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Nontrivial dynamics beyond the logarithmic shift in two-dimensional Fisher-KPP equations

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Abstract
We study the asymptotic behaviour, as time goes to infinity, of the Fisher-KPP equation $\partial_t u = \Delta u + u - u^2$ in spatial dimension 2, when the initial condition looks like a Heaviside function. Thus the solution is, asymptotically in time, trapped between two planar critical waves whose positions are corrected by the Bramson logarithmic shift. The issue is whether, in this reference frame, the solutions will converge to a travelling wave, or will exhibit more complex behaviours. We prove here that both convergence and nonconvergence may happen: the solution may converge towards one translate of the planar wave, or oscillate between two of its translates. This relies on the behaviour of the initial condition at infinity in the transverse direction.

Keywords: KPP equations, nontrivial dynamics, logarithmic shift
Mathematics Subject Classification numbers: 35K57, 35B40, 35B35, 35C07

1. Introduction

The paper is devoted to the large time behaviour of the solution of the reaction-diffusion equation

$$\begin{aligned} \partial_t u &= \Delta u + f(u), \quad t > 1\,,\ (x, y) \in \mathbb{R}^2 \\ u(1, x, y) &= u_0(x, y), \qquad\qquad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{1}$$

We will take

$$f(u) = u(1-u) \text{ if } u \in [0,1] \text{ and } f(u) = 0 \text{ if } u \notin [0,1];$$

thus f is, in reference to the celebrated paper [17], said to be of the Fisher-KPP type. The initial datum u_0 is in $\mathcal{C}(\mathbb{R}^2)$ and there exist $x_2 < x_1$ such that

$$1 - H(x - x_2) \leqslant u_0(x, y) \leqslant 1 - H(x - x_1) \tag{2}$$

where H is the Heaviside function. Then, since f is globally Lipschitz on $\mathbb{R}$, there exists (see for instance [16]) a unique classical solution $u(t, x, y)$ in $\mathcal{C}([1, +\infty[\times\mathbb{R}^2, (0, 1))$ to equation (1) emanating from such u_0.

The assumptions on f imply that zero and one are, respectively, unstable and stable equilibria for the ODE $\dot{\zeta} = f(\zeta)$. For the PDE (1), the state $u \equiv 1$ invades the state $u \equiv 0$. Equation (1) admits one-dimensional travelling fronts $U(x - ct)$ if and only if $c \geqslant c^* = 2$ where the profile U, depending on c, satisfies

$$U'' + c\,U' + f(U) = 0, \quad x \in \mathbb{R}, \tag{3}$$

together with the conditions at infinity

$$\lim_{x\to-\infty} U(x) = 1 \text{ and } \lim_{x\to+\infty} U(x) = 0. \tag{4}$$

Any solution U to (3) and (4) is a shift of a fixed profile U_c: $U(x) = U_c(x + s)$ with some fixed $s \in \mathbb{R}$. The profile U_{c^*} at minimal speed $c^* = 2$ satisfies

$$U_{c^*}(x) = (x + k)\,\mathrm{e}^{-x} + O(\mathrm{e}^{-(1+\delta_0)x}), \text{ as } x \to +\infty$$

for some universal constants $k \in \mathbb{R}$ and $\delta_0 > 0$, see [7] and [25].

1.1. Convergence for the KPP equation: related works

The large time behaviour of the one dimensional problem

$$\partial_t u = \partial_{xx} u + f(u), \quad t > 1\,, x \in \mathbb{R} \tag{5}$$

has a history of important contributions. One of the first, and perhaps most well-known one, is the pioneering KPP paper [17]. Kolmogorov, Petrovskii and Piskunov proved that the solution of (5), starting from $1 - H(x)$, converges to U_{c*} in shape: there is a function

$$\sigma_\infty(t) = 2t + o(t),$$

such that

$$\lim_{t\to+\infty} u(t, x + \sigma_\infty(t)) = U_{c^*}(x) \quad \text{uniformly in } x \in \mathbb{R}.$$

The main ingredient in [17] is the monotonicity of $\partial_x u$ on the level sets of u. This argument was recently revisited by Ducrot–Giletti–Matano [10], Nadin [19], for results in the same spirit, concerning one-dimensional inhomogeneous models.

The second one makes precise the $\sigma_\infty(t)$: in [5, 6], Bramson proves the following

Theorem 1.1. *There is a constant x_∞, depending on u_0, such that*

$$\sigma_\infty(t) = 2t - \frac{3}{2}\ln t - x_\infty + o(1), \text{ as } t \to +\infty.$$

The fact that there is a nontrivial logarithmic shift in σ_∞ can be understood as follows: from the maximum principle, we have $u(t,x) \leqslant \bar{u}(t,x)$ where $\bar{u}$ solves the linear PDE $\partial_t \bar{u} = \Delta \bar{u} + \bar{u}$ starting from $1 - H$. Since

$$\bar{u}(t,x) = e^t \int_{\mathbb{R}} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} (1 - H(y)) dy = \frac{e^t}{2} \left(1 - \mathrm{erf}\left(\frac{x}{\sqrt{4t}}\right)\right) \sim_{+\infty} \frac{\sqrt{t} e^t}{x\sqrt{\pi}} e^{-x^2/4t}$$

with $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$, we see that

$$\limsup_{t \to +\infty} \left(\max_{x \geqslant A} u(t, 2t - \frac{\ln t}{2} + A) \right) \longrightarrow 0 \text{ as } A \to +\infty \, .$$

This shows that for any $m \in (0,1)$, there exists $A \in \mathbb{R}$ such that the m-level set of u $S_m(t) = \{x > 0 \,|\, u(t,x) = m\}$ satisfies $\max S_m(t) \leqslant 2t - \frac{\ln t}{2} + A$ for A large enough. This yields that the positions of the level sets $S_m(t)$ should be corrected to the left of the $2t$ position by a term that grows at least logarithmically for large times. This heuristic argument does not yield an equivalent for the correction, and, in particular, does not prove that the correction should be logarithmic. However it explains why a nontrivial term has to be there. The precise computation of the delay is presented in section 2.

Theorem 1.1 was proved through elaborate probabilistic arguments. A natural question was thus to prove theorem 1.1 with purely PDE arguments. In that spirit, a weaker version, precise up to the $O(1)$ term, is the main result of [15] (which is actually the PDE counterpart of [5]): $\sigma_\infty(t) = 2t - \frac{3}{2} \ln t + O(1)$ *as* $t \to +\infty$. Bramson's theorem 1.1 is fully recovered in [20], with once again simple and robust PDE arguments. The dynamics beyond the shift has also been the subject of intense studies. Define $\sigma(t) = \sup\{x > 0 \,|\, u(t,x) = \frac{1}{2}\}$. The issue is to determine the asymptotic behaviour as time goes to infinity of $\sigma(t) - (2t - \frac{3}{2} \ln t - x_\infty)$ where the constant x_∞, depending only on u_0, is given by theorem 1.1. Let us mention the paper [11], which proposes a universal behaviour by means of formal asymptotic arguments. See also [26]. The universal correction $\sigma_\infty(t) = 2t - \frac{3}{2} \ln t + x_\infty - \frac{3\sqrt{\pi}}{\sqrt{t}} + O\left(\frac{1}{t^{1-\gamma}}\right)$ for any $\gamma \in (0, 1/8)$ is obtained, in a mathematically rigorous way, in [21].

When the initial datum u_0 is not compactly supported on the right (or, at least, decays at a sufficiently slow exponential rate), the behaviour of the solution may be quite different. In fact, it depends on the precise behaviour of u_0 at infinity. The main results are roughly the following: if u_0/U_{c^*} decays sufficiently fast as $x \to +\infty$, then $u(t,x)$ will still have the logarithmic delay (with a possibly different value if u_0/U_{c^*} has an algebraic decay). See [6], and [4] for a closely related free boundary problem. If u_0/U_{c^*} has a limit as $x \to +\infty$, then $u(t,x)$ will converge to U_{c^*} without any shift (Gallay [14]). If there is $c > c^*$ such that u_0/U_c has a limit, then $u(t,x)$ converges to U_c without any shift ([2]; when u_0/U_c converges to its limit exponentially, the result is much older and due to Sattinger [24]). If u_0 is trapped between two translates of U_c, a nontrivial behaviour occurs [2]. See also Berestycki–Hamel [3] for a general overview.

In several space dimensions, the theory has been pushed less far. The analogy with the one-dimensional situation is when the initial datum is compactly supported, and the first, and most general result, is due to Aronson–Weinberger [1]. The solution u spreads at the speed $c^* = 2\sqrt{f'(0)} = 2$ in the sense that

$$\min_{|x| \leqslant ct} u(t,x) \to 1 \text{ as } t \to +\infty \, , \text{ for all } 0 \leqslant c < c^*$$

and

$$\sup_{|x|\geqslant ct} u(t,x) \to 0 \text{ as } t \to +\infty\,, \text{ for all } c > c^*.$$

This estimate is made precise up to $O(1)$ terms in Gärtner [13]. If N is the space dimension, for every $\lambda \in (0,1)$, the level set $\{u = \lambda\}$ is trapped, for large times, between two spheres of radius

$$R(t) = c^* t - \frac{N+2}{c^*}\ln t + O_{t\to+\infty}(1).$$

The $O_{t\to+\infty}(1)$ terms are not studied. Gärtner's contribution is probabilistic, and a PDE proof of his result is provided by Ducrot: in [9], he adapts the ideas of [15] to the space dimension N. The forthcoming contribution [23] will make the $O_{t\to+\infty}(1)$ terms explicit.

1.2. Question and results

Let us come back to our two-dimensional case. Let $u_i(t,x)$, $i \in \{1,2\}$ be the solution of the one-dimensional problem (5) emanating from $u_i(1,x) = 1 - H(x - x_i)$. By the maximum principle we have $u_2(t,x) \leqslant u(t,x,y) \leqslant u_1(t,x)$. And so, there exist $x_{\infty,1} \geqslant x_{\infty,2}$ such that, if an arbitrary level set of $u(t,.)$ is represented by the graph $\{x = \sigma(t,y)\}$ - this is not always true, but certainly true if u_0 is nonincreasing in x (applying the maximum principle on u_x) there is a function $\sigma_\infty(t,y) \in [x_{2,\infty}, x_{1,\infty}]$ such that

$$\sigma(t,y) = 2t - \frac{3}{2}\ln t + \sigma_\infty(t,y). \tag{6}$$

The issue is: does this function σ_∞ converge for large times? In one space dimension (σ_∞ only depending on time), this is true. In order to realise that it is an issue in two space dimensions, let us make a parallel with the case where f is bistable, namely: there is $\theta \in (0,1)$ such that $f(u) < 0$ if $u \in (0,\theta)$ and $f(u) > 0$ on $(\theta,1)$. Contrary to the KPP case, the travelling wave problem (3) and (4) has a unique orbit (c_*, U_{c_*}). The speed c_* has the same sign as $\int_0^1 f(u)du$. If $u(1,x) = 1 - H(x)$, then (Fife-McLeod [12]) $u(t,x)$ converges exponentially fast to the wave profile; in other words there are $x_\infty \in \mathbb{R}$ and $\omega > 0$ such that

$$u(t,x) = U_{c_*}(x - c_* t + x_\infty) + O(\mathrm{e}^{-\omega t}) \text{ uniformly in } x \in \mathbb{R}.$$

However, under the assumption (2), and if $\sigma(t,y)$ denotes any level set of $u(t,.)$, both authors proved in [22] that there is a bounded function $\sigma_\infty(t,y)$ such that

$$\sigma(t,y) = c_* t + \sigma_\infty(t,y) + O(t^{-1/2}),$$

and, depending on the initial datum u_0, the function $\sigma_\infty(t,y)$ may or not converge as time goes to infinity. See also Matano–Nara–Taniguchi [18] for similar results, with a balanced bistable nonlinearity corresponding to $c_* = 0$. It is therefore legitimate to suspect a phenomenon of that kind here, and this is exactly what happens.

Let us now state and explain our results. We point out that the very same would hold in space dimension $N > 2$, provided that u_0 is trapped by two translates (in the same direction) of $1 - H$. This does not include situations that are more specific to space dimensions larger than 2, for instance if the level sets of u_0 are trapped between two cylinders. This will be treated elsewhere.

Our first result says that the large time dynamics of (1) is, in some sense, that of the heat equation.

Theorem 1.2. *Let u_0 satisfy assumption (2). For every small $\varepsilon > 0$, there is $T_\varepsilon > 0$ and a function a_0^ε, with $\|a_0^\varepsilon\|_\infty$ and $\|da_0^\varepsilon/\mathrm{d}y\|_\infty$ bounded in ε, such that the solution u of (1) emanating from u_0 satisfies*

$$u(t,x,y) = U_{c_*}\left(x - 2t + \frac{3}{2}\ln t - \ln(a^\varepsilon(t,y) + O(\varepsilon))\right) + O(t^{-1/2}), \quad \text{for } t \geqslant T_\varepsilon,$$

where the function $a^\varepsilon(t,y)$ solves the heat equation

$$(\partial_t - \partial_{yy})a^\varepsilon = 0\,, \quad t > 1\,,\, y \in \mathbb{R}\,, \qquad a^\varepsilon(1,y) = a_0^\varepsilon(y).$$

This explains that (1) has, beyond the logarithmic shift, a large time dynamics which mimics that of the heat equation. We point out that this result is optimal, since the solution of the heat equation does not, in general, converge to anything: see for instance Collet–Eckmann [8], Vàzquez–Zuazua [27]. We will, by the way, use those results to construct solutions that do not converge beyond the shift.

Theorem 1.2 is the most general one can prove. However, it does not really say whether the solution will, or not, converge to something, for the simple reason that it does not exclude a sequence $(a_0^\varepsilon)_\varepsilon$ such that the heat equation starting from a_0^ε will diverge for $\varepsilon = O(1)$, and converge to something as ε becomes very small. So, in the following result, we are going to show that both types of behaviour may happen: convergence to a single wave, or, on the contrary, nonconvergence. Let us not forget, though, that the asymptotic dynamics is that of the heat equation. So, nonconvergence will occur through infinitely slow oscillations between two waves. Assume, for definiteness, that u_0 is nonincreasing in x. This is by no means necessary but, since we are not aiming for utmost generality, this slight loss of generality will be compensated by a lighter formulation. Let $\sigma_\infty(t,y)$ be given by (6).

Theorem 1.3. *The following situations hold.*

1. *There are initial data $u_0(x,y)$, satisfying assumptions (2), such that $t \mapsto \sigma_\infty(t,0)$ does not converge as $t \to +\infty$.*
2. *Assume the existence of two functions $u_0^\pm(x)$, and $x_1 \leqslant x_2$, such that*

$$1 - H(x - x_1) \leqslant u_0^+(x),\ u_0^-(x) \leqslant 1 - H(x - x_2),$$

and such that

$$\lim_{y\to\pm\infty} u_0(x,y) = u_0^\pm(x), \ \text{ uniformly in } x \in \mathbb{R}.$$

If $u^\pm(t,x)$ is the solution of (5) emanating from $u_0^\pm(t,x)$, define $\sigma_\infty^\pm$ as:

$$u^\pm(t,x) = U_{c_*}\left(x - 2t + \frac{3}{2}\ln t + \sigma_\infty^\pm\right) + o_{t\to+\infty}(1).$$

Then we have

$$\lim_{t\to+\infty} \sigma_\infty(t,y) = -\ln\left(\frac{\mathrm{e}^{-\sigma_\infty^+} + \mathrm{e}^{-\sigma_\infty^-}}{2}\right),$$

uniformly on every compact set in y. If $\sigma_\infty^+ = \sigma_\infty^-$, the convergence is uniform in y.

3. *Assume the existence of $u_\infty(x,y)$, periodic in y, such that*

$$\lim_{y\to+\infty}\Big(u_0(x,y)-u_\infty(x,y)\Big)=0, \text{ uniformly in } x.$$

Then $\sigma_\infty(t,y)$ *converges to a constant as* $t\to+\infty$*, uniformly in* y*.*

We could of course imagine more situations, such as, for instance, the existence of two periodic functions $u^{\pm}_\infty(x,y)$ such that $u_0(x,y)$ resembles $u^{+}_\infty(x,y)$ (resp. $u^{-}_\infty(x,y)$) as $y\to+\infty$ (resp. $y\to-\infty$)... Another interesting question is a possible asymptotic expansion of $\sigma_\infty(t,y)$.

2. Strategy, discussion, organisation of the paper

2.1. Main ideas of the proof

There is a sequence of transformations that bring the equations under the (1) to a form that will be amenable to treatment.

1. We observe the equation (1) in the reference frame whose origin is $X(t)=2t-\frac{3}{2}\ln t$ and choose the change of variables $x'=x-X(t)$ and $u(t,x,y)=u_1(t,x-X(t),y)$. After dropping the primes and indexes, equation (1) becomes
$$\partial_t u=\Delta u+\left(2-\frac{3}{2t}\right)\partial_x u+u-u^2\,,\quad t>1\,,\quad (x,y)\in\mathbb{R}^2 \tag{7}$$
with initial datum $u(1,x,y)=u_0(x+2,y)$.
2. To follow the exponential decay of the wave U_{c^*}, it will be useful to take it out and set $u(t,x,y)=\mathrm{e}^{-x}v(t,x,y)$; (7) thus becomes
$$\partial_t v=\Delta v-\frac{3}{2t}\left(\partial_x v-v\right)-\mathrm{e}^{-x}v^2\,,\quad t>1\,,\quad (x,y)\in\mathbb{R}^2 \tag{8}$$
with initial datum $v(1,x,y)=\mathrm{e}^{x}u_0(x+2,y)$.
3. Finally, if we want to study (8) in the diffusive zone, i.e. the region $x\sim\sqrt{t}$, we introduce self similar variables $\xi=\frac{x}{\sqrt{t}}$, $\tau=\ln t$. The variable y is unchanged:
$$w(\tau,\xi,y)=w\left(\ln t,\frac{x}{\sqrt{t}},y\right)=\frac{1}{\sqrt{t}}v(t,x,y). \tag{9}$$
Then (8) becomes
$$\partial_\tau w=\mathcal{L}w+\mathrm{e}^{\tau}\partial_{yy}w-\frac{3}{2}\mathrm{e}^{-\frac{\tau}{2}}\partial_\xi w-e^{\frac{3}{2}\tau-\xi\mathrm{e}^{\frac{\tau}{2}}}w^2\,,\quad \tau>0\,,\quad (\xi,y)\in\mathbb{R}^2 \tag{10}$$
where
$$\mathcal{L}w=\partial_{\xi\xi}w+\frac{\xi}{2}\partial_\xi w+w$$
with initial datum $w(0,\xi,y)=\mathrm{e}^{\xi}u_0(\xi+2,y)$.

In the sequel, we will use the form that will be best suited to our purposes. Let us say a word about the strategy of the proof of theorem 1.2. In one space dimension, (10) becomes

$$\partial_\tau w=\mathcal{L}w-\frac{3}{2}\mathrm{e}^{-\frac{\tau}{2}}\partial_\xi w-e^{\frac{3}{2}\tau-\xi\mathrm{e}^{\frac{\tau}{2}}}w^2\,,\quad \tau>0\,,\quad \xi\in\mathbb{R}.$$

The main step of the proof in [20] was to prove the existence of a constant $\alpha_\infty > 0$ such that

$$w(\tau, \xi) \longrightarrow_{\tau\to+\infty} \alpha_\infty \xi^+ \mathrm{e}^{-\xi^2/4}, \quad \text{in } \{\xi \geqslant e^{-(\frac{1}{2}-\delta)\tau}\},$$

where $\delta > 0$ is arbitrarily small. We would then define the translation $\sigma_\infty(t)$ such that

$$U_{c_*}(x + \sigma_\infty(t))\Big|_{x=t^\delta} = \mathrm{e}^{-x} v(t, x)\Big|_{x=t^\delta}.$$

That is,

$$\sigma_\infty(t) = -\ln\alpha_\infty + O(t^{-\delta}). \tag{11}$$

We would then prove the uniform convergence to $U_{c_*}(x - \ln\alpha_\infty)$ by examining the difference

$$\tilde{v}(t, x) = \left|v(t, x) - U_{c_*}(x + \sigma_\infty(t))\right|$$

in the region $\{x < t^\delta\}$. It turned out that $\tilde{v}(t, x)$ was a subsolution of (a perturbation of) the heat equation

$$\begin{array}{rll} V_t = & V_{xx} + O(t^{1-\delta}), & t > 0,\, -t^\delta < x < t^\delta \\ V(t, -t^\delta) = & \mathrm{e}^{-t^\delta}, & t > 0 \\ V(t, t^\delta) = & 0, & t > 0. \end{array} \tag{12}$$

The condition at $x = -t^\delta$ simply comes from the fact that $v(t, x)$ decays, by definition, like e^x at $-\infty$. Although the domain looks very large, its first Dirichlet eigenvalue is of the order $t^{-2\delta}$, hence a much larger quantity than the right hand side of (12). Thus $V(t, x)$ goes to zero uniformly in x as $t \to +\infty$, which implies the sought for convergence result.

In what follows, we are going to adapt these ideas to our setting. The main additional difficulty is the transverse diffusion, which, in a very paradoxical way, does not help us. This is not a rhetorical argument: its presence is really what prevents convergence, in most cases. This implies that we will have to be quite careful with the estimates.

2.2. Comparison with the bistable case

Theorems 1.2 and 1.3 have similarities with those pertaining to the bistable state, that we proved in [22]. Both KPP and bistable dynamics show features of similar kinds: a one-dimensional dynamics at the leading order, followed by perturbations driven by diffusion at the lower order. This shows a certain universality of the phenomenon. However they are not at all proved in the same way, because the leading order dynamics is given by two very different phenomena. In the bistable case, the travelling wave U_{c_*} is at the heart of the study, in the sense that it dictates the whole asymptotic behaviour. In order to take this fact into account, we have devised an infinite Lyapunov–Schmidt reduction, and the heat equation arises as a bifurcation equation. In the Fisher-KPP case, the dynamics is driven by the tail of the solution. More precisely, the diffusive area $x \sim \sqrt{t}$ is crucial, and the heat equation arises directly from this area. The travelling wave behaviour in the region $x = O(1)$ is just a consequence of this diffusive dynamics. As a consequence, our results, although they do resemble those proved in [22], are quite different both in nature and in the methods that are used to prove them.

2.3. Organisation of the paper

The paper is organised as follows. In section 3, we explain how the behaviour of $u(t, x, y)$ in the half plane $\{x < t^\delta, y \in \mathbb{R}\}$ is slaved to that on the line $\{x = t^\delta, y \in \mathbb{R}\}$. In section 4, we characterise the asymptotic behaviour of a general linear equation that encompasses, in particular, equation (10). In section 5, we define sub and super solutions that will enable us to prove theorem 1.2. Finally, theorem 1.3 is proved in section 6.

3. Control of the solution by its value at t^δ

The goal of this section is to prove, as announced in the introduction, that controlling the solution slightly to the right of the $O(1)$ in x area implies, provided that the control is well-tailored, the control of the solution to the entire region to the left. From now on we consider $\delta \in (0, \frac{1}{2})$, that will be as small as we wish.

3.1. The basic result

Let $a(t, y)$ be a smooth function such that

– there are constants $0 < \underline{a}_0 \leqslant \overline{a}_0 < +\infty$ that bound a:

$$\forall t > 1\,,\ \forall y \in \mathbb{R}\,, \quad \underline{a}_0 \leqslant a(t, y) \leqslant \overline{a}_0, \tag{13}$$

– there is a constant $C_0 > 0$ depending on $\underline{a}_0$ and $\overline{a}_0$ that bounds the derivatives of a:

$$\forall t > 1\,,\ \forall y \in \mathbb{R}\,, \quad |\partial_y a(t, y)| \leqslant \frac{C_0}{\sqrt{t}}, \quad \max(|\partial_{yy} a(t, y)|, |\partial_t a(t, y)|) \leqslant \frac{C_0}{t}. \tag{14}$$

We define $\gamma(t, y)$ by the relation

$$U_{c_*}(t^\delta + \gamma(t, y)) = t^\delta \mathrm{e}^{-t^\delta - 1/4t^{1-2\delta}} \frac{a(t, y)}{\sqrt{2\sqrt{\pi}}} := u_+^a(t, y). \tag{15}$$

We have therefore, for large t and $\delta \in (0, \frac{1}{3})$:

$$\gamma(t, y) = -\ln\left(\frac{a(t, y)}{\sqrt{2\sqrt{\pi}}}\right) + O(t^{-\delta}).$$

More important we have, from the implicit functions theorem, that γ is at least C^1 in t and C^2 in y, and we have, for a universal constant C:

$$\begin{aligned} |\partial_y \gamma(t, y)| \leqslant&\ \ C|\partial_y a(t, y)| \\ |\partial_{yy} \gamma(t, y)| \leqslant&\ \ C\Big(|\partial_{yy} a(t, y)| + (\partial_y a(t, y))^2\Big) \\ |\partial_t \gamma(t, y)| \leqslant&\ \ C\Big(|\partial_t a(t, y)| + \tfrac{a(t,y)}{t^{1-\delta}}\Big). \end{aligned} \tag{16}$$

Let $u_a(t, x, y)$ be a solution of

$$\begin{aligned} \partial_t u_a = \Delta u_a + \left(2 - \frac{3}{2t}\right)\partial_x u_a + u_a - u_a^2 \quad & t > 1\,,\ x \leqslant t^\delta\,,\ y \in \mathbb{R} \\ u_a(t, t^\delta, y) = u_+^a(t, y) \quad & t \geqslant 1\,,\ x = t^\delta\,,\ y \in \mathbb{R} \\ \inf_{y\in\mathbb{R}} \liminf_{x\to-\infty} u_a(1, x, y) > 0. & \end{aligned} \tag{17}$$

Here is the main result of this section.

Theorem 3.1. *For $\delta \in (0, \frac{1}{4})$ and u_a solution to equation (17) where u_+^a is defined in (15) and a satisfies assumptions (13) and (14), we have for any $t > 1$*

$$\sup_{|x| \leqslant t^\delta} \sup_{y \in \mathbb{R}} \mathrm{e}^x \Big| u_a(t, x, y) - U_{c_*}(x + \gamma(t, y)) \Big| \leqslant \frac{C}{t^\lambda},$$

for some universal constant $C > 0$ and $\lambda \in (0, 1 - 4\delta)$.

Proof. We simply set

$$s(t, x, y) = \mathrm{e}^x \big(u_a(t, x, y) - U_{c_*}(x + \gamma(t, y))\big).$$

Then, for any $t > 1$, $x < t^\delta$, and $y \in \mathbb{R}$

$$\partial_t s - \Delta s + \frac{3}{2t}(\partial_x s - s) + s(u_a + U_{c_*}(x + \gamma)) = \mathrm{e}^x \left((\partial_{yy}\gamma - \partial_t\gamma)U' + (\partial_y\gamma)^2 U''\right)$$

so that by (16), we have

$$\begin{aligned} \partial_t s - \Delta s + \frac{3}{2t}(\partial_x s - s) + s(u_a + U_{c_*}(x + \gamma)) &= O\left(\frac{1}{t^{1-2\delta}}\right) \quad t > 1\,,\ x < t^\delta\,,\ y \in \mathbb{R} \\ s(t, t^\delta, y) &= 0 \quad t > 1\,,\ x = t^\delta\,,\ y \in \mathbb{R} \\ \sup_{y \in \mathbb{R}} s(t, -t^\delta, y) &= O\left(\mathrm{e}^{-t^\delta}\right) \quad t > 1\,,\ x = -t^\delta\,,\ y \in \mathbb{R}. \end{aligned} \tag{18}$$

The last equation comes from the definition of s, as the product of a bounded function by an exponential. As in [20], a super-solution to (18) is devised as

$$\bar{s}(t, x, y) = \frac{A}{t^\lambda} \cos\left(\frac{x}{t^{\delta + \tilde{\varepsilon}}}\right),$$

where $\delta \in (0, \frac{1}{4})$, $\lambda \in (0, 1 - 4\delta)$, $\tilde{\varepsilon} > 0$ is small enough such that $2\delta + 2\tilde{\varepsilon} + 1 - \lambda < 1 - 2\delta$ and $A > 0$ large enough. The idea is that the first Dirichlet eigenvalue of $(-\partial_{xx})$ in the interval $(-t^\delta, t^\delta)$ is of order $t^{-2\delta}$ (a nonintegrable power of t if δ is small enough), whereas the right hand side of (18) is of the order $t^{2\delta - 1}$, a much larger power. And so, $\bar{s}$ will dominate s, which proves the result. ■

3.2. Perturbative results

Consider $\varepsilon > 0$ and $b(t, y)$ a smooth function such that for any $t > 1$ and $y \in \mathbb{R}$:

$$|b(t, y)| \leqslant \varepsilon + \frac{C}{t^\delta}, \tag{19}$$

for some constant $C > 0$. Note that no assumption is made on the derivatives of b and, in particular, no assumption on a possible time decay of $\partial_t b$ or $\partial_y b$. Set, this time

$$u_+^{a+b}(t,y) = t^\delta e^{-t^\delta - 1/4t^{1-2\delta}} \frac{a(t,y)+b(t,y)}{\sqrt{2\sqrt{\pi}}}. \tag{20}$$

Theorem 3.1 perturbs into the following

Proposition 3.2. *For $\delta \in (0, \frac{1}{5})$, let u_a (resp. u_{a+b}) be a solution of the Dirichlet problem (17), with boundary condition $u_+^a(t,y)$ (resp. u_+^{a+b}). There exists $C>0$, depending on $u_a(1,.)$ and $u_{a+b}(1,.)$ such that for any $t>1$*

$$\sup_{|x|\leqslant t^\delta} \sup_{y\in\mathbb{R}} e^x |u_{a+b}(t,x,y) - u_a(t,x,y)| \leqslant C(\varepsilon + \frac{1}{t^\delta}).$$

Proof. Define $\underline{u}(t,x,y)$ (resp. $\overline{u}(t,x,y)$) as the solutions of (17) with the following data:

$$\begin{cases} \underline{u}(t,t^\delta,y) = & u_+^{a+b} - C(\varepsilon + t^{-\delta}), \quad \underline{u}(1,x,y) = \min\Big(u_a(1,x,y), u_{a+b}(1,x,y)\Big) \\ \overline{u}(t,t^\delta,y) = & u_+^{a+b} + C(\varepsilon + t^{-\delta}), \quad \overline{u}(1,x,y) = \max\Big(u_a(1,x,y), u_{a+b}(1,x,y)\Big). \end{cases}$$

Both $\overline{u}$ and $\underline{u}$ fall in the assumptions of theorem 3.1, thus $\overline{u}$ approaches $U_{c_*}(x+\overline{\gamma}(t,y))$ (resp. $\underline{u}$ approaches $U_{c_*}(x+\underline{\gamma}(t,y))$ like $t^{-\lambda}$ as $t\to+\infty$ with $\lambda\in(0,1-4\delta)$. The definition of $\overline{\gamma}$ and $\underline{\gamma}$ mimick that of γ in the preceding section; in other words the translation of U_{c_*} is adjusted to coincide with the solution at the boundary. Thus we have

$$|\overline{\gamma}(t,y) - \underline{\gamma}(t,y)| \leqslant C(\varepsilon + t^{-\delta}),$$

and the proposition follows since $1-4\delta > \delta$. ∎

4. A Dirichlet problem in the diffusive zone

Consider the following linear equation for $\varepsilon > 0$ small, and $\lambda > 0$:

$$\begin{aligned} \partial_\tau v = \mathcal{L}v + \frac{e^\tau}{\varepsilon^2}\partial_{yy}v + \varepsilon^{2\lambda}e^{-\lambda\tau}\left(\phi_\varepsilon(\tau)v + \psi_\varepsilon(\tau)\partial_\xi v + f_\varepsilon(\tau,\xi)\right), &\quad \tau>0,\ \xi>0,\ y\in\mathbb{R} \\ v(\tau,0,y) = 0, &\quad \tau>0,\ \xi=0,\ y\in\mathbb{R} \\ v(0,\xi,y) = v_0(\xi,y), &\quad \tau=0,\ \xi>0,\ y\in\mathbb{R}. \end{aligned} \tag{21}$$

In the sequel of the paper, we will need to use various versions of this equation, and this is why we have chosen to study it in its most general form. The factor $\varepsilon>0$ stands for the fact that, quite often, we start the study of the equation at an already large time, typically $\tau_\varepsilon = -\ln\varepsilon$. The real number λ is an exponent that is in general less than 1/2. In practice, it will stand for the decay of quantities that will be known to decay exponentially fast, but less than the critical exponent 1/2. The term f_ε is a forcing that will, in general, arise from inhomogeneous Dirichlet conditions.

4.1. Behaviour for general initial data

With no particular assumption on the behaviour of v_0 in the direction y apart from being bounded, we are going to prove the approximate stabilisation of the solutions of (21) to eigenfunctions of the heat operator, up to errors of the order ε. Let X be the space

$$X = \{v(\xi, y) \in L^2(\mathbb{R}_+, L^\infty(\mathbb{R})),\ \mathrm{e}^{\xi^2/8} v \in L^2(\mathbb{R}_+, L^\infty(\mathbb{R}))\}. \tag{22}$$

The result is the following.

Theorem 4.1. *Choose $\lambda > 0$ and $C_0 > 0$, as well as $v_0 \in X$. Let $(\phi_\varepsilon(\tau), \psi_\varepsilon(\tau), f_\varepsilon(\tau, \xi))_{0 \leqslant \varepsilon \leqslant 1}$ be a family of functions that are compactly supported in ξ, bounded by C_0 in τ, ξ and ε. There exist $\varepsilon_0 > 0$ and $C > 0$ depending only on λ and C_0 such that for any compact set $K \subset \mathbb{R}^+$, there is $C_K > 0$, and a a unique solution $v \in \mathcal{C}(\mathbb{R}^+, X)$ to (21) emanating from v_0, which satisfies*

$$\forall \tau > 0\,,\ \xi > 0\,,\ y \in \mathbb{R}\,, \quad v(\tau, \xi, y) = \xi \left(\frac{\mathrm{e}^{-\xi^2/4}}{\sqrt{2\sqrt{\pi}}} (\alpha_c(\tau, y) + \beta(\tau, y)) + \mathrm{e}^{-\frac{\lambda}{2}\tau} \tilde{v}(\tau, \xi, y) \right).$$

Moreover, for any $\tau > 0$, $y \in \mathbb{R}$ we have:

$$\begin{aligned}
&\partial_\tau \alpha_c = \frac{\mathrm{e}^\tau}{\varepsilon^2} \partial_{yy} \alpha_c\,, \quad \alpha_c(0, y) = \frac{1}{\sqrt{2\sqrt{\pi}}} \int_0^{+\infty} \xi v_0(\xi, y) \mathrm{d}\xi \\
&\|\beta(\tau)\|_{L^\infty(\mathbb{R})} \leqslant C\varepsilon^{2\lambda} \qquad \|\partial_\tau \beta(\tau)\|_{L^\infty} \leqslant C\varepsilon^{2\lambda} \\
&\|\partial_y \beta(\tau)\|_{L^\infty(\mathbb{R})} \leqslant C\varepsilon^{2\lambda+1} \mathrm{e}^{-\frac{\tau}{2}} \qquad \|\partial_{yy} \beta(\tau)\|_{L^\infty} \leqslant C\varepsilon^{2\lambda+2} \mathrm{e}^{-\tau},
\end{aligned}$$

and for any $\tau > 0$, $\xi \in K$, $y \in \mathbb{R}$

$$\begin{aligned}
&\max(|\tilde{v}(\tau, \xi, y)|, |\partial_\tau \tilde{v}|\,, |\partial_\xi \tilde{v}|\,, |\partial_{\xi\xi} \tilde{v}|) \leqslant C_K \varepsilon^\lambda \\
&|\partial_y \tilde{v}| \leqslant C_K \varepsilon^{\lambda+1} \mathrm{e}^{-\frac{\tau}{2}}\,, \quad |\partial_{yy} \tilde{v}| \leqslant C_K \varepsilon^{\lambda+2} \mathrm{e}^{-\tau}.
\end{aligned}$$

Proof of theorem 4.1. Let $\lambda > 0$ be given by equation (21) and $C_0 > 0$. Set $\varepsilon > 0$ and consider ϕ_ε, ψ_ε and f_ε uniformly bounded in τ and ε by C_0. Assume also f_ε is compactly supported in ξ. Let v be the solution to (21) emanating from $v_0 \in X$. Let us introduce the new function $w(\tau, \xi, y) = \mathrm{e}^{\frac{\xi^2}{8}} v(\tau, \xi, y)$. This new function solves for any $\tau > 0$, $\xi > 0$ and $y \in \mathbb{R}$.

$$\begin{aligned}
\partial_\tau w = \mathcal{M} w + \frac{\mathrm{e}^\tau}{\varepsilon^2} \partial_{yy} w + \varepsilon^{2\lambda} \mathrm{e}^{-\lambda\tau} \left((\phi_\varepsilon(\tau) - \frac{\xi}{4} \psi_\varepsilon(\tau)) w + \psi_\varepsilon(\tau) \partial_\xi w + \mathrm{e}^{\frac{\xi^2}{8}} f_\varepsilon(\tau, \xi) \right)& \\
w(\tau, 0, y) = 0 \quad \tau > 0\,, \xi = 0\,, y \in \mathbb{R}& \\
w(0, \xi, y) = w_0(\xi, y) = \mathrm{e}^{\frac{\xi^2}{8}} v_0(\xi, y) \quad \tau = 0\,, \xi > 0\,, y \in \mathbb{R}&
\end{aligned} \tag{23}$$

where $\mathcal{M} w = \partial_{\xi\xi} w + \left(\frac{3}{4} - \frac{\xi^2}{16}\right) w$. Thus $\mathcal{D}(\mathcal{M}) = \{w \in H_0^2(\mathbb{R}^+) \,|\, \xi^2 w \in L^2(\mathbb{R}^+)\}$, $\mathcal{M}$ is symmetric and its null space is generated by the unit eigenfunction $e_0(\xi) = \frac{1}{\sqrt{2\sqrt{\pi}}} \xi \mathrm{e}^{-\frac{\xi^2}{8}}$. This linear operator defines a quadratic form on $\{w \in H_0^1(\mathbb{R}^+) \,|\, \xi^2 w \in L^2(\mathbb{R}^+)\}$ as

$$q(w) = < -\mathcal{M} w, w >_{L^2(\mathbb{R}^+)} = \int_0^{+\infty} (\partial_\xi w)^2 + \left(\frac{\xi^2}{16} - \frac{3}{4}\right) w^2 \, \mathrm{d}\xi$$

which is nonnegative and satisfies

$$q(w) \geqslant \|w\|^2_{L^2(\mathbb{R}^+)} \text{ if } < w, e_0 >_{L^2(\mathbb{R}^+)} = 0.$$

■

Lemma 4.2. *There exist $\varepsilon_0 > 0$ (depending on λ and C_0) and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any w solution to (23) emanating from $w_0 \in L^2(\mathbb{R}^+, L^\infty(\mathbb{R}))$ satisfies*

$$\forall \tau \geqslant 0\,, \quad \|w(\tau)\|_{L^2(\mathbb{R}^+, L^\infty(\mathbb{R}))} \leqslant C(\|w_0\|_{L^2(\mathbb{R}^+, L^\infty(\mathbb{R}))} + \varepsilon^\lambda).$$

Proof of lemma 4.2. Taking the $L^2(\mathbb{R}^+)$ scalar product of (23) with w leads to

$$\partial_\tau \|w\|^2_{L^2(\mathbb{R}^+)} + 2q(w) = \frac{\mathrm{e}^\tau}{\varepsilon^2}\left(\partial_{yy}\|w\|^2_{L^2(\mathbb{R}^+)} - 2\|\partial_y w\|^2_{L^2(\mathbb{R}^+)}\right)$$
$$+ 2\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}\left(\phi_\varepsilon(\tau)\|w\|^2_{L^2(\mathbb{R}^+)} - \psi_\varepsilon(\tau)\int_0^{+\infty}\frac{\xi}{4}w^2\mathrm{d}\xi + \int_0^\infty \mathrm{e}^{\frac{\xi^2}{8}} f_\varepsilon(\tau, \xi) w\,\mathrm{d}\xi\right).$$

Note that

$$\int_0^{+\infty}\frac{\xi}{4}w^2\mathrm{d}\xi \leqslant \int_0^{+\infty}\left(\frac{\xi^2}{16} + \frac{1}{4}\right)w^2\mathrm{d}\xi \leqslant q(w) + \|w\|^2_{L^2(\mathbb{R}^+)}$$

whence, using Cauchy–Schwarz inequality,

$$\partial_\tau \|w\|^2_{L^2(\mathbb{R}^+)} + 2\left(1 - \varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}|\psi_\varepsilon|\right)q(w) \leqslant \frac{\mathrm{e}^\tau}{\varepsilon^2}\partial_{yy}\|w\|^2_{L^2(\mathbb{R}^+)}$$
$$+ 2\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}\left((|\phi_\varepsilon| + |\psi_\varepsilon| + \frac{1}{2})\|w\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{2}\|\mathrm{e}^{\frac{\xi^2}{8}} f_\varepsilon\|^2_{L^2}\right).$$

If $\varepsilon_0 > 0$ is small enough (depending on λ and C_0), $1 - \varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}|\psi_\varepsilon(\tau)| > 0$ for any $\tau \geqslant 0$ and $\varepsilon \in (0, \varepsilon_0)$, which combined with $q(w) \geqslant 0$ gives

$$\partial_\tau \|w\|^2_{L^2(\mathbb{R}^+)} \leqslant \frac{\mathrm{e}^\tau}{\varepsilon^2}\partial_{yy}\|w\|^2_{L^2(\mathbb{R}^+)} + \varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}(C_1\|w\|^2_{L^2(\mathbb{R}^+)} + C_2) \tag{24}$$

where C_1 only depends on $\sup\{|\phi_\varepsilon(\tau)|, |\psi_\varepsilon(\tau)|\,,\, \tau \geqslant 0\,,\, \varepsilon > 0\}$ while C_2 depends on f_ε. Let $h(\tau)$ be the solution to the ODE

$$\forall \tau \geqslant 0\,, \quad h'(\tau) = \varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}(C_1 h(\tau) + C_2)\,, \quad h(0) = \|w_0\|^2_{L^2(L^\infty)}$$

then h is a supersolution to (24) and for any $\tau \geqslant 0$,

$$h(\tau) = h(0)\mathrm{e}^{\frac{C_1}{\lambda}\varepsilon^{2\lambda}(1-\mathrm{e}^{-\lambda\tau})} + \frac{C_2}{C_1}\left(\mathrm{e}^{\frac{C_1}{\lambda}\varepsilon^{2\lambda}(1-\mathrm{e}^{-\lambda\tau})} - 1\right). \tag{25}$$

If ε_0 is small enough (compared to λ/C_1), we can bound the second term as follows:

$$\|w(\tau)\|^2_{L^2(L^\infty)} \leqslant C\left(\|w_0\|^2_{L^2(L^\infty)} + \frac{C_2}{\lambda}\varepsilon^{2\lambda}\right).$$

This concludes the proof of lemma 4.2. ■

Proof of theorem 4.1 (continued). We use the spectral property of $\mathcal{M}$ to decompose any solution w to (23) as

$$\forall(\tau, \xi, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}\,, \quad w(\tau, \xi, y) = \alpha(\tau, y)e_0(\xi) + r(\tau, \xi, y),$$

where $\alpha(\tau, y) =< w(\tau, \cdot, y), e_0 >_{L^2(\mathbb{R}^+)}$ so that r is a transverse perturbation: for any $(\tau, y) \in \mathbb{R}^+ \times \mathbb{R}$, $< r(\tau, \cdot, y), e_0 >_{L^2(\mathbb{R}^+)} = 0$. Projecting equation (23) on the null space of $\mathcal{M}$ gives

$$\partial_\tau \alpha = \frac{e^\tau}{\varepsilon^2}\partial_{yy}\alpha + \varepsilon^{2\lambda}e^{-\lambda\tau}\left((\phi_\varepsilon - \frac{\psi_\varepsilon}{\sqrt{\pi}})\alpha - \psi_\varepsilon < r, e_0' + \frac{\xi}{4}e_0 >_{L^2(\mathbb{R}^+)} + < e^{\frac{\xi^2}{8}}f_\varepsilon, e_0 >_{L^2(\mathbb{R}^+)}\right),$$

while the equation satisfied by r reads

$$\partial_\tau r = \mathcal{M}r + \frac{e^\tau}{\varepsilon^2}\partial_{yy}r + \varepsilon^{2\lambda}e^{-\lambda\tau}\left(\phi_\varepsilon r + \psi_\varepsilon Q(\partial_\xi r - \frac{\xi}{4}r) + \alpha\psi_\varepsilon\, Q(e_0' - \frac{\xi}{4}e_0) + Q(e^{\frac{\xi^2}{8}}f_\varepsilon)\right), \tag{26}$$

where $P = 1 - Q$ is the projection onto the null space of $\mathcal{M}$.

Since we have in mind that we will find a dynamics similar to that of the heat equation, we introduce α_c solution to

$$\partial_\tau \alpha_c = \frac{e^\tau}{\varepsilon^2}\partial_{yy}\alpha_c\,, \quad \alpha_c(0, y) = \alpha(0, y)$$

and set $\beta = \alpha - \alpha_c$ the difference. Then, we have $\beta(0, y) = 0$ and

$$\partial_\tau \beta = \frac{e^\tau}{\varepsilon^2}\partial_{yy}\beta + \varepsilon^{2\lambda}e^{-\lambda\tau}\left((\phi_\varepsilon - \frac{\psi_\varepsilon}{\sqrt{\pi}})(\alpha_c + \beta) - \psi_\varepsilon < r, e_0' + \frac{\xi}{4}e_0 > + < e^{\frac{\xi^2}{8}}f_\varepsilon, e_0 >\right). \tag{27}$$

We shall prove that β remains small for all time and that r decays exponentially fast to zero as time goes to infinity. Indeed, by the maximum principle and lemma 4.2, we get

$$\begin{aligned}\partial_\tau \beta \leqslant& \frac{e^\tau}{\varepsilon^2}\partial_{yy}\beta + \varepsilon^{2\lambda}e^{-\lambda\tau}\left(|\phi_\varepsilon| + \frac{\psi_\varepsilon}{\sqrt{\pi}}\right)|\beta| \\ &+ \varepsilon^{2\lambda}e^{-\lambda\tau}\left(\left(|\phi_\varepsilon| + \frac{\psi_\varepsilon}{\sqrt{\pi}}\right)\|\alpha_c(0)\|_{L^\infty} + |\psi_\varepsilon|\|e_0' + \frac{\xi}{4}e_0\|_{L^2}\|r(\tau)\|_{L^2(L^\infty)} + \|e^{\frac{\xi^2}{8}}f_\varepsilon\|_{L^2}\right).\end{aligned}$$

Define h as a solution to the ODE

$$h'(\tau) = \varepsilon^{2\lambda}e^{-\lambda\tau}(C_1|h(\tau)| + C_2)\,, \quad h(0) = 0,$$

where C_1 only depends on ϕ_ε and ψ_ε while C_2 depends on ϕ_ε, ψ_e, f_ε and $\|w\|_{L^2(L^\infty)}$. Then, h is a supersolution to (27) and dealing as in (25), we get for ε_0 small enough (compared to λ/C_1),

$$\forall \tau \geqslant 0\,, \quad \|\beta(\tau)\|_{L^\infty(\mathbb{R})} \leqslant |h(\tau)| \leqslant e\frac{C_2}{\lambda}\varepsilon^{2\lambda}. \tag{28}$$

We shall now apply parabolic regularity to get the same bounds on the derivatives of β. For any $y_0 \in \mathbb{R}$, set $\zeta = \varepsilon\, e^{-\frac{\tau}{2}}(y + y_0)$ and denote $B(\tau, \zeta) = B(\tau, \varepsilon e^{-\frac{\tau}{2}}(y + y_0)) = \beta(\tau, y)$. Then, by (27),

$$\partial_\tau B = \partial_{\zeta\zeta}B + \frac{\zeta}{2}\partial_\zeta B + \varepsilon^{2\lambda}e^{-\lambda\tau}\left((\phi_\varepsilon - \frac{\psi_\varepsilon}{\sqrt{\pi}})(\alpha_c + B) - \psi_\varepsilon\langle r, e_0' + \frac{\xi}{4}e_0 > + < e^{\frac{\xi^2}{8}}f_\varepsilon, e_0\rangle\right).$$

The above bound on β also gives B uniformly bounded by $\varepsilon^{2\lambda}$. By parabolic regularity applied in the range $|\zeta| < 1$, we get that the derivatives of B are uniformly bounded by $\varepsilon^{2\lambda}$. Coming back to β, we get the desired estimates since the bounds do not depend on y_0.

As far as r is concerned, we compute an energy estimate to benefit from the spectral gap in self similar variables. Taking the L^2 scalar product of (26) with r gives

$$\partial_\tau \|r\|^2_{L^2(\mathbb{R}^+)} + 2q(r) = \frac{\mathrm{e}^\tau}{\varepsilon^2}\left(\partial_{yy}\|r\|^2_{L^2(\mathbb{R}^+)} - 2\|\partial_y r\|^2_{L^2(\mathbb{R}^+)}\right) + 2\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}\phi_\varepsilon\|r\|^2_{L^2(\mathbb{R}^+)}$$
$$+ 2\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}\left(\psi_\varepsilon\langle Q(\partial_\xi r - \frac{\xi}{4}r) + \alpha Q(e_0' - \frac{\xi}{4}e_0), r\rangle_{L^2(\mathbb{R}^+)} + \langle Q(\mathrm{e}^{\frac{\xi^2}{8}}f_\varepsilon), r\rangle\right). \tag{29}$$

Since

$$\left|\langle Q(\partial_\xi r - \frac{\xi}{4}r), r\rangle_{L^2(\mathbb{R}^+)}\right| = \int_0^\infty \frac{\xi}{4}r^2\mathrm{d}\xi \leqslant \int_0^\infty\left(\frac{\xi^2}{16} + \frac{1}{4}\right)r^2\mathrm{d}\xi \leqslant q(r) + \|r\|^2_{L^2(\mathbb{R}^+)}$$

and

$$\left|\alpha\langle Q(e_0' - \frac{\xi}{4}e_0), r\rangle_{L^2(\mathbb{R}^+)}\right| \leqslant \|\alpha(\tau)\|_{L^\infty}\|e_0' - \frac{\xi}{4}e_0\|_{L^2}\|r\|_{L^2} \leqslant (\|\alpha_c\|_{L^\infty} + \|\beta\|_{L^\infty})\|r\|_{L^2},$$

we get

$$\partial_\tau\|r\|^2_{L^2(\mathbb{R}^+)} + 2(1 - \varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}|\psi_\varepsilon|)q(r) \leqslant \frac{\mathrm{e}^\tau}{\varepsilon^2}\partial_{yy}\|r\|^2_{L^2(\mathbb{R}^+)} + 2\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}(|\phi_\varepsilon| + |\psi_\varepsilon|)\|r\|^2_{L^2}$$
$$+ 2\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}\left(|\psi_\varepsilon|(\|\alpha_c\|_{L^\infty} + \|\beta\|_{L^\infty})\|r\|_{L^2} + \|\mathrm{e}^{\frac{\xi^2}{8}}f_\varepsilon\|_{L^2}\|r\|_{L^2}\right).$$

If ε_0 is small enough (depending on λ and C_0), then we have $1 - \varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}|\psi_\varepsilon| \geqslant \frac{3}{4}$ for any $\tau \geqslant 0$ and $\varepsilon \in (0, \varepsilon_0)$. Combined with $q(r) \geqslant \|r\|^2_{L^2}$, (28) and lemma 4.2, this gives

$$\partial_\tau\|r\|^2_{L^2(\mathbb{R}^+)} + \frac{3}{2}\|r\|^2_{L^2} \leqslant \frac{\mathrm{e}^\tau}{\varepsilon^2}\partial_{yy}\|r\|^2_{L^2(\mathbb{R}^+)} + C\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}.$$

Define h as the solution to the ODE

$$h'(\tau) + \frac{3}{2}h(\tau) = C\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}, \quad h(0) = \|r_0\|^2_{L^2(L^\infty)}.$$

Then, h is a supersolution to (29) and

$$\forall\tau \geqslant 0, \quad \|r(\tau)\|^2_{L^2(L^\infty)} \leqslant h(\tau) \leqslant C\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau} + e^{-\frac{3}{2}\tau}\|r_0\|^2_{L^2(L^\infty)}. \tag{30}$$

We shall now apply again parabolic regularity to get some bounds on r. For any $y_0 \in \mathbb{R}$, set $\zeta = \varepsilon\,\mathrm{e}^{-\frac{\tau}{2}}(y + y_0)$ and denote $R(\tau, \xi, \zeta) = R(\tau, \xi, \varepsilon\mathrm{e}^{-\frac{\tau}{2}}(y + y_0)) = r(\tau, \xi, y)$. Then, by (26),

$$\partial_\tau R = \mathcal{M}R + \partial_{\zeta\zeta}R + \frac{\zeta}{2}\partial_\zeta R + \varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}\left(\phi_\varepsilon R + \psi_\varepsilon Q(\partial_\xi R - \frac{\xi}{4}R) + \alpha\psi_\varepsilon Q(e_0' - \frac{\xi}{4}e_0) + Q(\mathrm{e}^{\frac{\xi^2}{8}}f_\varepsilon)\right).$$

Moreover, by (30), $\|R\|^2_{L^2(L^\infty)} \leqslant C\varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}$ and the parabolic regularity states that for any compact K of $\mathbb{R}^+$, there exists $C_K > 0$ independent of y_0 such that for any $\tau > 0$, $\xi \in K$ and $|\zeta| < 1$,

$$\max\ (|\partial_\tau R|\,, |\partial_\xi R|\,, |\partial_{\xi\xi} R|\,, |\partial_\zeta R|\,, |\partial_{\zeta\zeta} R|) \leqslant C_K\varepsilon^\lambda \mathrm{e}^{-\frac{\lambda}{2}\tau}.$$

Coming back to r, we get

$$\max\ (|\partial_\tau r|\,, |\partial_\xi r|\,, |\partial_{\xi\xi} r|) \leqslant C_K\varepsilon^\lambda \mathrm{e}^{-\frac{\lambda}{2}\tau},$$

while

$$|\partial_y r| \leqslant C_K\varepsilon^{\lambda+1}\mathrm{e}^{-\frac{\lambda+1}{2}\tau}\,, \quad |\partial_{yy} r| \leqslant C_K\varepsilon^{\lambda+2}\mathrm{e}^{-\frac{\lambda+2}{2}\tau}.$$

This implies the lemma with $\tilde{v}(\tau,\xi,y) = \frac{r(\tau,\xi,y)}{\xi}\mathrm{e}^{-\frac{\xi^2}{8}}\mathrm{e}^{\frac{\lambda}{2}\tau}$. ■

4.2. When the initial datum goes to zero as $|y|$ goes to infinity

The result that we are going to prove is much simpler than theorem 4.1. We could use this last result, but we prefer to give a direct approach.

Proposition 4.3. *Let v be a solution of (21), with initial datum v_0 satisfying*

1. $\sup_{y\in\mathbb{R}} \|\mathrm{e}^{\xi^2/8} v_0(\xi,y)\|_{L^2(\mathbb{R}^+)} < +\infty$,
2. $\lim_{y\to\pm\infty} v_0(\xi,y) = 0$, *uniformly in* $\xi \in \mathbb{R}^+$.

Then we have $v(\tau,\xi,y) = \xi\tilde{v}(\tau,\xi,y)$ *with*

$$\lim_{\tau\to+\infty} \|\tilde{v}(\tau,.)\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} = 0.$$

Proof. Let us first make the following simplifying assumption: there is $A > 0$ such that

$$v_0(\xi,y) = 0 \quad \text{if } |y| \geqslant A. \tag{31}$$

This allows us to pass to self-similar variables in y: $\zeta = \varepsilon\frac{y}{\sqrt{t}}$. And so, (21) becomes

$$\begin{aligned} \partial_\tau v = (\mathcal{L}+\mathcal{N})v + \varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}\,(\phi_\varepsilon(\tau)v + \psi_\varepsilon(\tau)\partial_\xi v + f_\varepsilon(\tau,\xi))\,, &\quad \tau>0\,, \xi>0\,, \zeta\in\mathbb{R} \\ v(\tau,0,y) = 0\,, &\quad \tau>0\,, \xi=0\,, \zeta\in\mathbb{R}, \end{aligned} \tag{32}$$

with $\mathcal{N} = \partial_{\zeta\zeta} + \frac{1}{2}\zeta\partial_\zeta$. The spectrum of $\mathcal{N}$, in the space $L^2(\mathbb{R}, \mathrm{e}^{\zeta^2/8}\mathrm{d}\zeta)$, is $\{\frac{k}{2}\,, k\in\mathbb{N}^*\}$. And so, writing $v(\tau,\xi,\zeta) = \mathrm{e}^{-(\xi^2+\zeta^2)/8} w(\tau,\xi,\zeta)$ we obtain the following equation for w:

$$\begin{aligned} &\partial_\tau w = (\mathcal{M}+\mathcal{P})w + \varepsilon^{2\lambda}\mathrm{e}^{-\lambda\tau}\left((\phi_\varepsilon(\tau) - \frac{\xi}{4}\psi_\varepsilon(\tau))w + \psi_\varepsilon(\tau)\partial_\xi w + \mathrm{e}^{\frac{\xi^2+\zeta^2}{8}} f_\varepsilon(\tau,\xi)\right) \\ &w(\tau,0,y) = 0 \quad \tau>0\,, y\in\mathbb{R}, \end{aligned} \tag{33}$$

where $\mathcal{P}w = \partial_{\zeta\zeta}w + \left(\frac{1}{4} - \frac{\xi^2}{16}\right)w$. We have, for all $w(\tau,\xi,\cdot) \in L^2(\mathbb{R})$:

$$\int_{\mathbb{R}} (\mathcal{P}w)w \, \mathrm{d}\zeta \geqslant \frac{1}{2}\|w\|^2_{L^2(\mathbb{R})}.$$

Arguing as in the proof of theorem 4.1, we obtain

$$\|w(\tau,.)\|_{L^2(\mathbb{R}^+\times\mathbb{R})} \leqslant \mathrm{e}^{-\tau/2}\|w(0,.)\|_{L^2(\mathbb{R}^+\times\mathbb{R})}. \tag{34}$$

This proves the convergence to 0 of v. In order to suppress assumption (31), let us notice that, for all $\delta > 0$, the function $(v_0(\xi, y) - \delta)^+$ satisfies (31). Moreover, due to the convexity of $v \mapsto (v - \delta)^+$, the function $(v(\tau, \xi, \zeta) - \delta)^+$ is a sub-solution of (33). And so, we have $v(\tau, \xi, \zeta) \leqslant \overline{v}^\delta(\tau, \xi, \zeta)$ where $\overline{v}(\tau, \xi, \zeta)$ solves (33) with initial datum $(v_0(\xi, y) - \delta)^+$. So $\overline{v}^\delta$ satisfies (34), which entails, by elliptic regularity, its convergence to 0 on every compact subset of $\mathbb{R}_+ \times \mathbb{R}$. Because the zero-order coefficients of the equation (32) are positive at infinity, the convergence holds in fact in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$. By elliptic regularity, this is also true for $\partial_\xi v$. The mean value theorem implies the result. ■

5. General large time asymptotics for the full KPP equation, proof of theorem 1.2

Let $u_0 \in \mathcal{C}(\mathbb{R}^2)$ satisfy assumption (2), i.e. trapped between two translates of $1 - H$. Denote u the unique classical solution to (1) emanating from u_0 at time $t = 1$.

As announced in the introduction, we shall construct two functions $\bar{u}(t, x, y)$ and $\underline{u}(t, x, y)$, defined for $t > 1$, $\{x \leqslant t^\delta\}$ (with δ small to be chosen later) and $y \in \mathbb{R}$, by the following procedure. We will solve equation (1) inside this region, and we will impose, as a Dirichlet condition at $\{x = t^\delta\}$, the value of a well chosen approximate solution of (1) in the diffusive zone. We will see, in the next sections, that the functions $\bar{u}(t, x, y)$ and $\underline{u}(t, x, y)$ actually mimic the behaviour of the true solution $u(t, x, y)$.

It will, however, be convenient to work in the self-similar coordinates. Let $w(\tau, \xi, \eta)$ be defined as in section 2. Recall that w satisfies (10) with initial condition $w(0, \xi, \eta) = \mathrm{e}^\xi u_0(\xi + 2, y)$.

We will need the following frame, borrowed from [20]. Under the assumption (2), there are functions $\eta_\pm(\tau)$ and $q_\pm(\tau)$, and constants $0 < \eta_0 < \eta_1$, depending only on x_1 and x_2, satisfying

$$\eta_0 \leqslant \eta_-(\tau) \leqslant \eta_+(\tau) \leqslant \eta_1, \quad q_\pm(\tau) = O(\mathrm{e}^{-\frac{\tau}{4}}),$$

and such that for any $\tau > 0$, $\xi > \xi_\delta$,

$$\eta_-(\tau)\xi\mathrm{e}^{-\frac{\xi^2}{4}} - q_-(\tau)\xi\mathrm{e}^{-\frac{\xi^2}{7}} \leqslant w(\tau, \xi, y) \leqslant \eta_+(\tau)\xi\mathrm{e}^{-\frac{\xi^2}{4}} + q_+(\tau)\xi\mathrm{e}^{-\frac{\xi^2}{7}}\mathrm{e}^{-\mathrm{e}^{\delta\tau}}. \tag{35}$$

To see it, it suffices to apply the paragraphs 'an upper barrier' in [20] to the solution of the 1D KPP equation emanating from $1 - H(x - x_1)$ and 'a lower barrier' to that emanating from $1 - H(x - x_2)$ and apply the comparison principle.

In the sequel, for every small $\varepsilon > 0$, we will set

$$T_\varepsilon = \varepsilon^{-2} \text{ and } \tau_\varepsilon = \ln T_\varepsilon \text{ (notice that } \varepsilon = \mathrm{e}^{-\frac{\tau_\varepsilon}{2}}). \tag{36}$$

In the next two sections, we will seek to apply theorem 4.1 with the initial datum

$$w(\tau_\varepsilon, \xi, y) = \mathrm{e}^\xi \mathrm{e}^{\frac{T_\varepsilon}{2}} u(T_\varepsilon, \xi + 2, y). \tag{37}$$

Due to (35), we will be able to control this initial condition.

5.1. Diffusive supersolution

For any $\delta \in (0, \frac{1}{2})$, define $\xi_\delta = \mathrm{e}^{-(\frac{1}{2}-\delta)\tau}$ which corresponds to $x = t^\delta$ in self similar coordinates. Let $\bar{w}$ the solution to

$$\begin{aligned}\partial_\tau \bar{w} = \mathcal{L}\bar{w} + \mathrm{e}^\tau \partial_{yy}\bar{w} - \frac{3}{2}\mathrm{e}^{-\frac{\tau}{2}}\partial_\xi \bar{w} \quad & \tau \geqslant \tau_\varepsilon\,,\ \xi > -\xi_\delta\,,\ y \in \mathbb{R} \\ \bar{w}(\tau, \xi_\delta, y) = \mathrm{e}^{-\mathrm{e}^{\delta\tau}} \quad & \tau \geqslant \tau_\varepsilon\,,\ \xi = -\xi_\delta\,,\ y \in \mathbb{R} \\ \bar{w}(\tau_\varepsilon, \xi, y) = w(\tau_\varepsilon, \xi, y) \quad & \tau = \tau_\varepsilon\,,\ \xi > -\xi_\delta\,,\ y \in \mathbb{R}.\end{aligned} \tag{38}$$

Then, $\bar{w}$ is a supersolution to (10) for $\xi > -\xi_\delta$. Indeed, by definition (7)

$$w(\tau, \xi, y) = \mathrm{e}^{-\frac{\tau}{2}+\xi \mathrm{e}^{\frac{\tau}{2}}} u_1(\mathrm{e}^\tau, \xi \mathrm{e}^{\frac{\tau}{2}}, y),$$

the function u_1 being strictly uniformly bounded by 0 and 1. It follows that

$$\forall \tau \geqslant 0\,,\ \forall y \in \mathbb{R}\,,\quad 0 < w(\tau, -\xi_\delta, y) < \mathrm{e}^{-\mathrm{e}^{\delta\tau}}.$$

We have $\partial_\tau w(\tau, -\xi_\delta, y) = \left(\partial_t u_1 \mathrm{e}^{\frac{\tau}{2}} - \frac{1}{2}(u_1 + \partial_x u_1)\mathrm{e}^{-(\frac{1}{2}-\delta)\tau} - \frac{1}{2}u_1 \mathrm{e}^{-\frac{\tau}{2}}\right) \mathrm{e}^{-\mathrm{e}^{\delta\tau}}$ gives for $\delta > 0$ small enough

$$\forall \tau \geqslant 0\,,\ \forall y \in \mathbb{R}\,,\quad |\partial_\tau w(\tau, -\xi_\delta, y)| \leqslant C\mathrm{e}^{-\delta \mathrm{e}^{\delta\tau}}.$$

To simplify the moving Dirichlet boundary $\xi = \xi_\delta = \mathrm{e}^{-(\frac{1}{2}-\delta)\tau}$, we introduce a change of variables:

$$\bar{w}(\tau, \xi, y) = \bar{p}(\tau - \tau_\varepsilon, \xi + \xi_\delta, y) + \mathrm{e}^{-\mathrm{e}^{\delta\tau}}\chi(\xi + \xi_\delta)$$

where τ_ε is defined in (36) and χ is a smooth monotonic function such that $\chi(\eta) = 1$ for $\eta \in [0, 1)$ and $\chi(\eta) = 0$ for $\eta > 2$. The function $\bar{p}(\tau', \eta, y)$ then satisfies (removing the primes) for any $\tau > 0$, $\eta > 0$ and $y \in \mathbb{R}$,

$$\begin{aligned}\partial_\tau \bar{p} = \mathcal{L}\bar{p} + \frac{\mathrm{e}^\tau}{\varepsilon^2}\partial_{yy}\bar{p} + \varepsilon^{1-2\delta}\mathrm{e}^{-(\frac{1}{2}-\delta)\tau}\left(-\left(\delta + \frac{3}{2}\varepsilon^{2\delta}\mathrm{e}^{-\delta\tau}\right)\partial_\eta \bar{p} + \Xi_\varepsilon(\tau, \eta)\right) & \\ \bar{p}(\tau, 0, y) = 0 \quad \tau > 0\,,\ \eta = 0\,,\ y \in \mathbb{R} & \\ \bar{p}(0, \eta, y) = w(\tau_\varepsilon, \eta - \varepsilon^{1-2\delta}, y) - \mathrm{e}^{-1/\varepsilon^{2\delta}}\chi(\eta) \quad \tau = 0\,,\ \eta > 0\,,\ y \in \mathbb{R} &\end{aligned} \tag{39}$$

where Ξ_ε is a smooth function supported in $\eta \in [0, 2]$ and uniformly bounded:

$$\exists C_\delta > 0 \,|\, \forall \varepsilon > 0\,,\ \forall \tau \geqslant 0\,,\ \forall \eta \geqslant 0\,,\quad |\Xi_\varepsilon(\tau, \eta)| \leqslant C_\delta.$$

Choose $\lambda = \frac{1}{2} - \delta > 0$, $\phi_\varepsilon = 0$, $\psi_\varepsilon(\tau) = -(\delta + \frac{3}{2}\varepsilon^{2\delta}\mathrm{e}^{-\delta\tau})$ uniformly bounded in τ and ε and $f_\varepsilon = \Xi_\varepsilon$ compactly supported in η and uniformly bounded in τ and ε. Then, applying theorem 4.1, we have for $\tau > \tau_\varepsilon$, $\xi > -\xi_\delta$, $y \in \mathbb{R}$,

$$\bar{w}(\tau, \xi, y) = (\xi + \xi_\delta)\left(\frac{\mathrm{e}^{-\frac{(\xi+\xi_\delta)^2}{4}}}{\sqrt{2\sqrt{\pi}}}\left(\bar{\alpha}_c(\tau - \tau_\varepsilon, y) + \bar{\beta}(\tau - \tau_\varepsilon, y)\right) + \mathrm{e}^{-\frac{\lambda}{2}(\tau - \tau_\varepsilon)}\tilde{p}(\tau - \tau_\varepsilon, \xi + \xi_\delta, y)\right)$$

where for any $\tau > 0$ and $y \in \mathbb{R}$

$$\partial_\tau \bar{\alpha}_c = \frac{\mathrm{e}^\tau}{\varepsilon^2}\partial_{yy}\bar{\alpha}_c\,,\quad \bar{\alpha}_c(0, y) = \frac{1}{\sqrt{2\sqrt{\pi}}}\int_0^{+\infty} \eta\left(w(\tau_\varepsilon, \eta - \varepsilon^{1-2\delta}, y) - \mathrm{e}^{-1/\varepsilon^{2\delta}}\chi(\eta)\right)\mathrm{d}\eta$$

and for any $\tau > 0$

$$\|\bar{\beta}(\tau)\|_{L^\infty(\mathbb{R})} \leqslant C\varepsilon^{1-2\delta}\,, \|\partial_\tau \bar{\beta}(\tau)\|_{L^\infty(\mathbb{R})} \leqslant C\varepsilon^{1-2\delta}$$
$$\|\partial_y \bar{\beta}(\tau)\|_{L^\infty} \leqslant C\varepsilon^{2-2\delta}\mathrm{e}^{-\frac{\tau}{2}}\,, \|\partial_{yy}\bar{\beta}(\tau)\|_{L^\infty} \leqslant C\varepsilon^{3-2\delta}\mathrm{e}^{-\tau}$$

and for any $\tau > 0$, $\xi \in K$ compact set of $\mathbb{R}^+$, $y \in \mathbb{R}$

$$\max\left(|\tilde{p}(\tau,\xi,y)|, |\partial_\tau \tilde{p}|\,, |\partial_\xi \tilde{p}|\,, |\partial_{\xi\xi}\tilde{p}|\right) \leqslant C_K \varepsilon^{\frac{1}{2}-\delta}$$
$$|\partial_y \tilde{p}| \leqslant C_K \varepsilon^{\frac{3}{2}-\delta}\mathrm{e}^{-\frac{\tau}{2}}\,, \quad |\partial_{yy}\tilde{p}| \leqslant C_K\varepsilon^{\frac{5}{2}-\delta}\mathrm{e}^{-\tau}.$$

5.2. Diffusive subsolution

Since $0 < w(\tau,\xi,y) \leqslant \bar{w}(\tau,\xi,y) \leqslant C(\xi+\xi_\delta)$ for some large $C > 0$ and $\tau \geqslant \tau_\varepsilon$, the nonlinear term in (10) can be bounded as follows: for any $\xi > \xi_\delta > \mathrm{e}^{-\frac{\tau}{2}}$

$$e^{\frac{3}{2}\tau - \xi \mathrm{e}^{\frac{\tau}{2}}} w^2 \leqslant C(\xi+\xi_\delta) e^{\frac{3}{2}\tau - \xi\mathrm{e}^{\frac{\tau}{2}}} w \leqslant 2Ce^{\frac{3}{2}\tau}\xi_\delta \mathrm{e}^{-\xi_\delta \mathrm{e}^{\frac{\tau}{2}}} \leqslant 2C\mathrm{e}^{(1+\delta)\tau}\mathrm{e}^{-\mathrm{e}^{\delta\tau}} w \leqslant C_0 e^{-(\frac{1}{2}-\delta)\tau} w$$

so that a subsolution to (10) is given by

$$\begin{aligned} \partial_\tau \underline{w} = \mathcal{L}\underline{w} + \mathrm{e}^\tau \partial_{yy}\underline{w} - \frac{3}{2}\mathrm{e}^{-\frac{\tau}{2}}\partial_\xi \underline{w} + C_0 e^{-(\frac{1}{2}-\delta)\tau}\underline{w}\,, &\quad \tau > \tau_\varepsilon\,,\ \xi > \xi_\delta\,,\ y \in \mathbb{R} \\ \underline{w}(\tau,\xi_\delta,y) = 0\,, &\quad \tau > \tau_\varepsilon\,,\ \xi = \xi_\delta\,,\ y \in \mathbb{R} \\ \underline{w}(\tau_\varepsilon,\xi,y) = w(\tau_\varepsilon,\xi,y)\,, &\quad \tau = \tau_\varepsilon\,,\ \xi > \xi_\delta\,,\ y\in\mathbb{R}. \end{aligned} \tag{40}$$

Let us study its beaviour as $\tau \to +\infty$. As in the previous section, we simplify the moving Dirichlet boundary by defining $\eta = \xi - \xi_\delta$, $\tau' = \tau - \tau_\varepsilon$ and set $\underline{w}(\tau,\xi,y) = \underline{p}(\tau',\eta,y) = \underline{p}(\tau-\tau_\varepsilon, \xi-\xi_\delta, y)$. Then, $\underline{p}$ satisfies (after dropping the primes) for any $\tau > 0$, $\eta > 0$ and $y \in \mathbb{R}$,

$$\begin{aligned} \partial_\tau \underline{p} = \mathcal{L}\underline{p} + \frac{\mathrm{e}^\tau}{\varepsilon^2}\partial_{yy}\underline{p} + \varepsilon^{1-2\delta} e^{-(\frac{1}{2}-\delta)\tau}\left(C_0\underline{p} + (\delta - \frac{3}{2}\varepsilon^{2\delta}\mathrm{e}^{-\delta\tau})\partial_\eta \underline{p}\right) \\ \underline{p}(\tau,0,y) = 0\,, \quad \tau \geqslant 0\,,\ \eta = 0\,,\ y\in\mathbb{R} \\ \underline{p}(0,\eta,y) = w(\tau_\varepsilon, \eta+\varepsilon^{1-2\delta}, y)\,, \quad \tau = 0\,,\ \eta > 0\,,\ y \in \mathbb{R}. \end{aligned} \tag{41}$$

Choose $\lambda = \frac{1}{2} - \delta > 0$, $\phi_\varepsilon = C_0$, $\psi_\varepsilon = \delta - \frac{3}{2}\varepsilon^{2\delta}\mathrm{e}^{-\delta\tau}$ uniformly bounded in τ and ε and $f_\varepsilon = 0$. Then, applying theorem 4.1, we have for $\tau > \tau_\varepsilon$, $\xi > \xi_\delta$ and $y \in \mathbb{R}$,

$$\underline{w}(\tau,\xi,y) = (\xi-\xi_\delta)\left(\frac{\mathrm{e}^{-(\xi-\xi_\delta)^2/4}}{\sqrt{2\sqrt{\pi}}}\left(\underline{\alpha}_c(\tau-\tau_\varepsilon,y) + \underline{\beta}(\tau-\tau_\varepsilon,y)\right) + \mathrm{e}^{-\frac{\lambda}{2}(\tau-\tau_\varepsilon)}\tilde{q}(\tau-\tau_\varepsilon,\xi-\xi_\delta,y)\right);$$

where for any $\tau > 0$ and $y \in \mathbb{R}$:

$$\partial_\tau \underline{\alpha}_c = \frac{\mathrm{e}^\tau}{\varepsilon^2}\partial_{yy}\underline{\alpha}_c\,, \quad \underline{\alpha}_c(0,y) = \frac{1}{\sqrt{2\sqrt{\pi}}}\int_0^{+\infty} \eta\, w(\tau_\varepsilon, \eta+\varepsilon^{1-2\delta}, y)\mathrm{d}\eta;$$

and for any $\tau > 0$,

$$\|\underline{\beta}(\tau)\|_{L^\infty(\mathbb{R})} \leqslant C\varepsilon^{1-2\delta}\,, \|\partial_\tau \underline{\beta}(\tau)\|_{L^\infty(\mathbb{R})} \leqslant C\varepsilon^{1-2\delta}$$
$$\|\partial_y \underline{\beta}(\tau)\|_{L^\infty} \leqslant C\varepsilon^{2-2\delta}\mathrm{e}^{-\frac{\tau}{2}}\,, \|\partial_{yy}\underline{\beta}(\tau)\|_{L^\infty} \leqslant C\varepsilon^{3-2\delta}\mathrm{e}^{-\tau};$$

and for any $\tau > 0$, $\xi \in K$ compact set of $\mathbb{R}^+$, $y \in \mathbb{R}$

$$\max\left(|\tilde{q}(\tau,\xi,y)|, |\partial_\tau \tilde{q}|, |\partial_\xi \tilde{q}|, |\partial_{\xi\xi}\tilde{q}|\right) \leqslant C_K \varepsilon^{\frac{1}{2}-\delta}$$
$$|\partial_y \tilde{q}| \leqslant C_K \varepsilon^{\frac{3}{2}-\delta} \mathrm{e}^{-\frac{\tau}{2}}, \quad |\partial_{yy}\tilde{q}| \leqslant C_K \varepsilon^{\frac{5}{2}-\delta} \mathrm{e}^{-\tau}.$$

5.3. The proof of theorem 1.2

It is now, just a matter of applying the preceding sections in the right order. Note that we have for any $\tau > \tau_\varepsilon$, $\xi > \xi_\delta$ and $y \in \mathbb{R}$,

$$0 \leqslant \overline{w}(\tau,\xi,y) - \underline{w}(\tau,\xi,y) \leqslant C\varepsilon^{1-2\delta}.$$

Define $\overline{u}_+$ and $\underline{u}_+$ the function corresponding to $\overline{w}(\tau,0,y)$ and $\underline{w}(\tau,2\xi_\delta,y)$ in the moving frame (see (7) to (9)):

$$\overline{u}_+(t,y) = \mathrm{e}^{-t^\delta} t^{1/2} \overline{w}(\ln t, 0, y), \quad \underline{u}_+(t,y) = \mathrm{e}^{-\delta} t^{1/2} \underline{w}(\ln t, 2t^{-(\frac{1}{2}-\delta)}, y).$$

Both $\overline{u}_+$ and $\overline{u}_-$ have the form (20), with estimate (19) and assumptions (13) and (14). Indeed, (dealing for instance with $\overline{u}_+$, and the same holds for $\underline{u}_+$)

$$\overline{u}_+(t,y) = t^\delta \mathrm{e}^{-t^\delta - 1/4t^{1-2\delta}} \frac{\overline{a}(t,y) + \overline{b}(t,y)}{\sqrt{2\sqrt{\pi}}}$$

where $\overline{a}(t,y) = \overline{\alpha}_c(\ln(t\varepsilon^2), y)$ satisfies $\partial_t \overline{a} = \partial_{yy}\overline{a}$ for any $t > 1$ with $\overline{a}(1,y) = \overline{\alpha}_c(0,y)$ and $|b(t,y)| \leqslant C(\varepsilon^{1-2\delta} + 1/t^{\frac{1}{4}-\delta/2})$. $\overline{a}$ satisfies (13) and (14) thanks to (35). Proposition 3.2 and theorem 3.1 therefore imply

$$U_{c_*}(x - \ln(\underline{a}(t,y) - C\varepsilon^{1-2\delta})) - \frac{C}{\sqrt{t}} \leqslant u(t,x,y) \leqslant U_{c_*}(x - \ln(\overline{a}(t,y) + C\varepsilon^{1-2\delta})) + \frac{C}{\sqrt{t}}$$
$$|\overline{a}(t,y) - \underline{a}(t,y)| \leqslant C\varepsilon^{1-2\delta}.$$

Now we choose

$$a_0^\varepsilon(y) = \underline{a}(1,y) = \underline{\alpha}_c(0,y) = \frac{1}{\sqrt{2\sqrt{\pi}}} \int_0^{+\infty} \eta\, w(\tau_\varepsilon, \eta + \varepsilon^{1-2\delta}, y) \mathrm{d}\eta,$$

this finishes the proof. ∎

6. Examples of convergence and nonconvergence

This section is devoted to the consequences of theorem 1.2, i.e the proof of theorem 1.3. We will first give an example of nonconvergence by exploiting the fact that some solutions of the heat equation do not converge to anything. In the next three sub-sections, we will give various cases of convergence: the simplest one is that of an initial datum tending, as $|y| \to \infty$, to a unique translate of $1 - H$. The next one is when the initial datum tends to a y-periodic translate of $1 - H$. The last one is when the initial datum tends to two different limits as $y \to \pm\infty$: here, we will still have convergence, but only on compact sets in y.

6.1. Suitably oscillating initial data

The starting point of our construction is the following solution to the standard heat equation—see [8, 27], where similar phenomena are discussed:

$$\partial_t a = \partial_{yy} a, \quad \text{or, with the change of variables } \tau = \ln(t): \quad \partial_\tau \alpha = e^\tau \partial_{yy} \alpha,$$

with initial datum $\alpha(0, y) = a(1, y) = \alpha_M(y)$, $M > 1$ will be chosen later. Consider two sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{Z}}$ satisfying the following five requirements:

1. $x_n = x_{-n}$ for $n \in \mathbb{N}$.
2. The sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are increasing.
3. $\lim\limits_{n\to+\infty} \frac{x_{n+1}}{x_n} = +\infty$.
4. $\lim\limits_{n\to+\infty} \frac{x_n^2}{t_n} = 0$, $\lim_{n\to+\infty} \frac{x_{n+1}^2}{t_n} = +\infty$.
5. For $n \in \mathbb{N}$, $\alpha_M \equiv 1$ on (x_{2n}, x_{2n+1}), $\alpha_M \equiv M$ on (x_{2n+1}, x_{2n+2}) and α_M even.

An example is $t_n = \sqrt{n}(n!)$, $x_n = \sqrt{n!}$. We then have

$$\lim_{n\to+\infty} a(t_{2n}, 0) = 1, \qquad \lim_{n\to+\infty} a(t_{2n+1}, 0) = M > 1. \tag{42}$$

Indeed, we have for $t > 1$ and $y \in \mathbb{R}$

$$a(t, y) = \frac{1}{\sqrt{4\pi(t-1)}} \int_{\mathbb{R}} e^{-(y-y')^2/4(t-1)} \alpha_M(y') dy' = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2} \alpha_M(y + 2z\sqrt{t-1}) dz,$$

and so, because α_M is even, this reduces to

$$a(t, 0) = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-z^2} \alpha_M(2z\sqrt{t-1}) dz.$$

Now, use the fact that $\alpha_M(y) = \bar{\alpha}_M \in \{1, M\}$ on (x_n, x_{n+1}):

$$\begin{aligned} a(t_n, 0) = \frac{2}{\sqrt{\pi}} \bar{\alpha}_M \int_{x_n/(2\sqrt{t_n-1})}^{x_{n+1}/(2\sqrt{t_n-1})} e^{-z^2} dz + \frac{2}{\sqrt{\pi}} \int_0^{x_n/(2\sqrt{t_n-1})} e^{-z^2} \alpha_M(2z\sqrt{t_n-1}) dz \\ + \frac{2}{\sqrt{\pi}} \int_{x_{n+1}/(2\sqrt{t_n-1})}^{+\infty} e^{-z^2} \alpha_M(2z\sqrt{t_n-1}) dz. \end{aligned}$$

Because of requirement four and the dominated convergence theorem, the last two terms go to zero as $n \to +\infty$. And so we have

$$a(t_n, 0) = \frac{2\bar{\alpha}_M}{\sqrt{\pi}} \int_{x_n/(2\sqrt{t_n-1})}^{x_{n+1}/(2\sqrt{t_n-1})} e^{-z^2} dz + o_{n\to+\infty}(1) = \bar{\alpha}_M + o_{n\to+\infty}(1).$$

This proves (42). Consider now the diffusive super and sub solutions $\overline{w}(\tau, \xi, y)$ and $\underline{w}(\tau, \xi, y)$ constructed in section 5, and respectively defined by (38) and (40), with the common initial datum at time $\tau = 0$

$$\overline{w}(0, \xi, y) = \underline{w}(0, \xi, y) = \lambda \alpha_M(y)(1 - H(\xi)),$$

where H is the Heaviside function, and $\lambda > 0$ will be adjusted as the discussion proceeds. We have

$$\Big(\underline{w}(\tau, \xi, y), \overline{w}(\tau, \xi, y)\Big) = \lambda \alpha(\tau, y) \Big(\underline{W}(\tau, \xi), \overline{W}(\tau, \xi)\Big)$$

where $\overline{W}$ and $\underline{W}$ solve, respectively, (38) and (40) with no term ∂_{yy}. From theorem 4.1, there is $0 < \underline{\Lambda}^\infty \leqslant \bar{\Lambda}^\infty$ such that

$$\left(\underline{W}(\tau,\xi), \bar{W}(\tau,\xi)\right) \to_{\tau\to+\infty} \left(\underline{\Lambda}^\infty, \bar{\Lambda}^\infty\right)\xi \mathrm{e}^{-\xi^2/4}.$$

Notice $\underline{\Lambda}^\infty > 0$ since we choose $\alpha_M \geqslant 1 > 0$. We choose $M > 0$ large enough so that

$$M\underline{\Lambda}^\infty > \bar{\Lambda}^\infty.$$

And, finally, we choose $\lambda > 0$ such that

$$u(1,x,y) = \mathrm{e}^{-x}\lambda\alpha_M(y)(1 - H(x)) \leqslant 1 - H(x).$$

So we have

$$\begin{aligned}
\bar{u}(t,1,0) = &\quad e^{-1}\sqrt{t}\,\bar{w}(\ln t, 1/\sqrt{t}, 0) \sim_{t\to+\infty} \bar{\Lambda}^\infty e^{-1}\lambda a(t,0)\\
\underline{u}(t,1,0) = &\quad e^{-1}\sqrt{t}\,\underline{w}(\ln t, 1/\sqrt{t}, 0) \sim_{t\to+\infty} \underline{\Lambda}^\infty e^{-1}\lambda a(t,0).
\end{aligned}$$

Because $\underline{u}(t,x,y) \leqslant u(t,x,y) \leqslant \bar{u}(t,x,y)$ we have, in the end:

$$\limsup_{n\to+\infty} u(t_{2n},1,0) \leqslant \lambda e^{-1}\bar{\Lambda}^\infty, \quad \liminf_{n\to+\infty} u(t_{2n+1},1,0) \geqslant \lambda e^{-1}M\underline{\Lambda}^\infty.$$

Thus,

$$\liminf_{n\to+\infty} u(t_{2n+1},1,0) > \limsup_{n\to+\infty} u(t_{2n},1,0),$$

which is our counterexample and proves theorem 1.3(1). ■

6.2. Initial data tending to a limit

Let us consider u_0 such that

$$\lim_{y\to\pm\infty} u_0(x,y) = u_0^+(x),$$

uniformly with respect to $x \in \mathbb{R}$. Recall that, for compatibility with (2), we should have

$$1 - H(x - x_2) \leqslant u_0^+(x) \leqslant 1 - H(x - x_1).$$

Let $u^+(t,x)$ be the one-dimensional solution of (7) emanating from u_0^+ and σ_∞ (see (11)) such that

$$u^+(t,x) \longrightarrow_{t\to+\infty} U_{c_*}(x + \sigma_\infty).$$

Standard arguments from the theory of semilinear parabolic equations yield

$$\lim_{y\to\pm\infty} u(t,x,y) = u^+(t,x),$$

uniformly in x and locally uniformly in t. Let $w(\tau,\xi,y)$ be defined by (9), and $w^+(\tau,\xi)$ be the corresponding 1D solution. We still have

$$\lim_{y\to\pm\infty} w(\tau,\xi,y) = w^+(\tau,\xi),$$

uniformly in ξ and locally uniformly in τ. Consider

$$\tilde{w}(\tau,\xi,y) = w(\tau,\xi,y) - w^+(\tau,\xi),$$

and $\varepsilon > 0$. For $\tau \geqslant \tau_\varepsilon = -2\ln\varepsilon$, the function $\tilde{w}$ falls in the assumptions of proposition 3.2. So,

$$\lim_{\tau\to+\infty} \tilde{w}(\tau,\xi,y) = 0,$$

uniformly in ξ and y. This translates to $\tilde{u}(t,x,y) = u(t,x,y) - u^+(t,x)$.

6.3. Initial data that are asymptotically periodic in y

Consider first an initial datum $u_0(x,y)$ that is periodic in y. The function $\alpha_c(\tau,y)$ defined in theorem 4.1 tends as $\tau\to+\infty$ to the average of its initial datum. The ω-limit set of u_0 for the full system (10) is therefore made up of functions of the form $\alpha\xi^+\mathrm{e}^{-\xi^2/4}$. Because of the stability of these functions under the asymptotic equation of (10), the set $\omega(u_0)$ is made up of only one of these functions, say $\alpha_\infty\xi^+\mathrm{e}^{-\xi^2/4}$.

Let now be $u_0(x,y)$ and $u_0^+(x,y)$ such that

$$\lim_{y\to\pm\infty}\Big(u_0(x,y) - u_0^+(x,y)\Big) = 0, \quad \text{uniformly in } x.$$

Let $u^+(t,x,y)$ be the solution emanating from $u_0^+(x,y)$ and, as before,

$$\tilde{u}(t,x,y) = u(t,x,y) - u^+(t,x,y).$$

Arguing as in the preceding section, we obtain the uniform convergence of $\tilde{u}$ to 0 as $t\to+\infty$ and prove theorem 1.3(3). ■

6.4. Initial data tending to two different limits

Let us consider u_0 such that

$$\lim_{y\to+\infty} u_0(x,y) = u_0^+(x), \qquad \lim_{y\to-\infty} u_0(x,y) = u_0^-(x),$$

uniformly with respect to $x\in\mathbb{R}$. Recall that, for compatibility with assumption (2), we should have

$$1 - H(x-x_2) \leqslant u_0^+(x), u_0^-(x) \leqslant 1 - H(x-x_1).$$

Let us come back to equation (10). We use the self-similar variable $\zeta = \frac{y}{\sqrt{t}}$, and discover that the function $\alpha_c(\tau,\zeta)$ tends, as $\tau\to+\infty$, to α_c^∞, the unique solution of

$$\begin{aligned} -\frac{\mathrm{d}^2\alpha_c^\infty}{\mathrm{d}\zeta^2} - \frac{\zeta}{2}\frac{\mathrm{d}\alpha_c^\infty}{\mathrm{d}\zeta} &= 0, \quad \zeta\in\mathbb{R} \\ \alpha_c(\pm\infty) &= \mathrm{e}^{-\sigma_\infty^\pm}. \end{aligned}$$

We have $\alpha_c(0) = \frac{\mathrm{e}^{-\sigma_\infty^+}+\mathrm{e}^{-\sigma_\infty^-}}{2}$, which is the pointwise limit of $\alpha_c(\tau,y)$. And from theorem 4.1, we have

$$\lim_{\tau\to+\infty} w(\tau,\xi,\zeta) = \alpha_c^\infty(\zeta)\xi^+\mathrm{e}^{-\zeta^2/4}.$$

Undoing this and reverting to u proves theorem 1.3(2). ■

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