NONTRIVIAL LARGE-TIME DYNAMICS IN REACTION-DIFFUSION EQUATIONS

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Solutions of bistable reaction-diffusion equations, posed on the whole real line, will converge to travelling waves (Fife, McLeod). When the same equation is posed on the plane, or in the N-dimensional space, sufficiently well-prepared initial data will still give raise to convergence to 1D travelling waves (Levermore-Xin; Kapitula). We explain here why, when the initial datum is slightly less well-prepared than in the Levermore-Xin setting, the solution will not converge to a single 1D wave; it will instead undergo some diffusive behaviour. We also discuss what happens when the level sets of the initial datum are conical-shaped. Finally, we describe what happens for KPP equations: nontrivial behaviour occurs even in one space dimension.

Keywords: Reaction-diffusion equations; travelling fronts; Pulsating fronts; nontrivial dynamics.

1. Introduction; motivation

Consider the reaction-diffusion equation

$$u_t - u_{xx} = f(u) \ \ (t>0,\ x\in I\!R), \qquad u(t,-\infty)=0,\ u(t,+\infty)=1 \quad (1)$$

where the function f is of the bistable, unbalanced type:

$$f(0)=f(1)=0;\ \ f'(0),f'(1)<0;\ \ f<0 \text{ on } (0,\theta),\ f>0 \text{ on } (\theta,1);\quad \int_0^1 f>0.$$

It is well-known[1] that the solutions of (1) will converge to travelling waves, i.e. solutions of the form $\phi_0(x+c_0t)$. The profile ϕ_0 satisfies

$$\phi_0''+c_0\phi_0'=f(\phi_0),\ (x\in I\!R);\qquad \phi_0(-\infty)=0,\ \phi_0(+\infty)=1 \qquad (2)$$

Suppose now that (1) is an equation in the plane $I\!R^2$, i.e.

$$u_t - \Delta u = f(u), \quad t > 0,\ (x, y) \in I\!R^2 \tag{3}$$

with an initial datum u_0 whose level sets are at bounded distance from horizontal straight lines. Under the assumption that $\varepsilon := \|u_0 - \phi_0\|_{H^1(I\!R^2)} << 1$, we have - see Xin,[2] Levermore-Xin,[3] Kapitula[4] - for some $\omega > 0$: $u(t, x, y) - \phi_0(y + c_0 t) = O(t^{-\omega})$, uniformly in $(x, y) \in I\!R^2$. Notice that the assumption $u_0 - \phi_0 \in H^1(I\!R^2)$ is somewhat more stringent than simply supposing level sets trapped between straight lines: it says that the level sets of u_0 are asymptotic to horizontal straight lines. A natural question is therefore whether something drastically different occurs if we relax it.

The goal of this paper is to review some recent results obtained by the authors of the present paper. To sum up: unless the initial datum of (3) is really of the Xin, Levermore-Xin or Kapitula form, convergence to a single wave is never likely to occur. The profile of the solution does converge to that of a wave, but with a spatial shift whih does not converge to any value as $t \to +\infty$. In Section 2, we review how two-dimensional effects in the bistable case (3) prevent convergence; in Section 3 we describe what happens when f is of the KPP type: here, non-convergence occurs even in the one-dimensional case.

2. Planar and conical waves for bistable equations

Let us start with almost planar initial data for (3).

Theorem 2.1. *(Roquejoffre, Roussier-Michon[5]) Given $u_0 \in C(I\!R^2)$, assume the existence of two reals $y_1 \le y_2$ such that*

$$\forall (x, y) \in I\!R^2 : \quad \phi_0(y + y_1) \le u_0(x, y) \le \phi_0(y + y_2),$$

where $\phi_0(y)$ is a solution of (2). Then there is $t_0 > 0$ and a function $s(t, x) \in C^2([t_0, +\infty) \times I\!R)$ such that the solution $u(t, x, y)$ of (3), emanating from u_0, satisfies, for all $\delta \in (0, 1)$:

$$\sup_{t \ge t_0, (x,y) \in I\!R^2} |u(t, x, y) - \phi_0(y + c_0 t + s(t, x))| = O(t^{\delta - 1}). \tag{4}$$

Moreover, for all $\delta \in (0, 1)$, there is $C_\delta(u_0) > 0$ such that the function $\sigma(t, x) := e^{c_0 s(t,x)/2}$ satisfies, for $t \ge t_0$:

$$|\sigma_t - \sigma_{xx}| \le \frac{C_\delta(u_0)}{(1 + t)^{2-2\delta}}. \tag{5}$$

Equation (5) indicates that the solution has a local shift that behaves, as $t \to +\infty$, as a solution of the heat equation. We know, since Collet-Eckmann,[6] that a lot of solutions of the heat equation do not converge as $t \to +\infty$, while all the derivatives go to 0. In our case it is possible[5] to construct initial data to (3) that will not converge to a single wave - while the large-time profile is that of the wave.

Equation (3) has genuinely nonplanar travelling wave solutions, still propagating downwards; of the form $u(t,x,y) = \phi(x, y+ct)$, solving

$$-\Delta\phi + c\partial_y\phi = f(\phi) \ \text{ in } \mathbb{R}^2; \quad \phi(x,-\infty) = 0,\ \phi(x,+\infty) = 1 \qquad (6)$$

where the limit is now taken *pointwise* in x. We restrict ourselves to those waves whose level sets are globally Lipschitz graphs; we then may[7,8] classify them all; moreover we have very precise stability properties for them The following properties of these waves are extracted from.[7–9] and[9] - see also[10] for the existence part:

(P1) For a solution ϕ of (6) whose level sets are globally Lipschitz graphs we have $c \geq c_0$ - the planar wave speed; moreover $0 < \phi < 1$ in $\mathbb{R}^2$,

(P2) $\phi(x,y) = \tilde{\phi}(|x|, y)$, $\partial_{|x|}\tilde{\phi} \geq 0$, $\partial_y\phi > 0$,

(P3) There is $\alpha \in (0, \frac{\pi}{2}]$ such that $c = \dfrac{c_0}{\sin\alpha}$, and the function ϕ satisfies

$$\limsup_{A\to+\infty,\ y\geq A-|x|\cot\alpha} (1-\phi(x,y)) = 0, \qquad \limsup_{A\to-\infty,\ y\leq A-|x|\cot\alpha} \phi(x,y) = 0.$$

(P4) the function ϕ is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^2$ such that $\tau_y < -\cos\alpha$,

(P5) there is exponential convergence of $\phi(x,y)$ to the planar fronts $\phi_0(\pm x\cos\alpha + y\sin\alpha)$ in the directions $(\pm\sin\alpha, -\cos\alpha)$; moreover the slopes of the level lines of ϕ converge exponentially, in the same directions, to $\mp\cot\alpha$.

(P6) Let $u_0(x,y)$ be a - two-dimensional - Cauchy datum to (3), satisfying $|u_0(x,y) - \phi(x,y)| = O(e^{-\omega(|x|+|y|)})$, where ω is some positive number, and $\phi(x,y)$ a solution of (6) - hence a conical-shaped solution. Then we have, for some $\gamma > 0$ uniformly in $(x,y) \in \mathbb{R}^2$: $u(t,x,y) - \phi(x, y+ct) = O(e^{-\gamma t})$.

Notice that the assumption on u_0 in (P5) is similar to the Xin-Levermore-Kapitula one, and we wonder what happens if this assumption is slightly relaxed. In what follows, $(X_\pm, Y_\pm)$ is the refrence frame obtained by rotating the axes (x,y) by an angle $\pm\alpha$.

Theorem 2.2. *Consider a smooth Cauchy datum u_0 such that: - there exist a small $\varepsilon > 0$ and a couple $(X_1, X_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that*

$$\phi((x,y)+X_1) \leq u_0(x,y) \leq \phi((x,y)+X_2), \quad |X_1 - X_2| \leq \varepsilon, \qquad (7)$$

- there holds $\partial_y u_0 > 0$. Moreover there is $\rho_\varepsilon > 0$, with $\lim_{\varepsilon\to 0} \frac{\rho_\varepsilon}{\varepsilon^4} = 0$, such that $\limsup_{X_\pm \to +\infty} \|\partial_{X_\pm X_\pm} u_0(X_\pm, .)\|_{L^\infty(\mathbb{R})} \le \rho_\varepsilon^4$.

Choose $\lambda \in (0,1)$, let the set $\{u_0(x,y) = \lambda\}$ be written as $\{Y_+ = s_0^+(X_+)\}$ - resp. $\{Y_- = s_0^-(X_-)\}$ in the right half-plane $\{x > 0\}$ - resp. in the left half-plane $\{x < 0\}$ (the dependence in λ is deleted for commodity). Define the functions $\sigma_0^\pm(X_\pm)$ as

$$\sigma_0^\pm(X_\pm) = \begin{cases} e^{c_0 s_0^\pm(X_\pm)/2} & \text{if } X_\pm \ge 1 \\ e^{c_0 s_0^\pm(1)/2} & \text{if } X_\pm \le 1 \end{cases} \tag{8}$$

Let $\sigma^\pm(t, X_\pm)$ be the solutions of the advection-diffusion equations

$$(\partial_t - \partial_{X_\pm X_\pm} - c\cos\alpha \partial_{X_\pm})\sigma^\pm = 0, \quad \sigma^\pm(0, X_\pm) = \sigma_0^\pm(X_\pm) \tag{9}$$

Let $u(t,x,y)$ be the solution of (3) emanating from u_0. For a given $\lambda \in (0,1)$, there exists $A > 0$ such that the set $\{u(t,x,y) = \lambda\}$ can be described as of the form $\{Y_+ = \chi^+(t, X_+)\}$ in the half-plane $\{x \ge A\}$ - resp. $\{Y_- = \chi^-(t, X_-)\}$ in the half-plane $\{x \le -A\}$. Moreover there is a constant $C_\varepsilon > 0$ - possibly going to $+\infty$ as $\varepsilon \to 0$ - and another constant $C > 0$ independent of ε, such that there holds, for all $\delta \in (0, \frac{1}{2})$, and uniformly in $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^2$:

$$|\chi^\pm(t, X_\pm) - \mathrm{Log}\sigma^\pm(t, X_\pm)| \le C_\varepsilon \left(\frac{1}{(1+t)^{1-2\delta}} + e^{-\omega(|x|+|y|)} \right) + C\rho_\varepsilon^{\delta/2}.$$

We believe that the size restrictions on the initial datum can be removed. However, Theorem 2.2 is still sufficient to exhibit nontrivial large-time dynamics.

3. Nontrivial dynamics for KPP type equations

We consider here the one-dimensional Cauchy problem

$$u_t - u_{xx} = f(x,u), \quad (t > 0,\ x \in \mathbb{R}) \qquad u(t, -\infty) = 0,\ u(t, +\infty) = 1 \tag{10}$$

where the nonlinear term f is 1-periodic in x, and of KPP (Kolmogorov-Petrovskii-Piskunov) type:

$$f(x,0) = f(x,1) = 0; \quad f(x,u) > 0 \text{ if } 0 < u < 1, \quad f_{uu}(x,.) < 0 \text{ on } [0,1].$$

Rather than that of travelling waves, the relevant notion is that of *pulsating waves*, i.e. solutions $\phi(t,x)$ of (10) for which there exists $c > 0$ such that $t \mapsto \phi(t, x - ct)$ is $\frac{1}{c}$-periodic. When f does not depend on x, existence and stability of travelling waves is an old problem dating back to;[11] see for instance.[12] In the x-dependent case, existence theory qualitative theory are very well developped.[13,14] Essentially: there is $c_* > 0$ such that (11) has pulsating wave solutions ϕ_c of velocity c if and only if $c \geq c_*$. Moreover, $\partial_t \phi_c > 0$. The minimal speed c_* is characterised as follows: set $\zeta(x) := f_u(x,0) > 0$; let $k(\lambda)$ be the least eigenvalue of the operator $L_\lambda := -d_{xx} - 2\lambda d_x - (\lambda^2 + \zeta(x))$, acting on the space of 1-periodic functions. Then $c_* = \min_{\lambda>0} \dfrac{k(\lambda)}{\lambda}$. Moreover each ϕ_c behaves at $-\infty$ like $e^{\lambda_c(x+ct)}\psi_\lambda(x)$, where λ_c the smallest (positive) root of $k(\lambda) = c\lambda$, and $(L_\lambda - c\lambda)\psi_\lambda = 0$. To study the large-time behaviour of (10), we need a slightly more precise information.

Proposition 3.1. [15,16] *For $c > c_*$, (10) has a unique (pulsating) wave solution $\phi_c(t,x)$ such that, as $x + ct \to -\infty$: $\phi_c(t,x) = e^{\lambda_c(x+ct)}\psi_\lambda(x) + O(e^{(\lambda_c+\delta)(x+ct)})$.*

Consider the case $c > c_*$. An easy follow-up of the argument of the preceding proposition proves the following: consider an initial datum $u_0(x)$ for (10), such that there is a pulsating wave ϕ_c and $\delta > 0$ satisfying

$$u_0(x) = \phi_c(0,x)(1 + O(e^{\delta x})) \text{ as } x \to -\infty. \tag{11}$$

Then, as $t \to +\infty$ we have, uniformly in x: $u(t,x) = \phi_c(t,x)(1 + O(e^{-\omega t})$ for some $\omega > 0$.

We wish to understand what happens when the - quite stringent - assumption (11) is slightly relaxed. A natural way to loosen it is to assume that u_0 is trapped between two translates - in time - of a pulsating wave; the beaviour of $u(t,x)$ is given by the following theorem - whose proof is much more difficult than that of Proposition 3.1:

Theorem 3.1. *Assume there is $M > 0$ such that $\phi_c(-M,x) \leq u_0(x) \leq \phi_c(M,x)$. Let $m(t,x)$ solve*

$$m_t - m_{xx} - 2(\lambda_c + \frac{\psi'_{\lambda_c}}{\psi_{\lambda_c}})m_x - c\lambda_c m_x^2 = 0; \qquad m(0,x) = \phi_c(.,x)^{-1}(u_0(x))$$

Then, as $t \to +\infty$: [i] we have $\|m_t(t,.)\|_\infty = O(t^{-1})$, $\|m_x(t,.)\|_\infty = O(t^{-1/2})$; [ii] we have, uniformly in $x \in \mathbb{R}$: $|u(t,x) - \phi_c(t + m(t,x),x)| = O(t^{-1/2})$.

The following remarks are in order. First, we point out that Theorem 3.1 is new, even in the x-independent case. Then, let us mention that nontrivial dynamics does occur: once again in the x-independent case, the Collet-Eckman example can be adapted. Part [i] of this theorem is the most involved: it relies on very precise asymptotics of a heat kernel that the current literature[17] cannot treat. Let us finally point out a consequence of this part: although we cannot treat the case $c = c_*$ in the same generality, we have

Theorem 3.2. [15,16] *Assume that, as* $x \to -\infty$: $u_0(x) = \phi_{c_*}(0,x)(1 + O(e^{\delta x}))$ *for some* $\delta > 0$. *Then, as* $t \to +\infty$: $\|u(t,x) - \phi_{c_*}(t,x)\|_\infty = O(t^{-1/2})$.

The stability of the wave with speed c_* is, as opposed to the case $c > c_*$, a delicate matter. In the x-independent case, the optimal result is due to Gallay.[18] When the initial datum does not decay exactly as the wave, investigation of the dynamics is currently going on.

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