A \longrightarrow B$ such that $f(s)$ is invertible for all $s \in S$ can be uniquely extended to an algebra homomorphism $S^{-1}A \longrightarrow B$. $S^{-1}A$ is called* the localization of A by S, *and the map $A \longrightarrow S^{-1}A$ is called* the localization map.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ S^{-1}A \arrow[ur, dotted] & \end{tikzcd}$$

If we identify an element with its image under the localization map, it follows directly from Ore's construction that every element of $S^{-1}A$ has the form $s^{-1}a$ for some $s \in S$ and $a \in A$. Likewise, any element of $S^{-1}A$ can also be written as bt^{-1} for some $t \in S$, $b \in A$.

Ore's localization condition may seem a bit arbitrary at first, but a more thorough investigation reveals the intuition behind it. The issue in question here is that in the non-commutative case we can no longer take the existence of common denominators for granted. However, the existence of common denominators is fundamental to the proof of the fact the field of fractions is a ring – it is used, for example, to define the sum of two elements in the field of fractions. We thus need to impose their existence for us to have any hope of defining consistent arithmetics in the localization of an algebra, and Ore's condition is actually equivalent to the existence of common denominators – see the discussion in the introduction of [RW04, ch. 6] for further details.

We should also point out that there are numerous other conditions – which may be easier to check than Ore's – known to imply Ore's condition. For instance...

Lemma 5.35. *Let $S \subseteq A$ be a multiplicative subset generated by finitely many locally* ad*-nilpotent elements – i.e. elements $s \in S$ such that for each $a \in A$ there exists $r > 0$ such that $\operatorname{ad}(s)^r a = [s, [s, \cdots [s, a]] \cdots] = 0$. Then S satisfies Ore's localization condition.*

In our case, we are more interested in formally inverting the action of F_α on M than in inverting F_α itself. To that end, we introduce one further construction, known as *the localization of a module*.

Definition 5.36. Let $S \subseteq A$ be a multiplicative subset satisfying Ore's localization condition and M be an A-module. The $S^{-1}A$-module $S^{-1}M = S^{-1}A \otimes_A M$ is called *the localization of M by S*, and the homomorphism of A-modules

$$\begin{aligned} M &\longrightarrow S^{-1}M \\ m &\longmapsto 1 \otimes m \end{aligned}$$

is called *the localization map of M*.

Notice that the $S^{-1}A$-module $S^{-1}M$ has the natural structure of an A-module, where the action of A is given by the localization map $A \longrightarrow S^{-1}A$.

It is interesting to observe that, unlike in the case of the field of fractions of a commutative domain, in general the localization map $A \longrightarrow S^{-1}A$ – i.e. the map $a \longmapsto \frac{a}{1}$ – may not be injective. For instance, if S contains a divisor of zero s, its image under the localization map is invertible and therefore cannot be a divisor of zero in $S^{-1}A$. In particular, if $a \in A$ is nonzero and such that $sa = 0$ or $as = 0$ then its image under the localization map has to be 0. However, the existence of divisors of zero in S turns out to be the only obstruction to the injectivity of the localization map, as shown in…

Lemma 5.37. *Let $S \subseteq A$ be a multiplicative subset satisfying Ore's localization condition and M be an A-module. If S acts injectively on M then the localization map $M \longrightarrow S^{-1}M$ is injective. In particular, if S has no zero divisors then A is a subalgebra of $S^{-1}A$.*

Again, in our case we are interested in inverting the actions of the F_α on M. However, for us to be able to translate between all weight spaces associated with elements of $\lambda + Q$, $\lambda \in \operatorname{supp} M$, we only need to invert the F_α's for α in some subset of Δ which spans all of $Q = \mathbb{Z}\Delta$. In other words, it suffices to invert F_β for all β in some basis Σ for Δ. We can choose such a basis to be well-behaved. For example, we can show…

Lemma 5.38. *Let M be a simple infinite-dimensional bounded $\mathfrak{g}$-module. There is a basis $\Sigma = \{\beta_1, \ldots, \beta_r\}$ for Δ such that the elements F_{β_i} all act injectively on M and satisfy $[F_{\beta_i}, F_{\beta_j}] = 0$.*

Remark. The basis Σ in Lemma 5.38 may very well depend on the representation M! This is another obstruction to the functoriality of our constructions.

The proof of the previous Lemma is quite technical and was deemed too tedious to be included in here. See Lemma 4.4 of [Mat00] for a full proof. Since F_α is locally ad-nilpotent for all $\alpha \in \Delta$, we can see…

Corollary 5.39. *Let Σ be as in Lemma 5.38 and $(F_\beta)_{\beta\in\Sigma} \subseteq \mathcal{U}(\mathfrak{g})$ be the multiplicative subset generated by the F_β's. The K-algebra $\Sigma^{-1}\mathcal{U}(\mathfrak{g}) = (F_\beta)^{-1}_{\beta\in\Sigma}\mathcal{U}(\mathfrak{g})$ is well defined. Moreover, if we denote by $\Sigma^{-1}M$ the localization of M by $(F_\beta)_{\beta\in\Sigma}$, the localization map $M \longrightarrow \Sigma^{-1}M$ is injective.*

From now on let Σ be some fixed basis for Δ satisfying the hypothesis of Lemma 5.38. We now show that $\Sigma^{-1}M$ is a weight $\mathfrak{g}$-module whose support is an entire Q-coset.

Proposition 5.40. *The restriction of the localization $\Sigma^{-1}M$ is a bounded $\mathfrak{g}$-module of degree d with $\operatorname{supp}\Sigma^{-1}M = Q + \operatorname{supp} M$ and $\dim \Sigma^{-1}M_\lambda = d$ for all $\lambda \in \operatorname{supp}\Sigma^{-1}M$.*

Proof. Fix some $\beta \in \Sigma$. We begin by showing that F_β and F_β^{-1} map the weight space $\Sigma^{-1}M_\lambda$ to $\Sigma^{-1}M_{\lambda-\beta}$ and $\Sigma^{-1}M_{\lambda+\beta}$, respectively. Indeed, given $m \in M_\lambda$ and $H \in \mathfrak{h}$ we have

$$H \cdot (F_\beta \cdot m) = ([H, F_\beta] + F_\beta H) \cdot m = F_\beta(-\beta(H) + H) \cdot m = (\lambda - \beta)(H) F_\beta \cdot m$$

On the other hand,

$$0 = [H, 1] = [H, F_\beta F_\beta^{-1}] = F_\beta[H, F_\beta^{-1}] + [H, F_\beta]F_\beta^{-1} = F_\beta[H, F_\beta^{-1}] - \beta(H) F_\beta F_\beta^{-1},$$

so that $[H, F_\beta^{-1}] = \beta(H) \cdot F_\beta^{-1}$ and therefore

$$H \cdot (F_\beta^{-1} \cdot m) = ([H, F_\beta^{-1}] + F_\beta^{-1} H) \cdot m = F_\beta^{-1}(\beta(H) + H) \cdot m = (\lambda + \beta)(H) F_\beta^{-1} \cdot m$$

From the fact that $F_\beta^{\pm 1}$ maps M_λ to $\Sigma^{-1}M_{\lambda\pm\beta}$ follows our first conclusion: since M is a weight module and every element of $\Sigma^{-1}M$ has the form $s^{-1}\cdot m = s^{-1}\otimes m$ for $s \in (F_\beta)_{\beta\in\Sigma}$ and $m \in M$, we can see that $\Sigma^{-1}M = \bigoplus_\lambda \Sigma^{-1}M_\lambda$. Furthermore, since the action of each F_β on $\Sigma^{-1}M$ is bijective and Σ is a basis for Q we obtain $\operatorname{supp}\Sigma^{-1}M = Q + \operatorname{supp} M$.

Again, because of the bijectivity of the F_β's, to see that $\dim \Sigma^{-1}M_\lambda = d$ for all $\lambda \in \operatorname{supp}\Sigma^{-1}M$ it suffices to show that $\dim\Sigma^{-1}M_\lambda = d$ for some $\lambda \in \operatorname{supp}\Sigma^{-1}M$. We may take $\lambda \in \operatorname{supp} M$ with $\dim M_\lambda = d$. For any finite-dimensional subspace $V \subseteq \Sigma^{-1}M_\lambda$ we can find $s \in (F_\beta)_{\beta\in\Sigma}$ such that $s\cdot V \subseteq M$. If $s = F_{\beta_{i_1}}\cdots F_{\beta_{i_r}}$, it is clear $s\cdot V \subseteq M_{\lambda-\beta_{i_1}-\cdots-\beta_{i_r}}$, so $\dim V = \dim s\cdot V \leqslant d$. This holds for all finite-dimensional $V \subseteq \Sigma^{-1}M_\lambda$, so $\dim\Sigma^{-1}M_\lambda \leqslant d$. It then follows from the fact that $M_\lambda \subseteq \Sigma^{-1}M_\lambda$ that $M_\lambda = \Sigma^{-1}M_\lambda$ and therefore $\dim\Sigma^{-1}M_\lambda = d$. ■

We now have a good candidate for a coherent extension of M, but $\Sigma^{-1}M$ is still not a coherent extension since its support is contained in a single Q-coset. In particular, $\operatorname{supp}\Sigma^{-1}M \neq \mathfrak{h}^*$ and $\Sigma^{-1}M$ is not a coherent family. To obtain a coherent family we thus need somehow extend $\Sigma^{-1}M$. To that end, we will attempt to replicate the construction of the coherent extension of the $\mathfrak{sl}_2(K)$-module $K[x,x^{-1}]$. Specifically, the idea is that if twist $\Sigma^{-1}M$ by an automorphism which shifts its support by some $\lambda \in \mathfrak{h}^*$, we can construct a coherent family by summing these modules over λ as in Example 5.22.

For $K[x,x^{-1}]$ this was achieved by twisting the $\operatorname{Diff}(K[x,x^{-1}])$-module $K[x,x^{-1}]$ by the automorphisms $\varphi_\lambda : \operatorname{Diff}(K[x,x^{-1}]) \longrightarrow \operatorname{Diff}(K[x,x^{-1}])$ and restricting the results to $\mathcal{U}(\mathfrak{sl}_2(K))$ via the map $\mathcal{U}(\mathfrak{sl}_2(K)) \longrightarrow \operatorname{Diff}(K[x,x^{-1}])$, but this approach is inflexible since not every $\mathfrak{sl}_2(K)$-module factors through $\operatorname{Diff}(K[x,x^{-1}])$. Nevertheless, we could just as well twist $K[x,x^{-1}]$ by automorphisms of $\mathcal{U}(\mathfrak{sl}_2(K))_f$ directly – where $\mathcal{U}(\mathfrak{sl}_2(K))_f = (f)^{-1}\mathcal{U}(\mathfrak{g})$ is the localization of $\mathcal{U}(\mathfrak{sl}_2(K))$ by the multiplicative subset generated by f.

In general, we may twist the $\Sigma^{-1}\mathcal{U}(\mathfrak{g})$-module $\Sigma^{-1}M$ by automorphisms of $\Sigma^{-1}\mathcal{U}(\mathfrak{g})$. For $\lambda = \beta \in \Sigma$ the map

$$\begin{aligned}\theta_\beta : \Sigma^{-1}\mathcal{U}(\mathfrak{g}) &\longrightarrow \Sigma^{-1}\mathcal{U}(\mathfrak{g})\\ u &\longmapsto F_\beta u F_\beta^{-1}\end{aligned}$$

is a natural candidate for such a twisting automorphism. Indeed, we will soon see that ${}^{\theta_\beta}(\Sigma^{-1}M)_\lambda = \Sigma^{-1}M_{\lambda+\beta}$. However, this is hardly useful to us, since $\beta \in Q$ and therefore $\beta + \operatorname{supp}\Sigma^{-1}M = \operatorname{supp}\Sigma^{-1}M$. If we want to expand the support of $\Sigma^{-1}M$ we will have to twist by automorphisms that shift its support by $\lambda \in \mathfrak{h}^*$ lying *outside* of Q.

The situation is much less obvious in this case. Nevertheless, it turns out we can extend the family $\{\theta_\beta\}_{\beta\in\Sigma}$ to a family of automorphisms $\{\theta_\lambda\}_{\lambda\in\mathfrak{h}^*}$. Explicitly…

Proposition 5.41. *There is a family of automorphisms $\{\theta_\lambda : \Sigma^{-1}\mathcal{U}(\mathfrak{g}) \longrightarrow \Sigma^{-1}\mathcal{U}(\mathfrak{g})\}_{\lambda\in\mathfrak{h}^*}$ such that*

(i) $\theta_{k_1\beta_1+\cdots+k_r\beta_r}(u) = F_{\beta_1}^{k_1}\cdots F_{\beta_r}^{k_r} u F_{\beta_r}^{-k_r}\cdots F_{\beta_1}^{-k_1}$ *for all $u \in \Sigma^{-1}\mathcal{U}(\mathfrak{g})$ and $k_1,\ldots,k_r \in \mathbb{Z}$.*

(ii) *For each $u \in \Sigma^{-1}\mathcal{U}(\mathfrak{g})$ the map*

$$\begin{aligned}\mathfrak{h}^* &\longrightarrow \Sigma^{-1}\mathcal{U}(\mathfrak{g})\\ \lambda &\longmapsto \theta_\lambda(u)\end{aligned}$$

is polynomial.

(iii) *If $\lambda,\mu \in \mathfrak{h}^*$, N is a $\Sigma^{-1}\mathcal{U}(\mathfrak{g})$-module whose restriction to $\mathcal{U}(\mathfrak{g})$ is a weight $\mathfrak{g}$-module and ${}^{\theta_\lambda}N$ is the $\Sigma^{-1}\mathcal{U}(\mathfrak{g})$-module N twisted by the automorphism θ_λ then $N_\mu = {}^{\theta_\lambda}N_{\mu+\lambda}$. In particular, $\operatorname{supp}{}^{\theta_\lambda}N = \lambda + \operatorname{supp} N$.*

Proof. Since the elements F_β, $\beta \in \Sigma$ commute with one another, the endomorphisms

$$\begin{aligned} \theta_{k_1\beta_1+\cdots+k_r\beta_r} : \Sigma^{-1}\mathscr{U}(\mathfrak{g}) &\longrightarrow \Sigma^{-1}\mathscr{U}(\mathfrak{g}) \\ u &\longmapsto F_{\beta_1}^{k_1}\cdots F_{\beta_r}^{k_r} u F_{\beta_1}^{-k_r}\cdots F_{\beta_r}^{-k_1} \end{aligned}$$

are well defined for all $k_1, \ldots, k_r \in \mathbb{Z}$.

Fix some $u \in \Sigma^{-1}\mathscr{U}(\mathfrak{g})$. For any $s \in (F_\beta)_{\beta\in\Sigma}$ and $k > 0$ we have $s^k u = \binom{k}{0}\operatorname{ad}(s)^0 u s^{k-0} + \cdots + \binom{k}{k}\operatorname{ad}(s)^k u s^{k-k}$. Now if we take ℓ such $\operatorname{ad}(F_\beta)^{\ell+1}u = 0$ for all $\beta \in \Sigma$ we find

$$\theta_{k_1\beta_1+\cdots+k_r\beta_r}(u) = \sum_{i_1,\ldots,i_r=1,\ldots,\ell} \binom{k_1}{i_1}\cdots\binom{k_r}{i_r}\operatorname{ad}(F_{\beta_1})^{i_1}\cdots\operatorname{ad}(F_{\beta_r})^{i_r} u F_{\beta_1}^{-i_1}\cdots F_{\beta_r}^{-i_r}$$

for all $k_1, \ldots, k_r \in \mathbb{N}$.

Since the binomial coefficients $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ can be uniquely extended to polynomial functions in $x \in K$, we may in general define

$$\theta_\lambda(u) = \sum_{i_1,\ldots,i_r\geqslant 0} \binom{\lambda_1}{i_1}\cdots\binom{\lambda_r}{i_r}\operatorname{ad}(F_{\beta_1})^{i_1}\cdots\operatorname{ad}(F_{\beta_r})^{i_r} r F_{\beta_1}^{-i_1}\cdots F_{\beta_r}^{-i_r}$$

for $\lambda_1, \ldots, \lambda_r \in K$, $\lambda = \lambda_1\beta_1 + \cdots + \lambda_r\beta_r \in \mathfrak{h}^*$.

It is clear that the θ_λ are endomorphisms. To see that the θ_λ are indeed automorphisms, notice $\theta_{-k_1\beta_1-\cdots-k_r\beta_r} = \theta^{-1}_{k_1\beta_1+\cdots+k_r\beta_r}$. The uniqueness of the polynomial extensions then implies $\theta_{-\lambda} = \theta_\lambda^{-1}$ in general: given $u \in \Sigma^{-1}\mathscr{U}(\mathfrak{g})$, the map

$$\begin{aligned} \mathfrak{h}^* &\longrightarrow \Sigma^{-1}\mathscr{U}(\mathfrak{g}) \\ \lambda &\longmapsto \theta_\lambda(\theta_{-\lambda}(u)) - u \end{aligned}$$

is a polynomial extension of the zero map $\mathbb{Z}\beta_1 \oplus \cdots \oplus \mathbb{Z}\beta_r \longrightarrow \Sigma^{-1}\mathscr{U}(\mathfrak{g})$ and is therefore identically zero.

Finally, let N be a $\Sigma^{-1}\mathscr{U}(\mathfrak{g})$-module whose restriction is a weight module. If $n \in N$ then

$$n \in {}^{\theta_\lambda}N_{\mu+\lambda} \iff \theta_\lambda(H)\cdot n = (\mu+\lambda)(H)n\,\forall H \in \mathfrak{h}$$

But

$$\theta_\beta(H) = F_\beta H F_\beta^{-1} = ([F_\beta, H] + HF_\beta)F_\beta^{-1} = (\beta(H) + H)F_\beta F_\beta^{-1} = \beta(H) + H$$

for all $H \in \mathfrak{h}$ and $\beta \in \Sigma$. In general, $\theta_\lambda(H) = \lambda(H) + H$ for all $\lambda \in \mathfrak{h}^*$ and hence

$$\begin{aligned} n \in {}^{\theta_\lambda}N_{\mu+\lambda} &\iff (\lambda(H) + H)\cdot n = (\mu+\lambda)(H)n\ \forall H \in \mathfrak{h} \\ &\iff H \cdot n = \mu(H)n\ \forall H \in \mathfrak{h} \qquad , \\ &\iff n \in N_\mu \end{aligned}$$

so that ${}^{\theta_\lambda}N_{\mu+\lambda} = N_\mu$. ■

It should now be obvious. . .

Proposition 5.42 (Mathieu). *There exists a coherent extension $\mathscr{M}$ of M.*

Proof. Take[4]

$$\mathscr{M} = \bigoplus_{\lambda+Q\in\mathfrak{h}^*/Q} {}^{\theta_\lambda}(\Sigma^{-1}M)$$

[4]Here we fix some $\lambda_\xi \in \xi$ for each Q-coset $\xi \in \mathfrak{h}^*/Q$. While there is a natural isomorphism ${}^{\theta_\lambda}(\Sigma^{-1}M) \xrightarrow{\sim} {}^{\theta_\mu}(\Sigma^{-1}M)$ for each $\mu \in \lambda + Q$, they are not the same $\mathfrak{g}$-modules strictly speaking. This is yet another obstruction to the functoriality of our constructions.

It is clear M lies in $\Sigma^{-1}M = {}^{\theta_0}(\Sigma^{-1}M)$ and therefore $M \subseteq \mathscr{M}$. On the other hand, $\dim \mathscr{M}_\mu = \dim {}^{\theta_\lambda}(\Sigma^{-1}M)_\mu = \dim \Sigma^{-1}M_{\mu-\lambda} = d$ for all $\mu \in \lambda + Q - \lambda$ standing for some fixed representative of its Q-coset. Furthermore, given $u \in \mathcal{U}(\mathfrak{g})_0$ and $\mu \in \lambda + Q$,

$$\operatorname{Tr}(u \restriction_{\mathscr{M}_\mu}) = \operatorname{Tr}(\theta_\lambda(u) \restriction_{\Sigma^{-1}M_{\mu-\lambda}})$$

is polynomial in μ because of the second item of Proposition 5.41. ∎

Lo and behold…

Theorem 5.43 (Mathieu)**.** *There exists a unique semisimple coherent extension $\mathscr{E}xt(M)$ of M. More precisely, if $\mathscr{M}$ is any coherent extension of M, then $\mathscr{M}^{\text{ss}} \cong \mathscr{E}xt(M)$. Furthermore, $\mathscr{E}xt(M)$ is an irreducible coherent family.*

Proof. The existence part should be clear from the previous discussion: it suffices to fix some coherent extension $\mathscr{M}$ of M and take $\mathscr{E}xt(M) = \mathscr{M}^{\text{ss}}$.

To see that $\mathscr{E}xt(M)$ is irreducible, recall from Corollary 5.29 that M is a $\mathfrak{g}$-submodule of $\mathscr{E}xt(M)$. Since the degree of M is the same as the degree of $\mathscr{E}xt(M)$, some of its weight spaces have maximal dimension inside of $\mathscr{E}xt(M)$. In particular, it follows from Lemma 5.30 that $\mathscr{E}xt(M)_\lambda = M_\lambda$ is a simple $\mathcal{U}(\mathfrak{g})_0$-module for some $\lambda \in \operatorname{supp} M$.

As for the uniqueness of $\mathscr{E}xt(M)$, fix some other semisimple coherent extension $\mathscr{N}$ of M. We claim that the multiplicity of a given simple $\mathfrak{g}$-module L in $\mathscr{N}$ is determined by its *trace function*

$$\begin{aligned} \mathfrak{h}^* \times \mathcal{U}(\mathfrak{g})_0 &\longrightarrow K \\ (\lambda, u) &\longmapsto \operatorname{Tr}(u \restriction_{\mathscr{N}_\lambda}) \end{aligned}$$

It is a well known fact of the theory of modules that, given an associative K-algebra A, a finite-dimensional semisimple A-module L is determined, up to isomorphism, by its *character*

$$\begin{aligned} \chi_L : A &\longrightarrow K \\ a &\longmapsto \operatorname{Tr}(a \restriction_L) \end{aligned}$$

In particular, the multiplicity of L in $\mathscr{N}$, which is the same as the multiplicity of L_λ in $\mathscr{N}_\lambda$, is determined by the character $\chi_{\mathscr{N}_\lambda} : \mathcal{U}(\mathfrak{g})_0 \longrightarrow K$. Since this holds for all simple weight $\mathfrak{g}$-modules, it follows that $\mathscr{N}$ is determined by its trace function. Of course, the same holds for $\mathscr{E}xt(M)$. We now claim that the trace function of $\mathscr{N}$ is the same as that of $\mathscr{E}xt(M)$. Clearly, $\operatorname{Tr}(u \restriction_{\mathscr{E}xt(M)_\lambda}) = \operatorname{Tr}(u \restriction_{M_\lambda}) = \operatorname{Tr}(u \restriction_{\mathscr{N}_\lambda})$ for all $\lambda \in \operatorname{supp}_{\text{ess}} M$, $u \in \mathcal{U}(\mathfrak{g})_0$. Since the essential support of M is Zariski-dense and the maps $\lambda \longmapsto \operatorname{Tr}(u \restriction_{\mathscr{E}xt(M)_\lambda})$ and $\lambda \longmapsto \operatorname{Tr}(u \restriction_{\mathscr{N}_\lambda})$ are polynomial in $\lambda \in \mathfrak{h}^*$, it follows that these maps coincide for all u.

In conclusion, $\mathscr{N} \cong \mathscr{E}xt(M)$ and $\mathscr{E}xt(M)$ is unique. ∎

A sort of "reciprocal" of Theorem 5.43 also holds. Namely…

Proposition 5.44. *Let $\mathscr{M}$ be a semisimple irreducible coherent family and $M \subseteq \mathscr{M}$ be an infinite-dimensional simple submodule. Then $\mathscr{M} \cong \mathscr{E}xt(M)$. In particular, all semisimple coherent families have the form $\mathscr{M} \cong \mathscr{E}xt(M)$ for some simple bounded $\mathfrak{g}$-module M.*

Proof. Since $M \subseteq \mathscr{M}$, M is bounded and $\operatorname{supp}_{\text{ess}} M$ is Zariski-dense. In addition, it follows from Lemma 5.32 that $U = \{\lambda \in \mathfrak{h}^* : \mathscr{M}_\lambda \text{ is a simple } \mathcal{U}(\mathfrak{g})_0\text{-module}\}$ is a Zariski-open subset – which is non-empty since $\mathscr{M}$ is irreducible.

Hence there is some $\lambda \in \operatorname{supp}_{\text{ess}} M \cap U$. In particular, there is some $\lambda \in \operatorname{supp}_{\text{ess}} M$ such that $M_\lambda = \mathscr{M}_\lambda$ and thus $\deg M = \dim \mathscr{M}_\lambda = \deg \mathscr{M}$. This implies that $\mathscr{M}$ is a coherent extension of M, so that by the uniqueness of semisimple irreducible coherent extensions we get $\mathscr{M} \cong \mathscr{E}xt(M)$. ∎

Having thus reduced the problem of classifying the cuspidal $\mathfrak{g}$-modules to that of understanding semisimple irreducible coherent families, the only remaining question for us to tackle is: what are the coherent $\mathfrak{g}$-families? This turns out to be a decently complicated question on its own, and we will require a full chapter to answer it. This will be the focus of our final chapter.

Chapter 6

Classification of Coherent Families

Proposition 6.1. *Suppose $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_r$ and let $\mathscr{M}$ be a semisimple irreducible coherent $\mathfrak{g}$-family. Then there are semisimple irreducible coherent $\mathfrak{s}_i$-families $\mathscr{M}_i$ such that*

$$\mathscr{M} \cong \mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r$$

Proof. Suppose $\mathfrak{h}_i \subseteq \mathfrak{s}_i$ are Cartan subalgebras, $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$ and $d = \deg \mathscr{M}$. Let $M \subseteq \mathscr{M}$ be any infinite-dimensional simple submodule, so that $\mathscr{M}$ is a semisimple coherent extension of M. By Example 5.8, there exists (unique) simple weight $\mathfrak{s}_i$-modules M_i such that $M \cong M_1 \otimes \cdots \otimes M_r$. Take $\mathscr{M}_i = \mathscr{E}xt(M_i)$. We will show that $\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r$ is a coherent extension of M.

It is clear that $\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r$ is a degree d bounded $\mathfrak{g}$-module containing M as a submodule. It thus suffices to show that $\mathscr{M}$ is a coherent family. By Example 5.11, $\operatorname{supp}_{\mathrm{ess}}(\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r) = \mathfrak{h}^*$. To see that the map

$$\begin{aligned} \mathfrak{h}^* &\longrightarrow K \\ \lambda &\longmapsto \operatorname{Tr}(u\restriction_{(\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r)_\lambda}) \end{aligned}$$

is polynomial, notice that the natural isomorphism of algebras

$$\begin{aligned} f : \mathcal{U}(\mathfrak{s}_1) \otimes \cdots \otimes \mathcal{U}(\mathfrak{s}_1) &\xrightarrow{\sim} \mathcal{U}(\mathfrak{g}) \\ u_1 \otimes \cdots \otimes u_r &\longmapsto u_1 \cdots u_r \end{aligned}$$

described in Example 1.43 is a $\mathfrak{g}$-homomorphism between the tensor product of the adjoint $\mathfrak{s}_i$-modules $\mathcal{U}(\mathfrak{s}_i)$ and the adjoint $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g})$.

Indeed, given $X = X_1 + \cdots + X_r \in \mathfrak{g}$ with $X_i \in \mathfrak{s}_i$ and $u_i \in \mathcal{U}(\mathfrak{s}_i)$,

$$\begin{aligned} f(X \cdot (u_1 \otimes \cdots \otimes u_r)) &= f([X_1, u_1] \otimes u_2 \otimes \cdots \otimes u_r) + \cdots + f(u_1 \otimes \cdots \otimes u_{r-1} \otimes [X_r, u_r]) \\ &= [X_1, u_1] u_2 \cdots u_r + \cdots + u_1 \cdots u_{r-1} [X_r, u_r] \\ ([X_i, u_j] = 0 \text{ for } i \neq j) &= [X_1, u_1 u_2 \cdots u_r] + \cdots + [X_r, u_1 \cdots u_{r-1} u_r] \\ &= [X, f(u_1 \otimes \cdots \otimes u_r)] \end{aligned}$$

Hence by Example 5.9 f restricts to an isomorphism of algebras $\mathcal{U}(\mathfrak{s}_1)_0 \otimes \cdots \otimes \mathcal{U}(\mathfrak{s}_r)_0 \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})_0$ with image $\mathcal{U}(\mathfrak{g})_0 = \mathcal{U}(\mathfrak{s}_1)_0 \cdots \mathcal{U}(\mathfrak{s}_r)_0$. More importantly, if we write $\lambda = \lambda_1 + \cdots + \lambda_r$ for $\lambda_i \in \mathfrak{h}_i^*$ it is clear from Example 5.7 that the $\mathcal{U}(\mathfrak{g})_0$-module $(\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r)_\lambda$ corresponds to exactly the $\mathcal{U}(\mathfrak{s}_1)_0 \otimes \cdots \otimes \mathcal{U}(\mathfrak{s}_r)_0$-module $(\mathscr{M}_1)_{\lambda_1} \otimes \cdots \otimes (\mathscr{M}_r)_{\lambda_r}$, so we can see that the value

$$\operatorname{Tr}(u_1 \cdots u_r \restriction_{(\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r)_\lambda}) = \operatorname{Tr}(u_1 \restriction_{(\mathscr{M}_1)_{\lambda_1}}) \cdots \operatorname{Tr}(u_r \restriction_{(\mathscr{M}_r)_{\lambda_r}})$$

varies polynomially with $\lambda \in \mathfrak{h}^*$ for all $u_i \in \mathcal{U}(\mathfrak{s}_i)_0$.

Finally, $\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r$ is a coherent extension of M. Since the $\mathscr{M}_i = \mathscr{E}xt(M_i)$ are semisimple, so is $\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r$. It thus follows from the uniqueness of semisimple coherent extensions that $\mathscr{M} \cong \mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_r$. ∎

This last result allows us to concentrate on focus exclusive on classifying coherent $\mathfrak{s}$-families for the simple Lie algebras $\mathfrak{s}$. In addition, it turns out that very few simple algebras admit irreducible coherent families at all. Namely...

Proposition 6.2 (Fernando)**.** *Let $\mathfrak{s}$ be a finite-dimensional simple Lie algebra and suppose there exists a infinite-dimensional cuspidal $\mathfrak{s}$-module. Then $\mathfrak{s} \cong \mathfrak{sl}_n(K)$ or $\mathfrak{s} \cong \mathfrak{sp}_{2n}(K)$ for some n.*

Corollary 6.3. *Let $\mathfrak{s}$ be a finite-dimensional simple Lie algebra and suppose there exists an irreducible coherent $\mathfrak{s}$-family. Then $\mathfrak{s} \cong \mathfrak{sl}_n(K)$ or $\mathfrak{s} \cong \mathfrak{sp}_{2n}(K)$ for some n.*

The problem of classifying the semisimple irreducible coherent $\mathfrak{g}$-families for some arbitrary semisimple $\mathfrak{g}$ can thus be reduced to a proof by exaustion: it suffices to classify coherent $\mathfrak{sl}_n(K)$-families and coherent $\mathfrak{sp}_{2n}(K)$-families. We will follow this path by analysing each case – $\mathfrak{sl}_n(K)$ and $\mathfrak{sp}_{2n}(K)$ – separately, classifying coherent families in terms of combinatorial invariants – as does Mathieu in [Mat00, sec. 8, sec. 9]. Alternatively, Mathieu also provides a more explicit "geometric" construction of the coherent families for both $\mathfrak{sl}_n(K)$ and $\mathfrak{sp}_{2n}$ in sections 11 and 12 of his paper.

Before we proceed to the individual case analysis, however, we would like discuss some further reductions to our general problem, the first of which is a crutial refinement to Proposition 5.44 due to Mathieu.

Proposition 6.4. *Let $\mathscr{M}$ be a semisimple irreducible coherent $\mathfrak{g}$-family. Then there exists some $\lambda \in \mathfrak{h}^*$ such that $L(\lambda)$ is bounded and $\mathscr{M} \cong \mathscr{E}xt(L(\lambda))$.*

Remark. I once had the opportunity to ask Olivier Mathieu himself how he first came across the notation of coherent families and what was his intuition behind it. Unfortunately, his responce was that he "did not remember." However, Mathieu was able to tell me that "the *trick* is that I managed to show that they all come from simple highest-weight modules, which were already well understood." I personally find it likely that Mathieu first considered the idea of twisting $L(\lambda)$ – for λ with $L(\lambda)$ bounded – by a suitable automorphism $\theta_\mu : \Sigma^{-1}\mathscr{U}(\mathfrak{g}) \xrightarrow{\sim} \Sigma^{-1}\mathscr{U}(\mathfrak{g})$, as in the proof of Proposition 5.42, and only after decided to agregate this data in a coherent family by summing over the Q-cosets $\mu + Q$, $\mu \in \mathfrak{h}^*$.

In case the significance of Proposition 6.4 is unclear, the point is that it allows is to reduce the problem of classifying the coherent $\mathfrak{g}$-families to that of aswering the following two questions:

(i) When is $L(\lambda)$ bounded?

(ii) Given $\lambda, \mu \in \mathfrak{h}^*$ with $L(\lambda)$ and $L(\mu)$ bounded, when is $\mathscr{E}xt(L(\lambda)) \cong \mathscr{E}xt(L(\mu))$?

These are the questions which we will attempt to answer for $\mathfrak{g} = \mathfrak{sl}_n(K)$ and $\mathfrak{g} = \mathfrak{sp}_{2n}(K)$. We begin by providing a partial answer to the second answer by introducing an invariant of coherent families, known as its *central character*.

To describe this invariant, we consider the Verma module $M(\lambda) = \mathscr{U}(\mathfrak{g}) \cdot m^+$. Given $\mu \in \mathfrak{h}^*$ and $m \in M(\lambda)_\mu$, it is clear that $u \cdot m \in M(\lambda)_\mu$ for all central $u \in \mathscr{U}(\mathfrak{g})$. In particular, $u \cdot m^+ \in M(\lambda)_\lambda = Km^+$ is a scalar multiple of m^+ for all $u \in Z(\mathscr{U}(\mathfrak{g}))$, say $\mathcal{X}_\lambda(u)m^+$ for some $\mathcal{X}_\lambda(u) \in K$. More generally, if we take any $m = v \cdot m^+ \in M(\lambda)$ we can see that

$$u \cdot m = v \cdot (u \cdot m^+) = \mathcal{X}_\lambda(u)\, v \cdot m^+ = \mathcal{X}_\lambda(u)m$$

Since every highest-weight module is a quotient of a Verma module, it follows that $u \in Z(\mathscr{U}(\mathfrak{g}))$ acts on a highest-weight module M of highest-weight λ via multiplication by $\mathcal{X}_\lambda(u)$. In addition, it is clear that the function $\mathcal{X}_\lambda : Z(\mathscr{U}(\mathfrak{g})) \longrightarrow K$ must be an algebra homomorphism. This leads us to the following definition.

Definition 6.5. Given a highest weight $\mathfrak{g}$-module M of highest weight λ, the unique algebra homomorphism $\mathcal{X}_\lambda : Z(\mathcal{U}(\mathfrak{g})) \longrightarrow K$ such that $u \cdot m = \mathcal{X}_\lambda(u)m$ for all $m \in M$ and $u \in Z(\mathcal{U}(\mathfrak{g}))$ is called *the central character of M* or *the central character associated with the weight λ*.

Since a simple highest-weight $\mathfrak{g}$-module is uniquely determined by is highest-weight, it is clear that central characters are invariants of simple highest-weight modules. We should point out that these are far from perfect invariants, however. Namely. . .

Theorem 6.6 (Harish-Chandra). *Given $\lambda, \mu \in \mathfrak{h}^*$, $\mathcal{X}_\lambda = \mathcal{X}_\mu$ if, and only if $\mu \in W \bullet \lambda$.*

This and much more can be found in [E H08, ch. 1]. What is interesting about all this to us is that, as it turns out, central character are also invariants of coherent families. More specifically. . .

Proposition 6.7. *Suppose $\lambda, \mu \in \mathfrak{h}^*$ are such that $L(\lambda)$ and $L(\mu)$ are both bounded and $\mathscr{Ext}(L(\lambda)) \cong \mathscr{Ext}(L(\mu))$. Then $\mathcal{X}_\lambda = \mathcal{X}_\mu$. In particular, $\mu \in W \bullet \lambda$.*

Proof. Fix $u \in \mathcal{U}(\mathfrak{g})_0$. It is clear that $\mathrm{Tr}(u \restriction_{\mathscr{Ext}(L(\lambda))_\nu}) = \mathrm{Tr}(u \restriction_{L(\lambda)_\nu}) = d\mathcal{X}_\lambda(u)$ for all $\nu \in \mathrm{supp}_{\mathrm{ess}} L(\lambda)$. Since $\mathrm{supp}_{\mathrm{ess}} L(\lambda)$ is Zariski-dense and the map $\nu \longmapsto \mathrm{Tr}(u \restriction_{\mathscr{Ext}(L(\lambda))_\nu})$ is polynomial, it follows that $\mathrm{Tr}(u \restriction_{\mathscr{Ext}(L(\lambda))_\nu}) = d\mathcal{X}_\lambda(u)$ for all $\nu \in \mathfrak{h}^*$. But by the same token

$$d\mathcal{X}_\lambda(u) = \mathrm{Tr}(u \restriction_{\mathscr{Ext}(L(\lambda))_\nu}) = \mathrm{Tr}(u \restriction_{\mathscr{Ext}(L(\mu))_\nu}) = d\mathcal{X}_\mu(u)$$

for any $\nu \in \mathrm{supp}_{\mathrm{ess}} L(\mu)$ and thus $\mathcal{X}_\lambda(u) = \mathcal{X}_\mu(u)$. ■

Central characters may thus be used to distinguished between two semisimple irreducible coherent families. Unfortunately for us, as in the case of simple highest-weight modules, central characters are not perfect invariants of coherent families: there are non-isomorphic semisimple irreducible coherent families which share a common central character. Nevertheless, Mathieu was able to at least establish a somewhat *precarious* version of the converse of Proposition 6.7. Namelly. . .

Lemma 6.8. *Let $\beta \in \Sigma$ and $\lambda \notin P^+$ be such that. $L(\lambda)$ is bounded and $\lambda(H_\beta) \notin \mathbb{N}$. Then $L(\sigma_\beta \bullet \lambda) \subseteq \mathscr{Ext}(L(\lambda))$. In particular, if $\sigma_\beta \bullet \lambda \notin P^+$ then $L(\sigma_\beta)$ is a bounded infinite-dimensional $\mathfrak{g}$-module and $\mathscr{Ext}(L(\sigma_\beta \bullet \lambda)) \cong \mathscr{Ext}(L(\lambda))$.*

Remark. We should point out that, while it may very well be that $\sigma_\beta \bullet \lambda \in P^+$, there is generally only a slight chance of such an event happening. Indeed, given $\lambda \in \mathfrak{h}^*$, its orbit $W \bullet \lambda$ meets P^+ precisely once, so that the probability of $\sigma_\beta \bullet \lambda \in P^+$ for some random $\lambda \in \mathfrak{h}^*$ is only $1/|W \bullet \lambda|$. With the odds stacked in our favor, we will later be able to exploit the second part of Lemma 6.8 without much difficulty!

While technical in nature, this lemma already allows us to classify all semisimple irreducible coherent $\mathfrak{sl}_2(K)$-families.

Example 6.9. Let $\mathfrak{g} = \mathfrak{sl}_2(K)$. It follows from Example 4.42 that $M(\lambda)$ is a bounded $\mathfrak{sl}_2(K)$ of degree 1, so that $L(\lambda)$ is bounded – with $\deg L(\lambda) = 1$ – for all $\lambda \in K \cong \mathfrak{h}^*$. In addition, a simple calculation shows $W \bullet \lambda = \{\lambda, -\lambda - 2\}$. This implies that if $\lambda, \mu \notin P^+ = \mathbb{N}$ are such that $\mathscr{Ext}(L(\lambda)) \cong \mathscr{Ext}(L(\mu))$ then $\mu = \lambda$ or $\mu = -\lambda - 2$. Finally, by Lemma 6.8 the converse also holds: if $\lambda, -\lambda - 2 \notin P^+$ then $\mathscr{Ext}(L(\lambda)) \cong \mathscr{Ext}(L(-\lambda - 2))$.

6.1 Coherent $\mathfrak{sp}_{2n}(K)$-families

Consider the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{sp}_{2n}(K)$ of diagonal matrices, as in Example 4.5, and the basis $\Sigma = \{\beta_1, \ldots, \beta_n\}$ for Δ given by $\beta_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$ and $\beta_n = 2\epsilon_n$. Here $\epsilon_i : \mathfrak{h} \longrightarrow K$ is the linear functional which yields the i-th entry of the diagonal of a given matrix, as described in Example 4.24. Also fix $\rho = 1/2\beta_1 + \cdots + 1/2\beta_n$.

Lemma 6.10. *Then $L(\lambda)$ is bounded if, and only if*

(i) *$\lambda(H_{\beta_i})$ is non-negative integer for all $i \neq n$.*

(ii) *$\lambda(H_{\beta_n}) \in \frac{1}{2} + \mathbb{Z}$.*

(iii) *$\lambda(H_{\beta_{n-1}} + 2H_{\beta_n}) \geqslant -2$.*

Proposition 6.11. *The map*

$$\begin{aligned} m : \mathfrak{h}^* &\longrightarrow K^n \\ \lambda &\longmapsto (\kappa(\epsilon_1, \lambda + \rho), \ldots, \kappa(\epsilon_n, \lambda + \rho)) \end{aligned}$$

is W-equivariant bijection, where the action $W \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ on $\mathfrak{h}^*$ is given by the dot action and the action of W on K^n is given my permuting coordinates and multiplying them by ± 1. A weight $\lambda \in \mathfrak{h}^*$ satisfies the conditions of Lemma 6.10 if, and only if $m(\lambda)_i \in 1/2 + \mathbb{Z}$ for all i and $m(\lambda)_1 > m(\lambda)_2 > \cdots > m(\lambda)_{n-1} > \pm m(\lambda)_n$.

Proof. The fact $m : \mathfrak{h}^* \longrightarrow K^n$ is a bijection is clear from the fact that $\{\epsilon_1, \ldots, \epsilon_n\}$ is an orthonormal basis for $\mathfrak{h}^*$. Veryfying that $L(\lambda)$ is bounded if, and only if $m(\lambda)_1 > m(\lambda)_2 > \cdots > m(\lambda)_{n-1} > \pm m(\lambda)_n$ is also a simple combinatorial affair.

The only part of the statement worth proving is the fact that m is an equivariant map, which is equivalent to showing the map

$$\begin{aligned} \mathfrak{h}^* &\longrightarrow K^n \\ \lambda &\longmapsto (\kappa(\epsilon_1, \lambda), \ldots, \kappa(\epsilon_n, \lambda)) \end{aligned}$$

is equivariant with respect to the natural action of W on $\mathfrak{h}^*$. But this also clear from the isomorphism $W \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, as described in Example 4.35: $(\sigma_i, (\bar{0}, \ldots, \bar{0})) = \sigma_{\beta_i}$ permutes ϵ_i and ϵ_{i+1} for $i < n$ and $(1, (\bar{0}, \ldots, \bar{0}, \bar{1})) = \sigma_{\beta_n}$ flips the sign of ϵ_n. Hence $m(\sigma_{\beta_i} \cdot \epsilon_j) = \sigma_{\beta_i} \cdot m(\epsilon_j)$ for all i and j. Since W is generated by the σ_{β_i}, this implies that the required map is equivariant. ∎

Definition 6.12. We denote by $\mathcal{B}$ the set of the $m \in (1/2 + \mathbb{Z})^n$ such that $m_1 > m_2 > \cdots > m_{n-1} > \pm m_n$. We also consider the canonical partition $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ where $\mathcal{B}^+ = \{m \in \mathcal{B} : m_n > 0\}$ and $\mathcal{B}^- = \{m \in \mathcal{B} : m_n < 0\}$.

Theorem 6.13 (Mathieu). *Given λ and μ satisfying the conditions of Lemma 6.10, $\mathcal{E}xt\,(L(\lambda)) \cong \mathcal{E}xt\,(L(\mu))$ if, and only if $m(\lambda)_i = m(\mu)_i$ for $i < n$ and $m(\lambda)_n = \pm m(\mu)_n$. In particular, the isomorphism classes of semisimple irreducible coherent $\mathfrak{sp}_{2n}(K)$-families are parameterized by $\mathcal{B}^+$.*

Proof. Let $\lambda, \mu \notin P^+$ be such that $L(\lambda)$ and $L(\mu)$, so that $m(\lambda), m(\mu) \in \mathcal{B}$.

Suppose $\mathcal{E}xt\,(L(\lambda)) \cong \mathcal{E}xt\,(L(\mu))$. By Proposition 6.7, $\mathcal{X}_\lambda = \mathcal{X}_\mu$. It thus follows from the Harish-Chandra Theorem that $\mu \in W \bullet \lambda$. Since m is equivariant, $m(\mu) \in W \cdot m(\lambda)$. But the only elements in of $\mathcal{B}$ in $W \cdot m(\lambda)$ are $m(\lambda)$ and $(m(\lambda)_1, m(\lambda)_2, \ldots, m(\lambda)_{n-1}, -m(\lambda)_n)$.

Conversely, if $m(\lambda)_i = m(\mu)_i$ for $i < n$ and $m(\mu)_n = -m(\lambda)_n$ then $m(\mu) = \sigma_{\beta_n} \cdot m(\lambda)$ and $\mu = \sigma_n \bullet \lambda$. Since $m(\lambda) \in \mathcal{B}$, $\lambda(H_{\beta_n}) \in 1/2 + \mathbb{Z}$ and thus $\lambda(H_{\beta_n}) \notin \mathbb{N}$. Hence by Lemma 6.8 $L(\mu) \subseteq \mathscr{E}xt(L(\lambda))$ and $\mathscr{E}xt(L(\mu)) \cong \mathscr{E}xt(L(\lambda))$.

For each semisimple irreducible coherent $\mathfrak{sp}_{2n}(K)$-family $\mathscr{M}$ there is some $m = m(\lambda) \in \mathcal{B}$ such that $\mathscr{M} = \mathscr{E}xt(L(\lambda))$. The only other $m' \in \mathcal{B}$ which generates the same coherent family as m is $m' = \sigma_{\beta_n} \cdot m$. Since m and m' lie in different elements of the partition $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$, the is a unique $m'' = m(\nu) \in \mathcal{B}^+$ – either m or m' – such that $\mathscr{M} \cong \mathscr{E}xt(L(\nu))$. ∎

6.2 Coherent $\mathfrak{sl}_n(K)$-families

Consider the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}_n(K)$ of diagonal matrices, as in Example 4.4, and the basis $\Sigma = \{\beta_1, \ldots, \beta_{n-1}\}$ for Δ given by $\beta_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$. Here $\epsilon_i : \mathfrak{h} \longrightarrow K$ is the linear functional which yields the i-th entry of the diagonal of a given matrix, as described in Example 4.23. Also fix $\rho = 1/2\beta_1 + \cdots + 1/2\beta_{n-1}$.

Lemma 6.14. *Let $\lambda \notin P^+$ and $A(\lambda) = \{i : \lambda(H_{\beta_i})$ is* not *a non-negative integer$\}$. Then $L(\lambda)$ is bounded if, and only if one of the following assertions holds.*

(i) *$A(\lambda) = \{1\}$ or $A(\lambda) = \{n-1\}$.*

(ii) *$A(\lambda) = \{i\}$ for some $1 < i < n-1$ and $(\lambda+\rho)(H_{\beta_{i-1}} + H_{\beta_i})$ or $(\lambda+\rho)(H_{\beta_i} + H_{\beta_{i+1}})$ is a positive integer.*

(iii) *$A(\lambda) = \{i, i+1\}$ for some $1 \leqslant i < n-1$ and $(\lambda+\rho)(H_{\beta_i} + H_{\beta_{i+1}})$ is a positive integer.*

Definition 6.15. A $\mathfrak{sl}_n$*-sequence* m is a n-tuple $m = (m_1, \ldots, m_n) \in K^n$ such that $m_1 + \cdots + m_n = 0$.

Definition 6.16. A k-tuple $m = (m_1, \ldots, m_k) \in K^k$ is called *ordered* if $m_i - m_{i+1}$ is a positive integer for all $i < k$.

Proposition 6.17. *The map*

$$\begin{aligned} m : \mathfrak{h}^* &\longrightarrow \{\mathfrak{sl}_n\text{-sequences}\} \\ \lambda &\longmapsto 2n(\kappa(\epsilon_1, \lambda+\rho), \ldots, \kappa(\epsilon_n, \lambda+\rho)) \end{aligned}$$

is W-equivariant bijection, where the action $W \cong S_n$ on $\mathfrak{h}^*$ is given by the dot action and the action of W on the space of $\mathfrak{sl}_n$-sequences is given my permuting coordinates. A weight $\lambda \in \mathfrak{h}^*$ satisfies the conditions of Lemma 6.14 if, and only if $m(\lambda)$ is not *ordered, but becomes ordered after removing one term.*

The proof of this result is very similar to that of Proposition 6.11 in spirit: the equivariance of the map $m : \mathfrak{h}^* \longrightarrow \{\mathfrak{sl}_n\text{-sequences}\}$ follows from the nature of the isomorphism $W \cong S_n$ as described in Example 4.34, while the rest of the proof amounts to simple technical verifications. The number $2n$ is a normalization constant chosen because $\lambda(H_\beta) = 2n\,\kappa(\lambda, \beta)$ for all $\lambda \in \mathfrak{h}^*$ and $\beta \in \Sigma$. Hence $m(\lambda)$ is uniquely characterized by the property that $(\lambda+\rho)(H_{\beta_i}) = m(\lambda)_i - m(\lambda)_{i+1}$ for all i, which is relevant to the proof of the equivalence between the contiditions of Lemma 6.14 and those explained in the last statement of Proposition 6.17.

Definition 6.18. We denote by $\mathcal{B}$ the set of $\mathfrak{sl}_n$-sequences m which are *not* ordered, but becomes ordered after removing one term. We also consider the *extremal* subsets $\mathcal{B}^+ = \{m \in \mathcal{B} : (\widehat{m_1}, m_2, \ldots, m_n) \text{ is ordered}\}$ and $\mathcal{B}^- = \{m \in \mathcal{B} : (m_1, \ldots, m_{n-1}, \widehat{m_n}) \text{ is ordered}\}$.

The issue here is that the relationship between $\lambda, \mu \in P^+$ with $m(\lambda), m(\mu) \in \mathcal{B}$ and $\mathscr{Ext}(L(\lambda)) \cong \mathscr{Ext}(L(\mu))$ is more complicated than in the case of $\mathfrak{sp}_{2n}(K)$. Nevertheless, Lemma 6.8 affords us a criteria for verifying that $\mathscr{Ext}(L(\lambda)) \cong \mathscr{Ext}(L(\mu))$. For $\sigma = \sigma_i$ and the weight $\lambda + \rho$, the hypothesis of Lemma 6.8 translates to $m(\lambda)_i - m(\lambda)_{i+1} = (\lambda + \rho)(H_{\beta_i}) \notin \mathbb{N}$. If $m(\lambda) \in \mathcal{B}$, this is equivalent to requiring that $m(\lambda)$ is not ordered, but becomes ordered after removing its i-th term. This discussions losely inspires the following definition, which endows the set $\mathcal{B}$ with the structure of a directed graph.

Definition 6.19. Given $m, m' \in \mathcal{B}$, say there is an arrow $m \longrightarrow m'$ if there some i such that $m_i - m_{i+1}$ is *not* a positive integer and $m' = \sigma_i \cdot m$.

It should then be obvious from Lemma 6.8 that...

Proposition 6.20. *Let $\lambda \notin P^+$ be such that $L(\lambda)$ is bounded – so that $m(\lambda) \in \mathcal{B}$ – and suppose that $\mu \in \mathfrak{h}^*$ is such that $m(\mu) \in \mathcal{B}$ and there is an arrow $m(\lambda) \longrightarrow m(\mu)$. Then $L(\mu)$ is also bounded and $\mathscr{Ext}(L(\mu)) \cong \mathscr{Ext}(L(\lambda))$.*

A weight $\lambda \in \mathfrak{h}^*$ is called *regular* if $(\lambda + \rho)(H_\alpha) \neq 0$ for all $\alpha \in \Delta$. In terms of $\mathfrak{sl}_n$-sequences, λ is regular if, and only if $m(\lambda)_i \neq m(\lambda)_j$ for all $i \neq j$. It thus makes sence to call a $\mathfrak{sl}_n$-sequence regular or singular if $m_i \neq m_j$ for all $i \neq j$ or $m_i = m_j$ for some $i \neq j$, respectively. Similarly, λ is integral if, and only if $m(\lambda)_i - m(\lambda)_j \in \mathbb{Z}$ for all i and j, so it makes sence to call a $\mathfrak{sl}_n$-sequence m integral if $m_i - m_j \in \mathbb{Z}$ for all i and j.

Lemma 6.21. *The connected component of some $m \in \mathcal{B}$ is given by the following.*

(i) *If m is regular and integral then there exists*[1] *a unique ordered $m' \in W \cdot m$, in which case the connected component of m is given by*

$$\begin{tikzcd} \sigma_1\sigma_2\cdots\sigma_i \cdot m' \arrow[r] & \sigma_2\cdots\sigma_i \cdot m' \arrow[r] & \cdots \arrow[r] & \sigma_{i-1}\sigma_i \cdot m' \arrow[llld, rounded corners] \\ \sigma_i \cdot m' & \sigma_{i+1}\sigma_i \cdot m' \arrow[l] & \cdots \arrow[l] & \sigma_{n-1}\cdots\sigma_i \cdot m' \arrow[l] \end{tikzcd}$$

for some unique i, with $\sigma_1 \cdots \sigma_i \cdot m' \in \mathcal{B}^+$ and $\sigma_{n-1}\cdots\sigma_i \cdot m' \in \mathcal{B}^-$.

(ii) *If m is singular then there exists unique $m' \in W \cdot m$ and i such that $m'_i = m'_{i+1}$ and $(m'_1, \cdots, m'_{i-1}, \widehat{m'_i}, m'_{i+1}, \ldots, m'_n)$ is ordered, in which case the connected component of m is given by*

$$\begin{tikzcd} \sigma_1\sigma_2\cdots\sigma_{i-1} \cdot m' \arrow[r] & \sigma_2\cdots\sigma_{i-1} \cdot m' \arrow[r] & \cdots \arrow[r] & \sigma_{i-1} \cdot m' \arrow[llld, rounded corners] \\ m' & \sigma_{i+1} \cdot m' \arrow[l] & \cdots \arrow[l] & \sigma_{n-1}\cdots\sigma_{i+1} \cdot m' \arrow[l] \end{tikzcd}$$

with $\sigma_1 \cdots \sigma_{i-1} \cdot m' \in \mathcal{B}^+$ and $\sigma_{n-1}\cdots\sigma_{i+1} \cdot m' \in \mathcal{B}^-$.

(iii) *If m is non-integral then there exists unique $m' \in W \cdot m$ with $m' \in \mathcal{B}^+$, in which case the connected component of m is given by*

$$m' \longleftrightarrow \sigma_1 \cdot m' \longleftrightarrow \sigma_2\sigma_1 \cdot m' \longleftrightarrow \cdots \longleftrightarrow \sigma_{n-1}\cdots\sigma_1 \cdot m'$$

with $\sigma_{n-1}\cdots\sigma_1 \cdot m' \in \mathcal{B}^-$.

Theorem 6.22 (Mathieu)**.** *Given $\lambda, \mu \notin P^+$ with $L(\lambda)$ and $L(\mu)$ bounded, $\mathcal{E}xt(L(\lambda)) \cong \mathcal{E}xt(L(\mu))$ if, and only if $m(\lambda)$ and $m(\mu)$ lie in the same connected component of $\mathcal{B}$. In particular, the isomorphism classes of semisimple irreducible coherent $\mathfrak{sl}_n(K)$-families are parameterized by the set $\pi_0(\mathcal{B})$ of the connected components of $\mathcal{B}$, as well as by $\mathcal{B}^+$.*

Proof. Let $\lambda, \mu \notin P^+$ be such that $L(\lambda)$ and $L(\mu)$, so that $m(\lambda), m(\mu) \in \mathcal{B}$.

It is clear from Proposition 6.20 that if $m(\lambda)$ and $m(\mu)$ lie in the same connected component of $\mathcal{B}$ then $\mathcal{E}xt(L(\lambda)) \cong \mathcal{E}xt(L(\mu))$. On the other hand, if $\mathcal{E}xt(L(\lambda)) \cong \mathcal{E}xt(L(\mu))$ then $\chi_\lambda = \chi_\mu$ and thus $\mu \in W \bullet \lambda$. We now investigate which elements of $W \bullet \lambda$ satisfy the conditions of Lemma 6.14. To do so, we describe the set $\mathcal{B} \cap W \cdot m(\lambda)$.

If λ is regular and integral then the only permutations of $m(\lambda)$ which lie in $\mathcal{B}$ are $\sigma_k\sigma_{k+1}\cdots\sigma_i \cdot m'$ for $k \leqslant i$ and $\sigma_k\sigma_{k-1}\cdots\sigma_i \cdot m'$ for $k \geqslant i$, where m' is the unique ordered element of $W \cdot m(\lambda)$. Hence by Lemma 6.21 $\mathcal{B} \cap W \cdot m(\lambda)$ is the union of the connected components of the $\sigma_i \cdot m'$ for $i \leqslant n$. On the other hand, if λ is singular or non-integral then the only permutations of $m(\lambda)$ which lie in $\mathcal{B}$ are the ones from the connected component of $m(\lambda)$ in $\mathcal{B}$, so that $\mathcal{B} \cap W \cdot m(\lambda)$ is exactly the connected component of $m(\lambda)$.

In both cases, we can see that if $B(\lambda)$ is the set of the $m' = m(\mu) \in \mathcal{B}$ such that $\mathcal{E}xt(L(\mu)) \cong \mathcal{E}xt(L(\lambda))$ then $B(\lambda) \subseteq \mathcal{B} \cap W \cdot m(\lambda)$ is contain in a union of connected components of $\mathcal{B}$ – including that of $m(\lambda)$ itself. We now claim that $B(\lambda)$ is exactly the connected component of $m(\lambda)$. This is already clear when λ is singular or non-integral, so we may assume that λ is regular and integral, in which case every other $\mu \in W \bullet \lambda$ is regular and integral.

In this situation, $m(\mu) \in \mathcal{B}^+$ implies $\mu(H_{\beta_1}) = m(\mu)_1 - m(\mu)_2 \in \mathbb{Z}$ is negative. But it follows from Lemma 6.8 that for each $\beta \in \Sigma$ there is at most one $\mu \notin P^+$ with $\mathcal{E}xt(L(\mu)) \cong \mathcal{E}xt(L(\lambda))$

[1] Notice that in this case $m' \notin \mathcal{B}$, however.

such that $\mu(H_\beta)$ is a negative integer – see Lemma 6.5 of [Mat00]. Hence there is at most one $m' \in \mathcal{B}^+ \cap W \cdot m(\lambda)$. Since every connected component of $\mathcal{B}$ meets $\mathcal{B}^+$ – see Lemma 6.21 – this implies $B(\lambda)$ is precisely the connected component of $m(\lambda)$.

Another way of putting it is to say that $\mathscr{Ext}(L(\lambda)) \cong \mathscr{Ext}(L(\mu))$ if, and only if $m(\lambda)$ and $m(\mu)$ lie in the same connected component – which is, of course, precisely the first part of our theorem! There is thus a one-to-one correspondance between $\pi_0(\mathcal{B})$ and the isomorphism classes of semisimple irreducible coherent $\mathfrak{sl}_n(K)$-families. Since every connected component of $\mathcal{B}$ meets $\mathcal{B}^+$ precisely once – again, see Lemma 6.21 – we also get that such isomorphism classes are parameterized by $\mathcal{B}^+$. ■

This construction also brings us full circle to the beginning of these notes, where we saw in Proposition 1.44 that $\mathfrak{g}$-modules may be understood as geometric objects. In fact, throughout the previous four chapters we have seen a tremendous number of geometrically motivated examples, which further emphasizes the connection between representation theory and geometry. I would personally go as far as saying that the beautiful interplay between the algebraic and the geometric is precisely what makes representation theory such a fascinating and charming subject.

Alas, our journey has come to an end. All it is left is to wonder at the beauty of Lie algebras and their representations.

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