

# A homogenization approach for the motion of motor proteins

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## Abstract

We consider the asymptotic behavior of a time dependent, weakly coupled, Fokker-Planck system of two equations set in a periodic environment. The magnitudes of the diffusion and the coupling are respectively proportional and inversely proportional to the size of the period. We prove that, as the period tends to zero, the solutions of the system either propagate (concentrate) with a fixed constant velocity (determined by the data) or do not move at all. The system arises in the modeling of motor proteins which can take two different states. Our result implies that, in the limit, the molecules either move along a fixed filament with a constant speed and direction or remain immobile.

**Key-Words:** Hamilton-Jacobi equations, homogenization, molecular motor, singular perturbation, viscosity solutions

**AMS Class. No:** 35B25, 35B27, 49L25, 92C05

## 1 Introduction

intro

We study the asymptotics, as  $\varepsilon \rightarrow 0$ , of the weakly coupled Fokker-Planck system

$$\begin{cases} n_{\varepsilon,t}^1 - \varepsilon \Delta n_{\varepsilon}^1 - \operatorname{div}_x (n_{\varepsilon}^1 D_y \psi(\frac{x}{\varepsilon})) + \frac{1}{\varepsilon} \nu^1(\frac{x}{\varepsilon}) n_{\varepsilon}^1 = \frac{1}{\varepsilon} \nu^2(\frac{x}{\varepsilon}) n_{\varepsilon}^2 \\ n_{\varepsilon,t}^2 - \varepsilon \Delta n_{\varepsilon}^2 + \frac{1}{\varepsilon} \nu^2(\frac{x}{\varepsilon}) n_{\varepsilon}^2 = \frac{1}{\varepsilon} \nu^1(\frac{x}{\varepsilon}) n_{\varepsilon}^1. \end{cases} \quad \text{in } \mathbb{R}^d \times (0, \infty), \quad (1)$$

eq:model

Systems like (1) have been proposed as models to describe the motion of motor proteins along molecular filaments or microtubules [13, 20, 2, 21, 9, 14]. The intracellular transport in eukaria is attributed to motor proteins that transform the chemical energy into mechanical motion. For example myosins, which are known for their role in the muscle contraction, move along actin filaments and kinesins move along microtubules

In (1) the molecules have the conformations 1 and 2 with densities  $n_{\varepsilon}^1$  and  $n_{\varepsilon}^2$  respectively and are influenced by the periodic potential  $\psi$  provided by the filaments (see [19]). The (periodic) functions  $\nu^1$  and  $\nu^2$  indicate the rates of change between the two states. The existence of traveling waves, the asymptotic speed of propagation for large time as well as the presence of concentration effects

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for models with diffusion and a periodic drift have been studied in many papers; for instance see [10, 8]. Systems like (1) also appear in other settings, like, for example, the stochastic Stokes' drift where particles are suspended in a liquid and are subjected to diffusion and a net drift due to the presence of a wave in the liquid [7]. From the mathematical point of view, we refer to [1] for a related homogenization problem as well as to [15] for the homogenization of a similar equation with parabolic scaling.

To formulate our result for the densities  $n_\varepsilon^1$  and  $n_\varepsilon^2$  next we introduce the assumptions we will be using throughout the paper. In addition to

$$\nu^1, \nu^2 \text{ and } \psi \text{ are smooth and 1-periodic and } \nu^1 > 0, \nu^2 > 0, \quad (2) \quad \boxed{\text{ass:coef}}$$

we assume that

$$n_\varepsilon^1(x, 0) > 0 \text{ and } n_\varepsilon^2(x, 0) > 0 \text{ for all } x \in \mathbb{R}^d, \quad (3) \quad \boxed{\text{as:pos}}$$

$$I_\varepsilon^0 = \int_{\mathbb{R}} n_\varepsilon^1(x, 0) dx + \int_{\mathbb{R}} n_\varepsilon^2(x, 0) dx \rightarrow I_0 > 0, \text{ as } \varepsilon \rightarrow 0, \quad (4) \quad \boxed{\text{mass:ini}}$$

$$\left\{ \begin{array}{l} \lim_{\varepsilon \rightarrow 0} \varepsilon \ln n_\varepsilon^1(y, 0) = \lim_{\varepsilon \rightarrow 0} \varepsilon \ln n_\varepsilon^2(y, 0) = -\infty \text{ locally uniformly in } \mathbb{R}^d \setminus \{0\}, \\ \text{and} \\ \limsup_{\substack{y \rightarrow 0 \\ \varepsilon \rightarrow 0}} \varepsilon \ln n_\varepsilon^i(y, 0) \leq 0 \text{ for } i = 1, 2, \end{array} \right. \quad (5) \quad \boxed{\text{R2ini2}}$$

and, there exist constants  $A > 0$  and  $B$  such that,

$$n_\varepsilon^i(x, 0) \leq e^{\varepsilon^{-1}(-A|x|+B)} \text{ for all } x \in \mathbb{R}^d \text{ and } i = 1, 2. \quad (6) \quad \boxed{\text{as:massf}}$$

Let  $\delta$  denote the Dirac mass at the origin. Our first result is

**thm:trans** **Theorem 1.1** *Assume (2), (3), (4), (5) and (6). There exists  $\bar{v} \in \mathbb{R}^d$  such that, as  $\varepsilon \rightarrow 0$  and in the sense of measures,*

$$n_\varepsilon^1(t, x) + n_\varepsilon^2(t, x) \rightharpoonup \delta(x - t\bar{v})I_0. \quad (7) \quad \boxed{\text{concentra}}$$

To prove Theorem 1.1 we analyze the behavior, as  $\varepsilon \rightarrow 0$ , of the functions  $R_\varepsilon^1, R_\varepsilon^2 : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$  which are obtained from  $n_\varepsilon^1$  and  $n_\varepsilon^2$  by the classical exponential change of variable (we show later in the paper that  $n_\varepsilon^1 > 0$  and  $n_\varepsilon^2 > 0$ )

$$n_\varepsilon^1 = \exp(-R_\varepsilon^1/\varepsilon) \text{ and } n_\varepsilon^2 = \exp(-R_\varepsilon^2/\varepsilon), \quad (8) \quad \boxed{\text{def:R}}$$

and solve the system

$$\left\{ \begin{array}{l} R_{\varepsilon,t}^1 - \varepsilon \Delta_x R_\varepsilon^1 + |D_x R_\varepsilon^1|^2 - D_y \psi(\frac{x}{\varepsilon}) \cdot D_x R_\varepsilon^1 + \Delta_y \psi(\frac{x}{\varepsilon}) + \nu^2(\frac{x}{\varepsilon}) \exp(\frac{R_\varepsilon^1 - R_\varepsilon^2}{\varepsilon}) = \nu^1(\frac{x}{\varepsilon}) \\ R_{\varepsilon,t}^2 - \varepsilon \Delta_x R_\varepsilon^2 + |D_x R_\varepsilon^2|^2 + \nu^1(\frac{x}{\varepsilon}) \exp(\frac{R_\varepsilon^2 - R_\varepsilon^1}{\varepsilon}) = \nu^2(\frac{x}{\varepsilon}). \end{array} \right. \text{ in } \mathbb{R}^d \times (0, \infty) \quad (9) \quad \boxed{\text{eq:R}}$$

To state the second main result we recall that the ‘‘half-relaxed’’ upper and lower limits, denoted  $\overline{R}^i$  and  $\underline{R}^i$  respectively, of the family  $(R_\varepsilon^i)_{\varepsilon > 0}$  are given, for  $(x, t) \in \mathbb{R}^d \times [0, \infty)$  and  $i = 1, 2$ , by

$$\underline{R}^i(x, t) = \liminf_{\substack{(y,s) \rightarrow (x,t) \\ \varepsilon \rightarrow 0}} R_\varepsilon^1(y, s) \quad \text{and} \quad \overline{R}^i(x, t) = \limsup_{\substack{(y,s) \rightarrow (x,t) \\ \varepsilon \rightarrow 0}} R_\varepsilon^1(y, s). \quad (10)$$

We have:

thm:homo

**Theorem 1.2** *Assume (2) and (3). Then*

(i)  $\overline{R}^1 = \overline{R}^2$  and  $\underline{R}^1 = \underline{R}^2$  in  $\mathbb{R}^d \times (0, \infty)$ .

(ii) *There exists a strictly convex  $\overline{H} \in C^1(\mathbb{R}^d)$  satisfying  $\overline{H}(0) = 0$  and  $\overline{H}(p) \geq |p|^2 - C$ , for some  $C > 0$ , such that  $\overline{R} = \overline{R}^1 = \overline{R}^2$  and  $\underline{R} = \underline{R}^1 = \underline{R}^2$  are respectively subsolution and supersolution of*

$$\overline{R}_t + \overline{H}(\overline{R}_x) \leq 0 \quad \text{and} \quad \underline{R}_t + \overline{H}(\underline{R}_x) \geq 0 \quad \text{in } \mathbb{R}^d \times (0, \infty). \quad (11)$$

(iii) *Assume, in addition, (5). Then*

$$\overline{R}^1 = \overline{R}^2 = \underline{R}^1 = \underline{R}^2 = +\infty, \quad \text{in } (\mathbb{R}^d \setminus \{0\}) \times \{0\}, \quad \underline{R}^1(0, 0) \geq 0, \quad \underline{R}^2(0, 0) \geq 0. \quad (12)$$

(iv) *The special direction  $\bar{v}$  in Theorem 1.1 is*

$$\bar{v} = D\overline{H}(0). \quad (13)$$

The asymptotic behavior of a time-independent version of (1) set in  $[0, 1]$ , which is also controlled by the same effective Hamiltonian  $\overline{H}$ , was studied in [27] where it was proved that, for asymmetric potentials, the mass concentrates at either  $x = 0$  or  $x = 1$ . The asymmetry condition of [27] is  $D\overline{H}(0) \neq 0$ . This behavior is in agreement with our study of the time dependent problem. Indeed we prove here that, if  $\bar{v} = D\overline{H}(0) \neq 0$ , the proteins (mass) move (spread) with constant velocity  $\bar{v}$ . Hence the mass concentrates, for large times, on one end point of the filament, if the latter is assumed to have finite length. We also refer to [25] for a large deviation approach for the asymptotic behavior of the stationary solution of a similar model but with two potentials and to [26] for the study of flashing ratchets.

By slight modifications of the proofs, all the results in this paper can be extended to the following case with two potentials

$$\begin{cases} n_{\varepsilon,t}^1 - \varepsilon \Delta n_\varepsilon^1 - \operatorname{div}_x(n_\varepsilon^1 D_y \psi^1(\frac{x}{\varepsilon})) + \frac{1}{\varepsilon} \nu^1(\frac{x}{\varepsilon}) n_\varepsilon^1 = \frac{1}{\varepsilon} \nu^2(\frac{x}{\varepsilon}) n_\varepsilon^2 \\ n_{\varepsilon,t}^2 - \varepsilon \Delta n_\varepsilon^2 - \operatorname{div}_x(n_\varepsilon^2 D_y \psi^2(\frac{x}{\varepsilon})) + \frac{1}{\varepsilon} \nu^2(\frac{x}{\varepsilon}) n_\varepsilon^2 = \frac{1}{\varepsilon} \nu^1(\frac{x}{\varepsilon}) n_\varepsilon^1, \end{cases} \quad \text{in } \mathbb{R}^d \times (0, \infty), \quad (14)$$

the only difference being in the value of the effective Hamiltonian  $\overline{H}(\cdot)$ . See Section 2.

Our work is inspired from the ideas in wavefront propagation and large deviations [17, 5], the method of perturbed test functions in homogenization [16] and the methods used in the study of the concentration effects [6, 24] and motor effects [25, 27, 26].

Throughout the paper solutions are taken to be classical if smooth or, otherwise, in the viscosity sense. We refer to [11, 4] for a general introduction to the theory of the latter. In addition we denote by  $C$  positive constants which are independent of  $\varepsilon$  but may change from line to line. Moreover,  $B_r(x)$  is the open ball in  $\mathbb{R}^d$  centered at  $x$  and of radius  $r > 0$  and  $\overline{B}_r(x)$  stands for its closure. When  $x = 0$  we simply write  $B_r$  and  $\overline{B}_r$  respectively. Finally many statements in the paper hold for  $i = 1, 2$  without any changes. Hence, unless necessary, we will not be repeating the ‘‘for  $i = 1, 2$ ’’.

The paper is organized as follows. In Section 2 we introduce the cell problem corresponding to (9), we recall that it has a solution and we introduce  $\overline{H}$ . In Section 3 we present some preliminary facts about the family  $(n_\varepsilon^i)_{\varepsilon>0}$  and study the properties of the families  $(R_\varepsilon^i)_{\varepsilon>0}$  that are needed to prove the convergence to  $\overline{R}^i, \underline{R}^i$ . Theorem 1.2 is proved in Section 4. Using the results on the asymptotic behavior of the family  $(R_\varepsilon^i)_{\varepsilon>0}$  we prove Theorem 1.1 in Section 5. The asymptotic behavior of the family  $(n_\varepsilon^i)_{\varepsilon>0}$  in a more general setting is analyzed in Section 6. In Section 7 we compare our results with the case of the parabolic scaling in [15]. Finally, for the convenience of the reader, we present in the Appendix a sketch of the proof of the solvability of the cell problem and the properties of  $\overline{H}$ .

## 2 The cell problem

cell

In view of the presence of the exponential terms in (9) it is natural to expect that the  $R_\varepsilon^1$ 's and  $R_\varepsilon^2$ 's converge, if at all, as  $\varepsilon \rightarrow 0$  to the same limit  $R$ . Following [27] we insert in (9) the formal expansion

$$R_\varepsilon^i(x, t) = R(x, t) + \varepsilon \phi^i\left(\frac{x}{\varepsilon}\right) + O^i(\varepsilon^2),$$

and, keeping only the terms multiplying  $\varepsilon^0$  and writing  $y$  for the fast variable  $\frac{x}{\varepsilon}$ , we conclude that

$$\begin{cases} R_t - \Delta_y \phi^1 + |D_y \phi^1 + D_x R|^2 - D_y \psi \cdot (D_y \phi^1 + D_x R) + \Delta_y \psi + \nu^2 \exp(\phi^1 - \phi^2) = \nu^1 \\ R_t - \Delta_y \phi^2 + |D_y \phi^2 + D_x R|^2 + \nu^1 \exp(\phi^2 - \phi^1) = \nu^2. \end{cases} \quad \text{in } \mathbb{R}^d \times (0, \infty)$$
eq:phi

The goal is then to come up with  $\phi^i$ 's so that (15) is independent of  $y$ . This leads to the problem to find, for each  $p \in \mathbb{R}^d$ , a unique constant  $\overline{H}(p)$  such that the system, which is usually called the cell problem,

$$\begin{cases} -\Delta_y \phi^1 + |D_y \phi^1 + p|^2 - D_y \psi \cdot (D_y \phi^1 + p) + \Delta_y \psi + \nu^2 \exp(\phi^1 - \phi^2) = \nu^1 + \overline{H}(p) \\ -\Delta_y \phi^2 + |D_y \phi^2 + p|^2 + \nu^1 \exp(\phi^2 - \phi^1) = \nu^2 + \overline{H}(p), \end{cases} \quad \text{in } \mathbb{R}^d, \quad (16)$$
eq:cell

admits an 1-periodic solution  $(\phi^1, \phi^2)$  called the corrector.

We have:

lem:cell

**Lemma 2.1** *For each  $p \in \mathbb{R}^d$  there exists a unique constant  $\overline{H}(p)$  such that (16) has an 1-periodic solution  $(\phi^1, \phi^2)$ . Moreover,  $\overline{H} \in C^1(\mathbb{R}^d)$ ,  $\overline{H}(0) = 0$ ,  $\overline{H}$  is strictly convex and there exists a constant  $C$  such that  $\overline{H}(p) \geq |p|^2 - C$ , and, hence,  $\overline{H}(p) \rightarrow \infty$  as  $|p| \rightarrow \infty$ .*

A proof of Lemma 2.1 for  $d = 1$  was included in [27]. For the convenience of the reader we sketch in the Appendix the proof in  $\mathbb{R}^d$ .

When considering the system (14) with two potentials, (16) is replaced by

$$\begin{cases} -\Delta_y \phi^1 + |D_y \phi^1 + p|^2 - D_y \psi^1 \cdot (D_y \phi^1 + p) + \Delta_y \psi^1 + \nu^2 \exp(\phi^1 - \phi^2) = \nu^1 + \overline{H}(p) \\ -\Delta_y \phi^2 + |D_y \phi^2 + p|^2 - D_y \psi^2 \cdot (D_y \phi^2 + p) + \Delta_y \psi^2 + \nu^1 \exp(\phi^2 - \phi^1) = \nu^2 + \overline{H}(p), \end{cases} \quad \text{in } \mathbb{R}^d.$$

### 3 Some preliminaries and the properties of $R_\varepsilon^i$

We summarize in the next lemma some basic properties of the families  $(n_\varepsilon^i)_{\varepsilon>0}$ . They are the conservation of mass, the strict positivity of the  $n_\varepsilon^i$ 's, a global upper bound yielding that, as  $\varepsilon \rightarrow 0$ , there is very little mass at infinity.

We have:

**Lemma 3.1** (i) For all  $t \geq 0$  and  $\varepsilon > 0$ ,

$$\int_{\mathbb{R}^d} n_\varepsilon^1(x, t) dx + \int_{\mathbb{R}^d} n_\varepsilon^2(x, t) dx = I_\varepsilon^0. \quad (17) \quad \text{conservat}$$

(ii) Assume (2) and (3). Then

$$0 < n_\varepsilon^i \text{ in } \mathbb{R}^d \times [0, +\infty). \quad (18) \quad \text{npos}$$

(iii) Assume, in addition, (6). There exists  $D > 0$  such that, for all  $(x, t) \in \mathbb{R}^d \times [0, +\infty)$ .

$$n_\varepsilon^i(x, t) \leq \exp\left(\frac{-A|x| + B + Dt}{\varepsilon}\right). \quad (19) \quad \text{nborn}$$

In particular, there is small mass at infinity, i.e., for all  $t \geq 0$ , there exists  $M = M(t) > 0$  such that

$$\int_{|x| \geq M} n_\varepsilon^i(t, x) dx \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (20) \quad \text{mass0}$$

**Proof.** The conservation of the mass follows from adding the equations in (1) and integrating over  $\mathbb{R}^d$ .

The form of (1) and (2) and (3) allow us to use maximum principle-type arguments to obtain (18) and (19).

Indeed let

$$F_{1,\varepsilon}(n) = n_t - \varepsilon \Delta n - \operatorname{div}_x(n D_y \psi(\frac{x}{\varepsilon})) + \frac{1}{\varepsilon} \nu^1(\frac{x}{\varepsilon}) n - \frac{1}{\varepsilon} \nu^2(\frac{x}{\varepsilon}) n,$$

and

$$F_{2,\varepsilon}(n) = n_t - \varepsilon \Delta n + \frac{1}{\varepsilon} \nu^2(\frac{x}{\varepsilon}) n - \frac{1}{\varepsilon} \nu^1(\frac{x}{\varepsilon}) n.$$

It is easy to verify that  $n_\varepsilon = \min(n_\varepsilon^1, n_\varepsilon^2)$  and  $N_\varepsilon = \max(n_\varepsilon^1, n_\varepsilon^2)$  satisfy in  $\mathbb{R}^d \times (0, \infty)$  respectively

$$\max(F_{1,\varepsilon}(n_\varepsilon), F_{2,\varepsilon}(n_\varepsilon)) \geq 0 \text{ and } \min(F_{1,\varepsilon}(N_\varepsilon), F_{2,\varepsilon}(N_\varepsilon)) \leq 0. \quad (21) \quad \text{maxF}$$

Since 0 is clearly a solution of the first inequality in (21) we use the strong maximum principle to get (18).

To prove (19) we observe that, for  $D$  sufficiently large,

$$G_\varepsilon(x, t) = \exp\left(\frac{-A|x| + B + Dt}{\varepsilon}\right)$$

is a viscosity supersolution of the second inequality in (21). Since, in view of (6), we also have

$$N_\varepsilon(x, 0) \leq G_\varepsilon(x, 0),$$

we use the comparison principle to conclude.

Finally using (19) and choosing the constant  $M$  large enough, we clearly get (20).  $\square$

We turn now to the properties of the  $R_\varepsilon^i$ 's which are presented in Theorem 3.2 below. The proof is rather long. The first part, which provides an one-sided Lipschitz-type continuity in time, is based on the classical Harnack inequality. The lower bound in part(ii) follows from part(i). The arguments leading to the upper bound (part(iii)) are more tedious and require as an intermediary step, namely, the construction, again using part(i), of an appropriate local upper bound.

We have:

**Theorem 3.2** (i) Assume (2) and (3). For all  $\delta > 0$ , there exists  $C_\delta > 0$ , such that, for all  $\varepsilon > 0$ ,  $|z - z'| \leq \varepsilon$ ,  $\varepsilon\delta \leq t_0$  and  $i, j = 1, 2$ ,

$$R_\varepsilon^j(z', t_0 + \varepsilon) - R_\varepsilon^i(z, t_0) \leq \varepsilon C_\delta. \quad (22) \quad \boxed{\text{harnackR}}$$

(ii) Assume (2), (3), (4) and (5). For any  $a \in (0, \infty)$ , there exists  $\varepsilon_0 = \varepsilon_0(a) > 0$  such that, for all  $\varepsilon \leq \varepsilon_0$ ,

$$R_\varepsilon^i \geq -a \quad \text{in } \mathbb{R}^d \times [0, \infty). \quad (23) \quad \boxed{\text{h}}$$

(iii) Assume (2), (3), (4) and (6). For any compact subset  $K$  of  $\mathbb{R}^d \times (0, \infty)$ , there exist  $C_K > 0$  and  $\varepsilon_1 = \varepsilon_1(K) > 0$  such that, for all  $\varepsilon \leq \varepsilon_1$  and  $(x, t) \in K$ ,

$$R_\varepsilon^i(x, t) \leq C_K. \quad (24) \quad \boxed{\text{k}}$$

**Proof.** We begin with the

**Proof of (22):** Observe that  $\tilde{n}^1(y, \tau) = n_\varepsilon^1(\varepsilon y, \varepsilon\tau)$  and  $\tilde{n}^2(y, \tau) = n_\varepsilon^2(\varepsilon y, \varepsilon\tau)$  are positive (recall (18)) solutions to

$$\begin{cases} \tilde{n}_\tau^1 - \Delta_y \tilde{n}^1 - \tilde{n}^1 \Delta_y \psi(y) - D_y \psi(y) \cdot D_y \tilde{n}^1 + \nu^1 \tilde{n}^1 = \nu^2 \tilde{n}^2 \\ \tilde{n}_\tau^2 - \Delta_y \tilde{n}^2 + \nu^2 \tilde{n}^2 = \nu^1 \tilde{n}^1, \end{cases} \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad (25) \quad \boxed{\text{eq:ntilde}}$$

a linear parabolic system with bounded, according to (2), coefficients. It follows from the classical Harnack inequality [23] that, for each  $\delta > 0$ , there exists, an independent of  $\varepsilon$ ,  $C_\delta > 0$  such that for all  $y_0 \in \mathbb{R}^d$ ,  $\tau_0 \geq \delta$  and  $i, j = 1, 2$ ,

$$\sup_{z \in B_1(y_0)} \tilde{n}^i(z, \tau_0) \leq C_\delta \inf_{z \in B_1(y_0)} \tilde{n}^j(z, \tau_0 + 1). \quad (26) \quad \boxed{\text{harnackt}}$$

Rewriting (26) in terms of  $n^1$  and  $n^2$  and in the original variables  $(x, t)$  we get

$$\sup_{z \in B_\varepsilon(x_0)} n_\varepsilon^i(z, t_0) \leq C_\delta \inf_{z \in B_\varepsilon(x_0)} n_\varepsilon^j(z, t_0 + \varepsilon), \quad \text{for } (x_0, t_0) \in \mathbb{R}^d \times [\varepsilon\delta, +\infty). \quad (27) \quad \boxed{\text{harnackn}}$$

Finally using (8) we obtain (22).

We continue with the

**Proof of the uniform bounds from below:** Arguing by contradiction we assume that for some  $a > 0$  there exist  $\varepsilon_k \rightarrow 0$  and  $(x_k, t_k) \in \mathbb{R}^d \times [0, \infty)$  such that

$$\min(R_{\varepsilon_k}^1(x_k, t_k), R_{\varepsilon_k}^2(x_k, t_k)) < -a.$$

Newt observe that, in view of (5) and (6), for  $\varepsilon \leq \varepsilon_a$  with  $\varepsilon_a$  small enough, we have

$$\min(R_{\varepsilon_k}^1(x_k, t_k), R_{\varepsilon_k}^2(x_k, t_k)) > -\frac{a}{2}.$$

Moreover  $\min(R_\varepsilon^1, R_\varepsilon^2)$  is a supersolution to

$$\begin{aligned} \max & (R_{\varepsilon,t}^1 - \varepsilon \Delta_x R_\varepsilon^1 + |D_x R_\varepsilon^1|^2 - D_y \psi(\frac{x}{\varepsilon}) \cdot D_x R_\varepsilon^1 + \Delta_y \psi(\frac{x}{\varepsilon}) + \nu^2(\frac{x}{\varepsilon}) - \nu^1(\frac{x}{\varepsilon}), \\ & R_{\varepsilon,t}^2 - \varepsilon \Delta_x R_\varepsilon^2 + |D_x R_\varepsilon^2|^2 + \nu^1(\frac{x}{\varepsilon}) - \nu^2(\frac{x}{\varepsilon})) \geq 0, \end{aligned}$$

which admits  $-\frac{a}{2} - ct$  as a subsolution provided  $c$  is chosen sufficiently large.

It follows that, for  $\varepsilon \leq \varepsilon_a$  and  $t \geq 0$ ,

$$\min(R_\varepsilon^1(\cdot, t), R_\varepsilon^2(\cdot, t)) \geq -\frac{a}{2} - ct, \quad \text{in } \mathbb{R}^d,$$

and therefore, if  $\delta = \frac{a}{2c}$ , for  $\varepsilon \leq \varepsilon_a$ ,

$$\min(R_\varepsilon^1, R_\varepsilon^2) \geq -a, \quad \text{in } \mathbb{R}^d \times [0, \delta].$$

As a result the sequence  $(x_k, t_k)$  chosen at the beginning of the proof must satisfy  $t_k > \delta$ .

Using (22) we deduce that there exists  $C_1 > 0$  such that, as  $k \rightarrow \infty$  and for all  $x$  such that  $|x - x_k| \leq \varepsilon_k$  and  $i = 1, 2$ ,

$$R_{\varepsilon_k}^i(x, t_k + \varepsilon_k) \leq R_{\varepsilon_k}^i(x_k, t_k) + C_1 \varepsilon_k \leq -a + C_1 \varepsilon_k.$$

It follows that

$$\int_{\mathbb{R}^d} n_{\varepsilon_k}^i(x, t_k + \varepsilon_k) dx \geq \int_{|x-x_k| \leq \varepsilon_k} e^{-\frac{R_{\varepsilon_k}^i(x, t_k + \varepsilon_k)}{\varepsilon_k}} dx \geq |B_{\varepsilon_k}(x_k)| e^{\frac{a}{\varepsilon_k} - C_1}.$$

The right hand side of this inequality blows up as  $k \rightarrow \infty$ , while the left hand side is bounded in view of (17) and (4), again a contradiction.

The last part of the proof is devoted to the

**Proof of the uniform upper bounds on compact:** Fix a compact subset  $K$  of  $\mathbb{R}^d \times (0, \infty)$ , observe that

$$t_0 = \inf\{s \in (0, \infty) : \text{there exists } x \in \mathbb{R}^d \text{ such that } (x, s) \in K\} > 0,$$

choose  $t_1 \in (0, t_0)$  and write  $\bar{t}_1 = t_1/2$  and  $\bar{t}_2 = t_1/4$ .

It follows from (4), (17) and (20) that there exist  $\varepsilon_1 > 0$  and  $M > 0$  both dependent on  $t_1$  such that, if  $\varepsilon \leq \varepsilon_1$ , then

$$\int_{|x| \leq M} n_\varepsilon^1(x, \bar{t}_1) + n_\varepsilon^2(x, \bar{t}_1) dx \geq \frac{I_0}{2},$$

and, hence, in view of (8), there exists some  $a > 0$  such that

$$\min_{\substack{|x| \leq M \\ i=1,2}} R_\varepsilon^i(x, \bar{t}_1) \leq b = -\ln\left(\frac{aI_0}{M^d}\right).$$

Assume next that the above minimum is achieved at some point  $x_\varepsilon \in \bar{B}_M$  and for  $i = i^*$ . Applying (22)  $L = \lfloor \frac{t_1}{2\varepsilon} \rfloor$  times with  $i = i^*$ ,  $j = 2$ ,  $x_0 = x_\varepsilon$  and  $\delta = \bar{t}_1$ , we find some  $C = C_\delta > 0$  such that, for all  $x \in B_{L\varepsilon}(x_\varepsilon)$ ,

$$R_\varepsilon^2(x, \bar{t}_1 + L\varepsilon) \leq b + CL\varepsilon \leq b + C\bar{t}_1. \quad (28) \quad \boxed{\text{bound1}}$$

Choose  $\gamma \geq b + C\bar{t}_1$  and for some  $\beta > 0$  to be fixed below define  $\phi^1 : B_{\bar{t}_1}(x_\varepsilon) \times (0, \infty) \rightarrow \mathbb{R}$  by

$$\phi^1(x, t) = \frac{1}{\bar{t}_1^2 - |x - x_\varepsilon|^2} + \beta t + \gamma$$

We claim that, for  $\varepsilon \leq \varepsilon_2 = \min(\bar{t}_2, \varepsilon_1)$ ,

$$R_\varepsilon^2 \leq \phi^1 \quad \text{in} \quad Q_\varepsilon = B_{\bar{t}_1}(x_\varepsilon) \times [\bar{t}_1 + L\varepsilon, +\infty), \quad (29) \quad \boxed{\text{bound2}}$$

and, therefore,

$$R_\varepsilon^2 \leq \phi^1 \quad \text{in} \quad Q_\varepsilon^1 = B_{\bar{t}_1}(x_\varepsilon) \times [t_1, +\infty). \quad (30) \quad \boxed{\text{bound3}}$$

To prove (29) we first notice that, in view of (28) and the choice of  $\gamma$

$$R_\varepsilon^2(\cdot, \bar{t}_1 + L\varepsilon) \leq b + C\bar{t}_1 \leq \gamma \leq \phi^1(\cdot, \bar{t}_1 + L\varepsilon) \quad \text{in} \quad B_{\bar{t}_1}(x_\varepsilon)$$

Moreover, if  $\beta$  is large enough, using (2), for all  $(x, t) \in Q_\varepsilon$ , we have

$$\phi_t^1 - \Delta_x \phi^1 + |D_x \phi^1|^2 = \beta - \varepsilon \left( \frac{2d}{(\bar{t}_1^2 - |x - x_\varepsilon|^2)^2} + \frac{8|x - x_\varepsilon|^2}{(\bar{t}_1^2 - |x - x_\varepsilon|^2)^3} \right) + \frac{4|x - x_\varepsilon|^2}{(\bar{t}_1^2 - |x - x_\varepsilon|^2)^4} > D = \max_{y \in \mathbb{R}^d} \nu^2(y).$$

The inequality above and (9) yield that, for all  $(x, t) \in Q_\varepsilon$ ,

$$\phi_t^1 - \varepsilon \Delta_x \phi^1 + |D_x \phi^1|^2 + \nu^1\left(\frac{\cdot}{\varepsilon}\right) \exp\left(\frac{R_\varepsilon^2 - R_\varepsilon^1}{\varepsilon}\right) > \nu^2.$$

Since  $R_\varepsilon^2 \leq \phi^1$  on the parabolic boundary of  $Q_\varepsilon$ , (29) follows from the maximum principle.

For  $\varepsilon \leq \varepsilon_2$  let  $Q^2 = (\mathbb{R}^d \setminus B_{\bar{t}_2}(x_\varepsilon)) \times (t_1, +\infty)$  and for positive constants  $\alpha, \eta, \zeta$  to be fixed below consider the map  $\phi^2 : (\mathbb{R}^d \setminus B_{\bar{t}_2}(x_\varepsilon)) \times (t_1, +\infty) \rightarrow \mathbb{R}$  given by

$$\phi^2(x, t) = \frac{\alpha|x - x_\varepsilon|^2}{t - t_1} + \eta t + \zeta.$$

We claim that

$$R_\varepsilon^2 \leq \phi^2 \quad \text{in} \quad Q^2. \quad (31) \quad \boxed{\text{bound4}}$$

As above we will show that, if we choose  $\alpha, \eta$  and  $\zeta$  appropriately,  $\phi^2$  is a supersolution in  $Q^2$  of the equation satisfied by  $R_\varepsilon^2$  and is above  $R_\varepsilon^2$  on the parabolic boundary of  $Q^2$ .

To this end notice that, in view of (30), we may select  $\zeta$  and  $\eta$  large enough so that, for  $\varepsilon \leq \varepsilon_2$ ,

$$R_\varepsilon^2 \leq \eta t + \zeta \leq \phi^2 \quad \text{on} \quad \partial B_{\bar{t}_2}(x_\varepsilon) \times (t_1, \infty).$$

Moreover, for sufficiently large  $\alpha, \eta$ , we have

$$\phi_t^1 - \Delta_x \phi^1 + |D_x \phi^1|^2 = \eta - \frac{\alpha|x - x_\varepsilon|^2}{(t - t_1)^2} - \frac{2\varepsilon\alpha d}{t - t_1} + \frac{4\alpha^2|x - x_\varepsilon|^2}{(t - t_1)^2} > D,$$



and, hence, in  $Q^2$ ,

$$\phi_t^2 - \varepsilon \Delta_x \phi^2 + |D_x \phi^2|^2 + \nu^1\left(\frac{\cdot}{\varepsilon}\right) \exp\left(\frac{R_\varepsilon^2 - R_\varepsilon^1}{\varepsilon}\right) > \nu^2\left(\frac{\cdot}{\varepsilon}\right).$$

Since clearly  $R_\varepsilon^2(\cdot, t_1) \leq \phi^2(\cdot, t_1)$  in  $(\mathbb{R}^d \setminus B_{t_2}(x_\varepsilon))$ , using again the maximum principle we find (31).

To conclude observe that (30) and (31) yield that the family  $(R_\varepsilon^2)_{\varepsilon>0}$  is uniformly bounded from above in any compact subset of  $\mathbb{R}^d \times (t_1, \infty[$  and thus, in particular, on  $K$ , for  $\varepsilon \leq \varepsilon_1$ . Finally, using again (22), we deduce that the family  $(R_\varepsilon^1)_{\varepsilon>0}$  is also uniformly bounded from above on  $K$ , for  $\varepsilon \leq \min(\varepsilon_1, t_0 - t_1)$ .  $\square$

## 4 Convergence to the Homogenized equation-The proof of Theorem 1.2

sec:conv

Before we begin the proof, we remark that, in view of the claimed properties of  $\overline{H}$  (convexity and coercivity), the equation

$$R_t + \overline{H}(R_x) = 0 \text{ in } \mathbb{R}^d \times (0, \infty), \quad (32)$$

eq:eff

admits a comparison principle even for initial data taking the value  $\infty$  (see [12]). However, this together with Theorem 1.2 do not lead to  $\overline{R} \leq \underline{R}$ . Indeed we show in the next section that, in addition to (12) in Theorem 1.2, the conservation of mass yields

$$\underline{R}(0, 0) = \underline{R}^1(0, 0) = \underline{R}^2(0, 0) = 0.$$

To be able to use the comparison principle it is necessary to have  $\overline{R}^1(0, 0) = \overline{R}^2(0, 0) \leq 0$ , which is not possible. Indeed, since  $\overline{R}^1$  and  $\overline{R}^2$  are upper semicontinuous, (12) implies  $\overline{R}^1(0, 0) = \overline{R}^2(0, 0) = +\infty$ .

We continue with the

**Proof of Theorem 1.2.** It is immediate from (22) that (i) holds. Moreover it follows from Theorem 3.2 (ii) that

$$0 \leq \underline{R}^1 \leq \overline{R}^1 \text{ and } 0 \leq \underline{R}^2 \leq \overline{R}^2 \text{ on } \mathbb{R}^d \times [0, \infty), \quad (33)$$

Rpos

and thus, in particular, the second part of (12).

To prove (11) we employ the so called perturbed test function method. Since the arguments are similar here we only show that  $\overline{R}$  is a subsolution of (11).

To this end we assume that, for some smooth  $\varphi$ ,  $\overline{R} - \varphi$  achieves a strict local maximum at  $(x_0, t_0) \in \mathbb{R}^d \times (0, \infty)$ . Following [27], we perturb  $\varphi$  using the solution  $(\phi^1, \phi^2)$  of the cell problem (16) corresponding to  $p = D\varphi(x_0, t_0)$ . It follows that there exists  $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^d \times (0, \infty) \rightarrow (x_0, t_0)$ , as  $\varepsilon \rightarrow 0$ , such that the  $\max_{i=1,2}(R_\varepsilon^i - \varphi - \varepsilon\phi^i(\frac{\cdot}{\varepsilon}))$  is attained at  $(x_\varepsilon, t_\varepsilon)$  and, without any loss of generality since the argument is identical, for  $i = 1$ . Hence,

$$R_\varepsilon^1(x_\varepsilon, t_\varepsilon) - \varepsilon\phi^1\left(\frac{x_\varepsilon}{\varepsilon}\right) \geq R_\varepsilon^2(x_\varepsilon, t_\varepsilon) - \varepsilon\phi^2\left(\frac{x_\varepsilon}{\varepsilon}\right). \quad (34)$$

order

That  $R_\varepsilon^1$  is a solution of (9) yields

$$\begin{aligned} & \varphi_t(x_\varepsilon, t_\varepsilon) - \varepsilon \Delta_x \varphi(x_\varepsilon, t_\varepsilon) - \Delta_y \phi^1\left(\frac{x_\varepsilon}{\varepsilon}\right) + |D_x \varphi(x_\varepsilon, t_\varepsilon) + D_y \phi^1\left(\frac{x_\varepsilon}{\varepsilon}\right)|^2 \\ & - D_y \psi\left(\frac{x_\varepsilon}{\varepsilon}\right) \cdot (D_x \varphi(x_\varepsilon, t_\varepsilon) + D_x \phi^1\left(\frac{x_\varepsilon}{\varepsilon}\right)) + \Delta_y \psi\left(\frac{x_\varepsilon}{\varepsilon}\right) + \nu^2\left(\frac{x_\varepsilon}{\varepsilon}\right) \exp\left(\frac{R_\varepsilon^1 - R_\varepsilon^2}{\varepsilon}\right) \leq \nu^1\left(\frac{x_\varepsilon}{\varepsilon}\right). \end{aligned}$$

Using the latter and (34) and writing  $p_\varepsilon = D\varphi(x_\varepsilon, t_\varepsilon)$  we get

$$\begin{aligned} \varphi_t(x_\varepsilon, t_\varepsilon) - \varepsilon \Delta_x \varphi(x_\varepsilon, t_\varepsilon) - \Delta_y \phi^1\left(\frac{x_\varepsilon}{\varepsilon}\right) + |p_\varepsilon + D_y \phi^1\left(\frac{x_\varepsilon}{\varepsilon}\right)|^2 - D_y \psi\left(\frac{x_\varepsilon}{\varepsilon}\right) \cdot (p_\varepsilon + D_x \phi^1\left(\frac{x_\varepsilon}{\varepsilon}\right)) \\ + \Delta_y \psi\left(\frac{x_\varepsilon}{\varepsilon}\right) + \nu^2\left(\frac{x_\varepsilon}{\varepsilon}\right) \exp(\phi^1\left(\frac{x_\varepsilon}{\varepsilon}\right) - \phi^2\left(\frac{x_\varepsilon}{\varepsilon}\right)) \leq \nu^1\left(\frac{x_\varepsilon}{\varepsilon}\right). \end{aligned}$$

It follows from the above inequality and the definition of the effective Hamiltonian (16) that, for some  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,

$$\varphi_t(x_\varepsilon, t_\varepsilon) - \varepsilon \Delta_x \varphi(x_\varepsilon, t_\varepsilon) + \overline{H}(D_x \varphi(x_\varepsilon, t_\varepsilon)) \leq o(1),$$

and the conclusion follows letting  $\varepsilon \rightarrow 0$ .

To prove the first part of (12), since the arguments are identical, here we only show that  $\underline{R}^1(\cdot, 0) = \overline{R}^1(\cdot, 0) = \infty$  in  $\mathbb{R}^d \setminus \{0\}$ . To this aim, we fix  $\kappa > 0$ , we select an auxiliary function  $\xi \in C^\infty(\mathbb{R}^d)$  such that  $0 < \xi(x) < 1$  for  $x \in \mathbb{R}^d \setminus \{0\}$  and  $\xi(0) = 0$ , and prove that

$$\max(\underline{R}_t^1 + \overline{H}(\underline{R}_x^1), \underline{R}^i - \kappa \xi) \geq 0 \quad \text{in } \mathbb{R}^d \times [0, \infty). \quad (35) \quad \boxed{\text{variation}}$$

Since we already know that (35) holds in  $\mathbb{R}^d \times (0, \infty)$ , to conclude we assume that, for a smooth  $\phi$ ,  $\underline{R}^1 - \phi$  achieves a (strict) local maximum in  $(x_0, 0)$  and we prove that either

$$\underline{R}^1(x_0, 0) \geq \kappa \xi(x_0),$$

or

$$\phi_t(x_0, 0) + \overline{H}(\phi_x(x_0, 0)) \geq 0. \quad (36) \quad \boxed{\text{super}}$$

If  $x_0 = 0$ , the former is clearly true since  $\underline{R}^1(0, 0) \geq 0 = \kappa \xi(0)$ .

So we assume that  $\underline{R}^1 - \phi$  has a local maximum in  $(x_0, 0)$  with  $x_0 \neq 0$  and, in addition, that  $\underline{R}^1(x_0, 0) < \kappa \xi(x_0)$ . Repeating the arguments used earlier in the proof we obtain, for some  $(x_\varepsilon, t_\varepsilon) \rightarrow (x_0, 0)$  as  $\varepsilon \rightarrow 0$ , we have

$$\varphi_t(x_\varepsilon, t_\varepsilon) - \varepsilon \Delta_x \varphi(x_\varepsilon, t_\varepsilon) + \overline{H}(D_x \varphi(x_\varepsilon, t_\varepsilon)) \geq o(1).$$

Indeed (5) and the facts that  $\underline{R}^1(x_0, 0) < \kappa \xi(x_0) < \kappa$  and  $\lim_{\varepsilon \rightarrow 0} R_\varepsilon^1(y, 0) = +\infty$  for all  $y$  near  $x_0$  yield that  $t_\varepsilon > 0$ . The claim now follows by letting  $\varepsilon \rightarrow 0$ .

Assume next that, for some  $x_0 \neq 0$ ,  $\underline{R}^1(x_0, 0) = b < +\infty$ . We fix  $\delta > 0$  and let

$$\mu^\delta(x, t) = -\frac{|x - x_0|^2}{\delta} - \gamma t,$$

for  $\gamma = \gamma(\delta) > 0$  to be determined later.

Since  $\underline{R}^1$  is lower semicontinuous,  $\underline{R}^1 - \mu^\delta$  attains a minimum at some  $(x_\delta, t_\delta) \in \mathbb{R}^d \times [0, \infty)$  such that, as  $\delta \rightarrow 0$ ,  $(x_\delta, t_\delta) \rightarrow (x_0, 0)$ , provided that  $\gamma \rightarrow \infty$  as  $\delta \rightarrow 0$ .

Observe next that the choice of  $(x_\delta, t_\delta)$  yields

$$\frac{|x_\delta - x_0|^2}{\delta} \leq \underline{R}^1(x_\delta, t_\delta) + \frac{|x_\delta - x_0|^2}{\delta} + \gamma t_\delta \leq \underline{R}^1(x_0, 0) = b, \quad (37) \quad \boxed{\text{Rdainf}}$$

and, hence,

$$|x_\delta - x_0| \leq \sqrt{b\delta}.$$

If  $t_\delta > 0$ , according to part (ii), we must have  $\mu_t^\delta(x_\delta, t_\delta) + \overline{H}(D\mu^\delta(x_\delta, t_\delta)) \geq 0$  and, hence,

$$-\gamma + \overline{H}\left(-\frac{2(x_\delta - x_0)}{\delta}\right) \geq 0, \quad (38) \quad \boxed{\text{contrad}}$$

which cannot be true if we choose  $\gamma > \sup_{|x| \leq \sqrt{b\delta}} \overline{H}(-\frac{2x}{\delta})$ .

Now we assume that  $t_\delta = 0$ . If  $\underline{R}^1(x_0, 0) < \kappa\xi(x_0)$ , then (37) yields  $\underline{R}^1(x_\delta, 0) < \kappa_\delta\xi(x_\delta)$  for some  $\kappa_\delta \rightarrow \kappa$  as  $\delta \rightarrow 0$ . Using (35) with  $\kappa_\delta$  at the point  $(x_\delta, 0)$  we obtain again (38) and thus a contradiction.

It follows that  $\underline{R}^1(x_0, 0) > \kappa\xi(x_0)$ , which also leads to a contradiction, since it holds for arbitrarily large  $\kappa$  and  $\underline{R}^1(x_0, 0) = b < +\infty$ .

The first part of (12) now follows.  $\square$

## 5 The transport of the concentration points - The proof of Theorem 1.1

We present here the

**Proof.** [Proof of Theorem 1.1] Using (11), (12), the standard optimal control formula [22, 18, 3] and a barrier argument similar to the one of Section 5 in [17], we obtain that  $\underline{R}$  satisfies

$$\underline{R}(x, t) \geq \inf_{\substack{(\zeta(s), s) \in \mathbb{R}^d \times (0, \infty) + \\ \zeta(0)=0, \zeta(t)=x}} \int_0^t H^*(\dot{\zeta}(s)) ds + \max(\underline{R}^1(0, 0), \underline{R}^2(0, 0)), \quad (39) \quad \boxed{\text{control}}$$

with  $H^*(p) = \sup_{q \in \mathbb{R}^d} (p \cdot q - \overline{H}(q))$ .

Observe next that, since Lemma 2.2 yields

$$\lim_{|q| \rightarrow +\infty} p \cdot q - \overline{H}(q) \geq \lim_{|q| \rightarrow +\infty} p \cdot q - |q|^2 + C = -\infty,$$

the maximum of  $(p \cdot q - \overline{H}(q))$  is attained at some  $q_p \in \mathbb{R}^d$  such that  $p = D\overline{H}(q_p)$ .

Moreover, since  $\overline{H}$  is strictly convex and  $\overline{H}(0) = 0$ , we have

$$D\overline{H}(q_p) \cdot q_p - \overline{H}(q_p) \geq 0 \text{ with equality only if } q_p = 0.$$

Hence we deduce that

$$H^*(p) > 0 \text{ for all } p \neq D\overline{H}(0) \text{ and } H^*(D\overline{H}(0)) = 0.$$

Therefore, using (39) and (12), we obtain that

$$\underline{R} \geq 0 \text{ in } \mathbb{R}^d \times \mathbb{R}^+ \text{ and if } \underline{R}(x, t) = 0, \text{ then } x = tD\overline{H}(0) \text{ and } \underline{R}^1(0, 0) = \underline{R}^2(0, 0) = 0. \quad (40) \quad \boxed{\text{tDH}}$$

It follows from the latter, (8) and (10) that, for  $i = 1, 2$ ,

$$\limsup_{\substack{(y, s) \rightarrow (x, t) \\ \varepsilon \rightarrow 0}} n_\varepsilon^i(y, s) = 0 \text{ in } \mathbb{R}^d \times [0, +\infty) \setminus \{(tD\overline{H}(0), t) \mid t \in [0, +\infty)\}.$$

Finally the last claim and (17) yield that the  $(n_\varepsilon^1 + n_\varepsilon^2)$ 's converge weakly along subsequences to a measure  $n$  with

$$\text{supp } n \subset \{(t\nabla\overline{H}(0), t) \mid t \in [0, \infty)\}.$$

Since we also know that, according to (20), no mass escapes to infinity as  $\varepsilon \rightarrow 0$ , we deduce (7) using (17). Moreover, in view of (40), we obtain that

$$\underline{R}^1(0,0) = \underline{R}^2(0,0) = 0.$$

□

## 6 The case with several Dirac masses initially

sec:sev

We proved Theorem 1.1 under assumptions (5) which imply that the densities  $n_\varepsilon^1$  and  $n_\varepsilon^2$  are both initially concentrated at the origin. The result can be generalized to densities concentrated at several points and probably not on the same points. If this is the case, the initial condition is written as

$$\left\{ \begin{array}{l} \lim_{\varepsilon \rightarrow 0} \varepsilon \ln n_\varepsilon^1(y,0) = \lim_{\varepsilon \rightarrow 0} \varepsilon \ln n_\varepsilon^2(y,0) = -\infty \text{ locally uniformly in } \mathbb{R}^d \setminus (\mathcal{A} \cup \mathcal{B}), \\ \text{and} \\ \limsup_{\substack{y \rightarrow x \\ \varepsilon \rightarrow 0}} \varepsilon \ln n_\varepsilon^i(y,0) \leq 0 \text{ for all } x \in \mathcal{A} \cup \mathcal{B} \end{array} \right. \quad (41)$$

multidira

We have:

thm:multi

**Theorem 6.1** *Assume (2), (3), (4), (6) and (41). Then, for  $i = 1, 2$  and as  $\varepsilon \rightarrow 0$ , along subsequences and in the sense of measures,  $n_\varepsilon^1 \rightharpoonup n^1$  and  $n_\varepsilon^2 \rightharpoonup n^2$ , with*

$$\text{supp}(n^1 + n^2)(\cdot, t) \subset \mathcal{C}(t) = \{x_1 + tD\overline{H}(0), \dots, x_n + tD\overline{H}(0)\} \cup \{y_1 + tD\overline{H}(0), \dots, y_m + tD\overline{H}(0)\}.$$

**Proof.** The proof of Theorem 6.1 follows along the same lines as the one of Theorem 1.1. The only difference is that (12) and (39) are replaced respectively by

$$\overline{R}^1 = \overline{R}^2 = \underline{R}^1 = \underline{R}^2 = +\infty \text{ in } (\mathbb{R}^d \setminus \mathcal{C}(0)) \times \{0\} \text{ and } \underline{R}^1 \geq 0 \text{ and } \underline{R}^2 \geq 0 \text{ in } \mathcal{C}(0) \times \{0\}$$

and

$$\underline{R}(x,t) \geq \inf_{\substack{(\zeta(s),s) \in \mathbb{R}^d \times (0,\infty) \\ \zeta(0) \in \mathcal{C}(0), \zeta(t)=x}} \left\{ \int_0^t H^*(\dot{\zeta}(s)) ds + \max(\underline{R}^1(\zeta(0),0), \underline{R}^2(\zeta(0),0)) \right\}. \quad (42)$$

controlb

Therefore, we have

$$\underline{R} \geq 0 \text{ in } \mathbb{R}^d \times (0, \infty) \text{ and if } \underline{R}(x,t) = 0, \text{ then } x \in \mathcal{C}(t).$$

The other parts of the proof are similar. □

## 7 A comparison with results for parabolic scaling

sec:comp

We describe here the connection between our result and the study in [15] of the asymptotics as  $\varepsilon \rightarrow 0$  of the solutions to

$$n_{\varepsilon,t} = (a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) n_{\varepsilon, x_j})_{x_i} + \frac{1}{\varepsilon} b_i(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) n_{\varepsilon, x_i} + \frac{1}{\varepsilon^2} c(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) n_\varepsilon \text{ in } \mathbb{R}^d \times (0, \infty). \quad (43)$$

ap

Notice that this is a single equation — not a system—obtained after a parabolic scaling  $(x, t) \mapsto (\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})$  rather than the hyperbolic scaling  $(x, t) \mapsto (\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$  of the problem we consider in this paper. Nevertheless as we explain below there are some similarities.

It is proved in [15] that the solution  $n_\varepsilon$  of (43) admits the expansion

$$n_\varepsilon(x, t) = w\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \exp\left(-\frac{\lambda_0 t}{\varepsilon^2}\right) v^0\left(x - \frac{\bar{b}}{\varepsilon} t, t\right) + o(1). \quad (44) \quad \boxed{\text{piat}}$$

The result we obtain here with the hyperbolic scaling formally gives

$$n_\varepsilon^i(x, t) = w^i\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \rho^i(t) \delta\left(x - D\bar{H}(0)t\right) + o(1). \quad (45) \quad \boxed{\text{compare}}$$

In particular we obtain a Dirac mass instead of the function  $v^0$  while the term  $\exp(-\frac{\lambda_0 t}{\varepsilon^2})$  disappears because we have conservation of mass. Moreover  $t/\varepsilon$  is replaced by  $t$ , because of the difference in the scaling. There is, however, as we explain below a close connection between the function  $w$  in (44) and  $w^i$  in (45).

Indeed  $w$  in [15] is the principal eigenvector of the periodic cell problem

$$w_s - (a_{ij}(z, s)w_{z_j})_{z_i} - b_i(z, s)w_{z_i} - c(z, s)w = \Lambda_0 w,$$

while we have

$$w^i = \exp(-\phi^i),$$

where  $(\phi^1, \phi^2)$  is the principle eigenvector for the cell problem (16) corresponding to  $p = 0$  with the corresponding eigenvalue  $\bar{H}(0) = 0$ . This is because for  $(\bar{x}, \bar{t})$  in the support of  $n^i$ , we have  $R(\bar{x}, \bar{t}) = 0$  and thus  $DR(\bar{x}, \bar{t}) = 0$ .

## A The proof of Lemma 2.1

app:cell

**Step 1:** We prove that, for all  $p \in \mathbb{R}^d$ , there exists a unique  $\bar{H}(p)$  with the properties stated in Lemma 2.1. Following [27] for  $i = 1, 2$  we define

$$\chi^i(y) = \exp(-p \cdot y - \phi^i(y)).$$

Multiplying the two equations in (16) by  $\chi^1$  and  $\chi^2$  respectively, we obtain

$$\begin{cases} -\Delta_y \chi^1 - \operatorname{div}_y(D_y \psi \chi^1) + \nu^1 \chi^1 - \nu^2 \chi^2 = -\bar{H}(p) \chi^1, \\ -\Delta_y \chi^2 + \nu^2 \chi^2 - \nu^1 \chi^1 = -\bar{H}(p) \chi^2. \end{cases} \quad (46) \quad \boxed{\text{eq:Chi}}$$

with the boundary condition

$$y \rightarrow e^{p \cdot y} \chi^i(y) \text{ is one-periodic and } \chi^i > 0. \quad (47) \quad \boxed{\text{period}}$$

We also impose the normalization

$$\int_0^1 (\chi^1(y) + \chi^2(y)) dy = 1. \quad (48) \quad \boxed{\text{normal}}$$

It follows from the Krein-Rutman Theorem that, for all  $p \in \mathbb{R}^d$ , there exists a unique constant  $\overline{H}(p)$  and functions  $(\chi^1, \chi^2)$  satisfying (46) together with (47) and (48).

**Step 2:** To prove that  $\overline{H} \in C^1(\mathbb{R}^d)$  we rewrite (46) in terms of  $w_p^i(y) = e^{p \cdot y} \chi^i(y)$ , which in view of (47) are 1-periodic, and obtain the new system

$$\begin{cases} -\Delta_y w_p^1 + 2p \cdot D_y w_p^1 - \operatorname{div}_y(D_y \psi w_p^1) + (-|p|^2 + p \cdot D_y \psi + \nu^1) w_p^1 - \nu^2 w_p^2 = -\overline{H}(p) w_p^1, \\ -\Delta_y w_p^2 + 2p \cdot D_y w_p^2 + (-|p|^2 + \nu^2) w_p^2 - \nu^1 w_p^1 = -\overline{H}(p) w_p^2. \end{cases} \quad (49) \quad \boxed{\text{SS}}$$

Assuming for the moment that  $w_p^1$  and  $w_p^2$  are differentiable with respect to  $p$ , after differentiating (49) we get

$$\begin{cases} -\Delta_y \partial_p w_p^1 + 2p \cdot D_y \partial_p w_p^1 - \operatorname{div}_y(D_y \psi \partial_p w_p^1) + (-|p|^2 + p \cdot \nabla_y \psi + \nu^1) \partial_p w_p^1 - \nu^2 \partial_p w_p^2 \\ + 2D_y w_p^1 + (-2p + D_y \psi) w_p^1 = -\overline{H}(p) \partial_p w_p^1 - \overline{H}'(p) w_p^1, \\ -\Delta_y \partial_p w_p^2 + 2p \cdot D_y \partial_p w_p^2 + (-|p|^2 + \nu^2) \partial_p w_p^2 - \nu^1 \partial_p w_p^1 + 2D_y w_p^2 - 2p w_p^2 = -\overline{H}(p) \partial_p w_p^2 \\ - \overline{H}'(p) w_p^2. \end{cases}$$

Let  $(w_{p,*}^1, w_{p,*}^2)$  be the solution to the adjoint system of (49) –note that Fredholm’s alternative implies the existence of such a solution. Multiplying the equations of (49) by  $w_{p,*}^1$  and  $w_{p,*}^2$ , integrating with respect to  $y$  and adding the two resulting equations we obtain

$$2 \int w_{p,*}^1 D_y w_p^1 dy + 2 \int w_{p,*}^2 D_y w_p^2 dy + \int D_y \psi w_p^1 w_{p,*}^1 dy = (2p - D\overline{H}(p)) \left( \int w_p^1 w_{p,*}^1 dy + \int w_p^2 w_{p,*}^2 dy \right),$$

which yields a formula for  $D\overline{H}(p)$ .

To prove the above claim rigorously, we write the system satisfied by difference quotients with respect to  $p$  and we use the same idea as above to prove that the difference quotients  $h^{-1}(\overline{H}(p + he_k) - \overline{H}(p))$  converge to the above formula, as  $h \rightarrow 0$ . Indeed from this formulation we first obtain that  $\overline{H}(p)$  is continuous with respect to  $p$ . Then we show that  $w_p^1$  and  $w_p^2$  are continuous with respect to  $p$  and, finally, we pass to the limit  $h \rightarrow 0$ .

To prove that  $w_p^1$  and  $w_p^2$  are continuous with respect to  $p$ , we show that, for any sequence  $(p_n)_{n \in \mathbb{N}}$  such that  $p_n \rightarrow p$ , the corresponding eigenfunctions  $(w_{p_n}^1, w_{p_n}^2)$  converge to  $(w_p^1, w_p^2)$  as  $n \rightarrow \infty$ . This follows, for example, from the stability of viscosity solutions, the continuity of  $\overline{H}$ , the uniqueness (up to a multiplicative constant) of the eigenfunctions  $(w_p^1, w_p^2)$  and the normalization condition (48).

**Step 3:** The proof of  $\overline{H}(0) = 0$ .

The adjoint system to (46) is

$$\begin{cases} -\Delta_y u^1 + D_y \psi \cdot D_y u^1 + \nu^1 u^1 = \nu^1 u^2 - \overline{H}(p) u^1, \\ -\Delta_y u^2 + \nu^2 u^2 = \nu^2 u^1 - \overline{H}(p) u^2, \end{cases} \quad (50) \quad \boxed{\text{SSS}}$$

with the condition

$$y \rightarrow e^{-p \cdot y} u^i(y) \text{ 1-periodic and } u^i > 0.$$

When  $p = 0$ , (50) admits the trivial solution  $(u^1, u^2) = (1, 1)$  and the constant  $\overline{H}(0) = 0$ . The claim follows from the uniqueness of the positive eigenvector.

**Step 4:** The proof of the strict convexity of  $\overline{H}(p)$ .

Arguing by contradiction we assume that there exist  $p_1, p_2$  such that

$$\frac{\overline{H}(p_1) + \overline{H}(p_2)}{2} \leq \overline{H}\left(\frac{p_1 + p_2}{2}\right).$$

Let  $(\phi_1^1, \phi_1^2)$  and  $(\phi_2^1, \phi_2^2)$  be solutions of (16) corresponding to  $p = p_1$  and  $p = p_2$  respectively. Adding the corresponding equations we find

$$\begin{aligned} & -\Delta_y\left(\frac{\phi_1^1 + \phi_2^1}{2}\right) + |D_y\left(\frac{\phi_1^1 + \phi_2^1}{2}\right) + \frac{p_1 + p_2}{2}|^2 - D_y\psi \cdot \left(D_y\left(\frac{\phi_1^1 + \phi_2^1}{2}\right) + \frac{p_1 + p_2}{2}\right) + \Delta_y\psi \\ & + \nu^2 \exp\left(\frac{\phi_1^1 + \phi_2^1 - \phi_1^2 - \phi_2^2}{2}\right) \\ & < -\Delta_y\left(\frac{\phi_1^1 + \phi_2^1}{2}\right) + \frac{1}{2}|D_y\phi_1^1 + p_1|^2 + \frac{1}{2}|D_y\phi_2^1 + p_2|^2 - D_y\psi \cdot \left(\frac{D_y\phi_1^1 + p_1}{2}\right) - D_y\psi \cdot \left(\frac{D_y\phi_2^1 + p_2}{2}\right) \\ & + \Delta_y\psi + \frac{\nu^2}{2} \exp(\phi_1^1 - \phi_1^2) + \frac{\nu^2}{2} \exp(\phi_2^1 - \phi_2^2) \\ & = \nu^1 + \frac{\overline{H}(p_1) + \overline{H}(p_2)}{2} \leq \nu^1 + \overline{H}\left(\frac{p_1 + p_2}{2}\right), \end{aligned}$$

and

$$-\Delta_y\left(\frac{\phi_1^2 + \phi_2^2}{2}\right) + |D_y\left(\frac{\phi_1^2 + \phi_2^2}{2}\right) + \frac{p_1 + p_2}{2}|^2 + \nu^1 \exp\left(\frac{\phi_1^2 + \phi_2^2 - \phi_1^1 - \phi_2^1}{2}\right) < \nu^2 + \overline{H}\left(\frac{p_1 + p_2}{2}\right).$$

It follows that the pair  $(\frac{\phi_1^1 + \phi_2^1}{2}, \frac{\phi_1^2 + \phi_2^2}{2})$  is a strict subsolution to the cell problem (16) corresponding to  $p = \frac{p_1 + p_2}{2}$ . This, however, contradicts the fact that  $\overline{H}(\frac{p_1 + p_2}{2})$  is the principal eigenvalue of the system.

**Step 5:** The proof of  $\overline{H}(p) \geq |p|^2 - C$ .

Rewriting (16) we find

$$\begin{cases} -\Delta_y\phi^1 + |D_y\phi^1|^2 - (D_y\psi - 2p) \cdot D_y\phi^1 - D_y\psi \cdot p + \Delta_y\psi + \nu^2 \exp(\phi^1 - \phi^2) = \nu^1 + \overline{H}(p) - |p|^2, \\ -\Delta_y\phi^2 + |D_y\phi^2|^2 + 2p \cdot D_y\phi^2 + \nu^1 \exp(\phi^2 - \phi^1) = \nu^2 + \overline{H}(p) - |p|^2. \end{cases}$$

Assume next that  $\max_{[0,1]}(\phi^1, \phi^2)$  is attained at the point  $\bar{y}$ . If  $\phi^1(\bar{y}) > \phi^2(\bar{y})$ , then

$$-\Delta\phi^1(\bar{y}) \geq 0 \quad \text{and} \quad D\phi^1(\bar{y}) = 0,$$

and, hence,

$$\overline{H}(p) \geq -\|\Delta\psi\|_{L^\infty} - |p| \|D_y\psi\|_{L^\infty} - \|\nu^1\|_{L^\infty} + |p|^2.$$

If  $\phi^1(\bar{y}) > \phi^2(\bar{y})$  then, similarly, we find

$$\overline{H}(p) \geq -\|\nu^1\|_{L^\infty} + |p|^2.$$

The claim now follows.

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