Definitio nova algebroidis verticiani

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Summary. An algebra of differential operators is the enveloping algebra of a Lie algebroid T of vector fields. Similarly, a vertex algebra of differential operators is the enveloping algebra of a vertex algebroid, which is a Lie algebroid equipped with certain complementary differential operators. These operators should satisfy some complicated identities, these identities being a corollary of the Borcherd's axioms of a vertex algebra.

In this note we attempt to shed some light at the definition of a vertex algebroid, by proposing a new, equivalent definition which has nothing to do with the axioms of a vertex algebra and uses only classical objects such as complexes of De Rham, Hochschild and Koszul. This point of view works nicely for Calabi–Yau structures as well and opens the way to higher dimensional generalisations.

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Prooemium

1. Sint $\mathfrak{k}$ anulus commutativus $\mathbb{Q}$ continens, A $\mathfrak{k}$-algebra commutativa et T A-algebroid Lietianus.
Posito $\Omega := \mathrm{Hom}_A(T, A)$, habemus derivatio canonica $d : A \longrightarrow \Omega$, ubi $\langle \tau, da\rangle = \tau(a)$, denotante per $\langle , \rangle : T \otimes_A \Omega \longrightarrow A$ copulationem canonicam.

2. Revocamus (vide commentatione [V. Gorbounov, F. Malikov, V. Schechtman, Gerbes of chiral differenial operators. II. Vertex algebroids, *Inventiones Mathematicae*, **155**, 605–680 (2004)], 1.4), *structura verticiana* super T triplex $\mathcal{A} = (\gamma, \langle , \rangle, c)$ est, ubi elementa $\gamma \in \mathrm{Hom}(A \otimes T, \Omega)$, $\langle , \rangle \in \mathrm{Hom}(S^2T, A)$ et $c \in \mathrm{Hom}(\Lambda^2 T, \Omega)$ aequationibus sequentibus satisfacit:

$$\gamma(a, b\tau) - \gamma(ab, \tau) + a\gamma(b, \tau) = -\tau(a)db - \tau(b)da \tag{A1}$$

$$\langle a\tau, \tau'\rangle = a\langle\tau, \tau'\rangle - \tau\tau'(a) + \langle\gamma(a, \tau), \tau'\rangle \tag{A2}$$

$$\begin{aligned} c(a\tau, \tau') &= ac(\tau, \tau') + \gamma(a, [\tau, \tau']) - \gamma(\tau'(a), \tau) \\ &\quad + \tau'\gamma(a, \tau) + \frac{1}{2}ad\langle\tau, \tau'\rangle - \frac{1}{2}d\langle a\tau, \tau'\rangle \end{aligned} \tag{A3}$$

$$\begin{aligned} \langle[\tau, \tau'], \tau''\rangle + \langle\tau', [\tau, \tau'']\rangle &= \tau\langle\tau', \tau''\rangle - \frac{1}{2}\tau'\langle\tau, \tau''\rangle - \frac{1}{2}\tau''\langle\tau, \tau'\rangle \\ &\quad + \langle\tau', c(\tau, \tau'')\rangle + \langle\tau'', c(\tau, \tau')\rangle \end{aligned} \tag{A4}$$

atque

$$\begin{aligned} &\mathrm{Cycle}_{\tau,\tau',\tau''}\left[c([\tau, \tau'], \tau'') - \tau c(\tau', \tau'') + \frac{1}{3}d\langle\tau, c(\tau', \tau'')\rangle\right] \\ &= -\frac{1}{6}\mathrm{Cycle}_{\tau,\tau',\tau''}d\langle\tau, [\tau', \tau'']\rangle, \end{aligned} \tag{A5}$$

ubi denotabimus, brevitatis gratia:

$$\mathrm{Cycle}_{\tau,\tau',\tau''}f(\tau, \tau', \tau'') := f(\tau, \tau', \tau'') + f(\tau', \tau'', \tau) + f(\tau'', \tau, \tau').$$

3. E (A2) prodit:

$$\frac{1}{2}ad\langle\tau, \tau'\rangle - \frac{1}{2}d\langle a\tau, \tau'\rangle = -\frac{1}{2}da\ \langle\tau, \tau'\rangle + \frac{1}{2}d\tau\tau'(a) - \frac{1}{2}d\langle\tau', \gamma(a, \tau)\rangle,$$

ergo (A3) ita exhiberi licet:

$$\begin{aligned} c(a\tau, \tau') &= ac(\tau, \tau') + \gamma(a, [\tau, \tau']) - \gamma(\tau'(a), \tau) + \tau'\gamma(a, \tau) \\ &\quad - \frac{1}{2}d\langle\tau', \gamma(a, \tau)\rangle - \frac{1}{2}da\ \langle\tau, \tau'\rangle + \frac{1}{2}d\tau\tau'(a). \end{aligned} \tag{A3$^{\mathrm{bis}}$}$$

4. Applicatio $h : \mathcal{A} \longrightarrow \mathcal{A}'$ elementum $h \in \mathrm{Hom}(T, \Omega)$ est, axiomatibus sequentibus satisfaciens:

$$h(a\tau) - ah(\tau) = \gamma(a, \tau) - \gamma'(a, \tau) \tag{$(Mor)_\gamma$}$$

$$\langle\tau, h(\tau')\rangle + \langle\tau', h(\tau)\rangle = \langle\tau, \tau'\rangle - \langle\tau, \tau'\rangle' \tag{$(Mor)_{\langle,\rangle}$}$$

et

$$\begin{aligned} &h(|\tau, \tau']) - \tau h(\tau') + \tau' h(\tau) + \frac{1}{2}d\langle\tau, h(\tau')\rangle - \langle\tau', h(\tau)\rangle\} \\ &= c(\tau, \tau') - c'(\tau, \tau'). \end{aligned} \tag{$(Mor)_c$}$$

5. Posito $\Omega^n := \mathrm{Hom}_A(\Lambda_A^n T, A)$, revocamus differentiale DE RHAMIANUM

$$d : \Omega^{n-1} \longrightarrow \Omega^n$$

ubi

$$d\omega(\tau_1, \tau_2, \dots) = \omega([\tau_1, \tau_2], \tau_3, \dots) - \dots + (-1)^{i+j+1}\omega([\tau_i, \tau_j], \tau_1, \dots) + \dots$$

$$- \tau_1\omega(\tau_2, \dots) + \dots + (-1)^i \tau_i \omega(\tau_1, \dots, \hat{\tau}_i, \dots) + \dots$$

$$= \mathrm{Alt}_{12\dots n} \left\{ \frac{1}{2(n-2)!}\omega([\tau_1, \tau_2], \tau_3, \dots) - \frac{1}{(n-1)!}\tau_1\omega(\tau_2, \dots) \right\}.$$

6. Quodque Ω^n in complexum HOCHSCHILDIANUM immergi potest:

$$0 \longrightarrow \Omega^n \longrightarrow \mathrm{Hom}(\Lambda^{n-1}T, \Omega) \xrightarrow{d_H} \mathrm{Hom}(A \otimes T \otimes \Lambda^{n-2}T, \Omega) \xrightarrow{d_H} \dots$$

$$\longrightarrow \dots \xrightarrow{d_H} \mathrm{Hom}(A^{\otimes i} \otimes T \otimes \Lambda^{n-2}T, \Omega) \xrightarrow{d_H} \dots,$$

ubi

$$d_H\omega(a, b, c, \dots, e, f, \tau_1, \dots) = a\omega(b, c, \dots, e, f, \tau_1, \dots)$$

$$- \omega(ab, c, \dots, e, f, \tau_1, \dots) + \dots \pm \omega(a, b, \dots, e, f\tau_1, \dots)$$

Manifesto, $\Omega^n = \mathrm{Ker}\, d_H$.

7. Rursus, sit V $\mathfrak{k}$-modulus, potestas extera sua in genum complexus KOSZULIANI immerseri potest:

$$0 \longrightarrow \Lambda^n V \longrightarrow \mathrm{Hom}(V^*, \Lambda^{n-1}V) \xrightarrow{Q} \mathrm{Hom}(S^2V^*, \Lambda^{n-2}V)$$

$$\xrightarrow{Q} \dots \xrightarrow{Q} \mathrm{Hom}(S^iV^*, \Lambda^{n-i}V) \xrightarrow{Q} \dots,$$

ubi $V^* := \mathrm{Hom}(V, \mathfrak{k})$ ac

$$Qc(\tau_1, \tau_2) = \langle \tau_1, c(\tau_2)\rangle + \langle \tau_2, c(\tau_1)\rangle;$$

$$Qc(\tau_1, \tau_2, \tau_3) = \langle \tau_1, c(\tau_2, \tau_3)\rangle + \langle \tau_2, c(\tau_3, \tau_1)\rangle + \langle \tau_3, c(\tau_1, \tau_2)\rangle,$$

etc. Manifesto, $\Lambda^n V = \mathrm{Ker}\, Q$.

8. In hac commentatione solum inspicimus partem complexus de Rhamiani

$$\Omega^{[2,5]} : \Omega^2 \longrightarrow \dots \longrightarrow \Omega^5$$

Copulatione complexuum Koszulianorum, Hochschildianorum de Rhamianorumque usa, definiamus complexum

$$W^{[2,5]} : W^2 \longrightarrow \dots \longrightarrow W^5,$$

de inclusione complexuum $\Omega^{[2,5]} \subset W^{[2,5]}$ atque de cocyclo canonico $\mathcal{E} \in W^4$ ornatum. Structura verticiana super T elementum $\mathcal{A} \in W^3$ est, aequationi $D\mathcal{A} = \mathcal{E}$ satisfaciens. Sagittula $\mathcal{A} \longrightarrow \mathcal{A}'$ elementum $h \in W^2$ est, talis ut fit $Dh = \mathcal{A} - \mathcal{A}'$, vide Caput Secundum, Pars Tertia.

Caput primum. Structurae calabi–yautianae

1. Revocatio

1.1.

Revocamus (vide op.cit., 11.1), *structura Calabi–Yautiana* super T est applicatio $c : T \longrightarrow A$, duabus proprietatibus sequentibus satisfaciens:

$$c(a\tau) = ac(\tau) + \tau(a) \qquad (CY1)$$

atque

$$c([\tau, \tau']) = \tau c(\tau') - \tau' c(\tau). \qquad (CY2)$$

2. Complexus Hochschild–De Rhamianus

(a)

2.1.

Definimus operator

$$d_{DR} : \operatorname{Hom}(T, A) \longrightarrow \operatorname{Hom}(\Lambda^2 T, A)$$

per formulam:

$$d_{DR} c(\tau_1, \tau_2) = c([\tau_1, \tau_2]) - \tau_1 c(\tau_2) + \tau_2 c(\tau_1).$$

2.2.

Rursus, inspicimus complexum Hochschildianum

$$0 \longrightarrow \operatorname{Hom}(T, A) \xrightarrow{d_H^0} \operatorname{Hom}(A \otimes T, A) \xrightarrow{d_H^1} \operatorname{Hom}(A^{\otimes 2} \otimes T, A),$$

ubi differentialia Hochschildiana per regulas definitur:

$$d_H^0 c(a, \tau) = c(a\tau) - ac(\tau)$$

ac

$$d_H^1 c(a, b, \tau) = ac(b, \tau) - c(ab, \tau) + c(a, b\tau)$$

Liquet quod fit $\Omega = \operatorname{Ker} d_H^0$.

2.3.

Simili modo, consideremus complexum Hochschildianum:

$$0 \longrightarrow \mathrm{Hom}(\Lambda^2 T, A) \xrightarrow{d_H^0} \mathrm{Hom}(A \otimes T^{\otimes 2}, A) \xrightarrow{d_H^1} \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A),$$

ubi differentialia Hochschildiana per formulas definiuntur:

$$d_H^0 c(a, \tau, \tau') = c(a\tau, \tau') - ac(\tau, \tau')$$

atque

$$d_H^1 c(a, b, \tau, \tau') = ac(b, \tau, \tau') - c(ab, \tau, \tau') + c(a, b\tau, \tau')$$

Nunc erit $\Omega^2 = \mathrm{Ker}\, d_H^0$.

2.4.

Porro, introducamus operator

$$d_{DR}: \ \mathrm{Hom}(A \otimes T, A) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 2}, A)$$

per regulam

$$d_{DR} c(a, \tau, \tau') = c(a, [\tau, \tau']) - c(\tau'(a), \tau) + \tau' c(a, \tau) = \mathrm{Lie}_{\tau'} c(a, \tau).$$

2.5. Lemma. $d_H^0 d_{DR} = d_{DR} d_H^0$.

2.6. Demonstratio. Pro elemento $c \in \mathrm{Hom}(T, A)$, habebimus

$$d_H^0 d_{DR} c(a, \tau, \tau') = d_{DR} c(a\tau, \tau') - a d_{DR} c(\tau, \tau'),$$

ubi

$$d_{DR} c(a\tau, \tau') = c([a\tau, \tau']) - a\tau c(\tau') + \tau' c(a\tau)$$

ac

$$-a d_{DR} c(\tau, \tau') = -ac([\tau, \tau']) + a\tau c(\tau') - a\tau' c(\tau),$$

unde

$$\begin{aligned} d_H^0 d_{DR} c(a, \tau, \tau') &= c(a[\tau, \tau']) - ac([\tau, \tau']) - c(\tau'(a)\tau) + \tau' c(a\tau) - a\tau' c(\tau) \\ &= c(a[\tau, \tau']) - ac([\tau, \tau']) - c(\tau'(a)\tau) + \tau' c(a\tau) \\ &\quad - \tau'\{ac(\tau)\} + \tau'(a) c(\tau) = d_{DR} d_H^0 c(a, \tau, \tau'), \end{aligned}$$

qed

(b)

2.7.

Definimus operator

$$d_{DR}: \ \mathrm{Hom}(A^{\otimes 2}\otimes T, A) \longrightarrow \mathrm{Hom}(A^{\otimes 2}\otimes T^{\otimes 2}, A)$$

per

$$\begin{aligned} d_{DR}c(a,b,\tau,\tau') = \mathrm{Lie}_{\tau'}c(a,b,\tau) &= \tau' c(a,b,\tau) - c(\tau'(a),b,\tau) \\ &- c(a,\tau'(b),\tau) - c(a,b,[\tau',\tau]). \end{aligned}$$

2.8. Lemma. $d_H d_{DR} = d_{DR} d_H$.

2.9. Demonstratio. Pro elemento $c\in \mathrm{Hom}(A\otimes T, A)$, fit

$$d_H d_{DR} c(a,b,\tau,\tau') = a d_{DR}c(b,\tau,\tau') - d_{DR}c(ab,\tau,\tau') + d_{DR}c(a,b\tau,\tau'),$$

ubi

$$\begin{aligned} a d_{DR}c(b,\tau,\tau') &= a\tau' c(b,\tau) - ac(\tau'(b),\tau) - ac(b,[\tau',\tau]); -d_{DR}c(ab,\tau,\tau') \\ &= -\tau' c(ab,\tau) + c(\tau'(ab),\tau) + c(ab,[\tau',\tau]) \\ &= -\tau' c(ab,\tau) + c(\tau'(a)b + a\tau'(b),\tau) + c(ab,[\tau',\tau]) \end{aligned}$$

atque

$$\begin{aligned} d_{DR}c(a,b\tau,\tau') &= \tau' c(a,b\tau) - c(\tau'(a),b\tau) - c(a,[\tau',b\tau]) \\ &= \tau' c(a,b\tau) - c(\tau'(a),b\tau) - c(a,\tau'(b)\tau - b[\tau',\tau]) \end{aligned}$$

Adde huc

$$0 = \tau'(a)c(b,\tau) - \tau'(a)c(b,\tau).$$

2.10.

Sed

$$\begin{aligned} &- ac(b,[\tau',\tau]) + c(ab,[\tau',\tau]) + c(a,b[\tau',\tau]) = -d_H c(a,b,[\tau',\tau]); \\ &a\tau' c(b,\tau) - \tau' c(ab,\tau) + \tau' c(a,b\tau) + \tau'(a)c(b,\tau) = \tau' d_H c(a,b,\tau); \\ &- ac(\tau'(b),\tau) + c(a\tau'(b),\tau) - c(a,\tau'(b)\tau) = -d_H c(a,\tau'(b),\tau) \end{aligned}$$

atque

$$c(\tau'(a)b,\tau) - c(\tau'(a),b\tau) - \tau'(a)c(b,\tau) = -d_H(\tau'(a),b,\tau)$$

Qua addendo obtinemus effatum lemmatis.

3. Cocyclus canonicus

3.1.

Inspicimus elementum $\epsilon \in \mathrm{Hom}(A \otimes T, A)$ per formulam definitum:

$$\epsilon(a, \tau) = \tau(a).$$

3.2. Lemma. $d_H\epsilon = d_{DR}\epsilon = 0$.

Habemus enim,

$$d_{DR}\epsilon(a, \tau, \tau') = \mathrm{Lie}_{\tau'}\epsilon(\tau, a) = \tau'\tau(a) - \tau\tau'(a) + [\tau, \tau'](a) = 0$$

(scilicet, ϵ operator invariens est).
Rursus,

$$d_H\epsilon(a, b, \tau) = a\tau(b) - \tau(ab) + b\tau(a) = 0.$$

3.3.

Aliter, ϵ cocyclum (bi)complexus Hochschild–De Rhamiani est.

3.4. Definitio altera. Structura Calabi–Yautiana est elementum $c \in \mathrm{Hom}(T, A)$, satisfaciens equationi $Dc = \epsilon$, denotanti per D differentiale complexus Hochschild–De Rhamiani.

Caput secundum. Structurae verticianae

Pars prima. Aedificium sinistrum

1. Koszul et de Rham

(a)

1.1.

Definimus operatores $Q : \mathrm{Hom}(T, \Omega) \longrightarrow \mathrm{Hom}(S^2T, A)$ per

$$Qh(\tau, \tau') = \langle\tau, h(\tau')\rangle + \langle\tau', h(\tau)\rangle,$$

ergo Ker $Q = \Omega^2$, atque $Q : \mathrm{Hom}(\Lambda^2T, \Omega) \longrightarrow \mathrm{Hom}(S^2T \otimes T, A)$ per

$$Qc(\tau, \tau', \tau'') = \langle\tau, c(\tau', \tau'')\rangle + \langle\tau', c(\tau, \tau'')\rangle,$$

ergo Ker $Q = \Omega^3$.

1.2.

Definimus operatores $d_{DR}:\ \mathrm{Hom}(T,\Omega)\longrightarrow \mathrm{Hom}(\Lambda^2 T,\Omega)$ per

$$d_{DR}h(\tau,\tau')=h([\tau,\tau'])-\tau\{h(\tau')\}+\tau'\{h(\tau)\}+\frac{1}{2}d\{\langle\tau,h(\tau')\rangle-\langle\tau',h(\tau)\rangle\}.$$

1.3.

atque

$$d_{DR}:\ \mathrm{Hom}(S^2T,A)\longrightarrow \mathrm{Hom}(S^2T\otimes T,A)$$

per

$$d_{DR}h(\tau,\tau',\tau'')=\mathrm{Sym}_{\tau,\tau'}\left[h(\tau,[\tau',\tau''])+\frac{1}{2}\tau''\{h(\tau,\tau')\}-\frac{1}{2}\tau\{h(\tau',\tau'')\}\right].$$

1.4. Lemma. $Qd_{DR}=d_{DR}Q$.

Demonstratio. Si $h\in\mathrm{Hom}(T,\Omega)$, habemus

$$\begin{aligned}Qd_{DR}h(\tau,\tau',\tau'')&=\mathrm{Sym}_{\tau,\tau'}\langle\tau,d_{DR}h(\tau',\tau'')\rangle\\&=\mathrm{Sym}_{\tau,\tau'}\Big\langle\tau,h([\tau',\tau''])-\tau'\{h(\tau'')\}+\tau''\{h(\tau')\}\\&\quad+\frac{1}{2}d\{\langle\tau',h(\tau'')\rangle-\langle\tau'',h(\tau')\rangle\}\Big\rangle\\&=\mathrm{Sym}_{\tau,\tau'}\Big[\langle\tau,h([\tau',\tau''])\rangle-\tau'(\langle\tau,h(\tau'')\rangle\\&\quad+\langle[\tau',\tau],h(\tau'')\rangle+\tau''(\langle\tau,h(\tau')\rangle-\langle[\tau'',\tau],h(\tau')\rangle\\&\quad+\frac{1}{2}\tau(\langle\tau',h(\tau'')\rangle)-\frac{1}{2}\tau(\langle\tau'',h(\tau')\rangle)\Big]\end{aligned}$$

Sed

$$\begin{aligned}&\mathrm{Sym}_{\tau,\tau'}[\langle\tau,h([\tau',\tau''])\rangle-\langle[\tau'',\tau],h(\tau')\rangle]\\&\quad=\mathrm{Sym}_{\tau,\tau'}[\langle\tau',h([\tau,\tau''])\rangle+\langle[\tau,\tau''],h(\tau')\rangle]=\mathrm{Sym}_{\tau,\tau'}Qh(\tau,[\tau',\tau'']);\\&\mathrm{Sym}_{\tau,\tau'}\langle[\tau',\tau],h(\tau'')\rangle=0,\\&\mathrm{Sym}_{\tau,\tau'}[\tau''(\langle\tau,h(\tau')\rangle]=\tau''\{Qh(\tau,\tau')\}\end{aligned}$$

ac

$$\begin{aligned}&\mathrm{Sym}_{\tau,\tau'}\left[-\tau'(\langle\tau,h(\tau'')\rangle+\frac{1}{2}\tau(\langle\tau',h(\tau'')\rangle)-\frac{1}{2}\tau(\langle\tau'',h(\tau')\rangle)\right]\\&\quad=-\frac{1}{2}\mathrm{Sym}_{\tau,\tau'}\left[\tau(\langle\tau',h(\tau'')\rangle+\tau(\langle\tau'',h(\tau')\rangle)\right]=-\frac{1}{2}\mathrm{Sym}_{\tau,\tau'}\tau\{Qh(\tau',\tau'')\},\end{aligned}$$

unde $Qd_{DR}h(\tau,\tau',\tau'')=d_{DR}Qh(\tau,\tau',\tau'')$, *quod erat demonstrandum.*

1.5.

Axioma (A4) etiam sic exhiberi potest:

$$Qc = d_{DR}\langle , \rangle \tag{$A4$}$$

(b)

1.6.

Definimus operatores

$$d_{DR} : \operatorname{Hom}(\Lambda^2 T, \Omega) \longrightarrow \operatorname{Hom}(\Lambda^3 T, \Omega)$$

per

$$\begin{aligned} d_{DR}c(\tau, \tau', \tau'') &= \operatorname{Cycle}_{\tau,\tau',\tau''}\left[c([\tau, \tau'], \tau'') - \tau\{c(\tau', \tau'')\} + \frac{1}{3}d\langle \tau, c(\tau', \tau'')\rangle\right] \\ &= d_{Lie}c(\tau, \tau', \tau'') + d'c(\tau, \tau', \tau''), \end{aligned}$$

ubi

$$d'c(\tau, \tau', \tau'') = \frac{1}{3}\operatorname{Cycle}_{\tau,\tau',\tau''} d\langle \tau, c(\tau', \tau'')\rangle$$

1.7.

atque

$$R : \operatorname{Hom}(S^2 T, A) \longrightarrow \operatorname{Hom}(\Lambda^3 T, \Omega)$$

per

$$Rh(\tau, \tau', \tau'') = -\frac{1}{6}\operatorname{Cycle}_{\tau,\tau',\tau''} dh([\tau, \tau'], \tau'').$$

1.8. Lemma. $d_{DR}^2 = RQ$.

1.9. Demonstratio. Fit

$$d_{DR}^2 = (d_{Lie} + d')^2 = d_{Lie}d' + d'd_{Lie} + d'^2$$

ob $d_{Lie}^2 = 0$.

Si $h \in \operatorname{Hom}(T, \Omega)$, habemus

$$\begin{aligned} d_{Lie}d'h(\tau, \tau', \tau'') &= \operatorname{Cycle}_{\tau,\tau',\tau''}[d'h([\tau, \tau'], \tau'') - \tau\{d'h(\tau', \tau'')\}] \\ &= \frac{1}{2}\operatorname{Cycle}_{\tau,\tau',\tau''}\big[d\{\langle [\tau, \tau'], h(\tau'')\rangle - \langle \tau'', h([\tau, \tau'])\rangle\} \\ &\quad - \tau d\{\langle \tau', h(\tau'')\rangle - \langle \tau', h(\tau'')\rangle\}\big]. \end{aligned}$$

Observamus:

$$-\frac{1}{2}\operatorname{Cycle}_{\tau,\tau',\tau''}\big[\tau d\{\langle \tau', h(\tau'')\rangle - \langle \tau', h(\tau'')\rangle\}\big] = -\frac{1}{2}\operatorname{Alt}_{\tau,\tau',\tau''}\tau d\langle \tau', h(\tau'')\rangle.$$

1.10.

Rursus

$$d' d_{Lie} h(\tau, \tau', \tau'') = \frac{1}{3}\mathrm{Cycle}_{\tau,\tau',\tau''} d\langle \tau, d_{Lie} h(\tau', \tau'')\rangle$$
$$= \frac{1}{3}\mathrm{Cycle}_{\tau,\tau',\tau''} d\langle \tau, h([\tau', \tau''])\rangle - \frac{1}{3}\mathrm{Alt}_{\tau,\tau',\tau''} d\langle \tau, \tau'\{h(\tau'')\}\rangle,$$

ubi

$$-\frac{1}{3}\mathrm{Alt}_{\tau,\tau',\tau''} d\langle \tau, \tau'\{h(\tau'')\}\rangle = -\frac{1}{3}\mathrm{Alt}_{\tau,\tau',\tau''} d[\tau'\langle \tau, h(\tau'')\rangle + \langle [\tau, \tau'], h(\tau'')\rangle]$$
$$= -\frac{1}{3}\mathrm{Alt}_{\tau,\tau',\tau''} d\tau'\langle \tau, h(\tau'')\rangle + \frac{2}{3}\mathrm{Cycle}_{\tau,\tau',\tau''} d\langle [\tau, \tau'], h(\tau'')\rangle.$$

1.11.

Denique,

$$d'^2 h(\tau, \tau', \tau'') = \frac{1}{6}\mathrm{Alt}_{\tau,\tau',\tau''} d\tau\{\langle \tau', h(\tau'')\rangle\}.$$

1.12.

Post summationem termini $d\tau\{\langle \tau', h(\tau'')\rangle\}$ exeunt, dum termini reliqui praebunt

$$d^2_{DR} h(\tau, \tau', \tau'') = -\frac{1}{6}\mathrm{Cycle}_{\tau,\tau',\tau''} d\big\{\langle [\tau, \tau'], h(\tau'')\rangle + \langle \tau'', h([\tau, \tau'])\rangle\big\}$$
$$= -\frac{1}{6}\mathrm{Cycle}_{\tau,\tau',\tau''} dQh([\tau, \tau'], \tau'') = RQh(\tau, \tau', \tau''), \qquad \text{qed}$$

1.13.

Axioma (A5) sic exhiberi potest:

$$d_{DR} c = R\langle , \rangle \qquad (A5)$$

(c)

1.14.

Determinamus operator

$$d_{DR} : \mathrm{Hom}(S^2 T \otimes T, A) \longrightarrow \mathrm{Hom}(S^2 T \otimes \Lambda^2 T, A)$$

per formulam

$$d_{DR} c(\tau_1, \tau_2, \tau_3, \tau_4) = -c(\tau_1, \tau_2, [\tau_3, \tau_4]) - \mathrm{Alt}_{3,4} \tau_4 c(\tau_1, \tau_2, \tau_3)$$
$$+ \mathrm{Sym}_{1,2} \mathrm{Alt}_{3,4} \left\{ c(\tau_1, [\tau_2, \tau_3], \tau_4) - \frac{1}{3}\tau_1 c(\tau_2, \tau_3, \tau_4) \right\}.$$

1.15. Lemma. *Fit* $d_{DR}Q = Qd_{DR}$.

1.16. Démonstratio. Si $c \in \mathrm{Hom}(\Lambda^2 T, \Omega)$, habemus

$$Qd_{DR}c(\tau_1, \tau_2, \tau_3, \tau_4) = \mathrm{Sym}_{1,2}\langle \tau_1, d_{DR}c(\tau_2, \tau_3, \tau_4)\rangle$$

$$= \mathrm{Sym}_{1,2}\mathrm{Cycle}_{2,3,4}\Big\langle \tau_1, c([\tau_2, \tau_3], \tau_4) - \tau_2 c(\tau_3, \tau_4)$$

$$+\frac{1}{3}d\langle \tau_2, c(\tau_3, \tau_4)\rangle\Big\rangle,$$

ubi

$$-\langle \tau_1, \tau_2 c(\tau_3, \tau_4)\rangle = \langle [\tau_2, \tau_1], c(\tau_3, \tau_4)\rangle - \tau_2\langle \tau_1, c(\tau_3, \tau_4)\rangle$$

atque

$$\langle \tau_1, d\langle \tau_2, c(\tau_3, \tau_4)\rangle\rangle = \tau_1\langle \tau_2, c(\tau_3, \tau_4)\rangle.$$

1.17.

Primo, fit

$$\mathrm{Sym}_{1,2}\mathrm{Cycle}_{2,3,4}\{\langle \tau_1, c([\tau_2, \tau_3], \tau_4)\rangle + \langle [\tau_2, \tau_1], c(\tau_3, \tau_4)\rangle\}$$

$$= \mathrm{Sym}_{1,2}\mathrm{Alt}_{3,4}\left\{\left\langle \tau_1, c([\tau_2, \tau_3], \tau_4) + \frac{1}{2}c([\tau_3, \tau_4], \tau_2)\right\rangle\right.$$

$$\left.+\frac{1}{2}\langle [\tau_2, \tau_1], c(\tau_3, \tau_4)\rangle + \langle [\tau_3, \tau_1], c(\tau_4, \tau_2)\rangle\right\}$$

$$= -Qc(\tau_1, \tau_2, [\tau_3, \tau_4]) + \mathrm{Sym}_{1,2}\mathrm{Alt}_{3,4}Qc(\tau_1, [\tau_2, \tau_3], \tau_4).$$

1.18.

Secundo,

$$\mathrm{Sym}_{1,2}\mathrm{Cycle}_{2,3,4}\left\{-\tau_2\langle \tau_1, c(\tau_3, \tau_4)\rangle + \frac{1}{3}\tau_1\langle \tau_2, c(\tau_3, \tau_4)\rangle\right\}$$

$$= \mathrm{Sym}_{1,2}\mathrm{Alt}_{3,4}\left\{-\frac{1}{2}\tau_2\langle \tau_1, c(\tau_3, \tau_4)\rangle - \tau_3\langle \tau_1, c(\tau_4, \tau_2)\rangle\right.$$

$$\left.+\frac{1}{6}\tau_1\langle \tau_2, c(\tau_3, \tau_4)\rangle + \frac{1}{3}\tau_1\langle \tau_3, c(\tau_4, \tau_2)\rangle\right\}$$

$$= \mathrm{Alt}_{3,4}\tau_3 Qc(\tau_1, \tau_2, \tau_4) - \frac{1}{3}\mathrm{Sym}_{1,2}\mathrm{Alt}_{3,4}\tau_1 Qc(\tau_2, \tau_3, \tau_4)$$

Hinc lemma nostra sponte sequitur.

(d) *Junctio*

1.19. Lemma. *Compositio*

$$d^2_{DR}: \operatorname{Hom}(S^2T, A) \longrightarrow \operatorname{Hom}(S^2T \otimes T, A) \longrightarrow \operatorname{Hom}(S^2T \otimes \Lambda^2 T, A)$$

aequat QR.

1.20. Demonstratio. Si $c \in \operatorname{Hom}(S^2T, A)$, fit

$$d^2_{DR}c(\tau_1, \tau_2, \tau_3, \tau_4) = -d_{DR}c(\tau_1, \tau_2, [\tau_3, \tau_4]) - \mathrm{Alt}_{3,4}\tau_4 d_{DR}c(\tau_1, \tau_2, \tau_3)$$
$$+ \mathrm{Sym}_{1,2}\mathrm{Alt}_{3,4}\left\{d_{DR}c(\tau_1, [\tau_2, \tau_3], \tau_4) - \frac{1}{3}\tau_1 d_{DR}c(\tau_2, \tau_3, \tau_4)\right\}.$$

1.21.

Primo,

$$-d_{DR}c(\tau_1, \tau_2, [\tau_3, \tau_4]) = -c(\tau_1, [\tau_2, [\tau_3, \tau_4]]) - c(\tau_2, [\tau_1, [\tau_3, \tau_4]])$$
$$- [\tau_3, \tau_4]c(\tau_1, \tau_2) + \frac{1}{2}\tau_1 c(\tau_2, [\tau_3, \tau_4]) + \frac{1}{2}\tau_2 c(\tau_1, [\tau_3, \tau_4])$$

Secundo,

$$\mathrm{Sym}_{1,2}\mathrm{Alt}_{3,4}d_{DR}c(\tau_1, [\tau_2, \tau_3], \tau_4) = \mathrm{Sym}_{1,2}\mathrm{Alt}_{3,4}\Big\{c(\tau_1, [[\tau_2, \tau_3], \tau_4])$$
$$+ c([\tau_2, \tau_3], [\tau_1, \tau_4]) + \tau_4 c(\tau_1, [\tau_2, \tau_3])$$
$$-\frac{1}{2}\tau_1 c([\tau_2, \tau_3], \tau_4) - \frac{1}{2}[\tau_2, \tau_3]c(\tau_1, \tau_4)\Big\}$$

Tertio,

$$-\mathrm{Alt}_{3,4}\tau_4 d_{DR}c(\tau_1, \tau_2, \tau_3) = -\mathrm{Alt}_{3,4}\Big\{\tau_4 c(\tau_1, [\tau_2, \tau_3]) + \tau_4 c(\tau_2, [\tau_1, \tau_3])$$
$$+ \tau_4\tau_3 c(\tau_1, \tau_2) - \frac{1}{2}\tau_4\tau_1 c(\tau_2, \tau_3)$$
$$-\frac{1}{2}\tau_4\tau_2 c(\tau_1, \tau_3)\Big\}$$

et quatro,

$$-\frac{1}{3}\mathrm{Sym}_{1,2}\mathrm{Alt}_{3,4}\tau_1 d_{DR}c(\tau_2, \tau_3, \tau_4) = -\frac{1}{3}\mathrm{Sym}_{1,2}\mathrm{Alt}_{3,4}\Big\{\tau_1 c(\tau_2, [\tau_3, \tau_4])$$
$$+ \tau_1 c(\tau_3, [\tau_2, \tau_4]) + \tau_1\tau_4 c(\tau_2, \tau_3)$$
$$-\frac{1}{2}\tau_1\tau_2 c(\tau_3, \tau_4) - \frac{1}{2}\tau_1\tau_3 c(\tau_2, \tau_4)\Big\}$$

1.22.

Termini formae $c([\tau_i, \tau_j], [\tau_k, \tau_l])$ evanescunt symmetrisatione causa, cum c symmetricos est.

Post summationem, termini cum triplicibus uncinis evanescunt Jacobi causa. Termini formae $\tau_i \tau_j c(\tau_k, \tau_l)$ quoque exire videri possunt.

Tandem termini formae $\tau_i c(\tau_j, [\tau_k, \tau_l])$ praebunt

$$\begin{aligned} -\frac{1}{6}\mathrm{Sym}_{1,2}\mathrm{Cycle}_{2,3,4}\tau_1 c(\tau_2, [\tau_3, \tau_4]) &= -\frac{1}{6}\mathrm{Sym}_{1,2}\mathrm{Cycle}_{2,3,4}\langle \tau_1, dc(\tau_2, [\tau_3, \tau_4])\rangle \\ &= QRc(\tau_1, \tau_2, \tau_3, \tau_4), \qquad \text{qed} \end{aligned}$$

(e) *Differentiale de Rhamianum tertium*

1.23.

Definimus operator

$$d_{DR} : \mathrm{Hom}(\Lambda^3 T, \Omega) \longrightarrow \mathrm{Hom}(\Lambda^4 T, \Omega)$$

ubi

$$\begin{aligned} d_{DR}c(\tau_1, \tau_2, \tau_3, \tau_4) &= c([\tau_1, \tau_2], \tau_3, \tau_4) - c([\tau_1, \tau_3], \tau_2, \tau_4) + \dots \\ &\quad - \tau_1 c(\tau_2, \tau_3, \tau_4) + \tau_2 c(\tau_1, \tau_3, \tau_4) - \dots \\ &\quad + \frac{1}{4} d\{\langle \tau_1, c(\tau_2, \tau_3, \tau_4)\rangle - \langle \tau_2, c(\tau_1, \tau_3, \tau_4)\rangle + \} \\ &= \mathrm{Alt}_{1234} \left[\frac{1}{4} c([\tau_1, \tau_2], \tau_3, \tau_4) - \frac{1}{6}\tau_1 c(\tau_2, \tau_3, \tau_4) \right. \\ &\quad \left. + \frac{1}{24} d\langle \tau_1, c(\tau_2, \tau_3, \tau_4)\rangle \right] = \{d_{Lie} + d'\} c(\tau_1, \tau_2, \tau_3, \tau_4), \end{aligned}$$

ubi

$$d'c(\tau_1, \tau_2, \tau_3, \tau_4) := \frac{1}{24}\mathrm{Alt}_{1234}\, d\langle \tau_1, c(\tau_2, \tau_3, \tau_4)\rangle,$$

confer artt. 1.2 et 1.6.

1.24.

Insuper introducamus operator

$$R : \mathrm{Hom}(S^2 T \otimes T, A) \longrightarrow \mathrm{Hom}(\Lambda^4 T, \Omega),$$

ubi

$$Rc(\tau_1, \tau_2, \tau_3, \tau_4) := -\frac{1}{24}\mathrm{Alt}_{1234}\, dc([\tau_1, \tau_2], \tau_3, \tau_4),$$

confer art. 1.7.

1.25. Lemma. *Fit* $d_{DR}^2 = RQ$.

1.26. Demonstratio. Primo, $d_{Lie}^2 = 0$.
Secundo, ostendetur methodo simili a 1.9.–1.12,

$$\{d_{Lie}d' + d'd_{Lie} + d'^2\}c(\tau_1, \tau_2, \tau_3, \tau_4)$$

$$= -\frac{1}{24}\mathrm{Alt}_{1234}\, d\big\{\langle[\tau_1, \tau_2], c(\tau_3, \tau_4)\rangle + \langle\tau_3, c([\tau_1, \tau_2], \tau_4)\rangle\big\}$$

$$= -\frac{1}{24}\mathrm{Alt}_{1234}\, dQc([\tau_1, \tau_2], \tau_3, \tau_4) = RQc(\tau_1, \tau_2, \tau_3, \tau_4), \qquad \text{qed}$$

1.27.

Consideremus duos operatores R ex artt. 1.7 et 1.24.

1.28. Lemma. *Fit* $d_{DR}R = Rd_{DR}$.

1.29. Demonstratio. Primo, habemus

$$\begin{aligned} d_{DR}Rc(\tau_1, \tau_2, \tau_3, \tau_4) = \mathrm{Alt}_{1234}\Big[&\frac{1}{4}Rc([\tau_1, \tau_2], \tau_3, \tau_4) \\ &-\frac{1}{6}\tau_1 Rc(\tau_2, \tau_3, \tau_4) + \frac{1}{24}d\langle\tau_1, Rc(\tau_2, \tau_3, \tau_4)\rangle\Big] \\ = \mathrm{Alt}_{1234}\Big[&-\frac{1}{24}d\{c([[\tau_1, \tau_2], \tau_3], \tau_4) + c([\tau_3, \tau_4], [\tau_1, \tau_2]) \\ &+c([\tau_4, [\tau_1, \tau_2]], \tau_3)\} + \frac{1}{36}d\tau_1\{c([\tau_2, \tau_3], \tau_4) \\ &+c([\tau_3, \tau_4], \tau_2) + c([\tau_4, \tau_2], \tau_3)\} - \frac{1}{144}d\langle\tau_1, d\{c([\tau_2, \tau_3], \tau_4) \\ &+ c([\tau_3, \tau_4], \tau_2) + c([\tau_4, \tau_2], \tau_3)\}\rangle\Big] \end{aligned}$$

(termini, uncinos triplices continentes, exeunt, relatione identica Jacobiana causa)

$$= \mathrm{Alt}_{1234}\left[-\frac{1}{24}dc([\tau_3, \tau_4], [\tau_1, \tau_2]) + \frac{1}{16}d\tau_1 c([\tau_2, \tau_3], \tau_4)\right].$$

1.30.

Rursus fit

$$Rd_{DR}c(\tau_1,\tau_2,\tau_3,\tau_4) = -\frac{1}{24}\mathrm{Alt}_{1234}\, dd_{DR}c([\tau_1,\tau_2],\tau_3,\tau_4)$$

$$= -\frac{1}{24}\mathrm{Alt}_{1234}\, d\Big[c([\tau_1,\tau_2],[\tau_3,\tau_4]) + c(\tau_3,[[\tau_1,\tau_2],\tau_4])$$

$$+\tau_4 c([\tau_1,\tau_2],\tau_3) - \frac{1}{2}[\tau_1,\tau_2]c(\tau_3,\tau_4) - \frac{1}{2}\tau_3 c([\tau_1,\tau_2],\tau_3)\Big]$$

(termini formae $[\tau_1,\tau_2]c(\tau_3,\tau_4)$ exibunt, quum c symmetricos est)

$$= -\frac{1}{24}\mathrm{Alt}_{1234}\, d\left\{c([\tau_1,\tau_2],[\tau_3,\tau_4]) - \frac{3}{2}\tau_4 c([\tau_1,\tau_2],\tau_3)\right\} = d_{DR}Rc(\tau_1,\tau_2,\tau_3,\tau_4),$$

qed

2. Pede plana

(a) *Paries recessus*

2.1.

Definimus operatores:

$$d_H : \mathrm{Hom}(T,\Omega) \longrightarrow \mathrm{Hom}(A\otimes T,\Omega)$$

per formulam

$$d_H c(a,\tau) = c(a\tau) - ac(\tau)$$

et

$$d_H : \mathrm{Hom}(\Lambda^2 T,\Omega) \longrightarrow \mathrm{Hom}(A\otimes T^{\otimes 2},\Omega)$$

per regulam:

$$d_H c(a,\tau,\tau') = c(a\tau,\tau') - ac(\tau,\tau').$$

2.2.

Introducamus operator:

$$d_{DR} : \mathrm{Hom}(A\otimes T,\Omega) \longrightarrow \mathrm{Hom}(A\otimes T^{\otimes 2},\Omega)$$

per regulam:

$$d_{DR}c(a,\tau,\tau') = c(a,[\tau,\tau']) - c(\tau'(a),\tau) + \tau' c(a,\tau) - \frac{1}{2}d\langle\tau', c(a,\tau)\rangle$$

$$= \mathrm{Lie}_{\tau'}c(a,\tau) - \frac{1}{2}d\langle\tau', c(a,\tau)\rangle.$$

2.3.

Insuper, operator:

$$M: \operatorname{Hom}(S^2T, A) \longrightarrow \operatorname{Hom}(A \otimes T^{\otimes 2}, \Omega)$$

definitur per regulam:

$$Mc(a, \tau, \tau') = -\frac{1}{2}c(\tau, \tau')da.$$

2.4. Lemma. *Fit* $d_H d_{DR} = d_{DR} d_H + MQ$.

2.5. Demonstratio. Elementum $c \in \operatorname{Hom}(T, \Omega)$ datum, habebimus

$$d_H d_{DR} c(a, \tau, \tau') = d_{DR} c(a\tau, \tau') - a d_{DR} c(\tau, \tau')$$

ubi

$$\begin{aligned} d_{DR} c(a\tau, \tau') &= c([a\tau, \tau']) - (a\tau)c(\tau') + \tau' c(a\tau) + \frac{1}{2}d\{\langle a\tau, c(\tau')\rangle - \langle \tau', c(a\tau)\rangle\} \\ &= c(a[\tau, \tau'] - \tau'(a)\tau) - a\tau c(\tau') - da\langle \tau, c(\tau')\rangle + \tau' c(a\tau) \\ &\quad + \frac{1}{2}da\langle \tau, c(\tau')\rangle + \frac{1}{2}ad\langle \tau, c(\tau')\rangle - \frac{1}{2}d\langle \tau', c(a\tau) - ac(\tau)\rangle \\ &\quad - \frac{1}{2}da\langle \tau', c(\tau)\rangle - \frac{1}{2}ad\langle \tau', c(\tau)\rangle, \end{aligned}$$

addemus huc:

$$0 = -\tau'\{ac(\tau)\} + \tau'(a)c(\tau) + a\tau' c(\tau)$$

Rursus,

$$-a d_{DR} c(\tau, \tau') = -ac([\tau, \tau']) + a\tau c(\tau') - a\tau' c(\tau) - \frac{1}{2}ad\{\langle \tau, c(\tau')\rangle - \langle \tau', c(\tau)\rangle\}.$$

2.6.

Sed

$$c(a[\tau, \tau']) - ac([\tau, \tau']) = d_H c(a, [\tau, \tau']);$$

$$-c(\tau'(a)\tau) + \tau'(a)c(\tau) = -d_H c(\tau'(a), \tau);$$

$$\tau' c(a\tau) - \tau'\{ac(\tau)\} = \tau' d_H c(a, \tau);$$

$$-\frac{1}{2}d\langle \tau', c(a\tau) - ac(\tau)\rangle = -\frac{1}{2}d\langle \tau', d_H c(a, \tau)\rangle,$$

terminos reliquos praebendo

$$-\frac{1}{2}da\{\langle \tau, c(\tau')\rangle + \langle \tau', c(\tau)\rangle\} = -\frac{1}{2}da\, Qc(\tau, \tau') = MQc(a, \tau, \tau'),$$

lemma nostrum sequitur.

(b) *Frons*

2.7.

Revocamus,

$$d_{DR}:\ \mathrm{Hom}(S^2T, A) \longrightarrow \mathrm{Hom}(S^2T \otimes T, A)$$

definitur per formulam:

$$\begin{aligned} d_{DR}c(\tau_1, \tau_2, \tau_3) &= c(\tau_1, [\tau_2, \tau_3]) + c(\tau_2, [\tau_1, \tau_3]) + \tau_3 c(\tau_1, \tau_2) \\ &\quad - \frac{1}{2}\tau_1 c(\tau_2, \tau_3) - \frac{1}{2}\tau_2 c(\tau_1, \tau_3) \\ &= \mathrm{Lie}_{\tau_3} c(\tau_1, \tau_2) - \frac{1}{2}\tau_1 c(\tau_2, \tau_3) - \frac{1}{2}\tau_2 c(\tau_1, \tau_3) \\ &= \{\mathrm{Lie} + d'_{DR}\} c(\tau_1, \tau_2, \tau_3), \end{aligned}$$

ubi ponamus

$$\mathrm{Lie}\ c(\tau_1, \tau_2, \tau_3) = \mathrm{Lie}_{\tau_3} c(\tau_1, \tau_2)$$

et

$$d'_{DR}c(\tau_1, \tau_2, \tau_3) = -\frac{1}{2}\tau_1 c(\tau_2, \tau_3) - \frac{1}{2}\tau_2 c(\tau_1, \tau_3)$$

Rursus, definimus

$$d_{DR}:\ \mathrm{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 3}, A)$$

per formulam:

$$\begin{aligned} d_{DR}c(a, \tau_1, \tau_2, \tau_3) &= \tau_3 c(a, \tau_1, \tau_2) - c(\tau_3(a), \tau_1, \tau_2) + c(a, [\tau_1, \tau_3], \tau_2) \\ &\quad + c(a, \tau_1, [\tau_2, \tau_3]) - \frac{1}{2}\tau_2 c(a, \tau_1, \tau_3) \\ &= \mathrm{Lie}_{\tau_3} c(a, \tau_1, \tau_2) - \frac{1}{2}\tau_2 c(a, \tau_1, \tau_3) \\ &= \{\mathrm{Lie} + d'_{DR}\} c(a, \tau_1, \tau_2, \tau_3), \end{aligned}$$

ubi ponamus

$$\mathrm{Lie}\ c(a, \tau_1, \tau_2, \tau_3) = \mathrm{Lie}_{\tau_3} c(a, \tau_1, \tau_2)$$

et

$$d'_{DR}c(a, \tau_1, \tau_2, \tau_3) = -\frac{1}{2}\tau_2 c(a, \tau_1, \tau_3)$$

Denique, introducamus

$$Q:\ \mathrm{Hom}(A \otimes T^{\otimes 2}, \Omega) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 3}, A)$$

per regulam:

$$Qc(a, \tau_1, \tau_2, \tau_3) = \langle \tau_2, c(a, \tau_1, \tau_3) \rangle.$$

2.8. Lemma. *Fit* $d_H d_{DR} = d_{DR} d_H + QM$.

2.9. Demonstratio. Primo, d_H commutat cum Lie. Si enim $c \in \mathrm{Hom}(S^2T, A)$, habemus:

$$d_H \mathrm{Lie}\, c(a, \tau_1, \tau_2, \tau_3) = \mathrm{Lie}\, c(a\tau_1, \tau_2, \tau_3) - a\mathrm{Lie}\, c(\tau_1, \tau_2, \tau_3)$$

ubi

$$\mathrm{Lie}\, c(a\tau_1, \tau_2, \tau_3) = \tau_3 c(a\tau_1, \tau_2) + c(a[\tau_1, \tau_3] - \tau_3(a)\tau_1, \tau_2) + c(a\tau_1, [\tau_2, \tau_3])$$

et

$$-a\mathrm{Lie}\, c(\tau_1, \tau_2, \tau_3) = -a\tau_3 c(\tau_1, \tau_2) - ac([\tau_1, \tau_3], \tau_2) - ac(\tau_1, [\tau_2, \tau_3])$$

Addemus huc:

$$0 = -\tau_3\{ac(\tau_1, \tau_2)\} + \tau_3(a)c(\tau_1, \tau_2) + a\tau_3 c(\tau_1, \tau_2)$$

Habebimus

$$\tau_3 c(a\tau_1, \tau_2) - \tau_3\{ac(\tau_1, \tau_2)\} = \tau_3 d_H c(a, \tau_1, \tau_2);$$

$$c(a[\tau_1, \tau_3], \tau_2) - ac([\tau_1, \tau_3], \tau_2) = d_H c(a, [\tau_1, \tau_3], \tau_2);$$

$$c(a\tau_1, [\tau_2, \tau_3]) - ac(\tau_1, [\tau_2, \tau_3]) = d_H c(\tau_1, [\tau_2, \tau_3])$$

et

$$-c(\tau_3(a)\tau_1, \tau_2) + \tau_3(a)c(\tau_1, \tau_2) = -d_H c(\tau_3(a), \tau_1, \tau_2)$$

unde

$$d_H \mathrm{Lie}\, c(a, \tau_1, \tau_2, \tau_3) = \mathrm{Lie}_{\tau_3} d_H c(a, \tau_1, \tau_2).$$

2.10.

Secundo,

$$d_H d'_{DR} c(a, \tau_1, \tau_2, \tau_3) = d'_{DR} c(a\tau_1, \tau_2, \tau_3) - a d'_{DR} c(\tau_1, \tau_2, \tau_3)$$

ubi

$$d'_{DR} c(a\tau_1, \tau_2, \tau_3) = -\frac{1}{2} a\tau_1 c(\tau_2, \tau_3) - \frac{1}{2}\tau_2 c(a\tau_1, \tau_3)$$

et

$$-a d'_{DR} c(\tau_1, \tau_2, \tau_3) = \frac{1}{2} a\tau_1 c(\tau_2, \tau_3) + \frac{1}{2} a\tau_2 c(\tau_1, \tau_3).$$

Addemus huc:

$$0 = \frac{1}{2}\tau_2\{ac(\tau_1, \tau_3)\} - \frac{1}{2}\tau_2(a)c(\tau_1, \tau_3) - \frac{1}{2} a\tau_2 c(\tau_1, \tau_3).$$

Nanciscemur:

$$d_H d'_{DR} c(a, \tau_1, \tau_2, \tau_3) = -\frac{1}{2}\tau_2 d_H c(a, \tau_1, \tau_3) - \frac{1}{2}\tau_2(a) c(\tau_1, \tau_3),$$

ubi

$$-\frac{1}{2}\tau_2(a) c(\tau_1, \tau_3) = \left\langle \tau_2, -\frac{1}{2} da\, c(\tau_1, \tau_3) \right\rangle = QMc(a, \tau_1, \tau_2, \tau_3)$$

unde hoc lemma sequitur.

(c) *Camera*

2.11.

Revocamus differentialia de Rhamiana:

$$d_{DR} : \operatorname{Hom}(A \otimes T, \Omega) \longrightarrow \operatorname{Hom}(A \otimes T^{\otimes 2}, \Omega)$$

definiuntur per formulam (vide art. 2.2):

$$d_{DR} c(a, \tau, \tau') = \operatorname{Lie}_{\tau'} c(a, \tau) - \frac{1}{2} d\langle \tau', c(a, \tau)\rangle$$

ac

$$d_{DR} : \operatorname{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \operatorname{Hom}(A \otimes T^{\otimes 3}, A)$$

definiuntur per regulam (vide art. 2.7):

$$d_{DR} c(a, \tau_1, \tau_2, \tau_3) = \operatorname{Lie}_{\tau_3} c(a, \tau_1, \tau_2) - \frac{1}{2}\tau_2 c(a, \tau_1, \tau_3).$$

2.12. Lemma. *Fit* $d_{DR} Q = Q d_{DR}$.

2.13. Demonstratio. Primo,

$$\operatorname{Lie} Q = Q \operatorname{Lie}$$

Habemus enim,

$$\begin{aligned}
&\operatorname{Lie}_{\tau_3} Qc(a, \tau_1, \tau_2) \\
&\quad = \tau_3 Qc(a, \tau_1, \tau_2) - Qc(\tau_3(a), \tau_1, \tau_2) + Qc(a, [\tau_1, \tau_3], \tau_2) + Qc(a, \tau_1, [\tau_2, \tau_3]) \\
&\quad = \tau_3\langle \tau_2, c(a, \tau_1)\rangle - \langle \tau_2, c(\tau_3(a), \tau_1)\rangle + \langle \tau_2, c(a, [\tau_1, \tau_3]\rangle + \langle [\tau_2, \tau_3], c(a, \tau_1)\rangle \\
&\quad = \langle \tau_2, \operatorname{Lie}_{\tau_3} c(a, \tau_1)\rangle.
\end{aligned}$$

2.14.

Secundo,

$$d'_{DR} Q = Q d'_{DR}$$

Computamus enim,

$$\begin{aligned} d'_{DR} Qc(a, \tau_1, \tau_2, \tau_3) &= -\frac{1}{2}\tau_2 Qc(a, \tau_1, \tau_3) = -\frac{1}{2}\tau_2 \langle \tau_3, c(a, \tau_1)\rangle \\ &= \left\langle \tau_2, -\frac{1}{2} d\langle \tau_3, c(a, \tau_1)\rangle \right\rangle \\ &= \langle \tau_2, d'_{DR} c(a, \tau_1, \tau_3)\rangle = Q d'_{DR} c(a, \tau_1, \tau_2, \tau_3), \end{aligned}$$

quod trahit effatum lemmatis.

(d) *Paries rectus*

2.15.

Revocamus operatores:

$$Q : \operatorname{Hom}(\Lambda^2 T, \Omega) \longrightarrow \operatorname{Hom}(S^2 T \otimes T, A)$$

definitum per formulam:

$$Qc(\tau_1, \tau_2, \tau_3) = \langle \tau_1, c(\tau_2, \tau_3)\rangle + \langle \tau_2, c(\tau_1, \tau_3)\rangle$$

atque

$$Q : \operatorname{Hom}(A \otimes T^{\otimes 2}, \Omega) \longrightarrow \operatorname{Hom}(A \otimes T^{\otimes 3}, A)$$

definitum per regulam:

$$Qc(a, \tau_1, \tau_2, \tau_3) = \langle \tau_2, c(a, \tau_1, \tau_3)\rangle.$$

2.16. Lemma. $d_H Q = Q d_H$.

Si enim $c \in \operatorname{Hom}(\Lambda^2 T, \Omega)$, habemus:

$$\begin{aligned} d_H Qc(a, \tau_1, \tau_2, \tau_3) &= Qc(a\tau_1, \tau_2, \tau_3) - aQc(a\tau_1, \tau_2, \tau_3) \\ &= \langle a\tau_1, c(\tau_2, \tau_3)\rangle + \langle \tau_2, c(a\tau_1, \tau_3)\rangle - a\langle \tau_1, c(\tau_2, \tau_3)\rangle \\ &\quad - a\langle \tau_2, c(\tau_1, \tau_3)\rangle \\ &= \langle \tau_2, d_H c(a, \tau_1, \tau_3)\rangle = Q d_H c(a, \tau_1, \tau_2, \tau_3) \end{aligned}$$

(e) *Paries sinister*

2.17.

Revocamus operatores:

$$Q: \ \mathrm{Hom}(T, \Omega) \longrightarrow \mathrm{Hom}(S^2 T, A)$$

definitur per regulam:

$$Qc(\tau_1, \tau_2) = \langle \tau_1, c(\tau_2)\rangle + \langle \tau_2, c(\tau_1)\rangle$$

atque

$$Q: \ \mathrm{Hom}(A \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 2}, A)$$

definitur per formulam:

$$Qc(a, \tau_1, \tau_2) = \langle \tau_2, c(a, \tau_1)\rangle.$$

2.18. Lemma. *Fit* $d_H Q = Q d_H$.

Quod probatur eadem ratione ut in art. 2.16.

3. Tabulatum primum

(a) *Paries recessus*

3.1.

Determinamus sagittulas

$$d_{DR}: \ \mathrm{Hom}(A^{\otimes 2} \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega)$$

per formulam

$$\begin{aligned} d_{DR} c(a, b, \tau, \tau') &= \tau' c(a, b, \tau) - c(\tau'(a), b, \tau) - c(a, \tau'(b), \tau) + c(a, b, [\tau, \tau']) \\ &\quad - \frac{1}{2} d\langle \tau', c(a, b, \tau)\rangle \\ &= \mathrm{Lie}_{\tau'} c(a, b, \tau) - \frac{1}{2} d\langle \tau', c(a, b, \tau)\rangle; \end{aligned}$$

(commodum est introducere operatores

$$\mathrm{Lie}\ c(a, b, \tau, \tau') := \mathrm{Lie}_{\tau'} c(a, b, \tau)$$

atque

$$d'_{DR} c(a, b, \tau, \tau') := -\frac{1}{2} d\langle \tau', c(a, b, \tau)\rangle,$$

ergo $d_{DR} = \mathrm{Lie} + d'_{DR}$);

3.2.

ac

$$d_H : \operatorname{Hom}(A \otimes T, \Omega) \longrightarrow \operatorname{Hom}(A^{\otimes 2} \otimes T, \Omega)$$

per regulam

$$d_H c(a, b, \tau) = ac(b, \tau) - c(ab, \tau) + c(a, b\tau);$$

deinde

$$d_H : \operatorname{Hom}(A \otimes T^{\otimes 2}, \Omega) \longrightarrow \operatorname{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega)$$

per formulam

$$d_H c(a, b, \tau, \tau') = ac(b, \tau, \tau') - c(ab, \tau, \tau') + c(a, b\tau, \tau');$$

3.3.

$$Q : \operatorname{Hom}(A \otimes T, \Omega) \longrightarrow \operatorname{Hom}(A \otimes T^{\otimes 2}, A)$$

per regulam

$$Qc(a, \tau, \tau') = \langle \tau', c(a, \tau) \rangle,$$

3.4.

denique

$$M : \operatorname{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \operatorname{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega)$$

per formulam

$$Mc(a, b, \tau, \tau') = \frac{1}{2} da\, c(b, \tau, \tau').$$

3.5. Lemma. *Fit* $d_H d_{DR} = d_{DR} d_H + MQ$.

3.6. Demonstratio. Pro $c \in \operatorname{Hom}(A \otimes T, \Omega)$ habebimus

$$d_H d_{DR} c(a, b, \tau, \tau') = d_H \{\text{Lie} + d'_{DR}\} c(a, b, \tau, \tau').$$

3.7.

Primo, derivatio Lietiana et differentiale Hochschildianum commutant:

$$d_H \text{Lie}\; c(a, b, \tau, \tau') = \text{Lie}\; d_H c(a, b, \tau, \tau').$$

3.8.

Secundo, fit:

$$d_H d'_{DR} c(a, b, \tau, \tau') = -\frac{1}{2}\{ad\langle\tau', c(b, \tau)\rangle - d\langle\tau', c(ab, \tau)\rangle + d\langle\tau', c(a, b\tau)\rangle\}.$$

Addemus huc:

$$0 = \frac{1}{2}\{-d\langle\tau', ac(b, \tau)\rangle + da\ \langle\tau', c(b, \tau)\rangle + ad\langle\tau', c(b, \tau)\}.$$

Adipiscemur:

$$d_H d'_{DR} c(a, b, \tau, \tau') = -\frac{1}{2} d\langle\tau', d_H c(a, b, \tau)\rangle + \frac{1}{2} da\ \langle\tau', c(b, \tau)\rangle$$

ubi manifesto

$$\frac{1}{2} da\ \langle\tau', c(b, \tau)\rangle = MQc(a, b, \tau, \tau')$$

unde lemma sequitur.

(b) *Paries sinister*

3.9.

Contemplemur operatores

$$Q : \ \mathrm{Hom}(A \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 2}, A),$$

per regulam

$$Qc(a, \tau, \tau') = \langle\tau', c(a, \tau)\rangle,$$

definitur, tamquam in art. 2.17, atque

$$Q : \ \mathrm{Hom}(A^{\otimes 2} \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A)$$

per formulam

$$Qc(a, b, \tau, \tau') = \langle\tau', c(a, b, \tau)\rangle$$

definitur.

3.10. Lemma. *Fit* $d_H Q = Qd_H$.

3.11. Demonstratio. Pro $c \in \mathrm{Hom}(A \otimes T, \Omega)$ habeatur

$$\begin{aligned} d_H Qc(a, b, \tau, \tau') &= aQc(b, \tau, \tau') - Qc(ab, \tau, \tau') + Qc(a, b\tau, \tau') \\ &= a\langle\tau', c(b, \tau)\rangle - \langle\tau', c(ab, \tau)\rangle + \langle\tau', c(a, b\tau)\rangle \\ &= \langle\tau', d_H c(a, b, \tau)\rangle = Qd_H c(a, b, \tau, \tau'), \end{aligned}$$ qed

(c) *Paries rectus*

3.12.

Contemplemur operatores:

$$Q: \ \mathrm{Hom}(A \otimes T^{\otimes 2}, \Omega) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 3}, A)$$

per regulam

$$Qc(a, \tau_1, \tau_2, \tau_3) = \langle \tau_2, c(a, \tau_1, \tau_3) \rangle,$$

definitur, et

$$Q: \ \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 3}, A)$$

per formulam

$$Qc(a, b, \tau_1, \tau_2, \tau_3) = \langle \tau_2, c(a, b, \tau_1, \tau_3) \rangle,$$

definitur.

3.13. Lemma. *Fit* $d_H Q = Q d_H$.

Quod probatur eodem modo ut in art. 3.11.

(d) *Camera*

3.14.

Contemplemur operatores: primo, sagittulam

$$d_{DR}: \ \mathrm{Hom}(A^{\otimes 2} \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega)$$

per formulam

$$d_{DR}c(a, b, \tau, \tau') = \mathrm{Lie}_{\tau'} c(a, b, \tau) - \frac{1}{2} d\langle \tau', c(a, b, \tau) \rangle$$

definitam (vide art. 3.1); secundo, sagittulam novam,

$$d_{DR}: \ \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 3}, A)$$

per formulam

$$d_{DR}c(a, b, \tau_1, \tau_2, \tau_3) = \mathrm{Lie}_{\tau_3} c(a, b, \tau_1, \tau_2) - \frac{1}{2} \tau_2 c(a, b, \tau_1, \tau_3)$$

definitam.

3.15. Lemma. *Fit* $Q d_{DR} = d_{DR} Q$.

3.16.

Primo, Q Lie $=$ Lie Q.

Habeatur enim,

$$\begin{aligned} Q\mathrm{Lie}c(a,b,\tau_1,\tau_2,\tau_3) &= \langle \tau_2, \mathrm{Lie}_{tau_3} c(a,b,\tau_1)\rangle \\ &= \langle \tau_2, \tau_3 c(a,b,\tau_1) - c(\tau_3(a),b,\tau_1) - c(a,\tau_3(b),\tau_1) \\ &\quad + c(a,b,[\tau_1,\tau_3])\rangle \\ &= \tau_3\langle \tau_2, c(a,b,\tau_1)\rangle + \langle [\tau_2,\tau_3], c(a,b,\tau_1)\rangle \\ &\quad + \langle \tau_2, -c(\tau_3(a),b,\tau_1) - c(a,\tau_3(b),\tau_1) + c(a,b,[\tau_1,\tau_3])\rangle \\ &= \tau_3 Qc(a,b,\tau_1,\tau_2) + Qc(a,b,\tau_1,[\tau_2,\tau_3]) \\ &\quad - Qc(\tau_3(a),b,\tau_1,\tau_2) - Qc(a,\tau_3(b),\tau_1,\tau_2) \\ &\quad + Qc(a,b,[\tau_1,\tau_3],\tau_2) \\ &= \mathrm{Lie}_{\tau_3} Qc(a,b,\tau_1,\tau_2), \qquad \text{qed} \end{aligned}$$

3.17.

Secundo,

$$\begin{aligned} Qd'_{DR}c(a,b,\tau_1,\tau_2,\tau_3) &= \left\langle \tau_2, -\frac{1}{2} d\langle \tau_3, c(a,b,\tau_1)\rangle \right\rangle \\ &= -\frac{1}{2}\tau_2\tau_3, c(a,b,\tau_1)\rangle\Bigg\rangle = -\frac{1}{2}\tau_2 Qc(a,b,\tau_1,\tau_3) \\ &= d'_{DR} Qc(a,b,\tau_1,\tau_2,\tau_3), \end{aligned}$$

unde lemma sequitur.

(e) *Frons*

3.18.

Revocamus sagittulas:

$$d_{DR}: \ \mathrm{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 3}, A)$$

per formulam

$$d_{DR}c(a,\tau_1,\tau_2,\tau_3) = \mathrm{Lie}_{\tau_3} c(a,\tau_1,\tau_2) - \frac{1}{2}\tau_2 c(a,\tau_1,\tau_3),$$

definitam, vide art. 2.7, atque

$$d_{DR}: \ \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 3}, A)$$

per regulam

$$d_{DR}c(a, b, \tau_1, \tau_2, \tau_3) = \mathrm{Lie}_{\tau_3} c(a, b, \tau_1, \tau_2) - \frac{1}{2}\tau_2 c(a, b, \tau_1, \tau_3),$$

definitam, vide art. 3.14, ac denique

$$M : \mathrm{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega)$$

per formulam

$$Mc(a, b, \tau, \tau') = \frac{1}{2} da\, c(b, \tau, \tau'),$$

definitam, vide art. 3.4.

3.19. Lemma. *Fit* $d_H d_{DR} = d_{DR} d_H + QM$.

3.20. Demonstratio. Primo, fit d_H Lie $=$ Lie d_H.

Habeatur enim:

$$\begin{aligned} d_H\, \mathrm{Lie} c(a, b, \tau_1, \tau_2, \tau_3) = {} & a\mathrm{Lie}_{\tau_3} c(b, \tau_1, \tau_2) - \mathrm{Lie}_{\tau_3} c(ab, \tau_1, \tau_2) \\ & + \mathrm{Lie}_{\tau_3} c(a, b\tau_1, \tau_2), \end{aligned}$$

ubi

$$\begin{aligned} a\mathrm{Lie}_{\tau_3} c(b, \tau_1, \tau_2) &= a\tau_3 c(b, \tau_1, \tau_2) - ac(\tau_3(b), \tau_1, \tau_2) + ac(b, [\tau_1, \tau_3], \tau_2) \\ &\quad + ac(b, \tau_1, [\tau_2, \tau_3]); \\ -\mathrm{Lie}_{\tau_3} c(ab, \tau_1, \tau_2) &= -\tau_3 c(ab, \tau_1, \tau_2) + c(\tau_3(a)b + a\tau_3(b), \tau_1, \tau_2) \\ &\quad - c(ab, [\tau_1, \tau_3], \tau_2) - c(ab, \tau_1, [\tau_2, \tau_3]) \end{aligned}$$

atque

$$\begin{aligned} \mathrm{Lie}_{\tau_3} c(a, b\tau_1, \tau_2) = {} & \tau_3 c(a, b\tau_1, \tau_2) - c(\tau_3(a), b\tau_1, \tau_2) + c(a, b[\tau_1, \tau_3] \\ & - \tau_3(b)\tau_1, \tau_2) + c(a, b\tau_1, [\tau_2, \tau_3]). \end{aligned}$$

Addemus huc:

$$0 = \tau_3\{ac(b, \tau_1, \tau_2)\} - \tau_3(a)c(b, \tau_1, \tau_2) + a\tau_3 c(b, \tau_1, \tau_2)$$

Post summationem, videamus statim:

$$\begin{aligned} d_H\, \mathrm{Lie} c(a, b, \tau_1, \tau_2, \tau_3) &= \tau_3 d_H c(a, b, \tau_1, \tau_2) - d_H c(\tau_3(a), b, \tau_1, \tau_2) \\ &\quad - d_H c(a, \tau_3(b), \tau_1, \tau_2) + d_H c(a, b, [\tau_1, \tau_3], \tau_2) \\ &\quad + d_H c(a, b, \tau_1, [\tau_2, \tau_3]) \\ &= \mathrm{Lie}_{\tau_3} d_H c(a, b, \tau_1, \tau_2), \end{aligned}$$

qed

3.21.

Secundo,

$$d_H d'_{DR} c(a, b, \tau_1, \tau_2, \tau_3) = -\frac{1}{2}\{a\tau_2 c(b, \tau_1, \tau_3) - \tau_2 c(ab, \tau_1, \tau_3) + \tau_2 c(a, b\tau_1, \tau_3)\}.$$

Addemus huc:

$$0 = -\frac{1}{2}\{\tau_2\{ac(b, \tau_1, \tau_3)\} - \tau_2(a)c(b, \tau_1, \tau_3) - a\tau_2 c(b, \tau_1, \tau_3)\}$$

Post summationem, adipiscemur:

$$d_H d'_{DR} c(a, b, \tau_1, \tau_2, \tau_3) = d'_{DR} d_H c(a, b, \tau_1, \tau_2, \tau_3) + \frac{1}{2}\tau_2(a)c(b, \tau_1, \tau_3),$$

ubi

$$\frac{1}{2}\tau_2(a)c(b, \tau_1, \tau_3) = \left\langle \tau_2, \frac{1}{2} da\ c(b, \tau_1, \tau_3)\right\rangle$$
$$= \langle \tau_2, Mc(b, \tau_1, \tau_2, \tau_3)\rangle = QMc(a, b, \tau_1, \tau_2, \tau_3) \qquad \text{qed}$$

(f) *Junctio*

3.22.

Revocamus sagittulas:

$$M : \operatorname{Hom}(S^2T, A) \longrightarrow \operatorname{Hom}(A \otimes T^{\otimes 2}, \Omega)$$

per regulam

$$Mc(a, \tau, \tau') = -\frac{1}{2} da\ c(\tau, \tau')$$

definitam, atque

$$M : \operatorname{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \operatorname{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega)$$

per formulam

$$Mc(a, b, \tau, \tau') = \frac{1}{2} da\ c(b, \tau, \tau')$$

definitam.

3.23. Lemma. *Fit* $d_H M = -M d_H$.

3.24. Demonstratio. Pro elemento $c \in \operatorname{Hom}(S^2T, A)$, habemus

$$d_H Mc(a, b, \tau, \tau') = aMc(b, \tau, \tau') - Mc(ab, \tau, \tau') + Mc(a, b\tau, \tau')$$
$$= -\frac{1}{2}\{adb\ c(\tau, \tau') - d(ab)c(\tau, \tau') + da\ c(b\tau, \tau')\}$$
$$- \frac{1}{2}\{-da\ bc(\tau, \tau') + da\ c(b\tau, \tau')\} = -\frac{1}{2} da d_H c(b, \tau, \tau')$$
$$= -M d_H c(a, b, \tau, \tau'), \qquad \text{qed}$$

4. Tabulatum secundum

(a) *Paries recessus*

4.1.

Revocamus sagittulam:

$$d_{DR}: \ \mathrm{Hom}(A^{\otimes 2} \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega)$$

per formulam

$$d_{DR}c(a, b, \tau, \tau') = \mathrm{Lie}_{\tau'}c(a, b, \tau) - \frac{1}{2}d\langle \tau', c(a, b, \tau)\rangle,$$

definitam (vide art. 3.1) atque introducamus sagittulam:

$$d_{DR}: \ \mathrm{Hom}(A^{\otimes 3} \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 3} \otimes T^{\otimes 2}, \Omega)$$

per regulam

$$d_{DR}c(a, b, c, \tau, \tau') = \mathrm{Lie}_{\tau'}c(a, b, c, \tau) - \frac{1}{2}d\langle \tau', c(a, b, c, \tau)\rangle$$

definitam.

4.2.

Determinamus sagittulas:

$$d_H: \ \mathrm{Hom}(A^{\otimes 2} \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 3} \otimes T, \Omega)$$

per formulam:

$$d_H c(a, b, c, \tau) = ac(b, c, \tau) - c(ab, c, \tau) + c(a, bc, \tau) - c(a, b, c\tau);$$

porro

$$d_H: \ \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 3} \otimes T^{\otimes 2}, \Omega)$$

per regulam:

$$d_H c(a, b, c, \tau, \tau') = ac(b, c, \tau, \tau') - c(ab, c, \tau, \tau') + c(a, bc, \tau, \tau') - c(a, b, c\tau, \tau').$$

4.3.

atque

$$M: \ \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A^{\otimes 3} \otimes T^{\otimes 2}, \Omega)$$

per formulam:

$$Mc(a, b, c, \tau, \tau') = \frac{1}{2}da\ c(b, c, \tau, \tau').$$

4.4.

Tandem, revocamus sagittulam:

$$Q:\ \mathrm{Hom}(A^{\otimes 2}\otimes T,\Omega)\longrightarrow \mathrm{Hom}(A^{\otimes 2}\otimes T^{\otimes 2},A)$$

per formulam

$$Qc(a,b,\tau,\tau')=\langle \tau', c(a,b,\tau)\rangle,$$

definitam, vide 3.9.

4.5. Lemma. *Fit* $d_H d_{DR} = d_{DR} d_H + MQ$.

Quod probatur eodem modo ut in arts. 3.7, 3.8.

(b) *Paries sinister*

4.6.

Revocamus sagittulam:

$$Q:\ \mathrm{Hom}(A^{\otimes 2}\otimes T,\Omega)\longrightarrow \mathrm{Hom}(A^{\otimes 2}\otimes T^{\otimes 2},A)$$

per regulam

$$Qc(a,b,\tau,\tau')=\langle \tau', c(a,b,\tau)\rangle,$$

definitam, vide art. 3.9, atque introducamus sagittulam novam:

$$Q:\ \mathrm{Hom}(A^{\otimes 3}\otimes T,\Omega)\longrightarrow \mathrm{Hom}(A^{\otimes 3}\otimes T^{\otimes 2},A)$$

per formulam

$$Qc(a,b,c,\tau,\tau')=\langle \tau', c(a,b,c,\tau)\rangle$$

definitam.

4.7. Lemma. *Fit* $d_H Q = Q d_H$.

Quod probatur eodem modo ut in art. 3.11.

(c) *Paries rectus*

4.8.

Revocamus operatorem:

$$Q:\ \mathrm{Hom}(A^{\otimes 2}\otimes T^{\otimes 2},\Omega)\longrightarrow \mathrm{Hom}(A^{\otimes 2}\otimes T^{\otimes 3},A),$$

definitum per formulam:

$$Qc(a,b,\tau_1,\tau_2,\tau_3)=\langle \tau_2, c(a,b,\tau_1,\tau_3)\rangle,$$

vide art. 3.12, ac determinamus operatorem novum:

$$Q:\ \mathrm{Hom}(A^{\otimes 3}\otimes T^{\otimes 2},\Omega)\longrightarrow \mathrm{Hom}(A^{\otimes 3}\otimes T^{\otimes 3},A)$$

per regulam:

$$Qc(a,b,c,\tau_1,\tau_2,\tau_3)=\langle \tau_2, c(a,b,c,\tau_1,\tau_3)\rangle.$$

4.9. Lemma. *Fit* $d_H Q = Q d_H$.

Quod etiam probatur eadem ratione ut in art. 3.11.
(d) *Camera*

4.10.

Contemplemur sagittulam:

$$d_{DR}: \operatorname{Hom}(A^{\otimes 3} \otimes T, \Omega) \longrightarrow \operatorname{Hom}(A^{\otimes 3} \otimes T^{\otimes 2}, \Omega)$$

per regulam

$$d_{DR}c(a, b, c, \tau, \tau') = \operatorname{Lie}_{\tau'} c(a, b, c, \tau) - \frac{1}{2} d\langle \tau', c(a, b, c, \tau)\rangle,$$

definitam, vide art. 4.1, atque introducamus sagittulam novam:

$$d_{DR}: \operatorname{Hom}(A^{\otimes 3} \otimes T^{\otimes 2}, A) \longrightarrow \operatorname{Hom}(A^{\otimes 3} \otimes T^{\otimes 3}, A)$$

per formulam:

$$d_{DR}c(a, b, c, \tau_1, \tau_2, \tau_3) = \operatorname{Lie}_{\tau_3} c(a, b, c, \tau_1, \tau_3) - \frac{1}{2} \tau_2 c(a, b, c, \tau_1, \tau_3).$$

4.11. Lemma. *Fit* $Q d_{DR} = d_{DR} Q$.

Quod probatur simili calculo ut in artt. 3.16, 3.17.

(e) *Frons*

4.12.

Revocamus sagittulas:

$$d_{DR}: \operatorname{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A) \longrightarrow \operatorname{Hom}(A^{\otimes 2} \otimes T^{\otimes 3}, A),$$

definitam per:

$$d_{DR}c(a, b, \tau_1, \tau_2, \tau_3) = \operatorname{Lie}_{\tau_3} c(a, b, \tau_1, \tau_2) - \frac{1}{2} \tau_2 c(a, b, \tau_1, \tau_3),$$

vide art. 3.14, porro:

$$d_{DR}: \operatorname{Hom}(A^{\otimes 3} \otimes T^{\otimes 2}, A) \longrightarrow \operatorname{Hom}(A^{\otimes 3} \otimes T^{\otimes 3}, A)$$

definitam per:

$$d_{DR}c(a, b, c, \tau_1, \tau_2, \tau_3) = \operatorname{Lie}_{\tau_3} c(a, b, c, \tau_1, \tau_2) - \frac{1}{2} \tau_2 c(a, b, c, \tau_1, \tau_3),$$

vide art 4.10, denique:

$$M : \operatorname{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A) \longrightarrow \operatorname{Hom}(A^{\otimes 3} \otimes T^{\otimes 2}, \Omega)$$

definitam per:

$$Mc(a, b, c, \tau, \tau') = \frac{1}{2} da\, c(b, c, \tau, \tau'),$$

vide art. 4.3.

4.13. Lemma. *Fit* $d_H d_{DR} = d_{DR} d_H + QM$.

Quod probatur omnino simili modo ut in artt. 3.20, 3.21.

(f) *Junctio*

4.14.

Revocamus operatores:

$$M : \operatorname{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \operatorname{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega)$$

definitum per:

$$Mc(a, b, \tau, \tau') = \frac{1}{2} da\, c(b, \tau, \tau'),$$

vide art. 3.4, et

$$M : \operatorname{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A) \longrightarrow \operatorname{Hom}(A^{\otimes 3} \otimes T^{\otimes 2}, \Omega)$$

definitum per:

$$Mc(a, b, c, \tau, \tau') = \frac{1}{2} da\, c(b, c, \tau, \tau'),$$

vide art. 4.3.

4.15. Lemma. *Fit* $d_H\, M = M\, d_H$.

Quod probatur eodem modo ut in art. 3.24.

Pars secunda. Aedificium rectum

1. Pede plana

(a) *Paries recessus*

1.1.

Revocamus operatores:

$$d_{DR} : \operatorname{Hom}(\Lambda^2 T, \Omega) \longrightarrow \operatorname{Hom}(\Lambda^3 T, \Omega)$$

definitum per formulam:

$$d_{DR}c(\tau,\tau',\tau'') = \mathrm{Cycle}_{\tau,\tau',\tau''}\left[c([\tau,\tau'],\tau'') - \tau\{c(\tau',\tau'')\} + \frac{1}{3}d\langle\tau, c(\tau',\tau'')\rangle\right]$$

$$= \mathrm{Alt}_{\tau',\tau''}\left\{c([\tau,\tau'],\tau'') - \frac{1}{2}c(\tau,[\tau',\tau'']) - \frac{1}{2}\tau c(\tau',\tau'') \right.$$

$$\left. + \tau' c(\tau,\tau'') + \frac{1}{6}d\langle\tau, c(\tau',\tau'')\rangle - \frac{1}{3}d\langle\tau', c(\tau,\tau'')\rangle\right\},$$

vide Pars Prima, art. 1.6, atque

$$Q:\ \mathrm{Hom}(\Lambda^2 T,\Omega) \longrightarrow \mathrm{Hom}(S^2T\otimes T, A)$$

definitum per:

$$Qc(\tau,\tau',\tau'') = \mathrm{Sym}_{\tau,\tau'}\langle\tau, c(\tau',\tau'')\rangle.$$

1.2.

Introducamus operatores:

$$d_{DR}:\ \mathrm{Hom}(A\otimes T^{\otimes 2},\Omega) \longrightarrow \mathrm{Hom}(A\otimes T\otimes\Lambda^2 T,\Omega)$$

per regulam:

$$d_{DR}c(a,\tau,\tau',\tau'') = \mathrm{Alt}_{\tau',\tau''}\left\{\tau' c(a,\tau,\tau'') - c(\tau'(a),\tau,\tau'') + c(a,[\tau,\tau'],\tau'') \right.$$

$$\left. - \frac{1}{2}c(a,\tau,[\tau',\tau'']) - \frac{1}{3}d\langle\tau', c(a,\tau,\tau'')\rangle\right\}$$

$$= \{\mathrm{Lie} + d'_{DR}\}c(a,\tau,\tau',\tau''),$$

ubi

$$\mathrm{Lie}c(a,\tau,\tau',\tau'') = \mathrm{Alt}_{\tau',\tau''}\left\{\tau' c(a,\tau,\tau'') - c(\tau'(a),\tau,\tau'') \right.$$

$$\left. + c(a,[\tau,\tau'],\tau'') - \frac{1}{2}c(a,\tau,[\tau',\tau''])\right\}$$

et

$$d'_{DR}c(a,\tau,\tau',\tau'') = -\frac{1}{3}\mathrm{Alt}_{\tau',\tau''}d\langle\tau', c(a,\tau,\tau'')\rangle,$$

1.3.

ac

$$M:\ \mathrm{Hom}(S^2T\otimes T, A)\longrightarrow \mathrm{Hom}(A\otimes T\otimes\Lambda^2T,\Omega)$$

definitam per:

$$Mc(a,\tau,\tau',\tau'')=-\frac{1}{3}da\ \mathrm{Alt}_{\tau',\tau''}c(\tau,\tau',\tau'').$$

1.4. Lemma. *Fit* $d_Hd_{DR}=d_{DR}d_H+MQ$.

1.5. Demonstratio. Habemus

$$d_Hd_{DR}c(a,\tau,\tau',\tau'')=d_{DR}c(a\tau,\tau',\tau'')-ad_{DR}c(\tau,\tau',\tau''),$$

ubi

$$\begin{aligned}d_{DR}c(a\tau,\tau',\tau'')&=\mathrm{Alt}_{\tau',\tau''}\bigg\{c([a\tau,\tau'],\tau'')-\frac{1}{2}c(a\tau,[\tau',\tau''])\\&\quad-\frac{1}{2}(a\tau)c(\tau',\tau'')+\tau'c(a\tau,\tau'')+\frac{1}{6}d\langle a\tau,c(\tau',\tau'')\rangle\\&\quad-\frac{1}{3}d\langle\tau',c(a\tau,\tau'')\rangle\bigg\}\\&=\mathrm{Alt}_{\tau',\tau''}\bigg\{c(a[\tau,\tau']-\tau'(a)\tau,\tau'')\\&\quad-\frac{1}{2}c(a\tau,[\tau',\tau''])-\frac{1}{2}a\tau c(\tau',\tau'')-\frac{1}{2}da\langle\tau,c(\tau',\tau'')\rangle\\&\quad+\tau'c(a\tau,\tau'')+\frac{1}{6}da\langle\tau,c(\tau',\tau'')\rangle+\frac{1}{6}ad\langle\tau,c(\tau',\tau'')\rangle\\&\quad-\frac{1}{3}d\langle\tau',c(a\tau,\tau'')\rangle\bigg\}\end{aligned}$$

ac

$$\begin{aligned}-ad_{DR}c(\tau,\tau',\tau'')&=-a\mathrm{Alt}_{\tau',\tau''}\bigg\{\tau'c(a,\tau,\tau'')-c(\tau'(a),\tau,\tau'')\\&\quad+c(a,[\tau,\tau'],\tau'')-\frac{1}{2}c(a,\tau,[\tau',\tau''])\\&\quad-\frac{1}{3}d\langle\tau',c(a,\tau,\tau'')\rangle\bigg\}.\end{aligned}$$

Addemus huc:

$$0 = -\mathrm{Alt}_{\tau',\tau''}\Big[\tau'\{ac(\tau,\tau'')\} - \tau'(a)c(\tau,\tau'') - a\tau'c(\tau,\tau'')$$
$$-\frac{1}{3}\{d\langle\tau', ac(\tau,\tau'')\rangle - da\langle\tau', c(\tau,\tau'')\rangle - ad\langle\tau', c(\tau,\tau'')\rangle\}\Big]$$

Post rationem parvam, effectus proditur.

(b) *Junctio camerae*

1.6.

Determinamus operatorem:

$$R: \ \mathrm{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A \otimes T \otimes \Lambda^2 T, \Omega)$$

per:

$$Rc(a,\tau,\tau',\tau'') = -\frac{1}{6}\mathrm{Alt}_{\tau',\tau''}d\Big[c(a,[\tau,\tau'],\tau'') + \frac{1}{2}c(a,\tau,[\tau',\tau''])$$
$$+ c(\tau''(a),\tau,\tau')\Big]$$

atque revocamus operatores:

$$d_{DR}: \ \mathrm{Hom}(A \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 2}, \Omega)$$

definitum per:

$$d_{DR}c(a,\tau,\tau') = \mathrm{Lie}_{\tau'}c(a,\tau) - \frac{1}{2}d\langle\tau', c(a,\tau)\rangle,$$

vide Pars Prima, art. 2.2, ac

$$Q: \ \mathrm{Hom}(A \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 2}, A)$$

definitum per:

$$Qc(a,\tau,\tau') = \langle\tau', c(a,\tau)\rangle,$$

vide art. 2.17.

1.7. Lemma. *Fit* $d_{DR}^2 = RQ$.

1.8. Demonstratio. Primo, $\mathrm{Lie}^2 = 0$. Vero,

$$\mathrm{Lie}^2 c(a,\tau,\tau',\tau'') = \mathrm{Alt}_{\tau',\tau''}\{\tau'\mathrm{Lie}_{\tau''}c(a,\tau) - \mathrm{Lie}_{\tau''}c(\tau'(a),\tau) + \mathrm{Lie}_{\tau''}c(a,[\tau,\tau'])\} - \mathrm{Lie}_{[\tau',\tau'']}c(a,\tau),$$

ubi

$$\mathrm{Alt}_{\tau',\tau''}\tau'\mathrm{Lie}_{\tau''}c(a,\tau) = \mathrm{Alt}_{\tau',\tau''}\{\tau'\tau''c(a,\tau) - \tau'c(\tau''(a),\tau) + \tau'c(a,[\tau,\tau''])\};$$

$$-\mathrm{Alt}_{\tau',\tau''}\mathrm{Lie}_{\tau''}c(\tau'(a),\tau) = \mathrm{Alt}_{\tau',\tau''}\{-\tau''c(\tau'(a),\tau) + c(\tau''\tau'(a),\tau) - c(\tau'(a),[\tau,\tau''])\};$$

$$\mathrm{Alt}_{\tau',\tau''}\mathrm{Lie}_{\tau''}c(a,[\tau,\tau']) = \mathrm{Alt}_{\tau',\tau''}\{\tau''c(a,[\tau,\tau']) - c(\tau''(a),[\tau,\tau']) + c(a,[[\tau,\tau'],\tau''])\}$$

ac

$$-\mathrm{Lie}_{[\tau',\tau'']}c(a,\tau) = -[\tau',\tau'']c(a,\tau) + c([\tau',\tau''](a),\tau) - c(a,[\tau,[\tau',\tau'']])$$

Addendo adipiscimur protenus 0.

1.9.

Secundo, fit

$$\begin{aligned}\mathrm{Lie}d'_{DR}c(a,\tau,\tau',\tau'') &= \mathrm{Alt}_{\tau',\tau''}\{\tau'd'_{DR}c(a,\tau,\tau'') \\ &\quad - d'_{DR}c(\tau'(a),\tau,\tau'') + d'_{DR}c(a,[\tau,\tau'],\tau'')\} \\ &\quad - d'_{DR}c(a,\tau,[\tau',\tau'']) \\ &= -\frac{1}{2}\big[\mathrm{Alt}_{\tau',\tau''}\{d\tau'\langle\tau'',c(a,\tau)\rangle \\ &\quad - d\langle\tau'',c(\tau'(a),\tau)\rangle + d\langle\tau'',c(a,[\tau,\tau'])\rangle\} \\ &\quad - d\langle[\tau',\tau''],c(a,\tau)\rangle\big]\end{aligned}$$

ubi

$$\tau'\langle\tau'',c(a,\tau)\rangle = \langle[\tau',\tau''],c(a,\tau)\rangle + \langle\tau'',\tau'c(a,\tau)\rangle$$

1.10.

Tertio, fit

$$\begin{aligned}d'_{DR}\mathrm{Lie}\,c(a,\tau,\tau',\tau'') &= -\frac{1}{3}\mathrm{Alt}_{\tau',\tau''}d\langle\tau',\mathrm{Lie}_{\tau''}c(a,\tau)\rangle \\ &= -\frac{1}{3}\mathrm{Alt}_{\tau',\tau''}d\langle\tau',\tau''c(a,\tau) - c(\tau''(a),\tau) + c(a,[\tau,\tau''])\rangle.\end{aligned}$$

1.11.

Denique fit quarto

$$d'^2_{DR}c(a,\tau,\tau',\tau'') = -\frac{1}{3}\mathrm{Alt}_{\tau',\tau''}d\langle\tau', d'_{DR}c(a,\tau,\tau'')\rangle$$
$$= \frac{1}{6}\mathrm{Alt}_{\tau',\tau''}d\langle\tau', d\langle\tau'', c(a,\tau)\rangle\rangle = \frac{1}{6}\mathrm{Alt}_{\tau',\tau''}d\tau'\langle\tau'', c(a,\tau)\rangle.$$

1.12.

Addendo obtenebimus:

$$d^2_{DR}c(a,\tau,\tau',\tau'') = \{\mathrm{Lie}d'_{DR} + d'_{DR}\mathrm{Lie} + d'^2_{DR}\}c(a,\tau,\tau',\tau'')$$
$$= -\frac{1}{6}d\langle[\tau',\tau''], c(a,\tau)\rangle + \frac{1}{6}\mathrm{Alt}_{\tau',\tau''}\Big\{d\langle\tau', c(a,[\tau,\tau''])\rangle$$
$$- d\langle\tau', c(\tau''(a),\tau)\rangle\Big\}$$
$$= \frac{1}{6}d\big[-Qc(a,\tau,[\tau',\tau''])$$
$$+ \mathrm{Alt}_{\tau',\tau''}\{Qc(a,[\tau,\tau''],\tau') - Qc(\tau''(a),\tau,\tau')\}\big]$$
$$= RQc(a,\tau,\tau',\tau''), \qquad \text{qed}$$

(c) *Junctiones...*

1.13.

Revocamus operatores:

$$R:\ \mathrm{Hom}(S^2T, A) \longrightarrow \mathrm{Hom}(\Lambda^3T, \Omega)$$

definitum per:

$$Rc(\tau,\tau',\tau'') = -\frac{1}{6}\mathrm{Cycle}_{\tau,\tau',\tau''}dc([\tau,\tau'],\tau'')$$
$$= -\frac{1}{6}\big[\mathrm{Alt}_{\tau',\tau''}dc([\tau,\tau'],\tau'') + dc([\tau',\tau''],\tau)\big],$$

vide Pars Prima, 1.7, atque

$$R:\ \mathrm{Hom}(A\otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A\otimes T\otimes \Lambda^2T, \Omega)$$

definitum per:

$$Rc(a, \tau, \tau', \tau'') = -\frac{1}{6}d\big[\mathrm{Alt}_{\tau',\tau''}\{c(a, [\tau, \tau'], \tau'') + c(\tau''(a), \tau, \tau')\} + c(a, \tau, [\tau', \tau''])\big],$$

vide art. 1.6.

1.14.

Porro,

$$M : \mathrm{Hom}(S^2T, A) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 2}, \Omega)$$

definitum per:

$$Mc(a, \tau, \tau') = -\frac{1}{2}c(\tau, \tau')da,$$

vide Pars Prima, art. 2.3, ac

$$M : \mathrm{Hom}(S^2T \otimes T, A) \longrightarrow \mathrm{Hom}(A \otimes T \otimes \Lambda^2T, \Omega)$$

definitum per

$$Mc(a, \tau, \tau', \tau'') = -\frac{1}{3}da\ \mathrm{Alt}_{\tau',\tau''}c(\tau, \tau', \tau''),$$

vide art. 1.3.

1.15. Lemma. *Fit* $d_H R = Rd_H + Md_{DR} + d_{DR}M$.

1.16. Demonstratio. Primo, habeatur

$$HRc(a, \tau, \tau', \tau'') = Rc(a\tau, \tau', \tau'') - aRc(\tau, \tau', \tau''),$$

ubi

$$Rc(a\tau, \tau', \tau'') = -\frac{1}{6}\big[\mathrm{Alt}_{\tau',\tau''}dc([a\tau, \tau'], \tau'') + dc([\tau', \tau''], a\tau)\big]$$

$$= -\frac{1}{6}\big[\mathrm{Alt}_{\tau',\tau''}dc(a[\tau, \tau'] - \tau'(a)\tau, \tau'') + dc([\tau', \tau''], a\tau)\big]$$

et

$$-aRc(\tau, \tau', \tau'') = -\frac{1}{6}a\big[\mathrm{Alt}_{\tau',\tau''}dc([\tau, \tau'], \tau'') + dc([\tau', \tau''], \tau)\big].$$

Addemus huc:

$$0 = \frac{1}{6}\mathrm{Alt}_{\tau',\tau''}[d\{ac([\tau, \tau'], \tau'')\} - da\ c([\tau, \tau'], \tau'') - adc([\tau, \tau'], \tau'')],$$

$$0 = -\frac{1}{6}\mathrm{Alt}_{\tau',\tau''}[d\{\tau'(a)c(\tau, \tau'')\} - d\tau'(a)\ c(\tau, \tau'') - \tau'(a)dc(\tau, \tau'')]$$

et

$$0 = \frac{1}{6}\mathrm{Alt}_{\tau',\tau''}[d\{ac([\tau',\tau''],\tau)\} - da\, c([\tau',\tau''],\tau) - a\, dc([\tau',\tau''],\tau)],$$

unde post summationem:

$$\begin{aligned}
d_H Rc(a,\tau,\tau',\tau'') = -\frac{1}{6}\Big\{&\mathrm{Alt}_{\tau',\tau''}[dd_H c(a,[\tau,\tau'],\tau'') - dd_H c(\tau'(a),\tau,\tau'')]\\
&+ dd_H c(a,\tau,[\tau',\tau'']) + \mathrm{Alt}_{\tau',\tau''}[da\, c([\tau,\tau'],\tau'')\\
&+ adc([\tau,\tau'],\tau'') - d\tau'(a)\, c(\tau,\tau'') - \tau'(a)dc(\tau,\tau'')]\\
&+ da\, c([\tau',\tau''],\tau) + a\, dc([\tau',\tau''],\tau)\Big\}\\
= Rd_H c(a,\tau,\tau',\tau'') &+ \mathrm{Alt}_{\tau',\tau''}[da\, c([\tau,\tau'],\tau'')\\
&+ adc([\tau,\tau'],\tau'') - d\tau'(a)\, c(\tau,\tau'') - \tau'(a)dc(\tau,\tau'')]\\
&+ da\, c([\tau',\tau''],\tau) + a\, dc([\tau',\tau''],\tau).
\end{aligned}$$

1.17.

Secundo autem,

$$Md_{DR}c(a,\tau,\tau',\tau'') = -\frac{1}{3}da\ \mathrm{Alt}_{\tau',\tau''}d_{DR}c(\tau,\tau',\tau'')$$

(vide Pars Prima, art. 1.3)

$$= -\frac{1}{3}da\ \mathrm{Alt}_{\tau',\tau''}\{c(\tau,[\tau',\tau'']) + c(\tau',[\tau,\tau'']) - \frac{1}{2}\tau' c(\tau,\tau'') + \tau'' c(\tau,\tau')\}$$

et

$$\begin{aligned}
d_{DR}Mc(a,\tau,\tau',\tau'') = \mathrm{Alt}_{\tau',\tau''}\Big\{&\tau' Mc(a,\tau,\tau'') - Mc(\tau'(a),\tau,\tau'')\\
&+ Mc(a,[\tau,\tau'],\tau'') - \frac{1}{3}d\langle\tau', Mc(a,\tau,\tau'')\rangle\Big\}\\
&- Mc(a,\tau,[\tau',\tau'']),
\end{aligned}$$

ubi

$$\tau' Mc(a,\tau,\tau'') = -\frac{1}{2}\{\tau' da\, c(\tau,\tau'') + da\ \tau' c(\tau,\tau'')\};$$

$$-Mc(\tau'(a),\tau,\tau'') = \frac{1}{2}d\tau'(a)\, c(\tau,\tau'');$$

$$Mc(a, [\tau, \tau'], \tau'') = -\frac{1}{2} da\, c([\tau, \tau'], \tau'');$$

$$-Mc(a, \tau, [\tau', \tau'']) = \frac{1}{2} da\, c(\tau, [\tau', \tau''])$$

et

$$-\frac{1}{3} d\langle \tau', Mc(a, \tau, \tau'')\rangle = \frac{1}{6} d\{\tau'(a) c(\tau, \tau'')\}$$
$$= \frac{1}{6}\{d\tau'(a)\, c(\tau, \tau'') + \tau'(a) dc(\tau, \tau'')\},$$

unde, post summationem,

$$\{Md_{DR} + d_{DR}M\}c(a, \tau, \tau', \tau'') = \mathrm{Alt}_{\tau', \tau''}[da\, c([\tau, \tau'], \tau'') + adc([\tau, \tau'], \tau'')$$
$$- d\tau'(a)\, c(\tau, \tau'') - \tau'(a) dc(\tau, \tau'')]$$
$$+ da\, c([\tau', \tau''], \tau) + a\, dc([\tau', \tau''], \tau),$$

unde lemma nostrum sponte sequitur.

(d) *Paries rectus*

1.18.

Revocamus operatorem:

$$Q:\ \mathrm{Hom}(\Lambda^3 T, \Omega) \longrightarrow \mathrm{Hom}(S^2 T \otimes \Lambda^2 T, A),$$

definitum per

$$Qc(\tau_1, \tau_2, \tau_3, \tau_4) = \mathrm{Sym}_{1,2}\langle \tau_1, c(\tau_2, \tau_3, \tau_4)\rangle$$

atque introducamus operatorem

$$Q:\ \mathrm{Hom}(A \otimes T \otimes \Lambda^2 T, \Omega) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 2} \otimes \Lambda^2 T, A)$$

per

$$Qc(a, \tau_1, \tau_2, \tau_3, \tau_4) = \langle \tau_2, c(a, \tau_1, \tau_3, \tau_4)\rangle.$$

1.19. Lemma. *Fit $d_H Q = Q d_H$.*

Demonstratio. Exstat

$$d_H Qc(a, \tau_1, \tau_2, \tau_3, \tau_4) = Qc(a\tau_1, \tau_2, \tau_3, \tau_4) - aQc(\tau_1, \tau_2, \tau_3, \tau_4),$$

ubi

$$Q(a\tau_1, \tau_2, \tau_3, \tau_4) = \langle a\tau_1, c(\tau_2, \tau_3, \tau_4)\rangle + \langle \tau_2, c(a\tau_1, \tau_3, \tau_4)\rangle$$
$$= a\langle \tau_1, c(\tau_2, \tau_3, \tau_4)\rangle + \langle \tau_2, c(a\tau_1, \tau_3, \tau_4)\rangle,$$

unde

$$d_H Qc(a, \tau_1, \tau_2, \tau_3, \tau_4) = \langle \tau_2, d_H c(\tau_1, \tau_3, \tau_4)\rangle$$

Hinc lemma nostra sequitur sponte.

(e) *Camera*

1.20.

Revocamus sagittulam

$$D_{DR} : \operatorname{Hom}(A \otimes T^{\otimes 2}, \Omega) \longrightarrow \operatorname{Hom}(A \otimes T \otimes \Lambda^2 T, \Omega)$$

per formulam

$$d_{DR}c(a, \tau, \tau', \tau'') = \operatorname{Alt}_{\tau',\tau''}\Big\{\tau' c(a, \tau, \tau'') - c(\tau'(a), \tau, \tau'')$$
$$+ c(a, [\tau, \tau'], \tau'') - \frac{1}{2}c(a, \tau, [\tau', \tau'']) - \frac{1}{3}d\langle \tau', c(a, \tau, \tau'')\rangle\Big\}$$

definitam, vide art. 1.2.

Eadem definitio etiam ita scriberi potest:

$$d_{DR}c(a, \tau, \tau', \tau'') = \operatorname{Alt}_{\tau',\tau''}\operatorname{Lie}_{\tau'}c(a, \tau, \tau'') + c(a, \tau, [\tau', \tau''])$$
$$- \frac{1}{3}\operatorname{Alt}_{\tau',\tau''} d\langle \tau', c(a, \tau, \tau'')\rangle.$$

Introducamus autem operatorem:

$$d_{DR} : \operatorname{Hom}(A \otimes T^{\otimes 3}, A) \longrightarrow \operatorname{Hom}(A \otimes T^{\otimes 2} \otimes \Lambda^2 T, A)$$

per formulam:

$$d_{DR}c(a, \tau_1, \tau_2, \tau_3, \tau_4) = \operatorname{Alt}_{3,4}\Big\{\operatorname{Lie}_{\tau_3}c(a, \tau_1, \tau_2, \tau_4) - \frac{1}{3}\tau_2 c(a, \tau_1, \tau_3, \tau_4)\Big\}$$
$$+ c(a, \tau_1, \tau_2, [\tau_3, \tau_4]).$$

1.21. Lemma. *Fit* $Qd_{DR} = d_{DR}Q$.

1.22. Demonstratio. Exstat

$$Qd_{DR}c(a,\tau_1,\tau_2,\tau_3,\tau_4) = \langle \tau_2, d_{DR}c(a,\tau_1,\tau_3,\tau_4)\rangle$$
$$= \mathrm{Alt}_{3,4}\Big\langle \tau_2, \mathrm{Lie}_{\tau_3}c(a,\tau_1,\tau_4) + \frac{1}{2}c(a,\tau_1,[\tau_3,\tau_4]) - \frac{1}{3}d\langle \tau_3, c(a,\tau_1,\tau_4)\rangle\Big\rangle.$$

Probatur sponte, primo:

$$\langle \tau_2, \mathrm{Lie}_{\tau_3}c(a,\tau_1,\tau_4)\rangle = \mathrm{Lie}_{\tau_3}Qc(a,\tau_1,\tau_2,\tau_4);$$

Secundo, manifesto:

$$\langle \tau_2, c(a,\tau_1,[\tau_3,\tau_4])\rangle = Qc(a,\tau_1,\tau_2,[\tau_3,\tau_4])$$

et

$$\langle \tau_2, d\langle \tau_3, c(a,\tau_1,\tau_4)\rangle\rangle = \tau_2\langle \tau_3, c(a,\tau_1,\tau_4)\rangle\rangle = \tau_2 Qc(a,\tau_1,\tau_3,\tau_4),$$

unde lemma nostrum statim sequitur.

(f) *Frons*

1.23. Lemma. *Fit* $d_H d_{DR} = d_{DR} d_H + QM$.

1.24. Demonstratio. Exstat:

$$d_H d_{DR}c(a,\tau_1,\tau_2,\tau_3,\tau_4) = d_{DR}c(a\tau_1,\tau_2,\tau_3,\tau_4) - a d_{DR}c(\tau_1,\tau_2,\tau_3,\tau_4)$$

Sed (vide Pars Prima, art. 1.14)

$$d_{DR}c(a\tau_1,\tau_2,\tau_3,\tau_4) = -c(a\tau_1,\tau_2,[\tau_3,\tau_4]) + \mathrm{Alt}_{3,4}\Big\{c(a\tau_1,[\tau_2,\tau_3],\tau_4) + c(\tau_2, a[\tau_1,\tau_3] - \tau_3(a)\tau_1,\tau_4) - \tau_4 c(a\tau_1,\tau_2,\tau_3) - \frac{1}{3}a\tau_1 c(\tau_2,\tau_3,\tau_4) - \frac{1}{3}\tau_2 c(a\tau_1,\tau_3,\tau_4)\Big\}.$$

Addemus huc:

$$0 = \mathrm{Alt}_{3,4}\{\tau_4\{ac(\tau_1,\tau_2,\tau_3)\} - \tau_4(a)c(\tau_1,\tau_2,\tau_3) - a\tau_4 c(\tau_1,\tau_2,\tau_3)\}$$

cum

$$0 = \frac{1}{3}\mathrm{Alt}_{3,4}\{\tau_2\{ac(\tau_1,\tau_3,\tau_4)\} - \tau_2(a)c(\tau_1,\tau_3,\tau_4) - a\tau_2 c(\tau_1,\tau_3,\tau_4)\}.$$

Post summationem obtinemus statim:

$$d_H d_{DR} c(a, \tau_1, \tau_2, \tau_3, \tau_4) = d_{DR} d_H c(a, \tau_1, \tau_2, \tau_3, \tau_4) - \frac{1}{3}\mathrm{Alt}_{3,4}\tau_2(a)c(\tau_1, \tau_3, \tau_4),$$

ubi

$$-\frac{1}{3}\mathrm{Alt}_{3,4}\tau_2(a)c(\tau_1, \tau_3, \tau_4) = \left\langle \tau_2, -\frac{1}{3}\mathrm{Alt}_{3,4} da\ c(\tau_1, \tau_3, \tau_4)\right\rangle = QMc(a, \tau_1, \tau_2, \tau_3, \tau_4), \qquad \text{qed}$$

(g) *Junctio camerae altera*

1.25.

Revocamus operatores:

$$d_{DR} : \mathrm{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 3}, A),$$

ubi

$$d_{DR} c(a, \tau_1, \tau_2, \tau_3) = \mathrm{Lie}_{\tau_3} c(a, \tau_1, \tau_2) - \frac{1}{2}\tau_2 c(a, \tau_1, \tau_3) =: \{L + d'_{DR}\}c(a, \tau_1, \tau_2, \tau_3)$$

(vide Pars Prima, art. 2.7),

$$d_{DR} : \mathrm{Hom}(A \otimes T^{\otimes 3}, A) \longrightarrow \mathrm{Hom}(A \otimes T^{\otimes 2} \otimes \Lambda^2 T, A),$$

ubi

$$d_{DR} c(a, \tau_1, \tau_2, \tau_3, \tau_4) = \mathrm{Alt}_{3,4}\mathrm{Lie}_{\tau_3} c(a, \tau_1, \tau_2, \tau_4) + c(a, \tau_1, \tau_2, [\tau_3, \tau_4]) - \frac{1}{3}\mathrm{Alt}_{3,4}\tau_2 c(a, \tau_1, \tau_3, \tau_4) =: \{L + d'_{DR}\}c(a, \tau_1, \tau_2, \tau_3),$$

vide art 1.20; tandem,

$$R : \mathrm{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A \otimes T \otimes \Lambda^2 T, \Omega),$$

ubi

$$Rc(a, \tau, \tau', \tau'') = -\frac{1}{6}\mathrm{Alt}_{\tau', \tau''} d\{c(a, [\tau, \tau'], \tau'') - c(\tau'(a), \tau, \tau'')\} - \frac{1}{6}c(a, \tau, [\tau', \tau'']),$$

vide art. 1.6.

1.26. Lemma. *Fit* $d_{DR}^2 = QR$.

1.27. Demonstratio. Habemus $d_{DR}^2 = \{L + d'_{DR}\}^2$.
Primo, ostendetur, posito

$$\mathrm{Lie}\, c(a, \tau_1, \tau_2, \tau_3) := \mathrm{Lie}_{\tau_3} c(a, \tau_1, \tau_2)$$

et

$$\mathrm{Lie}\, c(a, \tau_1, \tau_2, \tau_3, \tau_4) := \mathrm{Lie}_{\tau_4} c(a, \tau_1, \tau_2, \tau_3),$$

habebimus

$$\mathrm{Alt}_{3,4} \mathrm{Lie}^2 c(a, \tau_1, \tau_2) = -\mathrm{Lie}_{[\tau_3, \tau_4]} c(a, \tau_1, \tau_2),$$

Hinc subito fluit $L^2 = 0$.

1.28.

Secundo, videamus post rationem:

$$\begin{aligned}
&\{L d'_{DR} + d'_{DR} L + d'^2_{DR}\} c(a, \tau_1, \tau_2, \tau_3, \tau_4) \\
&\quad = -\frac{1}{6} \mathrm{Alt}_{3,4} \tau_2 \left\{ c(a, [\tau_1, \tau_3], \tau_4) - c(\tau_3(a), \tau_1, \tau_4) + \frac{1}{2} c(a, \tau_1, [\tau_3, \tau_4]) \right\} \\
&\quad = \left\langle \tau_2, -\frac{1}{6} \mathrm{Alt}_{3,4} d \left\{ c(a, [\tau_1, \tau_3], \tau_4) - c(\tau_3(a), \tau_1, \tau_4) + \frac{1}{2} c(a, \tau_1, [\tau_3, \tau_4]) \right\} \right\rangle \\
&\quad = QRc(a, \tau_1, \tau_2, \tau_3, \tau_4),
\end{aligned}$$

qed

2. Tabulatum primum

(a) *Paries recessus*

2.1.

Revocamus operatorem:

$$d_{DR} : \mathrm{Hom}(A \otimes T^{\otimes 2}, \Omega) \longrightarrow \mathrm{Hom}(A \otimes T \otimes \Lambda^2 T, \Omega),$$

ubi

$$\begin{aligned}
d_{DR} c(a, \tau, \tau', \tau'') = {}& \mathrm{Alt}_{\tau', \tau''} \mathrm{Lie}_{\tau'} c(a, \tau, \tau'') + c(a, \tau, [\tau', \tau'']) \\
& - \frac{1}{3} \mathrm{Alt}_{\tau', \tau''} d \langle \tau', c(a, \tau, \tau'') \rangle,
\end{aligned}$$

sive

$$d_{DR} c(a, \tau, \tau', \tau'') = \{L + d'_{DR}\} c(a, \tau, \tau', \tau'')$$

ubi
$$Lc(a,\tau,\tau',\tau'') = \mathrm{Alt}_{\tau',\tau''}\mathrm{Lie}_{\tau'}c(a,\tau,\tau'') + c(a,\tau,[\tau',\tau'']),$$
vide art. 1.20, et definimus operatores:
$$d_{DR}:\ \mathrm{Hom}(A^{\otimes 2}\otimes T^{\otimes 2},\Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 2}\otimes T\otimes \Lambda^2 T,\Omega),$$
ubi
$$d_{DR}c(a,b,\tau,\tau',\tau'') = \mathrm{Alt}_{\tau',\tau''}\mathrm{Lie}_{\tau'}c(a,b,\tau,\tau'') + c(a,b,\tau,[\tau',\tau'']) - \frac{1}{3}\mathrm{Alt}_{\tau',\tau''}d\langle\tau',c(a,b,\tau,\tau'')\rangle,$$
sive
$$d_{DR}c(a,b,\tau,\tau',\tau'') = \{L + d'_{DR}\}c(a,b,\tau,\tau',\tau'')$$
ubi
$$Lc(a,b,\tau,\tau',\tau'') = \mathrm{Alt}_{\tau',\tau''}\mathrm{Lie}_{\tau'}c(a,b,\tau,\tau'') + c(a,b,\tau,[\tau',\tau'']),$$
porro:
$$M:\ \mathrm{Hom}(A\otimes T^{\otimes 3},A) \longrightarrow \mathrm{Hom}(A^{\otimes 2}\otimes T\otimes \Lambda^2 T,\Omega),$$
ubi
$$Mc(a,b,\tau,\tau',\tau'') = \frac{1}{3}da\ \mathrm{Alt}_{\tau',\tau''}c(b,\tau,\tau',\tau''),$$
confer Pars Prima, art. 3.4.

2.2. Lemma. *Fit* $d_H d_{DR} = d_{DR} d_H + MQ$.

2.3. Demonstratio. Primo, posito
$$\mathrm{Lie}\ c(a,\tau,\tau',\tau'') = \mathrm{Lie}_{\tau'}c(a,\tau,\tau'')$$
et simili modo
$$\mathrm{Lie}\ c(a,b,\tau,\tau',\tau'') = \mathrm{Lie}_{\tau'}c(a,b,\tau,\tau'')$$
probatur, d_H Lie = Lie d_H, unde subito sequitur $Ld_H = d_H L$.

Secundo, fit
$$d_H d'_{DR}c(a,b,\tau,\tau',\tau'') = d'_{DR}d_H c(a,b,\tau,\tau',\tau'') + \frac{1}{3}da\ \mathrm{Alt}_{\tau',\tau''}\langle\tau',c(b,\tau,\tau'')\rangle,$$
ubi patet
$$\frac{1}{3}da\ \mathrm{Alt}_{\tau',\tau''}\langle\tau',c(b,\tau,\tau'')\rangle = MQc(a,b,\tau,\tau',\tau''),$$
unde manifesto lemma nostrum fluit.

(b) *Junctio camerae*

2.4.

Revocamus operatorem

$$d_{DR} : \ \mathrm{Hom}(A^{\otimes 2} \otimes T, \Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, \Omega)$$

ubi

$$d_{DR}c(a, b, \tau, \tau') = \tau' c(a, b, \tau) - c(\tau'(a), b, \tau) - c(a, \tau'(b), \tau) + c(a, b, [\tau, \tau'])$$
$$- \frac{1}{2} d\langle \tau', c(a, b, \tau)\rangle =:= \{L + d'_{DR}\}c(a, b, \tau, \tau'),$$

vide Pars Prima, art. 3.1, definimusque operatorem

$$R : \ \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T \otimes \Lambda^2 T, \Omega),$$

ubi

$$Rc(a, b, \tau, \tau', \tau'') = -\frac{1}{6} d\big[\mathrm{Alt}_{\tau', \tau''}\{-c(\tau'(a), b, \tau, \tau'') - c(a, \tau'(b), \tau, \tau'')$$
$$+ c(a, b, [\tau, \tau'], \tau'')\} + c(a, b, \tau, [\tau', \tau''])\big],$$

confer 1.6.

2.5. Lemma. *Fit* $d_{DR}^2 = RQ$.

2.6. Demonstratio. Primo, probatur, $L^2 = 0$.
Secundo, computatur:

$$\{Ld'_{DR} + d'_{DR}L + d'^2_{DR}\}c(a, b, \tau, \tau', \tau'')$$
$$= -\frac{1}{6} d\big[\langle [\tau', \tau''], c(a, b, \tau)\rangle + \mathrm{Alt}_{\tau', \tau''}\langle \tau'', c(a, b, [\tau, \tau'])$$
$$- c(\tau'(a), b, \tau) - c(a, \tau'(b), \tau)\rangle\big]$$
$$= RQc(a, b, \tau, \tau', \tau''),$$

unde lemma nostrum subito fluit.

(c) *Junctio duarum cellarum tabulati primi*

2.7.

Revocamus operatorem

$$R : \ \mathrm{Hom}(A \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A \otimes T \otimes \Lambda^2 T, \Omega)$$

ubi

$$Rc(a,\tau,\tau',\tau'') = -\frac{1}{6}d\Big[\mathrm{Alt}_{\tau',\tau''}\{c(a,[\tau,\tau'],\tau'') - c(\tau'(a),\tau,\tau'')\} + c(a,\tau,[\tau',\tau''])\Big],$$

vide art. 1.6 vel art. 1.13.

2.8. Lemma. *Fit* $d_H R = Rd_H + d_{DR}M + Md_{DR}$.

Confer art. 1.15.

2.9. Demonstratio. Fit

$$d_H Rc(a,b,\tau,\tau',\tau'') = aRc(a,b,\tau,\tau',\tau'') - Rc(ab,\tau,\tau',\tau'') + Rc(a,b\tau,\tau',\tau'').$$

Primo, computatur methodo simili ut in art. 1.16:

$$\{d_H R - Rd_H\}c(a,b,\tau,\tau',\tau'') = \frac{1}{6}da\; d[\mathrm{Alt}_{\tau',\tau''}\{c(b,[\tau,\tau'],\tau'') - c(\tau'(b),\tau,\tau'')\} - c(b,\tau,[\tau',\tau''])] - \frac{1}{6}\mathrm{Alt}_{\tau',\tau''}d\{\tau'(a)c(b,\tau,\tau'')\}$$

Secundo, ostendetur $\{d_{DR}M + Md_{DR}\}c(a,b,\tau,\tau',\tau'')$ eamdem responsionem praebere, unde lemma fluit.

(d) *Paries rectus*

2.10.

Introducamus operatorem:

$$Q:\ \mathrm{Hom}(A^{\otimes 2}\otimes T\otimes \Lambda^2 T,\Omega) \longrightarrow \mathrm{Hom}(A^{\otimes 2}\otimes T^{\otimes 2}\otimes \Lambda^2 T, A),$$

ubi

$$Qc(a,b,\tau_1,\tau_2,\tau_3,\tau_4) = \langle \tau_2, c(a,b,\tau_1,\tau_3,\tau_4)\rangle.$$

2.11. Lemma. *Fit* $d_H Q = Qd_H$.

(e) *Camera*

2.12.

Introducamus sagittulam:

$$d_{DR}:\ \mathrm{Hom}(A^{\otimes 2}\otimes T^{\otimes 3}, A) \longrightarrow \mathrm{Hom}(A^{\otimes 2}\otimes T^{\otimes 2}\otimes \Lambda^2 T, A),$$

ubi

$$d_{DR}c(a,b,\tau_1,\tau_2,\tau_3,\tau_4) = \mathrm{Alt}_{3,4}\Big\{\mathrm{Lie}_{\tau_3}c(a,b,\tau_1,\tau_2,\tau_4) - \frac{1}{3}\tau_2 c(a,b,\tau_1,\tau_3,\tau_4)\Big\} + c(a,b,\tau_1,\tau_2,[\tau_3,\tau_4]),$$

confer art. 1.20.

2.13. Lemma. *Fit* $Qd_{DR} = d_{DR}Q$.

Demonstratio. procedit ut in art. 1.22.

(f) *Frons*

2.14. Lemma. *Fit* $d_H d_{DR} = d_{DR} d_H + QM$.

Confer art. 1.23.

Demonstratio. Probatur eodem modo ut in art. 1.24:

$$\begin{aligned}\{d_H d_{DR} - d_{DR} d_H\}c(a,b,\tau_1,\tau_2,\tau_3,\tau_4) &= \frac{1}{3}\mathrm{Alt}_{3,4}\tau_2(a)c(b,\tau_1,\tau_3,\tau_4)\\ &= \Big\langle \tau_2, \frac{1}{3}da\ \mathrm{Alt}_{3,4}c(b,\tau_1,\tau_3,\tau_4)\Big\rangle\\ &= \langle \tau_2, Mc(a,b,\tau_1,\tau_3,\tau_4)\rangle\\ &= QMc(a,b,\tau_1,\tau_2,\tau_3,\tau_4),\end{aligned}$$

qed

(g) *Junctio camerae altera*

2.15.

Contemplemur compositio sagittulae

$$d_{DR}: \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 2}, A) \longrightarrow \mathrm{Hom}(A^{\otimes 2} \otimes T^{\otimes 3}, A),$$

ubi

$$d_{DR}c(a,b,\tau_1,\tau_2,\tau_3) = \mathrm{Lie}_{\tau_3}c(a,b,\tau_1,\tau_2) - \frac{1}{2}\tau_2 c(a,b,\tau_1,\tau_3),$$

vide Pars Prima, art. 3.14, cum sagittula d_{DR} ex art. 2.12.

2.16. Lemma. *Fit* $d_{DR}^2 = QR$.

Demonstratio. Eadem ut in art. 1.27.

Pars tertia. Finale

1. Cocyclus canonicus

(a)

1.1.

Contemplemur elementa $\epsilon \in \mathrm{Hom}(A \otimes T^{\otimes 2}, A)$, $\epsilon' \in \mathrm{Hom}(A \otimes T^{\otimes 2}, \Omega)$ atque $\epsilon'' \in \mathrm{Hom}(A^{\otimes 2} \otimes T, \Omega)$ definita per:

$$\epsilon(a, \tau, \tau') = \tau\tau'(a),\ \epsilon'(a, \tau, \tau') = -\frac{1}{2}d\tau\tau'(a)$$

et

$$\epsilon''(a, b, \tau) = -\tau(a)db - \tau(b)da.$$

1.2. Lemma. *Fit* $d_{DR}\epsilon = Q\epsilon'$.

1.3. Demonstratio. Constat:

$$d_{DR}\epsilon(a, \tau_1, \tau_2, \tau_3) = \mathrm{Lie}_{\tau_3}\epsilon(a, \tau_1, \tau_2) - \frac{1}{2}\tau_2\tau_1\tau_3(a),$$

vide Pars Prima, art. 2.11. Terminus primus evadit, quod ϵ operator invariens sit. Hinc

$$d_{DR}\epsilon(a, \tau_1, \tau_2, \tau_3) = -\frac{1}{2}\tau_2\tau_1\tau_3(a) = \langle\tau_2, \epsilon'(a, \tau_1, \tau_3)\rangle = Q\epsilon'(a, \tau_1, \tau_2, \tau_3),$$

1.4. Lemma. *Fit* $d_H\epsilon = Q\epsilon''$.

1.5. Demonstratio. Habemus:

$$\begin{aligned} d_H\epsilon(a, b, \tau, \tau') &= a\tau\tau'(b) - \tau\tau'(ab) + b\tau\tau'(a) \\ &= -\tau(a)\tau'(b) - \tau'(a)\tau(b) = \langle\tau', \epsilon''(a, b, \tau)\rangle = Q\epsilon''(a, b, \tau, \tau'), \end{aligned}$$

qed

1.6. Lemma. *Fit* $d_{DR}\epsilon'' = d_H\epsilon' - M\epsilon$.

1.7. Demonstratio. Habemus (vide Pars Prima, art. 2.7):

$$d_{DR}\epsilon''(a, b, \tau, \tau') = \mathrm{Lie}_{\tau'}\epsilon''(a, b, \tau) - \frac{1}{2}d\langle\tau', \epsilon''(a, b, \tau)\rangle$$

(cum ϵ'' invariens est)

$$= \frac{1}{2}d\{\tau(a)\tau'(b) + \tau'(a)\tau(b)\}$$

Rursus,

$$d_H\epsilon'(a,b,\tau,\tau') = -\frac{1}{2}\big[ad\tau\tau'(b) - d\tau\tau'(ab) + d\{b\tau\tau'(a)\}\big]$$
$$= \frac{1}{2}[d\{\tau(a)\tau'(b) + \tau'(a)\tau(b)\} + da\ \tau\tau'(b)]$$
$$= d_{DR}\epsilon''(a,b,\tau,\tau') + M\epsilon,$$

unde sequitur lemma.

1.8. Lemma. *Fit* $d_H\epsilon'' = 0$.

1.9. Demonstratio. Statuamus:

$$\epsilon''_0(a,b,\tau) := -\tau(a)db;\ \epsilon''_1(a,b,\tau) := -\tau(b)da,$$

ergo $\epsilon'' = \epsilon''_0 + \epsilon''_1$. Adipiscimur:

$$d_H\epsilon''_0(a,b,c,\tau) = a\epsilon''_0(b,c,\tau) - \epsilon''_0(ab,c,\tau) + \epsilon''_0(a,bc,\tau) - \epsilon''_0(a,b,c\tau)$$
$$= -a\tau(b)dc + \tau(ab)dc - \tau(a)d(bc) + c\tau(a)db = 0$$

Simili modo probatur, $d_H\epsilon''_1 = 0$, unde lemma fluit.

1.10. Lemma. $d_{DR}\epsilon' = R\epsilon$.

1.11. Demonstratio. Primo observamus, quod definitio sagittulae d_{DR} ex Parte Secunda, art. 1.2, ita exhiberi potest:

$$d_{DR}c(a,\tau,\tau',\tau'') = \mathrm{Alt}_{\tau',\tau''}\mathrm{Lie}_{\tau'}c(a,\tau,\tau'') + c(a,\tau,[\tau',\tau''])$$
$$-\frac{1}{3}\mathrm{Alt}_{\tau',\tau''}d\langle\tau', c(a,\tau,\tau'')\rangle,$$

unde, quia $\mathrm{Lie}_\tau\epsilon' = 0$, sequitur:

$$d_{DR}\epsilon'(a,\tau,\tau',\tau'') = \epsilon'(a,\tau,[\tau',\tau'']) - \frac{1}{3}\mathrm{Alt}_{\tau',\tau''}d\langle\tau',\epsilon'(a,\tau,\tau'')\rangle$$
$$= -\frac{1}{2}d\tau[\tau',\tau''](a) + \frac{1}{6}\mathrm{Alt}_{\tau',\tau''}d\tau'\tau\tau''(a)$$
$$= \mathrm{Alt}_{\tau',\tau''}\left\{\frac{1}{2}d\tau\tau''\tau'(a) + \frac{1}{6}d\tau'\tau\tau''(a)\right\}.$$

1.12.

Secundo, habemus

$$\begin{aligned} R\epsilon(a,\tau,\tau',\tau'') &= -\frac{1}{6}d[\mathrm{Alt}_{\tau',\tau''}\{\epsilon(\tau''(a),\tau,\tau')+\epsilon(a,[\tau,\tau'],\tau'')\}+\epsilon(a,\tau,[\tau'\tau''])] \\ &= -\frac{1}{6}d\mathrm{Alt}_{\tau',\tau''}\{\tau\tau'\tau''(a)+[\tau,\tau']\tau''(a)+\tau\tau'\tau''(a)\} \\ &= -\frac{1}{6}d\mathrm{Alt}_{\tau',\tau''}\{3\tau\tau'\tau''(a)-\tau'\tau\tau''(a)\} = d_{DR}\epsilon'(a,\tau,\tau',\tau''), \end{aligned}$$

qed

2. Definitio altera

(a)

2.1.

Primo, axioma (A1) structurae verticianae ita exhiberi potest:

$$d_H\gamma = \epsilon''. \tag{A1}$$

2.2.

Secundo, axioma (A2) ita scriberi licet:

$$d_H\langle,\rangle - Q\gamma = -\epsilon. \tag{A2}$$

2.3.

Tertio, axioma (A3)$^{\mathrm{bis}}$ ita exhiberi potest:

$$d_Hc - d_{DR}\gamma - M\langle,\rangle = -\epsilon'. \tag{A3$^{\mathrm{bis}}$}$$

2.4.

Quatro, axioma (A4) ita quoque exhiberi licet:

$$Qc = d_{DR}\langle,\rangle, \tag{A4}$$

confer Pars Prima, 1.5.

2.5.

Tandem axioma (A5) ita exhiberi licet:

$$d_{DR}c = R\langle , \rangle, \tag{$A5$}$$

confer Pars Prima, art. 1.13.

(b)

2.6.

Applicatio structurarum verticianarum

$$h: \mathcal{A} = (\gamma, \langle , \rangle, c) \longrightarrow \mathcal{A}' = (\gamma', \langle , \rangle', c')$$

elementum $h \in \mathrm{Hom}(T, \Omega)$ est, talis ut:

$$d_H h = \gamma - \gamma'; \;\; Qh = \langle , \rangle - \langle , \rangle'$$

atque

$$d_{DR}h = c - c'.$$

3. Complexus de Rham–Koszul–Hochschildianus

3.1.

Introducamus moduli: W^{ijk}, $i \geq 2$; $j = 0, 1$; $k \geq 0$, posito: $W^{200} := \mathrm{Hom}(T, \Omega)$, tractandoque indices: i tamquam gradum DE RHAMIANUM, j tamquam gradum KOSZULIANUM ac k tamquam gradum HOCHSCHILDIANUM, ergo:

$$W^{300} = \mathrm{Hom}(\Lambda^2 T, \Omega), \;\; W^{210} = \mathrm{Hom}(S^2 T, A), \;\; W^{201} = \mathrm{Hom}(A \otimes T, \Omega),$$

etc.

Statuimus $W^n := \oplus_{i+j+k=n} W^{ijk}$, ergo:

$$W^2 = W^{200};$$
$$W^3 = W^{300} \oplus W^{210} \oplus W^{201};$$
$$W^4 = W^{400} \oplus W^{310} \oplus W^{301} \oplus W^{211} \oplus W^{202}$$

et

$$W^5 = W^{500} \oplus W^{410} \oplus W^{401} \oplus W^{311} \oplus W^{302} \oplus W^{212} \oplus W^{203}.$$

3.2.

Introducucamus operatores $D = D_{DRQH} + \mathcal{R} + \mathcal{M} : W^i \longrightarrow W^{i+1}$, ubi

$$D_{DRQH} c^{ijk} = \{d_{DR} + (-1)^i Q + (-1)^{i+j} d_H\} c^{ijk},$$
$$\mathcal{R}c^3 = -Rc^{210};\ \mathcal{R}c^4 = Rc^{310} - Rc^{211}$$

atque

$$\mathcal{M}c^3 = Mc^{210};\ \mathcal{M}c^4 = Mc^{310} + Mc^{211}$$

Partium Primae Secundaeque summa significat, $D^2 = 0$, unde eruimus complexum

$$W^{[2,5]} :\ W^2 \xrightarrow{D} W^3 \xrightarrow{D} W^4 \xrightarrow{D} W^5,$$

de inclusione canonica complexuum: $\mathbf{\Omega}^{[2,5]} \longrightarrow W^{[2,5]}$ ornatum.

3.3.

Contemplemur elementum:

$$\mathcal{E} := (\epsilon', \epsilon, \epsilon'') \in W^{301} \oplus W^{211} \oplus W^{202} \subset W^4$$

Sectionis 1 summa significat, $D\mathcal{E} = 0$.

3.4.

Structura verticiana super T est elementum

$$\mathcal{A} = (c, \langle , \rangle, \gamma) \in W^{300} \oplus W^{210} \oplus W^{201} = W^3,$$

talis ut fit $D\mathcal{A} = \mathcal{E}$.

3.5.

Applicatio structurarum verticianarum $h : \mathcal{A} \longrightarrow \mathcal{A}'$ elementum $h \in W^{200} = W^2$ est, talis ut fit $Dh = \mathcal{A} - \mathcal{A}'$.