A Survey of Cocompactness Results

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Tools of Concentration-Compactness

Adimurthi et al. (2013)

1. Profile decompositions Gallagher (2001).
2. Cocompactness Tintarev 2010.
3. Liaponov-Schmidt reduction (critical exponent).

Ad Hoc

Early 80's lack of compacity (Palais-Smale condition): Aubin, Brezis, Coron, Esteban, Lieb, Lions, Nirenberg, Ulenbeck etc.

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Cocompactness: Lieb (1983) $H^1(\mathbb{R}^d) \overset{G}{\hookrightarrow} L^p(\mathbb{R}^d)$, $2 < p < 2^*$, $G =$ translations.

Lions (1985b) $\mathcal{D}^{1,2}(\mathbb{R}^d) \overset{G}{\hookrightarrow} L^{2^*}(\mathbb{R}^d)$, $G =$ dilations.

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Profile Decompositions

- Struwe (1984), (critical sequence) $\mathcal{D}^{1,2}(\mathbb{R}^d)$ $G =$ dilations.
- Lions (1987), $H^1(\mathbb{R}^d)$ $G =$ translations.
- Solimini (1995), includes $\mathcal{D}^{1,p}(\mathbb{R}^d) \overset{G}{\hookrightarrow} L^{p^*}(\mathbb{R}^d)$, $G =$dilations $\times$ translations.

Systematic Studies

- Lions (1985a,b), Palais-Smale condition: dichotomy, vanishing, or compactness in $H^1(\mathbb{R}^d)$ and $\mathcal{D}^{1,2}(\mathbb{R}^d)$. Extended to general measures by Chabrowski and Willem Chabrowski (1999).

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- **Wavelets**: Gérard (1996, 1998); Jaffard (1999); Bahouri et al. (2011): wavelet analysis, cocompactness via profile decomposition: $H^s(\mathbb{R}^d) \overset{G}{\hookrightarrow} L^p(\mathbb{R}^d)$, $p = 2d/(d-2s)$.

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- **Functional analytic**: Schindler and Tintarev (2001); Tintarev and Fieseler (2007); Solimini and Tintarev (2016). Abstract profile decomposition in Hilbert and Banach spaces.

G-weak convergence

Definition

Let B be a Banach space, $u_k \in B$, and G be a group of linear isometries acting on B. We say

$$u_k \overset{G}{\rightharpoonup} u,$$

if for all ϕ in B',

$$\lim_{k\to\infty} \sup_{g\in G} \langle g\phi, u_k - u\rangle = 0.$$

Example

$w \in H^1(\mathbb{R}^d)$, $d \geq 3$, $\|w\|_{H^1} = 1$.

1. $u_k = w(\cdot + ek)$. $u_k \rightharpoonup 0$, $u_k \overset{G}{\not\rightharpoonup} 0$
2. $v_k = 2^{\frac{d-2}{2}k} w(2^k \cdot)$. $v_k \rightharpoonup 0$, $v_k \overset{G}{\not\rightharpoonup} 0$.

cocompact

Definition

Let $B \hookrightarrow E$ Banach spaces. B is *cocompact* in E with respect to G, denoted $B \overset{G}{\hookrightarrow} E$, if $u_k \overset{G}{\rightharpoonup} w$ in $B \implies u_k \to w$ in E.

Wavelet profile decomposition

Definition

- *scale*: $h \stackrel{\text{def}}{=} (h_k) \in \mathbb{R}_+$, $h_k \to l$, $l \in \{0, 1, \infty\}$.
- *core*: $x \stackrel{\text{def}}{=} (x_k) \in \mathbb{R}^d$.
- *profile*: $w \in$ function space.

$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, $p = 2d/(d - 2s)$.

Gérard 96

Theorem

(u_k) bounded in $\dot{H}^s(\mathbb{R}^d)$, $\implies \exists\ h, x, (w_k^{(j)})$, and subsequece (u_k) such that $\forall l \geq 1$

$$u_k(x) = \sum_{j=1}^{l} \left(\frac{1}{h_k^{(j)}} \right)^{\frac{d}{p}} w^{(j)} \left(\frac{x - x_k^{(j)}}{h_k^{(j)}} \right) + v_k^{(l)}(x), \qquad (0.1)$$

where $\lim_{l\to\infty} \limsup_{k\to\infty} \|v_k\|_{L^p(\mathbb{R}^d)} \to 0$.

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Corollary

$\dot{H}^s(\mathbb{R}^d) \overset{G}{\hookrightarrow} L^p(\mathbb{R}^d)$

General profile decomposition

$(g_k) \in G$, $g_k \rightharpoonup 0 \iff g_k u \to 0 \ \forall u \in H$.

Theorem

Let H be a Hilbert space, G a group of linear isometries on H such that if $g_k \not\rightharpoonup 0 \implies g_k \to g_0$, and $u_k \in H$ a bounded sequence. $\exists$ a subsequece u_k, $w^{(i)} \in H$, $g_k^{(i)} \in G$ such that

1. $u_k - \sum_{i\geq 0} g_k^{(i)} w^{(i)} = v_k \overset{G}{\rightharpoonup} 0$.
2. ${g_k^{(i)}}^{-1} u_k \rightharpoonup w^{(i)}$.
3. $g_k^{(i)} {g_k^{(j)}}^{-1} \rightharpoonup 0$, $i \neq j$.
4. $\sum \|w^{(i)}\|^2 \leq \liminf \|u_k\|^2$.

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Banach spaces Solimini and Tintarev (2016), Δ-convergence.

Use with variational methods

$u_k \in X$ bounded, $J \in C^1(X \mapsto \mathbb{R})$.

(1) $J(u_k) \to c \neq 0$,

(2) $J'(u_k) \to 0$.

Steps:

1. Cocompactness + (1) + (4) $\implies v_k \to 0$
2. $c = J(u_k) + o(1) = \sum_i J(g_k^{(i)} w_k^{(i)}) + o(1)$.
3. $J'(u_k) = \sum_i J'(g_k^{(i)} w_k^{(i)}) + o(1) \to 0$.

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