Long-Time Asymptotics of Navier-Stokes and Vorticity Equations in a three-dimensional Layer

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Abstract

We study the long-time behavior of solutions of the Navier-Stokes equation in $\mathbb{R}^2 \times (0,1)$. After introducing self-similar variables, we compute the asymptotics of the vorticity up to second order and prove that they are governed by the two-dimensional Navier-Stokes equation. In particular, we show that the solutions converge towards Oseen vortices.

We consider the motion of an incompressible viscous fluid filling a three dimensional layer $\mathbb{R}^2 \times (0,1)$. If no external force is applied, the velocity field u and the pressure p of the fluid are given by the Navier-Stokes equation

$$\partial_t u + (u \cdot \nabla) u = \Delta u - \nabla p\,, \quad \text{div } u = 0\,, \quad (x,z) \in \mathbb{R}^2 \times (0,1)\,,\, t \geq 0\,, \tag{1}$$

where $u = u(x,z,t) \in \mathbb{R}^3$ and $p(x,z,t) \in \mathbb{R}$, together with initial and boundary conditions

$$\begin{aligned} u(x,z,0) &= u_0(x,z)\,, && (x,z) \in \mathbb{R}^2 \times (0,1)\,, \\ u(x,z+1,t) &= u(x,z,t)\,, && (x,z,t) \in \mathbb{R}^3 \times \mathbb{R}^+\,. \end{aligned}$$

For simplicity, the kinematic viscosity has been rescaled to 1. As no external force is applied, the velocity converges towards an equilibrium and we study its asymptotic behavior. The first key idea is to use the vorticity formulation. Setting $\omega = \text{rot } u$, equation (1) is transformed into

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = \Delta \omega\,, \quad \text{div } \omega = 0\,, \tag{2}$$

where the velocity field u can be reconstructed from ω via the Biot-Savart law (see V. Roussier [2]), together with initial and boundary conditions

$$\begin{aligned} \omega(x,z,0) &= \omega_0(x,z)\,, && (x,z) \in \mathbb{R}^2 \times (0,1)\,, \\ \omega(x,z+1,t) &= \omega(x,z,t)\,, && (x,z,t) \in \mathbb{R}^3 \times \mathbb{R}^+\,. \end{aligned}$$

The other key idea comes from methods of infinite dynamical systems and spectral projections: we express the vorticity ω and the velocity u in terms of self-similar variables (ξ, z, τ) defined by

$$\begin{aligned} \xi &= \tfrac{x}{\sqrt{1+t}}\,,\ \tau = \log(1+t)\,, \\ \omega(x,z,t) &= \tfrac{1}{1+t}\, w\left(\tfrac{x}{\sqrt{1+t}}, z, \log(1+t)\right), \\ u(x,z,t) &= \tfrac{1}{\sqrt{1+t}}\, v\left(\tfrac{x}{\sqrt{1+t}}, z, \log(1+t)\right). \end{aligned}$$

Then, w satisfies the following non-autonomous equation

$$\partial_\tau w = \Lambda(\tau)w + N(w)(\tau)\,, \quad \mathrm{div}_\tau w(\tau) = 0\,, \quad \tau \geq 0\,, \tag{3}$$

where

$$\begin{aligned}
\Lambda(\tau) &= L + e^\tau \partial_z^2 \\
L &= \Delta_\xi + \frac{1}{2}\xi \cdot \nabla_\xi + 1 \\
N(w)(\tau) &= (w \cdot \nabla_\tau)v - (v \cdot \nabla_\tau)w \\
\mathrm{div}_\tau w(\tau) &= \nabla_\tau \cdot w \\
\nabla_\tau &= (\partial_{\xi_1}, \partial_{\xi_2}, e^{\frac{\tau}{2}}\partial_z)^T
\end{aligned}$$

and the velocity field v is given in terms of w by the Biot-Savart law (see V. Roussier [2]). Existence and uniqueness of solutions of (3) for small initial data proceed from the following theorem:

Theorem 0.1 *Let $m > 1$. There exists $K_0 > 0$ such that, for all initial data $w_0 \in L^2(m)$ with div $w_0 = 0$ and $\|w_0\|_m \leq K_0$, equation (3) has a unique global solution $w \in C^0([0,+\infty); L^2(m))$ satisfying $w(0) = w_0$ and for any $\tau \geq 0$, $\mathrm{div}_\tau w(\tau) = 0$. In addition, there exists $K_1 > 0$ such that*

$$\|w(\tau)\|_m \leq K_1 \|w_0\|_m\,, \quad \tau \geq 0\,, \tag{4}$$

where the weighted Lebesgue space $L^2(m)$ is defined by

$$\begin{aligned}
L^2(m) &= \{f(\xi,z) : \mathbb{R}^3 \to \mathbb{R}^3 \,|\, f \textit{ is 1-periodic in } z,\ \|f\|_m < \infty\}\,, \\
\|f\|_m &= \left(\int_{\mathbb{R}^2}\int_0^1 (1+|\xi|^2)^m |f(\xi,z)|^2 dz d\xi\right)^{\frac{1}{2}}\,.
\end{aligned}$$

Additionally, the asymptotics of (3) are driven by the two-dimensional Navier-Stokes equation studied by Th. Gallay and C.E. Wayne [1]. Thanks to the previous manipulations, the operator L has in $L^2(m)$ remarkable spectral properties . For instance, if $m > 1$, 0 is a simple isolated eigenvalue of L with eigenfunction $\mathbf{G} = (0,0,G)^T$ where $G(\xi) = \frac{1}{4\pi}\exp(-\frac{|\xi|^2}{4})$ for $\xi \in \mathbb{R}^2$. The corresponding velocity is called Oseen vortex. For $m > 2$, $-\frac{1}{2}$ is an isolated eigenvalue of multiplicity 3 with eigenfunctions $\mathbf{F}_1 = (0,0,\partial_1 G)^T$, $\mathbf{F}_2 = (0,0,\partial_2 G)^T$ and $\mathbf{F}_3 = (-\partial_2 G, \partial_1 G, 0)^T$. Thus, the long-time asymptotics in a neigborhood of the origin are determined, up to second order, by a finite system of ordinary differential equations. Indeed, we prove in [2] the following theorems which drive the asymptotics of w.

Theorem 0.2 *Let $0 < \mu < \frac{1}{2}$ and $m > 1 + 2\mu$. There exists $K_0' > 0$ such that, for all initial data $w_0 \in L^2(m)$ with div $w_0 = 0$ and $\|w_0\|_m \leq K_0'$, equation (3) has a unique global solution $w \in C^0([0,+\infty); L^2(m))$ satisfying $w(0) = w_0$ and for any $\tau \geq 0$, $\mathrm{div}_\tau w(\tau) = 0$. In addition, there exists $K_2 > 0$ such that*

$$\|w(\tau) - \alpha \boldsymbol{G}\|_m \leq K_2\, e^{-\mu\tau} \|w_0\|_m\,, \quad \tau \geq 0\,,$$

where $\alpha = \int_{\mathbb{R}^2 \times (0,1)} (w_0)_3(\xi,z)\,dz\,d\xi$.

Theorem 0.3 *Let $\frac{1}{2} < \nu < 1$, $m > 1 + 2\nu$. There exists $K_0'' > 0$ such that, for all initial data $w_0 \in L^2(m)$ with div $w_0 = 0$, $\int_{\mathbb{R}^2 \times (0,1)} w_0 d\xi dz = 0$ and $\|w_0\|_m \leq K_0''$, equation (3) has a unique* *global solution $w \in C^0([0,+\infty); L^2(m))$ satisfying $w(0) = w_0$ and for any $\tau \geq 0$, $div_\tau w(\tau) = 0$, $\int_{\mathbb{R}^2\times(0,1)} w(\xi, z, \tau) d\xi dz = 0$. In addition, there exists $K_3 > 0$ such that*

$$\|w(\tau) - \sum_{i=1}^{3} \beta_i \boldsymbol{F}_i e^{-\frac{\tau}{2}}\|_m \leq K_3\, e^{-\nu\tau} \|w_0\|_m\,, \quad \tau \geq 0\,,$$

where

$$\begin{aligned}
\beta_1 &= \int_{\mathbb{R}^2\times(0,1)} \xi_1 (w_0)_3(\xi, z)\, dz\, d\xi\,,\\
\beta_2 &= \int_{\mathbb{R}^2\times(0,1)} \xi_2 (w_0)_3(\xi, z)\, dz\, d\xi\,,\\
\beta_3 &= \int_{\mathbb{R}^2\times(0,1)} \frac{1}{2}(\xi_1 (w_0)_2 - \xi_2 (w_0)_1)(\xi, z)\, dz\, d\xi\,.
\end{aligned}$$

References

[1] T. GALLAY & C.E. WAYNE, Invariant Manifolds and the Long-Time Asymptotics of the Navier-Stokes and Vorticity Equations on $\mathbb{R}^2$, Arch. Rational Mech. Anal. **163**, 209-258 (2002).

[2] V. ROUSSIER, Long-Time Asymptotics of Navier-Stokes and Vorticity Equations in a three-dimensional Layer, Preprint Orsay, (2003).