PENTAGRAMMA MIRIFICUM. II. MODULI

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1 Introduction

1.1 Main topics

This article is a continuation of [S]. We discuss here the following topics.

(a) The moduli space of miraculous pentagrams as the del Pezzo surface S_5 of degree 5.

(b) Relation to the Vinberg's most algebraic $K3$ surface, [V].

(c) Some particular miraculous pentagrams which we call *Fibonacci pentagrams.*

(d) Fermat's ascent and the Jacobi's proof of the Poncelet theorem.

1.2 The surface S_5 and the root system A_4

The del Pezzo surface S_5 of degree 5 may be defined as the blowup of $\mathbb{P}^2$ at 4 points, say $p_1, \ldots, p_4$, in general position, cf. [M], Ch. IV, §2, Thm. 2.5.

It contains 10 lines: 6 strict transforms of the lines passing through $p_i, p_j, i \neq j$, together with 4 exceptional divisors; denote these lines $\ell_i, 1 \leq i \leq 10$. They meet at 15 points; their incidence graph is the Petersen graph P, [P].

In the language of [M], Ch. IV the surface S_5 corresponds to the root system $R = R_4$ of type A_4, so

$$R \subset H^2(S_5; \mathbb{Z}) \subset H^2(S_5; \mathbb{R}) \cong \mathbb{R}^5.$$

(we identify $Pic(S_5)$ with $H^2(S_5; \mathbb{Z})$ using the first Chern class).

The Weil group $W(R)$ is the symmetric group Σ_5 which coincides with $Aut(S_5) = Aut(P)$.

In [M], Def. 1.7 and Thm. 1.8 the root system R and the set of exceptional curves I_4 are identified explicitly.

1.2.1 Exercise

Describe the set of exceprional curves

$$I_4 = \{\ell_1, \ldots, \ell_{10}\} \subset H^2(S_5; \mathbb{Z}) \cong \mathbb{Z}^5$$

explicitly using Manin's description in [M], Ch. IV, §1, Def. 1.7. See the Petersen graph therein.

Solution. See [M], Ch. IV, §4, Fig. 4. The Petersen graph is Γ_4, in a somewhat unusual shape.

1.3 Three modular interpretations

The surface S_5 admits three modular incarnations:

(a) it is a compatification of the space $\mathcal{M}_{0,5}$ of 5-tuples of points on $\mathbb{P}^1$;

(b) it is the moduli space $\mathcal{P}_5$ of *miraculous pentagrams* - certain 5-tuples of points on $\mathbb{P}^2$.

The isomorphism between these two models is provided by the Veronese embedding

$$v:\ \mathbb{P}^1 \longrightarrow \mathbb{P}^2,\ v(x:y) = (x^2 : xy : y^2).$$

(c) The surface $\mathcal{P}_5$ is the complexification of a surface $\mathbb{R}\mathcal{P}_5$ over $\mathbb{R}$. According to Gauss [Ga, S], $\mathbb{R}\mathcal{P}_5$ may be interpreted as the moduli space of *miraculous pentagrams* — certain 5-tuples of points on the sphere S^2. Such pentagram is defined by 5 numbers $\alpha, \beta, \gamma, \delta, \epsilon$ satisfying 5 simple equations, see (1) in 2.1 below. These equations define what is now called the *cluster algebra of type* A_2, cf. [FZ, Example 1.3].

For a point $p \in \mathcal{P}_5$ we call the numbers $\alpha, \beta, \gamma, \delta, \epsilon$ the *Gauss coordinates* of p.

Jacobi has associated to a point $p \in \mathcal{P}_5$ an elliptic curve $E(p)$; its explicit equation in the Legendre form was written down by Gauss. On the other hand we can see the same curve in (a) realization as well.

1.4 Vinberg $K3$ surface and the Poncelet-Jacobi pencil

There is a remarkable $K3$ surface X_4 introduced by Vinberg [V]; it can be defined as follows. Take four points in general position $p_1, \ldots, p_4 \in \mathbb{P}^2$. We have six straight lines in $\mathbb{P}^2$ connecting the pairs if distinct points p_i, p_j. Let $P \longrightarrow \mathbb{P}^2$ be the double covering branched at these six lines; P is singular at the inverse images of p_i. If we blowup them we get a nonsingular surface which is X_4, see 4.1 below.

Alternatively, we can consider the double covering of S_5 ramified at ten lines ℓ_i, and then blow up their 15 points of intersection; the resulting surface will be isomorphic to S_4.

Thus we have the projection

$$\pi : X_4 \longrightarrow S_5$$

For the analog of the Peterson graph in X_4 see 4.3 below.

1.4.1 Elliptic pencil in X_4

The del Pezzo S_5 admits a pencil of conics

$$\phi' : S_5 \longrightarrow \mathbb{P}^1,$$

cf. 3.3. Let

$$\phi = \phi'\pi : \quad X_4 \longrightarrow \mathbb{P}^1.$$

For each $p \in S_5$ we can identify the corresponding Jacobi elliptic curve $E(p)$ with the fiber

$$E(p) = \phi^{-1}\phi'(p) \subset X_4.$$

These curves form an elliptic pencil in X_4, the *Poncelet pencil*, or *pope*, with 3 degenerate fibers whose Kodaira type is the affine D_6.

1.5 Fibonacci pentagrams

Among the miraculous pentagrams there is a distinguished one p_0 which corresponds to the regular pentagon. The corresponding elliptic curve $E(p_0)$ is degenerate, its Legendre equation is

$$y^2 = 1 - x^2.$$

The Gauss coordinates of p_0 are all equal to the golden ratio

$$\alpha_0 = \frac{1+\sqrt{5}}{2}.$$

It admits the well-known expression as (the simplest among all existant) continuous fraction, whose convergeants are ratios of Fibonacci numbers.

In Section 5 we define a sequence of points $p_n \in \mathcal{P}_5(\mathbb{Q})$ such that

$$\lim_{n \longrightarrow \infty} p_n = p_0.$$

The Gauss coordinates of p_n admit very simple rational expressions in through the Fibonacci numbers, see (7).

The elliptic curve $E(p_n)$ is defined over a real quadratic extension of $\mathbb{Q}$ (depending on n).

1.6 Fermat and Poncelet

In Sections 6 - 12 we discuss the Fermat's ascent method for solving the Diophantus "double equations" and its relation to the Poncelet theorem.

1.7 Notation

We will use the notation from [S]. Our ground field will be $\mathbb{C}$. So $\mathbb{P}^2$ will denote the projective plane over $\mathbb{C}$; if necessary, we will consider $\mathbb{RP}^2 \subset \mathbb{P}^2$, etc.

1.8 Acknowledgement

I am grateful to Alexander Kuznetsov for numerous consultations. In particular the idea to realize the Jacobi-Poncelet elliptic curves inside the Vinberg wonderful $K3$ surface belongs to him. I am grateful to Gil Bor for the collaboration at the initial stage of this work.

2 Miraculous pentagrams and the del Pezzo surface S_5

2.1 A surface related to the Napier laws of spherical trigonometry

The laws of spherical trigonometry were expressed by John Napier in 1614 as follows. Given a spherical right triangle, one labels its 5 'parts' (2 angles and 3 side lengths) by $\alpha_1, \alpha_2', \alpha_3, \alpha_4', \alpha_5$, where $\alpha' := \pi/2 - \alpha$.

Define

$$x_i := \tan^2 \alpha_i, \quad i = 1, \ldots, 5.$$

Then one finds that

$$\begin{gathered} 1 + x_1 = x_3 x_4, \quad 1 + x_2 = x_4 x_5, \quad 1 + x_3 = x_5 x_1, \\ 1 + x_4 = x_1 x_2, \quad 1 + x_5 = x_2 x_3. \end{gathered} \tag{1}$$

Below we will use the notation

$$(\alpha, \beta, \gamma, \delta, \epsilon) = (x_1, x_2, x_3, x_4, x_5)$$

as well.

Note that these 5 formulas are obtained from any one of them by cyclic permutation of the five variables.

It follows that the same formulas hold for the four other triangles obtained from the original one by cyclically permuting the five parts of the original triangle. The five triangles form a *pentagramma mirificum* ('miraculous pentagram'), a spherical right-angled five-pointed star. The hypotenuses form a self-polar pentagon (each vertex is perpendicular to the opposite edge).

Theorem 1. *The space of solutions to equations* (1) *is a smooth surface in* $\mathbb{R}^5$. *Its closure* $S \subset \mathbb{RP}^5$ *is a smooth compact surface, whose complexification is a smooth complex compact surface* $S_{\mathbb{C}} \subset \mathbb{CP}^5$.

We leave the proof to the reader.

2.2 The del Pezzo surface of degree 5

Pasquale del Pezzo, Duke of Caianello and Marquis of Campodisola (1859 – 1936).

Definition 1. *A* del Pezzo surface *is a non-singular complex projective algebraic surface with an ample anti-canonical line bundle. Its degree is the self intersection number of its canonical class.*

All degree 5 del Pezzo surfaces S_5 are holomorphicaly equivalent. One way to contruct it is as the blow up $S_5 \to \mathbb{CP}^2$ of 4 points $p_1, \ldots, p_4$ in $\mathbb{CP}^2$ in general position (no 3 are collinear). This produces 10 rational curves ("lines") on S_5; 4 exceptional divisors (the inverse images of the p_i's) and the strict transforms of the 6 lines through these 4 pts. The incidence graph of these 10 lines is easily seen to be the Peterson graph. This construction can also be made over the reals. The result, $X_{\mathbb{R}}$, is the connected sum of 5 copies of $\mathbb{RP}^2$ ("crosscaps").

2.3 Three definitions of S_5

A del Pezzo surface is a smooth projective surface V with the very ample anti-canonical bundle $\omega = \Omega_V^{-1}$. The degree of V is $d = (\omega_V, \omega_V)$, cf. [M].

We will be interested in the del Pezzo surface S_5 of degree 5. It is unique up to an isomorphism.

It admits several explicit constructions.

2.3.1 First construction: blowing up

S_5 is the blow up of $\mathbb{P}^2$ at 4 points $p_1, \ldots, p_4$, no three of them being collinear. S_5 contains 10 straight lines ℓ_i, $i = 1, \ldots, 10$: 4 exceptional divisors and the 6 strict tranforms of the lines passing through p_i, p_j. Their incidence graph, i.e., the graph with 10 vertices v_i, with v_i being connected with v_j iff ℓ_i meets ℓ_j, is the *Petersen graph*, [P, D1, D3]. It has 15 edges.

2.3.2 Second construction: Plücker

Consider the Plücker embedding

$$Pl : G(2,5) \longrightarrow \mathbb{P}^9 = \mathbb{P}^{\binom{5}{2}-1}$$

We denote the homogeneous coordinates in $\mathbb{P}^9$ by $x_{ij}, 1 \le i < j \le 5$.

Its image is defined by five Plücker equations

$$x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} = 0 \tag{2}$$

for all $1 \le i < j < k < l \le 5$. So we have 10 Plücker coordinates.

Warning: $\dim G(2,5) = 2 \cdot 3 = 6$, so among the 5 equations (2) only three are independent.

Claim 1. *Let $L \subset \mathbb{P}^9$ be a 4-dimensional linear subspace such that*

$$S := L \cap Pl(G(2,5))$$

is smooth (such L form a Zarisky dense open subset in the Grassmanian of all linear subspaces of dimension 4). Then S is the del Pezzo surface S_5.

See [D3, Prop. 8.5.1] or [CKS, Lemma 2.29].

2.3.3 Third construction: the space of stable curves $\overline{M_{0,5}}$

Let $M_{0,5}$ be the moduli space of curves of genus 0 with 5 distinct ordered marked points, i.e.

$$M_{0,5} = ((\mathbb{P}^1)^5 \setminus \bigcup \text{diagonals})/PGL(2).$$

Let $p_1, \ldots, p_4 \in \mathbb{P}^2$ be in general position, and

$$S_5^o = S_5 \setminus \bigcup_{i=1}^{10} \ell_i \subset S_5$$

where ℓ_i are ten straight lines from section 2.3.1.

Given $p \in S_5^o$, there is a unique conic C passing through $p, p_1, \ldots, p_4$. This conic is isomorphic to $\mathbb{P}^1$ whereupon 5 distinct points are marked; after passing to the quotient by $PGL(2)$, we get a point on $M_{0,5}$.

We get a morphism

$$S_5^o \longrightarrow M_{0,5}$$

which is an isomorphism. It extends to the compactifications,

$$S_5 \stackrel{\sim}{=} \overline{M_{0,5}}$$

where $\overline{M_{0,5}}$ is the moduli space of stable curves of genus 0 with 5 marked points, see [D1], [K].

3 Poncelet-Jacobi curve

3.1 An elliptic curve related to a couple of plane conics

Let $C, D \subset \mathbb{P}^2$ be two conics in general position. They meet at four points $p_1, \ldots, p_4$. They give rise to the dual conics $C^*, D^* \subset \mathbb{P}^{2*}$ in the dual projective space, intersection at 4 points $p_1^*, \ldots p_4^*$, corresponding to 4 bitangents of C, D.

The Poncelet-Jacobi elliptic curve is

$$E = E(C, D) = \{(p, \ell)|\ p \in C,\ p \in \ell, \ell \text{ tangent to } D\} \subset C \times D^* \subset \mathbb{P}^2 \times \mathbb{P}^{2*},$$

cf. [GH1]. The two projections

$$C \longleftarrow C \times D^* \longrightarrow D^*$$

induce maps

$$C \overset{\pi_1}{\longleftarrow} E \overset{\pi_2}{\longrightarrow} D^* \tag{3}$$

which are both double coverings ramified at 4 points, whence we get two involutions $i_1, i_2 : E \overset{\sim}{\longrightarrow} E$ whose composition $i_1 i_2$ is a translation by an element $a \in E$. Here we have to choose an origin of the group law of E to be

$$(p_i, \ell_i)$$

for some i, cf. [GH1, §2].

If a happens to be of finite order n, the Poncelet process closes up at the n-th step.

3.2 Equation

An explicit equation for E (due to Cayley) is described in [GH1]. Namely, denote the homogeneous coordinates in $\mathbb{P}^2$ by x_1, x_2, x_3. Let two conics C_1, C_2 be given by the equations

$$\sum_{i,j=1}^{3} c_{ij} x_i x_j = 0, \ \sum_{i,j=1}^{3} d_{ij} x_i x_j = 0.$$

where $C = (c_{ij}), D = (d_{ij})$ are symmetric 3×3 matrices. Then $\{C + \lambda D\}, \lambda \in \mathbb{C}$ is a pencil of conics passing through $\{p_1, \ldots, p_4\}$.

Then an equation for E is

$$\mu^2 = \det(C + \lambda D). \tag{4}$$

To the regular pentagon there corresponds a degenerate curve E. If we write a generic curve in the Legendre form

$$y^2 = (1 - x^2)(1 - k^2 x^2),$$

then for the degenerate case $k = 0$.

3.3 Pencil of conics, and another elliptic curve

We have 5 forgetful maps

$$f_i : \ \overline{M_{0,5}} \longrightarrow \overline{M_{0,4}} = \mathbb{P}^1, i = 1, \ldots, 5.$$

Each f_i has 3 singular fibers, and the nonsingular fibers

$$f_i^{-1}(x) \subset \overline{M_{0,5}} = S_5$$

are conics, and they form a pencil of conics.

Three singular fibers correspond to three possible decompositions of the set $\{1, 2, 3, 4\}$ into two pairs of two-element subsets.

Let $p \in S_5^o$; through p passes the conic $C(p)$ of this pencil. It intersects 4 of the lines ℓ_i.

Let $c_1, \ldots c_4$ be the points of intersection of $C(p)$ with these four lines. The two-sheeted covering of $C(p)$ ramified at c_i is an elliptic curve, to be denoted $E'(p)$.

4 Vinberg's $K3$ mirificum

4.1 Definition

The Vinberg's "most algebraic $K3$ surface" X_4 has been introduced and studied in depth in [V]. X_4 is the Vinberg's notion; we will use the notation $V = X_4$ as well.

It may be defined as follows, cf. [DBGKKW, 2.1], [GT, 5.1].

We start with the union of 6 straight lines in $\mathbb{P}^2$ with homogeneous coordinates $(x : y : z)$ given by the equation

$$xyz(x - y)(x - z)(y - z) = 0;$$

we consider the double covering $\overline{P}$ of $\mathbb{P}^2$ branched along these six lines (remark that one meets this object in the theory of KZ equation).

These lines meet three at a time at four points

$$p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0), p_3 = (0 : 0 : 1), p_4 = (1 : 1 : 1),$$

and our surface X_4 is obtained by blowing up $\overline{P}$ at these points.

4.2 Relation to S_5

On the other hand we can act in the reverse order, so to say, cf. [DBGKKW].

We blow up $\mathbb{P}^2$ at 4 points to get S_5. Then we consider the double cover

$$\pi_1 : \bar{S}_5 \longrightarrow S_5$$

ramified along its ten straight lines ℓ_i. The divisor

$$\sum_{i=1}^{10} \ell_i$$

represents the class $-2K_{S_5}$ in $Pic(S_5)$.

The ten lines ℓ_i intersect at 15 points, and the surface $\bar{S}_5$ is singular at their inverse images, let us denote them $\bar{p}_i, i = 1, \ldots, 15$.

If we blow all $\bar{p}_i$ we get a smooth surface

$$\pi_2 : \tilde{S}_5 \longrightarrow \bar{S}_5$$

which is isomorphic to X_4.

4.3 Analog of Petersen graph on X_4

Originally X_4 was defined as the unique $K3$ surface whose lattice of transcendental cycles has the form

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

see [V], 2.1. Let

$$S = Pic(X_4) \subset H_2(X_4, \mathbb{Z})$$

be the lattice of algebraic cycles; it has the Minkowski signature $(1, 19)$, so

$$O(S) = O_{19,1}(\mathbb{Z}).$$

The structure of this group has been investigated in [VK]. Let $O_r(S) \subset O(S)$ be the subgroup generated by reflections with respect to hyperplanes. It admits 25 generators $s_1, \ldots, s_{25}$, and relations are described by the corresponding Coxeter scheme.

This Coxeter scheme can be seen on [V], Fig. 1, or on [VK], Fig. 2 (a). We see that this scheme is a refinement of the Petersen graph, so we may suspect that X_4 is somehow related to S_5, and this is so indeed, as we have seen above.

It is not difficult to see 25 lines in X_4 which correspond to reflections s_i. Namely, they are 10 strict transforms of the lines $\ell_i \subset S_5$ with respect to the projection

$$\pi = \pi_1 \pi_2 : X_4 \longrightarrow S_5,$$

together with 15 exceptional divisors.

4.4 V versus S_5

Cf. [DBGKKW, 2.1]. As in 3.4 let $p_1, \ldots, p_4 \in \mathbb{P}^2$ be four points in general position. The del Pezzo S_5 is their blowing up:

$$b: \ S_5 \longrightarrow \mathbb{P}_2.$$

Let $\{C_t\}$ be the pencil of conics in $\mathbb{P}^2$ passing through $p_1, \ldots, p_4$, and let $\tilde{C}_t \subset S_5$ be the strict transform of C_t; we get a pencil of conics $\{\tilde{C}_t\}$ in S_5.

Let

$$\tilde{p}_{t,i} = b^{-1}(p_i) = \tilde{C}_t \cap \ell_i \in \tilde{C}_t \subset S_5, \ i = 1, \ldots 4.$$

Denote $V = X_{-1}$. We have a commutative square

$$\begin{array}{ccc} V & \xrightarrow{f_2} & S'' \\ {\scriptstyle b_2}\downarrow & & \downarrow{\scriptstyle b_1} \\ S' & \xrightarrow{f_1} & S_5 \end{array}$$

Here b_1 is the blowup of S_5 at 15 points $q_1, \dots, q_{15}$ of intersection of lines $\ell_i,\ i = 1, \dots, 10$.

The map $f_1 : S' \longrightarrow S_5$ is the double cover of S_5 ramified along the divisor $D = \sum_{i=1}^{10} \ell_i$.

Let $\ell_i' \subset S''$ be the strict transform of ℓ_i. The lines ℓ_i' are disjoint.

The map $f_2 : V \longrightarrow S''$ is the double cover of S_5' ramified along the divisor $D' = \sum_{i=1}^{10} \ell_i'$.

The map b_2 is the blowup of S' at points $f_1^{-1}(q_i),\ 1 \leq i \leq 15$.

4.5 A pencil of elliptic curves in V

Let

$$C_t' := f_1^{-1}(\tilde{C}_t) \subset S'.$$

It is the double cover of $\tilde{C}_t$ ramified at $\tilde{p}_{t,i}, i = 1, \dots, 4$, i.e. an elliptic curve. Let $E_t \subset V$ be the strict transform of C_i' under b_1.

Alternatively we can go the other way around. Let $C_t'' \subset S''$ be the strict transform of $\tilde{C}_t'$, and $p_{t,i}'' = b_1^{-1}(\tilde{p}_{t,i}) \in C_t'', i = 1, \dots, 4$. Then

$$E_t = f_2^{-1}(C_t'')$$

appearing now as the double covering of C_t'' ramified at $p_{t,i}'', i = 1, \dots, 4$.

This way we get map

$$\phi :\ V \longrightarrow \mathbb{P}^1 \tag{5}$$

having as non-singular fibers the curves E_t, and three singular fibers.

The realisation of V as a double covering shows that V admits an involution $i : V \xrightarrow{\sim} V$ which induces the involution on each fiber E_t of the pencil ϕ.

Claim 2. *The elliptic pencil ϕ has three singular fibers of Kodaira type I_2^*, or $\tilde{D}_6$ in the modern notation.*

In other words, using the notations of [Ko, Thm. 6.2], the singular fibers of ϕ have the form

$$\Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 + \Theta_6$$

where $\Theta_i = \mathbb{P}^1$, and the intersection graph of this divisor has Dynkin type $\tilde{D}_6$ (the affine D_6), cf. [BHPV], V, 7.

4.6 A combinatorial "Poncelet curve"

Consider a regular n-gon P_n and a set E_n of pairs (v, e) where v is a vertex of P_n and e is an edge of P_n containing v.

The set E_n is a torsor under the dihedral group D_n, and we have two projections

$$V_n \overset{\pi_{n,1}}{\longleftarrow} E_n \overset{\pi_{n,2}}{\longrightarrow} F_n. \tag{6}$$

This diagram is a discrete analogue of the diagram (3).

4.7 Two different families of elliptic curves

The point $p_i^* \in \mathbb{P}^{2*}$ may be defined as the variety of straight lines passing through p_i.

(a) Let us fix C and vary D in the pencil, so we get a family (C, D_t^*) and corresponding family of elliptic curves $E_t \subset C \times D_t^*$. They are all isomorphic since they are double coverings of C ramified at $p_1, \ldots, p_4$.

(b) Let us fix D and vary C in the pencil, so we get a family of elliptic curves $\tilde{E}_t \subset C_t \times D^*$.

The curve $\tilde{E}_t$ is the double covering of D^* ramified at $p_{1,t}^*, \ldots, p_{4,t}^*$ where $p_{i,t}$ are straight lines in $\mathbb{P}^2$ tangent to C_t and D, so $p_{i,t}^*$ varies when t varies.

The curves $\tilde{E}_t$ are not isomorphic, in fact they form a universal family.

The isomorphism class of $\tilde{E}_t$ is the cross ratio of $p_{1,t}^*, \ldots, p_{4,t}^*$ in D^*.

5 Fibonacci pentagrams

The results of this Section are obtained in collaboration with Gil Bor.

5.1 Definition of Fibonacci pentagrams

A *regular* miraculous pentagram has $\phi := \alpha = \beta = \gamma = \delta = \epsilon$, so that $1+\phi = \phi^2$, hence ϕ is the *golden ratio*,

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Let

$$\phi_n := \frac{F_{n+1}}{F_n}, \quad n = 1, 2, 3, \ldots$$

be the n-th convergent of ϕ, where F_n are the Fibonacci numbers,

$$F_1 = F_2 = 1, F_n = F_{n-1} + F_{n-2}.$$

Let 2 of the 5 parts of a right triangle be ϕ_n, say γ_n, ϵ_n, then the other 3 parts are determined by equations (1)

$$(\alpha_n, \beta_n, \gamma_n, \delta_n, \epsilon_n) = (\phi_{n+1}, \phi_{n+1}, \phi_n, \frac{\phi_{n+2}\phi_{n+1}}{\phi_n}, \phi_n). \tag{7}$$

We call the resulting pentagram the n-th *Fibonacci pentagram* P_n. The cases $P_1, \ldots, P_4$ are:

$$(2,2,1,3,1), \left(\frac{3}{2}, \frac{3}{2}, 2, \frac{5}{4}, 2\right), \left(\frac{5}{3}, \frac{5}{3}, \frac{3}{2}, \frac{16}{9}, \frac{3}{2}\right), \left(\frac{8}{5}, \frac{8}{5}, \frac{5}{3}, \frac{39}{25}, \frac{5}{3}\right).$$

5.2 Central projection, Poncelet

Consider central projection from the sphere onto a plane tangent to the sphere. It maps great circles to straight lines but does not preserve angles, in general. There is an exception though.

Lemma 1. *Under central projection onto a plane tangent to a sphere, the image of two orthogonal great circles, one of which passes through the tangency point, is a pair of orthogonal lines.*

Now take a convex self-polar spherical pentagon (or any self-polar n-gon with odd n). The diagonals of this pentagon form a smaller pentagon inside it. Fix a point P inside this smaller pentagon and consider the central projection of the spherical polygon onto the tangent plane at P. A consequence of the last Lemma is

Corollary 1. *The altitudes of the projected planar pentagon are concurrent, intersecting at P.*

Such a pentagon is called *orthocentric.*

Question. Is the converse true? I.e., is every orthocentric pentagon a central projection of a self-polar pentagon?

5.3 Gauss equation

The projected planar pentagon is a Poncelet polygon, i.e. it is inscribed in an ellipse and circumscribes another one. To this pair of ellipses there corresponds an elliptic curve such that the associated Poncelet map has order 5.

Gauss writes down this curve explicitly. It is written in the Legendre form

$$y^2 = (1 - x^2)(1 - k^2x^2)$$

and will be denoted $E(k)$. The Poncelet map associated with the planar pentagon is given by translation of order 5 on $E(k)$.

Gauss gives a formula for k in terms of the eigenvalues of the quadratic form in $\mathbb{R}^3$ whose null cone circumscribes the given pentagon. The characteristic equation for the eigenvalues of the quadratic form defining this null-cone is

$$t(2t - 1)^2 = (t - 1)\omega, \tag{8}$$

where

$$\omega = \alpha\beta\gamma\delta\epsilon,$$

its roots (eigenvalues of the quadratic form) are $G < 0 < G' < G''$, in terms of which

$$k^2 = \frac{G'^{-2} - G''^{-2}}{G'^{-2} - G^{-2}}. \tag{9}$$

For our Fibonacci polygons P_n,

$$\omega_n = \phi_n \phi_{n+1}^3 \phi_{n+2} = \frac{F_{n+2}^2 F_{n+3}}{F_{n+1}^2 F_n}. \quad (10)$$

Computer evidence shows that there are simple closed form formulas for the 3 roots, that one of the positive ones is always rational and that $\lim_{n\to\infty} k^2 = 0$. The 1st 5 cases are shown in Table 1.

n	G	G'	G''	k^2
1	$\frac{1}{4}\left(-\sqrt{33}-1\right)$	$\frac{1}{4}\left(\sqrt{33}-1\right)$	$\frac{3}{2}$	$\frac{1}{2}+\frac{25}{18\sqrt{33}} \approx 0.74$
2	$\frac{1}{8}\left(-\sqrt{145}-1\right)$	$\frac{5}{4}$	$\frac{1}{8}\left(\sqrt{145}-1\right)$	$\frac{339457-19175\sqrt{145}}{248832} \approx 0.44$
3	$\frac{1}{6}\left(-2\sqrt{19}-1\right)$	$\frac{1}{6}\left(2\sqrt{19}-1\right)$	$\frac{4}{3}$	$\frac{1}{2}-\frac{697}{512\sqrt{19}} \approx 0.19$
4	$\frac{1}{60}\left(-\sqrt{7761}-9\right)$	$\frac{13}{10}$	$\frac{1}{60}\left(\sqrt{7761}-9\right)$	$\frac{21231415601-238114071\sqrt{7761}}{3276800000} \approx 0.08$
5	$\frac{1}{160}\left(-\sqrt{54705}-25\right)$	$\frac{1}{160}\left(\sqrt{54705}-25\right)$	$\frac{21}{16}$	$\frac{1}{2}-\frac{2422967}{22050\sqrt{54705}} \approx 0.03$

Table 1: Fibonacci polygons.

We observe that:

- G'' is rational for n odd, G' is rational for n even. The first 10 such rational roots are

$$\frac{3}{2}, \frac{5}{4}, \frac{4}{3}, \frac{13}{10}, \frac{21}{16}, \frac{17}{13}, \frac{55}{42}, \frac{89}{68}, \frac{72}{55}, \frac{233}{178}$$

  These can be rewritten simply as

$$\frac{F_{n+3}}{2F_{n+1}}$$

  and one can check that these satisfy equations (1), with $\omega = \omega_n$ given by equation (10).

- After dividing equation (8) by the linear factor given by the rational root, one obtains a quadratic equation for the remaining 2 roots:

$$2t^2 + \frac{F_n}{F_{n+1}} t - \frac{F_{n+1}}{F_n} - \frac{F_n}{F_{n+1}} - 2 = 0,$$

  whose roots are

$$-\frac{F_n}{4F_{n+1}} \pm \frac{1}{4}\sqrt{\frac{F_n^2}{F_{n+1}^2} + \frac{8F_n}{F_{n+1}} + \frac{8F_{n+1}}{F_n} + 16}.$$

  One can now take these explicit expressions for G, G', G'' and use equation (9) to get an explicit formula for k^2.

6 Part II. Fermat's ascent and Poncelet theorem

In this Part we compare the Pierre Fermat's ascent method for solving the double equations of Diophantus, as described by André Weil, with the Poncelet's process.

7 The double equations of Diophantus

The "double equations" appear in the work of Diophantus, and have the form

$$ax^2 + bx + c = u^2, \ a'x^2 + b'x + c' = v^2, \tag{11}$$

cf. [Di], Book IV, Problem 23, [Dif].

We argue that Pierre Fermat's "ascent method" for solving (11), as described by Andre Weil, [W] Chap. II, §XV, is very close to the Poncelet's construction.

In both contexts certain elliptic curve plays the key role. For double equations this curve is described in [W], Chap. II. For the Poncelet construction it was discovered by Jacobi, and is nicely described in [GH1].

8 Emile Borel's theorem: a rationality criterium

The following theorem has been used by Bernard Dwork, [D], in his proof of the rationality of the zeta function of algebraic varieties over finite fields.

Theorem 1. *(E. Borel) Let K be a field,*

$$f(t) = \sum_{i \geq 0} a_i t^i \in K[[t]].$$

For any $m, n \in \mathbb{Z}_{\geq 0}$ define an $(m+1) \times (m+1)$ Hankel matrix

$$A_{n,m} = \begin{pmatrix} a_n & a_{n+1} & \dots & a_{n+m} \\ a_{n+1} & a_{n+2} & \dots & a_{n+m+1} \\ . & . & . & \\ a_{n+m} & a_{n+m+1} & \dots & a_{n+2m} \end{pmatrix} \tag{12}$$

Let $N_{n,m} = \det A_{n,m}$. Then $f(t)$ is a ratio of two polynomials

$$f(t) = \frac{p(t)}{q(t)}, \ p(t), q(t) \in K[t],$$

iff there exist m, n_0 such that for all $n \geq n_0$ $N_{n,m} = 0$.

See [B], [K], Ch. 5, §5, Lemma 5.

9 Cayley's theorem on the Poncelet porism

Consider two smooth conics in $\mathbb{P}^2$ given by equations $C(x) = 0, D(x) = 0$, $x = (x_0 : x_1 : x_2)$, $C(x) = \sum c_{ij}x_ix_j,\ D(x) = \sum d_{ij}x_ix_j$. Consider the formal power series

$$f(t) = \sqrt{\det(tC + D)} = \sum_{n\geq 0} a_n t^n$$

the corresponding matrices $A_{n,m}$, (12), and their determinants $N_{n,m}$ as in the previous Subsection.

Theorem 2. *(A. Cayley) The Poncelet construction yields a finite polygon with k sides if and only if $N_{2,m-1} = 0$ for $k = 2m + 1$, or $N_{3,m-2} = 0$ for $k = 2m$.*

See [C] or [GH1].

Let

$$g(t) = \det(tC + D); \tag{13}$$

it is a cubic polynomial.

Let $D^* \subset \mathbb{P}^{2*}$ be the dual conic whose points are straight lines $\xi \subset \mathbb{P}^2$ tangent to D.

The Poncelet - Jacobi elliptic curve $E(C, D)$ is by definition

$$E(C, D) = \{(x, \xi) \in C \times D^* |\ x \in \xi\} \tag{14}$$

Theorem 3. *The curve*

$$y^2 = g(t)$$

is birationally equivalent to $E(C, D)$.

See [GH1]. Another proof will be given below, see 10.3.

10 Intersection of two quadrics in $\mathbb{P}^3$ and Diophantus double equations

10.1 Intersection of two quadrics in $\mathbb{P}^3$

Following [W], Chap. II, App. III consider two smooth quadrics $\mathfrak{C}$ and $\mathfrak{D}$ in $\mathbb{P}^3$ given by equations

$$\Phi(x) = 0, \Psi(x) = 0,\ x = (x_0 : x_1 : x_2 : x_3). \tag{15}$$

We suppose that $\mathfrak{C}$, $\mathfrak{D}$ are in general position; then $\Omega = \mathfrak{C} \cap \mathfrak{D}$ is a smooth curve.

Let Φ, Ψ denote the 4×4 symmetric matrices for $\mathfrak{C}$, $\mathfrak{D}$, and

$$F(\xi) = \det(\Phi - \xi\Psi).$$

Theorem 4. *Ω is birationally equivalent to the elliptic curve given by the equation*

$$\eta^2 = F(\xi).$$

See [W], Chap. II, App. III.

10.2 Double equations and Diophantus curve

Consider a particular case. Let $\mathfrak{C}$ be given by an equation

$$ax^2 + bxw + cw^2 = u^2, \tag{16}$$

and $\mathfrak{D}$ be given by an equation

$$a'x^2 + b'xw + c'w^2 = v^2 \tag{17}$$

in $\mathbb{P}^3$ with homogeneous coordinates $(w : x : u : v)$. These are Fermat's and Diophantus' "double equations", cf. [W], Chap. II, §XV, p. 105.

Let C (resp. D) be given by the equation (16) (resp. (17)) in $\mathbb{P}^2$ with coordinates $(w : x : u)$ (resp. $(w : x : v)$); these two guys are isomorphic to $\mathbb{P}^1$ of course.

We have two maps

$$C \overset{\alpha}{\longleftarrow} \Omega \overset{\beta}{\longrightarrow} D$$

where $\alpha(w : x : u : v) = (w : x : u)$ and $\beta(w : x : u : v) = (w : x : v)$, which make Ω a double covering of C (resp. D) branched at four points.

Let $i : \Omega \overset{\sim}{\longrightarrow} \Omega$ (resp. $j : \Omega \overset{\sim}{\longrightarrow} \Omega$ be the involution which interchanges the two branches of α (resp. of β). Let

$$p = i \circ j : \ \Omega \overset{\sim}{\longrightarrow} \Omega,$$

this is an analog of the Poncelet map.

10.3 Diophatnus curve is equal to Poncelet curve

Let us apply Theorem 4 to the particular case of double equations. We remark that $F(t)$ will coincide with $g(t)$ from (13).

We see that the Diophantus curve $E_{Diph} = \Omega$ and the Poncelet curve E_{Ponc}, cf. [GH1], p. 40, are given by the same equation (up to replacement $\xi \mapsto -\xi$).

11 Fermat ascent

Now let us look at Fermat's "ascent" method for solving double equations, as described in [W], Chap. II, §XV, pp. 105, 106, and on an example due to Euler in [W], Chap. II, App. V, p. 155 - 156.

Euler discusses an equivalent problem of finding solutions to "canonical equations", equations of the form

$$F(\xi, \zeta) = 0$$

where F is a polynomial of degree 2 in two variables ξ, ζ. Such an equation defines an elliptic curve $C \subset \mathbb{P}^2$. Euler was completely aware of the relation of this problem to elliptic integrals, cf. [Jlat].

Starting with some "trivial" solutions $P_0, P_1 \in C$, and finding a solution $R_0 \in C$, he obtains a sequence $R_n \in C$ such that

$$R_n \sim R_0 + n(P_1 - P_0)$$

where for two divisors D_1, D_2 on C the notation $D_1 \sim D_2$ means that there exists a rational function f on C such that

$$\text{div}(f) = D_1 - D_2.$$

The process delivers a finite number of solutions iff $P_1 - P_0$ is of finite order.

We see that this procedure is similar to the Poncelet process.

12 Exercise

Write down the double equations for a miraculous pentagram.

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