RANDOM WALKS ON THE CREMONA GROUP
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1. Lecture 1: Introduction to random walks

1.1. **Basic examples.** Consider a drunkard who moves in a city by tossing coins to decide whether to go North, South, East or West: can he/she get back home?

It depends on the topography of the city.

Example 1: Squareville

In Squareville, blocks form a square grid. What is the probability of coming back to where you start? Let us first consider the easier case where your world is just a line, and you can only go in two directions: left or right.

Definition 1.1. *A random walk* (X_n) *on* X *is* recurrent *if for any* $x \in X$*, the probability that* $X_n = x$ *infinitely often is* 1*:*

$$\mathbb{P}_x(X_n = x \text{ i.o.}) = 1.$$

Otherwise, it is said to be transient*.*

Let X be a graph, and suppose we are given probabilities $p(x,y) \geq 0$ for any two vertices x, y of X, so that $\sum_{y \in X} p(x,y) = 1$ for any $x \in X$ (this setup is usually called a (time-homogeneous) *Markov chain* on X). Let us denote as $p^n(x,y)$ the probability of being at y after n steps starting from x.

Lemma 1.2. *Let* $m := \sum_{n \geq 1} p^n(x,x)$ *be the "expected number of visits to* x*". Then the random walk is recurrent iff* $m = \infty$*.*

Exercise. Prove the Lemma.

Now, what is the probability of going back to where you start after N steps? If N is odd, the probability is zero, but if $N = 2n$ you get

$$p^{2n}(0,0) = \frac{1}{2^{2n}} \binom{2n}{n} \qquad \text{(choose } 2n \text{ ways to go right)}$$

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Is $\sum_{n\geq 1} \frac{1}{2^{2n}} \binom{2n}{n}$ convergent?

Apply Stirling's Formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$\frac{1}{2^{2n}}\binom{2n}{n} \sim \frac{1}{2^{2n}} \frac{\sqrt{n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right)^2} = \frac{1}{\sqrt{n}}$$

$\therefore$ our RW is recurrent.

Now, let us go to Squareville, i.e. the case of the 2-dimensional grid. Now we have 4 directions to choose from. One can check that

$p^{2n}(0,0) = \frac{1}{4^{2n}}\binom{2n}{n}^2 \sim \frac{1}{n}$ (match left & right and match up and down)

hence the random walk is recurrent.

In general, one has the following.

Theorem 1.3 (Polya)**.** *The simple random walk on $\mathbb{Z}^d$ is recurrent iff $d = 1, 2$.*

That is, "a drunk man will get back home, but a drunk bird will get lost".

Exercise. Prove Polya's theorem for $d = 3$. Moreover, for the simple random walk on $\mathbb{Z}^d$ one can show that $p^{2n}(0,0) \approx n^{-\frac{d}{2}}$.

Example 2: Tree City

In Tree City, the map has the shape of a 4-valent tree.

Theorem 1.4. *The simple random walk on a 4-valent tree is transient.*

We want to look at d_n = "distance of the n^{th} step of the RW from the origin".

If you give the position of the n^{th} step, then finding d_{n+1} is as follows: if $d_n > 0$ then

$$d_{n+1} = \begin{cases} d_n + 1 & \text{with } \mathbb{P} = \frac{3}{4} \\ d_n - 1 & \text{with } \mathbb{P} = \frac{1}{4} \end{cases}$$

and if $d_n = 0$ then

$$d_{n+1} = d_n + 1$$

$\therefore \mathbb{E}(d_{n+1} - d_n) \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

$\therefore \mathbb{E}\left(\frac{d_n}{n}\right) \geq \frac{1}{2}$

If we know that $\lim_{n\to\infty}\frac{d_n}{n}$ exists almost surely and is constant, then

$$\lim_{n\to\infty}\frac{d_n}{n}\geq\frac{1}{2}$$

a.s. as $\mathbb{E}\left(\frac{d_n}{n}\right)\geq\frac{1}{2}\Rightarrow$ RW is transient.

Exercise. A *radially symmetric tree* of valence $(a_1, a_2, \ldots)$ is a tree where all vertices at distance n from the base point have exactly a_{n-1} children. Prove that the simple random walk on a radially symmetric tree $(a_1, a_2, \ldots)$ is transient iff

$$\sum_{n\geq 1}\frac{1}{a_1\cdot a_2\cdots a_n}<\infty.$$

1.2. **General setup.** Let G be a group and (X, d) a metric space. The *isometry group of X* is the group of elements which preserve distance:

$$\mathrm{Isom}(X)=\{f : X\to X : d(x,y)=d(f(x),f(y)) \text{ for all } x,y\in X\}$$

Definition 1.5. *A* group action *of G on X is a homomorphism*

$$\rho : G\to \mathit{Isom}(X).$$

The first example is the group of reals acting on themselves by translations: here $X=\mathbb{R}$, $G=\mathbb{R}$ and the action $\rho:\mathbb{R}\to\mathrm{Isom}(\mathbb{R})$ is given by $\rho(t): x\mapsto x+t$.

Let us now consider a probability measure μ on G. Then we can draw a sequence (g_n) of elements of G, independently and with distribution μ. The sequence (g_n) is called the sequence of *increments*, and we are interested in the products

$$w_n := g_1\ldots g_n.$$

The sequence (w_n) is called a *sample path* for the random walk.

More formally, the *space of increments* is the infinite product space $(G^{\mathbb{N}},\mu^{\mathbb{N}})$, and its elements are sequences (g_n) of elements of G. Then consider the map $G^{\mathbb{N}}\to G^{\mathbb{N}}$

$$(g_n)\mapsto(w_n)$$

defined as where $w_n=g_1g_2\ldots g_n$ and define as the *sample space* as the space $(\Omega,\mathbb{P})$ where $\Omega=G^{\mathbb{N}}$ and $\mathbb{P}$ is the pushforward of $\mu^{\mathbb{N}}$ by the above map. Elements of Ω are denoted as (w_n) and are called *sample paths.*

If you fix a basepoint $x\in X$ (where X is the metric space) you can look at the sequence $(w_n\cdot x)\subseteq X$.

Examples

(1) The group $G=\mathbb{Z}$ acts by translations on $X=\mathbb{R}$. Let us take $\mu=\frac{\delta_{+1}+\delta_{-1}}{2}$, i.e. one moves forward by 1 with probability $\frac{1}{2}$ and moves backward by 1 with probability $\frac{1}{2}$. This is the simple random walk on $\mathbb{Z}$ as previously discussed.

(2) The same holds for $G = \mathbb{R}^d$ or $G = \mathbb{Z}^d$ acting by translations on $X = \mathbb{R}^d$. For $d = 2$ and $\mu = \frac{1}{4}\left(\delta_{(1,0)} + \delta_{(-1,0)} + \delta_{(0,1)} + \delta_{(0,-1)}\right)$ you get the simple random walk on $\mathbb{Z}^2$ (i.e. the random walk on Squareville).

(3) X = 4-valent tree

$G = \mathbb{F}_2 = \{\text{reduced words in the alphabet } \{a, b, a^{-1}, b^{-1}\}\}$

Here, *reduced* means that there are no redundant pairs, i.e. there is no a after a^{-1}, no a^{-1} after a, no b after b^{-1}, and no b^{-1} after b.

$\mu = \frac{1}{4}(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}})$

More generally, given a finitely generated group we can define its *Cayley graph*:

Definition 1.6. *Given a group G finitely generated by a set S, the* Cayley graph $\Gamma = Cay(G, S)$ *is a graph whose vertices are the elements of G and there is an edge $g \to h$ $(g, h \in G)$ if $h = gs$ where $s \in S$.*

Often one takes $S = S^{-1}$, so that $\text{Cay}(G, S)$ is an undirected graph.

Definition 1.7. *Given a finitely generated group G and a finite generating set S, we define the* word length *of $g \in G$ as*

$$\|g\| := \min\{k \ : \ g = s_1 s_2 \dots s_k, s_i \in S \cup S^{-1}\}.$$

Moreover, we define the word metric *or* word distance *between $g, h \in G$ as*

$$d(g, h) := \|g^{-1}h\|.$$

Note that with this definition left-multiplication by an element of G is an isometry for the word metric: for any $h \in G$, $d(gh_1, gh_2) = d(h_1, h_2)$.

If we let $G = \mathbb{F}_2$, $S = \{a, b\}$, then $\text{Cay}(\mathbb{F}_2, S)$ is the 4-valent tree. On the other hand, if $G = \mathbb{Z}^2$, $S = \{(1, 0), (0, 1)\}$ then $\text{Cay}(\mathbb{Z}^2, S)$ is the square grid.

Note that $\text{Cay}(\mathbb{Z}^2, S)$ has loops, since e.g. $(0, 1) + (-1, 0) + (0, -1) + (1, 0) = (0, 0)$, while we do not have a corresponding loop in $\mathbb{F}_2$ as the element in $\mathbb{F}_2$ that would correspond to the loop is $ab^{-1}a^{-1}b$ and is not trivial.

(4) Consider $G = SL_2(\mathbb{R}) = \{A \in M_2 : \det A = 1\}$ which acts on the hyperbolic plane $X = \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d}.$$

The group G acts by isometries for the *hyperbolic metric* $ds = \frac{dx}{y}$. Let $A, B \in SL_2(\mathbb{R}), \mu = \frac{1}{4}(\delta_A + \delta_{A^{-1}} + \delta_B + \delta_{B^{-1}})$. The boundary of $\mathbb{H}$ is $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. As we will see later, random walks of this type converge almost surely to the boundary. Equivalently, one can also use the Poincaré disc model. The disc has a natural topological boundary, i.e. the circle.

Questions

(1) Does a typical sample path escape to ∞ or it comes back to the origin infinitely often?

(2) If it escapes, does it escape with "positive speed"?

Definition 1.8. *We define the* drift *or* speed *or* rate of escape *of the random walk to be the limit*

$$L := \lim_{n\to\infty} \frac{d(w_n x, x)}{n} \qquad \text{(if it exists).}$$

A measure μ on G has *finite first moment* on X if for some (equivalently, any) $x \in X$

$$\int_G d(x, gx)\ d\mu(g) < +\infty.$$

Lemma 1.9. *If μ has finite first moment, then there exists a constant $L \in \mathbb{R}$ such that for a.e. sample path*

$$\lim_{n\to\infty} \frac{d(w_n x, x)}{n} = L.$$

Proof. For any $x \in X$, the function $a(n,\omega) := d(x, w_n(\omega)x)$ is a subadditive cocycle, because

$$d(x, w_{n+m}(\omega)x) \le d(x, w_n(\omega)x) + d(w_n(\omega)x, w_{n+m}(\omega)x) =$$

and since w_n is an isometry

$$= d(x, w_n(\omega)x) + d(x, g_{n+1}\dots g_{n+m}x) = d(x, w_n(\omega)x) + d(x, w_m(T^n\omega)x)$$

where T is the shift on the space of increments, hence the claim follows by Kingman's subadditive ergodic theorem (Theorem 5.3). □

(3) Does a sample path track geodesics in X? How closely?

Recall that the *law of large numbers (LLN)* states that, if (X_i) are i.i.d. real-valued random variables with $\ell := \mathbb{E}[X_1] < \infty$, then almost surely

$$\frac{X_1 + X_2 + \cdots + X_n}{n} = \ell.$$

This can be rephrased by saying that there exists a geodesic $\gamma : [0, \infty) \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{d(X_1 + X_2 + \cdots + X_n, \gamma(\ell n))}{n} = 0.$$

In general, we say the random walk driven by (G, μ) has the *sublinear tracking* property if for a.e. $\omega \in \Omega$ there exists a unit-speed geodesic ray $\gamma : [0, \infty) \to X$ such that $\gamma(0) = x$ and

$$\lim_{n\to\infty} \frac{d(w_n x, \gamma(\ell n))}{n} = 0.$$

(4) If X has a topological boundary ∂X, does a typical sample path converge to ∂X?

Definition 1.10. *If so, define the* hitting measure ν *on* ∂X *as*

$$\nu(A) = \mathbb{P}\left(\lim_{n\to\infty} g_n x \in A\right)$$

for any $A \subset \partial X$ *.*

(5) What are the properties of hitting measure? Is it the same as the geometric measure? For example, is it the same as the Lebesgue measure?

(6) What is the boundary of a group? This leads to the notion of *Poisson boundary.*

1.3. **Statement of the results. The Cremona group.**

The *Cremona group* is the group of birational transformations of the complex projective plane $\mathbb{CP}^2$. That is, every element is given by

$$f([x : y : z]) := [P(x, y, z) : Q(x, y, z) : R(x, y, z)]$$

where P, Q, R are three homogenous polynomials of the same degree, without common factors. The common degree of P, Q, R is called the *degree* of f.

The Cremona group acts on the *Picard-Manin space*, which is given by taking the cohomology of all possible blowups of $\mathbb{P}^2$, and preserves a quadratic form of signature $(1, \infty)$. Hence, the Cremona group acts by isometries on a hyperboloid $\mathbb{H}_{\mathbb{P}^2}$ in the Picard-Manin space, which is indeed a non-proper δ-hyperbolic space. For details, see [CL13], [MT2].

Moreover, an element is WPD if it is not conjugate to a monomial map, i.e. a map which is in affine coordinates of the form $f(x, y) := (x^a y^b, x^c y^d)$ where $ad - bc \neq 0$.

Let μ be a probability measure on G with countable support. Let Γ_μ denote the semigroup generated by the support of μ, which we assume to be a group.

Definition 1.11. *The* dynamical degree *of a birational transformation* $f : X \dashrightarrow X$ *is defined as*

$$\lambda(f) = \lim_{n\to\infty} \deg(f^n)^{1/n}.$$

Note that $\lambda(f) = \lambda(gfg^{-1})$ is invariant by conjugacy.

Moreover, the degree is related to the displacement in the hyperbolic space $\mathbb{H}_{\mathbb{P}^2}$: in fact, if $x = [H] \in \mathbb{H}_{\mathbb{P}^2}$. As a consequence, the dynamical degree $\lambda(f)$ of a transformation f is related to its translation length $\tau(f)$ by the equation ([CL13], Remark 4.5):

$$\tau(f) = \lim_{n\to\infty} \frac{d(x, f^n x)}{n} = \lim_{n\to\infty} \frac{\cosh^{-1} \deg(f^n)}{n} = \log \lambda(f).$$

A Cremona transformation f is *loxodromic* if and only if $\lambda(f) > 1$.

1.4. **The Picard-Manin space.** If X is a smooth, projective, rational surface the group

$$N^1(X) := H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R})$$

is called the *Néron-Severi group.* Its elements are Cartier divisors on X modulo numerical equivalence. The intersection form defines an integral quadratic form on $N^1(X)$. We denote $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$.

If $f : X \to Y$ is a birational morphism, then the pullback map $f^\star : N^1(Y) \to N^1(X)$ is injective and preserves the intersection form, so $N^1(Y)_{\mathbb{R}}$ can be thought of as a subspace of $N^1(X)_{\mathbb{R}}$.

A *model* for $\mathbb{CP}^2$ is a smooth projective surface X with a birational morphism $X \to \mathbb{CP}^2$. We say that a model $\pi' : X' \to \mathbb{CP}^2$ dominates the model $\pi : X \to \mathbb{CP}^2$ if the induced birational map $\pi^{-1} \circ \pi' : X' \dashrightarrow X$ is a morphism. By considering the set $\mathcal{B}_X$ of all models which dominate X, one defines the space of *finite Picard-Manin classes* as the injective limit

$$\mathcal{Z}(X) := \lim_{X' \in B_X} N^1(X')_{\mathbb{R}}.$$

In order to find a basis for $\mathcal{Z}(X)$, one defines an equivalence relation on the set of pairs (p, Y) where Y is a model of X and p a point in Y, as follows. One declares $(p, Y) \sim (p', Y')$ if the induced birational map $Y \dashrightarrow Y'$ maps p to p' and is an isomorphism in a neighbourhood of p. We denote the quotient space as $\mathcal{V}_X$. Finally, the *Picard-Manin space* of X is the L^2-completion

$$\mathcal{Z}(X) := \left\{ [D] + \sum_{p \in \mathcal{V}_X} a_p [E_p] \ : \ [D] \in N^1(X)_{\mathbb{R}}, a_p \in \mathbb{R}, \sum a_p^2 < +\infty \right\}.$$

In this paper, we will only focus on the case $X = \mathbb{P}^2(\mathbb{C})$. Then the Néron-Severi group of $\mathbb{CP}^2$ is generated by the class $[H]$ of a line, with self-intersection +1. Thus, the Picard-Manin space is

$$\overline{\mathcal{Z}}(\mathbb{P}^2) := \left\{ a_0[H] + \sum_{p \in \mathcal{V}_{\mathbb{CP}^2}} a_p[E_p], \quad \sum_p a_p^2 < +\infty \right\}.$$

It is well-known that if one blows up a point in the plane, then the corresponding exceptional divisor has self-intersection -1, and intersection zero with divisors on the original surface.

Thus, the classes $[E_p]$ have self-intersection -1, are mutually orthogonal, and are orthogonal to $N^1(X)$. Hence, the space $\overline{\mathcal{Z}}(\mathbb{P}^2)$ is naturally equipped with a quadratic form of signature $(1, \infty)$, thus making it a Minkowski space of uncountably infinite dimension. Thus, just as classical hyperbolic space can be realized as one sheet of a hyperboloid inside a Minkowski space, inside the Picard-Manin space one defines

$$\mathbb{H}_{\mathbb{CP}^2} := \{[D] \in \overline{\mathcal{Z}} \ : \ [D]^2 = 1, [H] \cdot [D] > 0\}$$

which is one sheet of a two-sheeted hyperboloid. The restriction of the quadratic intersection form to $\mathbb{H}_{\mathbb{CP}^2}$ defines a Riemannian metric of constant curvature -1, thus making $\mathbb{H}_{\mathbb{CP}^2}$ into an infinite-dimensional hyperbolic space. More precisely, the induced distance dist satisfies the formula

$$\cosh \operatorname{dist}([D_1], [D_2]) = [D_1] \cdot [D_2].$$

Each birational map f acts on $\overline{\mathcal{Z}}$ by orthogonal transformations. To define the action, recall that for any rational map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ there exist a surface X and morphisms $\pi, \sigma : X \to \mathbb{CP}^2$ such that $f = \sigma \circ \pi^{-1}$. Then we define $f^\star = (\pi^\star)^{-1} \circ \sigma^\star$, and $f_\star = (f^{-1})^\star$. Moreover, $f_\star$ preserves the intersection form, hence it acts as an isometry of $\mathbb{H}_{\mathbb{P}^2}$: in other words, the map $f \mapsto f_\star$ is a group homomorphism

$$\text{Bir } \mathbb{CP}^2 \to \text{Isom}(\mathbb{H}_{\mathbb{P}^2})$$

hence one can apply to the Cremona group the theory of random walks on groups acting on non-proper δ-hyperbolic spaces.

Given a measure μ, we define as Γ_μ the semigroup generated by the support of μ. Moreover, the *limit set* $\Lambda_\mu := \overline{\Gamma_\mu x} \cap \partial X$ is the limit set of Γ_μ. We consider the group

$$E_\mu := \{g \in G \ : \ g\xi = \xi \text{ for all } \xi \in \Lambda_\mu\}$$

Since E_μ is normal in Γ_μ, conjugation gives a homomorphism $\Gamma_\mu \to \text{Aut } E_\mu$, and we denote as H_μ the image of this automorphism. If Γ_μ contains a WPD element, then H_μ is a finite group, and we denote as $k(\mu)$ the cardinality of H_μ. We call $k(\mu)$ the *characteristic index* of μ.

We call a measure μ *admissible* if Γ_μ is a countable non-elementary subgroup which contains at least one WPD element, and the support of μ has bounded degree.

Theorem 1.12 (Abundance of normal subgroups [MT2])**.** *Let μ be an admissible probability measure on the Cremona group $G = \mathrm{Bir}\ \mathbb{CP}^2$ and let $k = k(\mu)$. For any sample path $\omega = (w_n)$, consider the normal closure $N_n(\omega) := \langle\langle w_n^k \rangle\rangle$. Then we have:*

(1) *for almost every sample path ω, the sequence*
$$(N_1(\omega), N_2(\omega), \dots, N_n(\omega), \dots)$$
contains infinitely many different normal subgroups of $\mathrm{Bir}\ \mathbb{CP}^2$*.*

(2) *Let the* injectivity radius *of a subgroup $H < G$ be defined as*
$$\mathrm{inj}(H) := \inf_{f \in H \setminus \{1\}} \deg f.$$
Then, for any $R > 0$ the probability that $\mathrm{inj}(N_n) \geq R$ tends to 1 as $n \to \infty$;

(3) *The probability that the normal subgroup $N_n(\omega)$ is free satisfies*
$$\mathbb{P}(\langle\langle w_n^k \rangle\rangle \text{ is free}) \to 1$$
as $n \to \infty$.

(4) *The probability that the normal closure $\langle\langle w_n \rangle\rangle$ of w_n in G is free satisfies*
$$\mathbb{P}(\langle\langle w_n \rangle\rangle \text{ is free}) \to \frac{1}{k(\mu)}$$
as $n \to \infty$.

Theorem 1.13 (Exponential growth [MT2])**.** *Let μ be a countable non-elementary probability measure on the Cremona group with finite first moment. Then there exists $L > 0$ such that for almost every random product $w_n = g_1 \dots g_n$ of elements of the Cremona group we have the limit*
$$\lim_{n \to \infty} \frac{1}{n} \log \deg(w_n) = L.$$
Moreover, if μ is bounded then for almost every sample path we have
$$\lim_{n \to \infty} \frac{1}{n} \log \lambda(w_n) = L.$$

Theorem 1.14 (Poisson boundary [MT2])**.** *Let μ be a non-elementary probability measure on the Cremona group with finite entropy and finite logarithmic moment, and suppose that Γ_μ contains a WPD element. Then the Gromov boundary of the hyperboloid $\mathbb{H}_{\mathbb{P}^2}$ with the hitting measure is a model for the Poisson boundary of (G, μ).*

2. Lecture 2: Gromov hyperbolic spaces

Hyperbolic spaces.

Let (X,d) be a geodesic, metric space, and let $x_0 \in X$ be a basepoint.

Define the *Gromov product* of x, y as :

$$(x,y)_{x_0} := \frac{1}{2}\left(d(x_0,x) + d(x_0,y) - d(x,y)\right)$$

Definition 2.1. *The geodesic metric space X is δ-hyperbolic if geodesic triangles are δ-thin.*

We will use the notation

$$A = B + O(\delta)$$

to mean that there exists C, which depends only on δ, for which $|A - B| \le C$.

If X is δ-hyperbolic $\implies (x,y)_{x_0} = d(x_0,[x,y]) + O(\delta)$

Example 2.2. *The following are δ-hyperbolic spaces:*

$X = \mathbb{R}\checkmark$

$X =$ *tree* $\checkmark$

$G = \mathbb{F}_2, X = Cay(\mathbb{F}_2, S)\checkmark$

Definition 2.3. *A group G is* word hyperbolic *if there is a finite set S of generators such that $Cay(G,S)$ is δ-hyperbolic.*

Note: The fact that G is word hyperbolic does not depend on the choice of S (**Exercise**: why?).

Definition 2.4. *The action of G on X is* properly discontinuous *if for any $x \in X, \exists U \ni x$ such that $\#\{g \in G : gU \cap U \neq \varnothing\}$ is finite.*

Let us now consider

$\underset{\text{countable}}{G} < \mathrm{Isom}(\mathbb{H}^n)$ (where $\mathbb{H}^n$ is an n-dimensional hyperbolic space)

If the action of G on $\mathbb{H}^n$ is properly discontinuous and cocompact, then G is word hyperbolic. This is a special case of the following:

Lemma 2.5 (Švarc-Milnor)**.** *If G acts properly discontinuously and cocompactly on a δ-hyperbolic space, then G is word hyperbolic.*

Example 2.6. *Let $S =$ surface of genus $g \ge 2$. Then $\pi_1(S)$ is word hyperbolic. In fact, $\widetilde{S} \simeq \mathbb{D} \simeq \mathbb{H}^2$, and there is a regular $4g$-gon in $\mathbb{H}^2$ with angles $\frac{2\pi}{4g}$. Then $\mathbb{H}^2/G = S$ where $G = \langle a_1, b_1, \dots a_g, b_g \mid [a_1,b_1]\cdot\dots\cdot[a_g,b_g] = 1\rangle = \pi_1(S)$, and the action of G on $\mathbb{H}^2$ is properly discontinuous and cocompact.*

Note: if $G < \mathrm{Isom}(\mathbb{H}^3)$ which acts properly discontinuously but not cocompactly, then G need not be word hyperbolic (it may contain $\mathbb{Z}^2$). The same is true for $n \geq 3$ (how about $n = 2$?)

The mapping class group.

Let S be a closed, orientable, surface of genus $g \geq 2$. The *mapping class group* of S is

$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S) \big/ \text{isotopy}$$

and is a countable, finitely generated group.

Note that $Mod(S)$ is *not* word hyperbolic. In fact, if you fix a curve α on S, you can define a *Dehn twist* D_α around this curve. Then, if α, β are disjoint then $\langle D_\alpha, D_\beta \rangle = \mathbb{Z}^2$.

However, the mapping class group *does* act on a δ-hyperbolic space (but this space is *not* proper!).

If S is a topological surface of finite genus g, possibly with finitely many boundary components, then the *curve graph* $\mathcal{C}(S)$ is a graph whose vertices are isotopy classes of essential[1] , simple closed curves on S, and there is an edge $\alpha \to \beta$ if α and β have disjoint representatives.

Theorem 2.7 (Masur-Minsky [MM99])**.** *The curve graph is δ-hyperbolic.*

In fact, one can also define the *curve complex* by considering the simplicial complex where every k-simplex represents a set of k disjoint curves on the surface. The curve graph is the 1-skeleton of the curve complex, and it is quasi-isometric to it. Thus, for most purposes, it is enough to work with the curve graph.

Exercise. Consider a closed surface of genus $g \geq 2$. Prove that the curve graph has diameter ≥ 2. In fact, prove that it has infinite diameter.

Exercise. Consider a surface of genus g with n punctures. Figure out for what values of g, n the curve graph is empty, and for what values of n it is disconnected. In the latter case, think of how to modify the definition in order to obtain a connected space.

Outer automorphisms of the free group.

Let F_n be a free group of rank n, and let $G = \mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$ the group of outer automorphisms of F_n. Then for $n \geq 2$, G is not a word hyperbolic group but it acts on several non-proper hyperbolic spaces.

In particular, the *free factor complex* $\mathcal{FF}(F_n)$ is a countable graph whose vertices are conjugacy classes of proper free factors of F_n, and simplices are determined by chains of nested free factors. (A *free factor* is a subgroup

[1] Recall that a curve on a surface is *essential* if it is not homotopic to either a point or a boundary component.

H of F_n such that there exists another subgroup K so that $F_n = H \star K$). The graph $\mathcal{F}$ is hyperbolic by Bestvina-Feighn [BF14]. Another hyperbolic space on which $\text{Out}(F_n)$ acts is the *free splitting complex* $\mathcal{FS}(F_n)$.

An element of $\text{Out}(F_n)$ is *fully irreducible* if for all proper free factors F of F_n and all $k > 0$, $f^k(F)$ is not conjugate to F. An element is loxodromic on $\mathcal{FF}(F_n)$ if and only if it is fully irreducible, and all fully irreducible elements satisfy the WPD property.

Exercise. What is $\text{Out}(F_2)$? How about its corresponding free factor complex?

Right-angled Artin groups.

Let Γ be a finite graph. Define the right-angled Artin group $A(\Gamma)$ as

$$A(\Gamma) = \langle v \in V(\Gamma) : vw = wv \text{ if } (v, w) \in E(\Gamma) \rangle.$$

Right-angled Artin groups act on X = extension graph where vertices are conjugacy classes of elements of $V(\Gamma)$, and there is an edge between v^g and u^h iff they commute. Acylindricality of the action is due to Kim-Koberda.

Relatively hyperbolic groups.

Let H be a finitely generated subgroup of a finitely generated group G, and fix a finite generated set S for G. Then consider the Cayley graph $X = \text{Cay}(G, S)$, and construct a new graph $\widehat{X}$ as follows. For each left coset gH of H in G, add a vertex $v(gH)$ to $\widehat{X}$, and add an edge from $v(gH)$ to each vertex representing an element of gH.

The group G is *hyperbolic relative to* H if the coned-off space $\widehat{X}$ is a δ-hyperbolic space; $\widehat{X}$ is not proper as long as H is infinite.

The Cremona group.

The *Cremona group* is the group of birational transformations of the complex projective plane $\mathbb{CP}^2$. That is, every element is given by

$$f([x : y : z]) := [P(x, y, z) : Q(x, y, z) : R(x, y, z)]$$

where P, Q, R are three homogenous polynomials of the same degree, without common factors. The common degree of P, Q, R is called the *degree* of f.

The Cremona group acts on the *Picard-Manin space*, which is given by taking the cohomology of all possible blowups of $\mathbb{P}^2$, and preserves a quadratic form of signature $(1, \infty)$. Hence, the Cremona group acts by isometries on a hyperboloid $\mathbb{H}_{\mathbb{P}^2}$ in the Picard-Manin space, which is indeed a non-proper δ-hyperbolic space. For details, see [CL13], [MT2].

Moreover, an element is WPD if it is not conjugate to a monomial map, i.e. a map which is in affine coordinates of the form $f(x, y) := (x^a y^b, x^c y^d)$ where $ad - bc \neq 0$.

Exercise. Read the definition of the Picard-Manin space in the appendix.

Exercise. Find an example of two Cremona transformations f, g such that $\deg f \circ g \neq \deg f \cdot \deg g$.

2.1. **The Gromov boundary.**

Definition 2.8. *A metric space is* proper *if closed balls (i.e. sets of the form* $\{y \in X \ : \ d(x,y) \leq r\}$*) are compact.*

Let X be a δ-hyperbolic metric space. If X is proper then we can give the following definition of the boundary of X. Fix a base point $x_0 \in X$. We declare two geodesic rays γ_1, γ_2 based at x_0 to be *equivalent* if $\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$ and we denote this as $\gamma_1 \sim \gamma_2$.

We define the *Gromov boundary* of X as

$$\partial X := \{\gamma \text{ geodesic rays based at } x_0\} / \sim .$$

Example 2.9. *Examples of Gromov boundaries.*

- $X = \mathbb{R}$ *and* $\partial X = \{-\infty, +\infty\}$.
- $X = \text{ladder}$ *and* $\partial X = \{-\infty, +\infty\}$.

In general (if X is not necessarily proper) we define the boundary using sequences.

A sequence $(x_n) \subset X$ is a *Gromov sequence* if $\liminf_{m,n \to \infty} (x_n \cdot x_m)_{x_0} = \infty$. Two Gromov sequences $(x_n), (y_n)$ are *equivalent* if $\liminf_{n \to \infty} (x_n, y_n)_{x_0} = \infty$. In general we define the boundary of X as

$$\partial X := \{(x_n) \text{ Gromov sequence }\} / \sim$$

where $\sim$ denotes equivalence of Gromov sequences.

Theorem 2.10. ∂X *is a metric space.*

In order to define the metric, let $\eta, \xi \in \partial X$. Then $\eta = [x_n], \xi = [y_n]$ for two Gromov sequences (x_n), (y_n). Then one defines

$$(\eta \cdot \xi)_{x_0} := \sup_{x_n \to \eta, y_n \to \xi} \liminf_{m,n} (x_m \cdot y_n)_{x_0}.$$

Pick $\epsilon > 0$, and set $\rho(\xi, \eta) := e^{-\epsilon (\eta \cdot \xi)_{x_0}}$. This is not yet a metric (no triangle inequality). To get an actual metric, you need to take

$$d(\xi, \eta) := \inf \sum_{i=1}^{n-1} \rho(\xi_i, \xi_{i+1})$$

where the inf is taken along all finite chains $\xi = \xi_0, \xi_1, \cdots, \xi_{n-1}, \eta = \xi_n$.

Lemma 2.11. *$\exists C = C(\epsilon)$ such that*

$$C\rho(\xi,\eta) \leq d(\xi,\eta) \leq \rho(\xi,\eta) \quad \forall \xi,\eta \in \partial X.$$

If X is proper, then ∂X is a compact metric space, but if X is not proper, then ∂X need not be compact.

Example 2.12. *$X = \mathbb{N} \times \mathbb{R}^{\geq 0} \big/ (n,0) \sim (m,0)$. Then $\partial X \simeq \mathbb{N}$ is not compact.*

Acylindricality. In order to obtain the Poisson boundary in the non-proper case, we need a weak notion of properness for the action. The first condition is called acylindricality.

Definition 2.13 (Sela; Bowditch)**.** *The action of G on X is* acylindrical *if for every $K > 0$ there are numbers N, R such that $\forall x, y \in X$: if $d(x,y) \geq R$, then*

$$\#\{g : d(x,gx) \leq K \text{ and } d(y,gy) \leq K\} \leq N.$$

The WPD condition. Since acylindricality does not hold for the action of the Cremona group on the Picard-Manin space, we need to replace it with a weaker notion of properness introduced by Bestvina and Fujiwara in the context of mapping class groups, and known as the *weak proper discontinuity (WPD)*. Intuitively, an element is WPD if it acts properly on its axis. In formulas, an element $g \in G$ is *WPD* if for any $x \in X$ and any $K \geq 0$ there exists $N > 0$ such that

$$\#\{h \in G \ : \ d(x,hx) \leq K \text{ and } d(g^N x, hg^N x) \leq K\} < +\infty. \tag{1}$$

In other words, the finiteness condition is not required of all pairs of points in the space, but only of points along the axis of a given loxodromic element.

Classification of hyperbolic isometries. Let (X,d) be a geodesic, δ-hyperbolic, separable metric space, and let G be a countable group of isometries of X.

Definition 2.14. *Given an isometry g of X and $x \in X$, we define its* translation length *of g as*

$$\tau(g) := \lim_{n\to\infty} \frac{d(g^n x, x)}{n}$$

where the limit is independent of the choice of x (why?).

Lemma 2.15 (Classification of isometries of hyperbolic spaces)**.** *Let g be an isometry of a δ-hyperbolic metric space X (not necessarily proper). Then either:*

1. *g has bounded orbits. Then g is called* elliptic*.*
2. *g has unbounded orbits and $\tau(g) = 0$. Then g is called* parabolic*.*
3. *$\tau(g) > 0$. Then g is called* hyperbolic *or* loxodromic*, and has precisely two fixed points on ∂X, one attracting and one repelling.*

Given a measure μ on a countable group, its *support* is the set of elements $g \in G$ with $\mu(g) > 0$. We will denote as Γ_μ or $\mathrm{sgr}(\mu)$ the semigroup generated by the support of μ.

Definition 2.16. *Two loxodromic elements are* independent *if their fixed point sets are disjoint. A probability measure $\mu \in P(G)$ is* non-elementary *if sgr(μ) contains 2 independent hyperbolic elements.*

3. Lecture 3: the Poisson boundary

The well-known *Poisson representation formula* expresses a duality between bounded harmonic functions on the unit disk and bounded functions on its boundary circle. Indeed, bounded harmonic functions admit radial limit values almost surely, while integrating a boundary function against the Poisson kernel gives a harmonic function on the interior of the disk. This picture is deeply connected with the geometry of $SL_2(\mathbb{R})$; then in the 1960's Furstenberg and others extended this duality to more general groups.

The classical Poisson representation formula. If $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is essentially bounded, then define its *harmonic extension* as

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) P_r(t-\theta)\ dt \tag{2}$$

where

$$P_r(t) := \frac{1-r^2}{1+r^2-2r\cos t}$$

is the *Poisson kernel*. Then u satisfies $\Delta u = 0$. This establishes a correspondence

$$h^\infty(\mathbb{D}) \leftrightarrow L^\infty(S^1, \lambda) \tag{3}$$

where $h^\infty(\mathbb{D}) := \{u : \mathbb{D} \to \mathbb{R} \ :\ \Delta u = 0,\ \sup|u| < +\infty\}$. The direction $\leftarrow$ is the representation formula, while $\to$ is by taking radial limits (which exist a.e.).

This formula is deeply connected with the geometry of $SL_2(\mathbb{R})$. Indeed, let $a = re^{i\theta}$, and choose $g \in \operatorname{Aut} \mathbb{D}$ with $g(0) = a$. For instance, take

$$g(z) = \frac{a-z}{1-\overline{a}z}$$

so

$$|g'(z)| = \frac{1-|a|^2}{|1-\overline{a}z|^2}$$

and if $z = e^{it}$,

$$|g'(e^{it})| = \frac{1-r^2}{|1-re^{i(t-\theta)}|^2}$$

so (2) becomes

$$\begin{aligned} u(re^{i\theta}) &= \int_{-\pi}^{\pi} f(e^{it})|g'(e^{it})|\ \frac{dt}{2\pi} \\ &= \int_{\partial\mathbb{D}} f(\xi)\frac{dg\lambda}{d\lambda}(\xi)\ d\lambda(\xi) \\ &= \int_{\partial\mathbb{D}} f(\xi)\ dg\lambda(\xi) \end{aligned}$$

where λ is the Lebesgue measure.

Definition 3.1. *A function $f : G \to \mathbb{R}$ is μ-*harmonic *if it satisfies the mean value property with respect to averaging using μ; that is, if*

$$f(g) = \sum_{h \in G} f(gh)\ \mu(h) \qquad \forall g \in G.$$

We denote the space of bounded, μ-harmonic functions as $H^\infty(G,\mu)$.

Following Furstenberg [Fu1, Fu2], a measure space (M,ν) on which G acts is then a boundary if there is a duality between bounded, μ-harmonic functions on G and L^∞ functions on M.

Let $\sigma : \Omega = G^{\mathbb{N}} \to \Omega$ be the shift map in the space of sample paths, i.e. $(\sigma(w_n))_n = w_{n+1}$.

Definition 3.2. *A μ-boundary of (G,μ) is a measure space (B,ν) such that there exists a σ-invariant map $\pi : (\Omega,\mathbb{P}) \to (B,\nu)$, i.e. such that $\pi \circ \sigma = \pi$.*

Note that as a consequence, the measure ν is μ-stationary; that is,

$$\nu = \int_G g_\star \nu \ d\mu(g).$$

The most important example of μ-boundary for our purpose arises if we know that the random walk converges a.s. to some point in ∂X. Then we can set

$$\pi((w_n)) := \lim_{n\to\infty} w_n x$$

and ν as the hitting measure.

Definition 3.3. *Given a μ-boundary (B,ν), one defines the* Poisson transform *as*

$$\Phi(f)(g) := \int_B f \ dg\nu.$$

This is a G-equivariant map $\Phi : L^\infty(B,\nu) \to H^\infty(G,\mu)$.

Definition 3.4. *A G-space B with a μ-stationary measure ν is the* Poisson boundary *if the Poisson transform is a bijection (hence, an isometric isomorphism).*

Remark 3.5. *The Poisson boundary is trivial (i.e., a point) if and only if every bounded μ-harmonic function is constant.*

Other interpretations:

(1) The *universal property.* Every σ-invariant map $(\Omega,\mathbb{P}) \to (M,\lambda)$, where (M,λ) is a μ-boundary, factors through $(\Omega,\mathbb{P}) \to (B,\nu) \to (M,\lambda)$. Thus, the Poisson boundary is the maximal μ-boundary.

(2) The *stationary boundary.* Let us consider the relation $\sim$ on $\Omega = G^{\mathbb{N}}$ defined by $(w_n) \sim (w'_n)$ if there exists k, k' such that $w_{n+k} = w'_{n+k'}$ for all $n \geq 0$. Now, the *measurable quotient* of $(\Omega,\mathbb{P})$ by this relation is the Poisson boundary.

(3) The *space of ergodic components*. Let $\sigma : \Omega \to \Omega$ the shift in the space of sample paths, so that $(\sigma((w_n)))_n = w_{n+1}$. Then (B, ν) is the space of *ergodic components* of $(\Omega, \mathbb{P})$ with respect to σ.

Examples.

(1) If G is abelian, then the Poisson boundary is trivial for *any* measure (Blackwell).

For instance, for $(\mathbb{Z}, \frac{1}{2}(\delta_{+1} + \delta_{-1})$ the simple random walk, it is easy to see that any harmonic function $f : \mathbb{Z} \to \mathbb{R}$ satisfies

$$f(n) = \frac{f(n-1) + f(n+1)}{2}$$

which implies $f(n) = \alpha n + \beta$ for some $\alpha, \beta \in \mathbb{R}$, hence in order for it to be bounded we need $\alpha = 0$, hence f is constant.

(2) Same if G is nilpotent (Dynkin-Malyutov).

(3) If G is a semisimple Lie group, then the Poisson boundary is the quotient G/P of G by a minimal parabolic subgroup P (Furstenberg).

(4) G is non-amenable if and only if the Poisson boundary is non-trivial for *any* generating measure (Kaimanovich-Vershik; Rosenblatt).

(5) If G is a hyperbolic group, then for any (finite entropy + finite log moment) measure the Poisson boundary is the Gromov boundary $(\partial G, \nu)$ (Kaimanovich).

(6) If G is the mapping class group, then the Poisson boundary is the (Thurston-)boundary of Teichmüller space (Kaimanovich-Masur).

(7) If $G = Out(F_n)$, then the Poisson boundary is the boundary of Outer space (Horbez).

If ν is the hitting measure for a random walk, then a fundamental question in the field is whether the pair $(\partial X, \nu)$ equals indeed the Poisson boundary of the random walk (G, μ), i.e. if all harmonic functions on G can be obtained by integrating a bounded, measurable function on ∂X.

The main theorem of this section is the following identification of the Poisson boundary for groups of isometries of δ-hyperbolic spaces containing at least one WPD element.

Theorem 3.6 (Poisson boundary for WPD actions, [MT2])**.** *Let G be a countable group which acts by isometries on a δ-hyperbolic metric space (X, d), and let μ be a non-elementary probability measure on G with finite logarithmic moment and finite entropy. Suppose that there exists at least one WPD element h in the semigroup generated by the support of μ. Then the Gromov boundary of X with the hitting measure is a model for the Poisson boundary of the random walk (G, μ).*

3.1. **Entropy criterion.** Given a measure μ on G, define its *entropy* as

$$H(\mu) := -\int_G \log \mu(g)\, d\mu(g).$$

Moreover, for any n denote as μ^n the nth step convolution of μ, which is the distribution of the nth step of the random walk:

$$\mu^n(A) := \mathbb{P}(w_n \in A).$$

If $H(\mu) < +\infty$, we define the *asymptotic entropy* as the limit

$$h(\mu) := \lim_{n\to\infty} \frac{H(\mu^n)}{n}.$$

We have the fundamental *entropy criterion.*

Theorem 3.7 (Derriennic; Kaimanovich-Vershik)**.** *If $H(\mu) < +\infty$, then the Poisson boundary of (G,μ) is trivial if and only if*

$$h(\mu) = 0.$$

Exercise. *Compute $h(\mu)$ for the simple random walk on $\mathbb{Z}$.*

Example. We compute for any $0 \le k \le 2n$,

$$\mathbb{P}(w_{2n} = 2k - 2n) = \binom{2n}{k} 2^{-2n}$$

so

$$H(\mu^n) = -\sum_{k=0}^{2n} \binom{2n}{k} 2^{-2n} \log\left(\binom{2n}{k} 2^{-2n}\right)$$

Conditional random walks. Suppose that the random walk converges almost surely to ∂X, and ν is the hitting measure. Then for almost every $\xi \in \partial X$ we can define the *conditional random walk*, which is the process obtained by conditioning the random walk to hit ξ at infinity.

Consider the boundary map $(\Omega,\mathbb{P}) \to (\partial X,\nu)$. Then we can disintegrate with respect to this map, that is for a.e. $\xi \in \partial X$ we have a conditional measure $\mathbb{P}_\xi$ on Ω such that

$$\mathbb{P} = \int_{\partial X} \mathbb{P}_\xi \, d\nu(\xi).$$

Now, for any n let us consider the projection to the nth coordinate $\pi_n : (\Omega,\mathbb{P}) \to G$ given by $\pi_n((w_i)) = w_n$, and define

$$h(\mathbb{P}_\xi) = \lim_{n\to\infty} \frac{1}{n} H((\pi_n)_\star \mathbb{P}_\xi).$$

Note moreover that $\mathbb{P}_\xi$ is the measure on Ω induced by the stochastic process on G defined by transition probabilities (for $g,h \in G$)

$$p^\xi(g,h) = \mu(g^{-1}h)\frac{dh\nu}{dg\nu}(\xi).$$

We call this process the *conditional random walk* associated to ξ (even though it is not quite a random walk, as the transition probabilities are not G-invariant).

Theorem 3.8 (Entropy criterion, conditional version; Kaimanovich)**.** *Suppose $H(\mu) < +\infty$. Then a μ-boundary (B, ν) is the Poisson boundary if and only if the entropy $h(\mathbb{P}_\xi)$ of the conditional random walk associated to ξ satisfies*

$$h(\mathbb{P}_\xi) = 0$$

for ν-almost every $\xi \in \partial X$.

The strip criterion. Recall that a measure μ has *finite logarithmic moment* if $\int_G \log^+ d(x, gx)\, d\mu(g) < \infty$. Let us denote as

$$B_G(g) := \{h \in G \ : \ d(x, hx) \le d(x, gx)\}.$$

We shall use the following *strip criterion* by Kaimanovich. Let us denote as $\check{\mu}(g) := \mu(g^{-1})$ the *reverse measure*, and let $\check{\nu}$ be the hitting measure of the random walk driven by $\check{\mu}$.

Theorem 3.9 (Strip criterion)**.** *Let μ be a probability measure with finite entropy on G, and let $(\partial X, \nu)$ and $(\partial X, \check{\nu})$ be μ- and $\check{\mu}$-boundaries, respectively. If there exists a measurable G-equivariant map S assigning to almost every pair of points $(\alpha, \beta) \in \partial X \times \partial X$ a non-empty "strip" $S(\alpha, \beta) \subset G$, such that*

$$\frac{1}{n} \log |S(\alpha, \beta) \cap B_G(w_n)| \to 0 \qquad \text{as } n \to \infty,$$

for $\nu \times \check{\nu}$-almost every $(\alpha, \beta) \in \partial X \times \partial X$, then $(\partial X, \nu)$ and $(\partial X, \check{\nu})$ are the Poisson boundaries of the random walks (G, μ) and $(G, \check{\mu})$, respectively.

Sketch of the proof of the criterion. If w_n belongs to the strip $S(\alpha, \beta) \cap B_G(w_n)$, we have by Jensen's inequality

$$H((\pi_n)_\star \mathbb{P}_\xi) \le \log \#|S(\alpha, \beta) \cap B_G(w_n)|$$

hence, since strips grow subexponentially,

$$h(\mathbb{P}_\xi) = \lim_n \frac{H_\xi(\mu^n)}{n} = 0.$$

3.2. Proof of identification of Poisson boundary for WPD actions.

Lemma 3.10. *Let G be a group acting on a Gromov hyperbolic space X, and let h be a WPD element in G. Then there are functions $M: \mathbb{R}_{\ge 0} \to \mathbb{N}$ and $N: \mathbb{R}_{\ge 0} \to \mathbb{N}$ such that for any $x \in X$, any $K \ge 0$, and for any $f \in G$ one has*

$$\#|Stab_K(fx, fh^{M(K)}x)| \le N(K).$$

Proof. By definition, note that

$$\mathrm{Stab}_K(fx, fy) = f\mathrm{Stab}_K(x, y)f^{-1}$$

hence the cardinality

$$\#|\mathrm{Stab}_K(fx, fh^M x)| = \#|f(\mathrm{Stab}_K(x, h^M x))f^{-1}| = \#|\mathrm{Stab}_K(x, h^M x)|$$

is finite and independent of f, proving the claim. □

Elements of bounded geometry. Recall that we define a *shadow* as

$$S_x(y, R) := \{z \in X \;:\; d(x, [y, z]) \geq d(x, y) - R\}.$$

We use the following.

Proposition 3.11. *Let G be a non-elementary, countable group acting by isometries on a Gromov hyperbolic space X, and let μ be a non-elementary probability distribution on G. Then there is a number R_0 such that if $g, h \in G$ are group elements such that h and $h^{-1}g$ lie in the semigroup generated by the support of μ, then*

$$\nu(\overline{S_{hx}(gx, R_0)}) > 0,$$

where $\overline{A}$ denotes the closure in $X \cup \partial X$.

Now, for any pair $(\alpha, \beta) \in \partial X \times \partial X$, with $\alpha \neq \beta$, define the set of *bounded geometry elements* as

$$\mathcal{O}(\alpha, \beta) := \{g \in G \;:\; \alpha \in \overline{S_{gvx}(gx, K)} \text{ and } \beta \in \overline{S_{gx}(gvx, K)}\}.$$

Note that for any $g \in G$ we have $\mathcal{O}(g\alpha, g\beta) = g\mathcal{O}(\alpha, \beta)$. Moreover, we define the ball in the group with respect to the metric on X as

$$B(y, r) := \{g \in G \;:\; d(y, gx) \leq r\}$$

where $y \in X$ and $r \geq 0$.

The most crucial property of bounded geometry elements is that their number in a ball grows linearly with the radius of the ball.

Proposition 3.12. *There exists a constant C such that for any radius $r > 0$ and any pair of distinct boundary points $\alpha, \beta \in \partial X$ one has*

$$|B(x, r) \cap \mathcal{O}(\alpha, \beta)| \leq Cr.$$

This fact follows from the next lemma, which uses the WPD property in a crucial way.

Lemma 3.13. *For any $K \geq 0$ there exists a constant N such that*

$$|B(z, 4K) \cap \mathcal{O}(\alpha, \beta)| \leq N$$

for any $z \in X$ and any pair of distinct boundary points α, β.

Proof. Let us consider two elements g, h which belong to $\mathcal{O}(\alpha,\beta)\cap B(z,4K)$. Then if we let $f = hg^{-1}$, then

$$d(gx, fgx) \leq 8K. \tag{4}$$

Let γ be a quasigeodesic which joins α and β, and denote $S_1 := \overline{S_{gvx}(gx,K)}$, $S_2 := \overline{S_{gx}(gvx,K)}$. By construction, α belongs to both S_1 and fS_1 hence both α and $f\alpha$ belong to fS_1; similarly, β and $f\beta$ belong to fS_2. Hence, the two quasigeodesics γ and $f\gamma$ have endpoints in fS_1 and fS_2, hence they must fellow travel in their middle: more precisely, they must pass within distance $2K$ from both fgx and $y := fgvx$. Hence, if we call q a closest point to $f\gamma$ to fgx, we have $d(fgx,q) \leq 2K$. Moreover, if we call p a closest point on γ to y, and p' a closest point on $f\gamma$ to y, we have

$$d(p,p') \leq d(p,y) + d(y,p') \leq 4K$$

Combining this with eq. (4) we get

$$|d(gx,p) - d(fgx,p')| \leq 12K$$

Moreover, since f is an isometry we have $d(fgx, fp) = d(gx,p)$, hence

$$|d(fgx,fp) - d(fgx,p')| \leq 12K \tag{5}$$

Now, the points q, p' and fp both lie on the quasigeodesic $f\gamma$; let us assume that fp lies in between q and p', and draw a geodesic segment γ' between q and p', and let p'' be a closest point projection of fp to γ' (the case where p' lies between q and fp is completely analogous). By fellow traveling, we have $d(fp,p'') \leq L$. Then, since p', p'' and q lie on a geodesic, we have

$$d(p',p'') = |d(q,p') - d(q,p'')| \leq$$

and by using eq. (5)

$$\leq |d(fgx,p') - d(fgx,fp)| + d(fgx,q) + d(fgx,q) + d(fp,p'') \leq 12K + 2K + 2K + L$$

hence

$$d(fp,p') \leq 16K + 2L$$

and finally

$$d(y,fy) \leq d(y,p') + d(p',fp) + d(fp,fy) \leq 20K + 2L$$

Thus, if we choose K large enough so that $L \leq K$ we have $d(gvx, fgvx) = d(fgvx, f^2gvx) \leq 22K$ hence

$$f \in \mathrm{Stab}_{22K}(gx, gvx)$$

so by Lemma 3.10 there are only N possible choices of f, as claimed. □

Proof of Proposition 3.12. Let γ be a quasi-geodesic in X which joins α and β. By definition, if g belongs to $\mathcal{O}(\alpha,\beta)$, then gx lies within distance $\leq 2K$ of γ. Then one can pick points $(z_n)_{n\in\mathbb{Z}}$ along γ such that any point of γ is within distance $\leq 2K$ of some z_n. Then, any ball of radius r contains at most cr of such z_n, where c depends only on K and the quasigeodesic constant of γ. The claim then follows from Lemma 3.13. □

We now turn to the proof of Theorem 3.6. By Theorem 4.1, we know that since both μ and its reflected measure $\check{\mu}$ are non-elementary, both the forward random walk and the backward random walk converge almost surely to points on the boundary of X. Thus, one defines the two boundary maps $\partial_\pm : (G^{\mathbb{Z}}, \mu^{\mathbb{Z}}) \to \partial X$ as follows. Let $\omega = (g_n)_{n\in\mathbb{Z}}$ be a bi-infinite sequence of increments, and define

$$\partial_+(\omega) := \lim_{n\to\infty} g_1 \dots g_n x, \qquad \partial_-(\omega) := \lim_{n\to\infty} g_0^{-1} g_{-1}^{-1} \dots g_{-n}^{-1} x$$

the two endpoints of, respectively, the forward random walk and the backward random walk. Then define

$$\mathcal{O}(\omega) := \mathcal{O}(\partial_+(\omega), \partial_-(\omega))$$

the set of bounded geometry elements along the "geodesic" which joins $\partial_+(\omega)$ and $\partial_-(\omega)$. Note that, if $T : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ is the shift in the space of increments, we have

$$\mathcal{O}(T^n\omega) = \mathcal{O}(w_n^{-1}\partial_+, w_n^{-1}\partial_-) = w_n^{-1}\mathcal{O}(\omega).$$

Now we will show that for almost every bi-infinite sample path ω the set $\mathcal{O}(\omega)$ is non-empty and has at most linear growth. In fact, by definition of bounded geometry

$$\mathbb{P}(1 \in \mathcal{O}(\omega)) = p = \nu(\overline{S})\check{\nu}(\overline{S'}) > 0$$

where $S = S_{vx}(x, K)$ and $S' = S_x(vx, K)$, and their measures are positive by Proposition 3.11. Moreover, since the shift map T preserves the measure in the space of increments, we also have for any n

$$\mathbb{P}(w_n \in \mathcal{O}(\omega)) = \mathbb{P}(1 \in \mathcal{O}(T^n\omega)) = p > 0.$$

Thus, by the ergodic theorem, the number of times w_n belongs to $\mathcal{O}(\omega)$ grows almost surely linearly with n: namely, for a.e. ω

$$\lim_{n\to\infty} \frac{\#|\{1 \le i \le n \ : \ w_i \in \mathcal{O}(\omega)\}|}{n} = p > 0.$$

Hence the set $\mathcal{O}(\omega)$ is almost surely non-empty (in fact, it contains infinitely many elements). On the other hand, by Proposition 3.12 the set $\mathcal{O}(\omega)$ has at most linear growth, i.e. there exists $C > 0$ such that

$$\#|\mathcal{O}(\omega) \cap B_G(z, r)| \le Cr \qquad \forall r > 0. \tag{6}$$

The Poisson boundary result now follows from the strip criterion (Theorem 3.9). Let $P(G)$ denote the set of subsets of G. Then, we define the strip map $S : \partial X \times \partial X \to P(G)$ as $S(\alpha, \beta) := \mathcal{O}(\alpha, \beta)$; hence, by equation (6)

$$|S(\alpha, \beta)g \cap B_G(w_n)| \le Cd(w_n x, x).$$

Then, since μ has finite logarithmic moment, one has almost surely

$$\lim_{n\to\infty} \frac{1}{n} \log d(w_n x, x) \to 0$$

which verifies the criterion of Theorem 3.9, establishing that the Gromov boundary of X is a model for the Poisson boundary of the random walk.

4. Convergence to the hyperbolic boundary

The main results we are going to discuss in this lecture are the following.

Theorem 4.1 (Maher-Tiozzo [MT1]). *Let G be a countable group of isometries of a (separable) δ-hyperbolic metric space X, such that the semigroup generated by the support of μ is non-elementary. Then:*

(1) *For a.e. (w_n) and every $x \in X$,*
$$\lim_{n\to\infty} w_n x = \xi \in \partial X \text{ exists}.$$

(2) *There exists $L > 0$ s.t.*
$$\liminf_{n\to\infty} \frac{d(w_n x, x)}{n} = L > 0.$$
If μ has finite 1^{st} moment then
$$\lim_{n\to\infty} \frac{d(w_n x, x)}{n} = L > 0 \text{ exists a.s.}$$

(3) *For any $\epsilon > 0$ we have*
$$\mathbb{P}(\tau(w_n) \geq n(L-\epsilon)) \to 1$$
as $n \to \infty$.

As a corollary, the probability that w_n is loxodromic converges to 1 as $n \to \infty$. This generalizes results of Maher and Rivin about genericity of pseudo-Anosovs in the mapping class group.

In the rest of this section, we will sketch the proof of the first point in the previous theorem, namely the almost sure convergence to the boundary. Such a result is due to Furstenberg for semisimple Lie groups and to Kaimanovich for proper hyperbolic spaces. We will show how to deal with non-proper hyperbolic spaces.

4.1. The horofunction boundary.

Pick a base point $x_0 \in X$. For any $z \in X$ we define the function $\rho_z : X \to \mathbb{R}$:
$$\rho_z(x) := d(x,z) - d(x_0, z).$$

Then $\rho_z(x)$ is 1-Lipschitz and $\rho_z(x_0) = 0$.

Consider space $\mathrm{Lip}^1_{x_0}(X) = \{f : X \to \mathbb{R} \text{ s.t. } |f(x)-f(y)| \leq d(x,y), f(x_0) = 0\}$

with the topology of pointwise convergence. Let us consider the map $\rho : X \to \mathrm{Lip}^1_{x_0}(X)$ given by
$$z \mapsto \rho_z.$$

Definition 4.2. *The* horofunction compactification *of* (X,d) *is the closure* $\overline{X}^h := \overline{\rho(X)}$ *in* $Lip^1_{x_0}(X)$.

Proposition 4.3. *If* X *is separable, then the horofunction compactification* $\overline{X}^h$ *is a compact metrizable space.*

Proof. If one picks $h \in \mathrm{Lip}^1_{x_0}(X)$, then

$$|h(x)| \le |h(x) - h(x_0)| \le d(x, x_0)$$

hence $\mathrm{Lip}^1_{x_0}(X) \subset \otimes_{x\in X}[-d(x,x_0), d(x,x_0)]$

which is compact by Tychonoff's theorem. Since X is separable, then $C(X)$ is second countable, hence $\overline{X}^h$ is second countable. Thus $\overline{X}^h$ is compact, Hausdorff, and second countable, hence metrizable.

□

Exercise. Prove that $C(X)$ is second countable and Hausdorff if X is separable. Prove that a Hausdorff, second countable, compact topological space is metrizable.

Define the action of G on $\overline{X}^h$ as

$$g.h(z) := h(g^{-1}z) - h(g^{-1}z_0)$$

for all $g \in G$ and $h \in \overline{X}^h$.

The action of G on X extends to an action by homeomorphisms on $\overline{X}^h$.

Example 4.4. $X = \mathbb{R}$ *with the euclidean metric, and* $x_0 = 0$. *Then all horofunctions for* X *are either:*

- $\rho(x) = |x-p| - |p|$ *for some* $p \in \mathbb{R}$*;*

  or
- $\rho(x) = \pm x$.

hence $\partial^h X = \overline{X}^h \setminus X = \{-\infty, +\infty\}$.

Example 4.5. *In the hyperbolic plane* $X = \mathbb{H}^2$, *pick* $\xi \in \partial\mathbb{H}^2$ *and consider a geodesic ray* $\gamma : [0,\infty) \to \mathbb{H}^2$ *with* $\gamma(0) = x_0$ *and* $\lim_{t\to+\infty}\gamma(t) = \xi$. *Then if* $z_n := \gamma(n)$ *we get for any* $x \in \mathbb{H}^2$

$$h_\xi(x) = \lim_{z_n\to\xi} \rho_{z_n}(x) = \lim_{t\to\infty}(d(\gamma(t),x) - t)$$

is the usual definition of horofunction, and level sets are horoballs.

Example 4.6. *Let* X = *"infinite tree" defined as* $X = \mathbb{Z}\times\mathbb{R}^{\ge 0} \big/ (n,0)\sim(m,0)$.

Then the Gromov boundary is $\partial X = \mathbb{Z}$.

On the other hand, if $z_n = [(n,n)]$ *then in the horofunction compactification one has* $\lim_n \rho_{z_n} = \rho_{z_0}$*. If you think about it, this is related to the fact that the set of infinite horofunctions is* not *closed.*

Proposition 4.7 (Classification of horunctions)**.** *Let* h *be a horofunction in* $\overline{X}^h$*, and let* γ *be a geodesic in* X*. Then there is a point* p *on* γ *such that the restriction of* h *to* γ *is equal to exactly one of the following two functions, up to bounded additive error:*

- *either*
$$h(x) = h(p) + d(p,x) + O(\delta)$$
- *or*
$$h(x) = h(p) + d^+_\gamma(p,x) + O(\delta)$$
where d^+ *is the oriented distance along the geodesic, for some choice of orientation of* γ*.*

For any horofunction $h \in \overline{X}^h$, let us consider
$$\inf(h) := \inf_{y\in X} h(y).$$

Definition 4.8. *The set of* finite horofunctions *is the set*
$$\overline{X}^h_F := \{h \in \overline{X}^h \ : \ \inf h > -\infty\}$$
and the set of infinite horofunctions *is the set*
$$\overline{X}^h_\infty := \{h \in \overline{X}^h \ : \ \inf h = -\infty\}.$$

The key geometric lemma relating the geometry of the horofunction boundary and the Gromov boundary is the following.

Lemma 4.9. *For each base point* $x_0 \in X$*, each horofunction* $h \in \overline{X}^h$ *and each pair of points* $x, y \in X$ *the following inequality holds:*
$$\min\{-h(x), -h(y)\} \le (x,y)_{x_0} + O(\delta).$$

Proof. Let $z \in X$. Then one has, by the triangle inequality
$$(x\cdot z)_{x_0} = \frac{d_X(x_0,x) + d_X(x_0,z) - d_X(x,z)}{2},$$
which implies
$$(x\cdot z)_{x_0} \geqslant d_X(x_0,z) - d_X(x,z),$$
and by definition, the right hand side is equal to $-\rho_z(x)$, which gives
$$(x\cdot z)_{x_0} \ge -\rho_z(x).$$

Now, by δ-hyperbolicity one has

$$(x \cdot y)_{x_0} \geqslant \min\{(x \cdot z)_{x_0}, (y \cdot z)_{x_0}\} - \delta,$$

hence, by combining it with the previous estimate,

$$(x \cdot y)_{x_0} \geqslant \min\{-\rho_z(x), -\rho_z(y)\} - \delta.$$

Since every horofunction is the pointwise limit of functions of type ρ_z, the claim follows. □

This has the following consequence.

Lemma 4.10. *Let $(x_n) \subseteq X$ be a sequence of points, and $h \in \overline{X}^h$ a horofunction. If $h(x_n) \to -\infty$, then (x_n) converges in the Gromov boundary, and*

$$\lim_{n\to\infty} x_n \in \partial X$$

does not depend on choice of (x_n) .

Definition 4.11. *The* local minimum map $\varphi : \overline{X}^h \to X \cup \partial X$ *is defined as follows.*

- *If $h \in \overline{X}^h_F$, then define*

$$\varphi(h) := \{x \in X \ : \ h(x) \leq \inf h + 1\}$$

- *If $h \in \overline{X}^h_\infty$, then choose a sequence (y_n) with $h(y_n) \to -\infty$ and set*

$$\varphi(h) := \lim_{n\to\infty} y_n$$

  be the limit point in the Gromov boundary.

Lemma 4.12. *There exists K, which depends only on δ, such that for each finite horofunction h we have*

$$\operatorname{diam} \varphi(h) \leq K.$$

Proof. Let $x, y \in \phi(h)$, for some $h \in \overline{X}^h$, and consider the restriction of h along a geodesic segment from x to y. By Proposition 4.7, the restriction has at most one coarse local minimum: hence, since x and y are coarse local minima of h, the distance between x and y is universally bounded in terms of δ. □

Corollary 4.13. *The local minimum map $\varphi : \overline{X}^h \to X \cup \partial X$ is well-defined and G-equivariant.*

Note: φ is not continuous but $\varphi|_{\overline{X}^h_\infty}$ is continuous. For instance, in the "infinite tree" case of Example 4.6, if $z_n := (n, n)$ then $\rho_{z_n} \to \rho_{x_0}$ but $\phi(\rho_{z_n}) = z_n \not\to x_0$.

4.2. **Stationary measures.**

Definition 4.14. *Let μ be a probability measure on a group G, and let M be a metric space on which G acts by homeomorphisms. A probability measure ν on M is* μ-stationary *(or just* stationary*) if*

$$\int_G g\nu \; d\mu(g) = \nu.$$

The pair (M, ν) is then called a (G, μ)-space.

Problem: Since ∂X need not be compact, you may not be able to find a stationary measure in $P(\partial X)$. Trick: Consider the horofunction compactification (which is always compact and metrizable).

Lemma 4.15. $P(\overline{X}^h)$ *is compact, so it contains a μ-stationary measure.*

Proposition 4.16. *Let M be a compact metric space on which the countable group G acts continuously, and ν a μ-stationary Borel probability measure on M. Then for $\mathbb{P}$-a.e. sequence (w_n) the limit*

$$\nu_\omega := \lim_{n\to\infty} g_1 g_2 \dots g_n \nu$$

exists in the space $P(M)$ of probability measures on M.

Proof. Apply the martingale convergence theorem. □

Proposition 4.17. *Let μ be a non-elementary probability measure on G, and let ν be a μ-stationary measure on $\overline{X}^h$. Then*

$$\nu(\overline{X}^h_F) = 0.$$

4.3. **End of proof of convergence.**

Proposition 4.18. *For $\mathbb{P}$-a.e. sample path (w_n) there exists a subsequence $(\rho_{w_n x_0})$ which converges to a horofunction in $\overline{X}^h$.*

As a corollary, $\mathbb{P}$-a.e. sample path (w_n) there exists a subsequence $(w_{n_k} x_0)$ which converges to a point in the Gromov boundary ∂X.

Proposition 4.19. *Let $\widetilde{\nu}$ be a μ-stationary measure on ∂X, and suppose that the sequence $(w_n \widetilde{\nu})$ converges to a δ-measure δ_λ on ∂X. Then $(w_n x_0)$ converges to λ in $X \cup \partial X$.*

Proof of Theorem 4.1 (1). Let $\nu \in P(\overline{X}^h)$ a μ-stationary measure, and denote $\widetilde{\nu} := \phi_* \nu \in P(\partial X)$. By the martingale convergence theorem, for a.e. w_n we have $(w_n)_* \nu \longrightarrow \nu_w \in P(\overline{X}^h)$. Then by pushing forward by φ_* one gets $(w_n)_*(\widetilde{\nu}) \longrightarrow (\widetilde{\nu})_w \in P(\partial X)$. By δ-hyperbolicity, if $w_n x \longrightarrow \xi \in \partial X$ then $w_n \widetilde{\nu} \longrightarrow \delta_\xi$. The sequence $w_n x$ has at least one limit point ξ in ∂X, and for each limit point ξ , $w_{n_k} \overline{\nu} \longrightarrow \delta_\xi$, but there can be only one limit point, as $\lim_{n\to\infty} w_n \overline{\nu}$ exists. □

5. Appendix

5.1. **Ergodic theorems.** In order to talk about asymptotic properties of random walks we need to have tools which assure us of the existence of various averages. Ergodic theorems provide such averages.

The most classical ergodic theorem is the *pointwise ergodic theorem* of Birkhoff.

Definition 5.1. *A transformation $T : (X, \mu) \to (X, \mu)$ of a measure space (X, μ) is* measure-preserving *if $\mu(A) = \mu(T^{-1}(A))$ for any measurable set A.*

Theorem 5.2 (Birkhoff)**.** *Let (X, μ) be a measure space with $\mu(X) = 1$, $f : X \to \mathbb{R}$ be a measurable function, and $T : X \to X$ a measure-preserving transformation. If $f \in L^1(X, \mu)$, then the limit*

$$\overline{f}(x) := \lim_{n\to\infty} \frac{f(x) + f(T(x)) + \cdots + f(T^n(x))}{n}$$

exists for μ-almost every $x \in X$.

We will derive Birkhoff's theorem from the more general *subadditive ergodic theorem* of Kingman.

A function $a : \mathbb{N} \times X \to \mathbb{R}$ is a *subadditive cocycle* if

$$a(n + m, x) \le a(n, x) + a(m, T^n x) \qquad \text{for any } n, m \in \mathbb{N}, x \in X.$$

The cocycle is *integrable* if for any n, the function $a(n, \cdot)$ belongs to $L^1(X, \mu)$. Assume moreover that

$$\inf \frac{1}{n} \int_X a(n, x)\ d\mu(x) > -\infty.$$

Then the following theorem holds.

Theorem 5.3 (Kingman)**.** *Under the previous assumptions, there is an integrable, a.t. T-invariant function $\overline{a}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} a(n, x) = \overline{a}(x)$$

for almost every $x \in X$. Moreover, the convergence also takes place in L^1.

Proof of Birhkoff's theorem. We now see that Birkhoff's ergodic theorem follows as a corollary. In fact, if we let $a(n, x) := \sum_{k=0}^{n-1} f(T^k x)$ then

$$a(n + m, x) = \sum_{k=0}^{n+m-1} f(T^k x) = a(n, x) + a(m, T^n x)$$

is actually an *additive cocycle*, thus it is subadditive. □

5.2. **Conditional expectation.**

Theorem 5.4 (Radon-Nikodym)**.** *Let $(X, \mathcal{A}, \mu)$ be a probability space, and let ν be a probability measure on $\mathcal{A}$ which is absolutely continuous with respect to μ. Then there exists a function $f \in L^1(X, \mathcal{A}, \mu)$ such that*

$$\nu(A) = \int_A f \, d\mu.$$

Let us now consider a probability space $(X, \mathcal{A}, \mu)$, and $\mathcal{B} \subset \mathcal{A}$ a smaller σ-algebra. Then the *conditional expectation* of a function $f \in L^1(X, \mathcal{A}, \mu)$ with respect to $\mathcal{B}$ is a function $g \in L^1(X, \mathcal{B}, \mu)$ (in particular, g is $\mathcal{B}$-measurable) such that

$$\int_B f \, d\mu = \int_B g \, d\mu \qquad \text{for all } B \in \mathcal{B}.$$

Usually one denotes such a g as $\mathbb{E}(f \mid \mathcal{B})$.

Proof. To prove the existence of conditional expectation, one considers the measure ν on $\mathcal{B}$ defined as

$$\nu(B) := \int_B f \, d\mu.$$

Then, by the Radon-Nikodym theorem, the measure ν is abs.cont. with respect to μ, hence the Radon-Nikodym derivative $g = \frac{d\nu}{d\mu}$ is a function in $L^1(X, \mathcal{B}, \mu)$ which satisfies

$$\int_B f \, d\mu = \int_B g \, d\mu \qquad \text{for all } B \in \mathcal{B}$$

as claimed. The uniqueness follows from the fact that two functions whose integrals agree on any set of the σ-algebra must agree almost everywhere (check this!). □

Given a set $\mathcal{F}$ of functions, we denote as $\sigma(\mathcal{F})$ the smallest σ-algebra for which all functions are measurable (i.e. the σ-algebra generated by all preimages of measurable sets) and denote

$$\mathbb{E}(f \mid \mathcal{F})$$

the conditional expectation of f with respect to $\sigma(\mathcal{F})$.

This has the intuitive interpretation of the expectation of f *once you know the values of the variables* $\mathcal{F}$. Consider the toin coss $(X_n) : \{0,1\}^{\mathbb{N}} \to \{+1, -1\}$ where each X_n is i.i.d. and is $+1$ with prob. $1/2$, and -1 with prob. $1/2$. Then the σ-algebra $\sigma(X_1, \dots, X_n)$ is the set of functions on Ω which *only depend* on the first n coordinates. Note that:

(1) If f is independent of $\mathcal{F}$, then $\mathbb{E}(f \mid \mathcal{F}) = \mathbb{E}(f)$.

(2) If f is $\mathcal{F}$-measurable, then $\mathbb{E}(f \mid \mathcal{F}) = f$.

Note that in particular if $T : X \to X$ is a measure-preserving system, then one can define the σ-algebra $\mathcal{F}_T$ of all T-invariant sets, and then the conditional expectation $\mathbb{E}(f \mid \mathcal{F}_T) = \overline{f}$ is precisely the time average given by the ergodic theorem.

5.3. **Martingales.**

Definition 5.5. *A sequence $(X_n) : \Omega \to \mathbb{R}$ of measurable functions is a* martingale *if for any n we have*

$$\mathbb{E}(X_{n+1} \mid X_1, \dots, X_n) = X_n.$$

A way to think of a martingale is that X_n is the payoff after n steps in a fair (i.e., zero-sum) game. That is, once you know the outcomes of the first n draws, the expected value of the payoff at step X_{n+1} is the previous payoff X_n.

In the example of the toin coss, $Y_n := X_1 + \cdots + X_n$ is a martingale. In fact

$$\mathbb{E}(Y_{n+1} \mid Y_1, \dots, Y_n) = \mathbb{E}(Y_n + X_{n+1} \mid Y_1, \dots, Y_n) = Y_n + \mathbb{E}(X_{n+1}) = Y_n.$$

5.4. **A bit of functional analysis.** Let M be a compact metric space. Then $P(M)$ is the space of probability measures on M. We define convergence in the space of measure by saying that (ν_n) converges to ν in the *weak-* topology* if for any continuous $f : M \to \mathbb{R}$, we have

$$\int f \, d\nu_n \to \int f \, d\nu.$$

Theorem 5.6 (Riesz-Markov-Kakutani)**.** *The dual to the space $C(M)$ of continuous functions on the compact metric space M is the space of signed Borel measures on M.*

Theorem 5.7. *The space $P(M)$ is compact with respect to the weak-$\star$ topology.*

Proof. It is a closed subspace in the unit ball of the dual space of $C(M)$, in particular

$$P(M) := \{\varphi \in C(M)^\star \ : \ \varphi \geq 0, \varphi(1) = 1\}.$$

We say a functional is *positive* if $\varphi(f) \geq 0$ whenever f is a non-negative function. □

Theorem 5.8 (Alaoglu-Banach)**.** *Let V be a normed vector space. Then the unit ball in its dual $V^\star$ is compact with respect to the weak-$\star$ topology.*

Proof. Recall that if $\varphi \in V^\star$ belongs to the unit ball, then $|\varphi(v)| \leq \|v\|$ for any $v \in V$. Denote as B the unit ball in V, and $B^\star$ the unit ball in the dual, and consider the map $F : B^\star \to [-1, 1]^B$ defined as

$$F(\varphi) := (\phi(v))_{v \in B}.$$

The map is injective as a functional is determined by its values on the unit ball. Moreover, by Tychonoff's theorem the cube $[-1,1]^B$ is compact as it is a product of compact spaces, and the image $F(B^\star)$ is closed in $[-1,1]^B$, hence it is also compact. □

5.5. **Stationary measures.** A metric space M is called a G-space if there exists an action of G on M by homeomorphisms, i.e. a homomorphism $\rho : G \to Homeo(M)$.

Lemma 5.9. *Let M be a compact, metric G-space, and μ a probability measure on G. Then there exists a μ-stationary measure ν on M.*

Lemma 5.10. *Let ν be a μ-stationary measure on a (G,μ)-space M. Then for any $f \in L^1(M,\nu)$, the sequence*

$$X_n := \int_M f \ d(g_n \nu)$$

is a martingale.

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