On a Stationary Schrödinger Equation with Periodic Magnetic Potential

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Outline

Profile decomposition (Schindler and Tintarev, 2001)

Theorem

Let $u_k \in H$ be a bounded sequence. Then there exists $w^{(n)} \in H$, $g_k^{(n)} \in D$, $k, n \in \mathbb{N}$ such that for a renumbered subsequence

$$g_k^{(n)^{-1}} u_k \rightharpoonup w^{(n)}, \tag{1}$$

$$g_k^{(n)^{-1}} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m, \tag{2}$$

$$\sum_{n \in \mathbb{N}} \|w^{(n)}\|^2 \leq \limsup \|u_k\|^2 \tag{3}$$

and

$$u_k - \sum_{n \in \mathbb{N}} g_k^{(n)} w^{(n)} \overset{cw}{\to} 0. \tag{4}$$

Possible application

$$-\Delta u + u = \lambda f(x,u) \,\mathrm{in}\, \mathbb{R}^N \tag{5}$$

$f(x+z,u) = f(x,u)$, $z \in Z^N$.
$g_y u = u(\cdot + y)$, $y \in Z^N$.
(Lions, 1985)

Homogeneous right hand side

$$(\frac{1}{i}\nabla + A(x))^2 u + V(x)u = |u|^{q-1}u \text{ in } \Omega \subset \mathbb{R}^3. \tag{6}$$

$u : \mathbb{R}^N \mapsto \mathbb{C}$, $2 < q < 2^* = \frac{2N}{N-2}$
$A : \mathbb{R}^N \mapsto \mathbb{R}^N$, $A \in L^2_{loc}$ magnetic potential
$B \stackrel{\text{def}}{=} \operatorname{curl} A$ magnetic field.
$V : \mathbb{R}^N \mapsto [\delta, \infty)$ electric potential
$g_y u = e^{i\phi_y} u(\cdot + y)$
$\Delta_A \stackrel{\text{def}}{=} (\frac{1}{i}\nabla + A(x))^2$

$$\int |\nabla_A u|^2 \geq \int |\nabla |u||^2 \tag{7}$$

(Esteban and Lions, 1989)
(Schindler and Tintarev, 2002)

minimax RHS

$$(\frac{1}{i}\nabla + A(x))^2 u + V(x)u = g(x, |u|)u \text{ in } \mathbb{R}^N. \tag{8}$$

$g(x+y, s) = g(x, s),\ y \in \mathbb{Z}^N.$
$V(x+y) = V(x)\ y \in \mathbb{Z}^N.$
$B(x+y) = B(x)\ y \in \mathbb{Z}^N.$
(Arioli and Szulkin, 2003)
$g_{y+z} \neq g_y + g_z.$

with P. Bégout

$$(-\mathrm{i}\,\nabla + A)^2 u + V(x)u = \lambda f(x, |u(x)|)\frac{u}{|u|}, \quad \text{in } \mathbb{R}^N, \tag{9}$$

$$u \in H^1_{A,V}(\mathbb{R}^N), \tag{10}$$

$$H^1_{A,V}(\mathbb{R}^N) \stackrel{\text{def}}{=} \Big\{u \in L^2(\mathbb{R}^N);\ V|u|^2 \in L^1(\mathbb{R}^N) \text{ and} \tag{11}$$

$$(\nabla + \mathrm{i}A)u \in L^2(\mathbb{R}^N \mapsto \mathbb{C}^N)\Big\}. \tag{12}$$

Assumptions on A ($N \geq 3$)

$$A \in L^N_{\mathrm{loc}}(\mathbb{R}^N \mapsto \mathbb{R}^N). \tag{13}$$

$$\alpha_A \stackrel{\text{def}}{=} \sup_{j \in \mathbb{N}} \|A\|_{L^N(Q_j)} < \infty, \tag{14}$$

$$\forall j \in \{1, \ldots N\}, \ \operatorname{curl} A(x + e_j) \stackrel{\mathcal{D}^p(\mathbb{R}^N)}{=} \operatorname{curl} A(x), \tag{15}$$

Consequences (Leinfelder, 1983)

$$\operatorname{curl} A(x+y) \overset{\mathcal{D}(\mathbb{R}^N)}{=} \operatorname{curl} A(x),\ y \in \mathbb{Z}^N \ .$$

$y \in \mathbb{Z}^N \implies \exists \varphi_y \in W^{1,N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$ such that

$x \in \mathbb{R}^N \implies A(x+y) = A(x) + \nabla\varphi_y(x)$ a.e.

Assumption V

$$V \in L^1_{\mathrm{loc}}(\mathbb{R}^N \mapsto \mathbb{R}) \text{ and } \nu \overset{\text{def}}{=} \operatorname*{ess\,inf}_{x \in \mathbb{R}^N} V(x) > 0. \tag{16}$$

inner product

$\forall u, v \in H^1_{A,V}(\mathbb{R}^N)$,

$$\langle u, v\rangle_{H^1_{A,V}(\mathbb{R}^N)} = \mathrm{Re}\int_{\mathbb{R}^N} Vu\,\overline{v}\,\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u.\overline{\nabla_A v}\mathrm{d}x,$$

$$\|u\|^2_{H^1_{A,V}(\mathbb{R}^N)} = \langle u, u\rangle_{H^1_{A,V}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} V|u|^2\mathrm{d}x + \|\nabla_A u\|^2_{L^2(\mathbb{R}^N)},$$

Reason:
$\langle \left(\|u\|^2\right)', \varphi\rangle = 2\left(\int_{\mathbb{R}^N} Vu\overline{\varphi}\mathrm{d}x + \int_{\mathbb{R}^N} \nabla_A u\overline{\nabla_A\varphi}\mathrm{d}x\right)$

$H^1(\mathbb{R}^N) = H^1_{A,V}(\mathbb{R}^N)$

$$
\begin{aligned}
\int_{\mathbb{R}^N} |Au|^2 \mathrm{d}x &= \sum_{j\in\mathbb{N}} \int_{Q_j} |Au|^2 \mathrm{d}x \\
&\le \sum_{j\in\mathbb{N}} \|A\|^2_{L^N(Q_j)} \|u\|^2_{L^{2^*}(Q_j)} \\
&\le C^2 \alpha_A^2 \sum_{j\in\mathbb{N}} \|\,|u|\,\|^2_{H^1(Q_j)} \\
&= C^2 \alpha_A^2 \|\,|u|\,\|^2_{H^1(\mathbb{R}^N)}.
\end{aligned}
$$

A periodic

$y \in \mathbb{Z}^N \implies \exists! \psi_y \in W^{1,N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N \mapsto \mathbb{R})$, such that

$$\psi_y(0) = 0, \tag{17}$$

$$\forall x \in \mathbb{R}^N,\ \psi_y(x-y) + \psi_{-y}(x) = \psi_y(-y) = \psi_{-y}(y), \tag{18}$$

$$A(x+y) = A(x) + \nabla\psi_y(x), \tag{19}$$

Reason

$$A(x-y) = A(x) - \nabla\psi_y(x-y) = A(x) + \nabla\psi_{-y}(x).$$

$\Longrightarrow$

$$\forall x \in \mathbb{R}^N,\ \psi_y(x-y) + \psi_{-y}(x) = c.$$

Substituting first $x = 0$, then $x = y$ and using (17) we obtain (18).

delocalisations

$$\varphi_y \overset{\text{def}}{=} \psi_y - \frac{1}{2}\psi_y(-y), \tag{20}$$

Then $\varphi_y \in C(\mathbb{R}^N; \mathbb{R})$ and verifies,

$$\forall x \in \mathbb{R}^N,\ \varphi_y(x-y) + \varphi_{-y}(x) = 0, \tag{21}$$

$$A(x+y) = A(x) + \nabla\varphi_y(x), \tag{22}$$

a.e. and $\varphi_0 = 0$ over $\mathbb{R}^N$
$g_y : u \longmapsto e^{i\varphi_y} u(\cdot + y)$

RHS

$f : \mathbb{R}^N \times \mathbb{R}_+ \mapsto \mathbb{R}$
$\forall \varepsilon > 0$, $\exists p_\varepsilon \in (2, 2^*)$ and $C_\varepsilon > 0$ such that $\forall t \geq 0$,

$$|f(x,t)| \leq \varepsilon(t + t^{2^*-1}) + C_\varepsilon t^{p_\varepsilon - 1}, \tag{23}$$

$\implies |\{\lambda > 0 | \text{no solution}\}| = 0$ (Schechter and Tintarev, 1991; Tintarev, 1991; Jeanjean and Toland, 1998)

Value function

$\gamma(t) \stackrel{\text{def}}{=} \sup_{\|u\|^2=t} \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x$ where $F(x,t) \stackrel{\text{def}}{=} \int_0^t f(x,s)\mathrm{d}s$.
$\forall 1/\lambda \in \{2\inf\gamma'(t), 2\sup\gamma'(t)\}\exists$ critical sequence.
$\lim_{t\to 0}\gamma(t)/t = 0$
$\lim_{t\to\infty}\gamma(t)/t = \infty$

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