

EXAMPLES OF INFINITE COVOLUME SUBGROUPS OF $\mathrm{PSL}(2, \mathbb{R})^r$ WITH BIG LIMIT SETS.

SLAVYANA GENINSKA

ABSTRACT. We provide examples of finitely generated infinite covolume subgroups of $\mathrm{PSL}(2, \mathbb{R})^r$ with a "big" limit set, e.g. that contains an open subset of the geometric boundary. They are given by the so called semi-arithmetic Fuchsian groups admitting modular embeddings.

0. INTRODUCTION

While lattices are studied very well, only little is known about discrete subgroups of infinite covolume of semi-simple Lie groups. The main class of examples are Schottky groups. In this paper we consider another class of examples for the semi-simple Lie group $\mathrm{PSL}(2, \mathbb{R})^r$ with $r \geq 2$.

We provide examples of finitely generated infinite covolume subgroups of $\mathrm{PSL}(2, \mathbb{R})^r$ with a "big" limit set, e.g. that contains an open subset of the geometric boundary. They are given by the so called semi-arithmetic Fuchsian groups admitting modular embeddings.

The semi-arithmetic Fuchsian groups constitute a specific class of Fuchsian groups which can be embedded up to commensurability in arithmetic subgroups of $\mathrm{PSL}(2, \mathbb{R})^r$ (see Schmutz Schaller and Wolfart [16]). These embeddings are of infinite covolume in $\mathrm{PSL}(2, \mathbb{R})^r$. A trivial example is the group $\mathrm{PSL}(2, \mathbb{Z})$ that can be embedded diagonally in any Hilbert modular group. Further examples are the other arithmetic Fuchsian groups, the triangle Fuchsian groups and the Veech groups of Veech surfaces.

It is a general question if certain classes of Fuchsian groups can be characterized by geometric means. In [8] was shown that a nonelementary finitely generated subgroup of arithmetic groups in $\mathrm{PSL}(2, \mathbb{R})^r$ with $r \geq 2$ has the smallest possible limit set if and only if its projection to one factor is a subgroup of an arithmetic Fuchsian group. This means that in this case the limit set gives information about the arithmetic properties of the group. In this article we show that the groups admitting modular embeddings that are not arithmetic are in a sense really very different to the arithmetic Fuchsian groups. It is

still an open question what are the limit sets of the embeddings of the other semi-arithmetic Fuchsian groups in arithmetic subgroups of $\mathrm{PSL}(2, \mathbb{R})^r$.

This article is organized as follows. Section 1 provides a detailed description of the geometric boundary of $(\mathbb{H}^2)^r$, which is the set of equivalence classes of asymptotic geodesic rays. We introduce the notion of the limit set as the part of the orbit closure $\overline{\Gamma(x)}$ in the geometric boundary where x is an arbitrary point in $(\mathbb{H}^2)^r$. We also state a natural structure theorem for the regular limit set \mathcal{L}_Γ^{reg} of discrete nonelementary groups Γ due to Link: \mathcal{L}_Γ^{reg} is the product of the Furstenberg limit set F_Γ and the projective limit set P_Γ .

In Section 2 we prove the following criterion for Zariski dense subgroups of arithmetic groups in $\mathrm{PSL}(2, \mathbb{R})^r$ (Corollary 2.2 in the text). For a group S we denote by $S^{(2)}$ its subgroup generated by the set $\{g^2 \mid g \in S\}$.

Proposition. *Let Γ be a nonelementary subgroup of an irreducible arithmetic group Δ in $\mathrm{PSL}(2, \mathbb{R})^r$. Then Γ is Zariski dense in $\mathrm{PSL}(2, \mathbb{R})^r$ if and only if the fields generated by $\mathrm{Tr}(p_i(\Gamma^{(2)}))$ and $\mathrm{Tr}(p_i(\Delta^{(2)}))$ are equal for one and hence for all $i \in \{1, \dots, r\}$, where p_i denotes the projection to the i -th factor.*

In Section 3, we provide examples of “small” groups with a “big” limit set, namely groups of infinite covolume for which the projection of the regular limit set into the Furstenberg boundary is the whole Furstenberg boundary. We define semi-arithmetic groups admitting a modular embedding. A cofinite Fuchsian group S that is commensurable to a subgroup of the projection to the first factor of an irreducible arithmetic group Δ in $\mathrm{PSL}(2, \mathbb{R})^r$ is said to have a *modular embedding* if for the natural embedding $f : S \rightarrow \Delta$ there exists a holomorphic embedding $F : \mathbb{H}^2 \rightarrow (\mathbb{H}^2)^r$ with

$$F(Tz) = f(T)F(z), \quad \text{for all } T \in S \quad \text{and all } z \in \mathbb{H}^2.$$

Examples of semi-arithmetic groups admitting modular embeddings are Fuchsian triangle groups (see Cohen and Wolfart [4]).

The next theorem is the main result of this paper. It is a compilation of Lemma 3.2, Theorem 3.4 and Theorem 3.8 in the text.

Theorem. *Let Γ be a subgroup of an irreducible arithmetic group in $\mathrm{PSL}(2, \mathbb{R})^r$ with $r \geq 2$ such that $p_j(\Gamma)$ is a semi-arithmetic Fuchsian group admitting a modular embedding and r is the smallest power for which $p_j(\Gamma)$ has a modular embedding in an irreducible arithmetic subgroup of $\mathrm{PSL}(2, \mathbb{R})^r$. Then*

- (i) the Furstenberg limit set F_Γ is the whole Furstenberg boundary $(\partial\mathbb{H}^2)^r$,
- (ii) the limit set \mathcal{L}_Γ contains a subset homeomorphic to D^{2r-1} (the $(2r-1)$ -dimensional ball).

In Section 4 we consider the case with parabolic elements. In the remark after Proposition 4.9 we determine the exact limit set of the embeddings in $\mathrm{PSL}(2, \mathbb{R})^2$ of the triangle groups with signature $(2, 5, \infty)$ and $(5, \infty, \infty)$.

I would like to thank the advisor of my thesis Enrico Leuzinger. I would also like to thank Yves Benoist and George Tomanov for the discussion about the algebraic subgroups of $\mathrm{SL}(2, \mathbb{R})^r$ and to the anonymous referee for the useful comments.

1. BACKGROUND

In this section we provide some basic facts and notations that are needed in the rest of this paper.

We will change freely between matrices in $\mathrm{SL}(2, \mathbb{R})$ and their action as fractional linear transformations, namely as elements in $\mathrm{PSL}(2, \mathbb{R})$.

For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{R})$ we set $\mathrm{tr}(g) = |a + d|$.

For a subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ we call

$$\mathrm{Tr}(\Gamma) = \{\mathrm{tr}(g) \mid g \in \Gamma\}$$

the *trace set* of Γ .

The *translation length* $\ell(g)$ of a hyperbolic isometry g is the distance between a point x on the geodesic fixed by g and its image $g(x)$ under g . If g is elliptic, parabolic or the identity, we define $\ell(g) := 0$.

The following notion of “smallness” for subgroups Γ of $\mathrm{PSL}(2, \mathbb{R})$ is important in the subsequent discussion. The group Γ is *elementary* if it has a finite orbit in its action on $\mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$. Otherwise it is said to be *nonelementary*. Every nonelementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ contains infinitely many hyperbolic elements, no two of which have a common fixed point (see Theorem 5.1.3 in the book of Beardon [1]).

A Schottky group is a finitely generated free subgroup of $\mathrm{PSL}(2, \mathbb{R})$ that contains only hyperbolic isometries except for the identity. We will mainly deal with two-generated Schottky groups.

For each two hyperbolic isometries without common fixed points, we can find powers of them that generate a Schottky group. This means that every nonelementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ has a subgroup that is a Schottky group. A proof of this fact can be found in [7].

A Schottky group contains isometries without common fixed points because it is nonelementary.

1.1. The geometric boundary of $(\mathbb{H}^2)^r$. For $i = 1, \dots, r$, we denote by $p_i : (\mathbb{H}^2)^r \rightarrow \mathbb{H}^2$, $(z_1, \dots, z_r) \mapsto z_i$ the i -th projection of $(\mathbb{H}^2)^r$ into \mathbb{H}^2 . The curve $\gamma : [0, \infty) \rightarrow (\mathbb{H}^2)^r$ is a geodesic ray in $(\mathbb{H}^2)^r$ if and only if $p_i \circ \gamma$ is a geodesic ray or a point in \mathbb{H}^2 for each $i = 1, \dots, r$. A geodesic γ is *regular* if $p_i \circ \gamma$ is a nonconstant geodesic in \mathbb{H}^2 for each $i = 1, \dots, r$.

Two unit speed geodesic rays γ and δ in $(\mathbb{H}^2)^r$ are said to be asymptotic if there exists a positive number c such that $d(\gamma(t), \delta(t)) \leq c$ for all $t \geq 0$. This is an equivalence relation on the unit speed geodesic rays of $(\mathbb{H}^2)^r$. For any unit speed geodesic γ of $(\mathbb{H}^2)^r$ we denote by $\gamma(+\infty)$ the equivalence class of its positive ray.

We denote by $\partial((\mathbb{H}^2)^r)$ the set of all equivalence classes of unit speed geodesic rays of $(\mathbb{H}^2)^r$. We call $\partial((\mathbb{H}^2)^r)$ the *geometric boundary* of $(\mathbb{H}^2)^r$. The *regular boundary* $\partial((\mathbb{H}^2)^r)_{reg}$ of $(\mathbb{H}^2)^r$ consists of the equivalence classes of regular geodesics.

The geometric boundary $\partial((\mathbb{H}^2)^r)$ with the cone topology is homeomorphic to the unit tangent sphere of a point in $(\mathbb{H}^2)^r$ (see Eberlein [6], 1.7). (For example $\partial\mathbb{H}^2$ is homeomorphic to S^1 .) The homeomorphism is given by the fact that for each point x_0 and each unit speed geodesic ray γ in $(\mathbb{H}^2)^r$ there exists a unique unit speed geodesic ray δ with $\delta(0) = x_0$ which is asymptotic to γ .

The group $\mathrm{PSL}(2, \mathbb{R})^r$ acts on $(\mathbb{H}^2)^r$ by isometries in the following way. For $g = (g_1, \dots, g_r) \in \mathrm{PSL}(2, \mathbb{R})^r$

$$g : (\mathbb{H}^2)^r \rightarrow (\mathbb{H}^2)^r, \quad (z_1, \dots, z_r) \mapsto (g_1 z_1, \dots, g_r z_r),$$

where $z_i \mapsto g_i z_i$ is the usual action given by linear fractional transformation, $i = 1, \dots, r$.

The action of $\mathrm{PSL}(2, \mathbb{R})^r$ can be extended naturally to $\partial((\mathbb{H}^2)^r)$. Let g be in $\mathrm{PSL}(2, \mathbb{R})^r$ and ξ be a point in the boundary $\partial((\mathbb{H}^2)^r)$. If γ is a representative of ξ , then $g(\xi)$ is the equivalence class of the geodesic ray $g \circ \gamma$.

We call g *elliptic* if all g_i are elliptic isometries, *parabolic* if all g_i are parabolic isometries and *hyperbolic* if all g_i are hyperbolic isometries. In all the other cases we call g *mixed*.

If at least one $\ell(g_i)$ is different from zero, then we define the *translation direction* of g as $L(g) := (\ell(g_1) : \dots : \ell(g_r)) \in \mathbb{RP}^{r-1}$.

1.2. Decomposition of the geometric boundary of $(\mathbb{H}^2)^r$. In this section we show a natural decomposition of the geometric boundary of $(\mathbb{H}^2)^r$ and in particular of its regular part. This is a special case of

a general construction for a large class of symmetric spaces (see e.g. Leuzinger [11] and Link [12]). This decomposition plays a main role in this article.

Let $x = (x_1, \dots, x_r)$ be a point in $(\mathbb{H}^2)^r$. We consider the *Weyl chambers* with vertex x in $(\mathbb{H}^2)^r$ given by the product of the images of the geodesics $\delta_i : [0, \infty) \rightarrow \mathbb{H}^2$ with $\delta_i(0) = x_i$ for $i = 1, \dots, r$. The isotropy group in $\mathrm{PSL}(2, \mathbb{R})^r$ of x is $\mathrm{PSO}(2)^r$. It acts simply transitively on the Weyl chambers with vertex x .

Let W be a Weyl chamber with vertex x . In W , two unit speed geodesics $\gamma(t) = (\gamma_1(t), \dots, \gamma_r(t))$ and $\tilde{\gamma} = (\tilde{\gamma}_1(t), \dots, \tilde{\gamma}_r(t))$ are different if and only if the corresponding projective points

$$(d_H(\gamma_1(0), \gamma_1(1)) : \dots : d_H(\gamma_r(0), \gamma_r(1))) \text{ and} \\ (d_H(\tilde{\gamma}_1(0), \tilde{\gamma}_1(1)) : \dots : d_H(\tilde{\gamma}_r(0), \tilde{\gamma}_r(1)))$$

are different. Here d_H denotes the hyperbolic distance in \mathbb{H}^2 . The point in \mathbb{RP}^{r-1} given by $(d_H(\gamma_1(0), \gamma_1(1)) : \dots : d_H(\gamma_r(0), \gamma_r(1)))$ is a direction in the Weyl chamber and it is the same as $(\|v_1\| : \dots : \|v_r\|)$, where $v = (v_1, \dots, v_r) := \gamma'(0)$ is the unit tangent vector of γ in 0 .

Let v be a unit tangent vector at x in $(\mathbb{H}^2)^r$. We denote by v_i the i -th projection of v on the tangent spaces at x_i , $i = 1, \dots, r$. Then all the vectors w in the orbit of v under Iso_x have $\|w_i\| = \|v_i\|$. Even more, the orbit of v under the group $\mathrm{PSO}(2)^r$ consists of all unit tangent vectors w at x such that $\|w_i\| = \|v_i\|$ for $i = 1, \dots, r$. Therefore if v is tangent to a regular geodesic, then the orbit of v is homeomorphic to $(S^1)^r \cong (\partial\mathbb{H}^2)^r$.

The *regular boundary* $\partial((\mathbb{H}^2)^r)_{reg}$ of $(\mathbb{H}^2)^r$ consists of the equivalence classes of regular geodesics. Hence it is identified with $(\partial\mathbb{H}^2)^r \times \mathbb{RP}_+^{r-1}$ where

$$\mathbb{RP}_+^{r-1} := \{(x_1 : \dots : x_r) \in \mathbb{RP}^{r-1} \mid x_1 > 0, \dots, x_r > 0\}.$$

Here x_1, \dots, x_r can be thought as the norms of the projections of the regular unit tangent vectors on the simple factors of $(\mathbb{H}^2)^r$.

$(\partial\mathbb{H}^2)^r$ is called the *Furstenberg boundary* of $(\mathbb{H}^2)^r$.

We note that the decomposition of the boundary into orbits under the group Iso_x is independent of the point x .

1.3. The limit set of a group. Let x be a point and $\{x_n\}_{n \in \mathbb{N}}$ a sequence of points in $(\mathbb{H}^2)^r$. We say that $\{x_n\}_{n \in \mathbb{N}}$ converges to a point $\xi \in \partial((\mathbb{H}^2)^r)$ if $\{x_n\}_{n \in \mathbb{N}}$ is discrete in $(\mathbb{H}^2)^r$ and the sequence of geodesic rays starting at x and going through x_n converges towards ξ in the cone topology. With this topology, $(\mathbb{H}^2)^r \cup \partial((\mathbb{H}^2)^r)$ is a compactification of $(\mathbb{H}^2)^r$.

Let Γ be a subgroup of $\mathrm{PSL}(2, \mathbb{R})^r$. We denote by $\Gamma(x)$ the orbit of x under Γ and by $\overline{\Gamma(x)}$ - its closure. The *limit set* of Γ is $\mathcal{L}_\Gamma := \overline{\Gamma(x)} \cap \partial((\mathbb{H}^2)^r)$. The limit set is independent of the choice of the point x in $(\mathbb{H}^2)^r$. The *regular limit set* is $\mathcal{L}_\Gamma^{reg} := \mathcal{L}_\Gamma \cap \partial((\mathbb{H}^2)^r)_{reg}$ and the *singular limit set* is $\mathcal{L}_\Gamma^{sing} := \mathcal{L}_\Gamma \setminus \mathcal{L}_\Gamma^{reg}$.

We denote by F_Γ the projection of \mathcal{L}_Γ^{reg} on the Furstenberg boundary $(\partial\mathbb{H}^2)^r$ and by P_Γ the projection of \mathcal{L}_Γ^{reg} on \mathbb{RP}_+^{r-1} . The projection F_Γ is the *Furstenberg limit set* of Γ and P_Γ is the *projective limit set* of Γ .

Let $h \in \Gamma$ be a hyperbolic element or a mixed one with only hyperbolic or elliptic components. There is a unique unit speed geodesic γ in $(\mathbb{H}^2)^r$ such that $h \circ \gamma(t) = \gamma(t + T_h)$ for a fixed $T_h \in \mathbb{R}_{>0}$ and all $t \in \mathbb{R}$. For $y \in \gamma$, the sequence $h^n(y)$ converges to $\gamma(+\infty)$. Hence also for every $x \in (\mathbb{H}^2)^r$, the sequence $h^n(x)$ converges to $\gamma(+\infty)$. Thus $\gamma(+\infty)$ is in \mathcal{L}_Γ . The sequence $h^{-n}(x)$ converges to $\gamma(-\infty) := -\gamma(+\infty)$ and therefore $\gamma(-\infty)$ is also in \mathcal{L}_Γ . The points $\gamma(+\infty)$ and $\gamma(-\infty)$ are the only fixed points of h in \mathcal{L}_Γ . The point $\gamma(+\infty)$ is the *attractive* fixed point of h and the point $\gamma(-\infty)$ - the *repulsive* fixed point of h .

If h is hyperbolic, then for all $i = 1, \dots, r$, the projection $p_i \circ \gamma$ is not a point. Hence γ is regular and $\gamma(+\infty) \in \mathcal{L}_\Gamma^{reg}$. The point $\gamma(+\infty)$ can be written as (ξ_F, ξ_P) in our description of the regular geometric boundary where

$$\xi_F := (p_1 \circ \gamma(+\infty), \dots, p_r \circ \gamma(+\infty))$$

is in the Furstenberg boundary and

$$\xi_P := (d_H(p_1 \circ \gamma(0), p_1 \circ \gamma(1)) : \dots : d_H(p_r \circ \gamma(0), p_r \circ \gamma(1)))$$

is in the projective limit set. Here we note that ξ_P is also equal to

$$(d_H(p_1 \circ \gamma(0), p_1 \circ \gamma(T_h)) : \dots : d_H(p_r \circ \gamma(0), p_r \circ \gamma(T_h))),$$

which is exactly the translation direction of h .

Thus the translation direction of each hyperbolic isometry h in Γ determines a point in the projective limit set P_Γ . This point does not change after conjugation with h or after taking a power h^m of h , because in these cases the translation direction remains unchanged.

Recall that following Maclachlan and Reid [13], we call a subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ *elementary* if there exists a finite Γ -orbit in $\overline{\mathbb{H}^2} := \mathbb{H}^2 \cup \partial\mathbb{H}^2$ and *nonelementary* if it is not elementary. Since \mathbb{H}^2 and $\partial\mathbb{H}^2$ are Γ -invariant, any Γ -orbit of a point in $\overline{\mathbb{H}^2}$ is either completely in \mathbb{H}^2 or completely in $\partial\mathbb{H}^2$.

We call a subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})^r$ *nonelementary* if for all $i = 1, \dots, r$, $p_i(\Gamma)$ is nonelementary, and if for all $g \in \Gamma$ that are mixed, the projections $p_i \circ g$ are either hyperbolic or elliptic of infinite order. Since for all $i = 1, \dots, r$, $p_i(\Gamma)$ is nonelementary, Γ does not contain only elliptic isometries and thus \mathcal{L}_Γ is not empty.

This definition of nonelementary is more restrictive than the one given by Link in [12]. By Lemma 1.2 in [8] if a subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})^r$ is nonelementary (according to our definition), then it is nonelementary in the sense of Link's definition in [12].

The next theorem is a special case of Theorem 3 from the introduction of [12]. It describes the structure of the regular limit set of nonelementary discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})^r$.

Theorem 1.1 ([12]). *Let Γ be a nonelementary discrete subgroup of the group $\mathrm{PSL}(2, \mathbb{R})^r$ acting on $(\mathbb{H}^2)^r$. If $\mathcal{L}_\Gamma^{\mathrm{reg}}$ is not empty, then F_Γ is a minimal closed Γ -invariant subset of $(\partial\mathbb{H}^2)^r$, the regular limit set equals the product $F_\Gamma \times P_\Gamma$ and P_Γ is equal to the closure in \mathbb{RP}_+^{r-1} of the set of translation directions of the hyperbolic isometries in Γ .*

1.4. Irreducible arithmetic groups in $\mathrm{PSL}(2, \mathbb{R})^r$. In this section, following Schmutz and Wolfart [16] and Borel [3], we describe the irreducible arithmetic subgroups of $\mathrm{PSL}(2, \mathbb{R})^r$.

Let K be a totally real algebraic number field of degree $n = [K : \mathbb{Q}]$ and let ϕ_i , $i = 1, \dots, n$, be the n distinct embeddings of K into \mathbb{R} , where $\phi_1 = \mathrm{id}$.

Let $A = \left(\frac{a,b}{K}\right)$ be a quaternion algebra over K such that for $1 \leq i \leq r$, the quaternion algebra $\left(\frac{\phi_i(a), \phi_i(b)}{\mathbb{R}}\right)$ is *unramified*, i.e. isomorphic to the matrix algebra $M(2, \mathbb{R})$, and for $r < i \leq n$, it is *ramified*, i.e. isomorphic to the Hamilton quaternion algebra \mathbf{H} . In other words, the embeddings

$$\phi_i : K \longrightarrow \mathbb{R}, \quad i = 1, \dots, r$$

extend to embeddings of A into $M(2, \mathbb{R})$ and the embeddings

$$\phi_i : K \longrightarrow \mathbb{R}, \quad i = r + 1, \dots, n$$

extend to embeddings of A into \mathbf{H} . Note that the embeddings ϕ_i , $i = 1, \dots, r$, of A into the matrix algebra $M(2, \mathbb{R})$ are not canonical.

Let \mathcal{O} be an order in A and \mathcal{O}^1 the group of units in \mathcal{O} . Define $\Gamma(A, \mathcal{O}) := \phi_1(\mathcal{O}^1) \subset \mathrm{SL}(2, \mathbb{R})$. The canonical image of $\Gamma(A, \mathcal{O})$ in $\mathrm{PSL}(2, \mathbb{R})$ is called a group *derived from a quaternion algebra*. The group $\Gamma(A, \mathcal{O})$ acts by isometries on $(\mathbb{H}^2)^r$ as follows. An element $g = \phi_1(\varepsilon)$ of $\Gamma(A, \mathcal{O})$ acts via

$$g : (z_1, \dots, z_r) \mapsto (\phi_1(\varepsilon)z_1, \dots, \phi_r(\varepsilon)z_r),$$

where $z_i \mapsto \phi_i(\varepsilon)z_i$ is the usual action by linear fractional transformation, $i = 1, \dots, r$.

For a subgroup S of $\Gamma(A, \mathcal{O})$ we denote by S^* the group

$$\{g^* := (\phi_1(\varepsilon), \dots, \phi_r(\varepsilon)) \mid \phi_1(\varepsilon) = g \in S\}.$$

Instead of $(\phi_1(\varepsilon), \dots, \phi_r(\varepsilon))$, we will usually write $(\phi_1(g), \dots, \phi_r(g))$ or, since ϕ_1 is the identity, even $(g, \phi_2(g), \dots, \phi_r(g))$. The elements $\phi_1(g), \dots, \phi_r(g)$ are called ϕ -conjugates.

Note that g^* and S^* depend on the chosen embeddings ϕ_i of A into $M(2, \mathbb{R})$. On the other hand, the type of g^* is determined uniquely by the type of g . This is given by the following lemma.

Lemma 1.2 ([8]). *Let S be a subgroup of $\Gamma(A, \mathcal{O})$ and S^* be defined as above. For an element $g \in S$ the following assertions are true.*

1. *If g is the identity, then g^* is the identity.*
2. *If g is parabolic, then g^* is parabolic.*
3. *If g is elliptic of finite order, then g^* is elliptic of the same order.*
4. *If g is hyperbolic, then g^* is either hyperbolic or mixed such that, for $i = 1, \dots, r$, $\phi_i(g)$ is either hyperbolic or elliptic of infinite order.*
5. *If g is elliptic of infinite order, then its ϕ -conjugates are hyperbolic or elliptic of infinite order.*

Hence the mixed isometries in this setting have components that are only hyperbolic or elliptic of infinite order. This justifies the condition in our definition of nonelementary that the projections of all mixed isometries can be only hyperbolic or elliptic of infinite order.

By Borel [3], Section 3.3, all *irreducible arithmetic subgroups* of the group $\mathrm{PSL}(2, \mathbb{R})^r$ are commensurable to a $\Gamma(A, \mathcal{O})^*$. They have finite covolume. By Margulis [14], for $r \geq 2$, all irreducible discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})^r$ of finite covolume are arithmetic, which shows the importance of the above construction.

P. Schmutz and J. Wolfart define in [16] arithmetic groups acting on $(\mathbb{H}^2)^r$. An *arithmetic group acting on $(\mathbb{H}^2)^r$* is a group G that is commensurable to a $\Gamma(A, \mathcal{O})$. It is finitely generated because it is commensurable to the finitely generated group $\Gamma(A, \mathcal{O})$. Then by Corollary 3.3.5 from [13], the quaternion algebra $AG^{(2)} = \{\sum a_i g_i \mid a_i \in \mathbb{Q}(\mathrm{Tr}(G^{(2)})), g_i \in G^{(2)}\}$ of the group $G^{(2)}$ generated by the set $\{g^2 \mid g \in G\}$ is an invariant of the commensurability class of G . Hence $AG^{(2)}$ is isomorphic to A . By Exercise 3.2, No. 1, in [13], $\mathcal{O}G^{(2)} = \{\sum a_i g_i \mid a_i \in \mathcal{O}_{\mathbb{Q}(\mathrm{Tr}(G^{(2)}))}, g_i \in G^{(2)}\}$ is an order in $AG^{(2)}$. Therefore $G^{(2)}$ is a subgroup of $\Gamma(AG^{(2)}, \mathcal{O}G^{(2)})$ and we can define $(G^{(2)})^*$. This explains “acting on $(\mathbb{H}^2)^r$ ” in the name.

The interest of this approach is that it allows us to consider a subgroup S of $G \subset \mathrm{PSL}(2, \mathbb{R})$ and then the corresponding subgroups $S^{(2)*}$ and $G^{(2)*}$ of $\mathrm{PSL}(2, \mathbb{R})^r$. If S is a Fuchsian group of finite covolume, then it is called *semi-arithmetic Fuchsian group*.

1.5. Properties of nonelementary subgroups of $\mathrm{PSL}(2, \mathbb{R})^r$. In this section we cite several results that are used later in the article and whose proofs can be found in [8].

Lemma 1.3 ([8]). *Let Γ be a nonelementary subgroup of $\mathrm{PSL}(2, \mathbb{R})^r$. Further let g and h be two hyperbolic isometries in Γ . Then there are hyperbolic isometries g' and h' in Γ with $L(g) = L(g')$ and $L(h) = L(h')$ such that the groups generated by the corresponding components are all Schottky groups (with only hyperbolic isometries).*

Lemma 1.4 ([8]). *Let Γ be a subgroup of $\mathrm{PSL}(2, \mathbb{R})^r$ such that all mixed isometries in Γ have only elliptic and hyperbolic components and $p_j(\Gamma)$ is nonelementary for one $j \in \{1, \dots, r\}$. Then Γ is nonelementary and \mathcal{L}_Γ^{reg} is not empty.*

We define the limit cone of Γ to be the closure in \mathbb{RP}^{r-1} of the set of the translation directions of the hyperbolic and mixed isometries in Γ .

Theorem 1.5 ([8]). *Let Γ be a nonelementary subgroup of an irreducible arithmetic group in $\mathrm{PSL}(2, \mathbb{R})^r$ with $r \geq 2$. Then P_Γ is convex and the closure of P_Γ in \mathbb{RP}^{r-1} is equal to the limit cone of Γ and in particular the limit cone of Γ is convex.*

2. ZARISKI DENSE SUBGROUPS OF $\mathrm{PSL}(2, \mathbb{R})^r$

In this section we give a number-theoretical characterization of Zariski dense nonelementary subgroups of an arithmetic group in $\mathrm{PSL}(2, \mathbb{R})^r$ (Corollary 2.2). In order to do this we need a classification of all nonelementary Zariski closed subgroups of $\mathrm{PSL}(2, \mathbb{R})^r$.

Proposition 2.1. *Let Γ be a nonelementary subgroup of $\mathrm{PSL}(2, \mathbb{R})^r$. Then the following holds for the Zariski closure of Γ :*

$$\overline{\Gamma}^Z = \prod_{i=1}^n \mathrm{Diag}_{k_i}(\mathrm{PSL}(2, \mathbb{R})),$$

where $\mathrm{Diag}_{k_i}(\mathrm{PSL}(2, \mathbb{R}))$ is a conjugate of the diagonal embedding of $\mathrm{PSL}(2, \mathbb{R})$ in $\mathrm{PSL}(2, \mathbb{R})^{k_i}$ and where $k_1 + \dots + k_n = r$.

Proof. Let G be the Zariski closure of Γ and p_i , $i = 1, \dots, r$, the projection of G into the i -th factor.

First we show that $p_i(G) = \mathrm{PSL}(2, \mathbb{R})$ for all $i = 1, \dots, r$. Let G_0 be the connected component of the identity. Note that the group $p_i(G_0)$ is connected and nontrivial. As in the proof of the criterion in [5] we remark that $p_i(G_0)$ is normalized by the nonelementary (and hence Zariski dense) $p_i(\Gamma)$. Hence the projection of the Lie algebra of G_0 is normalized by the adjoint action of a Zariski dense group and therefore it is the whole $\mathfrak{sl}(2, \mathbb{R})$. Thus p_i is open and therefore surjective.

We will prove that every Zariski closed subgroup G of $\mathrm{PSL}(2, \mathbb{R})^r$ such that its projections to the different simple factors are $\mathrm{PSL}(2, \mathbb{R})$ has the form $\prod_{i=1}^n \mathrm{Diag}_{k_i}(\mathrm{PSL}(2, \mathbb{R}))$ of the statement of the proposition.

Obviously this statement is true for $r = 1$. Let us assume that it is true for $r = 1, \dots, m - 1$. We prove it for $r = m$.

We consider the projection p_1 on the first factor. The kernel $H := \ker(p_1)$ of p_1 is a Zariski closed subgroup of $\mathrm{PSL}(2, \mathbb{R})^m$.

Claim. The group $p_i(H)$ is either trivial or the whole group $\mathrm{PSL}(2, \mathbb{R})$.

Let us assume that there is an element $h = (h_1, \dots, h_m)$ in H such that $h_i \neq id$. We choose $g = (g_1, \dots, g_m) \in G$ such that g_i and h_i do not have common fixed points. For some powers l_g, l_h , the elements $h_i^{l_h}$ and $g_i^{-l_g} h_i^{l_h} g_i^{l_g}$ generate a Schottky group. This Schottky group is even a subgroup of H because H is a normal subgroup of G . Hence there is a Zariski dense subgroup of $\mathrm{PSL}(2, \mathbb{R})$ which normalizes $p_i(H)$ and thus $p_i(H) = \mathrm{PSL}(2, \mathbb{R})$.

Without loss of generality, let $p_i(H) = \{id\}$ for $i = 2, \dots, k$ and $\mathrm{PSL}(2, \mathbb{R})$ for $i = k + 1, \dots, m$. Hence H is canonically isomorphic to a Zariski closed subgroup of $\mathrm{PSL}(2, \mathbb{R})^{m-k}$ and by induction it is equal to the group $S := \prod_{i=2}^n \mathrm{Diag}_{k_i}(\mathrm{PSL}(2, \mathbb{R}))$ where $k_2 + \dots + k_n = m - k$.

The next step is to show that $G = D \times S$ where D denotes the projection of G on the first k factors. For simplicity of the notation we give the proof just in the case $n = 2$ and $k_2 = 2$. The general case is analogous.

Let us assume the converse, namely that there is an element $(d, s, t) \in G$ such that $d \in D$ and $(s, t) \notin S$, i.e (s, t) is different from the element $(s, s_2) \in S$. Hence $a := (d, id, u := ts_2^{-1})$ is in G and $u \neq id$. We choose $v \in \mathrm{PSL}(2, \mathbb{R})$ that does not commute with u and consider the corresponding element $b := (id, v_1, v)$ in H . The commutator $[a, b] = (id, id, uvu^{-1}v^{-1})$ is an element in H that contradicts its structure.

The final step is to show that D is a conjugate of the diagonal embedding of $\mathrm{PSL}(2, \mathbb{R})$ in $\mathrm{PSL}(2, \mathbb{R})^k$. Since $\ker(p_1|_D) = \{id\}$, $\mathrm{PSL}(2, \mathbb{R})$ is isomorphic to D . And since for each $i = 1, \dots, k$ the projection p_i is surjective, we have that $D = \{(x, \phi_2(x), \dots, \phi_k(x))\}$ where ϕ_i are

surjective homomorphisms. Even more, ϕ_i are isomorphisms because if the kernel is not trivial it can only be $\mathrm{PSL}(2, \mathbb{R})$, which is obviously not possible.

The projections $p_i|_D$ are continuous isomorphisms and open maps as remarked in the beginning of the proof. Hence ϕ_i are continuous isomorphisms and hence conjugations of $\mathrm{PSL}(2, \mathbb{R})$ by elements in $\mathrm{GL}(2, \mathbb{R})$. \square

For a group S we denote as usual by $S^{(2)}$ its subgroup generated by the set $\{g^2 \mid g \in S\}$. If S is a finitely generated nonelementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ then $S^{(2)}$ is a finite index normal subgroup of S . The reason to look at $S^{(2)}$ instead of S is that for a finitely generated nonelementary subgroup S of $\mathrm{PSL}(2, \mathbb{R})$ the field generated by $\mathrm{Tr}(S^{(2)})$ is an invariant of the commensurability class of S (see [13], Lemma 3.3.4). This is not true in general for the field generated by $\mathrm{Tr}(S)$.

Corollary 2.2. *Let Γ be a finitely generated nonelementary subgroup of an irreducible arithmetic group Δ in $\mathrm{PSL}(2, \mathbb{R})^r$. Then Γ is Zariski dense in $\mathrm{PSL}(2, \mathbb{R})^r$ if and only if the fields generated by $\mathrm{Tr}(p_i(\Gamma^{(2)}))$ and $\mathrm{Tr}(p_i(\Delta^{(2)}))$ are equal for one and hence for all $i \in \{1, \dots, r\}$.*

Proof. Since Δ is arithmetic, it is commensurable with an arithmetic group derived from a quaternion algebra $\Gamma(A, \mathcal{O})^*$. Hence there is $k \in \mathbb{N}$ such that, for each $g = (g_1, \dots, g_{q+r})$ in Γ , g^k is in $\Gamma(A, \mathcal{O})^*$.

There is a subgroup S of $\Gamma(A, \mathcal{O})$ such that $S^* = \Gamma \cap \Gamma(A, \mathcal{O})^*$. The group Γ is commensurable with its subgroup S^* . Hence $p_1(\Gamma)$ and S are also commensurable. The group S^* is finitely generated because it is a finite index subgroup of the finitely generated group Γ . (This follows from the Schreier Index Formula, see for example the book of Stillwell [18], 2.2.5.) The group S^* is also nonelementary because Γ is nonelementary: Let g and h be two loxodromic isometries that generate a Schottky group in Γ . The isometries g^k and h^k are in S^* . Then g^k and h^k generate a Schottky subgroup of S^* .

Thus S is a finitely generated nonelementary subgroup of $\Gamma(A, \mathcal{O})$.

We prove now the statement for S^* and $\Gamma(A, \mathcal{O})^*$. We prove it for the first projection, i.e. we show that S^* is Zariski dense if and only if the trace field of $S^{(2)}$ is the same as the trace field of $\Gamma(A, \mathcal{O})^{(2)}$. The general case is analogous.

First we assume that $(S^{(2)})^*$ is not Zariski dense. By Proposition 2.1 there is in its Zariski closure at least one block $\mathrm{Diag}_k(\mathrm{PSL}(2, \mathbb{R}))$ with $k \geq 2$. Hence there are ϕ_i and ϕ_j with $i \neq j$ such that $\phi_i|_{\mathrm{Tr}(S^{(2)})} = \phi_j|_{\mathrm{Tr}(S^{(2)})}$. Since ϕ_i and ϕ_j are different for the trace field of $\Gamma(A, \mathcal{O})^{(2)}$,

it follows that the trace field of $S^{(2)}$ is not the same as the trace field of $\Gamma(A, \mathcal{O})^{(2)}$.

In order to prove the converse we assume that the trace field F of $S^{(2)}$ is not the same as the trace field K of $\Gamma(A, \mathcal{O})^{(2)}$, i.e. it is a nontrivial subfield. Hence there is a $\phi_i \neq id$ such that $\phi_i|_F = id|_F$. Since $\phi_i(S^{(2)})$ is nonelementary, the group $\phi_i(\Gamma(A, \mathcal{O})^{(2)})$ is also nonelementary and hence $i \leq r$. Therefore $\phi_i|_{S^{(2)}}$ is a conjugation by a matrix in $GL(2, \mathbb{R})$ (see for example Sampson [17]). This implies that $(S^{(2)})^*$ is not Zariski dense. \square

3. THE LIMIT SET OF MODULAR EMBEDDINGS OF FUCHSIAN GROUPS

As the following theorem shows the projection of the regular limit set into the Furstenberg boundary of the embedding in a lattice in $PSL(2, \mathbb{R})^r$ of an arithmetic Fuchsian group is homeomorphic to a circle. This is an example of a “small” group with a “small” limit set.

Theorem 3.1 ([8]). *Let Δ be an irreducible arithmetic subgroup of $PSL(2, \mathbb{R})^r$ with $r \geq 2$ and Γ a finitely generated nonelementary subgroup of Δ . Then \mathcal{L}_Γ is homeomorphically embedded in a circle if and only if $p_j(\Gamma)$ is contained in an arithmetic Fuchsian group for some $j \in \{1, \dots, r\}$.*

In this section we provide examples of “small” groups with a “big” limit set, i.e. groups for which the projection of the regular limit set into the Furstenberg boundary is the whole Furstenberg boundary and even more for which the limit set is of nonempty interior.

Let S be a semi-arithmetic subgroup of $PSL(2, \mathbb{R})$ that is a subgroup of some $\Gamma(A, \mathcal{O})$ as defined in §1.4. We set $\Gamma := S^*$ a subgroup of $PSL(2, \mathbb{R})^r$. Then S is said to have a *modular embedding* if for the natural embedding $f : S \rightarrow S^* = \Gamma$ there exists a holomorphic embedding $F : \mathbb{H}^2 \rightarrow (\mathbb{H}^2)^r$ with

$$F(Tz) = f(T)F(z), \quad \text{for all } T \in S \quad \text{and all } z \in \mathbb{H}^2.$$

Examples of strictly semi-arithmetic groups admitting modular embeddings are Fuchsian triangle groups (see Cohen and Wolfart [4]).

According to Theorem 3 by Schmutz and Wolfart in [16], if ϕ_i is not a conjugation then $\phi_i(S)$ is not a Fuchsian group. Since every nondiscrete subgroup of $PSL(2, \mathbb{R})$ has elliptic elements of infinite order (see Theorem 8.4.1 in Beardon [1]), the nondiscrete group $\phi_i(S)$ contains elliptic elements of infinite order. (Here the ϕ_i are the same as the ones from the definition of S^* .)

Additionally, Corollary 5 by Schmutz and Wolfart [16] states that for all $g = (g_1, \dots, g_r)$ in Γ such that g_1 is hyperbolic, we have the inequality $\text{tr}(g_i) < \text{tr}(g_1)$ for $i = 2, \dots, r$. Hence the Galois conjugates of the traces of hyperbolic elements are different, i.e. $|\phi_i(\text{tr}(g_1))| \neq |\phi_j(\text{tr}(g_1))|$ if $i \neq j$. Hence no trace in $\text{Tr}(S)$ of a hyperbolic element is contained in a proper subfield of the trace field $\mathbb{Q}(\text{Tr}(S))$ of S . In particular there are no hyperbolic elements in S with integer traces.

Lemma 3.2. *Let Γ be a subgroup of an irreducible arithmetic subgroup of $\text{PSL}(2, \mathbb{R})^r$ with $r \geq 2$ such that $p_j(\Gamma)$ is a semi-arithmetic Fuchsian group admitting a modular embedding and r is the smallest power for which $p_j(\Gamma)$ has a modular embedding in an irreducible arithmetic subgroup of $\text{PSL}(2, \mathbb{R})^r$. Then for all the other $i = 1, \dots, r$, $i \neq j$, the group $p_i(\Gamma)$ is not discrete and for every $g_j \in p_j(\Gamma)$ hyperbolic $\mathbb{Q}(\text{tr}(g_j^2)) = \mathbb{Q}(\text{Tr}(p_j(\Gamma^{(2)})))$.*

3.1. The Furstenberg limit set. The main result of this section is Theorem 3.4. It proves in particular for the groups satisfying the conditions of Lemma 3.2 that their Furstenberg limit set is the whole Furstenberg boundary. First we need to prove the following technical lemma.

Lemma 3.3. *Let Γ be a nonelementary subgroup of an arithmetic group in $\text{PSL}(2, \mathbb{R})^r$ with $r \geq 2$ and let $g = (g_1, \dots, g_r) \in \Gamma$ be a mixed isometry with g_1, \dots, g_{k-1} hyperbolic and g_k, \dots, g_r elliptic of infinite order where $2 \leq k \leq r - 1$ and such that $\mathbb{Q}(\text{tr}(g_1)) = \mathbb{Q}(\text{tr}(g_1^n))$ for every integer $n \neq 0$. If $\text{tr}(g_k) \neq \text{tr}(g_{k+1})$, then there exists a mixed isometry $\tilde{g} \in \Gamma$ such that $\tilde{g}_1, \dots, \tilde{g}_k$ are hyperbolic and \tilde{g}_{k+1} is elliptic of infinite order.*

Proof. Since Γ is nonelementary, by Lemma 1.4 and by the fact that the fixed points of hyperbolic isometries are dense in \mathcal{L}_Γ , there is $h \in \Gamma$ that is a hyperbolic isometry. By Lemma 1.3, we can assume without loss of generality that for all $i = 1, \dots, k - 1$, h_i and g_i generate a Schottky group.

The idea is to find $m \in \mathbb{N}$ such that $g_k^m h_k$ is hyperbolic and $g_{k+1}^m h_{k+1}$ is elliptic (of infinite order).

By §7.34 in the book of Beardon [1], the isometries h_i , $i = k, k + 1$, can be represented as $h_i = \sigma_{i,2} \sigma_{i,1}$ where $\sigma_{i,j}$ are the reflections in geodesics $L_{i,j}$ that are orthogonal to the axis of h_i and the distance between them equals $\ell(h_i)/2$. By §7.33 in [1], the elliptic isometries g_i^m can be represented as $g_i^m = \sigma_{i,4} \sigma_{i,3}$ where $\sigma_{i,j}$ are the reflections in $L_{i,j}$ that pass through the fixed point of g_i^m and the angle between them equals the half of the angle of rotation of g_i . We can choose $L_{i,2}$

and $L_{i,3}$ to be the same geodesic by choosing them to be the unique geodesic passing through the fixed point of g_i^m that is orthogonal to the axis of h_i . Then $\sigma_{i,2} = \sigma_{i,3}$ and

$$g_i^m h_i = (\sigma_{i,4}\sigma_{i,3})(\sigma_{i,2}\sigma_{i,1}) = \sigma_{i,4}\sigma_{i,1}.$$

Hence if $L_{i,1}$ and $L_{i,4}$ do not intersect and do not have a common point at infinity, the isometry $g_i^m h_i$ is hyperbolic, and if they intersect, $g_i^m h_i$ is elliptic. By the next claim we can choose m so that g_k^m is a rotation of a very small angle such that $L_{k,4}$ does not intersect $L_{k,1}$ and g_{k+1}^m is a rotation such that $L_{k+1,4}$ intersects $L_{k+1,1}$.

Claim. Every orbit of (g_k, g_{k+1}) on the 2-torus $(\partial\mathbb{H}^2)^2$ is dense.

Since $\text{tr}(g_k) \neq \text{tr}(g_{k+1})$ and g_k and g_{k+1} are elliptic of infinite order, they are rotations of angles $2\alpha\pi$ and $2\beta\pi$ respectively, i.e. $\text{tr}(g_k) = |2\cos(\alpha\pi)|$ and $\text{tr}(g_{k+1}) = |2\cos(\beta\pi)|$, where α and β are different irrational numbers in the interval $[0, \pi)$ and $\cos(\alpha\pi)$ is a Galois conjugate of $\cos(\beta\pi) =: \phi(\cos(\alpha\pi))$. The orbit of a point (ξ_k, ξ_{k+1}) in $(\partial\mathbb{H}^2)^2$ under (g_k, g_{k+1}) can be represented as the orbit of a point in the flat torus $S^1 \times S^1$ under translations with translation vector $(2\alpha, 2\beta)$. By Theorem 442 in the book of Hardy and Wright [10], the orbit of a point is dense if 2α , 2β and 1 are linearly independent over \mathbb{Q} .

Let us assume that 2α , 2β and 1 are linearly dependent over \mathbb{Q} , i.e. $2m\alpha + 2n\beta = q$ for some integers q, m, n .

For a positive integer l , we have $\cos(lx) = P_l(\cos x)$ where P_l is a polynomial of degree l . Hence

$$\begin{aligned} \phi(\cos(l\alpha\pi)) &= \phi(P_l(\cos(\alpha\pi))) = P_l(\phi(\cos(\alpha\pi))) \\ &= P_l(\cos(\beta\pi)) = \cos(l\beta\pi). \end{aligned}$$

This means that a conjugate of $\cos(n\alpha\pi)$ is

$$\begin{aligned} \phi(\cos(n\alpha\pi)) &= \cos(n\beta\pi) = \cos((q - m\alpha)\pi) \\ &= \cos(q\pi) \cos(m\alpha\pi) + \sin(q\pi) \sin(m\alpha\pi). \end{aligned}$$

Up to multiplying by two the equality $2m\alpha + 2n\beta = q$, one can assume that q is even and hence $\sin(q\pi) = 0$ and $\cos(q\pi) = 1$. Thus $\phi(\cos(n\alpha\pi)) = \cos(m\alpha\pi)$.

If $m = n$, then $\phi(\cos(n\alpha\pi)) = \cos(n\alpha\pi)$. Hence $\phi_k(\text{tr}(g_1^n)) = |2\cos(n\alpha\pi)| = \phi_{k+1}(\text{tr}(g_1^n))$, which is impossible because $\phi_k(\text{tr}(g_1)) = |2\cos(\alpha\pi)| \neq \phi_{k+1}(\text{tr}(g_1))$ and $\mathbb{Q}(\text{tr}(g_1)) = \mathbb{Q}(\text{tr}(g_1^n))$. Therefore $m \neq n$.

We will show that $\phi(\cos(n\alpha\pi)) = \cos(m\alpha\pi)$ is impossible by finding c different conjugates of $\cos(n^c\alpha\pi)$ where c is an arbitrary integer.

$$\begin{aligned}\phi(\cos(n^c\alpha\pi)) &= \phi(P_{n^{c-1}}(\cos(n\alpha\pi))) = P_{n^{c-1}}(\phi(\cos(n\alpha\pi))) \\ &= P_{n^{c-1}}(\cos(m\alpha\pi)) = \cos(mn^{c-1}\alpha\pi).\end{aligned}$$

By induction we see that for every integer d with $0 < d \leq c$,

$$\phi^d(\cos(n^c\alpha\pi)) = \cos(m^d n^{c-d}\alpha\pi).$$

All $\phi^d(\cos(n^c\alpha\pi))$ are conjugate and different because $\cos(m^d n^{c-d}\alpha\pi)$ are different for different d . This contradicts the fact that $\mathbb{Q}(\text{Tr}(\Gamma))$ is a finite extension of \mathbb{Q} .

Thus 2α , 2β and 1 are linearly independent over \mathbb{Q} . Therefore the orbit of (ξ_k, ξ_k) under (g_k, g_{k+1}) is dense in $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. \square

Theorem 3.4. *Let Γ be a subgroup of an irreducible arithmetic group Δ in $\text{PSL}(2, \mathbb{R})^r$ with $r \geq 2$ such that, for one $j \in \{1, \dots, r\}$, $p_j(\Gamma)$ is a cofinite Fuchsian group and for all the others $i = 1, \dots, r$, $i \neq j$, the group $p_i(\Gamma)$ is not discrete and such that, for every $g_j \in p_j(\Gamma)$ hyperbolic, $\mathbb{Q}(\text{tr}(g_j^2)) = \mathbb{Q}(\text{Tr}(p_j(\Gamma^{(2)})))$.*

Then F_Γ is the whole Furstenberg boundary $(\partial\mathbb{H}^2)^r$.

Proof. Without loss of generality we can assume that $j = 1$.

We denote $\Gamma_i := p_i(\Gamma)$, for $i = 1, \dots, r$.

First we remark that by Lemma 1.4 the regular limit set of Γ is not empty and hence there is $\xi = (\xi_1, \dots, \xi_r) \in F_\Gamma$. Without loss of generality, we can assume that ξ is the projection in the Furstenberg boundary of the attractive fixed point of a hyperbolic isometry in Γ . By Theorem 1.1, F_Γ is the minimal closed Γ -invariant subset of the Furstenberg boundary $(\partial\mathbb{H}^2)^r$. Therefore $F_\Gamma = \overline{\Gamma(\xi)}$. The idea is to show that $\overline{\Gamma(\xi)}$ is the whole Furstenberg boundary $(\partial\mathbb{H}^2)^r$ and hence $F_\Gamma = (\partial\mathbb{H}^2)^r$.

We will mainly use the fact that except for Γ_1 all the other Γ_i are non discrete and in particular that then they contain elliptic elements of infinite order. An elliptic element e of infinite order acts on $\partial\mathbb{H}^2$ as a "rotation" of irrational angle. That is why the orbit of a point in $\partial\mathbb{H}^2$ under the action of e is dense in $\partial\mathbb{H}^2$.

We further note that the restriction of every ϕ_i , $i = 2, \dots, r$, to $\text{Tr}(\Gamma_1^{(2)})$ is not the identity. Therefore, since $\mathbb{Q}(\text{tr}(g_1^2)) = \mathbb{Q}(\text{Tr}(\Gamma_1^{(2)}))$ for every $g_1 \in \Gamma_1$ hyperbolic, the Galois conjugates of every hyperbolic element $g_1 \in \Gamma_1$ are different.

We will show by induction that $(\partial\mathbb{H}^2)^r = \overline{\Gamma(\xi)}$. By abuse of notation, when it is clear from the context, we will note the projection of Γ to the first m factors also with Γ .

Case $m = 2$: Let $g = (g_1, g_2) \in \Gamma$ be a transformation such that g_2 is an elliptic transformation of infinite order. Such g_2 exists because Γ_2 is not discrete. The isometry g_1 is hyperbolic because Γ_1 is discrete.

Let η_1 be the attractive fixed point of g_1 . First we note that $\{\eta_1\} \times \partial\mathbb{H}^2$ is in $\overline{\Gamma(\xi)}$. The reason is that for any point $\eta_2 \in \mathbb{H}^2$ there is a sequence $\{n_k\}$ of powers such that $g_2^{n_k}(\xi_2) \rightarrow \eta_2$ when $n_k \rightarrow \infty$ and additionally, because of the dynamics of the hyperbolic isometries, $g_1^{n_k}(\xi_1) \rightarrow \eta_1$ when $n_k \rightarrow \infty$. The only problem could be that ξ_1 is the repelling fixed point of g_1 . In this case we consider g^{-1} instead of g .

Now let $\zeta = (\zeta_1, \zeta_2)$ be a point in $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. Since Γ_1 is a cofinite Fuchsian group, \mathcal{L}_{Γ_1} is the whole boundary of \mathbb{H}^2 . It is also the minimal closed Γ_1 -invariant subset of $\partial\mathbb{H}^2$ (see Theorem 5.3.7 in [1]). Therefore $\mathcal{L}_{\Gamma_1} = \overline{\Gamma_1(\eta_1)}$ and there is a sequence of elements $\{h_n\}$ in Γ such that $(h_n)_1(\eta_1) \xrightarrow{n \rightarrow \infty} \zeta_1$.

The points $(\eta_1, (h_n)_2^{-1}(\zeta_2))$ are points in $\overline{\Gamma(\xi)}$ because it contains $\{\eta_1\} \times \partial\mathbb{H}^2$. Therefore all the points $((h_n)_1(\eta_1), \zeta_2)$ are in $\overline{\Gamma(\xi)}$ and hence their limit $\zeta = \lim_{n \rightarrow \infty} ((h_n)_1(\eta_1), \zeta_2)$ is also in $\overline{\Gamma(\xi)}$. Thus $\overline{\Gamma(\xi)} = (\partial\mathbb{H}^2)^2$.

Case $m \geq 3$: Let us assume that $(\partial\mathbb{H}^2)^{m-1} = \overline{\Gamma((\xi_1, \dots, \xi_{m-1}))}$. The aim is to prove that $(\partial\mathbb{H}^2)^m = \overline{\Gamma((\xi_1, \dots, \xi_m))}$.

First we show that there exists $g = (g_1, \dots, g_m) \in \Gamma$ such that g_1, \dots, g_{m-1} are hyperbolic and g_m is elliptic of infinite order. Indeed, since Γ_m is not discrete, there is $g' = (g'_1, \dots, g'_m) \in \Gamma$ such that g'_m is elliptic of infinite order. If all the other g'_i are hyperbolic, then $g = g'$. Otherwise, since $\mathbb{Q}(\text{tr}(g_1'^2)) = \mathbb{Q}(\text{tr}(g_1'^{2n}))$ for every integer $n \neq 0$ and also all the Galois conjugates of the hyperbolic element $g_1'^2$ are different, we can apply Lemma 3.3 to $g_1'^2$. In this way we find $\tilde{g}' \in \Gamma$ such that \tilde{g}'_m is elliptic and among the other \tilde{g}'_i there are strictly more hyperbolic elements than in the initial isometry. Thus, after possibly using Lemma 3.3 several times, we can find $g = (g_1, \dots, g_m) \in \Gamma$ such that g_1, \dots, g_{m-1} are hyperbolic and g_m is elliptic of infinite order.

Let η_i denote the attractive fixed point of g_i for $i = 1, \dots, m$. We can assume that ξ_i is none of the fixed points of g_i . The reason is that ξ is the projection on the Furstenberg boundary of the attractive fixed point of an element h in Γ . By Lemma 1.3, we can find h' such that h'_i and g_i do not have any common fixed point for $i = 1, \dots, m-1$. Then instead of the attractive fixed point of h we can take the attractive fixed point of h' .

As in the previous case we see that $\{\eta_1\} \times \dots \times \{\eta_{n-1}\} \times \partial\mathbb{H}^2$ is a subset of $\overline{\Gamma(\xi)}$ and that for any point $(\zeta_1, \dots, \zeta_m)$ in the closure of the Γ -orbit of $\{\eta_1\} \times \dots \times \{\eta_{m-1}\} \times \partial\mathbb{H}^2$, the points in $\{\zeta_1\} \times \dots \times \{\zeta_{m-1}\} \times \partial\mathbb{H}^2$ are also in the closure of the orbit. Hence by the induction hypothesis, $(\partial\mathbb{H}^2)^m = \overline{\Gamma((\xi_1, \dots, \xi_m))}$. \square

Corollary 3.5. *Let Γ be as in Theorem 3.4. Then \mathcal{L}_Γ is connected.*

Proof. Since \mathcal{L}_Γ is the closure of the attractive fixed points of the hyperbolic isometries in Γ , it is also the closure of \mathcal{L}_Γ^{reg} . Hence if \mathcal{L}_Γ^{reg} is connected, then \mathcal{L}_Γ is connected too. $\mathcal{L}_\Gamma^{reg} = F_\Gamma \times P_\Gamma$ is connected because F_Γ is a torus and P_Γ is convex by Theorem 1.5. \square

Corollary 3.6. *Let Δ be an irreducible arithmetic subgroup of $\mathrm{PSL}(2, \mathbb{R})^r$ with $r \geq 2$ and Γ a subgroup of Δ such that $p_j(\Gamma)$ is a Fuchsian group admitting a modular embedding. Then F_Γ is a q -dimensional torus, where q is the smallest power for which $p_j(\Gamma)$ has a modular embedding in an irreducible arithmetic subgroup of $\mathrm{PSL}(2, \mathbb{R})^q$.*

Proof. Without loss of generality we can assume that $j = 1$. We denote $S := p_1(\Gamma)$.

Let $l := \mathbb{Q}(\mathrm{Tr}(S^{(2)}))$ and $\sigma_1, \dots, \sigma_m$ its Galois isomorphisms. Since $q \leq m$ is the smallest power for which $S^{(2)}$ can be embedded in an irreducible arithmetic subgroup of $\mathrm{PSL}(2, \mathbb{R})^q$, we have that $\sigma_i(\mathrm{Tr}(S^{(2)}))$ is not bounded for $i = 1, \dots, q$ and bounded for $i = q + 1, \dots, m$. Hence for every σ_a with $a \in \{1, \dots, q\}$ there is a ϕ_b with $b \in \{1, \dots, r\}$ such that $\phi_b|_l = \sigma_a$ and also for every ϕ_b with $b \in \{1, \dots, r\}$ the restriction $\phi_b|_l$ equals σ_a for some $a \in \{1, \dots, q\}$.

The rest of the proof is analogous to the proof of Lemma 5.16 in [8], which proves the statement in the case when $p_j(\Gamma)$ is an arithmetic Fuchsian group. \square

3.2. The projective limit set. Recall that by Corollary 5 by Schmutz and Wolfart [16] for all $g = (g_1, \dots, g_r)$ in Γ such that g_1 is hyperbolic, we have the inequality $\mathrm{tr}(g_i) < \mathrm{tr}(g_1)$ for $i = 2, \dots, r$. Hence for every hyperbolic (or mixed) g in Γ the inequalities $\ell(g_i) < \ell(g_1)$ for $i = 2, \dots, r$ hold. Thus the translation direction $L(g)$ is in $\{(x_1 : \dots : x_r) \mid x_1 > x_i \text{ for all } i = 2, \dots, r\}$.

Since Γ is nonelementary, by Theorem 1.1 it follows that the translation directions of the hyperbolic isometries in Γ are dense in P_Γ . So we have the following proposition.

Proposition 3.7. *Let $\Gamma < \mathrm{PSL}(2, \mathbb{R})^r$ be a modular embedding of a semi-arithmetic Fuchsian group S . Then*

$$P_\Gamma \subseteq \{(x_1 : \dots : x_r) \mid x_1 \geq x_i \text{ for all } i = 2, \dots, r\}.$$

Remark. (i) In the above proposition, r is not necessarily the smallest power for which Γ has a modular embedding in an irreducible arithmetic subgroup of $\mathrm{PSL}(2, \mathbb{R})^r$

(ii) There are examples of semi-arithmetic groups that do not admit modular embeddings and for which the proposition is not true. E.g. the strictly semi-arithmetic examples from Theorem 1 in [16].

As this proposition already shows, P_Γ is not \mathbb{RP}_+^{r-1} and hence the limit set \mathcal{L}_Γ is not the whole geometric boundary of $(\mathbb{H}^2)^r$. Nevertheless, we can still ask the question whether P_Γ contains an open subset of \mathbb{RP}_+^{r-1} or not.

The next theorem gives sufficient conditions for the limit set to be of nonempty interior.

Theorem 3.8. *Let Γ be a subgroup of an irreducible arithmetic group Δ in $\mathrm{PSL}(2, \mathbb{R})^r$ with $r \geq 2$ such that, for one $j \in \{1, \dots, r\}$, $p_j(\Gamma)$ is a cofinite Fuchsian group and for all the others $i = 1, \dots, r$, $i \neq j$, the group $p_i(\Gamma)$ is not discrete and such that, for every $g_j \in p_j(\Gamma)$ hyperbolic, $\mathbb{Q}(\mathrm{tr}(g_j^2)) = \mathbb{Q}(\mathrm{Tr}(p_j(\Gamma^{(2)})))$. Then \mathcal{L}_Γ contains a subset homeomorphic to D^{2r-1} (the $(2r - 1)$ -dimensional ball).*

Proof. It is enough to show that the regular limit set $\mathcal{L}_\Gamma^{reg} = F_\Gamma \times P_\Gamma$ contains a subset homeomorphic to D^{2r-1} . By Theorem 3.4, the Furstenberg limit set equals the Furstenberg boundary and hence contains an r -dimensional ball. It remains to show that P_Γ contains an $(r - 1)$ -dimensional ball. By Benoist's theorem in Section 1.2 in [2], for Zariski dense groups Γ , the projective limit set P_Γ is of nonempty interior. Hence it suffices to prove that Γ is Zariski dense.

Without loss of generality we assume that $j = 1$. For each $i = 2, \dots, r$, there is a hyperbolic element h in $p_1(\Gamma)$ such that $\phi_i(h)$ is elliptic of infinite order and thus $\mathrm{tr}(h) \neq \phi_i(\mathrm{tr}(h))$. Additionally, for $i = r + 1, \dots, n$, $\phi_i(\mathrm{Tr}(p_1(\Gamma^{(2)}))) \subseteq [-2, 2]$. Therefore, for $i = 2, \dots, n$, $\phi_i|_{\mathrm{Tr}(p_1(\Gamma^{(2)}))} \neq id$ and so the fields generated by $\mathrm{Tr}(p_1(\Gamma^{(2)}))$ and $\mathrm{Tr}(p_1(\Delta^{(2)}))$ coincide. Hence by Corollary 2.2, the group Γ is Zariski dense. \square

4. SUBGROUPS WITH PARABOLIC ELEMENTS

The existence of parabolic elements in the group gives additional information on the projective limit set (see Proposition 4.9). As a corollary, we propose an alternative proof in this case of Theorem 3.8 without using the result of Benoist [2].

Proposition 4.9. *Let Δ be an irreducible arithmetic subgroup of $\mathrm{PSL}(2, \mathbb{R})^r$ with $r \geq 2$ and Γ a nonelementary subgroup of Δ . If Γ contains a parabolic element, then P_Γ contains the point $(1 : \dots : 1)$.*

Proof. Since Δ is arithmetic, it is commensurable with an arithmetic group derived from a quaternion algebra $\Gamma(A, \mathcal{O})^*$ where $\Gamma(A, \mathcal{O})$ is a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Hence there is $k \in \mathbb{N}$ such that, for each $g = (g_1, \dots, g_r)$ in Γ , g^k is in $\Gamma(A, \mathcal{O})^*$.

There is a subgroup S of $\Gamma(A, \mathcal{O})$ such that $S^* = \Gamma \cap \Delta \cap \Gamma(A, \mathcal{O})^*$. The group Γ is commensurable with the subgroup S^* and hence S is nonelementary. The group S^* also contains a parabolic isometry.

We will show that $\mathcal{L}_{S^*}^{reg}$ is not empty and P_{S^*} contains $(1 : \dots : 1)$.

Let T_u be a hyperbolic isometry in S . The connection between the translation length of T_u and its trace is given by $\mathrm{tr}(T_u) = 2 \cosh(\ell(T_u)/2)$. The translation length of T_u is equal to the length of the only simple closed geodesic in $\langle T_u \rangle \backslash \mathbb{H}^2$. For a Fuchsian group Γ there is a bijection between its hyperbolic elements and the closed geodesics in $\Gamma \backslash \mathbb{H}^2$. Hence for the length of every closed geodesic in $\Gamma \backslash \mathbb{H}^2$ there is at least one hyperbolic transformation in Γ with this translation length.

A *Y-piece* is a surface of constant curvature -1 and of signature $(0, 3)$, i.e. homeomorphic to a topological sphere with three points removed. By Leuzinger and the autor [9], Corollary 2.3, which is true not only for Fuchsian groups but also for nonelementary (non-discrete) subgroups of $\mathrm{PSL}(2, \mathbb{R})$, an element $T_v \in S$ exists such that $\langle T_u, T_v \rangle \backslash \mathbb{H}^2$ contains a Y-piece with one cusp and two boundary geodesics of length $\ell(T_u)$ and $\ell(T_v)$. By Schmutz [15], Lemma 1, for all $n \in \mathbb{N}$, this Y-piece contains a closed geodesic such that the corresponding hyperbolic isometry T_n satisfies

$$\mathrm{tr}(T_n) = n(\mathrm{tr}(T_u) + \mathrm{tr}(T_v)) - \mathrm{tr}(T_u).$$

For n big enough, all ϕ -conjugates $\phi_i(T_n)$ of T_n are hyperbolic because

$$\mathrm{tr}(\phi_i(T_n)) = |\phi_i(\mathrm{tr}(T_n))| = |n(\phi_i(\mathrm{tr}(T_u)) + \phi_i(\mathrm{tr}(T_v))) - \phi_i(\mathrm{tr}(T_u))|$$

is unbounded when n is unbounded. (In this way we found a hyperbolic element in S^* , i.e. a hyperbolic element in S such that all its ϕ -conjugates are hyperbolic.)

We set $A := \mathrm{tr}(T_u) + \mathrm{tr}(T_v)$ and $B := \mathrm{tr}(T_u)$. Then

$$\ell(T_n) = 2 \operatorname{arccosh} \frac{\mathrm{tr}(T_n)}{2} = 2 \operatorname{arccosh} \frac{nA - B}{2} \stackrel{n \rightarrow \infty}{\approx} 2 \ln(nA - B).$$

Hence for big n , the translation direction of T_n is asymptotic to

$$(2 \ln(n) + 2 \ln(A) : \dots : 2 \ln(n) + 2 \ln(|\phi_i(A)|)),$$

which converges towards $(1 : \dots : 1)$. Thus $(1 : \dots : 1)$ is in P_{S^*} and hence $(1 : \dots : 1)$ is in P_Γ . \square

Remark. Examples of groups with $r = 2$ are the triangle groups $(5, \infty, \infty)$ and $(2, 5, \infty)$. For them we have also that their limit cone C_Γ is a subset of $P := \{(x_1 : x_2) \mid x_1 \geq x_2\}$. Even more, $C_\Gamma = P$ because $(1 : 1) \in C_\Gamma$ (Proposition 4.9), $(1 : 0) \in C_\Gamma$ (the second projection of the modular embedding contains an elliptic isometry of infinite order) and the two points are connected (Theorem 1.5). Therefore their limit set is “half” of the geometric boundary of $(\mathbb{H}^2)^2$.

Alternative proof of Theorem 3.8 in the case with parabolic isometries. As in the proof of Theorem 3.8 it is enough to show that P_Γ contains an $(r - 1)$ -dimensional ball.

Using Lemma 3.3 we can construct elements $g_2, \dots, g_r \in \Gamma$ such that g_{ii} is elliptic for $i = 2, \dots, r$ and g_{ik} is hyperbolic for $i \neq k$. Hence each translation directions $L(g_i)$, which is in the limit cone of Γ , has a coordinate 0 at the i -th place.

By Proposition 4.9, P_Γ and therefore the limit cone of Γ contains the point $(1 : \dots : 1)$. We can not represent $(1 : \dots : 1)$ as a linear combination of $L(g_2), \dots, L(g_r)$ because, as remarked before, for $L(g_i) = (x_{i1} : \dots : x_{ir})$ we have $x_{i1} > x_{ij}$ for all $j = 2, \dots, r$ and $i = 2, \dots, r$. Therefore the limit cone of Γ , which is convex by Lemma 1.5, contains the convex hull of r linearly independent points and hence its interior P_Γ contains an $(r - 1)$ -dimensional ball. \square

Remark. If Γ does not contain parabolic elements, then from the above proof it follows that \mathcal{L}_Γ contains a subset homeomorphic to D^{2r-2} .

REFERENCES

- [1] A. Beardon, *The Geometry of discrete groups*, Graduate Texts in Mathematics **91**, Springer-Verlag, New York, 1995.
- [2] Y. Benoist, *Propriétés asymptotiques des groupes linéaires*, *Geom. Funct. Anal.* **7** (1997), 1-47.
- [3] A. Borel, *Commensurability classes and volumes of hyperbolic 3-manifolds* *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **8** (1981), 1-33.
- [4] P. Cohen and J. Wolfart, *Modular embeddings for some non-arithmetic Fuchsian groups*, *Acta Arith.* **61** (1990), 93-110.
- [5] F. Dal’Bo and I. Kim, *A criterion of conjugacy for Zariski dense subgroups*, *C. R. Acad. Sci. Paris*, 330 série I (2000), 647-650.
- [6] P. Eberlein, *Geometry of nonpositively curved manifolds*, *Chicago Lectures in Math.*, University of Chicago Press, Chicago and London, 1996.
- [7] S. Geninska, *The limit set of subgroups of a class of arithmetic groups*, <http://digbib.ubka.uni-karlsruhe.de/volltexte/1000014024>, Dissertation, Karlsruhe, 2009.

- [8] S. Geninska, *The limit set of subgroups of arithmetic groups in $\mathrm{PSL}(2, \mathbb{C})^q \times \mathrm{PSL}(2, \mathbb{R})^r$* , arXiv:1001.1720.
- [9] S. Geninska and E. Leuzinger, *A geometric characterization of arithmetic Fuchsian groups*, *Duke Math. J.* **142** (2008), 111-125.
- [10] G. Hardy and E. Wright, *An introduction to the theory of numbers*, Oxford University Press, London, fourth edition, 1962.
- [11] E. Leuzinger, *On the trigonometry of symmetric spaces*, *Comment. Math. Helvetici* **67** (1992), 252-286.
- [12] G. Link, *Limit Sets of Discrete Groups acting on Symmetric Spaces*, <http://digbib.ubka.uni-karlsruhe.de/volltexte/7212002>, Dissertation, Karlsruhe, 2002.
- [13] C. Maclachlan and A. Reid, *The Arithmetic of Hyperbolic 3-Manifolds*, Graduate Texts in Mathematics **219**, Springer-Verlag, 2003.
- [14] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, Berlin, 1991.
- [15] P. Schmutz, *Arithmetic groups and the length spectrum of Riemann surfaces*, *Duke Math. J.* **84** (1996), 199-215.
- [16] P. Schmutz Schaller and J. Wolfart, *Semi-arithmetic Fuchsian groups and modular embeddings*, *J. London Math. Soc. (2)* **61** (2000), 13-24.
- [17] J. H. Sampson, *Sous-groupes conjugués d'un groupe linéaire*, *Ann. Inst. Fourier* **26** (2)(1976), 1-6.
- [18] J. Stillwell, *Classical Topology And Combinatorial Group Theory*, no. 72 in Graduate Texts in Math., Springer-Verlag, second ed., 1993.

KARLSRUHER INSTITUT FÜR TECHNOLOGIE, KAISERSTR. 89-93, 76133 KARLSRUHE, GERMANY

E-mail address: `slavyana.geninska@kit.edu`

UNIVERSITÉ DE PROVENCE, 39 RUE F. JOLIOT CURIE, 13453 MARSEILLE, FRANCE

E-mail address: `geninska@latp.univ-mrs.fr`