A note on holomorphic families of complex projective structures

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Abstract

The aim of this paper is to prove that a compact non-isotrivial holomorphic family of compact Riemann surfaces cannot support any holomorphic family of complex projective structres.

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In this paper $(\mathrm{PGL}(2,\mathbf{C}),\mathbb{P}^1)$-structure and complex projective structure stand for the same notion. It is well known that $(\mathrm{PGL}(2,\mathbf{C}),\mathbb{P}^1)$-structures on a compact Riemann surface, i.e. a closed oriented surface with fixed complex structure, are parametrized by a complex vector space. As a holomorphic map from a compact complex manifold to a complex vector space must be constant, there is no nontrivial compact holomorphic family of $(\mathrm{PGL}(2,\mathbf{C}),\mathbb{P}^1)$-structures on a given compact Riemann surface. There exists compact holomorphic families of compact Riemann surfaces non isomorphic to each other, first discovered by Kodaira (cf. [Kod67]). It is thus natural to consider the situation where the complex structure of the surface varies along with the $(\mathrm{PGL}(2,\mathbf{C}),\mathbb{P}^1)$-structure, and to ask whether there exists nontrivial families. The answer is still no:

Theorem 0.1 *Let $\pi : \mathscr{X} \to C$ be a holomorphic family of compact Riemann surfaces parametrized by a compact Riemann surface C. Suppose that π supports a holomorphic family of $(\mathrm{PGL}(2,\mathbf{C}),\mathbb{P}^1)$-structures on the fibers. Then the fibers are all isomorphic Riemann surfaces and the complex projective structures are the same.*

Though Theorem 0.1 follows from some classical works in the existing literature, it is not well known to many people working with complex projective structures. Therefore the aim of this note is to give a presentation that hopefully would be useful for the community.

Variations of $(\mathrm{PGL}(2,\mathbf{C}),\mathbb{P}^1)$-structures (when the complex structure of the surface varies too) are parametrized by a moduli space which has been studied by Hubbard

[Hub81] and Tyurin [Tyu78]. The geometry of this moduli space implies Theorem 0.1. A more recent paper concerning this moduli space is [BKN17]. One key ingredient to understand the above mentioned moduli space is the construction of some families of complex projective structures from Theta functions on Riemann surfaces. This can be found in the work of Fay [Fay73] and the formulas can be traced back to [HS66] and even to Klein [Kle90]. The references [Fay73], [Hub81] and [Tyu78] are the main sources of our paper.

In the first section we define complex projective structures. In Sections 2 and 3 we explain its relations with quadratic differentials. In Section 4 we present some preliminaries on Theta functions and in Section 5 we explain how to use them to construct complex projective structures. Since this construction is the key of the proof of Theorem 0.1, we will try to make our presentation self-contained assuming a few big theorems and some background knowledge in complex algebraic geometry, most of which can be found in [GH78]. In Sections 6 and 7 we define holomorphic families of complex projective structures and present the properties of the moduli space that allow us to obtain Theorem 0.1.

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1 Projective structures

Definition 1.1 *A complex projective structure, or a* $(\mathrm{PGL}_2(\mathbf{C}),\mathbb{P}^1(\mathbf{C}))$*-structure on a orientable real surface* Σ *is a maximal atlas* $\{(U_i,\varphi_i)\}$ *of charts such that*

- $\varphi_i : U_i \to \mathbb{P}^1$ *is a homeomorphism onto its image;*
- *the transition functions* $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ *are restrictions of elements of* $\mathrm{PGL}_2(\mathbf{C})$.

A complex affine structure, or a $(\mathrm{Aff}_1(\mathbf{C}),\mathbf{C})$*-structure on* Σ_g *is a maximal atlas* $\{(U_i,\varphi_i)\}$ *of charts such that*

- $\varphi_i : U_i \to \mathbf{C}$ *is a homeomorphism onto its image;*
- *the transition functions* $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ *are restrictions of elements of* $\mathrm{Aff}_1(\mathbf{C})$.

A complex affine structure induces a complex projective structure. A complex projective structure induces a complex structure on Σ, we denote by X the corresponding Riemann surface. The sphere has a unique complex structure $\mathbb{P}^1(\mathbf{C})$, and $\mathbb{P}^1(\mathbf{C})$ has an obvious complex projective structure (we will soon see that it is unique). An elliptic curve is the quotient of $\mathbf{C}$ by some lattice $\mathbf{Z}1+\mathbf{Z}\tau$, thus has naturally a complex affine structure. The following examples show that every compact Riemann surface is the underlying Riemann surface of some complex projective structure.

Example 1.2 (Kleinian groups) Let Γ be a finitely generated discrete subgroup of $\mathrm{PGL}_2(\mathbf{C})$ without torsion, acting freely and properly discontinuously on a non-empty open subset of $\mathbb{P}^1(\mathbf{C})$. There is a unique maximal such open subset U, called the *discontinuity set of* Γ. Suppose that U^0 is a component invariant under Γ. Then $X = U_0/\Gamma$

is a Riemann surface with a complex projective structure induced by the covering map $U \to X$. If the limit set L is a round circle, then Ω is called a *Fuchsian group*. Poincaré-Koebe's uniformization theorem says that every hyperbolic Riemann surface is the quotient by some Fuchsian group, thus has a complex projective structure; this unique determined complex projective structure will be called *Fuchsian*. If L is a Jordan curve, then it is a *quasi-circle* and Ω is called a *quasi-Fuchsian group*. Quasi-Fuchsian groups are obtained as deformations of Fuchsian groups. In general Ω is called a *Kleinian group* and L can be very complicated. See [Mas88] for Kleinian groups.

Definition/Proposition 1.3 *Let X be a Riemann surface with a complex projective structure. Denote by $\tilde{X}$ the universal cover of X and π the quotient map. There exist a homomorphism $\mathrm{Hol} : \pi_1(X) \to \mathrm{PGL}_2(\mathbf{C})$ and a $\pi_1(X)$-equivariant holomorphic map $\mathrm{Dev} : \tilde{V} \to \mathbb{P}^1(\mathbf{C})$ such that*

$$\forall \gamma \in \pi_1(X), \mathrm{Dev} \circ \gamma = \mathrm{Hol}(\gamma) \circ \mathrm{Dev}\,.$$

If $(\mathrm{Hol}', \mathrm{Dev}')$ is another such pair, then there exists $\sigma \in \mathrm{PGL}_2(\mathbf{C})$ such that $\mathrm{Hol}' = \sigma \, \mathrm{Hol} \, \sigma^{-1}$ and $\mathrm{Dev}' = \sigma \circ \mathrm{Dev}$. The morphism Hol *is called* holonomy representation *and* Dev *is called* developing map. *A complex projective structure is determined by its holonomy representation and developing map (up to composition).*

For complex projective structures obtained from simply connected invariant components of Kleinian groups (for example Fuchsian or quasi-Fuchsian groups), the developing map is the injective and the holonomy is the Kleinian group itself. A complex projective structure on $\mathbb{P}^1(\mathbf{C})$ gives rise to a developing map $\mathbb{P}^1(\mathbf{C}) \to \mathbb{P}^1(\mathbf{C})$ which is locally biholomorphic thus must be an automorphism. This proves that the obvious complex projective structure on $\mathbb{P}^1(\mathbf{C})$ is the unique one. With a little bit more work, we can prove that a complex projective structure on an elliptic curve is always reduced to a complex affine structure, and that a complex projective structure on a compact hyperbolic Riemann surface is never reduced to an affine one. Furthermore the monodromy representation associated with a complex projective structure on a compact hyperbolic Riemann surface always lifts to a representation into $\mathrm{SL}_2(\mathbf{C})$. We refer the reader to [Gun67] for these assertions and to [Dum09] for a general survey on complex projective structures.

2 Schwarzian derivative

Let $f(z)$ be a holomorphic function with nowhere vanishing derivative defined in a domain $D \subset \mathbf{C}$. The *Schwarzian derivative* of f is the holomorphic function

$$S(f;z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The following properties are not difficult to prove and can be found in any reference on Schwarzian derivative.

Proposition 2.1

- *Under a change of variables* $z = g(t)$, *we have* $S(f(g);t)(dt)^2 = S(f;z)(dz)^2 + S(g;t)(dt)^2$.
- $S(z;f)(df)^2 = -S(f;z)(dz)^2$.
- $S\left(f; \frac{az+b}{cz+d}\right) \frac{(ad-bc)^2}{(cz+d)^4} = S(f;z)$.
- *Let* $q(z)$ *be a holomorphic function of* z. *Then any solution* f *of the Schwarzian differential equation* $S(f;z) = q(z)$ *equals to* g_1/g_2 *where* (g_1, g_2) *is a pair of independant solutions of the linear differential equation* $g'' + \frac{q}{2}g = 0$. *Conversely if* (g_1, g_2) *is a pair of independant solutions of* $g'' + \frac{q}{2}g = 0$, *then* g_1/g_2 *is a solution of the Schwarzian equation.*
- $S(f;z) = 0$ *if and only if* $f(z) = \frac{az+b}{cz+d}$.

The third property says that the Schwarzian derivative transforms as a quadratic differential under fractional-linear transformation of the domain. The fourth property implies in particular that a Schwarzian equation always has a solution and that the solution is unique up to post-composition by a fractional-linear transformation. The fifth property is a consequence of the fourth and it means that in some sense the Schwarzian derivative measures to which extent a function is different from being fractional-linear.

For later use we now present some seemingly contoured computations (cf. [BS51], [Tyu78]) that lead to the mysterious expression of Schwarzian derivative. We expand $\frac{f(x)-f(y)}{x-y}$ as a series in $x_1 = x - z, y_1 = y - z$:

$$\frac{f(x) - f(y)}{x - y} = f'(z) + f''(z)\frac{1}{2}(x_1 + y_1) + \frac{1}{6}f'''(z)(x_1^2 + x_1 y_1 + y_1^2) + \dots.$$

We expand also $\log\left(\frac{f(x)-f(y)}{x-y}\right)$:

$$\log\left(\frac{f(x) - f(y)}{x - y}\right) = \log\left(f'(z)\right) + \frac{f''(z)}{2f'(z)}(x_1 + y_1) + \\ + \left(\frac{f'''(z)}{6f'(z)} - \frac{1}{8}\left(\frac{f''(z)}{f'(z)}\right)^2\right)(x_1^2 + y_1^2) - \left(\frac{f'''(z)}{6f'(z)} - \frac{1}{4}\left(\frac{f''(z)}{f'(z)}\right)^2\right)x_1 y_1 + \dots. \quad (1)$$

We apply $\frac{\partial^2}{\partial x \partial y}$ to both sides of Equation (1):

$$\frac{f'(x)f'(y)}{(f(x) - f(y))^2} - \frac{1}{(x - y)^2} = \frac{f'''(z)}{6f'(z)} - \frac{1}{4}\left(\frac{f''(z)}{f'(z)}\right)^2 + R(x_1, y_1), \quad (2)$$

where $R(x_1, y_1)$ is a sum of terms in x_1, y_1 of degree > 0. We denote by $p = f(x), q = f(y)$ the target variables and by $S(p, q; x, y)$ the symmetric expression

$$S(p, q; x, y) = \frac{f'(x)f'(y)}{(f(x) - f(y))^2} - \frac{1}{(x - y)^2}. \quad (3)$$

If $x = g(v), y = g(w)$ are themselves the target variables of a holomorphic function g, then we have the additional formula:

$$S(p,q;v,w)dvdw = S(p,q;x,y)dxdy + S(x,y;v,w)dvdw. \tag{4}$$

Let $x = y = z$ in Equation (2), we get the Schwarzian derivative:

$$S(p,p;x,x) = \frac{1}{6}\left(\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right) = \frac{1}{6}\left(\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right) = \frac{1}{6}S(f;z). \tag{5}$$

3 Projective connections

Definition 3.1 *Let X be a Riemann surface and let $U_I = \{U_i, z_i\}_{i\in I}$ be an atlas of holomorphic coordinates. A projective connection on X with respect to U_I is a collection of holomorphic functions $\{h_i\}$ such that on each $U_i \cap U_j$ we have*

$$S(z_i;z_j) = h_i\left(\frac{dz_i}{dz_j}\right)^2 - h_j. \tag{6}$$

For $\{h_i\}$ and $\{h_i'\}$ two projective connections with respect to $(U)_I$, their difference is a quadratic differential on X because $(h_i - h_i')(dz_i)^2 = (h_j - h_j')(dz_j)^2$ on $U_i \cap U_j$. For two projective connections with respect to different atlases $(U)_I$ and $(V)_J$, we consider both of them as projective connections with respect to $(U)_I \cap (V)_J$ and obtain their difference as a quadratic differential in the same way; they are said to be *equivalent* if the difference is zero.

Definition 3.2 *A projective connection on a Riemann surface X is an equivalence class of projective connections with respect to different atlases.*

Proposition 3.3 *The set of projective connections on a Riemann surface X is in bijection with the set of complex projective structures on X.*

Proof Let $\{h_i\}$ be a projective connection with respect to $(U)_I$. In U_i let g_i be any solution of the equation $S(g_i;z_i) = -h_i$. The holomorphic function g_i has nowhere vanishing derivative so that we can assume it is injective up to shrinking U_i. Then the new coordinates $\{U_i, g_i \circ z_i\}$ define the same complex structure on X. The Schwarzian derivatives $S(g_i \circ z_i; g_j \circ z_j)$ are easily seen to be zero by Proposition 2.1. This implies, again by Proposition 2.1, that $\{U_i, g_i \circ z_i\}$ is an atlas of complex projective structure. A different choice of atlas or a different collection of solutions g_i' would define the same complex projective structure.

Conversely if $\{V_j, f_j\}$ is an atlas of complex projective structure, then for any atlas $(U)_I$ of holomorphic coordinates, $\{V_j \cap U_i, S(f_j;z_i)\}$ defines a projective connection.□

Denote by K_X the canonical bundle of the Riemann surface X; it is equal to the cotangent bundle because X has complex dimension one. The difference between two projective connections is a quadratic differential, i.e. a holomorphic section of the line bundle K_X^2. Conversely we can add a quadratic differential to a given complex projective connection to obtain a new complex projetive connection. Since we know that each Riemann surface has at least one complex projective structure, the one given by Poincaré-Koebe uniformization, we have the following:

Proposition 3.4 *The set of projective connections on a Riemann surface X, or the set of complex projective structures on X, has a structure of principal homogeneous space over the vector space* $\mathrm{H}^0(X, K_X^2)$.

Let $\{U_i, z_i\}$ be an arbitrary atlas of holomorphic coordinates on X. The collection $\{S(z_i; z_j)\}$ defines a cocycle with values in K_X^2 because $S(z_i; z_k) = S(z_i; z_j)\left(\frac{dz_j}{dz_k}\right)^2 + S(z_j; z_k)$ by Proposition 2.1. If X is a compact hyperbolic Riemann surface, then $\mathrm{H}^1(X, K_X^2) = \{0\}$ (cf. [GH78]) so that the cocyle $\{S(z_i; z_j)\}$ is necessarily exact, that is, there is a projective connection $\{h_i\}$ such that $S(z_i; z_j) = h_i\left(\frac{dz_j}{dz_k}\right)^2 - h_j$. This gives a non-constructive proof that every compact hyperbolic Riemann surface admits a complex projective structure. Proposition 3.4 shows that projective connections on X form an affine space for the vector space $\mathrm{H}^0(X, K_X^2)$; by Riemann-Roch's Theorem (cf. [GH78]) the complex dimension of $\mathrm{H}^0(X, K_X^2)$ is $3g-3$ provided with $g > 1$.

Definition 3.5 *Let X be a compact Riemann surface. A meromorphic 2-form $\omega(x,y)$ on $X \times X$ is called a bidifferential of the second kind if for any fixed $x_0 \in X$ the form $\omega(x_0, y)$ induces a 1-form on X with a single pole of order two at x_0, and if the same things holds for the second variable. It is called symmetric if $\omega(x,y) = \omega(y,x)$.*

We denote by S the product surface $X \times X$, and by Δ the diagonal divisor in S. The canonical bundle of S is $K_S = p_1^* K_X \otimes p_2^* K_X$ where p_1, p_2 are projections onto the two factors. A bidifferential of the second kind ω is a section of the line bundle $K_S(2\Delta)$. In local coordinates around the diagonal ω has the form

$$\frac{\alpha dxdy}{(x-y)^2} + H(x,y)dxdy, \tag{7}$$

where $\alpha \in \mathbf{C}^*$ and $H(x,y)$ is holomorphic. The complex number α does not depend on the coordinates and is called the *biresidue* of ω. We will see in Section 5 that there always exists a symmetric bidifferential of second kind with biresidue 1.

Proposition 3.6 *Let ω be a bidifferential of the second kind with biresidue* 1 *and let $\{H_i(x,y)\}$ be the collection of its regular parts in local coordinates as in Equation* (7). *The the functions $h_i^\omega(z) = -6H_i(z,z)$ form a projective connection on X.*

Proof Let $x = g(v), y = g(w)$ be a change of coordinates. Then using Formula (3) we can write

$$H_j(v,w)dvdw = H_i(x,y)dxdy + S(x,y;v,w)dvdw.$$

Putting $v = w = z_j$ and $x = y = z_i$, by Equation (5) we get $h_j^\omega(z_j)dz_j^2 = h_i^\omega(z_i)dz_i^2 + S(z_i; z_j)dz_j^2$. This is the formula in the definition of a projective connection. □

Corollary 3.7 *Two bidifferentials of the second kind with biresidue* 1 *define the same projective connections if and only if their difference vanishes on the diagonal* Δ*.*

Let us remark that, in order to get a projective connection, we only need the bidifferential $\omega \in \mathrm{H}^0(S, K_S(2\Delta))$ to be defined locally in a neighborhood of Δ. In fact by Corollary 3.7, we only need ω to be some section of the line bundle $K_S(2\Delta)$ on the scheme-theoretic infinitesimal neighborhood 3Δ. Such a nowhere non-vanishing section defines a trivialisation of $K_S(2\Delta)$ on 3Δ while the condition that the bidifferential is of second kind with biresidue 1 means that when restricted to 2Δ the trivialisation is a fixed one given by the term $1/(x-y)^2$ in (7). From the exact sequence

$$0 \to K_X^2 \to K_S(2\Delta)|_{3\Delta} \to K_S(2\Delta)|_{2\Delta} \to 0$$

we see that the set of trivialisations of $K_S(2\Delta)|_{3\Delta}$ which induce a fixed trivialisation of $K_S(2\Delta)|_{2\Delta}$ is an affine space over $\mathrm{H}^0(X, K_X^2)$. The above discussion thus shows

Proposition 3.8 (Biswas-Raina [BR96]) *The affine space of all trivialisations of* $K_S(2\Delta)|_{3\Delta}$ *which on restriction to* 2Δ *give the trivialisation corresponding to* $dxdy/(x-y)^2$ *is isomorphic to the affine space of complex projective structures on* X*.*

Remark 3.9 A crucial point in the above discussion is the existence of a canonical bidifferential of the second kind with biresidue 1, which gives the canonical trivialisation $dxdy/(x-y)^2$ over 2Δ. This is non-trivial and will be shown in Section 5.

The difference of two bidifferentials of the second kind with biresidue 1 is a section of K_S and the space of bidifferentials of the second kind with biresidue 1 on S is an affine space over $\mathrm{H}^0(S, K_S)$. The trivialisations of $K_S(2\Delta)|_{3\Delta}$ coming from global bidifferentials of the second kind with biresidue 1 form an affine space over the image vector space of the restriction map

$$\mathrm{H}^0(X, K_X)^2 = \mathrm{H}^0(S, K_S) \to \mathrm{H}^0(\Delta, K_S|_\Delta) = \mathrm{H}^0(X, K_X^2).$$

This restriction map can be identified with the product map from $\mathrm{H}^0(X, K_X)^2$ to $\mathrm{H}^0(X, K_X^2)$, which is surjective if X is non-hyperelliptic by Noether's Theorem (see [GH78]). Thus we obtain

Proposition 3.10 (Tyurin [Tyu78]) *If* X *is a non-hyperelliptic compact Riemann surface, then every complex projective structure on* X *is induced by a bidifferential of the second kind with biresidue* 1*.*

4 Theta functions

Let $A = \mathbf{C}^g/(Id_g\mathbf{Z}^g + \tau\mathbf{Z}^g)$ be a principally polarized abelian variety, where τ is a symmetric $g \times g$-matrix with positive definite imaginary part. The set of such matrices is the Siegel half space $\mathscr{H}_g$. Riemann's Theta function is the following holomorphic function on $\mathbf{C}^g \times \mathscr{H}_g$:

$$\vartheta(z, \tau) = \sum_{m \in \mathbf{Z}^g} \exp(i\pi m^\mathsf{T} \tau m + 2i\pi m^\mathsf{T} z).$$

For $a, b \in \mathbf{R}^g$, the Theta function with characteristics $\begin{bmatrix} a \\ b \end{bmatrix}$ is

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_{m \in \mathbf{Z}^g} \exp\left[i\pi (m+a)^{\mathsf{T}} \tau (m+a) + 2i\pi (m+a)^{\mathsf{T}} (z+b) \right].$$

Then $\vartheta(z, \tau) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau)$. For $u, v \in \mathbf{Z}^g$, we have

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \exp[i\pi a^{\mathsf{T}} \tau a + 2i\pi a^{\mathsf{T}} (z+b)] \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z+b+\tau a, \tau)$$
$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z+v+\tau u, \tau) = \exp[2i\pi (a^{\mathsf{T}} v - u^{\mathsf{T}} (z+b)) - i\pi u^{\mathsf{T}} \tau u] \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau)$$

For fixed τ, the Theta function ϑ is a multi-valued holomorphic function on the abelian variety A which can be viewed as a section of some line bundle L_ϑ on A. The zero divisor Θ of ϑ is well defined on A and is called the *Theta divisor*; we have $L_\vartheta = \mathscr{O}_A(\Theta)$. The multi-valued function $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau)$ is a section of the line bundle obtained from L_ϑ by a translation $z \mapsto z + \tau a + b$. We refer to [Mum83] for the following transformation formula for Theta functions when we transform τ into $\tau' = (A\tau + B)(C\tau + D)^{-1}$ with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{Z})$:

$$\vartheta \begin{bmatrix} a' \\ b' \end{bmatrix} (Mz, \tau') = \kappa_\tau (\det M)^{1/2} \exp\left(\frac{1}{2} \sum_{j \le k} z_j z_k \frac{\partial \log \det M}{\partial \tau_{jk}} \right) \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) \tag{8}$$

with $\kappa_\tau \in \mathbf{C}^*$, $M = C\tau + D$ and

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \mathrm{diagonal}(CD^{\mathsf{T}}) \\ \mathrm{diagonal}(AB^{\mathsf{T}}) \end{pmatrix} \tag{9}$$

where diagonal takes the diagonal of a square matrix as a column vector.

We are particularly interested in those characteristics $\begin{bmatrix} a \\ b \end{bmatrix}$ with $a, b \in \frac{1}{2}\mathbf{Z}^g$. They correspond to 2-torsion points on A and are called *half-period* characteristics. We will denote the 4^g half-period characteristics by $\delta_1, \cdots, \delta_{4^g}$, and the corresponding Theta functions by $\vartheta[\delta_i]$. A Theta function with half-period characteristics is either even or odd; a half-period is called *even or odd* if the corresponding function is so. There are $2^g(2^g+1)$ even half-periods and $2^g(2^g-1)$ odd ones. We will need the following two embedding theorems concerning the Theta functions with half-period characteristics, that we state without giving proofs. Basically the two theorems say that either for fixed z or for fixed τ, the Theta functions with half-period characteristics form a very ample linear system.

Theorem 4.1 (Lefschetz embedding theorem [Mum83]) *The map*

$$A \to \mathbb{P}^{4^g-1}(\mathbf{C}), z \mapsto [\vartheta[\delta_1](4z, 4\tau); \cdots; \vartheta[\delta_{4^g}](4z, 4\tau)]$$

is an embedding.

Theorem 4.2 (Jun-Ichi Igusa [Igu72]) *The map*

$$\mathscr{H}_g \to \mathbb{P}^{4^g-1}(\mathbf{C}), \tau \mapsto [\vartheta[\delta_1](0,\tau);\cdots;\vartheta[\delta_{4^g}](0,\tau)]$$

induces an embedding from $\mathscr{H}_g/\Gamma$ into $\mathbb{P}^{4^g-1}(\mathbf{C})$. Here Γ is a non-principal congruence subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$ so that $\mathscr{H}_g/\Gamma$ is a finite cover of the moduli space of principally polarized abelian varieties.

Now let X be a compact Riemann surface of genus $g > 0$. Fix a Torelli marking on X, that is, a basis $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$ of $\mathrm{H}_1(X, \mathbf{Z})$ such that the intersection matrix has the form $\begin{pmatrix} 0 & -Id_g \\ Id_g & 0 \end{pmatrix}$ in this base. Let $v_1, \cdots, v_g$ be a basis of $\mathrm{H}^0(X, K_X)$ such that the period matrix with respect to $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$ is (Id_g, τ) with $\tau \in \mathscr{H}_g$. The Jacobian variety $J(X)$ is the principally polarized abelian variety $\mathbf{C}^g/(Id_g\mathbf{Z}^g + \tau\mathbf{Z}^g)$ which is identified with the group of divisors on X of degree 0 modulo principal divisors, and also with the group of line bundles on X of degree 0. For $d \in \mathbf{Z}$, denote by $J_d(X)$ the set of line bundles on X of degree d; it is a principal homogeneous space over $J(X)$. For a fixed $x \in X$ the Abel-Jacobi map from X to $J(X)$ is $y \mapsto [y-x]$, where $[y-x]$ denote the class of the divisor $y-x$. We can pull back Theta functions on $J(X)$ to get multi-valued functions on X. We refer to [GH78] for the following important theorem:

Theorem 4.3 (Riemann) *There is a divisor class $\Xi \in J_{g-1}(X)$ such that $2\Xi = K_X \in J_{2g-2}(X)$, and we have for any $x \in X$, $e \in J(X)$:*

1. *If $\vartheta(e,\tau) \neq 0$, then the zero divisor of $\vartheta([y-x]-e,\tau)$, as a function in y, is an effective divisor C of degree g on X such that $\dim \mathrm{H}^0(X, \mathscr{O}(C)) = 1$ and*

   $$e = [C-x] - \Xi.$$

2. *$\vartheta(e,\tau) = 0$ if and only if there exists an effective divisor D of degree $g-1$ on X such that*

   $$e = [D] - \Xi.$$

   Furthermore e is a smooth point of the Theta divisor Θ if and only if $\dim \mathrm{H}^0(X, \mathscr{O}(D)) = 1$. If $\dim \mathrm{H}^0(X, \mathscr{O}(D)) = 1$ and $\dim \mathrm{H}^0(X, \mathscr{O}(D+x)) = 1$, then the divisor of $\vartheta([y-x]-e,\tau)$ is $x+D$; otherwise $\vartheta([y-x]-e,\tau)$ vanishes identically on X.

Corollary 4.4 *Let $\delta \in J(X)$ be an odd half-period characteristic. There exists a divisor D_δ such that $\delta = [D_\delta] - \Xi$ and $[2D_\delta] = K_X$.*

Proof The odd characteristic δ is on the Theta divisor because $\vartheta(\delta,\tau) = \vartheta[\delta](0,\tau) = 0$. The equality $\delta = [D_\delta] - \Xi$ follows from the second situation in Theorem 4.3. Since $2\delta = 0$ and $2\Xi = K_X$, we obtain $[2D_\delta] = K_X$. □

By Theorem 4.1 there exists at least one odd half-period characteristic δ such that $d\vartheta[\delta]|_{z=0} \neq 0$. This means that $d\vartheta|_{z=\delta} \neq 0$, that is, δ is a non-singular point of the Theta divisor.

Proposition 4.5 *Let $\delta \in J(X)$ be a non-singular odd half-period characteristic and D_δ the corresponding divisor. Then $2D_\delta$ is the divisor of the holomorphic differential*

$$\omega_\delta = \sum_{j=1}^{g} \frac{\partial \vartheta}{\partial z_j}(\delta, \tau) v_j.$$

Proof By Theorem 4.3 $\vartheta([D]-\Xi)$ vanishes for all effective divisor $D = x_1 + \cdots + x_{g-1}$. For all k differentiating with respect to x_k we get

$$\sum_{j=1}^{g} \frac{\partial \vartheta}{\partial z_j}([D]-\Xi, \tau) v_j(x_k) = 0.$$

Putting $[D_\delta]-\Xi = \delta$ in the equality we deduce that ω_δ vanishes on D_δ. By Theorem 4.3 and the fact that δ is not a singular point, we have $\dim \mathrm{H}^0(X, \mathscr{O}(D_\delta)) = 1$. By Riemann-Roch's Theorem, we have $\dim \mathrm{H}^0(X, \mathscr{O}(K_X - D_\delta)) = 1$. This means that the divisor D_δ does not move in a linear system and that up to multiplication by a constant ω_δ is the only holomorphic differential vanishing on D_δ. The conclusion follows. $\square$

Definition 4.6 (John Fay [Fay73]) *Let δ be a non-singular odd half-period. Let r_δ be the section of $\mathscr{O}(D_\delta)$ such that $r_\delta^2 = \omega_\delta$. The prime form is the following multi-valued function on $X \times X$:*

$$E(x,y) = \frac{\vartheta[\delta]([y-x], \tau)}{r_\delta(x) r_\delta(y)}$$

which is a section of the line bundle $p_1^* \mathscr{O}(D_\delta)^{-1} \otimes p_2^* \mathscr{O}(D_\delta)^{-1} \otimes \xi^*(L_\vartheta)$ where ξ is the map from $X \times X$ to $J(X)$ sending (x,y) to $[y-x]$.

Proposition 4.7

1. *$E(x,y) = -E(y,x)$.*
2. *The divisor of E is the diagonal Δ.*
3. *The multi-valued function E is invariant under cycles $\alpha_1, \cdots, \alpha_g$; along β_k it transforms as*

$$E(\beta_k(x), y) = \exp(-i\pi\tau_{kk} - 2i\pi \int_x^y v_k) E(x,y).$$

4. *It does not depend on the non-singular odd characteristic δ.*
5. *For $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$, the divisor of the meromorphic function $\prod_{j=1}^n \frac{E(x,y_j)}{E(x,x_j)}$ is the divisor $\sum_{j=1}^n y_j - \sum_{j=1}^n x_j$.*

Proof When x tends to y, $\vartheta[\delta]([y-x], \tau)$ is equivalent to $\vartheta(\delta - [y-x])$. By Theorem 4.3 the divisor of the latter are the diagonal Δ and also the $\{x_j\} \times X, X \times \{x_j\}$ where $\sum_{j=1}^{g-1} x_j = D_\delta$. However $r_\delta(x)$ vanishes exactly on D_δ so that for E the only remaining zero is Δ. This proves the second assertion.

The first assertion holds because $\vartheta[\delta]$ is an odd function. The third assertion follows from the transformation formulas for Theta functions. The fourth assertion follows from the third one. The fifth follows from the second one. $\square$

5 Bergman and Wirtinger projective connections

Though the prime form $E(x,y)$ is only a multi-valued function on $X \times X$, the partial derivative

$$\omega_f(x,y) = \frac{\partial^2}{\partial x \partial y} \log E(x,y) dx dy$$

is a well-defined meromorphic differential on $X \times X$, by the third formula in Proposition 4.7. For a non-singular point $e \in \Theta$, it is also equal to (cf. [Fay73], we will not need this formula)

$$\frac{\partial^2}{\partial x \partial y} \log \vartheta([y-x]-e,\tau) dx dy.$$

We call ω_f the *fundamental bidifferential*; such expressions appeared already in [Kle90]. Moreover Proposition 4.7 implies:

Proposition 5.1 *$\omega_f(x,y)$ is a symmetric bidifferential of the second kind with biresidue* 1*. For fixed $x \in X$ and all j*

$$\int_{\alpha_j} \omega_f(x,y) = 0 \quad \textit{and} \quad \int_{\beta_j} \omega_f(x,y) = v_j(x).$$

Remark that ω_f depends on the period matrix τ, that is, it depends on the Torelli marking fixed on X. If we change the Torelli marking by a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{Z})$, then ω_f becomes

$$\omega'_f(x,y) = \omega_f(x,y) - \frac{1}{2} \sum_{j \le k} \frac{\partial}{\partial \tau_{jk}} \log \det(C\tau + D)[v_j(x)v_k(y) + v_k(x)v_j(y)], \qquad (10)$$

which follows from the corresponding transformation formulas for Theta functions (see Equation (8)).

By Proposition 3.6, the fundamental bidifferential gives a projective connection h_B on X; the projective connection h_B is called the *Bergman projective connection*. For sake of completeness we mention the following explicit expression of h_B that can be found in [Fay73]:

$$h_B(z)dz^2 = S\left(\int_{z_0}^{z} T_1^e; z\right) dz^2 + \frac{3}{2}\left(\frac{T_2^e}{T_1^e}\right)^2 (z) - 2\frac{T_3^e}{T_1^e}(z)$$

where e is an arbitrary non-singular point of the Theta divisor and

$$T_1^e(z) = \sum_{j=1}^{g} \frac{\partial \vartheta}{\partial z_j}(e) v_j(z)$$

$$T_2^e(z) = \sum_{j,k=1}^{g} \frac{\partial^2 \vartheta}{\partial z_j \partial z_k}(e) v_j(z) v_k(z)$$

$$T_3^e(z) = \sum_{j,k,l=1}^{g} \frac{\partial^3 \vartheta}{\partial z_j \partial z_k \partial z_l}(e) v_j(z) v_k(z) v_l(z).$$

The Bergman projective connection h_B depends on the Torelli marking too; the corresponding transformation formula follows from Equation (10):

$$h'_B(z) = h_B(z) - \frac{1}{2}\sum_{j,k=1}^{g} v_j(z)v_k(z)\frac{\partial}{\partial \tau_{jk}}\log\det(C\tau + D) \tag{11}$$

Formula (9) gives an action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on the set of 4^g half-period characteristics. For δ a half-period characteristic, we denote by Γ_δ the subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$ fixing δ. It follows from Equations (8) and (11) that

$$h_\delta = h_B + \sum_{j,k=1}^{g}\left(\frac{\partial^2}{\partial z_j \partial z_k}\log\vartheta[\delta](0,\tau)\right) v_j v_k$$

is a projective connection invariant under changes of marking by Γ_δ. We call h_δ a *partial Wirtinger projective connection*; it is only defined for even δ such that $\vartheta[\delta](0,\tau) \neq 0$. Since the partial derivatives of an even function vanish at $z = 0$, we obtain by Equation (8) the following expression of the difference between two partial Wirtinger connections:

$$h_\delta - h_{\delta'} = \sum_{j,k=1}^{g} v_j v_k \left(\frac{\frac{\partial^2}{\partial z_j z_k}\vartheta[\delta](0,\tau)}{\vartheta[\delta](0,\tau)} - \frac{\frac{\partial^2}{\partial z_j z_k}\vartheta[\delta'](0,\tau)}{\vartheta[\delta'](0,\tau)}\right) \tag{12}$$

The *Wirtinger projective connection* (cf. [Wir44]) is the following connection invariant under the whole group $\mathrm{Sp}_{2g}(\mathbf{Z})$:

$$\begin{aligned} h_W &= h_B + \frac{2}{4^g + 2^g}\sum_{j,k=1}^{g}\left(\frac{\partial^2}{\partial z_j \partial z_k}\log\left(\prod_{\delta \text{ even}}\vartheta[\delta](0,\tau)\right)\right) v_j v_k \\ &= \frac{2}{4^g + 2^g}\sum_{\delta \text{ even}} h_\delta; \end{aligned}$$

it is only defined on those Riemann surfaces X such that $\vartheta[\delta](0,\tau) \neq 0$ for all even δ.

Example: elliptic curves. Let us consider the case where $z, \tau \in \mathbf{C}$ and $\Im\tau > 0$. Let $E = \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ be the corresponding elliptic curve. There are four Theta functions with half-period characteristics:

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}(z,\tau) = \vartheta(z,\tau) = \sum_{j\in\mathbf{Z}}\exp(i\pi j^2\tau + 2i\pi jz)$$

and $\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(z,\tau), \vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(z,\tau), \vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z,\tau)$. Each theta function has exactly one zero on E. The zeros of $\vartheta\begin{bmatrix}0\\0\end{bmatrix}, \vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}, \vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}, \vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$ are respectively $\frac{1}{2} + \frac{\tau}{2}, \frac{\tau}{2}, \frac{1}{2}$ and 0. The only odd half-period is $\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$. We have

$$\frac{\partial^2}{\partial z^2}\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z,\tau) = \wp(z) + \varepsilon$$

where $\wp$ is the Weierstrass function and $\varepsilon \in \mathbf{C}$ is a constant. The fundamental bidifferential is

$$\frac{\partial^2}{\partial x \partial y} \log \vartheta([y-x], \tau) dx dy = (\wp(y-x) + \varepsilon) dx dy.$$

A computation using formulas of Theta functions shows that the bidifferential which gives the invariant Wirtinger connection is

$$\wp(x-y) dx dy.$$

Since $\wp(z) = \frac{1}{z^2} + O(z^2)$, we see by Corollary 3.7 that the invariant Wirtinger connection is trivial with respect to the original coordinate on E, that is, the complex projective structure determined by the invariant Wirtinger connection is nothing else but the natural complex affine structure on E.

6 Moduli space of projective connections

Let $\pi : \mathscr{X} \to \mathscr{B}$ be a holomorphic submersion having maximal rank everywhere and whose set-theoretic fibers are compact Riemann surfaces. Here $\mathscr{B}$ may be a complex orbifold obtained as the quotient of a complex manifold by a group action with finite stabilizers, e.g. the moduli space of curves obtained as the quotient of Teichmüller space by the mapping class group. We refer to [Sat56] for bundles and cohomology for orbifolds. For $b \in \mathscr{B}$ we denote by X_b the fiber. We want to define how complex projective structures vary in a holomorphic way when the underlying complex structures vary.

Definition 6.1

- *A relative complex projective structure on $\mathscr{X}$ is a maximal relative atlas $\{U_i, w_i\}$ where the U_i form an open cover of $\mathscr{X}$ and the $w_i : U_i \to \mathbb{P}^1(\mathbf{C})$ are holomorphic maps which are biholomorphism onto their images when restricted to fibers of π, such that for each $b \in \mathscr{B}$, the fiber restriction $\{U_i|_b, w_i|_b\}$ is an atlas of complex projective structure on X_b.*
- *A relative projective connection on $\mathscr{X}$ is given by a collection of local holomorphic functions H_i so that when restricted to X_b for all $b \in \mathscr{X}$ they form a projective connection on X_b.*

Our Theorem 0.1 formulated in this language is as follows:

Theorem 6.2 *If $\pi : \mathscr{X} \to \mathscr{B}$ is a Kodaira fibration, then $\mathscr{X}$ does not have any relative complex projective structure.*

We will denote by $\mathscr{Q}_{\mathscr{B}}(\mathscr{X})$ the bundle of quadratic differentials; it is an orbifold vector bundle over $\mathscr{B}$ such that the fiber over $b \in \mathscr{B}$ is $\mathrm{H}^0(X_b, K_{X_b}^2)$.

Definition/Proposition 6.3 (Hubbard [Hub81]) *Let $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$ bet the set of pairs (b, h) where $b \in \mathscr{B}$ and h is a projective connection on X_b. Then there is a unique complex structure (manifold or orbifold) on $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$ such that*

- *the projection $\rho : \mathscr{P}_{\mathscr{B}}(\mathscr{X}) \to \mathscr{B}, (b, h) \mapsto b$ is holomorphic;*

- *relative projective connections on $\mathscr{X}$ correspond to holomorphic sections of ρ;*
- *the action of $\mathscr{Q}_{\mathscr{B}}(\mathscr{X})$ on $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$ given by $((b,q),(b,h)) \mapsto (b,h+q)$ makes $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$ into a holomorphic affine bundle for the vector bundle $\mathscr{Q}_{\mathscr{B}}(\mathscr{X})$.*

Proof The Bergman projective connections on the fibers vary holomorphically in coordinates and give a relative projective connection, at least locally over some open sets of $\mathscr{B}$. In other words Bergaman connections give rise to local holomorphic sections of the affine bundle. Then the action of $\mathscr{Q}_{\mathscr{B}}(\mathscr{X})$ on $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$ can be used to transport locally the complex structure on $\mathscr{Q}_{\mathscr{B}}(\mathscr{X})$ to $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$. The proofs of other statements are left to the reader. ☐

Remark 6.4 Quasi-fuchsian projective structures give a global holomorphic section over the Teichmüller space while Fuchsian ones only give a real section.

An affine bundle is determined by the corresponding vector bundle and a class in the first cohomology group of the vector bundle; we denote by $\zeta_{\mathscr{X}} \in \mathrm{H}^1(\mathscr{B}, \mathscr{Q}_{\mathscr{B}}(\mathscr{X}))$ the cohomology class determining $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$. A representative cocycle can be constructed as follows: let $\{V_i\}$ be an open covering of $\mathscr{B}$ such that for all i there is a holomorphic section s_i of $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$ over U_i, then $\{s_i - s_j\}$ is the desired cocycle. Thus we have

Proposition 6.5 *$\mathscr{X}$ has a relative complex projective structure if and only if $\zeta_{\mathscr{X}} = 0$.*

Given a relative projective structure on $\mathscr{X}$ and a section of π, there are a holomorphic map D from the universal covering of $\mathscr{X}$ to $\mathbb{P}^1(\mathbf{C})$ and a holomorphic family of representations r_b from the fundamental group of a fiber into $\mathrm{PGL}_2(\mathbf{C})$ parametrized by the universal covering of $\mathscr{B}$, such that D_b, r_b are the developing map and holonomy representation of the complex projective structure on X_b.

There is a tautological family over $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$ with a tautological relative projective connection where the fiber over (b,h) is X_b equipped with the projective connection h. This tautological family satisfies a universal property: the map which associates to any holomorphic mapping $f : T \to \mathscr{P}_{\mathscr{B}}(\mathscr{X})$ the pulled-back family of projective connections on the family of Riemann surfaces pulled back by $\pi \circ f$ is a bijection from the set of holomorphic maps from T to $\mathscr{P}_{\mathscr{B}}(\mathscr{X})$ onto the set of relative projective connection on $(\pi \circ f)^* \mathscr{X}$.

We will be interested in the case where $\mathscr{B} = \mathscr{M}_g$ is the moduli space of compact Riemann surfaces of genus g and $\mathscr{X} = \mathscr{X}_g$ is the universal curve. We will denote $\mathscr{P}_{\mathscr{M}_g}(\mathscr{X}_g)$ simply by $\mathscr{P}_g$; it is an affine bundle for the bundle of quadratic differentials $\mathscr{Q}_g$ which is identified with the cotangent bundle $\mathscr{T}_g^*$ of $\mathscr{M}_g$ via Kodaira-Spencer theory. We denote by $\zeta_g \in \mathrm{H}^1(\mathscr{M}_g, \mathscr{Q}_g)$ the class determining the affine bundle structure of $\mathscr{P}_g$.

7 Determine the cocycle

We prove Theorem 0.1 in this section. Let $\pi : \mathscr{X} \to C$ be a non-isotrivial family of compact Riemann surfaces of genus g over a compact Riemann surface C. The family is induced by a non-constant morphism $f : C \to \mathscr{M}_g$. To prove Theorem 0.1, it suffices to prove $\zeta_{\mathscr{X}} \neq 0$ by Proposition 6.5. The class $\zeta_{\mathscr{X}} \in \mathrm{H}^1(C, \mathscr{Q}_C(\mathscr{X})) = \mathrm{H}^1(C, f^*\mathscr{T}_g^*)$ is obtained by pulling back $\zeta_g \in \mathrm{H}^1(\mathscr{M}_g, \mathscr{Q}_g) = \mathrm{H}^1(\mathscr{M}_g, \mathscr{T}_g^*)$ by f.

If a line bundle L on a variety Y is given by a cocycle $\{\alpha_{ij}\} \in \mathrm{H}^1(Y, \mathscr{O}_Y^*)$, then its *caracteristic class* $c(L)$ is the element of $\mathrm{H}^1(Y, \mathscr{T}_Y^*)$ represented by the cocylce $\{\alpha_{ij}^{-1} d\alpha_{ij}\}$.

Lemma 7.1 *If $\zeta_g = dc(L)$ for some non-zero number d and for an ample line bundle L on $\mathscr{M}_g$, then $\zeta_{\mathscr{X}} \neq 0$.*

Proof We have a morphism $\nu : \mathrm{H}^1(C, f^*\mathscr{T}_g^*) \to \mathrm{H}^1(C, \mathscr{T}_C^*)$ induced by the bundle map $\mathscr{T}_C \to \mathscr{T}_g$. We have $\nu(\zeta_{\mathscr{X}}) = \nu(f^*(\zeta_g)) = \nu(df^*c(L))$. Note that $c(L)$ is represented by a cocycle of the form $\{\alpha_{ij}^{-1} d\alpha_{ij}\}$ where the α_{ij} are local functions on $\mathscr{M}_g$. Thus the $\alpha_{ij} \circ f$ are local functions on C and $f^*c(L)$ is represented by $\{(\alpha_{ij} \circ f)^{-1} d(\alpha_{ij} \circ f)\}$ which lives in $\mathrm{H}^1(C, \mathscr{T}_C^*)$. Therefore $\nu(\zeta_{\mathscr{X}}) = \nu(df^*c(L)) = dc(f^*L)$. Since $f : C \to \mathscr{M}_g$ is non-constant, f^*L is ample on C. As C is compact, the characteristic class of an ample line bundle in $\mathrm{H}^1(C, \mathscr{T}_C^*) = \mathrm{H}^{1,1}(C, \mathbf{C})$ is non-zero. ☐

Therefore Theorem 0.1 is a consequence of the following proposition

Proposition 7.2 *ζ_g is proportional to $c(L)$ for some ample line bundle L on $\mathscr{M}_g$.*

We give two computations which lead to Proposition 7.2. The two are basically the same: the first one is straightforward and the second one is more explicit. We need

Proposition 7.3 (Ahlfors-Rauch Formula [Ahl60] [Rau59]) *Consider the entry τ_{jk} of the priod matrix as a local function on $\mathscr{M}_g$. Under the identification of the cotangent bundle of $\mathscr{M}_g$ with the bundle of quadratic differentials, the differential $d\tau_{jk}$ is the family of quadratic differentials $v_j v_k$.*

Using Bergman connections. By Equation 11 the class ζ_g can be represented by the Čech cocyle $\{\lambda_{\tau\tau'}\}$ where

$$\lambda_{\tau\tau'} = \frac{1}{2} \sum_{j,k=1}^{g} v_j(z) v_k(z) \frac{\partial}{\partial \tau_{jk}} \log\det(C\tau + D).$$

Using Ahlfors-Rauch Formula, we get

$$\lambda_{\tau\tau'} = \frac{1}{2} \frac{d\det(C\tau + D)}{\det(C\tau + D)}$$

(this formula is obtained in [BKN17]) and $\zeta_g = \frac{1}{2} c(\iota^* L)$ where $\iota : \mathscr{M}_g \to \mathscr{A}_g$ is the Torelli map from $\mathscr{M}_g$ into the moduli space of principally polarized abelian varieties, and L is the line bundle on $\mathscr{A}_g$ represented by the cocyle $\{\det(C\tau + D)\}$ whose sections are Siegel modular forms with weight one half.

Using partial Wirtinger connections. This computation is made by Tyurin in [Tyu78]. Consider the Torelli map from $\iota : \mathscr{M}_g \to \mathscr{A}_g$. Theorem 4.2 gives an embedding of $\mathscr{A}'_g$, a finite cover of $\mathscr{A}_g$, into some $\mathbb{P}^n$ by using Theta constants. Composing this embedding with the Torelli map, we get an injective morphism $F : \mathscr{M}'_g \to \mathbb{P}^n$ from a finite cover of $\mathscr{M}_g$ into $\mathbb{P}^n$. We will deal with $\zeta'_g \in \mathrm{H}^1(\mathscr{M}'_g, \mathscr{T}^*\mathscr{M}'_g)$, the class pulled back from ζ_g.

Let us consider the line bundle $L = F^*\mathscr{O}(1)$. It is determined by the Čech cocyle $\{\alpha_{\delta\delta'}\}$ where δ, δ' are even half-period characteristics and

$$\alpha_{\delta\delta'} = \frac{\vartheta[\delta](0,\tau)}{\vartheta[\delta'](0,\tau)}.$$

The Chern class $c(L)$ is represented by the Čech cocyle $\{\alpha^{-1}_{\delta\delta'} d\alpha_{\delta\delta'}\}$ where

$$\alpha^{-1}_{\delta\delta'} d\alpha_{\delta\delta'} = \sum_{j,k=1}^{g} \frac{\partial}{\partial \tau_{jk}} \left(\log \frac{\vartheta[\delta](0,\tau)}{\vartheta[\delta'](0,\tau)} \right) d\tau_{jk}.$$

The Theta functions satisfy the heat equation (see [Mum83]):

$$\frac{\partial}{\partial \tau_{jk}} \vartheta[\delta](0,\tau) = \frac{\partial^2}{\partial z_j \partial z_k} \vartheta[\delta](0,\tau).$$

Therefore

$$\alpha^{-1}_{\delta\delta'} d\alpha_{\delta\delta'} = \sum_{j,k=1}^{g} \left(\frac{\frac{\partial^2}{\partial z_j z_k} \vartheta[\delta](0,\tau)}{\vartheta[\delta](0,\tau)} - \frac{\frac{\partial^2}{\partial z_j z_k} \vartheta[\delta'](0,\tau)}{\vartheta[\delta'](0,\tau)} \right) d\tau_{jk}.$$

Using Ahlfors-Rauch Formula and Equation (12), we see that this is exactly the cocyle determined by the differences of partial Wirtinger connections. Therefore we have $c(L) = \zeta'_g$.

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