# On the order of certain characteristic classes of the Hodge bundle of semi-abelian schemes

Vincent Maillot and Damian Roessler

Institut de Mathématiques de Jussieu, Université Paris 7 Denis Diderot, C.N.R.S., Case Postale 7012, 2 place Jussieu, F-75251 Paris Cedex 05, France.
vmaillot@math.jussieu.fr and dcr@math.jussieu.fr

**Summary.** We give a new proof of the fact that the even terms (of a multiple of) the Chern character of the Hodge bundles of semi-abelian schemes are torsion classes in Chow theory and we give explicit bounds for almost all the prime powers appearing in their order. These bounds appear in the numerators of modified Bernoulli numbers. We also obtain similar results in an equivariant situation.

## 1 Introduction

Let $g \geqslant 1$ (resp. $n \geqslant 4$) be an integer (resp. an even integer) and let $\mathbf{A}_{g,n}$ be the fine moduli scheme of principally polarised abelian varieties over $\mathbb{C}$ with an $n$-level structure (see [CF, chap. I]). By $\overline{\mathbf{A}}_{g,n}$ we denote a toroidal compactification of Faltings-Chai type (see [CF, chap. IV]). Let $G \to \overline{\mathbf{A}}_{g,n}$ be the universal semi-abelian scheme over $\overline{\mathbf{A}}_{g,n}$. We set $\mathbb{E} := e^*\Omega_{G/\overline{\mathbf{A}}_{g,n}}$, where $e\colon \overline{\mathbf{A}}_{g,n} \to G$ is the zero-section. For any integer $k \geqslant 0$, we shall write $\mathrm{ch}_0^k(V)$ for the additive characteristic class on vector bundles $V$, such that $\mathrm{ch}_0^k(V) := c_1(V)^k$ when $V$ is a line bundle. Furthermore for any integer $l \geqslant 2$ we shall write $B_l'$ for the numerator of the rational number $(2^l-1)B_l$, where $B_l$ is the $l$-th Bernoulli number. Recall that the Bernoulli numbers are defined by the formula

$$\frac{t}{\exp(t)-1} = \sum_{j\geqslant 0} B_j \frac{t^j}{j!}\,.$$

**Theorem 1.** *Let $b\colon \widetilde{\mathbf{A}}_{g,n} \to \overline{\mathbf{A}}_{g,n}$ be any desingularisation and let $l \geqslant 2$ be an even integer. Then:*

(1) *The characteristic class $\mathrm{ch}_0^l(b^*\mathbb{E}) \in \mathrm{CH}^l(\widetilde{\mathbf{A}}_{g,n})$ is a torsion class.*
(2) *Let $t \geqslant 1$ be the smallest natural number such that $t\cdot \mathrm{ch}_0^l(b^*\mathbb{E}) = 0$ and let $p$ be a prime number such that $p > l$. If $q \geqslant 0$ is the largest integer such that $p^q|t$ then $p^q|B_l'$.*

Here are some numerical examples. Case $l=2$: the class $\mathrm{ch}_0^2(b^*\mathbb{E})$ is a torsion class of order a power of 2, since $(2^2-1)B_2 = 3/6 = 1/2$. Case $l=12$: there is an integer $r \geqslant 0$ such that $691 \cdot 2310^r \cdot \mathrm{ch}_0^{12}(b^*\mathbb{E}) = 691 \cdot (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)^r \cdot \mathrm{ch}_0^{12}(b^*\mathbb{E}) = 0$, since $(2^{12} - 1)B_{12} = -2073/2 = -3 \cdot 691/2$.

Recall that $\mathrm{CH}(\widetilde{\mathbf{A}}_{g,n})$ refers to the Chow intersection ring of $\widetilde{\mathbf{A}}_{g,n}$ (see [F]). The ring $\mathrm{CH}(\widetilde{\mathbf{A}}_{g,n})$ carries a natural ring grading and $\mathrm{CH}^l(\widetilde{\mathbf{A}}_{g,n})$ refers to its $l$-th graded term. Characteristic classes with values in a cohomology theory factor via the cycle class map through their counterparts with values in the Chow ring. The Chow ring is thus a universal target for characteristic classes.

If one replaces $\overline{\mathbf{A}}_{g,n}$ by $\mathbf{A}_{g,n}$ in Theorem 1 (so that $b$ becomes an isomorphism) then the statement that the characteristic class $\mathrm{ch}_0^l(b^*\mathbb{E})$ is a torsion class was proven by van der Geer in [VDG]. Prompted by his work, Esnault and Viehweg then proved (1) in [EV1]. The original contribution of Theorem 1 thus consists in the information (2) given about the order of the torsion.

For $z$ belonging to the unit circle $S^1$, we define the Lerch's $\zeta$-function $\zeta_L(z,s) := \sum_{k\geqslant 1} z^k/k^s$ for $s \in \mathbb{C}$ such that $\Re(s) > 1$, and using analytic continuation, we extend it to a meromorphic function of $s$ over $\mathbb{C}$.

In the next theorem, $n$ is an integer $\geqslant 1$ and $D$ is any Dedekind ring containing $\mathcal{O}_{\mathbb{Q}(\mu_n)}$ as a subring. Recall that $\mathcal{O}_{\mathbb{Q}(\mu_n)}$ is the ring of integers of the subfield $\mathbb{Q}(\mu_n)$ of $\mathbb{C}$ generated by the $n$-th roots of unity. Let $C$ be a smooth quasi-projective scheme over $\mathrm{Spec}\, D[\frac{1}{n}]$. Let furthermore $\mathcal{C} \to C$ be a polarized abelian scheme and let $\iota$ be an automorphism of finite order $n$ of $\mathcal{C}$ over $C$. Suppose that the fixed point scheme $\mathcal{C}_\iota$ of $\iota$ is finite and flat over $C$. Let $\mathcal{H} := H^1_{\mathrm{dR}}(\mathcal{C}/C)$. The automorphism $\iota$ induces an automorphism of finite order of $\mathcal{H}$, which we also denote by $\iota$. For each $u \in \mu_n(D)$, let $\mathcal{H}_u := \mathrm{Ker}(\iota - u \cdot \mathrm{Id})$.

**Theorem 2.** *Let $l \geqslant 1$ be an integer. The meromorphic function $\zeta_L(u,z)$ is regular at $z = 1 - l$ and the complex number $\zeta_L(u, 1-l)$ lies in $\mathcal{O}_{\mathbb{Q}(\mu_n)}[\frac{1}{n\cdot l!}]$. The equality*

$$\sum_{u\in\mu_n(D)} \zeta_L(u, 1-l)\, \mathrm{ch}_0^l(\mathcal{H}_u) = 0$$

*holds in $\mathrm{CH}^l(C) \otimes \mathcal{O}_{\mathbb{Q}(\mu_n)}[\frac{1}{n\cdot l!}]$.*

Theorem 2 is compatible with Theorem 1 in the following sense. Let $C = \mathbf{A}_{g,n}$ and let $\mathcal{C}$ be the restriction of $G$ to $\mathbf{A}_{g,n}$. Let $\iota$ be the automorphism of order 2 of $\mathcal{C}$ over $C$ given by taking the inverse in the group scheme. Then the equality statement in Theorem 2 is equivalent to Theorem 1 with $\overline{\mathbf{A}}_{g,n}$ replaced by $\mathbf{A}_{g,n}$.

Theorem 2 overlaps with Stickelberger's theorem; this is explained at the end of subsection 4.2.

We shall now describe our methods of proof. Theorem 1 and Theorem 2 are both proved by applying a relative coherent Lefschetz fixed point formula (see subsection 2.3) to certain vector bundles and certain fibrations. A formula involving the extended Hodge bundle $b^*\mathbb{E}$ (resp. the first Gauss-Manin bundle $H^1_{\mathrm{dR}}$) is then obtained and Theorem 1 (resp. Theorem 2) is deduced from this formula, using some linear algebra and some facts relating the exponential function and the Lerch zeta-function.

In the paper by Esnault and Viehweg quoted above [EV1], the Grothendieck-Riemann-Roch is applied to a quotient of a compactification of the group scheme $G$ to prove that $\mathrm{ch}_0^l(b^*\mathbb{E})$ is a torsion class. This method is conceptually close to ours but its seems difficult to obtain fine information about denominators using it, because it involves the Chow group of the compactification, where the denominators of the Chern character can become large, when the dimension of the compactification is large. By contrast, the relative Lefschetz formula only involves the Chow group of the fixed point set, which has the same dimension as the base. Another advantage of the fixed point formula is that it involves less denominators at the outset whereas the Grothendieck-Riemann-Roch theorem would probably have to be replaced by the Adams-Riemann-Roch theorem to make control of denominators possible.

The authors were led to the theorems 1 and 2 and the methods of proof presented here by a conjecture on characteristic classes of Hodge bundles in the context of Arakelov theory. For this we refer to subsection 4.2.

The structure of the article is as follows. In the second section, we describe some results from the book of Chai and Faltings on the toroidal compactification of the universal semi-abelian family, as presented in the article [EV1]; we use these results to relate the sheaf of relative differentials with logarithmic singularities to the normal bundle of the fixed point set of $-1$ (see Proposition 2). We then proceed to describe the relative fixed point formula which will be our main tool in the proof. In the third section, we first prove Theorem 2 by applying the fixed point formula to the relative de Rham complex of the relevant abelian scheme; second we prove Theorem 1 by applying the fixed point formula to the relative logarithmic de Rham complex. In the fourth section, we shall discuss some consequences of the above theorems, as well as some conjectures to which they lead. A salient consequence of Theorem 2 is Corollary 1, which concerns abelian schemes with complex multiplications but possibly no automorphisms of finite order other than $-1$.

*Acknowledgements.* The authors thank G. van der Geer and B. Moonen for encouraging them to write up the results of this paper and more generally for their interest. Our thanks also go to C. Soulé, for interesting discussions and suggestions. Finally, we are especially grateful to H. Esnault and E. Viehweg for taking the time to discuss their paper [EV1] with us. This paper was the technical basis for the present article.

## 2 Preliminaries

### 2.1 Differentials with Logarithmic Singularities

In this subsection we shall review the definition of a sheaf of differentials with logarithmic singularities along a divisor with normal crossings, as well as its basic properties. Our basic reference is [EV2, chap. 2].

Let $Z$ be a quasi-projective non-singular variety over $\mathbb{C}$ and let $D$ be a normal crossings divisor in $Z$. Let $d$ be the dimension of $Z$. We set $U := Z\backslash D$ and denote by $j : U \hookrightarrow Z$ be the inclusion map. We shall write $\Omega^*_Z(\log\ D)$ for the complex of sheaves of differential forms with logarithmic singularities along $D$. The complex $\Omega^*_Z(\log\ D)$ is a subcomplex of the complex $j_*\Omega^*_U$ and it has the following defining property: if $V \subseteq Z$ is an open set, $p \geqslant 0$ is an integer and and $\omega \in j_*\Omega^p_U(V) = \Omega^p_Z(U \cap V)$ then $\omega \in \Omega^p_Z(\log\ D)(V)$ iff $\omega$ and $d\omega$ have simple poles along $D \cap V$. To say that $\omega$ has a simple pole along $D \cap V$ in the latter situation means the following: in any affine open subscheme $W \subseteq V$ such that the ideal of $D \cap W$ in $\mathcal{O}_Z(W)$ is principal, the section $e \cdot \omega|_{U\cap W}$ lies in the image of the restriction map $\Omega^p_Z(W) \to \Omega^p_Z(U \cap W)$ for any generator $e \in \mathcal{O}_Z(W)$ of the ideal of $D \cap W$ (this condition does not depend on the choice of $e$). The definition of $\Omega^*_Z(\log\ D)$ implies that for each $p \geqslant 0$ the sheaf $\Omega^p_Z(\log\ D)$ has the structure of $\mathcal{O}_Z$-module, which is compatible with the injection $\Omega^p_Z(\log\ D) \hookrightarrow j_*\Omega^p_U$; furthermore, $\Omega^p_Z(\log\ D)$ is then locally free for this $\mathcal{O}_Z$-module structure.

Abusing language, we shall write $\mathcal{O}_{D_0}$ for $\oplus_{i=1}^n \mathcal{O}_{C_i}$, where the $C_i$ run over the irreducible components of $D$. Let $P \in Z$ and let $\mathbf{m}$ be the maximal ideal of the local ring $\mathcal{O}_P$. We write $s$ for the number of irreducible components of $D$ which contain $P$. We may suppose without restriction of generality that the components $C_1, \dots, C_s$ contain $P$. We denote by $f_1, \dots, f_s \in \mathbf{m}$ generators of the ideals of the components $C_1, \dots, C_s$. Since by definition $f_1, \dots, f_s$ form a regular sequence in $\mathcal{O}_P$ and since $\mathcal{O}_P$ is a regular ring, we may find (see [M, Par. 14, Th.14.2]) elements $f_{s+1}, \dots, f_d \in \mathbf{m}$, such that the elements $f_1, \dots, f_d$ form a regular system of parameters in $\mathcal{O}_P$. There is then a neighborhood $V$ of $P$, such that the elements $\mathrm{d}f_1, \dots, \mathrm{d}f_d$ form a basis of $\Omega_Z(V)$ as a $\mathcal{O}_Z(V)$-module. It is shown in [EV2, chap. 2, 2.2 (c), p. 11] that in this situation the elements $\mathrm{d}f_1/f_1, \dots \mathrm{d}f_s/f_s, \mathrm{d}f_{s+1}, \dots \mathrm{d}f_d$ form a basis of $\Omega_Z(\log\ D)(V)$ as an $\mathcal{O}_Z(V)$-module.

Furthermore, there is a canonical exact sequence

$$0 \longrightarrow \Omega^1_Z \longrightarrow \Omega^1_Z(\log\ D) \xrightarrow{r} \mathcal{O}_{D_0} \longrightarrow 0$$

where the morphism $r$ has the following description. We shall use the terminology of the last paragraph. The homomorphism $\Omega_Z(\log\ D)(V) \to \mathcal{O}_{D_0}(V) = \oplus_{i=1}^s \mathcal{O}_{C_i}(V)$ sends $\alpha \cdot \mathrm{d}f_i/f_i$ (resp. $\alpha \cdot \mathrm{d}f_i$), where $\alpha \in \mathcal{O}_Z(V)$ and $1 \leqslant i \leqslant s$ (resp. $d \geqslant i > s$), to the image of $\alpha$ in $\mathcal{O}_{C_i}(V)$ (resp. on 0).

Now let $Z'$ be a non-singular quasi-projective variety over $\mathbb{C}$ and $g\colon Z' \to Z$ be a morphism over $\mathbb{C}$. Let $D' := (g^*(D))_{\mathrm{red}}$ and suppose that $D'$ is a divisor with normal crossings. We write $U'$ for its complement and let $j'\colon U' \hookrightarrow Z'$ be the inclusion morphism. Notice that by adjunction, there is a natural morphism of coherent sheaves $j^*\Omega_Z(\log\ D) \to j^*\Omega_Z = \Omega_U$; this induces a morphism $g|_U^*(j^*\Omega_Z(\log\ D)) \to g|_U^*(j^*\Omega_Z)$ and since $j \circ g|_U = g \circ j'$, we obtain a morphism $g^*\Omega_Z(\log\ D) \to j'_*j'^*g^*\Omega_Z$ by adjunction. Composing with the natural morphism $g^*\Omega_Z \to \Omega_{Z'}$, we finally obtain a morphism $g^*\Omega_Z(\log\ D) \to j'_*j'^*\Omega_{Z'} = j'_*\Omega_{U'}$.

**Lemma 1.** *The image of the morphism of coherent sheaves* $g^*\Omega_Z(\log\ D) \to j'_*j'^*\Omega_{Z'} = j'_*\Omega_{U'}$ *just given lies inside* $\Omega_{Z'}(\log\ D')$.

We shall prove Lemma 1 together with the Lemma 2 which we first describe. Consider first the following diagram:

$$\begin{array}{ccccccccc} & & g^*\Omega_Z & \longrightarrow & g^*\Omega_Z(\log\ D) & \longrightarrow & g^*\mathcal{O}_{D_0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \Omega_{Z'} & \longrightarrow & \Omega_{Z'}(\log\ D') & \longrightarrow & \mathcal{O}_{D'_0} & \longrightarrow & 0\,, \end{array}$$

where the middle vertical arrow is defined via Lemma 1. By construction this diagram is commutative and its existence shows that there is a unique morphism $g^*\mathcal{O}_{D_0} \to \mathcal{O}_{D'_0}$ such that the completed diagram

$$\begin{array}{ccccccccc} & & g^*\Omega_Z & \longrightarrow & g^*\Omega_Z(\log\ D) & \longrightarrow & g^*\mathcal{O}_{D_0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{Z'} & \longrightarrow & \Omega_{Z'}(\log\ D') & \longrightarrow & \mathcal{O}_{D'_0} & \longrightarrow & 0\,, \end{array}$$

commutes.

**Lemma 2.** *If $D$ and $g^*(D)$ are normal schemes then the morphism $g^*\mathcal{O}_{D_0} \to \mathcal{O}_{D'_0}$ just described is an isomorphism.*

*Proof of lemmas 1 and 2.* Since the lemmas 1 and 2 are both local statements on $Z$ and $Z'$, we may assume that $Z' = \mathrm{Spec}\ A$, $Z = \mathrm{Spec}\ B$ and that the morphism $g$ is induced by a ring morphism $g_0 \colon A \to B$. We may also suppose that $\Omega_Z(\log\ D)(Z)$ is free over $A$ and that a basis of $\Omega_Z(\log\ D)(Z)$ is given by $\mathrm{d}f_1/f_1, \dots \mathrm{d}f_s/f_s, \mathrm{d}f_{s+1}, \dots \mathrm{d}f_d$, where the elements $f_1, \dots, f_s$ are generators of the ideals of the irreducible components $C_1, \dots, C_s$ of $D$ and the elements $\mathrm{d}f_1, \dots, \mathrm{d}f_d$ form a basis of $\Omega_A$ over $A$. Similarly, we may also suppose that a basis of $\Omega_{Z'}(\log\ D')(Z')$ over $B$ is given by $\mathrm{d}f'_1/f'_1, \dots \mathrm{d}f'_{s'}/f'_{s'}, \mathrm{d}f'_{s'+1}, \dots \mathrm{d}f'_{d'}$, where the elements $f'_1, \dots, f'_s$ are generators of the ideals of the irreducible components $C'_1, \dots, C'_{s'}$ of $D'$ and the elements $\mathrm{d}f_1, \dots, \mathrm{d}f_{d'}$ form a basis of $\Omega_B$ over $B$. We may also suppose that $g_0(f_k) = \prod_{r=1}^{s'} u_{k,r} {f'_r}^{m_{k,r}}$, where $u_{k,r} \in B^\times$, $m_{k,r} \in \mathbb{Z}^{\geqslant 0}$ and $1 \leqslant k \leqslant s$ (this follows from the fact that the local rings of $Z'$ are regular rings and hence unique factorization domains).

Notice that we have a canonical isomorphism $j_*\Omega_U(Z) \simeq \Omega_{A, f_1 \cdots f_s}$ (resp. $j'_*\Omega_{U'}(Z') \simeq \Omega_{B, f'_1 \cdots f'_{s'}}$). If we follow the steps of the definition of the morphism $g^*\Omega_Z(\log\ D) \to j'_*\Omega_{U'}$, we see that it corresponds to the morphism $\Omega_Z(\log\ D)(Z) \otimes_A B \to \Omega_{B, f'_1 \cdots f'_{s'}}$ of $B$-modules such that

$$\frac{\mathrm{d}f_k}{f_k} \mapsto \sum_{r=1}^{s'} \Big( \frac{\mathrm{d}u_{k,r}}{u_{k,r}} + m_{k,r} \frac{\mathrm{d}f'_r}{f'_r} \Big) \tag{1}$$

for $1 \leqslant k \leqslant s$ and

$$\mathrm{d}f_k \mapsto \mathrm{d}(g_0(f_k)) \tag{2}$$

for $s < k \leqslant d$. Since the expressions appearing after the arrows in (1) and (2) are both linear combinations over $B$ of the elements

$$\frac{\mathrm{d}f_1}{f_1}, \dots, \frac{\mathrm{d}f'_{s'}}{f'_{s'}}, \mathrm{d}f'_{s'+1}, \dots, \mathrm{d}f'_{d'} ,$$

we have proven Lemma 1.

To prove Lemma 2, notice first that we may assume without loss of generality in the situation of Lemma 2 that $D$ and $g^*(D)$ are integral. We then have $s = 1$, $s' = 1$ and $m_{k,r} = 1$ for all $k, r$. The module associated to the coherent sheaf $g^*\mathcal{O}_{D_0}$ (resp. $\mathcal{O}_{D'_0}$) is then $A/(f_1) \otimes_A B$ (resp. $B/(g_0(f_1))$). Furthermore, if $a \otimes_A b \in A/(f_1) \otimes_A B$ (resp. $b' \in B/(g_0(f_1))$), then $a \otimes_A b$ is by definition the image of the element $a \cdot \mathrm{d}f_1/f_1 \otimes b \in g^*\Omega_Z(\log\ D)(Z')$ (resp. $b' \cdot \mathrm{d}f'_1/f'_1 \in \Omega_{Z'}(\log\ D')(Z')$). Looking at the diagram (2.1), we see that the morphism $A/(f_1) \otimes_A B \to B/(g_0(f_1))$ sends $a \otimes_A b$ to the image of $g(a)b \cdot (\mathrm{d}f'_1/f'_1 + \mathrm{d}u_{1,1}/u_{1,1})$, i.e. $g(a)b$. Thus the morphism $g^*\mathcal{O}_{D_0} \to \mathcal{O}_{D'_0}$ is given by the natural isomorphism $A/(f_1) \otimes_A B \simeq B/(g_0(f_1))$. $\square$

## 2.2 The Toroidal Compactification of the Universal Abelian Scheme

We shall need the following result, whose proof can be found in [CF, chap. I, prop. 2.7].

**Proposition 1 (Raynaud).** *Let $S$ be a noetherian normal scheme and let $U \subseteq S$ be an open dense subset. Let $\mathcal{B} \to U$ be an abelian scheme. If there is a semi-abelian scheme $\widetilde{\mathcal{B}} \to S$ extending $\mathcal{B}$, then it is unique up to unique isomorphism.*

We now quote a theorem stated in [EV1, Th 3.1], which sums up some results that can be found in [CF, chap. VI, par. 1]. Recall that $n \geqslant 4$ is an even integer.

**Theorem 3.** *There exists a cartesian diagram of morphisms of schemes*

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, "\overline{f}"] \\ \mathbf{A}_{g,n} \arrow[r, hook] & B \end{tikzcd}$$

*where $f\colon \mathcal{A} \to \mathbf{A}_{g,n}$ is the universal abelian scheme, such that*

(1) *the horizontal morphisms are open immersions;*
(2) *the closed set $T := B \backslash \mathbf{A}_{g,n}$, endowed with its reduced induced subscheme structure, is a normal crossings divisor;*
(3) *the closed subscheme $Y := (\overline{f}^* T)_{\mathrm{red}}$ is a normal crossings divisor;*
(4) *$X$ and $B$ are projective smooth varieties over $\mathbb{C}$;*
(5) *there exists a semi-abelian scheme $\widetilde{\mathcal{A}} \to B$ which extends the universal abelian scheme;*
(6) *the $n$-level structure sections $S_i\colon \mathbf{A}_{g,n} \to \mathcal{A}$ ($i \in (\mathbb{Z}/n\mathbb{Z})^{2g}$) extend to pairwise disjoint sections of $X$ over $B$;*
(7) *the action of the inversion on $\mathcal{A}$ extends to an involution $\alpha$ of $X$ over $B$ whose fixed point scheme factors through $\coprod_{i \in (\mathbb{Z}/n\mathbb{Z})^{2g}} S_i$;*
(8) *let $\widetilde{e}\colon B \to \widetilde{A}$ be the zero-section and let $\widetilde{\mathbb{E}} := \widetilde{e}^* \Omega_{\widetilde{A}/B}$; there is a natural isomorphism*

$$\overline{f}^* \widetilde{\mathbb{E}} \simeq \Omega_X(\log\ Y)/\overline{f}^*(\Omega_B(\log\ T)) =: \Omega_{X/B}(\log)\,,$$

*where*

(9) *there is a natural isomorphism*

$$R^q \overline{f}_*(\mathcal{O}_X) \simeq \wedge^q(\widetilde{\mathbb{E}}^\vee)$$

*for all $q \geqslant 0$.*

Notice that the statement (7) implies that $X_\alpha = \coprod_{i \in (\mathbb{Z}/n\mathbb{Z})^{2g},\ 2\cdot i = 0} S_i$. The conormal sheaf of $X_\alpha$ in $X$ is locally free since both $X_\alpha$ and $X$ are regular and we denote the dual of the conormal sheaf by $N$.

**Proposition 2.** *There is a natural isomorphism $N^\vee \simeq \Omega_{X/B}(\log)|_{X_\alpha}$.*

For the proof, we shall need the following lemma.

**Lemma 3.** *There exists an open neighborhood $V$ of $X_\alpha$ such that $\overline{f}|_V$ is smooth. Furthermore, the natural map $N^\vee \to \Omega_{X/B}|_{X_\alpha}$ is an isomorphism.*

*Proof.* Recall that there is an exact sequence

$$N^\vee \longrightarrow \Omega_{X/B}|_{X_\alpha} \longrightarrow \Omega_{X_\alpha/X} \longrightarrow 0$$

(see [H, II, Prop. 8.12]). Using the determination of $X_\alpha$ given above we deduce that $\Omega_{X_\alpha/X} = 0$. Furthermore, the restriction of the above sequence to $X_\alpha \cap \mathcal{A}$ is exact, since $\mathcal{A} \to \mathbf{A}_{g,n}$ is smooth (cf. [FL, IV, Par. 3, Prop. 3.7, (b)]). Hence $\mathrm{rk}(N) = g$. Now let $r\colon X \to \mathbb{Z}$ be the function $r(x) := \dim_{\kappa(x)} \Omega_{X/B,x} \otimes_{\kappa(x)} \kappa(x)$, where $\kappa(x)$ is the residue field at $x$. This function is upper semi-continuous (see [H, II, Ex. 5.8, (a)]) and thus reaches its minimum at $g$, which is the rank of $\Omega_{X/B}$ on the open dense subset $\mathcal{A}$ of $X$. The set $V := \{x \in X | r(x) = g\}$ is open and the restriction of $\Omega_{X/B}$ to $V$ is locally free of rank $g$ (see [H, II, Ex. 5.8, (c)]). The existence of the surjection $N^\vee \to \Omega_{X/B}|_{X_\alpha}$ implies that $r(x) \leqslant g$ when $x \in X_\alpha$ and thus $r(x) = g$ on $X_\alpha$. Thus $X_\alpha \subseteq V$. The restriction $\overline{f}|_V$ is smooth, since $V$ and $B$ are non-singular varieties over $\mathbb{C}$ and $\dim(V) - \dim(B) = g$ (see [H, III, Prop. 10.4]). The morphism $N^\vee \to \Omega_{X/B}|_{X_\alpha}$ is a surjection of locally free sheaves of the same rank and is thus an isomorphism. This concludes the proof. □

*Proof of proposition 2.* Consider the commutative diagram with exact rows on $X$:

$$\begin{array}{ccccccc}
 & & \overline{f}^*\Omega_B & \longrightarrow & \overline{f}^*\Omega_B(\log\ T) & \longrightarrow & \overline{f}^*\mathcal{O}_{T_0} & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \Omega_X & \longrightarrow & \Omega_X(\log\ Y) & \longrightarrow & \mathcal{O}_{Y_0} & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & \Omega_{X/B} & \longrightarrow & \Omega_{X/B}(\log) & \longrightarrow & \mathcal{O}_{Y_0}/\overline{f}^*\mathcal{O}_{T_0} & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}$$

where the morphism $\overline{f}^*\mathcal{O}_{T_0} \to \mathcal{O}_{Y_0}$ is defined by the two vertical morphisms on its left side. Let $V \subseteq T$ be the open subset of $T$ which consists of all the points which do not lie at the intersection of two irreducible components of $T$. The set $V$ has smooth disjoint irreducible components. Let $U_0$ be the open neighborhood of $X_\alpha$ provided by Lemma 3. Let now $V_B$ be an open subset of $B$ such that $V_B \cap T = V$ and let $U_B := \mathbf{A}_{g,n} \cup V_B$. Finally, let $U := \overline{f}^{-1}(U_B) \cap U_0$. Using Lemma 2 and the snake lemma, we see that the restriction of the last diagram to $U$ has the following appearance:

$$\begin{array}{ccccccc}
 & & \overline{f}^*\Omega_B|_U & \longrightarrow & \overline{f}^*\Omega_B(\log\ T)|_U & \longrightarrow & \overline{f}^*\mathcal{O}_{T_0}|_U & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \| & \\
0 & \longrightarrow & \Omega_X|_U & \longrightarrow & \Omega_X(\log\ Y)|_U & \longrightarrow & \mathcal{O}_{Y_0}|_U & \\
 & & \downarrow & & \downarrow & & & \\
 & & \Omega_{X/B}|_U & \simeq & \Omega_{X/B}(\log)|_U & & & \\
 & & \downarrow & & \downarrow & & & \\
 & & 0 & & 0 & & &
\end{array}$$

Consider now an $n$-level section $\sigma\colon B \to X$ whose image is an irreducible component of $X_\alpha$. By the last diagram and the second statement in Lemma 3, we have $\sigma|_{U_B}^* \Omega_{X/B}(\log) \simeq \sigma|_{U_B}^* N^\vee$. Since $B\backslash U_B$ has codimension 2 in $B$ by construction and since $B$ is regular, there is a unique extension of the isomorphism $\sigma|_{U_B}^* \Omega_{X/B}(\log) \simeq \sigma|_{U_B}^* N^\vee$ to an isomorphism $\sigma^* \Omega_{X/B}(\log) \simeq \sigma^* N^\vee$. This concludes the proof. □

### 2.3 A Relative Lefschetz Fixed Point Formula

In this subsection, we shall review a relative fixed point formula which is a corollary of a formula in Arakelov theory proved in [KR1]. Let $S$ be a noetherian affine scheme. Let $Z$ be a regular scheme which is quasi-projective over $S$. Let $\mu_n$ be the diagonalisable group scheme over $S$ which corresponds to $\mathbb{Z}/n\mathbb{Z}$. Suppose that $Z$ carries a $\mu_n$-action over $S$; furthermore, suppose that there is an ample line bundle on $Z$, which carries a $\mu_n$-equivariant structure compatible with the $\mu_n$-equivariant structure of $Z$ (see [T2, par. 1.2] for more details about the latter notion). We shall write $K_0^{\mu_n}(Z)$ for the Grothendieck group of locally free sheaves on $Z$ which carry a compatible $\mu_n$-equivariant structure. Replacing locally free sheaves by coherent sheaves in the latter definition leads to a naturally isomorphic group (see [T2, lemme 3.3]). If the $\mu_n$-equivariant structure of $Z$ is trivial, then the datum of a (compatible) $\mu_n$-equivariant structure on a locally free sheaf $E$ on $Z$ is equivalent to the datum of a $\mathbb{Z}/n\mathbb{Z}$-grading of $E$. The group $K_0^{\mu_n}(Z)$ carries a $\lambda$-ring structure such that for any $\mu_n$-equivariant locally free sheaf $E$, the element $\lambda^k(E)$ is represented in $K_0^{\mu_n}(Z)$ by the $k$-th exterior power of $E$, endowed with its natural $\mu_n$-equivariant structure (see [K, lemma 3.4]). For any $\mu_n$-equivariant locally free sheaf $E$ on $Z$, we write $\lambda_{-1}(E)$ for $\sum_{k=0}^{\mathrm{rk}(E)}(-1)^k\lambda^k(E) \in K_0^{\mu_n}(Z)$. There is a unique isomorphism of rings $K_0^{\mu_n}(S) \simeq K_0(S)[T]/(1-T^n)$ with the following property: it maps the structure sheaf of $S$ endowed with a homogenous $\mathbb{Z}/n\mathbb{Z}$-grading of weight one to $T$ and it maps any locally free sheaf carrying a trivial equivariant structure to the corresponding element of $K_0(S)$ $(= K_0^{\mu_1}(S))$.

The functor of fixed points associated to $Z$ is by definition the functor

$$\textbf{Schemes}/S \to \textbf{Sets}$$

described by the rule

$$T \ \mapsto \ Z(T)_{\mu_n(T)}\,.$$

Here $Z(T)_{\mu_n(T)}$ is the set of elements of $Z(T)$ which are fixed under each element of $\mu_n(T)$. The functor of fixed points is representable by a scheme $Z_{\mu_n}$ and the canonical morphism $Z_{\mu_n} \to Z$ is a closed immersion (see [SGA3, VIII, 6.5 d]). Furthermore, the scheme $Z_{\mu_n}$ is regular (see [T, Prop. 3.1]). We shall denote the immersion $Z_{\mu_n} \hookrightarrow Z$ by $i$. Write $N^\vee$ for the dual of the conormal sheaf of the closed immersion $Z_{\mu_n} \hookrightarrow Z$. It is locally free on $Z_{\mu_n}$ and carries a natural $\mu_n$-equivariant structure. This structure corresponds to a $\mu_n$-grading, since $Z_{\mu_n}$ carries the trivial $\mu_n$-equivariant structure and it can be shown that the weight 0 term of this grading is 0 (see [T, Prop. 3.1]).

Let $W$ be a regular scheme which is quasi-projective over $S$ and suppose that $W$ carries a $\mu_n$-action over $S$. Let $h\colon Z \to W$ be a projective $S$-morphism which respects the $\mu_n$-actions and write $h_{\mu_n}$ for the induced morphism $Z_{\mu_n} \to W$. The morphism $h$ induces a direct image map $Rh_*\colon K_0^{\mu_n}(Z) \to K_0^{\mu_n}(W)$, which is a homomorphism of groups described by the formula $Rh_*(E) := \sum_{k\geqslant 0}(-1)^k R^k h_*(E)$ for a $\mu_n$-equivariant coherent sheaf $E$ on $Z$. Here $R^k h_*(E)$ refers to the $k$-th higher direct image sheave of $E$ under $h$; the sheaves $R^k h_*(E)$ are coherent and carry a natural $\mu_n$-equivariant structure. The morphism $h$ also induces a pull-back map $Lh^*\colon K_0^{\mu_n}(W) \to K_0^{\mu_n}(Z)$; this is a ring morphism which sends a $\mu_n$-equivariant locally free sheaf $E$ on $W$ on the locally free sheaf $h^*(E)$ on $Z$, endowed with its natural $\mu_n$-equivariant structure. For any elements $z \in K_0^{\mu_n}(Z)$ and $w \in K_0^{\mu_n}(W)$, the projection formula $Rh_*(z \cdot Lh^*(w)) = w \cdot Rh_*(z)$ holds. This implies that the group homomorphism $Rh_*$ is a morphism of $K_0^{\mu_n}(S)$-modules, if the group $K_0^{\mu_n}(Z)$ (resp. $K_0^{\mu_n}(W)$) is endowed with the $K_0^{\mu_n}(S)$-module structure induced by the pull-back map $K_0^{\mu_n}(S) \to K_0^{\mu_n}(Z)$ (resp. $K_0^{\mu_n}(S) \to K_0^{\mu_n}(W)$).

Let $\mathcal{R}$ be a $K_0^{\mu_n}(S)$-algebra such that $1 - T^k$ is a unit in $R$ for all $k$ such that $1 \leqslant k < n$.

We shall refer to the following hypothesis as (H): $S$ is the spectrum of a Dedekind ring which can be embedded in $\mathbb{C}$, $Z$ and $W$ are flat over $S$ and $Z_{\mu_n}$ is flat over $S$.

**Theorem 4.** *Let the hypothesis (H) hold. The element $\lambda_{-1}(N^\vee)$ is a unit in the ring $K_0^{\mu_n}(Z_{\mu_n}) \otimes_{K_0^{\mu_n}(S)} R$. If the $\mu_n$-equivariant structure on $W$ is trivial, then for any element $z \in K_0^{\mu_n}(Z)$, the equality*

$$Rh_*(z) = Rh_{\mu_n,*}\big((\lambda_{-1}(N^\vee))^{-1} \cdot \mathrm{Li}^*(z)\big)$$

*holds in $K_0^{\mu_n}(W) \otimes_{K_0^{\mu_n}(S)} R$.*

*Proof.* The theorem is a consequence of [KR1, Par. 6, Th. 6.1] if the morphism $h$ is an immersion. Furthermore, the theorem is a consequence of [KR1, Par. 4, Th. 4.4] if $W = S$, $Z = \mathbf{P}_S^k$ for some $k \geqslant 0$ and $h$ is the structural morphism $\mathbf{P}_S^k \to S$. These two cases combined with the projection formula and the determination of $K_0^{\mu_n}(\mathbf{P}_S^k)$ given in [T2, Th. 3.1] imply the full statement. $\square$

*Remarks.* (1) The Theorem 4, without the hypothesis (H) but with the hypothesis that $S$ is the spectrum of an algebraically closed field of characteristic not dividing $n$, is proved in [BFM].

(2) The Theorem 4, without the hypothesis (H) but with the requirement that $R$ is a field is a consequence of [T, Th. 3.5].

(3) The proof of Theorem 4 given above only apparently refers to Arakelov theory; its underlying structure is purely algebraic and is a variant of the proof of the main result of [BFM]. This variant does not in fact use hypothesis (H). In particular, Theorem 4 is true without hypothesis (H).

# 3 Proof of theorems 1 and 2

The proofs of theorems 1 and 2 are similar and proceed in two steps. In the first one, we apply Theorem 4 to a certain geometrical situation and in the second one, we transform the resulting expression using some combinatorics. The first subsection contains the combinatorial statements we shall need and in the second one the computations leading to the proofs are given.

### 3.1 Combinatorics

Let's consider the two following formal series

$$\exp(x) := \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

and

$$\log(1+x) := \sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^j}{j} .$$

For $l \geqslant 1$, we shall write $\mathbb{C}[x]^{\leqslant l}$ for the quotient of the ring $\mathbb{C}[[x]]$ by the ideal of formal series divisible by $x^r$, where $r > l$. We then define $\exp^{\leqslant l}(x) \in \mathbb{Z}[\frac{1}{l!}][x]$ and $\log^{\leqslant l}(1+x) \in \mathbb{Z}[\frac{1}{l}][x]$ as the only polynomials of degree $l$ representing the above formal series $\exp(x)$ and $\log(1+x)$ in $\mathbb{C}[x]^{\leqslant l}$.

About these polynomials, we have the following lemma.

**Lemma 4.** *Let $u \in \mu_n(\mathbb{C})$, $u \neq 1$. Then the equality*

$$\log^{\leqslant l}\left(\frac{1 - u \cdot \exp^{\leqslant l}(x)}{1-u}\right) = -\sum_{j=1}^{l} \zeta_L(u, 1-j)\, \frac{x^j}{j!}$$

*holds in $\mathbb{C}[x]^{\leqslant l}$. In particular, the values $\zeta_L(u, 1-j)x^j/j!$ lie in $\mathcal{O}_{\mathbb{Q}(\mu_n)}[\frac{1}{n \cdot l!}]$ when $1 \leqslant j \leqslant l$.*

*Proof.* It will be sufficient to prove the identity

$$\log\left(\frac{1 - u \cdot \exp(x)}{1-u}\right) = -\sum_{j=1}^{\infty} \zeta_L(u, 1-j)\, \frac{x^j}{j!} \tag{3}$$

in $\mathbb{C}[[x]]$. In [MR2, (6), proof of lemma 3.1], the identity of complex power series

$$\frac{u \cdot \exp(x)}{1 - u \cdot \exp(x)} = \sum_{j=1}^{\infty} \zeta_L(u, -j)\, \frac{x^j}{j!} \tag{4}$$

is proven. If one takes the formal derivative of both sides of equation (3) (for $x$), one obtains equation (4). Hence it is sufficient to show that the constant terms of the power series on both sides of (3) coincide. Since both constant terms can be seen to vanish, we are done. □

The following Lemma will be used in the proof of Theorem 1.

**Lemma 5.** *The equality $\zeta_L(-1, 1-l) = -(2^l - 1)B_l/l$ holds for all $l \geqslant 2$.*

*Proof.* Let $s \in \mathbb{C}$ be such that $\Re(s) > 1$. By definition, we have

$$\zeta_L(-1, s) = \sum_{k \geqslant 1} \frac{(-1)^k}{k^s}$$

and

$$\zeta_{\mathbb{Q}}(s) = \sum_{k \geqslant 1} \frac{1}{k^s}$$

where $\zeta_{\mathbb{Q}}$ is Riemann's $\zeta$-function. From these equalities, we deduce that $\zeta_L(-1, s) = \zeta_{\mathbb{Q}}(s)(2^{1-s} - 1)$. Now $\zeta_{\mathbb{Q}}(1-l) = -B_l/l$ (see for example [W, chap. 4, Th. 4.2]), whence the lemma. □

If $Z$ is a scheme which is smooth over a Dedekind ring, we shall write $\mathrm{CH}(Z)^{\leqslant l}$ for the ring $\mathrm{CH}(Z)/\oplus_{j=l+1}^{\infty}\mathrm{CH}^j(Z)$. If $E$ is a locally free sheaf on $Z$, we shall write $\mathrm{ch}^{\leqslant l}(E)$ ("truncated Chern character") for the element of $\mathrm{CH}(Z)^{\leqslant l}\otimes\mathbb{Z}[\frac{1}{l!}]$ given by the formula $\mathrm{ch}^{\leqslant l}(E):=\sum_{j=0}^{l}\frac{1}{j!}\mathrm{ch}_0^j(E)$. The proof of the following lemma is similar to the proof of the multiplicativity and additivity of the Chern character and we shall omit it.

**Lemma 6.** *The map* $\mathrm{ch}^{\leqslant l}$ *factors through a ring homomorphism*

$$K_0(Z)\to\mathrm{CH}(Z)^{\leqslant l}\otimes\mathbb{Z}[\frac{1}{l!}]\,.$$

Let $\mathrm{CH}(Z)^{\leqslant l,*}$ be the multiplicative subgroup of $\mathrm{CH}(Z)^{\leqslant l}$ consisting of elements of the form $1+z$, where $z$ has the property that its degree 0 part vanishes.

The following lemma is a consequence of the fact that $\log((1+x)(1+y))=\log(1+x)+\log(1+y)$ in the ring of power series $\mathbb{C}[[x,y]]$.

**Lemma 7.** *The polynomial* $\log^{\leqslant l}$ *defines a map* $\mathrm{CH}(Z)^{\leqslant l,*}\to\mathrm{CH}(Z)^{\leqslant l}\otimes\mathbb{Z}[\frac{1}{l}]$ *which is a group homomorphism.*

### 3.2 Final Computations

We shall now prove Theorem 2.

**Lemma 8.** *Let* $\xi$ *be a primitive* $n$*-th root of unity in* $\mathcal{O}_{\mathbb{Q}(\mu_n)}$*. Then the elements* $1-\xi^k$ *are units in* $\mathcal{O}_{\mathbb{Q}(\mu_n)}[\frac{1}{n}]$ *for every integer* $k$ *such that* $1\leqslant k<n$*.*

*Proof.* Recall the polynomial identity $\prod_{j=1}^{n-1}(X-\xi^j)=X^{n-1}+\cdots+X+1$. This identity implies that the inverse of $1-\xi^k$ is given by $n^{-1}\prod_{r=1,r\neq k}^{n-1}(1-\xi^r)$. □

If $A$ is a $D[\frac{1}{n}]$-algebra such that Spec $A$ is connected and non-empty, then $A$ contains exactly $n$ distinct $n$-th roots of unity, all of which are images of roots of unity contained in $D[\frac{1}{n}]$. This is a consequence of the last lemma and of the Chinese remainder theorem. Fix a primitive root of unity $\zeta$. This choice fixes an isomorphism $\mathbb{Z}/n\mathbb{Z}\simeq\mu_n(D[\frac{1}{n}])$ and hence for each $D[\frac{1}{n}]$-algebra $A$, there is a canonical isomorphism of groups $\mu_n(A)\simeq\prod_{C\in\mathcal{CC}(A)}\mathbb{Z}/n\mathbb{Z}$, where $\mathcal{CC}(A)$ is the set of connected components of $\mathrm{Spec}(A)$. We have thus described a $D[\frac{1}{n}]$-isomorphism between the constant group scheme over $D[\frac{1}{n}]$ associated to $\mathbb{Z}/n\mathbb{Z}$ and the group scheme $\mu_n$ over $D[\frac{1}{n}]$.

Let $W$ be a scheme which is smooth over Spec $D[\frac{1}{n}]$ and which carries the trivial $\mu_n$-equivariant structure. For each $\mu_n$-equivariant locally free sheaf $E$ on $C$, define

$$\mathrm{ch}_{\mu_n}^{\leqslant l}(E):=\sum_{k\in\mathbb{Z}/n\mathbb{Z}}\zeta^k\cdot\mathrm{ch}^{\leqslant l}(E_k)\,.$$

We can see from the definitions that $\mathrm{ch}_{\mu_n}^{\leqslant l}$ induces a ring morphism

$$\mathrm{ch}_{\mu_n}^{\leqslant l}:K_0^{\mu_n}(W)\to\mathrm{CH}^{\leqslant l}(W)\otimes\mathcal{O}_{\mathbb{Q}(\mu_n)}[\frac{1}{l!}]\,.$$

We shall now apply Theorem 4. Let us denote the morphism $\mathcal{C}\to C$ by $c$ and its relative dimension by $d$. The automorphism $\iota$ defines a $\mathbb{Z}/n\mathbb{Z}$-action on $\mathcal{C}$ over $C$

and using the isomorphism described above, we obtain a $\mu_n$-equivariant structure on $\mathcal{C}$ over $C$. The fixed point scheme $c_{\mu_n}:\mathcal{C}_{\mu_n}\to C$ (which coincides with the fixed point scheme of $\iota$) is then étale over $C$. To see this, notice that we have an exact sequence of coherent sheaves

$$N^\vee \longrightarrow \Omega_{\mathcal{C}/C}|_{\mathcal{C}_{\mu_n}} \longrightarrow \Omega_{\mathcal{C}_{\mu_n}/C} \longrightarrow 0\,,$$

where $N$ is the normal bundle of the immersion $\mathcal{C}_{\mu_n}\hookrightarrow\mathcal{C}$ and all the maps respect the natural $\mu_n$-actions on the sheaves. The two first sheaves in this sequence are locally free, since $\mathcal{C}$ and $\mathcal{C}_{\mu_n}$ are regular and the map $c$ is smooth. Hence the first morphism in the sequence is injective, because it is injective on the dense open subset where $c_{\mu_n}$ is étale. If we now consider the weight 0 part of the sequence, we obtain an isomorphism $(\Omega_{\mathcal{C}/C}|_{\mathcal{C}_{\mu_n}})_0\to\Omega_{\mathcal{C}_{\mu_n}/C}$ and thus $\Omega_{\mathcal{C}_{\mu_n}/C}$ is locally free which in turn implies that $\Omega_{\mathcal{C}_{\mu_n}/C}=0$ since $c$ is finite. Hence $c$ is étale.

We now compute:

$$\begin{aligned}
\mathrm{ch}^{\leqslant l}_{\mu_n}\Big(Rc_*(\sum_{k=0}^{d}(-1)^k\wedge^k(\Omega_{\mathcal{C}/C}))\Big) &\overset{(1)}{=} \mathrm{ch}^{\leqslant l}_{\mu_n}\Big(\sum_{k=0}^{2d}(-1)^k\wedge^k(\mathcal{H})\Big)\\
&\overset{(2)}{=} \mathrm{ch}^{\leqslant l}_{\mu_n}\big(Rc_{\mu_n *}((\lambda_{-1}(N^\vee))^{-1}\lambda_{-1}(\Omega_{\mathcal{C}/C}|_{\mathcal{C}_{\mu_n}}))\big)\\
&\overset{(3)}{=} \mathrm{ch}^{\leqslant l}_{\mu_n}\big(Rc_{\mu_n,*}(\lambda_{-1}(\Omega_{\mathcal{C}_{\mu_n}/C}))\big)\\
&\overset{(4)}{=} f_0\otimes 1\in \mathrm{CH}^0(C)\otimes\mathcal{O}_{\mathbb{Q}(\mu_n)}[\frac{1}{n\cdot l!}]\,.
\end{aligned}$$

Here $f_0\in\mathrm{CH}^0(C)=\mathbb{Z}$ is the degree of the finite morphism $c_{\mu_n}$. The equality (1) is justified by the fact that the Hodge to de Rham spectral sequence of $c$ degenerates and the fact that there is a natural isomorphism $H^r_{\mathrm{dR}}(\mathcal{C}/C)\simeq\wedge^r(H^1_{\mathrm{dR}}(\mathcal{C}/C))$ for all $r\in\mathbb{Z}^{\geqslant 0}$ (see [BBM, 2.5.2]). The equality (2) is provided by Theorem 4, applied in the case where $S=\mathrm{Spec}\ D[\frac{1}{n}]$, $Z=\mathcal{C}$, $h=c$, the $\mu_n$-equivariant structure on $\mathcal{C}$ is the one described above and $z=\lambda_{-1}(\Omega_{\mathcal{C}/C})$. The equality (3) is justified by the fact that $c_{\mu_n}$ is étale and the multiplicativity of $\lambda_{-1}$. The equality (4) derives from the fact that $\Omega_{\mathcal{C}_{\mu_n}/C}=0$. We shall now rewrite the resulting equality

$$\mathrm{ch}^{\leqslant l}_{\mu_n}\Big(\sum_{k=0}^{2d}(-1)^k\wedge^k(\mathcal{H})\Big)=f_0\otimes 1\in\mathrm{CH}^0(C)\otimes\mathcal{O}_{\mathbb{Q}(\mu_n)}[\frac{1}{n\cdot l!}] \tag{5}$$

using the combinatorics of the first subsection.

By the splitting principle, we may suppose without restriction of generality that $\mathcal{H}=\sum_{k=1}^{2d}h_k$ in $K_0^{\mu_n}(C)$, where $h_k$ is a line bundle which carries a homogenous $\mathbb{Z}/n\mathbb{Z}$-grading. Write $t_k$ for the first Chern class $c_1(h_k)$ of $h_k$. Let $w(h_k)\in\mathbb{Z}/n\mathbb{Z}$ be the weight of $h_k$ and let $u_k:=\zeta^{w(h_k)}$. The equality (5) implies that

$$\prod_{k=1}^{2d}(1-u_k)=f_0\,.$$

In particular, $u_k\neq 1$ for all $k$. We now have the following reformulation of (5):

$$\frac{1}{f_0}\mathrm{ch}^{\leqslant l}_{\mu_n}\Big(\sum_{k=0}^{2d}(-1)^k\wedge^k(\mathcal{H})\Big) = \prod_{k=1}^{2d}\frac{\mathrm{ch}^{\leqslant l}_{\mu_n}(1-h_k)}{(1-u_k)}$$
$$= \prod_{k=1}^{2d}\frac{1-u_k\cdot\exp^{\leqslant l}(t_k)}{1-u_k} = 1\,.$$

If we apply the $\log^{\leqslant l}$ map to the members of the last string of equalities and use the Lemma 4, we obtain

$$\begin{aligned}\sum_{k=1}^{2d}\log^{\leqslant l}\left(\frac{1-u_k\cdot\exp^{\leqslant l}(t_k)}{1-u_k}\right) &= -\sum_{k=1}^{2d}\sum_{j=1}^{l}\zeta_L(u_k,1-j)\,\frac{t_k^j}{j!}\\ &= -\sum_{j=1}^{l}\sum_{k=1}^{2d}\zeta_L(u_k,1-j)\,\frac{t_k^j}{j!}\\ &= -\sum_{j=1}^{l}\sum_{u\in\mu_n(D)}\zeta_L(u,1-j)\,\frac{\mathrm{ch}_0^j(\mathcal{H}_u)}{j!}\\ &= 0 \end{aligned} \tag{6}$$

which implies the result.

We now turn to the proof of Theorem 1. We use the notation of Theorem 3. We shall apply Theorem 4 to the situation where $Z = X$, $h = \overline{f}$, $n = 2$, the action of $\mu_2$ is given by the involution $\alpha$ which extends the action of the inversion on $\mathcal{A}$ (notice that over $D[\frac{1}{2}]$ there is a unique isomorphism between $\mu_2$ and the constant group scheme associated to $\mathbb{Z}/2\mathbb{Z}$) and $z = \lambda_{-1}(\Omega_{X/B}(\log))$. Let $N$ be the dual of the conormal sheaf of the immersion $X_\alpha = X_{\mu_2} \hookrightarrow X$.

We compute

$$\begin{aligned}&\mathrm{ch}^{\leqslant l}_{\mu_2}\Big(R\overline{f}_*\big(\sum_{k=0}^{g}(-1)^k\wedge^k(\Omega_{X/B}(\log))\big)\Big)\\ &\overset{(1)}{=}\mathrm{ch}^{\leqslant l}_{\mu_2}\Big(\sum_{k=0}^{2g}(-1)^k\wedge^k(\widetilde{\mathbb{E}}\oplus\widetilde{\mathbb{E}}^\vee)\Big)\\ &\overset{(2)}{=}\mathrm{ch}^{\leqslant l}_{\mu_2}\big(R\overline{f}_{\mu_2*}((\lambda_{-1}(N^\vee))^{-1}\lambda_{-1}(\Omega_{X/B}(\log)|_{X_{\mu_2}}))\big)\\ &\overset{(3)}{=}f_0\otimes 1\in\mathrm{CH}(C)\otimes\mathbb{Z}[\frac{1}{2\cdot l!}]\,.\end{aligned}$$

Here $f_0 \in \mathrm{CH}^0(C) = \mathbb{Z}$ is the degree of the finite morphism $\overline{f}_{\mu_n}$. The equality (1) is justified by Theorem 3 (9). The equality (2) is provided by Theorem 4, applied in the situation just described. The equality (3) is justified by Lemma 3. Let us define $\mathcal{H} := \widetilde{\mathbb{E}}\oplus\widetilde{\mathbb{E}}^\vee$. We can now repeat the computations from (5) to (6) verbatim, setting $n = 2$. We obtain the equation

$$\sum_{j=1}^{l}\sum_{u\in\mu_2(D)}\zeta_L(u,1-j)\,\frac{\mathrm{ch}_0^j(\mathcal{H}_u)}{j!} = 0$$

in $\mathrm{CH}(B)\otimes\mathbb{Z}[\frac{1}{2\cdot l!}]$. In other words

$$\sum_{j=1}^{l}\zeta_L(-1,1-j)\,\frac{\mathrm{ch}_0^j(\mathcal{H})}{j!}=2\sum_{j=1}^{l}\zeta_L(-1,1-j)\,\frac{\mathrm{ch}_0^j(\widetilde{\mathbb{E}})}{j!}=0$$

in $\mathrm{CH}(B)\otimes\mathbb{Z}[\frac{1}{2\cdot l!}])$. Now notice that $\zeta_L(-1,1-l)=-(2^l-1)B_l/l$ by Lemma 5. We have thus proven an analog of Theorem 1, where $b^*\mathbb{E}$ is replaced by $\widetilde{\mathbb{E}}$. To deduce the Theorem 1 as stated from it, we shall make the following construction. Let $\Delta\colon \mathbf{A}_{g,n}\hookrightarrow\widetilde{\mathbf{A}}_{g,n}\times B$ be the diagonal immersion and let $\widetilde{\mathbf{A}}'_{g,n}\to\mathrm{Zar}(\Delta(\mathbf{A}_{g,n}))$ be a desingularisation of the Zariski closure of $\Delta(\mathbf{A}_{g,n})$. Let $p_1$ (resp. $p_2$) be the map obtained by composing the natural map $\widetilde{\mathbf{A}}'_{g,n}\hookrightarrow\widetilde{\mathbf{A}}_{g,n}\times B$ and the first (resp. second) projection map $\widetilde{\mathbf{A}}_{g,n}\times B\to\widetilde{\mathbf{A}}_{g,n}$ (resp. $\widetilde{\mathbf{A}}_{g,n}\times B\to B$). Let $b':=b\circ p_1$. The map $b'$ is a also a desingularisation of $\overline{\mathbf{A}}_{g,n}$. By Proposition 1, we have an isomorphism $p_2^*\mathcal{A}\simeq b'^*G$ on $\widetilde{\mathbf{A}}'_{g,n}$. Hence

$$2\sum_{j=1}^{l}\zeta_L(-1,1-j)\,\frac{\mathrm{ch}_0^j(p_2^*\widetilde{\mathbb{E}})}{j!}=2\sum_{j=1}^{l}\zeta_L(-1,1-j)\,\frac{\mathrm{ch}_0^j(b'^*\mathbb{E})}{j!}=0$$

in $\mathrm{CH}(\widetilde{\mathbf{A}}'_{g,n})\otimes\mathbb{Z}[\frac{1}{2\cdot l!}]$. Now notice that

$$\begin{aligned}p_{1,*}\Big(2\sum_{j=1}^{l}\zeta_L(-1,1-j)\,\frac{\mathrm{ch}_0^j(b'^*\mathbb{E})}{j!}\Big)&=p_{1,*}p_1^*\Big(2\sum_{j=1}^{l}\zeta_L(-1,1-j)\,\frac{\mathrm{ch}_0^j(b^*\mathbb{E})}{j!}\Big)\\&=p_{1,*}(1)\cdot 2\sum_{j=1}^{l}\zeta_L(-1,1-j)\,\frac{\mathrm{ch}_0^j(b^*\mathbb{E})}{j!}\end{aligned}$$

in $\mathrm{CH}(\widetilde{\mathbf{A}}_{g,n})\otimes\mathbb{Z}[\frac{1}{2\cdot l!}]$. We have used the projection formula for the last equality. Since $p_1$ is birational, we have $p_{1,*}(1)=1$ and we have thus completely proven Theorem 1.

# 4 Consequences and Conjectures

## 4.1 A Corollary of Theorem 2

Let $c\colon\mathcal{C}\to C$ be a polarized abelian scheme, where $C$ is a regular and quasi-projective variety over $\mathbb{C}$. Let $K$ be a finite abelian extension of $\mathbb{Q}$. Suppose that there is an embedding of rings $\mathcal{O}_K\hookrightarrow\mathrm{End}_C(\mathcal{C})$. Let $\mathcal{H}:=H^1_{\mathrm{dR}}(\mathcal{C}/C)$. The coherent sheaf $\mathcal{H}$ carries a ring action of $K$. Choose an element $k_0\in K$ such that $K=\mathbb{Q}(k_0)$ (a simple element of $K$ over $\mathbb{Q}$). For each $\sigma\in\mathrm{Hom}(K,\mathbb{C})$, define

$$\mathcal{H}_\sigma:=\mathrm{Ker}\big(k_0-\sigma(k_0)\cdot\mathrm{Id}\big)$$

The natural morphism $\oplus_{\sigma\in\mathrm{Hom}(K,\mathbb{C})}\mathcal{H}_\sigma\to\mathcal{H}$ is an isomorphism, as can be seen by considering its restriction to closed points of $C$. Furthermore, the sheaves $\mathcal{H}_\sigma$ do not depend on the choice of $k_0$.

Now let $\chi\colon\mathrm{Gal}(K|\mathbb{Q})\to S^1$ be a one-dimensional character of $K$. We shall show that the following Proposition is a consequence of Theorem 2.

**Proposition 3.** *The equality*

$$\sum_{\sigma\in\mathrm{Hom}(K,\mathbb{C})}\chi(\sigma)\mathrm{ch}(\mathcal{H}_\sigma)=\sum_{\sigma\in\mathrm{Hom}(K,\mathbb{C})}\chi(\sigma)\mathrm{rk}(\mathcal{H}_\sigma)$$

*holds in* $\mathrm{CH}(C)\otimes\overline{\mathbb{Q}}$.

Notice that there is a non-canonical isomorphism $\mathrm{Hom}(K,\mathbb{C})\simeq\mathrm{Gal}(K|\mathbb{Q})$. The equality in the proposition is true for any choice of such an isomorphism.

In the following, the use of $\mathrm{Hom}(\cdot,\mathbb{C})$ instead of $\mathrm{Gal}(\cdot)$ always implies that the corresponding statement is independent of the choice of an identification of $\mathrm{Hom}(\cdot,\mathbb{C})$ and $\mathrm{Gal}(\cdot)$.

**Corollary 1.** *The equality* $\mathrm{ch}(\mathcal{H}_\sigma)=\mathrm{rk}(\mathcal{H}_\sigma)$ *in* $\mathrm{CH}(C)\otimes\overline{\mathbb{Q}}$ *is true for all* $\sigma\in\mathrm{Hom}(K,\mathbb{C})$.

*Proof of corollary 1.* The content of Proposition 3 is that as functions of $\sigma$, all the Fourier coefficients of $\mathrm{ch}(\mathcal{H}_\sigma)$ and $\mathrm{rk}(\mathcal{H}_\sigma)$ coincide. Hence the conclusion follows from the uniqueness of the Fourier decomposition. □

Before coming to the full proof of Proposition 3, we shall prove the following weaker statement:

**Proposition 4.** *Proposition 3 holds for* $K=\mathbb{Q}(\mu_n)$ *for some* $n\geqslant 2$.

In the proof of Proposition 4, we shall need the following lemma, which is surprisingly difficult to prove. The hypotheses and the terminology of Proposition 4 are in force.

**Lemma 9.** *Let* $u_0:=\exp(2i\pi/n)$ *and let* $l\geqslant 1$ *be an integer.*

(1) *The following equalities of meromorphic functions of* $s\in\mathbb{C}$ *hold. If* $\chi$ *is an even character then*

$$\sum_{\sigma\in\mathrm{Hom}(\mathbb{Q}(\mu_n),\mathbb{C})}\chi(\sigma)\zeta_L(\sigma(u_0),s)=n^{1-s}\pi^{s-1/2}\,\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\,L(\chi,1-s)\,,$$

*while if* $\chi$ *is an odd character then*

$$\sum_{\sigma\in\mathrm{Hom}(\mathbb{Q}(\mu_n),\mathbb{C})}\chi(\sigma)\zeta_L(\sigma(u_0),s)=i\cdot n^{1-s}\pi^{s-1/2}\,\frac{\Gamma(1-s/2)}{\Gamma((s+1)/2)}\,L(\chi,1-s)\,.$$

(2) *We have*

$$\sum_{\sigma\in\mathrm{Hom}(K,\mathbb{C})}\chi(\sigma)\zeta_L\big(\sigma(u_0),1-l\big)\neq 0$$

*when either (a)* $\chi$ *is an even character and* $l$ *is an even integer, or (b)* $\chi$ *is an odd character and* $l$ *is an odd integer.*

Recall that a character $\chi$ as above is odd (resp. even) if the image of complex conjugation under $\chi$ is $-1$ (resp. 1). The symbol $L(\chi,s)$ refers to the meromorphic function of $s\in\mathbb{C}$ which is defined by the formula

$$L(\chi,s):=\sum_{k=1}^{\infty}\frac{\chi(k)}{k^s}$$

for $\Re(s)>1$. Notice that the character $\chi$ may be non-primitive. If the character $\chi$ is primitive, the equalities in (1) are consequences of the functional equation of Dirichlet $L$-functions.

*Proof of lemma 9.* The second equality in (1) is the content of [KR2, Lemma 5.2]. The proof of the first equality is similar and we shall omit it. Before beginning with the proof of (2) we recall that the function $1/\Gamma(s)$ has zeros at the points $0, -1, -2, \ldots$ and is $\neq 0$ for all the other values of $s$. Recall also that $L(\chi, s)$ has an Euler product expansion when $\Re(s) > 1$ and thus $L(\chi, s) \neq 0$ when $\Re(s) > 1$. To prove (2), (a), we compute (with $\chi$ and $l$ even)

$$\sum_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} \chi(\sigma)\zeta_L\big(\sigma(u_0), 1-l\big) = n^l \pi^{1/2-l} \frac{\Gamma(l/2)}{\Gamma((1-l)/2)} L(\chi, l)\,.$$

Using the remarks made before the computation, we can conclude the proof of (2), (a). For the proof of (2), (b), we make a similar computation:

$$\begin{aligned}
&\sum_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} \chi(\sigma)\zeta_L(\sigma(u_0), 1-l) \\
&\quad = n^l \pi^{1/2-l} i\, \frac{\Gamma(1/2+l/2)}{\Gamma(1-l/2)} L(\chi, l) \\
&\quad = n^l \pi^{1/2-l} i\, \frac{\Gamma(1/2+l/2)}{\Gamma(1-l/2)} L(\chi_{\mathrm{prim}}, l) \prod_{p|n} (1 - \chi_{\mathrm{prim}}(p)p^{-l})\,.
\end{aligned}$$

Here $\chi_{\mathrm{prim}}$ is the primitive Dirichlet character associated to $\chi$. It is shown in [W, chap. 4, Cor. 4.4]) that $L(\chi_{\mathrm{prim}}, 1) \neq 0$ (it is only to treat the case $l = 1$ that we introduced $\chi_{\mathrm{prim}}$). Using this fact and again the remarks made before the proof of (2), (a), we can conclude. □

*Proof of proposition 4.* Let $u_0 := \exp(2i\pi/n)$. Let $\tau \in \mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})$; the root of unity $\tau(u_0)$ acts on $\mathcal{C}$ as an automorphism of finite order $n$ over $C$. The fixed point scheme of $\tau(u_0)$ on $\mathcal{C}$ can be shown to be finite and flat in this situation. We leave this as an exercise to the reader. Applying Theorem 2 to this situation (with similar notations and $\iota$ given by $\tau(u_0)$), we obtain the equation

$$\sum_{\sigma \in \mathrm{Hom}(\mathbb{Q}(\mu_n),\mathbb{C})} \zeta_L\big(\sigma(\tau(u_0)), 1-l\big)\, \mathrm{ch}^l(\mathcal{H}_\sigma) = 0 \tag{7}$$

in $\mathrm{CH}^l(C) \otimes \overline{\mathbb{Q}}$, for any $l \geqslant 1$. We now identify $\mathrm{Hom}(\mathbb{Q}(\mu_n), \mathbb{C})$ and $\mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})$ via the natural embedding $\mathbb{Q}(\mu_n) \hookrightarrow \mathbb{C}$ and we evaluate at $\overline{\chi} = \chi^{-1}$ the Fourier transform of the left side of (7) for the variable $\tau \in \mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})$. We obtain

$$\begin{aligned}
&\sum_{\tau \in \mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \overline{\chi}(\tau) \Big[ \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \zeta_L\big(\sigma(\tau(u_0)), 1-l\big)\, \mathrm{ch}^l(\mathcal{H}_\sigma)\Big] \\
&\quad = \sum_\tau \chi(\tau) \Big[ \sum_\sigma \zeta_L\big(\sigma(u_0), 1-l\big)\, \mathrm{ch}^l(\mathcal{H}_{\sigma\tau})\Big] \\
&\quad = \sum_\tau \chi(\tau) \Big[ \sum_\sigma \zeta_L\big((\sigma\tau^{-1})(u_0), 1-l\big)\, \mathrm{ch}^l(\mathcal{H}_\sigma)\Big] \\
&\quad = \sum_\sigma \Big( \sum_\tau \chi(\tau)\zeta_L\big((\sigma\tau^{-1})(u_0), 1-l\big)\Big) \mathrm{ch}^l(\mathcal{H}_\sigma) \\
&\quad = \Big( \sum_\tau \overline{\chi}(\tau)\zeta_L\big(\tau(u_0), 1-l\big)\Big) \Big( \sum_\sigma \chi(\sigma)\mathrm{ch}^l(\mathcal{H}_\sigma)\Big) \\
&\quad = 0\,.
\end{aligned}$$

Now suppose that $l$ is an even integer (resp. odd integer) and that $\chi$ is an even character (resp. odd character). Then, using Lemma 9, we deduce that

$$\sum_{\sigma} \chi(\sigma)\mathrm{ch}^l(\mathcal{H}_\sigma) = 0$$

which is the equality to be proven. If $l$ is even (resp. odd) and $\chi$ is odd (resp. even) then $\mathrm{ch}^l(\mathcal{H}_\sigma)$ is an even function of $\sigma$ (resp. odd function of $\sigma$), which again implies that

$$\sum_{\sigma} \chi(\sigma)\mathrm{ch}^l(\mathcal{H}_\sigma) = 0\,.$$

Indeed, the change of variables $\sigma \mapsto \sigma^{-1}$ then changes the sign of the expression $\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})} \chi(\sigma)\mathrm{ch}^l(\mathcal{H}_\sigma)$. The fact that $\mathrm{ch}^l(\mathcal{H}_\sigma)$ is an even function of $\sigma$ (resp. odd function of $\sigma$) when $l$ is even (resp. odd) follows from the fact that there is a $u_0$-equivariant isomorphism $\mathcal{H} \simeq \mathcal{H}^\vee$. This in turn follows from relative Lefschetz and Poincaré duality and the fact that there exists an ample invertible sheaf $\mathcal{L}$ on $\mathcal{C}$, which carries a $u_0$-equivariant structure. To obtain such a sheaf, start with an ample invertible sheaf $\mathcal{L}'$ on $\mathcal{C}$ and let $\mathcal{L} := \otimes_{n=0}^{n-1}(u_0^k)^*\mathcal{L}'$. The sheaf $\mathcal{L}$ carries a natural $u_0$-equivariant structure.

Combining the two last equations, we can conclude the proof. □

We shall need the following lemma in the proof of Proposition 3.

**Lemma 10.** *Let $L'|L$ be a finite extension of number fields such that $\mathcal{O}_{L'}$ is free over $\mathcal{O}_L$ and that $L'$ is abelian over $\mathbb{Q}$. Let $\chi_L$ be a one-dimensional character of $L$. If Proposition 3 holds for $K = L'$ and $\chi = \mathrm{Ind}_L^{L'}(\chi_L)$ then it holds also for $K = L$ and $\chi = \chi_L$.*

Recall that by definition $\mathrm{Ind}_L^{L'}(\chi_L)$ is a one-dimensional character of $L'$ such that $\mathrm{Ind}_L^{L'}(\chi_L)(\sigma_{L'}) = \chi(\sigma_{L'}|_L)$ for all $\sigma_{L'} \in \mathrm{Gal}(L'|\mathbb{Q})$.

*Proof.* Let $r := [L' : L]$ and let $x_1, \dots, x_r$ be a basis of $\mathcal{O}_{L'}$ over $\mathcal{O}_L$. The mapping $\varphi\colon \mathcal{O}_{L'} \to M_r(\mathcal{O}_L)$ of $\mathcal{O}_{L'}$ into the $r \times r$-matrices with entries in $\mathcal{O}_L$ which maps an element of $\mathcal{O}_{L'}$ to the matrix representation in this basis of the corresponding $\mathcal{O}_L$ linear map $\mathcal{O}_{L'} \to \mathcal{O}_{L'}$, is an embedding of rings. Via the map $\varphi$, we obtain an embedding of rings $\mathcal{O}_{L'} \hookrightarrow \mathrm{End}_C(\mathcal{C}^r)$. There is a natural isomorphism of coherent sheaves

$$\bigoplus_{j=1}^{r} H^1_{\mathrm{dR}}(\mathcal{C}/C) \simeq H^1_{\mathrm{dR}}(\mathcal{C}^r/C)$$

and under this isomorphism, there is a decomposition

$$\bigoplus_{j=1}^{r} H^1_{\mathrm{dR}}(\mathcal{C}/C)_{\sigma_L} \simeq \bigoplus_{\sigma_{L'}|_L=\sigma_L} H^1_{\mathrm{dR}}(\mathcal{C}^r/C)_{\sigma_{L'}}$$

for any $\sigma_L \in \mathrm{Gal}(L|\mathbb{Q})$. Now choose an embedding $L' \hookrightarrow \mathbb{C}$ to identify $\mathrm{Hom}(L', \mathbb{C})$ and $\mathrm{Gal}(L'|\mathbb{Q})$. We compute

$$
\begin{aligned}
\sum_{\sigma_{L'}} & \operatorname{ch}\big(H^1_{\mathrm{dR}}(\mathcal{C}^r/C)_{\sigma_{L'}}\big)\operatorname{Ind}_L^{L'}(\chi)(\sigma_{L'}) \\
&= \sum_{\sigma_L}\chi(\sigma_L)\sum_{\sigma_{L'}|_L=\sigma_L}\operatorname{ch}\big(H^1_{\mathrm{dR}}(\mathcal{C}^r/C)_{\sigma_{L'}}\big) \\
&= \sum_{\sigma_L}\chi(\sigma_L)\sum_{j=1}^{r}\operatorname{ch}\big(H^1_{\mathrm{dR}}(\mathcal{C}/C)_{\sigma_L}\big) \\
&= r\cdot\sum_{\sigma_L}\operatorname{ch}\big(H^1_{\mathrm{dR}}(\mathcal{C}/C)_{\sigma_L}\big)\,\chi(\sigma_L)\,,
\end{aligned}
$$

from which the conclusion follows. □

*Proof of proposition 3.* Class field theory implies that $K$ can be embedded in $\mathbb{Q}(\mu_n)$ for some $n\geqslant 2$. We claim that $\mathcal{O}_{\mathbb{Q}(\mu_n)}$ is free over $\mathcal{O}_K$. To see this, let $u_0\in\mathcal{O}_{\mathbb{Q}(\mu_n)}$ be a primitive $n$-th root of 1 and let $k$ be the degree of the minimal polynomial of $u_0$ over $K$. The elements $1,\dots,u_0^{k-1}$ are then linearly independent over $K$, hence over $\mathcal{O}_K$. The minimal polynomial of $u_0$ over $K$ is of the form $a_0+a_1t+\dots+a_{k-1}t^{k-1}+t^k$, where $\{a_0,\dots,a_k\}\subseteq\mathcal{O}_K$. Hence $u_0^{k+s}=-a_0u_0^s-a_1u_0^{s+1}-\dots-a_{k-1}u_0^{k-1+s}$ for all integers $s\geqslant 0$. Applying induction over $s$, we see that all the elements $1,\dots,u_0^{n-1}$ are contained in the $\mathcal{O}_K$ module generated by $1,\dots,u_0^{k-1}$. Since the elements $1,\dots,u_0^{n-1}$ generate $\mathcal{O}_{\mathbb{Q}(\mu_n)}$ as an $\mathcal{O}_K$-module, we see that the elements $1,\dots,u_0^{k-1}$ generate $\mathcal{O}_{\mathbb{Q}(\mu_n)}$ as an $\mathcal{O}_K$-module and thus form a basis of $\mathcal{O}_{\mathbb{Q}(\mu_n)}$ as an $\mathcal{O}_K$-module.

Now using Lemma 10, we see that we may assume without restriction of generality that $K=\mathbb{Q}(\mu_n)$. In that case, Proposition 4 is equivalent to Proposition 3 and this concludes the proof. □

### 4.2 Conjectures and Speculations

Let $C$ be a smooth quasi-projective scheme over $\mathbb{C}$. Let furthermore $\mathcal{C}\to C$ be a polarized semi-abelian scheme and let $K$ be a number field which is Galois over $\mathbb{Q}$. Suppose that there is an embedding $K\hookrightarrow\operatorname{End}_C(\mathcal{C})\otimes\mathbb{Q}$. Let $\mathcal{H}:=e^*\Omega_{\mathcal{C}/C}\oplus e^*\Omega^\vee_{\mathcal{C}/C}$ where $e\colon C\to\mathcal{C}$ is the zero section. The coherent sheaf $\mathcal{H}$ carries a ring action of $K$. Choose an element $k_0\in K$ such that $K=\mathbb{Q}(k_0)$. For each $\sigma\in\operatorname{Hom}(K,\mathbb{C})$, define

$$\mathcal{H}_\sigma:=\operatorname{Ker}(k_0-\sigma(k_0)\cdot\operatorname{Id})\,.$$

The natural morphism $\oplus_{\sigma\in\operatorname{Hom}(K,\mathbb{C})}\mathcal{H}_\sigma\to\mathcal{H}$ is then an isomorphism, as before and the sheaves $\mathcal{H}_\sigma$ do not depend on the choice of $k_0$.

Let now $\chi\colon\operatorname{Hom}(K,\mathbb{C})\to\mathbb{C}$ be a simple Artin character of $K$. We make the following conjecture.

*Conjecture 1.* The equality

$$\sum_{\sigma\in\operatorname{Hom}(K,\mathbb{C})}\chi(\sigma)\operatorname{ch}(\mathcal{H}_\sigma)=\sum_{\sigma\in\operatorname{Hom}(K,\mathbb{C})}\chi(\sigma)\operatorname{rk}(\mathcal{H}_\sigma)$$

holds in $\operatorname{CH}(C)\otimes\overline{\mathbb{Q}}$.

An even stronger conjecture is

*Conjecture 2.* The equality $\mathrm{ch}(\mathcal{H}_\sigma) = \mathrm{rk}(\mathcal{H}_\sigma)$ holds for all $\sigma \in \mathrm{Hom}(K, \mathbb{C})$.

Notice that unlike in the case where $K$ is an abelian extension of $\mathbb{Q}$, Conjecture 2 is not a consequence of Conjecture 1.

The Conjecture 1 is a consequence of the Conjecture 2.1 in [MR1], which can be considered as a "lifting" of Conjecture 1 to Arakelov geometry.

We would also like to point out a general conjecture on Gauss-Manin bundles, which overlaps with Conjecture 1 and is a consequence of Conjecture 3.1 in [MR1].

*Conjecture 3.* Let $X$ and $Y$ be smooth quasi-projective varieties over $\mathbb{C}$. Let $f\colon X \to Y$ be a smooth and projective morphism. Then $\mathrm{ch}(H^l_{\mathrm{dR}}(X/Y)) = \mathrm{rk}(H^l_{\mathrm{dR}}(X/Y))$ in $\mathrm{CH}^l(Y) \otimes \mathbb{Q}$, for all $l \geqslant 1$.

The Conjecture 3 can be related to a conjecture of Bloch and Beilinson.

Suppose that $Y$ is projective over $\mathbb{C}$ and has a model $Y_0$ over a number field. Let $\mathrm{CH}(Y_0)_0$ be the subgroup of $\mathrm{CH}(Y_0) \otimes \mathbb{Q}$ consisting of homologically trivial cycles. Recall that there is a map from $\mathrm{CH}(Y_0)_0$ to the product of the intermediate Jacobians of $Y(\mathbb{C})$, called the Abel-Jacobi map. The conjecture of Bloch and Beilinson is that the Abel-Jacobi map is injective in this situation (see [BB, after lemma 5.6]).

Suppose now furthermore that there is a morphism $X_0 \to Y_0$, such that the morphism obtained after a field extension to $\mathbb{C}$ coincides with $f$. Notice that the classes $\mathrm{ch}(H^l_{\mathrm{dR}}(X_0/Y_0)) - \mathrm{rk}(H^l_{\mathrm{dR}}(X_0/Y_0))$ lie in $\mathrm{CH}(Y_0)_0$, because the bundles $H^l_{\mathrm{dR}}(X_0/Y_0)$ carry an algebraic connection, the Gauss-Manin connection. The Abel-Jacobi map can be described using Cheeger-Simons characteristic classes (see [S, Prop. 2]) and it has been shown by Corlette and Esnault (see [CE]) that the Cheeger-Simons classes of Gauss-Manin bundles vanish. All in all, this implies that the image of the classes $\mathrm{ch}(H^l_{\mathrm{dR}}(X_0/Y_0)) - \mathrm{rk}(H^l_{\mathrm{dR}}(X_0/Y_0))$ under the Abel-Jacobi map vanish and thus Conjecture 3 is implied by the conjecture of Bloch and Beilinson in this situation. The result of Corlette and Esnault could also have been replaced by a general result of Reznikov (see [R]) in this setup.

Finally, we shall indicate how Theorem 2 overlaps with Stickelberger's theorem. Let $K \subseteq \overline{\mathbb{Q}}$ be an abelian extension of $\mathbb{Q}$ and suppose that the conductor of $K$ is $n$. Let $G := \mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$. By class field theory, we have an inclusion $K \subseteq \mathbb{Q}(\mu_n)$ and the group $G$ thus acts on $K$ by restriction. Via this action, we obtain a $\mathbb{Z}[G]$-module structure on the multiplicative group of the ideals of $\mathcal{O}_K$. If $A$ is an ideal in $\mathcal{O}_K$ and $\upsilon \in \mathbb{Z}[G]$, we write $A^\upsilon$ for the image of $A$ under $\upsilon$. We write

$$\theta(K) = \theta := \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \{\frac{a}{n}\}\sigma_a^{-1} \in \mathbb{Q}[G]\,,$$

where $\{\cdot\}$ denotes the fractional part of a real number. The element $\theta(K)$ is called the *Stickelberger element.* Let $\beta \in \mathbb{Q}[G]$ and suppose that $\beta\cdot\theta \in \mathbb{Z}[G]$. Stickelberger's theorem asserts that if $A$ is an ideal of $\mathcal{O}_K$, then $A^{\beta\theta}$ is a principal ideal. In particular $A^{n\theta}$ is principal. Let now $\chi\colon G \to S^1$ be an odd primitive Dirichlet character and let

$$\epsilon_\chi := \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \in \overline{\mathbb{Z}}_{\mathrm{ab}}[G]\,.$$

Here $\overline{\mathbb{Z}}_{\mathrm{ab}}$ is the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}_{\mathrm{ab}}$, the subfield of $\overline{\mathbb{Q}}$ generated by all the roots of unity. Let $L(\chi, s)$ be the $L$-function of $\chi$, which is a meromorphic function of $s \in \mathbb{C}$ (see subsection 4.1 or [W, chap. 4] for the definition). We have $L(\chi, 1) = -B_{1,\chi} := -\sum_{a\in(\mathbb{Z}/n\mathbb{Z})^\times}\{\frac{a}{n}\}\chi(a)$. We compute

$$\epsilon_\chi\theta = \sum_{a\in(\mathbb{Z}/n\mathbb{Z})^\times}\{\frac{a}{n}\}\epsilon_\chi\sigma_a^{-1} = \Big[\sum_{a\in(\mathbb{Z}/n\mathbb{Z})^\times}\{\frac{a}{n}\}\overline{\chi}(\sigma_a)\Big]\epsilon_\chi = B_{1,\overline{\chi}}\,\epsilon_\chi\,.$$

Now identify $\mathrm{CH}^1(\mathcal{O}_K)$ with the class group of $\mathcal{O}_K$. The last computation shows that Stickelberger's theorem implies that $nB_{1,\overline{\chi}}\epsilon_\chi$ annihilates any element of $CH^1(\mathcal{O}_K)\otimes\overline{\mathbb{Z}}_{\mathrm{ab}}$.

On the other hand consider the situation of Theorem 2. With $u_0 := \exp(2i\pi/n)$ Theorem 2 says in particular that

$$\sum_{a\in\mathbb{Z}/n\mathbb{Z}}\zeta_L(u_0^a, 0)\,\mathrm{c}^1(\mathcal{H}_{u_0^a}) = 0$$

in $\mathrm{CH}^1(C)\otimes\overline{\mathbb{Z}}_{\mathrm{ab}}[\frac{1}{n}]$. More generally, let $b\in(\mathbb{Z}/n\mathbb{Z})^\times$ and apply Theorem 2 again, with $\iota^{-b}$ in place of $\iota$. We obtain the identity

$$\sum_{a\in\mathbb{Z}/n\mathbb{Z}}\zeta_L(u_0^{ab}, 0)\,\mathrm{c}^1(\mathcal{H}_{u_0^a}) = 0 \qquad (8)$$

in $\mathrm{CH}^1(C)\otimes\overline{\mathbb{Z}}_{\mathrm{ab}}[\frac{1}{n}]$.

Define the Gauss sum

$$\tau(\chi) := \sum_{a\in\mathbb{Z}/n\mathbb{Z}}\chi(a)u_0^a\,.$$

It is shown in [W, chap. 4, lemma 4.7] that $\sum_{a\in\mathbb{Z}/n\mathbb{Z}}\chi(a)u_0^{ab} = \tau(\chi)\overline{\chi}(b)$ holds for any $b\in\mathbb{Z}$. This implies that

$$\sum_{b\in\mathbb{Z}/n\mathbb{Z}}\overline{\chi}(b)\zeta_L(u_0^{ab}, 0) = \chi(a)\tau(\overline{\chi})L(\chi, 0)\,.$$

We shall now exploit (8). We compute

$$\begin{aligned}\sum_{b\in\mathbb{Z}/n\mathbb{Z}}\overline{\chi}(b)\Big(\sum_{a\in\mathbb{Z}/n\mathbb{Z}}\zeta_L(u_0^{ab}, 0)\,\mathrm{c}^1(\mathcal{H}_{u_0^a})\Big) &= \sum_{a\in\mathbb{Z}/n\mathbb{Z}}\sum_{b\in\mathbb{Z}/n\mathbb{Z}}\overline{\chi}(b)\,\zeta_L(u_0^{ab}, 0)\,\mathrm{c}^1(\mathcal{H}_{u_0^a})\\ &= \tau(\overline{\chi})L(\chi, 0)\sum_{a\in\mathbb{Z}/n\mathbb{Z}}\chi(a)\,\mathrm{c}^1(\mathcal{H}_{u_0^a})\end{aligned}$$

in $\mathrm{CH}^1(C)\otimes\overline{\mathbb{Z}}_{\mathrm{ab}}[\frac{1}{n}]$. Since $\tau(\overline{\chi})\overline{\tau(\overline{\chi})} = |\tau(\overline{\chi})|^2 = n$ (see [W, chap. 4, lemma 4.8]), $\tau(\overline{\chi})$ is a unit in $\overline{\mathbb{Z}}_{\mathrm{ab}}[\frac{1}{n}]$. Hence

$$-L(\chi, 0)\sum_{a\in\mathbb{Z}/n\mathbb{Z}}\chi(a)\,\mathrm{c}^1(\mathcal{H}_{u_0^a}) = B_{1,\chi}\sum_{a\in\mathbb{Z}/n\mathbb{Z}}\chi(a)\,\mathrm{c}^1(\mathcal{H}_{u_0^a}) = 0$$

in $\mathrm{CH}^1(C)\otimes\overline{\mathbb{Z}}_{\mathrm{ab}}[\frac{1}{n}]$. Now suppose furthermore that $D = \mathcal{O}_{\mathbb{Q}(\mu_n)}$ and that the fibration $\mathcal{C}\to C$ and the automorphism $\iota$ have models over $\mathbb{Z}[\frac{1}{n}]$. Fix such models. We then obtain

$$B_{1,\chi}\epsilon_{\overline{\chi}}\,\mathrm{c}^1(\mathcal{H}_{u_0}) = 0$$

where $\mathrm{CH}^1(C)\otimes\overline{\mathbb{Z}}_{\mathrm{ab}}[\frac{1}{n}]$ is considered as $\mathrm{Gal}(\mathbb{Q}(\mu_n)|\mathbb{Q})$-module via the given model of $C$.

Stickelberger's theorem and Theorem 2 thus lead to similar annihilation statements. It is even possible to construct a geometrical situation where Theorem 2 is implied by Stickelberger's theorem. This is left as an exercise to the reader.

One is thus led to speculate whether (the Fourier transform of) Theorem 2 is not a special case of a theorem generalising Stickelberger's theorem to the Chow groups of the various Shimura varieties classifying abelian varieties with complex multiplications.

## References


[BFM] Baum, P.; Fulton, W.; Quart, G.: Lefschetz-Riemann-Roch for singular varieties. Acta Math. **143**, 193–211 (1979).

[BB] Beĭlinson, A. A.: Height pairing between algebraic cycles. $K$-theory, arithmetic and geometry (Moscow, 1984–1986), 1–25, Lecture Notes in Math., 1289, Springer, Berlin (1987).

[BBM] Berthelot, P.; Breen, L.; Messing, W.: Théorie de Dieudonné cristalline. II. Lecture Notes in Mathematics, 930. Springer-Verlag, Berlin (1982).

[CE] Corlette, K.; Esnault, H.: Classes of local systems of hermitian vector spaces. Appendix to "Eta invariants, differential characters and flat vector bundles" by J.-M. Bismut. Preprint (1995).

[EV1] Esnault, H.; Viehweg, E.: Chern classes of Gauss-Manin bundles of weight 1 vanish. $K$-Theory **26**, no. 3, 287–305 (2002).

[EV2] Esnault, H.; Viehweg, E.: Lectures on vanishing theorems. DMV Seminar, 20. Birkhäuser Verlag, Basel (1992).

[CF] Faltings, G.; Chai, C.-L.: Degeneration of abelian varieties. With an appendix by David Mumford. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 22. Springer-Verlag, Berlin (1990).

[F] Fulton, W.: Intersection theory. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2. Springer-Verlag, Berlin (1998).

[FL] Fulton, W.; Lang, S.: Riemann-Roch algebra. Grundlehren der Mathematischen Wissenschaften, 277. Springer-Verlag, New York (1985).

[SGA3] Grothendieck, A. et al.: Schémas en groupes. III: Structure des schémas en groupes réductifs. Séminaire de Géomtrie Algébrique du Bois Marie 1962/64. Lecture Notes in Mathematics, Vol. 153. Springer-Verlag, Berlin-New York (1962/1964).

[H] Hartshorne, R.: Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg (1977).

[K] Köck, B.: The Grothendieck-Riemann-Roch theorem for group scheme actions. Ann. Sci. École Norm. Sup.(4) **31**, no. 3, 415–458 (1998).

[KR1] Köhler, K.; Roessler, D.: A fixed point formula of Lefschetz type in Arakelov geometry I: statement and proof. Invent. Math. **145**, no. 2, 333–396 (2001).

[KR2] Köhler, K.; Roessler, D.: A fixed point formula of Lefschetz type in Arakelov geometry IV: the modular height of C.M. abelian varieties. J. reine angew. Math. **556**, 127–148 (2003).

[L] Lang, S.: Algebra. Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York (2002).

[MR1] Maillot, V.; Roessler, D.: Conjectures sur les dérivées logarithmiques des fonctions $L$ d'Artin aux entiers négatifs. Math. Res. Lett. **9**, no. 5-6, 715–724 (2002).

[MR2] Maillot, V.; Roessler, D.: On the periods of motives with complex multiplication and a conjecture of Gross-Deligne. Annals of Math. **160**, 727–754 (2004).

[M] Matsumura, H.: Commutative ring theory. Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge (1989).

[R] Reznikov, A.: All regulators of flat bundles are torsion. Annals of Math. (2) **141**, no. 2, 373–386 (1995).

[S] Soulé, C. : Classes caractéristiques secondaires des fibrés plats. Séminaire Bourbaki, Vol. 1995/96. Astérisque No. **241**, Exp. No. 819, 5, 411–424 (1997).

[T] Thomason, R. W.: Une formule de Lefschetz en $K$-théorie équivariante algébrique. Duke Math. J. **68**, no. 3, 447–462 (1992).

[T2] Thomason, R. W.: Algebraic $K$-theory of group scheme actions. Algebraic topology and algebraic $K$-theory (Princeton, N.J., 1983), 539–563, Ann. of Math. Stud., 113, Princeton Univ. Press, Princeton, NJ (1987).

[VDG] van der Geer, G.: Cycles on the moduli space of abelian varieties. Moduli of curves and abelian varieties, 65–89, Aspects Math., E33, Viehweg, Braunschweig (1999).

[W] Washington, L. C.: Introduction to cyclotomic fields. Second edition. Graduate Texts in Mathematics, 83. Springer-Verlag, New York (1997).