Differential systems

Exercise 1. 1. Compute the solution (y_1, y_2) to the system

$$\begin{cases} y_1' = 2y_1, & y_1(0) = 1, \\ y_2' = y_2, & y_2(0) = 1. \end{cases}$$

Prove that $y_1(t) = y_2(t)^2$ and give a graphical representation in the plan of $t \mapsto (y_1(t), y_2(t))$ (called the trajectory of the solution).

Solution: We can first observe that the differential system is made of two *decoupled* equations (i.e. each equation is involving only one unkown function). As a consequence, we may solve each equation separetly. We find for the general solutions of each equations

$$y_1(t) = C_1 e^{2t}, \quad y_2(t) = C_2 e^t, \quad C_1, C_2 \in \mathbb{R}.$$

Then, we use the initial conditions to specify the constants C_1 and C_2. We have for the general solution

$$y_1(0) = C_1 e^0 = C_1,$$

hence we must have

$$C_1 = 1.$$

Similarly, we obtain

$$C_2 = 1.$$

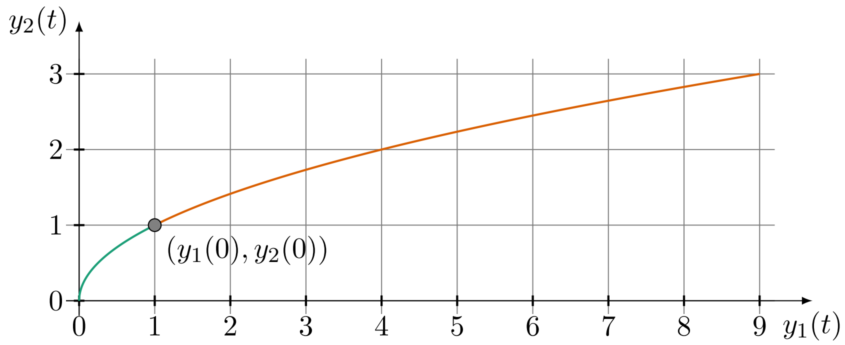
Therefore, the solution of the system with initial conditions is

$$(y_1, y_2)(t) = \left(e^{2t}, e^t\right).$$

We indeed have the relation

$$y_1(t) = e^{2t} = \left(e^t\right)^2 = y_2(t)^2.$$

The curve described by (y_1, y_2) is therefore the part of a parabola, as is represented on the following picture



The green part corresponds to $t < 0$ and the orange part corresponds to $t > 0$.

2. Compute the solution (y_1, y_2) of the system

$$\begin{cases} y_1' = 2y_1, & y_1(0) = 1, \\ y_2' = -y_2, & y_2(0) = -1. \end{cases}$$

and represent the trajectory.

Solution: We can first observe that the differential system is made of two *decoupled* equations (i.e. each equation is involving only one unkown function). As a consequence, we may solve each equation separetly. We find for the general solutions of each equations

$$y_1(t) = C_1 e^{2t}, \quad y_2(t) = C_2 e^{-t}, \quad C_1, C_2 \in \mathbb{R}.$$

Then, we use the initial conditions to specify the constants C_1 and C_2. We have for the general solution

$$y_1(0) = C_1 e^0 = C_1,$$

hence we must have

$$C_1 = 1.$$

Similarly, we obtain

$$C_2 = -1.$$

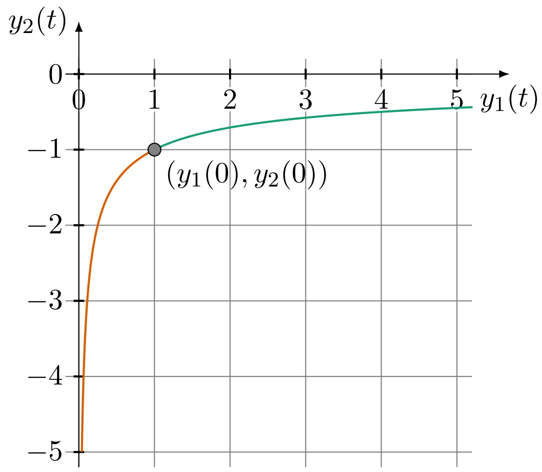
Therefore, the solution of the system with initial conditions is

$$(y_1, y_2)(t) = \left(e^{2t}, -e^{-t}\right).$$

We now have the relation

$$y_1(t) = e^{2t} = \left(-e^{-t}\right)^{-2} = y_2(t)^{-2}.$$

The curve described by (y_1, y_2) is represented on the following picture



The green part corresponds to $t < 0$ and the orange part corresponds to $t > 0$.

Exercise 2. 1. Give the matrix diagonalization of $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$.

Solution: The diagonalization of a matrix requires to find the eigenvalues and the corresponding eigenvectors.

We start by finding the eigenvalues as the roots of the characteristic polynomial of the matrix. The polynomial in the variable λ is given by

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2).$$

Therefore, the eigenvalues of A are 1 and 2.

We now find the eigenvectors, starting with the eigenvectors associated with the eigenvalue $\lambda = 1$. That is, we search for all vectors $V = \binom{\alpha}{\beta}$, $V \neq 0$ such that

$$AV = V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = V \iff \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff \begin{pmatrix} 3\alpha - 2\beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff \begin{cases} 3\alpha - 2\beta = \alpha, \\ \alpha = \beta. \end{cases}$$

This last system reduces in fact to the single equation

$$\alpha = \beta.$$

Hence all eigenvectors V of A associated with 1 are of the form

$$V = \begin{pmatrix} \beta \\ \beta \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \beta \in \mathbb{R} \setminus \{0\}.$$

The same procedure may be followed to find the eigenvectors associated with 2. Indeed, we now search for all vectors $V = \binom{\alpha}{\beta}$, $V \neq 0$ such that

$$AV = 2V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = V \iff \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff \begin{pmatrix} 3\alpha - 2\beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 2\alpha \\ 2\beta \end{pmatrix} \iff \begin{cases} 3\alpha - 2\beta = 2\alpha, \\ \alpha = 2\beta. \end{cases}$$

As before this last system reduces in fact to the single equation

$$\alpha = 2\beta.$$

Hence all eigenvectors V of A associated with 2 are of the form

$$V = \begin{pmatrix} 2\beta \\ \beta \end{pmatrix} = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \beta \in \mathbb{R} \setminus \{0\}.$$

To conclude the diagonalization procedure, we construct an invertible matrix with the eigenvectors and a diagonal matrix with the eigenvalues. It is easy in the present case, as the eigenvalues are all real and simple. We may therefore define

$$P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then we have

$$A = PDP^{-1},$$

which concludes the diagonalization procedure for A.

2. Deduce the general expression of the solutions to the system

$$\begin{cases} y_1' = 3y_1 - 2y_2, \\ y_2' = y_1. \end{cases}$$

Solution: It should first be observed that the system may be reinterpreted as a matrix differential equation. Indeed, define

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then the system is equivalent to

$$Y' = AY.$$

We introduce the auxilliary unknown

$$Z = P^{-1}Y.$$

Then, using the diagonalization of A, we may rewrite the matrix equation in the following way :

$$Y' = AY \iff Y' = PDP^{-1}Y \iff P^{-1}Y' = DP^{-1}Y \iff (P^{-1}Y)' = DP^{-1}Y \iff Z' = DZ.$$

Denote the components of Z by z_1 and z_2. Then we may reinterpret the matrix equation in Z as a system :

$$Z' = DZ \iff \begin{cases} z_1' = z_1, \\ z_2' = 2z_2. \end{cases}$$

Compare to the original system, this one has the advantage of being decoupled and therefore each composing equation may be solved separately. We find

$$z_1(t) = C_1 e^t, \quad C_1 \in \mathbb{R}, z_2(t) = C_2 e^{2t}, \quad C_2 \in \mathbb{R}.$$

To find the solutions of the original system, it is now sufficient to go back to the original variable using P. Indeed, we have

$$Z = P^{-1}Y \iff Y = PZ.$$

Therefore,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = P \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} z_1(t) + 2z_2(t) \\ z_1(t) + z_2(t) \end{pmatrix} = \begin{pmatrix} C_1 e^t + 2C_2 e^{2t} \\ C_1 e^t + C_2 e^{2t} \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}.$$

3. Compute the solution corresponding to the initial conditions $y_1(0) = 3$, $y_2(0) = 2$ and represent its trajectory.

Solution: At $t = 0$, the general solution previously computed gives

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} C_1 + 2C_2 \\ C_1 + C_2 \end{pmatrix},$$

hence in order to satisfy the initial conditions we request that C_1, C_2 are such that

$$\begin{cases} C_1 + 2C_2 = 3, C_1 + C_2 = 2. \end{cases}$$

This system may be solved by elementary manipulations : substracting the second line to the first gives

$$C_2 = 1,$$

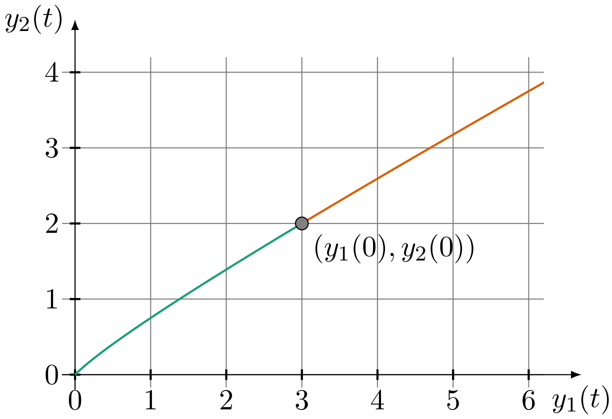
and by substitution in one of the equation we obtain

$$C_1 = 1.$$

As a consequence, the solution with initial $y_1(0) = 3$, $y_2(0) = 2$ is

$$y_1(t) = e^t + 2e^{2t}, \quad y_2(t) = e^t + e^{2t}.$$

The curve described by (y_1, y_2) is represented on the following picture.



The green part corresponds to $t < 0$ and the orange part corresponds to $t > 0$.

4. ** Using the same strategy employed in the previous questions compute and represent in the plan the solution of

$$\begin{cases} y_1' = 5y_1 - 6y_2, & y_1(0) = 3, \\ y_2' = 3y_1 - 4y_2, & y_2(0) = 2. \end{cases}$$

Solution: It should first be observed that the system may be reinterpreted as a matrix differential equation. Indeed, define

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}.$$

Then the system is equivalent to

$$Y' = AY.$$

As for the previous case, we diagonalize the matrix A. We start by finding the eigenvalues as the roots of the characteristic polynomial of the matrix. The polynomial in the variable λ is given by

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -6 \\ 3 & -4-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$$

Therefore, the eigenvalues of A are -1 and 2.
We now find the eigenvectors, starting with the eigenvectors associated with the eigenvalue $\lambda = -1$. That is, we search for all vectors $V = \binom{\alpha}{\beta}$, $V \neq 0$ such that

$$AV = -V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = V \iff \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff \begin{pmatrix} 5\alpha - 6\beta \\ 3\alpha - 4\beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \iff \begin{cases} 5\alpha - 6\beta = -\alpha, \\ 3\alpha - 4\beta = -\beta. \end{cases}$$

This last system reduces in fact to the single equation

$$\alpha = \beta.$$

Hence all eigenvectors V of A associated with 1 are of the form

$$V = \begin{pmatrix} \beta \\ \beta \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \beta \in \mathbb{R} \setminus \{0\}.$$

The same procedure may be followed to find the eigenvectors associated with 2. Indeed, we now search for all vectors $V = \binom{\alpha}{\beta}$, $V \neq 0$ such that

$$AV = 2V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = V \iff \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff \begin{pmatrix} 5\alpha - 6\beta \\ 3\alpha - 4\beta \end{pmatrix} = \begin{pmatrix} 2\alpha \\ 2\beta \end{pmatrix} \iff \begin{cases} 5\alpha - 6\beta = 2\alpha, \\ 3\alpha - 4\beta = 2\beta. \end{cases}$$

As before this last system reduces in fact to the single equation

$$3\alpha = 6\beta.$$

Hence all eigenvectors V of A associated with 2 are of the form

$$V = \begin{pmatrix} 2\beta \\ \beta \end{pmatrix} = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \beta \in \mathbb{R} \setminus \{0\}.$$

To conclude the diagonalization procedure, we construct an invertible matrix with the eigenvectors and a diagonal matrix with the eigenvalues. It is easy in the present case, as the eigenvalues are all real and simple. We may therefore define

$$P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then we have

$$A = PDP^{-1},$$

which concludes the diagonalization procedure for A.
We now introduce the auxilliary unknown

$$Z = P^{-1}Y.$$

Then, using the diagonalization of A, we may rewrite the matrix equation in the following way :

$$Y' = AY \iff Y' = PDP^{-1}Y \iff P^{-1}Y' = DP^{-1}Y \iff (P^{-1}Y)' = DP^{-1}Y \iff Z' = DZ.$$

Denote the components of Z by z_1 and z_2. Then we may reinterpret the matrix equation in Z as a system :

$$Z' = DZ \iff \begin{cases} z_1' = -z_1, \\ z_2' = 2z_2. \end{cases}$$

Compare to the original system, this one has the advantage of being decoupled and therefore each composing equation may be solved separately. We find

$$z_1(t) = C_1 e^{-t}, \quad C_1 \in \mathbb{R}, z_2(t) = C_2 e^{2t}, \quad C_2 \in \mathbb{R}.$$

To find the solutions of the original system, it is now sufficient to go back to the original variable using P. Indeed, we have

$$Z = P^{-1}Y \iff Y = PZ.$$

Therefore,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = P \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} z_1(t) + 2z_2(t) \\ z_1(t) + z_2(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{-t} + 2C_2 e^{2t} \\ C_1 e^{-t} + C_2 e^{2t} \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}.$$

At $t = 0$, the general solution previously computed gives

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} C_1 + 2C_2 \\ C_1 + C_2 \end{pmatrix},$$

hence in order to satisfy the initial conditions we request that C_1, C_2 are such that

$$\begin{cases} C_1 + 2C_2 = 3, \\ C_1 + C_2 = 2. \end{cases}$$

This system may be solved by elementary manipulations : substracting the second line to the first gives

$$C_2 = 1,$$

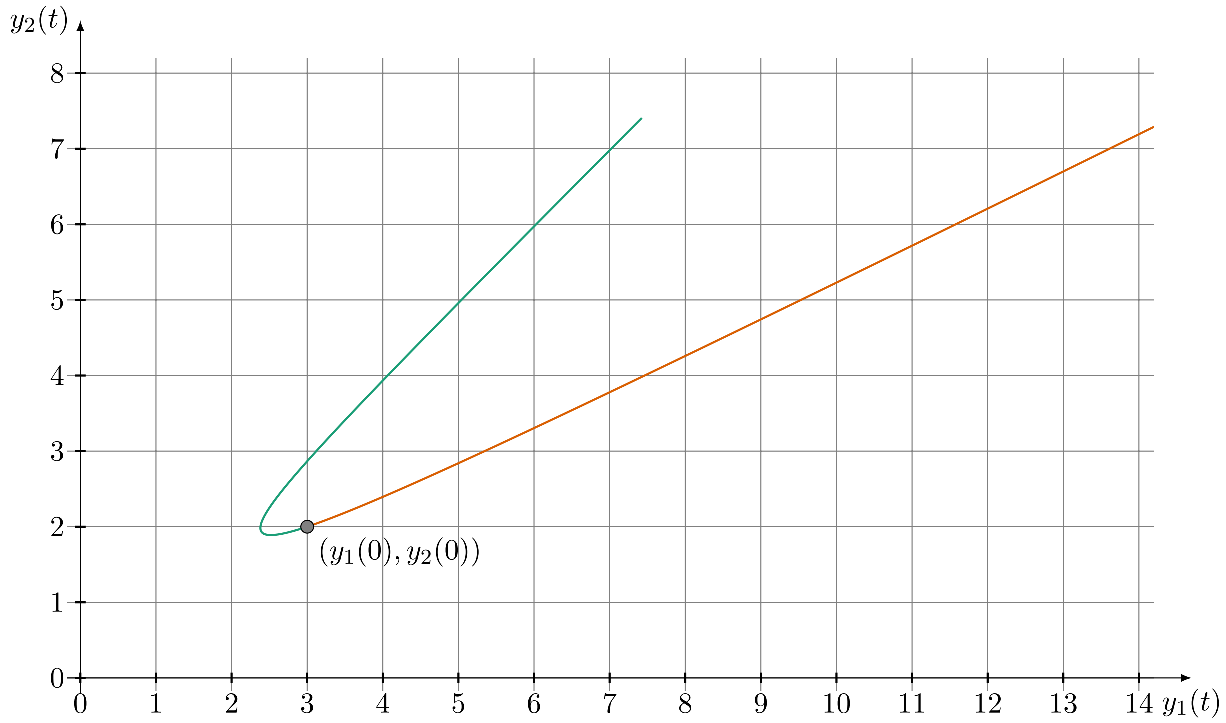
and by substitution in one of the equation we obtain

$$C_1 = 1.$$

As a consequence, the solution with initial $y_1(0) = 3$, $y_2(0) = 2$ is

$$y_1(t) = e^{-t} + 2e^{2t}, \quad y_2(t) = e^{-t} + e^{2t}.$$

The curve described by (y_1, y_2) is represented on the following picture.



The green part corresponds to $t < 0$ and the orange part corresponds to $t > 0$.

Exercise 3. We consider the system

$$(\mathrm{S}) \begin{cases} y_1' = y_2, & y_1(0) = 0, \\ y_2' = y_3, & y_2(0) = 1, \\ y_3' = 2y_1 + y_2 - 2y_3, & y_3(0) = -3. \end{cases}$$

1. Give the matrix A such that the system (S) is formulated under the form $Y' = AY$. Then give the diagonalization of A.

Solution: Define the vector unknown Y and the matrix A by

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

Then the system is indeed equivalent to

$$Y' = AY.$$

We now proceed to the diagonalization of A.

We start by finding the eigenvalues as the roots of the characteristic polynomial of the matrix. The polynomial in the variable λ is given by

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & 1 & -2-\lambda \end{vmatrix}.$$

This determinant might be computed using the Sarrus rule or by developing along one line or one column (forgot how to do that ? check `https://youtu.be/4xFIi0JF2AM`). We find

$$\det(A-\lambda I) = -\lambda^3 - 2\lambda^2 + \lambda + 2.$$

This is a cubic polynomial and we have to find its roots. It turns out that 1 is clearly a root and we may then factorize the characteristic polynomial as follows

$$\det(A-\lambda I) = -(\lambda-1)(\lambda^2+3\lambda+2) = -(\lambda-1)(\lambda+2)(\lambda+1).$$

Therefore, the eigenvalues of A are -2, -1 and 1.
We now find the eigenvectors, starting with the eigenvectors associated with the eigenvalue $\lambda = -2$. That is, we search for all vectors $V = \left(\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right)$, $V \neq 0$ such that

$$AV = -2V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$\begin{aligned} AV = -2V \iff & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = -2 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \\ & \iff \begin{pmatrix} \beta \\ \gamma \\ 2\alpha+\beta-2\gamma \end{pmatrix} = \begin{pmatrix} -2\alpha \\ -2\beta \\ -2\gamma \end{pmatrix} \iff \begin{cases} \beta = -2\alpha, \\ \gamma = -2\beta, \\ 2\alpha+\beta-2\gamma = -2\gamma, \end{cases} \end{aligned}$$

This last system reduces in fact to a system of two equations

$$\begin{cases} \beta = -2\alpha, \\ \gamma = -2\beta, \end{cases}$$

which we may reformulate in order to express the parameters β and γ in terms of α :

$$\begin{cases} \beta = -2\alpha, \\ \gamma = 4\alpha, \end{cases}$$

Hence all eigenvectors V of A associated with -2 are of the form

$$V = \begin{pmatrix} \alpha \\ -2\alpha \\ 4\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \quad \alpha \in \mathbb{R} \setminus \{0\}.$$

The same procedure may be followed to find the eigenvectors associated with -1. Indeed, we now search for all vectors $V = \left(\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right)$, $V \neq 0$ such that

$$AV = -V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = -V \iff \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\iff \begin{pmatrix} \beta \\ \gamma \\ 2\alpha + \beta - 2\gamma \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \\ -\gamma \end{pmatrix} \iff \begin{cases} \beta = -\alpha, \\ \gamma = -\beta, \\ 2\alpha + \beta - 2\gamma = -\gamma, \end{cases}$$

This last system reduces in fact to a system of two equations

$$\begin{cases} \beta = -\alpha, \\ \gamma = -\beta, \end{cases}$$

which we may reformulate in order to express the parameters β and γ in terms of α :

$$\begin{cases} \beta = -\alpha, \\ \gamma = \alpha, \end{cases}$$

Hence all eigenvectors V of A associated with -1 are of the form

$$V = \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \alpha \in \mathbb{R} \setminus \{0\}.$$

We follow again the same procedure to find the eigenvectors associated with 1. We now search for all vectors $V = \left(\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right)$, $V \neq 0$ such that

$$AV = V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = V \iff \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\iff \begin{pmatrix} \beta \\ \gamma \\ 2\alpha + \beta - 2\gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \iff \begin{cases} \beta = \alpha, \\ \gamma = \beta, \\ 2\alpha + \beta - 2\gamma = \gamma, \end{cases}$$

This last system reduces in fact to a system of two equations

$$\begin{cases} \beta = \alpha, \\ \gamma = \beta, \end{cases}$$

which we may reformulate in order to express the parameters β and γ in terms of α :

$$\begin{cases} \beta = \alpha, \\ \gamma = \alpha, \end{cases}$$

Hence all eigenvectors V of A associated with 1 are of the form

$$V = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \alpha \in \mathbb{R} \setminus \{0\}.$$

To conclude the diagonalization procedure, we construct an invertible matrix P with the eigenvectors and a diagonal matrix D with the eigenvalues. It is easy in the present case, as the eigenvalues are all real and simple. We may therefore choose any of the eigenvectors associated with each of the eigenvalues to construct P. We define

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$A = PDP^{-1},$$

which concludes the diagonalization procedure for A.

2. Compute the general solutions of (S).

Solution: We introduce the auxilliary unknown

$$Z = P^{-1}Y.$$

Then, using the diagonalization of A, we may rewrite the matrix equation in the following way :

$$Y' = AY \iff Y' = PDP^{-1}Y \iff P^{-1}Y' = DP^{-1}Y \iff (P^{-1}Y)' = DP^{-1}Y \iff Z' = DZ.$$

Denote the components of Z by z_1, z_2 and z_3. Then we may reinterpret the matrix equation in Z as a system :

$$Z' = DZ \iff \begin{cases} z_1' = -2z_1, \\ z_2' = z_2 \\ z_3' = z_3. \end{cases}$$

Compare to the original system, this one has the advantage of being decoupled and therefore each composing equation may be solved separately. We find

$$z_1(t) = C_1e^{-2t},\ C_1 \in \mathbb{R}, \quad z_2(t) = C_2e^{-t},\ C_2 \in \mathbb{R}, \quad z_3(t) = C_3e^{t},\ C_3 \in \mathbb{R}.$$

To find the solutions of the original system, it is now sufficient to go back to the original variable using P. Indeed, we have

$$Z = P^{-1}Y \iff Y = PZ.$$

Therefore,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = P \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} = \begin{pmatrix} z_1(t) + z_2(t) + z_3(t) \\ -2z_1(t) - z_2(t) + z_3(t) \\ 4z_1(t) + z_2(t) + z_3(t) \end{pmatrix} = \begin{pmatrix} C_1e^{-2t} + C_2e^{-t} + C_3e^{t} \\ -2C_1e^{-2t} - C_2e^{-t} + C_3e^{t} \\ 4C_1e^{-2t} + C_2e^{-t} + C_3e^{t} \end{pmatrix}, \quad C_1, C_2, C_3 \in \mathbb{R}.$$

3. Give the expression of the solution corresponding to the initial conditions $y_1(0) = 0$, $y_2(0) = 1$ and $y_3(0) = -3$.

Solution: At $t = 0$, the general solution previously computed gives

$$\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} C_1 + C_2 + C_3 \\ -2C_1 - C_2 + C_3 \\ 4C_1 + C_2 + C_3 \end{pmatrix},$$

hence in order to satisfy the initial conditions we request that C_1, C_2 and C_3 are such that

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ -2C_1 - C_2 + C_3 = 1 \\ 4C_1 + C_2 + C_3 = -3 \end{cases}$$

This system may be solved by one of the available techniques to solve linear systems (don't remember any of those ? check `https://en.wikipedia.org/wiki/System_of_linear_equations`). We proceed to a first part of Gaussian elimination followed by a back-substitution. The augmented matrix of the system is reduced as follows :

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & -1 & 1 & 1 \\ 4 & 1 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & -3 & -3 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 6 & 0 \end{array}\right).$$

The last row of the augmented matrix gives

$$C_3 = 0.$$

and back-substitution leads to

$$C_2 = 1, \quad C_1 = -1.$$

As a consequence, the solution with initial data $y_1(0) = 0$, $y_2(0) = 1$ and $y_3(0) = -3$ is

$$y_1(t) = -e^{-2t} + e^{-t}, \quad y_2(t) = 2e^{-2t} - e^{-t}, \quad y_3(t) = -4e^{-2t} + e^{-t}.$$

4. ** Using the same strategy employed in the previous questions, compute the solution of

$$\begin{cases} y_1' = y_2 + y_3, & y_1(0) = 1, \\ y_2' = y_1 + y_3, & y_2(0) = 1, \\ y_3' = y_1 + y_2, & y_3(0) = 1. \end{cases}$$

Solution: Define the vector unknown Y, the vector initial data Y_0 and the matrix A by

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad Y_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then the system is equivalent to

$$Y' = AY, \quad Y(0) = Y_0.$$

We now proceed to the diagonalization of A.
We start by finding the eigenvalues as the roots of the characteristic polynomial of the matrix. The polynomial in the variable λ is given by

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}.$$

This determinant has the particularity that each row contains the same coefficients in a different order, a situation in which it is convenient to sum all columns on the first one and then factor out the appearing term :

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2-\lambda & -\lambda & 1 \\ 2-\lambda & 1 & -\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}.$$

The remaining 3×3 determinant may then be reduced be substracting the first column to the other two :

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix}.$$

The obtained determinant is lower triangular, hence it may be calculated by doing the product of the diagonal terms :

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2.$$

To summarize, we have (arranging the signs) established that

$$\det(A - \lambda I) = -(\lambda - 2)(\lambda + 1)^2,$$

and therefore the eigenvalues of A are -1 and 2, the eigenvalue -1 being double (i.e. it appears with a power 2 in the characteristic polynomial).
We now find the eigenvectors, starting with the eigenvectors associated with the eigenvalue $\lambda = -1$. That is, we search for all vectors $V = \left(\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right)$, $V \neq 0$ such that

$$AV = -V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = V \iff \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \iff \begin{pmatrix} \beta+\gamma \\ \alpha+\gamma \\ \alpha+\beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \\ -\gamma \end{pmatrix} \iff \begin{cases} \beta+\gamma = -\alpha \\ \alpha+\gamma = -\beta \\ \alpha+\beta = -\gamma \end{cases}$$

This last system reduces in fact to a single equation

$$\alpha + \beta + \gamma = 0.$$

This equation has three unknowns, hence two of them may be chosen as free parameters which will determine the last one. Thus all eigenvectors V of A associated with -1 are of the form

$$V = \begin{pmatrix} \alpha \\ \beta \\ -\alpha-\beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad (\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Observe right now that we have here two non-colinear (hence independent) eigenvectors for -1, which are $\left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \\ -1 \end{smallmatrix}\right)$.
The same procedure may be followed to find the eigenvectors associated with 2. Indeed, we now search for all vectors $V = \left(\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right)$, $V \neq 0$ such that

$$AV = 2V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = 2V \iff \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 2 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \iff \begin{pmatrix} \beta+\gamma \\ \alpha+\gamma \\ \alpha+\beta \end{pmatrix} = \begin{pmatrix} 2\alpha \\ 2\beta \\ 2\gamma \end{pmatrix} \iff \begin{cases} \beta+\gamma = 2\alpha, \\ \alpha+\gamma = 2\beta, \\ \alpha+\beta = 2\gamma. \end{cases}$$

Hence we have to solve the system

$$\begin{cases} -2\alpha + \beta + \gamma = 0, \\ \alpha - 2\beta + \gamma = 0, \\ \alpha + \beta - 2\gamma = 0. \end{cases}$$

We may start a Gaussian elimination procedure by adding 1/2 of the first line to the second and the third to get

$$\begin{cases} -2\alpha + \beta + \gamma = 0, \\ -\frac{3}{2}\beta + \frac{3}{2}\gamma = 0, \\ \frac{3}{2}\beta - \frac{3}{2}\gamma = 0. \end{cases}$$

Hence the system reduces to a system of two equations :

$$\begin{cases} -2\alpha + \beta + \gamma = 0, \\ \beta = \gamma, \end{cases}$$

which may then be rearranged to get

$$\begin{cases} \alpha = \gamma, \\ \beta = \gamma. \end{cases}$$

Hence all eigenvectors V of A associated with 2 are of the form

$$V = \begin{pmatrix} \gamma \\ -\gamma \\ \gamma \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma \in \mathbb{R} \setminus \{0\}.$$

To conclude the diagonalization procedure, we construct an invertible matrix P with the eigenvectors and a diagonal matrix D with the eigenvalues. We have two non-colinear eigenvectors for -1 and may choose any eigenvector associated with 2 to construct P. We define

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then we have

$$A = PDP^{-1},$$

which concludes the diagonalization procedure for A.
We introduce the auxilliary unknown

$$Z = P^{-1}Y.$$

Then, using the diagonalization of A, we may rewrite the matrix equation in the following way :

$$Y' = AY \iff Y' = PDP^{-1}Y \iff P^{-1}Y' = DP^{-1}Y \iff (P^{-1}Y)' = DP^{-1}Y \iff Z' = DZ.$$

Denote the components of Z by z_1, z_2 and z_3. Then we may reinterpret the matrix equation in Z as a system :

$$Z' = DZ \iff \begin{cases} z_1' = -z_1, \\ z_2' = -z_2 \\ z_3' = 2z_3. \end{cases}$$

Compare to the original system, this one has the advantage of being decoupled and therefore each composing equation may be solved separately. We find

$$z_1(t) = C_1 e^{-t}, \quad C_1 \in \mathbb{R}, \quad z_2(t) = C_2 e^{-t}, \quad C_2 \in \mathbb{R}. z_3(t) = C_3 e^{2t}, \quad C_3 \in \mathbb{R}.$$

To find the solutions of the original system, it is now sufficient to go back to the original variable using P. Indeed, we have

$$Z = P^{-1}Y \iff Y = PZ.$$

Therefore,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = P \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} = \begin{pmatrix} z_1(t) + z_3(t) \\ z_2(t) + z_3(t) \\ -z_1(t) - z_2(t) + z_3(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{-t} + C_3 e^{2t} \\ C_2 e^{-t} + C_3 e^{2t} \\ -C_1 e^{-t} - C_2 e^{-t} + C_3 e^{2t} \end{pmatrix}, \quad C_1, C_2, C_3 \in \mathbb{R}.$$

At $t = 0$, the general solution previously computed gives

$$\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} C_1 + C_3 \\ C_2 + C_3 \\ -C_1 - C_2 + C_3 \end{pmatrix},$$

hence in order to satisfy the initial conditions we request that C_1, C_2 and C_3 are such that

$$\begin{cases} C_1 + C_3 = 1 \\ C_2 + C_3 = 1 \\ -C_1 - C_2 + C_3 = 1 \end{cases}$$

This system may be solved by one of the available techniques to solve linear systems (don't remember any of those ? check `https://en.wikipedia.org/wiki/System_of_linear_equations`). We proceed to a first part of Gaussian elimination followed by a back-substitution. The augmented matrix of the system is reduced as follows :

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right).$$

The last row of the augmented matrix gives

$$C_3 = 1.$$

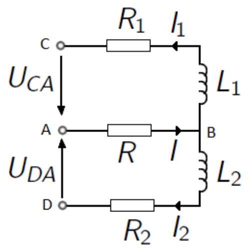
and back-substitution leads to

$$C_2 = 0, \quad C_1 = 0.$$

As a consequence, the solution with initial data $y_1(0) = 1$, $y_2(0) = 1$ and $y_3(0) = 1$ is

$$y_1(t) = e^{2t}, \quad y_2(t) = e^{2t}, \quad y_3(t) = e^{2t}.$$

Exercise 4. We consider an electrical circuit given by



The intensities I_1 and I_2 are solutions of the differential system

$$\begin{cases} \dfrac{dI_1}{dt} = -\left(\dfrac{R + R_1}{L_1}\right) I_1 - \left(\dfrac{R_2}{L_1}\right) I_2 + \dfrac{U_{CA}}{L_1}, \\ \dfrac{dI_2}{dt} = -\left(\dfrac{R_1}{L_2}\right) I_1 - \left(\dfrac{R + R_2}{L_2}\right) I_2 + \dfrac{U_{DA}}{L_2}. \end{cases}$$

In the following, we consider the case where $L_1 = L_2 = 1$, $R = 1$, $R_1 = 2$, $R_2 = 3$.

1. Prove that the system verifies by the couple (I_1, I_2) is

$$\begin{cases} \dfrac{dI_1}{dt} = -3I_1 - 3I_2 + U_{CA}, \\ \dfrac{dI_2}{dt} = -2I_1 - 4I_2 + U_{DA}. \end{cases}$$

Solution: The assertion follows immediately from the fact that, with the given values of the parameters, we have

$$-\left(\frac{R + R_1}{L_1}\right) = -3, \quad -\left(\frac{R_2}{L_1}\right) = -3, \quad \frac{1}{L_1} = 1, \quad -\left(\frac{R_1}{L_2}\right) = -2, \quad -\left(\frac{R + R_2}{L_2}\right) = -4, \quad \frac{1}{L_2} = 1.$$

2. Compute the associated general solution of the homogenous system (i.e. for $U_{CA} = 0$ and $U_{DA} = 0$).

Solution: We want to find the general solution of

$$\begin{cases} \dfrac{dI_1}{dt} = -3I_1 - 3I_2, \\ \dfrac{dI_2}{dt} = -2I_1 - 4I_2. \end{cases}$$

The system may be reinterpreted as a matrix differential equation. Indeed, define

$$I = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \quad A = \begin{pmatrix} -3 & -3 \\ -2 & -4 \end{pmatrix}.$$

Then the system is equivalent to

$$I' = AI.$$

We diagonalize the matrix A. We start by finding the eigenvalues as the roots of the characteristic polynomial of the matrix. The polynomial in the variable λ is given by

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -3 \\ -2 & -4-\lambda \end{vmatrix} = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1)$$

Therefore, the eigenvalues of A are -6 and -1.
We now find the eigenvectors, starting with the eigenvectors associated with the eigenvalue $\lambda = -6$. That is, we search for all vectors $V = \binom{\alpha}{\beta}$, $V \neq 0$ such that

$$AV = -6V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = V \iff \begin{pmatrix} -3 & -3 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -6 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff \begin{pmatrix} -3\alpha - 3\beta \\ -2\alpha - 4\beta \end{pmatrix} = \begin{pmatrix} -6\alpha \\ -6\beta \end{pmatrix} \iff \begin{cases} -3\alpha - 3\beta = -6\alpha, \\ -2\alpha - 4\beta = -6\beta. \end{cases}$$

This last system reduces in fact to the single equation

$$\alpha = \beta.$$

Hence all eigenvectors V of A associated with 1 are of the form

$$V = \begin{pmatrix} \beta \\ \beta \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \beta \in \mathbb{R} \setminus \{0\}.$$

The same procedure may be followed to find the eigenvectors associated with -1. Indeed, we now search for all vectors $V = \binom{\alpha}{\beta}$, $V \neq 0$ such that

$$AV = -V.$$

This matrix equation can easily be converted into a system for the coordinates :

$$AV = -V \iff \begin{pmatrix} -3 & -3 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff \begin{pmatrix} -3\alpha - 3\beta \\ -2\alpha - 4\beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \iff \begin{cases} -3\alpha - 3\beta = -\alpha, \\ -2\alpha - 4\beta = -\beta. \end{cases}$$

As before this last system reduces in fact to the single equation

$$-2\alpha = 3\beta.$$

Hence all eigenvectors V of A associated with 2 are of the form

$$V = \begin{pmatrix} -3\beta/2 \\ \beta \end{pmatrix} = \frac{\beta}{2} \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \quad \beta \in \mathbb{R} \setminus \{0\}.$$

To conclude the diagonalization procedure, we construct an invertible matrix with the eigenvectors and a diagonal matrix with the eigenvalues. It is easy in the present case, as the eigenvalues are all real and simple. We may therefore define

$$P = \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} -6 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we have

$$A = PDP^{-1},$$

which concludes the diagonalization procedure for A.

We now introduce the auxilliary unknown

$$J = P^{-1}I.$$

Then, using the diagonalization of A, we may rewrite the matrix equation in the following way :

$$I' = AI \iff I' = PDP^{-1}I \iff P^{-1}I' = DP^{-1}I \iff (P^{-1}I)' = DP^{-1}I \iff J' = DJ.$$

Denote the components of J by j_1 and j_2. Then we may reinterpret the matrix equation in J as a system :

$$J' = DJ \iff \begin{cases} j_1' = -6j_1, \\ j_2' = -j_2. \end{cases}$$

Compare to the original system, this one has the advantage of being decoupled and therefore each composing equation may be solved separately. We find

$$j_1(t) = C_1 e^{-6t}, \quad C_1 \in \mathbb{R}, j_2(t) = C_2 e^{-t}, \quad C_2 \in \mathbb{R}.$$

To find the solutions of the original system, it is now sufficient to go back to the original variable using P. Indeed, we have

$$J = P^{-1}I \iff I = PJ.$$

Therefore,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = P \begin{pmatrix} j_1(t) \\ j_2(t) \end{pmatrix} = \begin{pmatrix} j_1(t) - 3j_2(t) \\ j_1(t) + 2j_2(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{-6t} - 3C_2 e^{-t} \\ C_1 e^{-6t} + 2C_2 e^{-t} \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}.$$

3. We suppose that the tensions U_{CA} and U_{DA} are constants and equal to $U_{CA} = 3$ and $U_{DA} = 8$. Moreover we suppose that the initial intensities $I_1(0)$ and $I_2(0)$ are equal to 0. Compute the corresponding solution (I_1, I_2).

Solution: From the superposition principle, the general solution of the system with second member is the sum of the general solution of the homogeneous system (which we already know) and a particular solution of the system with second member. Here, the second member is

$$\begin{pmatrix} U_{CA} \\ U_{DA} \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}.$$

Since the second member is given by constants, we expect to find a particular solution of the system as constants. We search for $(i_1, i_2) \in \mathbb{R}^2$ such that the function defined by $I_1(t) = i_1$ and $I_2(t) = i_2$ verify the system. This is equivalent to request that

$$\begin{cases} 0 = -3i_1 - 3i_2 + 3, \\ 0 = -2i_1 - 4i_2 + 8. \end{cases} \iff \begin{cases} 3i_1 + 3i_2 = 3, \\ 2i_1 + 4i_2 = 8. \end{cases}$$

This system may be solved in the following way. Calling $L1$ and $L2$ the first and second lines, do $2 \times L1 - 3 \times L2$ to get

$$-6i_2 = -18 \iff i_2 = 3.$$

By substitution in $L1$ or $L2$, we readily get

$$i_1 = -2.$$

Hence the constants functions equal to $(3, 2)$ are a particular solution of the system, and the general solution of the system is therefore given by

$$\begin{cases} I_1(t) = C_1 e^{-6t} - 3C_2 e^{-t} - 2, \\ I_2(t) = C_1 e^{-6t} + 2C_2 e^{-t} + 3. \end{cases}$$

To find the solution satisfying $I_1(0) = I_2(0) = 0$, we first observe that at $t = 0$ the general solution verifies

$$\begin{cases} I_1(0) = C_1 - 3C_2 - 2, \\ I_2(0) = C_1 + 2C_2 + 3. \end{cases}$$

Hence we have to find C_1, C_2 such that

$$\begin{cases} 0 = C_1 - 3C_2 - 2, \\ 0 = C_1 + 2C_2 + 3. \end{cases} \iff \begin{cases} C_1 - 3C_2 = 2, \\ C_1 + 2C_2 = -3. \end{cases}$$

This system may be solved in the following way. We first substract the first line to the second to find

$$5C_2 = -5 \iff C_2 = -1.$$

Then by substitution in any of the equations we get

$$C_1 = -1.$$

Hence the solutions of the differential system that we were looking for are

$$\begin{cases} I_1(t) = -e^{-6t} + 3e^{-t} - 2, \\ I_2(t) = -e^{-6t} - 2e^{-t} + 3. \end{cases}$$