Free groups and group presentations

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General plan of the talk:

- Quick reminder on the dihedral group D_n and the symmetric group S_n with some natural set of generators.
- The free group over two generators: reduced words in two letters a and b and composition law "combine and cancel". Generalize to any set S of "letters".
- Definition of a presentation by generators and relations.
- Generators and relation for S_n (and maybe D_n).

Exercises chosen from the book:

Exercise 5. Show that every word in a, b, a^{-1}, and b^{-1} can be reduced to a unique reduced word.

Exercise 6. Check the details of the construction of the free group of rank 2, in particular the assertion that the multiplication is associative.

Exercise 13. Find (nice) group presentations for F_n, D_n, and S_n.

An elementary cancellation in a word is the cancellation of ss^{-1} or $s^{-1}s$ for some $s \in S$, reducing the length of the word by 2. A word is reduced if it does not admit any elementary cancellation.

Proposition 1. *Let w be a word in some letters in a set S. Then any sequence of elementary cancellations of w produces the same reduced word.*

Proof. We proceed by induction on the length of w, the case of the empty word being trivial.

Let $w = \dots \underline{ss^{-1}} \dots$ be a word, with $\underline{ss^{-1}}$ a distinguished cancelling pair inside w (the case $s^{-1}s$ is similar). Let $w_0 = w \to w_1 \to \cdots \to w_n$ be a sequence of elementary cancellations, with w_n reduced. By induction hypothesis, it is sufficient to show that it is possible to get w_n by starting with the cancellation of the distinguished cancelling pair.

First case: if the cancellation of $\underline{ss^{-1}}$ is one of the steps $w_i \to w_{i+1}$, then using the fact that two cancellations with disjoint support commute, we can move the cancellation $w_i \to w_{i+1}$ in first position.

Second case: otherwise, there is a step $w_i \to w_{i+1}$ that cancels exactly one letter of the distinguished pair. Either

$$\dots s^{-1}\underline{ss^{-1}} \cdots \to \dots s^{-1} \dots$$

or

$$\dots \underline{ss^{-1}}s \cdots \to \dots s \dots$$

We remark that we can interpret $w_i \to w_{i+1}$ as the cancellation of the distinguished pair, and we are back to the first case. ☐

Corollary 2. *The law "combine and cancel" on reduced words is associative.*

Proof. if u, v, w are three (reduced) words, $(uv)w$ and $u(vw)$ are equal to the unique reduced word obtained from uvw. ☐

(Proposition and Corollary taken from Artin - Algebra)

Proposition 3. *The dihedral group D_n (of order $2n$) admits the presentation*

$$D_n = \langle s, t \mid s^2 = t^2 = (st)^n = 1 \rangle$$

Proof. Let τ, σ be the symmetries with respect to the line through 1 and $e^{i\pi/n}$ respectively. Then $\sigma\tau$ is the rotation of angle $2\pi/n$, and so the group $\Gamma_n = \langle s, t \mid s^2 = t^2 = (st)^n = 1 \rangle$ surjects on D_n (by sending $s \to \sigma$ and $t \to \tau$). It is sufficient to show that $|\Gamma_n| \leqslant 2n$. Set $u = st$, and observe that $su = u^{-1}s(= t)$, so that any element in Γ_n can be written $s^a u^b$ with $0 \leqslant a \leqslant 1$ and $0 \leqslant b \leqslant n-1$. □

Proposition 4. *The symmetric group S_n admits the presentation*

$$S_n = \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1 \textit{ for } |j - i| \geqslant 2 \rangle$$

Proof. The transpositions $\tau_i = (i, i+1)$ generate S_n and satisfy the above relations. So if we set

$$\Gamma_n = \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1 \text{ for } |j - i| \geqslant 2 \rangle$$

we have surjective group homomorphisms

$$F_{n-1} \twoheadrightarrow \Gamma_n \twoheadrightarrow S_n$$

and to prove that $\Gamma_n \twoheadrightarrow S_n$ is an isomorphism it is sufficient to prove $|\Gamma_n| \leqslant n!$.

This is clear for $n = 2$ since $\Gamma_2 = \langle s_1 \mid s_1^2 = 1 \rangle \simeq \mathbf{Z}/2$, and we now proceed by induction.

Assume $|\Gamma_n| \leqslant n!$, and consider $H \subset \Gamma_{n+1}$ the subgroup generated by $s_1, \dots, s_{n-1}$. This is a priori a quotient of Γ_n, in any case $|H| \leqslant |\Gamma_n| \leqslant n!$, so to prove $|\Gamma_{n+1}| \leqslant (n+1)!$ it is sufficient to prove that the number of left cosets Γ_{n+1}/H is at most $n+1$. For $i = 1, \dots, n$, we denote

$$H_i = s_i s_{i+1} \dots s_{n-1} s_n H$$

So with the trivial coset H we get a collection of $n+1$ cosets, a priori not all distinct, but it does not matter since we only aim at an inequality. To prove that these cosets cover all elements of Γ_{n+1}/H, it is sufficient to prove that left multiplication by an element $g \in \Gamma_{n+1}$ permutes these cosets. Finally, it is sufficient to check this claim for each generator s_i. So finally we want to prove: for all $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n+1$, $s_i H_j$ is again one of the $n+1$ cosets of the collection (with the convention $H_{n+1} = H$ is the trivial coset). By definition $s_i H_i = H_{i+1}$ and $s_i H_{i+1} = H_i$. Now we are going to prove that if $j \neq i$ and $j \neq i+1$ then $s_i H_j = H_j$, and this will end the proof.

First subcase: $j \geqslant i+2$. Then $s_i \in H$ and commutes with $s_j, \dots, s_n$, so

$$s_i H_j = s_i s_j s_{j+1} \dots s_n H = s_j s_{j+1} \dots s_n s_i H = s_j s_{j+1} \dots s_n H = H_j.$$

Second subcase: $j \leqslant i-1$. Then

$$s_i H_j = s_i s_j s_{j+1} \dots s_n H = s_j \dots s_{i-2} (s_i s_{i-1} s_i) s_{i+1} \dots s_n H$$
$$= s_j \dots s_{i-2} (s_{i-1} s_i s_{i-1}) s_{i+1} \dots s_n H = s_j \dots s_n s_{i-1} H = H_j \quad □$$

(Taken from notes on the webpage of Dan Ciubotaru at Utah University, the same argument is also in Ramis-Warusfel “Algèbre et géométrie”.)