VANISHING CYCLES AND DOLD - KAN CORRESPONDENCE

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Abstract

We discuss analogies between normalized chains and vanishing cycles

These notes are a complement to previous joint works with Mikhail Kapranov and Michael Finkelberg.

Introduction

Solomon Lefschetz (1884 - 1972) was the author of several fundamental concepts in topology and algebraic geometry.

One of them is a notion of vanishing cycles (*cycles évanouissants*) which appeared in [L] (based on the previous work by Émile Picard, cf. [PS]) in what is called now the Picard - Lefschetz formula, see Figures 4 and 5 below.

§1. Hyperbolic sheaves and vanishing cycles

1.1. Moebius inversion is a rule of inverting a triangular matrix with 1's as their nonzero elements. Symbolically:

$$M = \sum L,$$
$$L = \sum \pm M$$

("inclusion - exclusion formula").

Example. Let

$$\gamma : M(0) \longrightarrow M(1)$$

be an epimorphism of vector spaces.

Define $L(1) = M(1)$ and $L(0) = \operatorname{Ker}\gamma$. So we have a resolution of $L(0)$

$$0 \longrightarrow L(0) \longrightarrow M(0) \xrightarrow{\gamma} M(1) \longrightarrow 0.$$

Once we choose a left inverse to γ, i.e. $\delta : M(1) \longrightarrow M(0)$ such that $\gamma\delta = 1_{M(1)}$, we get an isomorphism $M(0) \cong L(0) \oplus L(1)$.

Such objects appear in linear algebra descriptions of perverse sheaves and of their Fourier transforms.

1.2. Hyperbolic sheaves. Let $\mathcal{H} = \{H_i,\ i \in I\}$ be a finite collection of real hyperplanes in $V = \mathbb{R}^n$.

For each $J \subset I$ denote

$$H_J := \bigcap_{i \in J} H_i,\ \ H_J^o := H_J \setminus \bigcup_{H_{J'} \subset H_J,\ H_{J'} \neq H_J} H_{J'}$$

Let us call *a face* (or a cell) a connected component of H_J^o; the set of faces $\mathcal{C}$ is a poset: we write $A \leq B$ if A is contained in the closure of B, $A \subset \overline{B}$.

We have

$$V = \cup_{A \in \mathcal{C}} A$$

Example. $V = \mathbb{R}^2$, $\mathcal{H} = \{\ell_i,\ 1 \leq i \leq 3\}$; $\mathcal{C}$ has 13 cells.

Let $\mathcal{A}$ be a category. A *bisheaf* on $\mathcal{C}$ with values in $\mathcal{A}$ is a collection of objects $\{E(A) \in \mathcal{A},\ A \in \mathcal{C}\}$ and morphisms

$$\gamma_{AB} :\ E(A) \longrightarrow E(B),\ \delta_{BA} :\ E(B) \longrightarrow E(A)\ A \leq B$$

such that $\{\gamma_{AB}\}$ (resp. $\{\delta_{BA}\}$) is a functor $\gamma: \mathcal{C} \longrightarrow \mathcal{A}$ (resp. $\delta: \mathcal{C}^{opp} \longrightarrow \mathcal{A}$).

A *hyperbolic sheaf* on $\mathcal{C}$ with values in $\mathcal{A}$ is a bisheaf enjoying the following properties:

(Mon) For all $A \leq B$

$$\gamma_{AB}\delta_{BA} = \mathrm{Id}_{E(B)}.$$

This allows to define for all A, B a map

$$\phi_{AB} := \gamma_{CB}\delta_{AC}: \ E(A) \longrightarrow E(B)$$

where C is any cell such that $C \leq A$ and $C \leq B$.

Let us call three cells A, B, C *collinear* if there exist points $x \in A, y \in B, z \in C$ lying on one straight line.

(Tran) If A, B, C are collinear then

$$\phi_{AC} = \phi_{BC}\phi_{AB}.$$

(Inv) Let A, B be two d-dimensional cells belonging to the same d-dimensional linear subspace $L = H_J \subset V$ lying on the opposite sides of a $(d-1)$-dimensional cell C, $C < A, C < B$. Then the map

$$\phi_{AB} = \gamma_{CB}\delta_{AC}$$

is an isomorphism.

We denote by $\mathcal{H}yp(\mathcal{C}; \mathcal{A})$ the category of hyperbolic sheaves.

1.3. Complexification. Inside $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ consider the collection of complex hyperplanes $\{H_{i\mathbb{C}}, \ i \in I\}$. Similarly to the above, it gives rise to a stratification

$$V = \bigcup H^o_{J\mathbb{C}}$$

where

$$H_{J\mathbb{C}} := \bigcap_{i \in J} H_{i\mathbb{C}}, \ H^o_{J\mathbb{C}} := H_{J\mathbb{C}} \setminus \bigcup_{H_{J'\mathbb{C}} \subset H_{J\mathbb{C}}, \ H_{J'\mathbb{C}} \neq H_{J\mathbb{C}}} H_{J'\mathbb{C}}.$$

The strata $H^o_{J\mathbb{C}}$ are complex linear subspaces without some hyperplanes. We denote by $\mathcal{S} = \mathcal{C}_{\mathbb{C}}$ the set of complex strata. We have an obvious map

$$\mathcal{C} \longrightarrow \mathcal{S}.$$

Let $\mathbf{k}$ be a field, $\mathcal{A}(\mathbf{k})$ the category of $\mathbf{k}$-vector spaces, $\mathcal{A}^f(\mathbf{k}) \subset \mathcal{A}(\mathbf{k})$ the full subcategory of finite dimensional spaces.

Let $\mathcal{P}erv(V_{\mathbb{C}}, \mathcal{C}_{\mathbb{C}}; \mathcal{A}(\mathbf{k}))$ be the category of $\mathcal{A}(\mathbf{k})$-valued perverse sheaves over $V_{\mathbb{C}}$ smooth along $\mathcal{S}$.

The main result of [KS16] says that we have an equivalence of categories

$$\mathcal{Q} : \ \mathcal{P}erv(V_{\mathbb{C}}, \mathcal{C}_{\mathbb{C}}; \mathcal{A}(\mathbf{k})) \xrightarrow{\sim} \mathcal{H}yp(\mathcal{C}; \mathcal{A}^f(\mathbf{k})).$$

For $\mathcal{M} \in \mathcal{P}erv(V_{\mathbb{C}}, \mathcal{C}_{\mathbb{C}}; \mathcal{A}(\mathbf{k}))$

$$\mathcal{Q}(\mathcal{M}) = (E(\mathcal{M}, A), \gamma_{AB}, \delta_{BA})$$

where

$$E(\mathcal{M}, A) = R\Gamma(A, i_A^* i_V^! \mathcal{M})), \ i_A : \ A \hookrightarrow V, \ i_V : \ V \hookrightarrow V_{\mathbb{C}}$$

(these finite dimensional spaces are called *hyperbolic stalks* of $\mathcal{M}$).

1.4. Vanishing cycles. Let us suppose that $\cap_{i\in I} H_i = \{0\}$.

Let $f : \ V \longrightarrow \mathbb{R}$ be a linear function such that the hyperplane

$$H_f = \{x \in V |\ f(x) = 0\}$$

is in general position to all subspaces H_J.

Let $f_{\mathbb{C}} : \ V_{\mathbb{C}} \longrightarrow \mathbb{C}$ be the complexification of f.

For any $\mathcal{M} \in \mathcal{P}erv(V_{\mathbb{C}}, \mathcal{C}_{\mathbb{C}}; \mathcal{A}(\mathbf{k}))$ the sheaf of vanishing cycles

$$\Phi_{f_{\mathbb{C}}}(\mathcal{M}) \in \mathcal{P}erv(H_{f\mathbb{C}}; \mathcal{A}(\mathbf{k}))$$

is supported at 0. Let us denote by $\Phi(\mathcal{M})$ its stalk at 0.

The main result of [FKS] describes $\Phi(\mathcal{M})$ in terms of the linear algebra data $\mathcal{Q}(\mathcal{M})$.

We shall describe it for two particular cases.

1.4.1. Example. A disc. $V = \mathbb{R}, \ \mathcal{H} = \{0\}$. There are three cells, 0, A^+ and A^-, see Fig. 1 below.

Let $\mathcal{M} \in \mathcal{P}erv(V_{\mathbb{C}}, 0; \mathcal{A}(\mathbf{k}))$. The hyperbolic sheaf $\mathcal{Q}(\mathcal{M})$ consists of three spaces

$$M_0 = E(0), \ M_+ = E(A^+), \ M_- = E(A^-)$$

and four linear maps

$$\gamma_\pm : \ M_0 \longrightarrow M_\pm, \ \delta_\pm : \ M_\pm \longrightarrow M_0$$

such that $\gamma_\pm \delta_\pm = \mathrm{Id}_{M_\pm}$, and two maps

$$\phi_\pm = \gamma_\mp \delta_\pm : \ M_\pm \longrightarrow M_\mp$$

are isomorphisms.

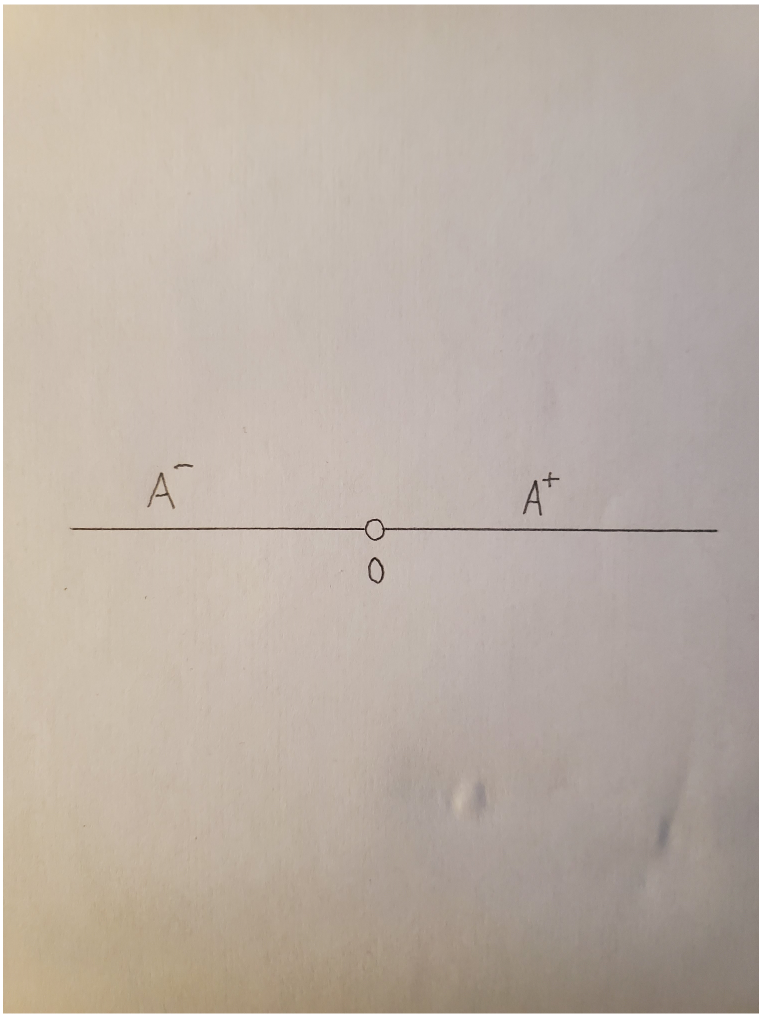


Fig. 1. A line

Let $f = \mathrm{Id}: \ V \longrightarrow \mathbb{R}$. The space

$$L_0 = \Phi_f(\mathcal{M})$$

may be identified with $\mathrm{Ker}(\gamma_+)$. Thus we have a right resolution of L_0

$$0 \longrightarrow L_0 \longrightarrow M_0 \xrightarrow{\gamma_+} M_+ \longrightarrow 0 \tag{1.4.1}$$

(note that γ_+ is surjective since $\gamma_+\delta_+ = \mathrm{Id}_{M_+}$).

A dual way to describe the same space is by introducing $L_0' = \mathrm{Coker}(\delta_+)$, so that we have a left resolution of it

$$0 \longrightarrow M_+ \xrightarrow{\delta_+} M_0 \longrightarrow L_0' \longrightarrow 0. \tag{1.4.2}$$

A map

$$\mathrm{Id} - \delta_+\gamma_+ : M_0 \longrightarrow M_0$$

induces an isomorphism

$$L_0' \xrightarrow{\sim} L_0$$

We can dock two complexes (1.4.1) and (1.4.2) together and get an acyclic *Janus* complex

$$0 \longrightarrow M_+ \longrightarrow M_0 \longrightarrow M_0 \longrightarrow M_+ \longrightarrow 0, \tag{1.4.3}$$

see Fig. 2 below.

Fig. 2. Janus

1.4.2. Define maps

$$u : M_- \longrightarrow L_0$$

as the composition

$$M_- \xrightarrow{\delta_-} M_0 \xrightarrow{p} L_0$$

where $p = \mathrm{Id} - \delta_+\gamma_+$, and

$$v : L_0 \longrightarrow M_-$$

as the composition

$$L_0 \hookrightarrow M_0 \xrightarrow{\gamma_-} M_-.$$

Then

$$vu = \mathrm{Id}_{M_-} - \phi_+\phi_-.$$

The quadruple $(L_0, L_- = M_-, v, u)$ forms the classical description of perverse sheaves over $\mathbb{C}$ with one possible singularity at 0.

We may denote $L_+ := M_+$, and we have

$$M_0 \overset{\sim}{=} L_0 \oplus L_1 \tag{1.4.4}$$

1.5. Example. Three lines on the plane. $V = \mathbb{R}^2, \mathcal{H} = \{\ell_1, \ell_2, \ell_3\}$. There are 13 cells:

0, six 1-dimensional ones $\ell_i^\pm, 1 \leq i \leq 3$, and six 2-dimensional ones $A_{12}^\pm, A_{23}^\pm, A_{31}^\pm$, see Fig. 3 below.

Let $f : V \longrightarrow \mathbb{R}$ be a linear function in general position such that for $x \in \ell_i^+$ we have $f(x) > 0$.

Let $\mathcal{M} \in \mathcal{P}erv(V_\mathbb{C}, \mathcal{S}; \mathcal{A}(\mathbf{k}))$, with

$$\mathcal{Q}(\mathcal{M}) = (M(0), M(\ell_i^\pm), M(A_{ij}^\pm), \gamma_{AB}, \delta_{BA}).$$

According to [FKS] the space $\Phi(\mathcal{M})$ admits a right resolution

$$0 \longrightarrow \Phi(\mathcal{M}) \longrightarrow M(0) \longrightarrow \oplus_{i=1}^3 M(\ell_i^+) \longrightarrow M(A_{12}^+) \oplus M(A_{23}^+) \longrightarrow 0 \tag{1.5.1}$$

where the matrix elements of the differential are $\pm\gamma_{AB}$.

Dually, $\Phi(\mathcal{M})$ admits a left resolution

$$0 \longrightarrow M(A_{12}^+) \oplus M(A_{23}^+) \longrightarrow \oplus_{i=1}^3 M(\ell_i^+) \longrightarrow M(0) \longrightarrow \Phi(\mathcal{M}) \longrightarrow 0 \tag{1.5.2}$$

where the matrix elements of the differential are $\pm\delta_{BA}$.

We can dock these two resolutions together and form an acyclic Janus complex.

Let us denote

$$L(0) := \Phi(\mathcal{M}),\ M(\ell_i^+) := \mathrm{Ker}(\gamma : M(\ell_i^+) \longrightarrow M(A_{i,i+1}^+)),\ L(A_{i,i+1}^+) := M(A_{i,i+1}^+)$$

Then

$$M(\ell_i^+) \cong L(\ell_i^+) \oplus L(A_{i,i+1}^+).$$

Note that in the Grothendieck group $K_0(\mathcal{A}^f(\mathbf{k}))$ all classes $[L(A_{i,i+1}^+)]$ are equal; let us denote them $[L(A)]$.

We have

$$[M(0)] = [L(0)] + \sum_{i=1}^{3}[L(\ell_i)] + [L(A)] \tag{1.5.3}$$

The complex (1.5.1) (or (1.5.2)) and (1.5.3) is an example of a Moebius inversion.

The summands here are in bijection with the *complex* strata ("Takeuchi formula"). This is a general phenomenon, cf. [T], [KS16], 4.C.1, [KS19], 1.3.

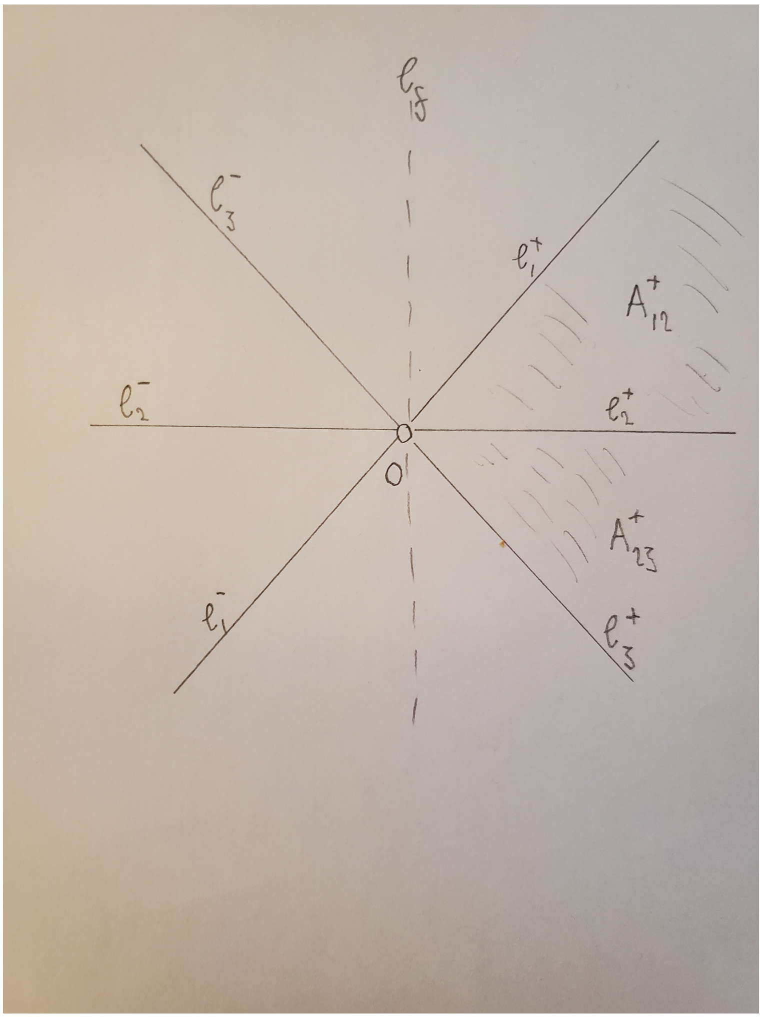


Fig. 3. Three lines on a plane

1.6. General case is similar. Let $\mathcal{M} \in \mathcal{P}erv(V, \mathcal{S}; \mathcal{A}(\mathbf{k}))$ with

$$\mathcal{Q}(\mathcal{M}) = (M(A), \gamma_{AB}, \delta_{BA});$$

let

$$f : V \longrightarrow \mathbb{R}$$

a linear function in general position to $\mathcal{H}$.

Let $\mathcal{C}^+ \subset \mathcal{C}$ be the subset of cells contained in the half-space $V^+ = \{x \in V |\ f(x) \geq 0\}$, and for each i let $\mathcal{C}_i^+ \subset \mathcal{C}^+$ be the subset of cells of dimension i.

The space

$$\Phi(\mathcal{M}) = \Phi_{f\mathbb{C}}(\mathcal{M})_0$$

admits a right resolution

$$0 \longrightarrow \Phi(\mathcal{M}) \longrightarrow M(0) \longrightarrow \oplus_{A \in \mathcal{C}_1^+} M(A) \longrightarrow \oplus_{A \in \mathcal{C}_2^+} M(A) \longrightarrow \ldots \oplus_{A \in \mathcal{C}_n^+} M(A) \longrightarrow 0 \tag{1.6.1}$$

where $n = \dim V$. The matrix elements of the differential are $\pm\gamma$.

Dually the same space admits a left resolution

$$0 \longrightarrow \oplus_{A \in \mathcal{C}_n^+} M(A) \longrightarrow \ldots \longrightarrow \oplus_{A \in \mathcal{C}_1^+} M(A) \longrightarrow M(0) \longrightarrow \Phi(\mathcal{M}) \longrightarrow 0, \tag{1.6.2}$$

the matrix elements of the differential being $\pm\delta$. Both complexes may be glued together into an acyclic Janus complex.

§2. Deriving normalized chains

2.1. Normalized chains. We will follow the notations of [DK].

Let $\mathbf{\Delta}^o$ be the category whose objects are (n), $n \in \mathbb{Z}_{\geq 0}$, and maps

$$d_i : (n) \longrightarrow (n-1),\ 0 \leq i \leq n,$$

$$s_i : (n) \longrightarrow (n+1),\ 0 \leq i \leq n,$$

subjects to the usual relations

$$d_i d_j = d_{j-1} d_i,\ i < j$$

$$s_i s_j = s_{j+1} s_i,\ i \leq j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ 1 & \text{if } i = j, j+1, \\ s_j d_{i-1} & i > j+1 \end{cases}$$

Let $\mathcal{A}$ be an abelian category, and $\mathbf{\Delta}^o\mathcal{A}$ be the category of simplicial objects of $\mathcal{A}$, i.e. of functors $A : \mathbf{\Delta}^o \longrightarrow \mathcal{A}$.

Normalized chains

Let $M = (M_0, M_1, \ldots) \in \mathbf{\Delta}^o\mathcal{A}$. There are two dual ways to define the normalized chains.

(a) As subobjects. We define

$$L_n = \bigcap_{i=1}^{n} \mathrm{Ker}(d_i : M_n \longrightarrow M_{n-1}) \subset M_n \tag{2.1.1}$$

(b) As quotient objects. We define

$$L'_n = M_n / \sum_{i=0}^{n} s_i(M_{n-1}) \tag{2.1.2}$$

Both ways give the same answer: the composition

$$L_n \hookrightarrow M_n \longrightarrow L'_n$$

is an isomorphism.

The above definitions suggest the idea that maybe (2.1.1) (resp. (2.1.2)) is the beginning of a right (resp. left) resolution of L_n by objects $M_i, i \leq n$.

2.1.1. Example. For $n = 0$, $L_0 = M_0$. For $n = 1$ we have an exact sequence

$$0 \longrightarrow L_1 \longrightarrow M_1 \xrightarrow{d_1} M_0 \longrightarrow 0$$

d_1 is surjective since $d_1 s_0 = \mathrm{Id}_{M_0}$.

Dually, we have an exact sequence

$$0 \longrightarrow M_0 \xrightarrow{s_0} M_1 \longrightarrow L'_1 \longrightarrow 0$$

2.2. Dold - Kan correspondence. Cf. [DK], 3.1.

Let $M = (M_0, M_1, \ldots) \in \mathbf{\Delta}^o\mathcal{A}$. For each $n \geq 0$ denote by

$$B_n = \sum_{i=0}^{n-1} s_i(M_{n-1}) \subset M_n$$

the subobject of degenerate simplices.

For each sequence $0 \leq p_1 < \ldots < p_i \leq n-1$ the composition

$$s_{p_i} \ldots s_{p_1} : \ L_{n-i} \longrightarrow M_n$$

is a monomorphism; denote its image

$$L_{n-i}^{p_1 \ldots p_i} \subset B_n$$

Dold - Kan affirms that we have an isomorphism

$$M_n \overset{\sim}{=} L_n \oplus (\oplus_{i=1}^n \oplus_{0 \leq p_1 < \ldots < p_i \leq n-1} L_{n-i}^{p_1 \ldots p_i}) \qquad (2.2.1)$$

So in this sum for each $0 \leq i \leq n$ we have $\binom{n}{i}$ copies of L_i:

$$M_n \overset{\sim}{=} \oplus_{i=1}^n L_i^{\binom{n}{i}}$$

2.2.1. Example. $n = 2$

$$M_2 = L_2 \oplus s_0 L_1 \oplus s_1 L_1 \oplus s_1 s_0 L_0$$

2.3. Moebius inversion: two cubes. It follows that each L_n admits two resolutions by the modules M_i:

(a) a right one:

$$0 \longrightarrow L_n \longrightarrow M_n \longrightarrow M_{n-1}^n \longrightarrow M_{n-2}^{\binom{n}{2}} \longrightarrow \ldots \longrightarrow M_0 \longrightarrow 0$$

whose differential should have various $\pm d_i$, $1 \leq i \leq n$ as matrix elements.

In other words we can put the objects M_i, $0 \leq i \leq n$ into the vertices of an n-dimensional cube.

Denote

$$L_n^\bullet : 0 \longrightarrow M_n \longrightarrow M_{n-1}^n \longrightarrow M_{n-2}^{\binom{n}{2}} \longrightarrow \ldots \longrightarrow M_0 \longrightarrow 0$$

which we regard as a complex concentrated in degrees $[0, n]$. So we have a quasiisomirphism

$$L_n \xrightarrow{\sim} L_n^\bullet$$

(b) a left one:

$$0 \longrightarrow M_0 \longrightarrow M_1^n \longrightarrow \ldots \longrightarrow M_{n-1}^n \longrightarrow M_n \longrightarrow L'_n \longrightarrow 0$$

whose differential should have various $\pm s_i$ $0 \leq i \leq n-1$ as matrix elements.

In other words we can put the objects M_i, $0 \leq i \leq n$ into the vertices of an n-dimensional cube.

We denote

$$L'_{n\bullet} : 0 \longrightarrow M_0 \longrightarrow M_1^n \longrightarrow \ldots \longrightarrow M_{n-1}^n \longrightarrow M_n \longrightarrow 0$$

which we regard as a complex concentrated in degrees $[-n, 0]$.

Thus we have a quasiisomirphism

$$L'_{n\bullet} \xrightarrow{\sim} L'_n$$

(c) Two-sided acyclic Janus complexes.

The composition

$$\psi_n : \ L_n \hookrightarrow M_n \longrightarrow L'_n$$

is an isomorphism. We use its inverse to define a gluing map

$$g : M_n \longrightarrow L'_n \overset{\psi_n}{\xrightarrow{\sim}} L_n \longrightarrow M_n$$

We use g to glue the complexes $L'_{n\bullet}$ (b) and $L^\bullet_n$ (a) to get an acyclic complex:

$$0 \longrightarrow M_0 \longrightarrow M_1^n \longrightarrow \ldots \longrightarrow M_{n-1}^n \longrightarrow M_n \overset{g}{\longrightarrow}$$

$$\longrightarrow M_n \longrightarrow M_{n-1}^n \longrightarrow M_{n-2}^{\binom{n}{2}} \longrightarrow \ldots \longrightarrow M_0 \longrightarrow 0$$

in the left (resp. right) part the differentials are various s_i (resp. d_i).

2.4. Example. $n = 2$

$$0 \longrightarrow L_2 \longrightarrow M_2 \overset{\binom{d_2}{d_1}}{\longrightarrow} \begin{matrix} M_1 \\ \oplus \\ M_1 \end{matrix} \overset{(d_1\ -d_1)}{\longrightarrow} M_0 \longrightarrow 0$$

Exatness at M_2 and M_0 is clear.

Let us prove the exactness at M_1^2. If we have

$$\begin{pmatrix} x \\ y \end{pmatrix} \in M_1^2$$

such that $d_1 x - d_1 y = 0$ then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_2 \\ d_1 \end{pmatrix} ((s_1 - s_0)x + s_0 y).$$

2.5. Derived normalized complex.

The normalized chains form a complex

$$\ldots \longrightarrow L_2 \overset{d_0}{\longrightarrow} L_1 \overset{d_0}{\longrightarrow} L_0 \longrightarrow 0$$

where the differential is induced by d_0.

Let us replace L_i by their resolutions $L_i^\bullet$. As was remarked by M.Kapranov, it is natural to expect that these complexes form a *twisted complex.*

This means that we can lift the maps $d_0 : L_i \longrightarrow L_{i-1}$ to morphisms of complexes

$$d_0^\bullet : L_i^\bullet \longrightarrow L_{i-1}^\bullet$$

but the composition $d_0^\bullet \circ d_0^\bullet$ will not be 0. However, one can write down a homotopy h between $(d_0^\bullet)^2$ and 0, etc.

2.6. Example. $n = 2$.

$$\begin{array}{ccccc}
M_2 & & & & \\
\uparrow & & & & \\
M_1 \oplus M_1 & \xrightarrow{(0,d_2)} & M_0 & & \\
(d_1, d_2) \uparrow & & \uparrow d_1 & & \\
M_2 & \xrightarrow{d_0} & M_1 \xrightarrow{d_0} & M_0 \longrightarrow 0 &
\end{array}$$

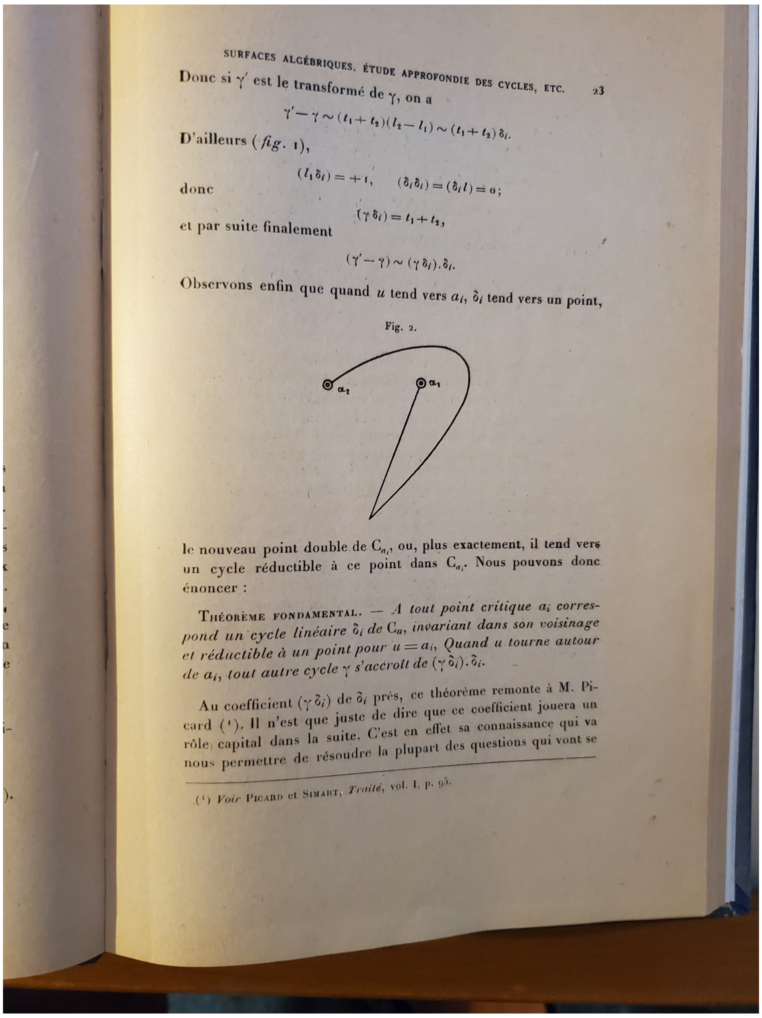
A component of the homotopy:

$$h = (d_0, 0) : \ M_1 \oplus M_1 \longrightarrow M_0.$$

This might be related to [D].

2.7. Complements. Moebius dual Kostka numbers appear in [FPS].

Some infinite Janus complexes related to chiral algebras are discussed in [MS].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Donc si γ' est le transformé de γ, on a

$$\gamma' - \gamma \sim (t_1 + t_2)(l_2 - l_1) \sim (t_1 + t_2)\,\delta_i.$$

D'ailleurs (*fig.* 1),

$$(l_1 \delta_i) = +1, \qquad (\delta_i \delta_i) = (\delta_i l) = 0;$$

donc

$$(\gamma \delta_i) = t_1 + t_2,$$

et par suite finalement

$$(\gamma' - \gamma) \sim (\gamma \delta_i).\delta_i.$$

Observons enfin que quand u tend vers a_i, δ_i tend vers un point,

Fig. 2.

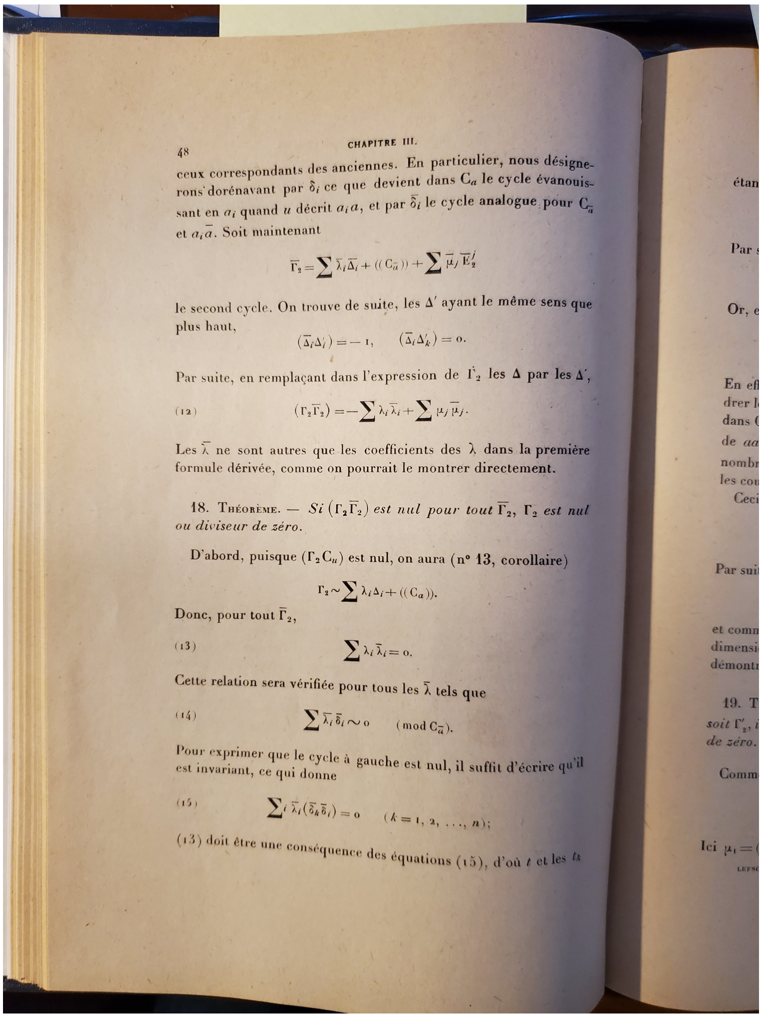
le nouveau point double de C_{a_i}, ou, plus exactement, il tend vers un cycle réductible à ce point dans C_{a_i}. Nous pouvons donc énoncer :

Théorème fondamental. — *A tout point critique a_i correspond un cycle linéaire δ_i de C_u, invariant dans son voisinage et réductible à un point pour $u = a_i$. Quand u tourne autour de a_i, tout autre cycle γ s'accroît de $(\gamma \delta_i).\delta_i$.*

Au coefficient $(\gamma \delta_i)$ de δ_i près, ce théorème remonte à M. Picard (1). Il n'est que juste de dire que ce coefficient jouera un rôle capital dans la suite. C'est en effet sa connaissance qui va nous permettre de résoudre la plupart des questions qui vont se

(1) *Voir* Picard et Simart, *Traité*, vol. I, p. 95.

Fig. 4. Picard - Lefschetz formula



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ceux correspondants des anciennes. En particulier, nous désignerons dorénavant par δ_i ce que devient dans C_a le cycle évanouissant en a_i quand u décrit $a_i a$, et par $\bar{\delta}_i$ le cycle analogue pour $C_{\bar{a}}$ et $a_i \bar{a}$. Soit maintenant

$$\bar{\Gamma}_2 = \sum \bar{\lambda}_i \bar{\Delta}_i + ((C_{\bar{a}})) + \sum \bar{\mu}_j \bar{E}_2^j$$

le second cycle. On trouve de suite, les Δ' ayant le même sens que plus haut,

$$(\bar{\Delta}_i \Delta'_i) = -1, \qquad (\bar{\Delta}_i \Delta'_k) = 0.$$

Par suite, en remplaçant dans l'expression de Γ_2 les Δ par les Δ',

(12) $$(\Gamma_2 \bar{\Gamma}_2) = -\sum \lambda_i \bar{\lambda}_i + \sum \mu_j \bar{\mu}_j.$$

Les $\bar{\lambda}$ ne sont autres que les coefficients des λ dans la première formule dérivée, comme on pourrait le montrer directement.

18. Théorème. — *Si $(\Gamma_2 \bar{\Gamma}_2)$ est nul pour tout $\bar{\Gamma}_2$, Γ_2 est nul ou diviseur de zéro.*

D'abord, puisque $(\Gamma_2 C_u)$ est nul, on aura (n° 13, corollaire)

$$\Gamma_2 \sim \sum \lambda_i \Delta_i + ((C_a)).$$

Donc, pour tout $\bar{\Gamma}_2$,

(13) $$\sum \lambda_i \bar{\lambda}_i = 0.$$

Cette relation sera vérifiée pour tous les $\bar{\lambda}$ tels que

(14) $$\sum \bar{\lambda}_i \bar{\delta}_i \sim 0 \qquad (\mathrm{mod}\ C_{\bar{a}}).$$

Pour exprimer que le cycle à gauche est nul, il suffit d'écrire qu'il est invariant, ce qui donne

(15) $$\sum_i \bar{\lambda}_i (\bar{\delta}_k \bar{\delta}_i) = 0 \qquad (k = 1, 2, \ldots, n);$$

(13) doit être une conséquence des équations (15), d'où t et les t_k

Fig. 5. Cycles évanouissants

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