POLYNOMIAL AUTOMORPHISMS
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Abstract. Polynomial automophisms of the affine plane (or space) can be viewed as birational maps. This subgroup is sufficiently rich to share many properties with the full Cremona group, but at the same time many more specific combinatorial tools are available to study them.

Contents

Lecture 1. Amalgamated product structure 2
1.1. Polynomial automorphisms 2
1.2. Amalgamated product and tree 2
1.3. Amalgamated product structure on $\mathrm{Aut}(\mathbb{A}^2)$ 3
1.4. Linearisation of finite subgroups 3
Lecture 2. A birational proof of Jung's theorem 5
2.1. Birational extension of a polynomial automorphism 5
2.2. The induction 5
Lecture 3. Non simplicity 11
3.1. The question of simplicity 11
3.2. The WPD property on trees 11
3.3. Application to polynomial automorphisms 13
Lecture 4. A glimpse at higher dimension 15
4.1. Three conjectures 15
4.2. About the relation between Conjectures 1 and 2 15
4.3. About Conjecture 3 15
4.4. About Conjecture 1 16
Exercises 17
References 18

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Lecture 1. Amalgamated product structure

1.1. **Polynomial automorphisms.** A polynomial automorphism is an automorphism of the affine space $\mathbb{A}^n$.

Since $\mathbb{A}^n = \operatorname{Spec} \mathbf{k}[x_1, \dots, x_n]$, a polynomial automorphism g can be seen as a $\mathbf{k}$-automorphism of the algebra $\mathbf{k}[x_1, \dots, x_n]$. We use the notation

$$g = (g_1, \cdots, g_n), \text{ with } g_i \in \mathbf{k}[x_1, \dots, x_n]$$

to denote such an automophism, where the condition on the g_i is

$$\mathbf{k}[g_1, \cdots, g_n] = \mathbf{k}[x_1, \cdots, x_n]$$

The automophism $g \in \operatorname{Aut}_{\mathbf{k}} \mathbf{k}[x_1, \dots, x_n]$ induces a bijective polynomial map from $\mathbb{A}^n$ to $\mathbb{A}^n$, which we still denote by g. We will use the notation $\operatorname{Aut}(\mathbb{A}^n)$ to denote such maps. Observe the groups $\operatorname{Aut}(\mathbb{A}^n)$ and $\operatorname{Aut}_{\mathbf{k}} \mathbf{k}[x_1, \dots, x_n]$ are *anti*-isomorphic, which causes endless confusion when adepts of one convention try to understand the work of one of the other side.

We define the **tame automorphism group**

$$\operatorname{Tame}(\mathbb{A}^n) = \langle \mathrm{GL}_n, E_n \rangle \subseteq \operatorname{Aut}(\mathbb{A}^n)$$

as the subgroup generated by linear and elementary automorphisms, where

$$E_n = \big\{(x_1, x_2, \dots, x_n) \mapsto (x_1 + P(x_2, \dots, x_n), x_2, \dots, x_n) \mid P \in \mathbf{k}[x_2, \dots, x_n]\big\}.$$

Observe that:

- Translations are tame automorphisms. Indeed $\tau = (x_1 + 1, x_2, \dots, x_n) \in E_n$, and any translation is conjugate to τ by an element in GL_n.
- The affine group A_n is a subgroup of the tame group.
- Triangular automorphisms of the form

$$(a_1 x_1 + P_1(x_2, \dots, x_n),\ a_2 x_2 + P_2(x_3, \dots, x_n), \dots, a_n x_n)$$

form a subgroup of $\operatorname{Tame}(\mathbb{A}^n)$, that we denote by B_n (B for Borel).

Observe also that polynomial automophism are a special kind of birational maps:

$$\operatorname{Aut}(\mathbb{A}^n) \subseteq \operatorname{Bir}(\mathbb{P}^n).$$

1.2. **Amalgamated product and tree.** Let G be a group, A, $B \subseteq G$ two subgroups such that $\langle A, B \rangle = G$. Consider the abstract free product $A * B$, and denote by $i_A \colon A \cap B \to A * B$, $i_B \colon A \cap B \to A * B$ the two natural injective morphisms. Then we construct N the normal subgroup in $A * B$ generated by all products $i_A(h) i_B(h)^{-1}$, where $h \in A \cap B$. There is a natural surjective morphism $\varphi \colon A * B \to G$, and $N \subseteq \ker \varphi$. We say that G is the **amalgamated product** of its subgroups A, B along their intersection, denoted $G = A *_{A \cap B} B$, if G is isomorphic to $(A * B)/N$, that is, if $N = \ker \varphi$. Informally, one can say that all relations in $G = \langle A, B \rangle$ come from the relations inside A or B. In particular, given choices of representatives (a_i), (b_j) of the non-trivial left cosets $A/(A \cap B)$, $B/(A \cap B)$, any element $g \in G$ admits a unique factorization of the form $g = wc$, where w (for *word*) is a finite alternate composition of some a_i and b_j, and $c \in A \cap B$.

Given such a structure we can construct an abstract simplicial tree $\mathcal{T}$ on which the group G acts: this is the simplest instance of what is known as Bass-Serre theory. The construction is as follows. The vertices of $\mathcal{T}$ are of two types, namely the left cosets G/A and G/B, and the edges are the left cosets $G/(A \cap B)$, with the following gluing rule: for each $f \in G$, the vertices fA and fE are joined by the edge $f(A \cap E)$. This defines a graph, and the fact that G is the amalgamated product of A and B translates in the property that this graph has no loop, that is, this is a tree. The group G acts naturally by isometry on this tree. If $f, g \in G$, we have

$$g \cdot fA = (g \circ f)A, \quad g \cdot fB = (g \circ f)B, \quad \text{and } g \cdot f(A \cap B) = (g \circ f)(A \cap B).$$

Moreover the action is without inversion: if an edge is invariant it is point-wise fixed. This follows from the fact that the two types of vertices are preserved by the action.

1.3. **Amalgamated product structure on** $\mathrm{Aut}(\mathbb{A}^2)$**.** We want to show that any relation in the group $\mathrm{Aut}[\mathbb{A}^2]$ is induced by the relations in the groups $A = A_2$ and $B = B_2$. This is equivalent to show that a composition

$$h = a_1 \circ b_1 \circ \cdots \circ a_n \circ b_n \text{ with } a_i \in A \setminus B, b_i \in B \setminus A$$

can never be equal to the identity. Note that we can restrict ourselves to such compositions h, that is of even length and beginning by an affine automorphism. Indeed if h is of odd length (and ≥ 3 : if h is of length 1 there is nothing to do) we can reduce the length of h by a suitable conjugation. Furthermore if h is of even length and begins by a triangular automorphism, we just consider h^{-1} instead.

Now it is easy to check that each automorphism b_i, view as a birational map on $\mathbb{P}^2$, contracts the line at infinity to the point $[1 : 0 : 0]$ (because we suppose $b_i \notin A$). Furthermore, $a_i \notin B$ is equivalent to say that the point $[1 : 0 : 0]$ is not a fixed point of a_i. By computing the images of a general point p at infinity under b_n, $a_n \circ b_n$, $b_{n-1} \circ a_n \circ b_n$, etc, we deduce that the extension of h to $\mathbb{P}^2$ contracts the line at infinity to the point $a_1([1 : 0 : 0])$, which contradicts that h is distinct from the identity.

1.4. **Linearisation of finite subgroups.**

Proposition 1.1. *Assume the base field has characteristic zero. Let $\Gamma \subseteq \mathrm{Aut}(\mathbb{A}^2)$ be a finite subgroup. Then Γ is linearisable, that is, there exists $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ such that*

$$\varphi \Gamma \varphi^{-1} \subseteq \mathrm{GL}_2(\mathbf{k}).$$

Proof. The first part of the proof is to use the action on the tree, and to observe that a finite group of isometries on a tree has a global fixed point. Any finite set of points has a circumcenter: pick any pair realizing the max of the distance, the middle point is the circumcenter.

The second part of the proof is to show that any finite subgroup of A_2 or B_2 is linearisable. We can apply the following quite general lemma. □

Lemma 1.2 ([BFL14, Lemma 5.1], see also [Fur83, Proposition 4])**.** *Let G be a group of transformations of a vector space V that admits a semi-direct product structure $G = M \rtimes L$. Assume that M is stable by mean (i.e. for any finite sequence $m_1, \ldots, m_r$ in M, the mean $\frac{1}{r}\sum_1^r m_i$ is in M) and that L is linear (i.e. $L \subseteq \mathrm{GL}(V)$). Then any finite subgroup $\Gamma \subseteq G$ is conjugate by an element of M to a subgroup of L.*

Proof. Consider the morphism of groups

$$\begin{aligned} \varphi\colon G = M \rtimes L &\to L \\ g = m \circ \ell &\mapsto \ell \end{aligned}$$

For any $g \in G$ we have $\varphi(g)^{-1} \circ g \in M$. Given a finite group $\Gamma \subseteq G$, define $m = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \varphi(g)^{-1} \circ g$. By the mean property, $m \in M$. Then, for each $f \in \Gamma$, we compute:

$$\begin{aligned} m \circ f &= \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \varphi(g)^{-1} \circ g \circ f \\ &= \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \varphi(f) \circ [\varphi(f)^{-1} \circ \varphi(g)^{-1}] \circ g \circ f \\ &= \varphi(f) \circ m. \end{aligned}$$

Hence $m\Gamma m^{-1}$ is equal to $\varphi(\Gamma)$, which is a subgroup of L. □

Example 1.3. The lemma applies to the following groups:

- The affine group A_n, with respect to the product

$$A_n = \mathbf{k}^n \rtimes \mathrm{GL}_n(k).$$

- The triangular group B_n with respect to the product

$$B_n = \{(x_1 + P_1(x_2, \ldots, x_n), x_2 + P_2(x_3, \ldots, x_n), \ldots, x_n + P_n)\} \rtimes \{(a_1 x_1, \ldots, a_n x_n)\}.$$

Lecture 2. A birational proof of Jung's theorem

We follow [Lam02]. $\mathbf{k}$ is an algebraically closed field.

2.1. **Birational extension of a polynomial automorphism.** In the sequel we consider $g\colon X \dashrightarrow \mathbb{P}^2$ coming from a polynomial automorphism of $\mathbb{A}^2$. By this we mean that we have a partition $X = \mathbb{A}^2 \cup D$ where D is an union of irreducible curves (called the divisor at infinity), and a partition $\mathbb{P}^2 = \mathbb{A}^2 \cup L$ where L is a line (line at infinity), such that g induces an isomorphism from $X \setminus D$ to $\mathbb{P}^2 \setminus L$. This situation implies some strong restrictions on the base points of g.

Lemma 2.1. *Let X be a surface and g be a birational map from X to $\mathbb{P}^2$ coming from a polynomial automorphism of $\mathbb{A}^2$. Assume that g is not a morphism. Then*

(1) *g admits a unique proper base point, located on the divisor at infinity of X;*
(2) *g admits some base points $p_1, \cdots, p_s$ $(s \geq 1)$ such that*
   (*i*) *p_1 is the unique proper base point;*
   (*ii*) *for all $i = 2, \cdots, s$, the point p_i is located on the divisor produced by the blow-up of p_{i-1};*
(3) *Every irreducible curve contained in the divisor at infinity of X is contracted to a point by g;*
(4) *the first contracted curve of π_2 is the strict transform of a curve contained in the divisor at infinity of X;*
(5) *in particular, if $X = \mathbb{P}^2$, the first contracted curve by π_2 is the strict transform of the line at infinity in X.*

Proof. We know that if p is a proper base point of g then there exists a curve contracted to p by g^{-1}. In our situation the only curve of $\mathbb{P}^2$ candidate to be contracted is the line at infinity; so there is at most one proper base point for g in X. As we suppose that g is not a morphism, g admits exactly one proper base point.

Assertion (2) then comes from a straightforward induction. Furthermore, each curve in the divisor at infinity in X either is contracted to a point, or is sent onto the line at infinity in $\mathbb{P}^2$. Since g^{-1} contracts the line at infinity to a point, this latter possibility is excluded : we just proved (3).

From the argument above we see that the divisor at infinity in M is composed from the divisor with self-intersection -1 produced by the blow-up of p_s, from the other divisors produced by the sequence of blow-ups, all of which with self-intersection less or equal to -2, and finally from the strict transform of the divisor at infinity in X. The first curve contracted by π_2 must have self-intersection -1, and can not be the last curve produced by π_1 (this would contradict the minimality of the resolution), thus we see that the first curve contracted by π_2 is the strict transform of a curve contained in the divisor at infinity in X.

The last assertion is just another formulation of (4), in the case $X = \mathbb{P}^2$. □

2.2. **The induction.** We consider g a polynomial automorphism of $\mathbb{A}^2$, that we extend as a birational self-map (still denoted g) of $\mathbb{P}^2$. If g is written

$$g : (x, y) \mapsto (g_1(x, y), g_2(x, y))$$

and if n is the degree of g (*i.e.* the maximum of the degrees of g_1 and g_2), then the extension of g to $\mathbb{P}^2$ is written in homogeneous coordinates as

$$g : [x : y : z] \dashrightarrow [z^n g_1(x/z, y/z) : z^n g_2(x/z, y/z) : z^n].$$

The line at infinity in $\mathbb{P}^2$ is the line of equation $z = 0$. We want to prove that g is a composition of affine and triangular automorphisms. The proof will proceed by induction on the number $\#(g)$ of base points of g.

By Lemma 2.1(1) the extension $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ admits a unique base point located on the line at infinity. Composing g by an affine automorphism we can assume that this point is $[1 : 0 : 0]$. In other words we have a commutative diagram:

$$\begin{tikzcd} & \mathbb{P}^2 \arrow[dr, dashed, "g_0 = g \circ a^{-1}"] & \\ \mathbb{P}^2 \arrow[ur, "a"] \arrow[rr, dashed, "g"'] & & \mathbb{P}^2 \end{tikzcd}$$

where a is affine and g_0 admits $[1 : 0 : 0]$ as base point. Obviously we have

$$\#(g_0) = \#(g).$$

We are now going to prove that there exists a diagram

$$\begin{tikzcd} & \mathbb{P}^2 \arrow[dr, dashed, "g_0 \circ \varphi^{-1}"] & \\ \mathbb{P}^2 \arrow[ur, dashed, "\varphi"] \arrow[rr, dashed, "g_0"'] & & \mathbb{P}^2 \end{tikzcd}$$

where φ is the extension of an triangular automorphism of $\mathbb{A}^2$, and such that

$$\#(g_0 \circ \varphi^{-1}) < \#(g_0).$$

Our strategy will be to consider the diagram given par Zariski's theorem

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "g_0"'] & & \mathbb{P}^2 \end{tikzcd}$$

and to reorganize the blow-ups occurring in the sequences π_1 and π_2. Thus, in the four following steps, φ will be constructed from some of the blow-ups of the sequence π_1 and some of the contractions of the sequence π_2.

First step : blow-up of $[1 : 0 : 0]$**.** The point $[1 : 0 : 0]$ is the first point blown-up in π_1; so we consider the surface F_1 obtained by blowing-up $\mathbb{P}^2$ at the point $[1 : 0 : 0]$. This surface is a completion of $\mathbb{A}^2$ which is naturally endowed with a rational fibration coming from the lines $y = constant$. The divisor at infinity is composed of two rational curves (*i.e.* isomorphic to $\mathbb{P}^1$) meeting transversally in one point. On one hand we have the strict transform of the line at infinity in $\mathbb{P}^2$; this is a fiber that we will denote f_∞. On the other hand we have the exceptional divisor of the blow-up, which is a section for the fibration : it will be denoted s_∞. We have $f_\infty^2 = 0$ and $s_\infty^2 = -1$. More generally for all $n \geq 1$ we will denote by F_n a completion of $\mathbb{A}^2$ with a structure of a rational fibration, such that the divisor at infinity is composed of two transverse rational curves : one fiber f_∞ and one section s_∞ with self-intersection $-n$. These surfaces are classically called Hirzebruch surfaces.

Now we come back to the map g_0. We have a commutative diagram:

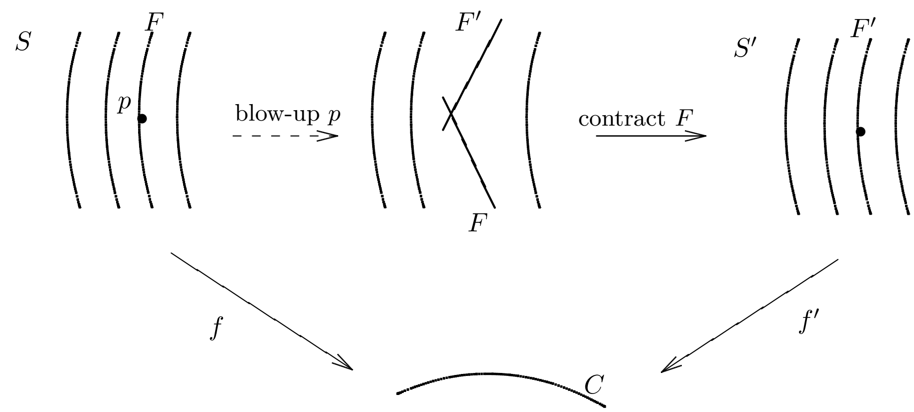
$$\begin{tikzcd} & F_1 \arrow[dr, dashed, "g_1"] & \\ \mathbb{P}^2 \arrow[ur, dashed, "\varphi_1"] \arrow[rr, dashed, "g_0"'] & & \mathbb{P}^2 \end{tikzcd} \tag{$D1$}$$

where φ_1^{-1} is the blow-up map at the point $[1:0:0]$. we have

$$\#(g_1) = \#(g_0) - 1.$$

We consider again the diagram obtained from Zariski's theorem applied to g_0. From Lemma 2.1(5) the first contracted curve in π_2, which must be a curve with self-intersection -1 in M, is the strict transform of the line at infinity. This is the fiber f_∞ in F_1. Now in F_1 we have $f_\infty^2 = 0$. The self-intersection of this curve still has to drop by one, thus the base point p of g_1 is located on f_∞. Furthermore we know by Lemma 2.1(2) that this same point p belongs to the curve s_∞ produced by the blow-up φ_1^{-1}. We conclude that p is precisely the intersection point of f_∞ and s_∞.

Second step : rising induction. In the following reasoning we use some maps between ruled surfaces generally called "elementary transformations". These transformations are defined as the composition of one blow-up at a point p, followed by the contraction of the strict transform of the rule containing p. In this way we obtain a new ruled surface.



In the proof of Lemmas 2.2 and 2.3 we will use such transformations, applied to ruled surfaces with basis C isomorphic to $\mathbb{P}^1$.

Lemma 2.2. *Let $n \geq 1$, and let h be a birational map from F_n to $\mathbb{P}^2$ coming from a polynomial automorphism of $\mathbb{A}^2$. Suppose that the unique proper base point of h is the intersection point p of $f_\infty(F_n)$ and $s_\infty(F_n)$. Consider the commutative diagram*

$$\begin{tikzcd} & F_{n+1} \arrow[dr, dashed, "h'"] & \\ F_n \arrow[ur, dashed, "\varphi"] \arrow[rr, dashed, "h"'] & & \mathbb{P}^2 \end{tikzcd}$$

where φ is the blow-up of p followed by the contraction of the strict transform of f_∞. Then the birational map $h' = h \circ \varphi^{-1}$ satisfies the following two properties:

(1) $\#(h') = \#(h) - 1$;
(2) *the proper base point of h' is located on $f_\infty(F_{n+1})$.*

Proof. We consider the decomposition of h as a sequence of blow-ups:

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F_n \arrow[rr, dashed, "h"'] & & \mathbb{P}^2 \end{tikzcd}$$

The (strict) transform of $s_\infty(F_n)$ in M has self-intersection less or equal to -2; then Lemma 2.1(assertion 4) allows us to conclude that the first contracted curve in π_2 is the transform of $f_\infty(F_n)$. Thus the transform of $f_\infty(F_n)$ in M has self-intersection -1; on the other hand in F_n we have $f_\infty(F_n)^2 = 0$. It follows that after the blow-up of p the rest of the sequence of blow-ups π_1 is performed on points outside of f_∞. Instead of doing these blow-ups before contracting the transform of $f_\infty(F_n)$ we can reverse the order, that is we can first contract $f_\infty(F_n)$ and then realize the rest of the blow-up sequence.

After the blow-up of p and contraction of $f_\infty(F_n)$, the resulting surface is of type F_{n+1}. The blow-up p drops by one the number of base points, and the contraction of $f_\infty(F_n)$ do not create a new one : we have $\#(h') = \#(h) - 1$. Furthermore the base point of h' is located on the curve produced blowing-up p, that is $f_\infty(F_{n+1})$. $\square$

At the end of the first step we can apply Lemma 2.2, with $n = 1$. The lemma then gives a map $h' : F_2 \dashrightarrow \mathbb{P}^2$ whose proper base point is located on the fiber $f_\infty(F_2)$. If this point is precisely the intersection point with the section at infinity, we can apply again the lemma. Iterating this process as long as we remain under the hypotheses of Lemma 2.2, we obtain a diagram

$$\begin{tikzcd} & F_n \arrow[dr, dashed, "g_2"] & \\ F_1 \arrow[ur, dashed, "\varphi_2"] \arrow[rr, dashed, "g_1"'] & & \mathbb{P}^2 \end{tikzcd} \tag{D2}$$

where φ_2 is obtained by applying $n-1$ times Lemma 2.2. Furthermore we have

$$\#(g_2) = \#(g_1) - n + 1.$$

Finally the base point of g_2 is located on $f_\infty(F_n)$, and is not the intersection point with $s_\infty(F_n)$ (otherwise we could apply the lemma once more).

Third step : descending induction. We are going to apply the following lemma, which is similar to Lemma 2.2 (except that now we suppose $n \geq 2$).

Lemma 2.3. *Let $n \geq 2$, and let h be a birational map of F_n to $\mathbb{P}^2$ that comes from a polynomial automorphisms of $\mathbb{A}^2$. Suppose that the unique proper base point p of h is located on f_∞, but is not the intersection point of f_∞ and s_∞. Consider the commutative diagram*

$$\begin{tikzcd} & F_{n-1} \arrow[dr, dashed, "h'"] & \\ F_n \arrow[ur, dashed, "\varphi"] \arrow[rr, dashed, "h"'] & & \mathbb{P}^2 \end{tikzcd}$$

where φ is the blow-up of p followed by the contraction of the strict transform of $f_\infty(F_n)$. Then the birational map h' satisfies the following two properties:

(1) $\#(h') = \#(h) - 1$;
(2) *the proper base point of h' is located on $f_\infty(F_{n-1})$ and is not the intersection point of $f_\infty(F_{n-1})$ and $s_\infty(F_{n-1})$.*

Proof. Consider the decomposition of h given by Zariski's theorem :

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F_n \arrow[rr, dashed, "h"'] & & \mathbb{P}^2 \end{tikzcd}$$

The transform of $s_\infty(F_n)$ in M has self-intersection $-n$, since $n \geq 2$ we deduce (lemma 2.1) that the first contracted curve of π_2 is the transform of $f_\infty(F_n)$. The surface obtained by blowing-up p and contracting the transform of f_∞ is of type F_{n-1}. The equality $\#(h') = \#(h) - 1$ is straightforward. Denoting by F' the divisor produced by the blow-up of p, h admits a (non proper) base point located on F'. Furthermore this point can not be the intersection point of F' and of the transform of $f_\infty(F_n)$, because otherwise we would have $\pi_1^{-1}(f_\infty(F_n))$ with self-intersection less or equal to -2 and this contradicts that it should be the first contracted curve of π_2. Finally this point is the proper base point of h', is located on $f_\infty(F_{n-1})$ and is not the intersection point of $f_\infty(F_{n-1})$ and $s_\infty(F_{n-1})$. □

After the second step we are under the hypotheses of Lemma 2.3. Furthermore if $n \geq 3$ then the map h' produced by the lemma still satisfies the hypotheses of this same lemma. After applying $n-1$ times Lemma 2.3 we obtain a diagram

$$\begin{tikzcd} & F_1 \arrow[dr, dashed, "g_3"] & \\ F_n \arrow[ur, dashed, "\varphi_3"] \arrow[rr, dashed, "g_2"'] & & \mathbb{P}^2 \end{tikzcd} \tag{D3}$$

with

$$\#(g_3) = \#(g_2) - n + 1.$$

Finally, the proper base point of g_3 is located on $f_\infty(F_1)$, and is not the intersection point of $f_\infty(F_1)$ and $s_\infty(F_1)$.

Fourth step : last contraction. Applying Zariski's theorem to g_3 we get a diagram :

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F_1 \arrow[rr, dashed, "g_3"'] & & \mathbb{P}^2 \end{tikzcd}$$

Lemma 2.1(4) ensures that the first contracted curve in π_2 is the strict transform by π_1 of either f_∞ or s_∞. Suppose this is the transform of f_∞. Then after the sequence of blow-ups π_1 and the contraction of this curve, the transform of s_∞ has self-intersection 0 and thus will not be contracted in the sequel of π_2; this contradicts the third assertion of Lemma 2.1. So the first contracted curve must be the transform of s_∞, and we can contract the latter straight away to obtain the following diagram:

$$\begin{tikzcd} & \mathbb{P}^2 \arrow[dr, dashed, "g_4"] & \\ F_1 \arrow[ur, "\varphi_4"] \arrow[rr, dashed, "g_3"'] & & \mathbb{P}^2 \end{tikzcd} \tag{D4}$$

The morphism φ_4 is the blow-up map with exceptional divisor s_∞, and since it is defined up to isomorphism we can decide that the point onto which we contract is $[1:0:0]$. Furthermore we have

$$\#(g_3) = \#(g_4).$$

Conclusion.

We can add up the four diagrams $(D1), \cdots, (D4)$ into only one

$$\begin{tikzcd} & F_1 \arrow[r, dashed, "\varphi_2"] \arrow[ddr, dashed, "g_1"] & F_n \arrow[r, dashed, "\varphi_3"] \arrow[dd, dashed, "g_2"] & F_1 \arrow[ddl, dashed, "g_3"] \arrow[dr, "\varphi_4"] & \\ \mathbb{P}^2 \arrow[ur, dashed, "\varphi_1"] \arrow[drr, dashed, "g_0"'] & & & & \mathbb{P}^2 \arrow[dll, dashed, "g_4"] \\ & & \mathbb{P}^2 & & \end{tikzcd}$$

or in more compact form :

$$\begin{tikzcd} & \mathbb{P}^2 \arrow[dr, dashed, "g_4"] & \\ \mathbb{P}^2 \arrow[ur, dashed, "\varphi_4\circ\varphi_3\circ\varphi_2\circ\varphi_1"] \arrow[rr, dashed, "g_0"'] & & \mathbb{P}^2 \end{tikzcd}$$

with

$$\#(g_4) = \#(g_0) - 2n + 1 \text{ (where } n \geq 2).$$

We now just have to check that $\varphi = \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ is a triangular automorphism. This is equivalent to show that φ preserve the foliation $y = constant$, that is that φ preserves the pencil of lines passing through $[1:0:0]$. Now this fact is clear : the blow-up φ_1 sends the lines through $[1:0:0]$ to the fibers of F_1, φ_2 and φ_3 preserve the fibrations of F_1 and F_n, and finally the contraction φ_4 sends the fibers of F_1 to the lines passing through $[1:0:0]$. Thus the map g_4 is an automorphism of $\mathbb{A}^2$ obtained from g by composing first with an affine automorphism and then with a triangular automorphism, and we have the inequality:

$$\#(g_4) < \#(g).$$

By induction on $\#(g)$, this ends the demonstration.

Lecture 3. Non simplicity

We follow [MO15].

3.1. **The question of simplicity.** We define the special automorphism group of the affine plane $\mathrm{SAut}(\mathbb{A}^2)$ as the subgroup of automorphisms with Jacobian 1. This is the derived subgroup of $\mathrm{Aut}(\mathbb{A}^2)$, that is, the subgroup generated by commutators, and a natural question is whether this group is simple or not. As an application of classical small cancellation theory, it was proved by Danilov [Dan74] that $\mathrm{SAut}(\mathbb{A}^2)$ is not simple; see also [FL10] where this result was revisited.

By recent versions of the small cancellation theory (see [DGO11, Cou14]), to obtain the existence of proper normal subgroups in a given group G, it is sufficient to find an action of G on a δ-hyperbolic space X and an element $g \in G$ that acts "Weakly Properly Discontinuously" on X (WPD property for short, see definition below).

Here is a more precise (but not optimal) statement:

Proposition 3.1. *Let G acting by isometries on a hyperbolic space X, and assume $g \in G$ is a hyperbolic isometry satisfying the WPD property. Then for sufficiently large n the normal subgroup $N = \langle\langle g^n \rangle\rangle \subseteq G$ is a free proper normal subgroup of G.*

This strategy was successfully applied to the following groups:

- The mapping class group of a surface of genus ≥ 2: Dahmani & Guirardel [DGO11].
- $\mathrm{Aut}(\mathbb{A}^2)$: Minasyan & Osin [MO15].
- Cremona group $\mathrm{Bir}(\mathbb{P}^2)$: Lonjou [Lon16].
- $\mathrm{Tame}(V)$ where V is an affine quadric 3-fold: Martin [Mar15].
- $\mathrm{Tame}(\mathbb{A}^3)$: Lamy & Przytycki ([LP16], see poster!)

3.2. **The WPD property on trees.** Let G be a group acting by isometries on a metric space X. Let $A \subseteq X$ any subset, and $\varepsilon \geq 0$. We define the pointwise ε-stabilizer of A as the subset

$$\mathrm{Stab}_\varepsilon A = \{g \in G;\ d(a, ga) \leq \varepsilon \text{ for all } a \in A\}.$$

Observe that a priori $\mathrm{Stab}_\varepsilon A$ is not stable under composition: this is not a subgroup in general!

We leave the following lemma as an exercise.

Lemma 3.2. *Let G be a group acting by isometries on a metric space X, and let $g \in G$. The following are equivalent:*

(1) $\exists x \in X$, $\forall R \geq 0$, *there exists* $M \in \mathbb{N}$ *such that* $\mathrm{Stab}_R\{x, g^M x\}$ *is finite.*
(2) $\forall y \in X$, $\forall R \geq 0$, *there exists* $M \in \mathbb{N}$ *such that* $\mathrm{Stab}_R\{y, g^M y\}$ *is finite.*

Any element g satisfying one of the equivalent properties of Lemma 3.2 is called WPD (Weakly Properly Discontinuous). This notion is mostly relevant for a hyperbolic element g in the context of an action on a hyperbolic space. We focus now on the case of a group acting on a tree.

For any isometry f of a tree $\mathcal{T}$ we denote by $\mathrm{Min}(f)$ the subtree of points minimizing the distance $d(x, f(x))$, and $\ell(f)$ this minimum (translation length). So $\mathrm{Min}(f)$ corresponds to $\mathrm{Fix}(f)$ or $\mathrm{Ax}(f)$ according whether f is elliptic ($\ell(f) = 0$) or hyperbolic ($\ell(f) > 0$).

Recall the classical lemma (which also justifies that the infimum in the definition of $\ell(f)$ is indeed a minimum)

Lemma 3.3. *Let g be an isometry d'un arbre $\mathcal{T}$.*

(1) *If gx is the middle point of the segment $[x, g^2x]$ then the segment $[x, g^2x]$ is contained in* $\mathrm{Min}(g)$*.*
(2) *For all point $x \in \mathcal{T}$, the middle point m of the segment $[x, gx]$ is contained in* $\mathrm{Min}(g)$*.*

Proof. (1) If $gx = x$ there is nothing to show. Otherwise, the segments $[g^ix, g^{i+1}x]$ form an infinite geodesic Γ on which g acts as a translation of length $\ell(g) = d(x, gx)$. If y is another point, and z is the projection of y on Γ, then $d(y, gy) = \ell(g)+2d(y, z)$. Thus $\Gamma = \mathrm{Min}(g)$, and by definition Γ contains the segment $[x, g^2x]$.

(2) Let $[x, p]$ be the maximal subsegment of $[x, m]$ such that $[gp, gx] \subseteq [x, gx]$. Then gp is the middle point of $[p, g^2p]$ (which also contains m), and we can apply the previous point. ☐

We deduce the following result, which allows to construct hyperbolic isometries with an axis containing a prescribed segment.

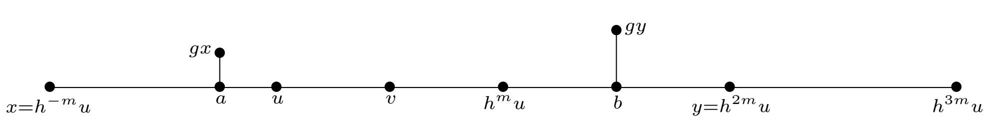
Lemma 3.4. *Let g, h be two isometries of a tree $\mathcal{T}$. Assume* $\mathrm{Min}(g) \cap \mathrm{Min}(h) = \emptyset$*, and denote by S the unique segment joining* $\mathrm{Min}(g)$ *and* $\mathrm{Min}(h)$*. Then $g \circ h$ is an hyperbolic isometry whose axis contains S.*

Proof. Assume that g and h are elliptic, the other cases are similar. We have $S = [x, y]$ with $x \in \mathrm{Stab}(g)$ and $y \in \mathrm{Stab}(h)$. The point x is the middle of $[y, (g \circ h)y]$, and y is the middle point of $[h^{-1}x, x]$, so we can conclude by Lemma 3.3. ☐

Now we restrict ourselves to the case of simplicial trees, endowed with a metric such that each edge is isometric to the segment $[0, 1]$. However such trees might very well be non locally compact.

Lemma 3.5. [MO15, Lemma 4.1 and Corollary 4.2] *Let G acting on a simplicial tree $\mathcal{T}$, and $h \in G$ hyperbolic. Suppose there exist $u, v \in$* $\mathrm{Ax}(h)$ *such that* $\mathrm{Stab}\{u, v\}$ *is finite. Then h is WPD.*

Proof. Let $\varepsilon \geq 0$. Without loss in generality, we can assume $\varepsilon > d(u, v)$, and that h translates from u to v along $\mathrm{Ax}(h)$. Pick $m \in \mathbb{N}$ such that $d(u, h^m u) > \varepsilon$, and define $x = h^{-m}u$, $y = h^{2m}u$. We are going to show that $\mathrm{Stab}_\varepsilon\{u, h^{3m}u\}$ is finite, which will show that h is WPD: see Lemma 3.2(1). Translating by h^{-m}, it is equivalent to show that $\mathrm{Stab}_\varepsilon\{x, y\}$ is finite.



Let $g \in \mathrm{Stab}_\varepsilon\{x, y\}$ (so also $g^{-1} \in \mathrm{Stab}_\varepsilon\{x, y\}$). Denote by a, b the projections of gx, gy on the segment $[x, y]$. We have $a \in [x, u]$ and $b \in [h^m u, y]$. In particular $[u, v] \subseteq [a, b] = [x, y] \cap [gx, gy]$. Similarly, working with g^{-1} instead of g, we obtain $[u, v] \subseteq [x, y] \cap [g^{-1}x, g^{-1}y]$, and so translating by g we get $g[u, v] \subseteq [a, b] = [x, y] \cap [gx, gy]$.

There exist some $t_i \in G$, $i = 1, \dots, r$, such that any image by an element of G of the segment $[u, v]$ contained in $[x, y]$ is equal to one of the $[t_i u, t_i v]$. Moreover by assumption $\mathrm{Stab}\{u, v\} = \{f_j;\ j = 1, \dots, s\}$ for some $f_j \in G$. By definition $g[u, v] = t_i[u, v]$ for some i, and so $t_i^{-1} g \in \mathrm{Stab}\{u, v\}$ is equal to one of the f_j. Finally $\mathrm{Stab}_\varepsilon\{x, y\} \subseteq \{t_i f_j\}$ is finite, as expected. □

Observe that Lemma 3.5 is false for the action of $\mathbb{R}$ on itself by translation: the assumption that $\mathcal{T}$ is simplicial is crucial... Let us mention (we will not use this fact) that conversely one can characterise WPD isometries on simplicial trees:

Proposition 3.6. [MO15, Proposition 4.7] *Let G be a non virtually cyclic group acting minimaly (no invariant subtree) on a simplicial tree $\mathcal{T}$ with at least 3 vertices. The following are equivalent:*

(1) *G contains a WPD hyperbolic element (with respect to the action of G on $\mathcal{T}$);*
(2) *G does not fix any end of $\mathcal{T}$, and there exist 2 vertices $v_1, v_2 \in \mathcal{T}$ such that* $\mathrm{Stab}\{v_1, v_2\}$ *is finite.*

3.3. **Application to polynomial automorphisms.** Let $\mathbf{k}$ be any field. We want to apply the previous criterion to find WPD elements in the group $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$, using the action on the Bass-Serre tree associated with the amalgamated product structure

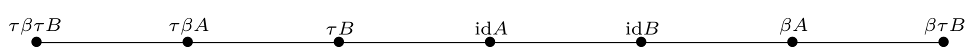
$$\mathrm{Aut}(\mathbb{A}^2) = A *_{A \cap B} B.$$

Observe that the tree is not locally finite (even if the base field $\mathbf{k}$ is the field with two elements!). We shall use the two involutions

$$\beta = (-x + y^2, y) \in B \smallsetminus A \text{ and } \tau = (y, x) \in A \smallsetminus B.$$

Proposition 3.7 (Compare with [MO15, Lemma 4.22])**.** *Assume that* $\mathrm{char}\,\mathbf{k} \neq 2$. *Then* $\beta\tau = (-y + x^2, x) \in \mathrm{Aut}(\mathbb{A}^2)$ *is a hyperbolic isometry satisfying the WPD property.*

Proof. We study automorphisms that fix the following path of length 6 (which is included in the axis of $\beta\tau$).



Let f an element stabilizing the edge between $\mathrm{id}A$ and $\mathrm{id}B$, that is, $f \in A \cap B$ can be written as

$$f = (ax + by + c, dy + e).$$

Now f fixes the vertex τB if and only if $\tau^{-1} f \tau \in B$. We compute:

$$\tau^{-1} f \tau = (dx + e, bx + ay + c).$$

So f fixes the vertex τB if and only if $b = 0$, that we assume from now. Observe also that $ad \neq 0$ since f is invertible.

Similarly, f fixes the vertices βA and $\beta\tau B$ if and only if $\beta^{-1} f \beta \in A$ and $\tau^{-1}\beta^{-1} f \beta\tau \in B$, which is equivalent to $\beta^{-1} f \beta$ of the form $(a'x + c', d'y + e')$. We compute:

$$\begin{aligned}
\beta^{-1} f \beta &= (-x + y^2, y) \circ (ax + c, dy + e) \circ (-x + y^2, y) \\
&= (-x + y^2, y) \circ (-ax + ay^2 + c, dy + e) \\
&= (ax - ay^2 - c + (dy + e)^2, dy + e).
\end{aligned}$$

So f fixes βA and $\beta\tau B$ imply $d^2 = a$ and $2de = 0$. Since $\operatorname{char} \mathbf{k} \neq 2$, this implies $e = 0$.

Now f fixes pointwise the path of length 4 from $\tau\beta\tau B$ to $\mathrm{id}B$ if and only if $\tau f \tau = (dx, ay + c)$ fixes the path of length 4 from τB to $\beta\tau B$. By the above discussion, we obtain the conditions $a^2 = d$ and $c = 0$. Finally the only automorphisms that fix the path of length 6 from $\tau\beta\tau B$ to $\beta\tau A$ are the (ax, dy) with $a^2 = d$ and $d^2 = a$, which implies that a, d are cubic roots of unity. So this is a group of order at most 3, and we can conclude by Lemma 3.5. $\square$

In characteristic 2, the automorphism $g = (-y + x^2, x)$ is definitely not WPD, and the normal subgroup generated by any iterate of g is the whole group $\operatorname{Aut}(\mathbb{A}^2)$ (see [Lon16, p. 2024]). However we have:

Proposition 3.8. *Assume* $\operatorname{char} \mathbf{k} = 2$. *The element* $(y + x^3, x) \in \operatorname{Aut}(\mathbf{k}^2)$, *which is a hyperbolic isometry on the tree, satisfies the WPD property.*

Proof. The proof is similar, working now with the involution $\beta = (x + y^3, y)$. The computation for $\beta^{-1} f \beta$ becomes:

$$\begin{aligned}
\beta^{-1} f \beta &= (x + y^3, y) \circ (ax + c, dy + e) \circ (x + y^3, y) \\
&= (x + y^3, y) \circ (ax + ay^3 + c, dy + e) \\
&= (ax + ay^3 + c + d^3y^3 + d^2ey^2 + de^2y + e^3, dy + e).
\end{aligned}$$

We obtain $e = 0$ and $a = d^3$. By symmetry $c = 0$ and $d = a^3$, and again the subgroup of elements fixing pointwise this path of length 6 is finite. $\square$

Lecture 4. A glimpse at higher dimension

4.1. **Three conjectures.** The results discussed in the previous lectures become conjectures in higher dimension.

Conjecture 1. Any finite subgroup $\Gamma \subseteq \mathrm{Tame}(\mathbb{A}^n)$ is linearisable (here $\mathrm{char}\,\mathbf{k} = 0$).

Conjecture 2. For $n \geq 3$, $\mathrm{Tame}(\mathbb{A}^n) \subsetneq \mathrm{Aut}(\mathbb{A}^n)$.

Conjecture 3. The group $\mathrm{STame}(\mathbb{A}^n)$ is not simple.

4.2. **About the relation between Conjectures 1 and 2.** First we should mention that Conjecture 2 is settled in the case $n = 3$ and $\mathrm{char}\,\mathbf{k} = 0$, by a result of Shestakov and Umirbaev [SU04], see also [Kur10, Lam15] where the proof was simplified. All other cases ($n = 3$ in positive characteristic, or $n \geq 4$ in any characteristic) are open.

In relation with Conjecture 1 let us mention the following result:

Theorem 4.1 ([FMJ02]). *Let $\mathbf{k}$ be a field of characteristic zero. Then there exists a subgroup $\Gamma \in \mathrm{Aut}(\mathbb{A}^n)$ isomorphic to the symetric group S_3, which is not linearisable*

In view of this theorem, we observe that for $n \geq 4$ and $\mathrm{char}\,\mathbf{k} = 0$, Conjecture 1 implies Conjecture 2[1].

4.3. **About Conjecture 3.** Following the strategy explained during the third lecture, we need to find:

- A δ-hyperbolic space X with an action $\mathrm{Tame}(\mathbb{A}^n) \curvearrowright X$;
- A WPD element $g \in \mathrm{Tame}(\mathbb{A}^n)$ with respect to this action.

We describe a candidate to be such a hyperbolic space X, with a construction that imitate the Bass-Serre tree. For any $1 \leq r \leq n$, define $H_r \subset \mathrm{Tame}(\mathbb{A}^n)$ the subgroup of elements of the form $(f_1, \ldots, f_r, f_{r+1}, \ldots, f_n)$ with $(f_1, \ldots, f_r) \in \mathrm{GL}_r(\mathbf{k}) \rtimes \mathbf{k}^r$.

Example 4.2. For $n = 3$, we obtain the three subgroups:

$$H_3 = A_3 \text{ the affine group;}$$
$$H_2 = \left\{ \big(ax_1 + bx_2 + c, a'x_1 + b'x_2 + c', \alpha x_3 + P(x_1, x_2)\big) \right\};$$
$$H_1 = \left\{ \big(ax_1 + b, f_2, f_3\big) \right\};$$

Then we define a $(n-1)$-dimensional simplicial complex $\mathcal{C}_n$ as follows:

- vertices: left cosets $\mathrm{Tame}(\mathbb{A}^n)/H_i$, $i = 1, \ldots, n$;
- edges: left cosets $\mathrm{Tame}(\mathbb{A}^n)/(H_i \cap H_j)$;
- 2-facets: left cosets $\mathrm{Tame}(\mathbb{A}^n)/(H_i \cap H_j \cap H_k)$;
- ...
- $(n-1)$-facets: left cosets $\mathrm{Tame}(\mathbb{A}^n)/(H_1 \cap H_2 \cap \cdots \cap H_n)$.

The tame group acts by left translation on this simplicial complex. When $n = 2$, this is exactly the definition of the Bass-Serre tree. When $n = 3$ and $\mathrm{char}\,\mathbf{k} = 0$, one can show that the previously mentioned result by Shestakov-Umirbaev and Kuroda implies that $\mathcal{C}_3$ is simply connected ([Lam15]). This is the starting point for the proof of the following:

[1] It was pointed out by several participants of the conference that in fact Conjecture 1 and [FMJ02] would imply the existence of non-stably tame automorphisms. This might be an indication that Conjecture 1 is too optimistic...

Theorem 4.3 ([LP16]). *Assume* $\operatorname{char}\mathbf{k} = 0$. *The 2-dimensional simplicial complex* $\mathcal{C}_3$ *is hyperbolic and there exist WPD elements with respect to the action* $\mathrm{STame}(\mathbb{A}^3) \curvearrowright \mathcal{C}_3$. *In particular,* $\mathrm{STame}(\mathbb{A}^3)$ *is not simple.*

4.4. **About Conjecture 1.** To imitate the proof given in the first lecture, we need a fixed point theorem. Observe that the existence of circumcenter for a bounded set, which is the property we used on the Bass-Serre tree, also holds for Euclidean space $\mathbb{R}^n$, and more generally for complete CAT(0) spaces: see [BH99, p.179].

The complex $\mathcal{C}_3$ described above is not CAT(0), for any choice of Euclidean metric on the triangles. Here is another space X on which $\mathrm{Tame}(\mathbb{A}^n)$ acts, which might be a good candidate to be a CAT(0) space.

A degree on $\mathbf{k}[x_1, \dots, x_n]$ is a function $d\colon \mathbf{k}[x_1, \dots, x_n] \to \{-\infty\} \cup [0, +\infty[$ such that (for any P_1, P_2):

- $d(P) = -\infty$ if and only if $P = 0$;
- $d(P_1 + P_2) \leq \max d(P_1), d(P_2)$;
- $d(P_1 P_2) = d(P_1) + d(P_2)$.

We denote by $\mathcal{D}$ the space of such degree, up to a multiplicative constant ($d \sim \lambda d$ for $\lambda > 0$). The group $\mathrm{Tame}(\mathbb{A}^n)$ acts on $\mathcal{D}$ by precomposition:

$$f \cdot d(P) := d(P \circ f^{-1}).$$

We define the set Δ of monomial degree as follows. For any choice of weight $\alpha = (\alpha_1, \dots, \alpha_2) \in (\mathbb{R}_{>0})^n$, and any polynomial with support S:

$$P = \sum_{(i_1, \dots, i_n) \in S} c_{i_1, \dots, i_n} x_1^{i_1} \dots x_n^{i_n},$$

we define the degree d_α:

$$d_\alpha(P) := \max_{(i_1, \dots, i_n) \in S} i_1\alpha_1 + \cdots + i_n\alpha_n.$$

Since we only consider degrees up to a constant, we can normalize by adding the condition $\sum \alpha_i = 1$, and then Δ identifies with the (open) simplex in $\mathbb{R}^n$. Finally we define $X \subset \mathcal{D}$ as the orbit of Δ under the action of $\mathrm{Tame}(\mathbb{A}^n)$.

In the case $n = 2$, X is a tree, and in dimension n, it might be a CAT(0) space. In any case the stabilizers for the action $\mathrm{Tame}(\mathbb{A}^n) \curvearrowright X$ admit a semi-direct structure as in Lemma 1.2, so Conjecture 1 might not be completely out of reach.

Exercises

Exercise 1. Let $n \geq 1$. Find a bijective polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is not a polynomial automorphism. Then meditate on that for a minute or two.

Exercise 2. Let $\mathbf{k}$ be the field with two elements. Your first instinct might be that $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ is a finite group. However, the exercise is to show that $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ contains a free group over two generators!

Exercise 3.
1. Let T be a tree, and f, g two elliptic isometries of T without a common fixed point. Prove that $\langle f, g \rangle = \langle f \rangle * \langle g \rangle$.
2. Let $\mathbf{k}$ be your favourite field. Find an example of two involutions $f, g \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ such that you can apply the previous question to the action on the Bass-Serre tree.

Exercise 4. Let $E = \{(x, y) \mapsto (x+P(y), y) \mid P \in \mathbf{k}[X]\}$ be the elementary group, and let E_λ be the conjugate of E by $a_\lambda \colon (x, y) \to (\lambda x + y, x)$, where $\lambda \in \mathbf{k}$. Prove that the subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ generated by the E_λ is a free product (and observe that if the field $\mathbf{k}$ is uncountable, this is a free product over uncountably many factors...):

$$\langle E_\lambda \mid \lambda \in \mathbf{k} \rangle = \mathop{\ast}_{\lambda \in k} E_\lambda \subseteq \mathrm{Aut}(\mathbb{A}^2) \subseteq \mathrm{Bir}(\mathbb{P}^2).$$

Exercise 5. Let $\mathbf{k}$ be your favourite field again. Give an explicit example of a polynomial automorphism $g \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ with exactly 8 base points.

Exercise 6. We know by the classical theorem of Noether & Castelnuovo that any birational self-map g of $\mathbb{P}^2$ can be decomposed as a product of quadratic maps σ_i, with each σ_i with three proper base points (that is, no infinitely near base point). Moreover, any polynomial automorphism of $\mathbb{A}^2$ can be naturally extended as a birational map of $\mathbb{P}^2$. The question is: what is the minimal number of such quadratic maps that you will need to factorize the polynomial automorphism $g \colon (x, y) \mapsto (x + y^3, y)$?

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