

# ON ARITHMETIC FUCHSIAN GROUPS AND THEIR CHARACTERIZATIONS

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ABSTRACT. This is a small survey paper about connections between the arithmetic and geometric properties in the case of arithmetic Fuchsian groups.

## 0. INTRODUCTION

In this small survey paper we discuss the connections between the arithmetic and geometric properties in the case of arithmetic Fuchsian groups. The paper is based on the works of the author [7] (with Enrico Leuziger), [8] and [9].

This paper is not meant to be a general introduction to lattices and in particular arithmetic lattices. Since they are classical objects, there are some very good introductory books on the field. See for example “Fuchsian Groups” by Svetlana Katok [13], “The Arithmetic of Hyperbolic 3-Manifolds” by Colin Maclachlan and Alan Reid [16], “The geometry of discrete groups” by Alan Beardon for Fuchsian and Kleinian groups and “Introduction to Arithmetic Groups” by Dave Witte Morris [18] for lattices in Lie groups in all dimensions.

The paper is organized as follows. First we give some general information about arithmetic lattices. In the second section we concentrate on characterizations of arithmetic Fuchsian groups. Further, in the third and last section, we discuss the limit sets of subgroups of irreducible arithmetic lattices in  $\mathrm{PSL}(2, \mathbb{R})^r$  with  $r \geq 2$ .

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## 1. SOME FACTS ABOUT ARITHMETIC LATTICES

Let  $G$  be a connected linear real semisimple Lie group with finite center. For simplicity, we assume that  $G$  has no compact factors.

A subgroup  $\Gamma$  of  $G$  is an *arithmetic lattice* if and only if there exist

- a closed connected semisimple subgroup  $G'$  of some  $\mathrm{SL}(l, \mathbb{R})$  such that  $G'$  is defined over  $\mathbb{Q}$ ,
- an isomorphism  $f : G \rightarrow G'$  up to a compact factor,

such that  $f(\Gamma)$  is commensurable with the subgroup of integer points  $G'_\mathbb{Z} = G' \cap \mathrm{SL}(l, \mathbb{Z})$  of  $G'$ .

A more precise formulation of this definition of arithmetic lattices can be found in the book of Dave Witte Morris [18] (Definition 5.16).

As the next theorem shows, arithmetic lattices are very important because in many Lie groups the only irreducible lattices are arithmetic. This was proven by Margulis (see for example [17], Theorem A, p. 298) for semi-simple Lie groups with  $\mathbb{R}$ -rank at least two. For  $\mathbb{R}$ -rank one, semi-simple Lie groups, Corlette [4] and Gromov and Schoen [11] proved that this also holds for lattices in  $\mathrm{Sp}(n, 1)$  and  $F_4^{-20}$ . Combining these results, we have the following theorem.

**Theorem 1.1** ([17], [4], [11]). *If  $G$  is different from  $\mathrm{SO}(1, n)$  and  $\mathrm{SU}(1, n)$  up to finite covers and  $\Gamma$  is an irreducible lattice in  $G$ , then  $\Gamma$  is arithmetic.*

It is still an open question if there are nonarithmetic lattices in  $\mathrm{SU}(1, n)$  for  $n \geq 4$ . Nonarithmetic lattices in  $\mathrm{SO}(1, n)$  were constructed by Gromov and Piatetski-Shapiro [10] for each  $n$ , whereas nonarithmetic lattices in  $\mathrm{SU}(1, n)$  were constructed only for  $n \leq 3$ : the first examples were constructed by Mostow [19] for  $n = 2$  and the list was expanded by Deligne and Mostow in [6] for  $n \leq 3$ .

Arithmetic lattices have characteristic properties that become especially interesting in the cases where there are also nonarithmetic lattices. The following commensurability criterion for arithmeticity is due to Margulis (see for example [17]).

**Theorem 1.2** ([17]). *Let  $\Gamma$  be an irreducible lattice in  $G$ . Then  $\Gamma$  is arithmetic if and only if*

$$\mathrm{Comm}_G(\Gamma) := \{g \in G \mid g^{-1}\Gamma g \text{ and } \Gamma \text{ are commensurable}\}$$

*is dense in  $G$ .*

Recently Kapovich [12] proved the following characterization of arithmetic lattices.

**Theorem 1.3** ([12]). *Let  $\Gamma$  be an irreducible lattice in  $G$ . Then  $\Gamma$  is arithmetic if and only if  $\Gamma$  admits an irreducible faithful self-similar action on a regular rooted tree (of finite valency).*

In the next section, we concentrate on the characterizations of arithmetic lattices in  $\mathrm{PSL}(2, \mathbb{R})$ .

## 2. CHARACTERIZATIONS OF ARITHMETIC FUCHSIAN GROUPS

Let  $\Gamma$  be a cofinite Fuchsian group, i.e. a discrete subgroup of  $\mathrm{PSL}(2, \mathbb{R})$  of finite covolume. Such a  $\Gamma$  acts properly discontinuously

and isometrically on the hyperbolic plane  $\mathbb{H}$  and  $M = \Gamma \backslash \mathbb{H}$  is a Riemann surface. The *trace set* of  $\Gamma$  is defined to be

$$\mathrm{Tr}(\Gamma) := \{\mathrm{tr}(g) \mid g \in \Gamma\}$$

and encodes in a natural way the set of lengths of closed geodesics on  $M$ . Specifically, (see for instance the book of Maclachlan and Reid [16], p. 384)

$$2 \cosh \left( \frac{\ell(c_g)}{2} \right) = \pm \mathrm{tr}(g),$$

where  $\ell(c_g)$  is the length of the unique closed geodesic  $c_g$  associated to the  $\Gamma$ -conjugacy class of a hyperbolic element  $g$ .

It is a general question if certain classes of Fuchsian groups can be characterized by means of their trace set. There is a classical characterization of arithmetic Fuchsian groups due to Takeuchi which is based on number theoretical properties of their trace sets [24].

**Theorem 2.1** ([24]). *Let  $\Gamma$  be a cofinite Fuchsian group. Then  $\Gamma$  is arithmetic if and only if  $\Gamma$  satisfies the following two conditions:*

- (i)  *$K := \mathbb{Q}(\mathrm{Tr}(\Gamma))$  is an algebraic number field of finite degree and  $\mathrm{Tr}(\Gamma)$  is contained in the ring of integers  $\mathcal{O}_K$  of  $K$ .*
- (ii) *Let  $K_2$  be the field  $\mathbb{Q}(\mathrm{tr}(g)^2 \mid g \in \Gamma)$ . For any embedding  $\varphi$  of  $K$  into  $\mathbb{C}$ , which is not the identity if restricted to  $K_2$ , the set  $\varphi(\mathrm{Tr}(\Gamma))$  is bounded in  $\mathbb{C}$ .*

Luo and Sarnak pointed out large scale properties of the behaviour of the trace set of arithmetic Fuchsian groups. We say that the trace set of  $\Gamma$  satisfies the *bounded clustering* or *B-C property* if there exists a constant  $B(\Gamma)$  such that for all integers  $n$  the set  $\mathrm{Tr}(\Gamma) \cap [n, n + 1]$  has less than  $B(\Gamma)$  elements.

In [15] (Lemma 2.1) Luo and Sarnak made a first step towards a new geometric characterization of arithmetic Fuchsian groups by proving the following result:

**Theorem 2.2** ([15]). *If  $\Gamma$  is arithmetic, then  $\mathrm{Tr}(\Gamma)$  satisfies the B-C property.*

Sarnak conjectured that the converse of this theorem also holds.

**Conjecture 2.3** (Sarnak [20]). *If  $\mathrm{Tr}(\Gamma)$  satisfies the B-C property, then  $\Gamma$  is arithmetic.*

In [21] Schmutz makes an even stronger conjecture using the linear growth of a trace set instead of the B-C property.

**Conjecture 2.4** (Schmutz [21]). *If  $\mathrm{Tr}(\Gamma)$  has linear growth, then  $\Gamma$  is arithmetic.*

In [21] Schmutz proposed a proof of Conjecture 2.4 in the case when  $\Gamma$  contains at least one parabolic element. But unfortunately the proof contains a gap as we point out in [7]. In [7] we prove Sarnak's conjecture in the case when  $\Gamma$  contains a parabolic element.

Sarnak's conjecture is still open in the cocompact case. It is also unknown if Schmutz's conjecture holds even in the case when  $\Gamma$  contains a parabolic element. There are also other natural open questions related to the growth rate of the trace set, e.g. if there are Fuchsian groups whose trace set grows more quickly than linearly but more slowly than quadratic.

Now we give a sketch of the proof in [7] of Sarnak's conjecture in the case with parabolic elements.

*Sketch of the proof in [7].* This proof is based on the proof proposed by Schmutz in [21].

First we note that in order to show that  $\Gamma$  is arithmetic in the case with parabolic elements, it is enough to show that  $\text{tr}(g)^2$  are integers for all  $g \in \Gamma$ . Indeed, the fact that  $\Gamma$  contains parabolic elements means that without loss of generality we can assume that  $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is an element of  $\Gamma$ . For any element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  of  $\Gamma$  with  $c \neq 0$  (such an element exists because  $\Gamma$  is nonelementary), we consider the product

$$g_n := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & an + b \\ c & cn + d \end{pmatrix}.$$

Hence for any embedding  $\varphi$  of  $K := \mathbb{Q}(\text{Tr}(\Gamma))$  into  $\mathbb{C}$ , we have that  $\varphi(\text{tr}(g_n))$  goes to infinity when  $n$  goes to infinity.

By the characterization of Takeuchi (Theorem 2.1),  $\Gamma$  is arithmetic if and only if

- (ii) all embeddings  $\varphi$  of  $K$  into  $\mathbb{C}$  are the identity if restricted to  $\{\text{tr}(g)^2 \mid g \in \Gamma\}$ , i.e.  $\{\text{tr}(g)^2 \mid g \in \Gamma\} \subset \mathbb{Q}$ , and
- (i)  $\text{Tr}(\Gamma)$  is contained in the ring of integers of  $K$ .

Hence  $\Gamma$  is arithmetic if and only if  $\{\text{tr}(g)^2 \mid g \in \Gamma\} \subset \mathbb{Z}$ . This is equivalent to the well known fact that all non-cocompact arithmetic Fuchsian lattices are commensurable with  $\text{PSL}(2, \mathbb{Z})$ .

So our new objective is to prove that if  $\text{Tr}(\Gamma)$  satisfies the B-C property, then  $\{\text{tr}(g)^2 \mid g \in \Gamma\} \subset \mathbb{Z}$ .

The idea of Schmutz is to consider the free subgroups of 2 generators of  $\Gamma$ , where at least one of the generators is a parabolic isometry. The corresponding surface is at Y-piece (i.e. topologically a sphere with 3 holes) with a least one cusp. Remark that if  $\text{Tr}(\Gamma)$  satisfies the B-C

property, then the trace set of every subgroup of  $\Gamma$  satisfies the B-C property.

For each nonparabolic  $g \in \Gamma$ , there is a power  $n$  such that the group  $\langle g, P^n \rangle$  generated by  $g$  and  $P^n$  is free. Hence it is enough to prove that if  $\text{Tr}(\langle g, P^n \rangle)$  satisfies the B-C property, then  $\text{tr}(g)^2$  is an integer.

By using the proof of Proposition 4 in [21], we reduce the problem to proving the following key lemma:

**Lemma 2.5.** *Let  $T = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$  with  $t \geq 2$  and  $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Then the trace set  $\text{Tr}(\langle T, P \rangle)$  satisfies the B-C property if and only if  $t$  is an integer.*

If  $t$  is an integer, then  $\text{Tr}(\langle T, P \rangle)$  clearly satisfies the B-C property.

If  $t$  is not rational, all the elements

$$\text{tr}(TP^mT^{-n}) = mnt^2 - 2(m-n)t - 2, \text{ for } m, n \in \mathbb{N},$$

are different and we show that the number of elements in the corresponding set  $A$  grows at rate  $O(N \log N)$  and therefore  $A$  and hence  $\text{Tr}(\langle T, P \rangle)$  does not satisfy the B-C property.

If  $t$  is a rational number, not all of the above elements are different and it is easy to see that the corresponding set  $A$  satisfies the B-C property. So in order to show that if  $t \in \mathbb{Q} \setminus \mathbb{Z}$ , then  $\text{Tr}(\langle T, P \rangle)$  does not satisfy the B-C property, we need to consider more elements than the elements in  $A$ .

In [7] we show that the set  $B := \{mt^{2^k} - 2 \mid m, k \in \mathbb{N}\}$ , which is a subset of  $\text{Tr}(\langle T, P \rangle)$ , does not satisfy the B-C property.  $\square$

*Remark.* In his paper [21] Schmutz tries to prove a stronger version of Lemma 2.5, namely that the trace set  $\text{Tr}(\langle T, P \rangle)$  has linear growth if and only if  $t$  is an integer. For this, he considers an even bigger set of traces than  $B$  but we show in [7] that in many cases this set has only linear growth. So maybe we need to consider almost all elements in  $\text{Tr}(\langle T, P \rangle)$  in order to prove Schmutz's conjecture.

Concerning Sarnak's conjecture, it seems very difficult to use similar elementary methods for the cocompact case. As also Schmutz indicates in his survey paper [22], in this case we do not have the very convenient parabolic element  $P$ .

A corollary of the above proof is the following characterization of arithmetic Fuchsian groups. It was formulated by Schmutz in [21] but formally proven in [7].

**Theorem 2.6.** *A cofinite Fuchsian group  $\Gamma$  containing parabolic elements is arithmetic if and only if every subgroup of  $\Gamma$  generated by two*

parabolic elements with different fixed points is conjugated to a group generated by  $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$  with  $t$  an integer.

### 3. FUCHSIAN GROUPS EMBEDDED IN IRREDUCIBLE LATTICES IN $\mathrm{PSL}(2, \mathbb{R})^r$

The so called semi-arithmetic Fuchsian groups constitute a specific class of Fuchsian groups which can be embedded up to commensurability in arithmetic subgroups of  $\mathrm{PSL}(2, \mathbb{R})^r$  (see Schmutz Schaller and Wolfart [23]). These embeddings are of infinite covolume in  $\mathrm{PSL}(2, \mathbb{R})^r$  if  $r \geq 2$ . A trivial example is the group  $\mathrm{PSL}(2, \mathbb{Z})$  that can be embedded diagonally in any Hilbert modular group. Further examples are the other arithmetic Fuchsian groups and the triangle Fuchsian groups. It is an interesting question if the semi-arithmetic Fuchsian groups can be characterized by geometric means, for example by their trace set.

**An example of a semi-arithmetic Fuchsian group.** A Hecke group is a triangle group of type  $(2, m, \infty)$ . The Hecke groups are strictly semi-arithmetic (i.e. not arithmetic) except for  $m = 3, 4, 6$ . A Hecke group  $S$  of type  $(2, m, \infty)$  is generated (in  $\mathrm{PSL}(2, \mathbb{R})$ ) by  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 2 \cos(\pi/m) \\ 0 & 1 \end{pmatrix}$ , see Katok [13]. Hence all elements in  $S$  have entries that are algebraic integers in  $\mathbb{Q}(\cos(\pi/m))$ . Therefore  $S$  is a subgroup of  $\mathrm{PSL}(2, \mathcal{O}_F)$  where  $\mathcal{O}_F$  is the ring of integers in  $F := \mathbb{Q}(\cos(\pi/m))$ .

The field  $F$  is totally real. Let  $\phi_1, \dots, \phi_r$  with  $\phi_1 = id$  be all the embeddings of  $F$  into  $\mathbb{R}$ . The natural extensions of these embeddings to the matrices in  $\mathrm{PSL}(2, F)$  give us the group

$$\Delta := \{(\phi_1(g), \dots, \phi_r(g)) \mid g \in \mathrm{PSL}(2, \mathcal{O}_F)\}.$$

This group is an example of a Hilbert modular group and it is a cofinite (but not cocompact) irreducible lattice in  $\mathrm{PSL}(2, \mathbb{R})^r$ .

So we see that the group

$$\Gamma := \{(\phi_1(s), \dots, \phi_r(s)) \mid s \in S\}$$

is a subgroup of  $\Delta$  and that the projection of  $\Gamma$  to the first factor is  $S$ .

**“Small” limit sets of subgroups of lattices in  $\mathrm{PSL}(2, \mathbb{R})^r$ .** The semi-arithmetic Fuchsian groups give rise to discrete subgroups of infinite covolume of arithmetic groups in  $\mathrm{PSL}(2, \mathbb{R})^r$ . While lattices are fairly well understood, little is known about discrete subgroups of infinite covolume of semi-simple Lie groups. The main goal of [8] is to connect the arithmetic and the geometric properties of subgroups of

irreducible lattices in  $\mathrm{PSL}(2, \mathbb{C})^q \times \mathrm{PSL}(2, \mathbb{R})^r$ ,  $r + q \geq 2$ . We are in particular interested in those groups whose projection to one of the factors is (a subgroup of) an arithmetic Fuchsian (or Kleinian) group. We prove that these are exactly the nonelementary groups with the “smallest” possible limit set. In order to explain this, first we need several definitions.

The *geometric boundary* of  $(\mathbb{H}^2)^r$  is the set of equivalence classes of asymptotic geodesic rays. The *regular geometric boundary* of  $(\mathbb{H}^2)^r$  consists of the equivalence classes of regular geodesic rays, i.e. geodesic rays whose projections to all factors are nonconstant geodesic rays. The regular geometric boundary has a natural structure as the product of the Furstenberg boundary  $(\partial\mathbb{H}^2)^r$  and the projective part  $\mathbb{RP}_{>0}^{r-1}$ .

The *limit set* is the part of the orbit closure  $\overline{\Gamma(x)}$  in the geometric boundary where  $x$  is an arbitrary point in  $(\mathbb{H}^2)^r$ . A natural structure theorem for the regular limit set  $\mathcal{L}_\Gamma^{reg}$  of discrete nonelementary groups  $\Gamma$  due to Link ([14], Theorem 4.15) is the following:  $\mathcal{L}_\Gamma^{reg}$  is the product of the Furstenberg limit set  $F_\Gamma$  and the projective limit set  $P_\Gamma$ .

The next result is a compilation of results in [8]. It is an example of a connection between the geometric properties of a group with its arithmetic ones.

**Theorem 3.1.** *Let  $\Gamma$  be a finitely generated nonelementary subgroup of an irreducible lattice of  $\mathrm{PSL}(2, \mathbb{R})^r$  with  $r \geq 2$ . Then the following are equivalent:*

- (i) *The projective limit set  $P_\Gamma$  consists of exactly one point.*
- (ii)  *$\Gamma$  is a conjugate by an element in  $\mathrm{GL}(2, \mathbb{R})^r$  of*

$$\mathrm{Diag}(S) := \{(s, \dots, s) \mid s \in S\},$$

*where  $S$  is a subgroup of  $\mathrm{PSL}(2, \mathbb{R})$ .*

- (iii) *There exists  $j \in \{1, \dots, r\}$  such that the projection  $p_j(\Gamma)$  is contained in an arithmetic Fuchsian group.*
- (iv) *The limit set  $\mathcal{L}_\Gamma$  is embedded homeomorphically in a circle.*
- (v) *There is a totally geodesic embedding of  $\mathbb{H}^2$  in  $(\mathbb{H}^2)^r$  that is left invariant by the action of  $\Gamma$ .*

The main ingredients of the proof are the characterization of cofinite arithmetic Fuchsian groups by Takeuchi [24], the criterion for Zariski density of Dal’Bo and Kim [5] and a theorem of Benoist [2] stating that for Zariski dense subgroups of  $\mathrm{PSL}(2, \mathbb{R})^r$  the projective limit cone has nonempty interior.

**“Big” limit sets of subgroups of lattices in  $\mathrm{PSL}(2, \mathbb{R})^r$ .** In [8], it was shown that a nonelementary finitely generated subgroup of arithmetic groups in  $\mathrm{PSL}(2, \mathbb{R})^r$  with  $r \geq 2$  has the smallest possible limit set if and only if its projection to one factor is a subgroup of an arithmetic Fuchsian group. One could ask the question if all embeddings of semi-arithmetic Fuchsian groups have relatively small limit sets. The answer is “no” as shown in [9].

A cofinite Fuchsian group  $S$  that is commensurable to a subgroup of the projection to the first factor of an irreducible arithmetic group  $\Delta$  in  $\mathrm{PSL}(2, \mathbb{R})^r$  is said to have a *modular embedding* if for the natural embedding  $f : S \rightarrow \Delta$  there exists a holomorphic embedding  $F : \mathbb{H}^2 \rightarrow (\mathbb{H}^2)^r$  with

$$F(Tz) = f(T)F(z), \quad \text{for all } T \in S \text{ and all } z \in \mathbb{H}^2.$$

Examples of semi-arithmetic groups admitting modular embeddings are Fuchsian triangle groups (see Cohen and Wolfart [3]).

**Theorem 3.2** ([9]). *Let  $\Gamma$  be a subgroup of an irreducible arithmetic group in  $\mathrm{PSL}(2, \mathbb{R})^r$  with  $r \geq 2$  such that  $p_j(\Gamma)$  is a semi-arithmetic Fuchsian group admitting a modular embedding and  $r$  is the smallest power for which  $p_j(\Gamma)$  has a modular embedding in an irreducible arithmetic subgroup of  $\mathrm{PSL}(2, \mathbb{R})^r$ . Then*

- (i) *the Furstenberg limit set  $F_\Gamma$  is the whole Furstenberg boundary  $(\partial\mathbb{H}^2)^r$ ,*
- (ii) *the limit set  $\mathcal{L}_\Gamma$  contains an open subset of the geometric boundary of  $(\mathbb{H}^2)^r$ .*

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