Mathematical justification of a nonlinear integro-differential equation for the propagation of spherical flames

Claudia LEDERMAN

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 - Buenos Aires, Argentina

Jean-Michel ROQUEJOFFRE

UFR-MIG, UMR CNRS 5640
Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex, France

Noemi WOLANSKI

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 - Buenos Aires, Argentina

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Abstract. – This paper is devoted to the justification of an asymptotic model for quasisteady three-dimensional spherical flames proposed by G. Joulin [17]. The paper [17] derives, by means of a three-scale matched asymptotics, starting from the classical thermo-diffusive model with high activation energies, an integro-differential equation for the flame radius. In the derivation, it is essential for the Lewis Number - i.e. the ratio between thermal and molecular diffusion - to be strictly less than unity. If ε is the inverse of the - reduced - activation energy, the idea underlying the construction of [17] is that (i) the time scale of the radius motion is ε^{-2}, and that (ii) at each time step, the solution is ε-close to a steady solution.

In this paper, we give a rigorous proof of the validity of this model under the restriction that the Lewis number is close to 1 - independently of the order of magnitude of the activation energy. The method used comprises three steps: (i) a linear stability analysis near a steady - or quasi-steady - solution, which justifies the fact that the relevant time scale is ε^{-2}; (ii) the rigorous construction of an approximate solution; (iii) a nonlinear stability argument.

Key words: half derivatives, high activation energies, linear and nonlinear stability, combustion.
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1. Introduction

In an important paper, G. Joulin [17] derives a nonlinear integro-differential equation for the slow motion of the radius of three-dimensional flame balls. This equation is derived from the classical thermo-diffusive model for flame propagation, with simple chemistry $A \to B$; namely

$$(1.1) \qquad \left\{ \begin{array}{rl} T_t - \Delta T = & Y f_\varepsilon(T) \\ Y_t - \dfrac{\Delta Y}{Le} = & -Y f_\varepsilon(T) \end{array} \right. \qquad (I\!R^3)$$

$$(1.2) \qquad T(t, r = +\infty) = 0, \qquad Y(t, r = +\infty) = 1.$$

The notations are classical: $T(t, x)$ is the temperature, $Y(t, x)$ is the mass fraction of the reactant. The number $Le > 0$ is the Lewis number, i. e. the ratio between thermal and molecular diffusion. The function $f_\varepsilon(T)$ is the classical Arrhenius term

$$(1.3) \qquad f_\varepsilon(T) = \frac{1}{\varepsilon^2} \exp \frac{T - Le^{-1}}{\varepsilon}$$

for T close to Le^{-1}. To avoid the cold boundary difficulty, the function f_ε is assumed to vanish for $T \leq \theta$, with $0 < \theta < Le^{-1}$. The only slight difference with the most classical expressions is that, here, the flame temperature has been normalized to Le^{-1}.

The goal of this paper is to give a mathematically rigorous proof of Joulin's derivation [17] of the asymptotic equation

$$(1.4) \qquad (1 - \sqrt{Le})\, \partial_{1/2}\, \rho = 2Le \mathrm{Log}(\sqrt{Le^3}\, \rho) + \phi(\tau)$$

for $R_\varepsilon(\tau/\varepsilon^2)$ where $R_\varepsilon(t)$ is the radius of the flame ball. Here $\phi(\tau)$ is a forcing term depending on the initial datum. The operator $\partial_{1/2}$ is the classical fractional derivative of order $1/2$. For a C^1 function ρ it is defined by

$$\partial_{1/2}\, \rho = \frac{1}{\sqrt{\pi}} \int_0^\tau \frac{\dot{\rho}(s)}{\sqrt{\tau - s}}\, ds.$$

The method that is devised in [17] to derive (1.4) is to notice that, when $Le < 1$, one can identify a slow time scale of the order ε^{-2}. This allows an explicit computation of the temperature and mass fraction at finite distance; the slow motion of the flame is then controlled by the temperature and mass fraction fields at $|x| = +\infty$: if the flame were really steady, then the temperature - resp. the mass fraction - would not, in general, be 0 - resp. 1 - at $|x| = +\infty$, thus violating the conditions (1.2). This imposes the introduction of a large spatial scale, and the computation of the variations of the temperature and mass fraction on this large scale yields, to the first order in ε, an evolution equation for the radius of the flame. These considerations are typical of $Le < 1$; they indeed break down for $Le > 1$: it is conjectured in [17], and proved in the present paper in the context of the evolution of (1.1) around a steady solution, that the characteristic time scale is ε^2.

One interesting aspect of Joulin's method is that it has a wide applicability: it is possible to add complex chemistry, heat loss effects in the burnt or fresh gases, heat losses induced by a wall, heat losses due to turbulence... of course, the more numerous an effect one adds, the richer the dynamics of the asymptotic equation is. A whole class of models is thus available: [18] - preferential diffusion in the presence of more than one chemical reaction; [7] - heat losses, collective effets [2], heat losses trough convection and nonlinear periodic effects [19]... On the mathematical side, the dynamics of models with heat losses have been investigated [23] and numerically simulated [3]. In this particular case, the model under study is

$$\partial_{1/2} R = \mathrm{Log} R - \lambda R^2 + \phi(t), \tag{1.5}$$

where λ may be time-dependent. Nontrivial bounded time-global solutions - i.e. solutions defined for $t \in I\!R$ are constructed.

All in all, the only condition for the retrieval of a nontrivial relevant asymptotic equation is the existence of this characteristic ε^{-2} time scale.

Joulin's method is based on formal asymptotic expansions, i.e. explicit expressions for T and Y are sought, and the higher order terms in ε are dropped. The question that one therefore may ask is the rigorous derivation of all these models, namely: do we have the right to drop the higher order terms in ε? A possible answer is that, if the approximations were not justified, then the limiting equation would certainly not be well-posed. In our case, the asymptotic equation is well-posed if $Le < 1$, and ill-posed if $Le > 1$. Another - and perhaps more interesting - way to pose the problem is the following: can we understand mathematically what makes the difference between the well behaved case $Le < 1$ with the case $Le > 1$, where the high activation asymptotics simply seems to be irrelevant from the dynamical point of view? In other words, what is the mechanism responsible for the well-posedness of the limiting problem when $Le < 1$, and for its ill-posedness when $Le > 1$?

This is a mathematical question that the formal asymptotics alone cannot explain. Therefore, given the breadth and variety of the different asymptotic models and nonlinear effects that can be retrieved from (1.1)-(1.2) - or slight modifications of it - it is worth spending some effort in trying to understand what properties of the original equations trigger the result. If the properties of the equations that come into play are general enough, we will have the hope to justify a whole class of models in which integral operators of the half derivative family come into play.

Needless to say, a lot of the ideas present or underlying in [17] will be of constant help: in particular, the Ariadne tread of the paper is the fact that, in our problem, the time-derivatives - which are usually a handicap in parabolic free boundary problems, because they are very difficult to control - will help us:

they will in general be one order of magnitude - in ε - more than the unknowns themselves.

Let us, for the whole paper, assume that the functions are radially symmetric and let us give the following definition.

Definition 1.1 *Let $A > 0$ be large - for instance $A = 10^{10}$. For any solution (T, Y) of (1.1), the radius $R(t)$ of the solution - in other words, of the flame - is the largest $r > 0$ such that*

$$T(t, r) = \frac{1}{Le} + A\varepsilon Log\varepsilon. \tag{1.6}$$

A first consequence of the fact that we are looking at a problem in $I\!R^3$ is the presence of steady solutions to (1.1); this is a fact dating back to Zeldovich [24]. We will choose in Section 2 to investigate a class of such solutions, parametrized by the real number w, and satisfying namely

$$\begin{cases} -\Delta T = Y f_\varepsilon(T) \\ -\dfrac{\Delta Y}{Le} = -Y f_\varepsilon(T) \end{cases} \qquad (I\!R^3) \tag{1.7}$$

$$T(r = +\infty) = \varepsilon w, \qquad Y(r = +\infty) = 1. \tag{1.8}$$

We will prove the existence of a C^1 family $(T_w^\varepsilon, Y_w^\varepsilon)$ of solutions to the above problem. This being under control, the main result of this paper is the following.

Theorem 1.1 *Set $Le = 1 - \delta$ with $0 < \delta < 1$. Let $(T^\varepsilon(t, r), Y^\varepsilon(t, r))$ be a solution of (1.1), with initial datum $(T_0^\varepsilon, Y_0^\varepsilon)$ such that*

- *there exist $w_0 \in I\!R$, $\nu \in]0, 1[$ and a constant $C > 0$, such that*

  $$|(T_0^\varepsilon, Y_0^\varepsilon)(x) - (T_{w_0}^\varepsilon, Y_{w_0}^\varepsilon)(x)|_{C^1} \leq C\varepsilon^4 \quad \text{for } |x| \leq \varepsilon^{-\nu} \tag{1.9}$$

- *$(T_0^\varepsilon, Y_0^\varepsilon)$ converges to $(0, 1)$ at a certain rate at $r = +\infty$, as $\varepsilon \to 0$,*
- *there exist two constants $C > 0$ and $\omega > 0$ such that*

  $$|(T_0^\varepsilon, Y_0^\varepsilon)(x) - (T_{w_0}^\varepsilon, Y_{w_0}^\varepsilon)(x)| \leq Ce^{-(R_0 - |x|)/\varepsilon} \quad \text{for } |x| \leq R_0 \tag{1.10}$$

  where R_0 is any radius at which $T_{w_0}^\varepsilon$ is $\varepsilon|Log\varepsilon|$ close to $\dfrac{1}{Le}$,
- *$(T_0^\varepsilon, Y_0^\varepsilon)$ converges to $(0, 1)$ at a certain rate at $r = +\infty$, as $\varepsilon \to 0$,*

Set $\tau = \varepsilon^2 t$. Then there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ and a smooth function $\phi(\tau)$, determined by the limiting behavior of the rescaled initial datum at infinity, such that, if $\delta < \delta_0$ and $\varepsilon < \varepsilon_0$, there exists $0 < \tau_{max} \leq +\infty$ such that, if $0 < \tau_0 < \tau_{max}$, the radius $R^\varepsilon(t)$ of the solution $(T^\varepsilon, Y^\varepsilon)$ satisfies that

$$R^\varepsilon\left(\frac{\tau}{\varepsilon^2}\right) \to \overline{R}(\tau), \qquad \textit{uniformly on } [0, \tau_0],$$

and $\overline{R}(\tau)$ is the solution of

$$(1.11) \qquad (1-\sqrt{Le})\,\partial_{1/2}\overline{R} = 2LeLog(\sqrt{Le^3}\,\overline{R}) + \phi(\tau),$$

with

$$(1.12) \qquad R(0) = \frac{1}{\sqrt{Le^3}}\exp\Big(-\frac{w_0}{2}\Big).$$

Remark 1.1 *1. The actual hypotheses on $(T_0^\varepsilon, Y_0^\varepsilon)$ is that it be at distance at most $O(\varepsilon^4)$ in $I\!R^3$ of a pair (T_0, Y_0) constructed in Theorem 4.1. In particular, a pair that coincides with the Zeldovich flame $T_{w_0}^\varepsilon$ in $B(0,\varepsilon^{-\nu})$.*
2. τ_{max} is the lifetime of the maximal solution to (1.11)-(1.12).

The formulation of this result is a good example of what can be expected for such a problem: an $O(\varepsilon)$ perturbation of a steady configuration at $O(\varepsilon^{-1})$ distance implies nontrivial $O(1)$ changes on the ε^{-2} time scale.

The merit of this theorem is to give a justification of equation (1.11), but it is only a first step to a complete understanding of the dynamics of system (1.1), and some questions linger around.

- We have not treated the case of a point source initiated flame; i.e. when $T(0,.) = 0$, $Y(0,.) = 1$ and a source term of the form $q(t)\delta_{x=0}$ is added in the equation for T. This should be tractable with the method that we have devised in this paper, provided that we treat the interaction of the flame sheet with the close field of the heat source at small times. This will be studied in a subsequent work.

- It would be desirable to remove the assumption "Lewis number close to 1". This is far from obvious, for it involves a stability analysis in the flame sheet that is completely open.

- It would be interesting to investigate the large time behaviour of the solutions of (1.1); namely for times much larger than ε^{-2}. A possible scenario is that, when the flame radius is large enough, then a travelling wave behaviour is picked up. The whole process evolves on a time of the order $e^{-\varepsilon^{-1}}$.

  This scenario is backed by formal computations in [6]. It would be very nice to prove - or disprove - such a result.

It is to be noted here that the situation under study differs radically from the case $Le = 1$. When $Le = 1$, the system becomes, setting $W = T + Y - 1$ and $g_\varepsilon(T) = (1-T)f_\varepsilon(T)$:

$$(1.13) \qquad \left\{ \begin{array}{rl} T_t - \Delta T = & g_\varepsilon(T) + W f_\varepsilon(T) \\ W_t - \Delta W = & 0 \end{array} \right. \qquad (I\!R^3)$$

If $W(0,x)=0$, then $W(t,x)=0$ and (1.13) reduces to the scalar equation

$$T_t - \Delta T = g_\varepsilon(T). \tag{1.14}$$

The limit $\varepsilon \to 0$ in (1.14) is studied in [8], [9], [10] for very general initial data. The outcome is that the solutions T_ε of (1.14) converge, in a suitable sense, to viscosity solutions of the free boundary problem

$$\left\{ \begin{array}{rll} T_t - \Delta T = & 0 & (t>0, x \in \Omega_t := \{x:\ T(t,x)<1\}) \\ T=1, \quad \partial_\nu T = & C & (t>0, x \in \partial\Omega_t) \end{array} \right. \tag{1.15}$$

where ν is the outer normal vector to Ω_t and C is a constant determined by the function g_ε. The limit $\varepsilon \to 0$ in (1.13) is studied in [11], when $W(0,x)=\varepsilon w_0(x)$. The solutions T_ε of (1.13) converge, in a suitable sense, to viscosity solutions of (1.15), but the free boundary relation $T_\nu = C$ has to be replaced by a relation of the form $T_\nu = h[w_0]$, where h is a continuous functional.

In any case, the correct time scale for (1.13) and (1.14) is 1, contrary to what happens in the case that we are going to study. The only case where evolution on an $O(1)$ time is rigorously proved for the $Le \neq 1$ case is the classical paper by Berestycki, Nicolaenko and Scheurer, where a 1D travelling wave analysis is carried out for (1.1). It is proved in [4] that the propagation speed is $O(1)$. See also [13], where a study for very general singular terms f_ε is performed, and which discusses thoroughly the value of the velocity.

Coming back to the full problem with $Le \neq 1$, we note that the complete characterisation of the behaviour of the solutions to (1.1) for $\varepsilon \to 0$ is still a very much open problem, and that we are exploring only a small subset of what may happen. In particular, it would be quite interesting to know whether a reasonable singular limit can be derived; from our first conclusions this is far from obvious.

Let us end this introduction by a brief description of the strategy used to prove Theorem 1.1. The first thing to do is to produce steady solutions; this is done in Section 2, where we give a rigorous proof of the existence and uniqueness of Zeldovich solutions.

In Section 3, we study the stability of the Zeldovich solution. Due to the highly singular behaviour of the nonlinearity in terms of ε, unstable eigenvalues of the order ε^{-2} are *a priori* possible. We prove that, when Le is less than unity but close enough - independently of ε - to 1, then the only unstable eigenvalue is a negative eigenvalue of order ε^2: this justifies the ε^{-2} time scale. On the other hand, when Le is more than unity but close enough - independently of ε - to 1, there is an unstable eigenvalue of order ε^{-2}: this explains why the case $Le > 1$ cannot produce quasi-steady solutions. The method that is used here is a bootstrap method which was originally devised in [14] - Theorem 2.1.

In Section 4 we rederive Joulin's equation. In fact, using Joulin's ideas, we construct in a rigorous fashion a class of functions $(T_J(t,x), Y_J(t,x))$ satisfying the system

$$
(1.16)\quad \begin{cases} T_t - \Delta T = & Yf_\varepsilon(T) + O(\varepsilon^4)\mathbf{1}_{|x|\le\varepsilon^{-\nu}} + O(\varepsilon^4)\delta_{|x|=\varepsilon^{-\nu}} \\ Y_t - \dfrac{\Delta Y}{Le} = & -Yf_\varepsilon(T) + O(\varepsilon^4)\mathbf{1}_{|x|\le\varepsilon^{-\nu}} + O(\varepsilon^4)\delta_{|x|=\varepsilon^{-\nu}} \end{cases} \qquad (I\!R^3)
$$

$$
(1.17)\qquad T(t, r = +\infty) = 0, \qquad Y(t, r = +\infty) = 1.
$$

In fact, a slightly less accurate approximate solution will be needed; our precise needs will be stated in Section 4. In the end, the solution that is constructed is Lipschitz in space and time and, if $R_J(t)$ is its radius, then $R_J(\tau/\varepsilon^2)$ satisfies the Joulin equation (1.4) up to $O(\varepsilon)$ errors.

The next step is then to show that, on time intervals of order ε^{-2}, the solutions of (1.1) remain ε^2-close to the ones of (1.16), provided the initial data are ε^2-close. This is a nonlinear stability result, with the additional inconvenient that the essential spectrum of the linearized operator around a Zeldovich steady solution - or a frozen Joulin solution - contains a segment $[0, M]$. This forces us to devise very precise decay estimates on the linear semigroup associated to the linearized operator around a frozen Joulin solution; once we have them the fact that the linear operator evolves on the ε^{-2} time scale allows us to conclude. These estimates, as well as the strategy of the proof at this point, are - although we formulate them in a different way - very close in spirit to the work of Grenier and Rousset [15]: indeed, in [15], the stability of viscous boundary layers is investigated through a Green's functions approach - an approach initiated by Zumbrun and Howard [25] for the study of the stability of travelling waves for viscous systems of conservation laws.

In order to apply our stability argument, two sections are needed for this task: Section 5 presents decay estimates on the linear semigroup associated to the linearized operator around a a frozen Joulin solution, and the convergence proof is terminated in Section 6.

All the results described here were announced in [21], which may serve th ereader as a guideline.

Note. *In the whole paper, all the functions that will be considered shall have the spherical symmetry; hence they will only depend on r. However, to keep the notations lighter, we will sometimes use functions of several variables notations, such as Δ, ∇...*

2. The basis of the analysis: the Zeldovich flame ball

Let w be a parameter. We look for a solution $(T_w^\varepsilon, Y_w^\varepsilon)$ of

$$
(2.1)\qquad \begin{cases} -\Delta T = Yf_\varepsilon(T) \\ -\dfrac{\Delta Y}{Le} = -Yf_\varepsilon(T) \end{cases} \qquad (I\!R^3)
$$

$$(2.2) \qquad T(r=+\infty)=\varepsilon w, \qquad Y(r=+\infty)=1.$$

The Zeldovich flame ball is usually the solution with $T(+\infty)=0$. Notice that, if $\varepsilon>0$ is small enough so that $\varepsilon|w|$ is below θ, Problem (2.1)-(2.2) is not ill-posed. In all cases we have $W:=T+\dfrac{Y}{Le}-\dfrac{1}{Le}=\varepsilon w$, and T solves

$$(2.3) \qquad \left\{ \begin{array}{ll} -\Delta T = Le\,(\varepsilon w - T + Le^{-1}) f_\varepsilon(T) & (I\!\!R^3) \\ T(|x|=+\infty)=\varepsilon w & \end{array} \right.$$

Proposition 2.1 *(Zeldovich; [24]) For any given $M>0$ large enough, there is $\varepsilon_0>0$ such that, for all $\varepsilon\le\varepsilon_0$, Problem (2.3) has a unique solution $T_w^\varepsilon(r)$, which is C^3 with respect to w - in the topology of $C^2(I\!\!R^3)$, decreasing with respect to r, and whose radius is between $\dfrac{1}{M}$ and M.*

Let $R_{\varepsilon,w}$ be the radius of this solution; we have

$$\lim_{\varepsilon\to 0} R_{\varepsilon,w}=\frac{1}{\sqrt{Le^3}}\exp(-\frac{w}{2}).$$

Also, we point out that T_w^ε has a lot more regularity than stated, but we will not need more.

Remark 2.1 *There should exist spurious solutions not exceeding $\dfrac{1}{Le}+A\varepsilon Log\varepsilon$; they can probably be obtained by a degree argument. Should they exist, they would be unstable and will not be considered here.*

The proposition will be proved - as is classical - by a rigorous matching of the derivatives at a point where the temperature is $\dfrac{1}{Le}+O(\varepsilon\mathrm{Log}\varepsilon)$. The computation of the inner derivative relies on the following theorem, which is in fact more interesting than the proposition itself. Set

$$(2.4) \qquad g_w(p)=Le(w-p)e^p$$

and, for $R>0$ let us consider the problem

$$(2.5) \qquad \begin{array}{lll} -\Delta p & = g_w(p) & (|x|\le\dfrac{R}{\varepsilon}) \\ p(\dfrac{R}{\varepsilon}) & = A\mathrm{Log}\varepsilon & \end{array}$$

Recall that $A>>1$ is defined in the introduction.

Theorem 2.1 *For $\varepsilon>0$ small enough, Problem (2.5) has a unique solution $p_{w,R}$, which is C^3 with respect to w and R in the topology of $C^2(I\!\!R^3)$, and which takes the value $\dfrac{-C+w}{2}$ somewhere in $B_{R/\varepsilon}\backslash B_{R/2\varepsilon}$, where C is a constant strictly between $ALog\varepsilon$ and w. Moreover*

[i]. The C^3 estimates for $p_{w,R}$ are of order $Log^2\varepsilon$ at most,
[ii]. the operator

$$\mathcal{U}_{\varepsilon,R} := -\Delta - g'_w(p_{w,R}) \tag{2.6}$$

with Dirichlet conditions on $\{r = \dfrac{R}{\varepsilon}\}$ is an isomorphism between its domain and $C(B_{R/\varepsilon})$; the norm of its inverse is of order at most $|Log\varepsilon|$; this estimate is uniform for R bounded and bounded away from 0.

Proof. We break it into several steps. In the whole proof, the constants will depend on R, but will be uniform for R bounded and bounded away from 0, and w bounded. This will always be understood to hold.

1. A family of subsolutions. Let us consider the annulus

$$C_\varepsilon = \{\frac{R}{\varepsilon} + B\text{Log}\varepsilon \leq |x| \leq \frac{R}{\varepsilon}\}. \tag{2.7}$$

with $B > 0$ large. Choose $\delta > 0$ small; we claim the existence of a solution $\underline{p}_\varepsilon$ to the problem

$$\begin{array}{rl} -\Delta p = & -Le(p + \delta - w)e^p \qquad (C_\varepsilon) \\ p(\dfrac{R}{\varepsilon}) = & A\log\varepsilon, \quad p(\dfrac{R}{\varepsilon} + B\text{Log}\varepsilon) = w - \delta \end{array} \tag{2.8}$$

Indeed, there exists a unique solution $P_{w,R}(r)$ to the problem

$$\begin{array}{rl} -\dfrac{d^2P}{dr^2} = & -Le(P - w)e^P \qquad (-\infty \leq r \leq \dfrac{R}{\varepsilon}) \\ \\ P(\dfrac{R}{\varepsilon}) = & A\text{Log}\varepsilon, \quad P(-\infty) = w \end{array} \tag{2.9}$$

which is additionally decreasing in r: we have

$$\frac{1}{2}(P'_{w,R})^2 - G_w(P_{w,R}) = 0, \tag{2.10}$$

where

$$G_w(p) = \int_p^w g_w(q)\ dq.$$

Also, we have the estimate

$$P_{w,R}(r) - w = O(e^{r - R/\varepsilon - BLog\varepsilon}) \tag{2.11}$$

for $r \leq \dfrac{R}{\varepsilon}$. To see it, we first prove with the aid of (2.9) that $P_{w,R}$ becomes larger than $w - \delta$ for $r \leq \dfrac{R}{\varepsilon} + B\text{Log}\varepsilon$, then we conclude by the stable manifold theorem.

Then, a slight modification of the proof of [14], Prop. 2.4 implies that the linear operator

$$\tilde{L}_0^\varepsilon := \frac{d^2}{dr^2} - (P_{w,R} + 1 - w)e^{\underline{q}_{w,R}} \tag{2.12}$$

around $P_{w,R}$, with domain in $C_0^2(]-\infty, \frac{R}{\varepsilon}])$ - Dirichlet condition at $r = \frac{R}{\varepsilon}$ - is an isomorphism with inverse norm of order $|\mathrm{Log}\varepsilon|$ at most; this deteriorates only if $|w|$ becomes very large. We have set

$$C_0^2(]-\infty, \frac{R}{\varepsilon}]) = \{u \in C^2([\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon), \frac{R}{\varepsilon}]), \quad u(\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon) = u(\frac{R}{\varepsilon}) = 0\}$$

and $C(C_\varepsilon)$.

Consider now the solution of (2.9) with $w - \delta$ instead of w, for some small $\delta > 0$. Let L_δ^ε be the same linearized operator as in (2.12), but with $P_{w,R}$ replaced by $P_{w-\delta,R}$. From the equation for $\underline{p}$ in (2.8), we see that we are looking for a solution $\underline{p}$ of an equation of the form

$$\underline{p} - P_{w-\delta,R} = (L_0^\varepsilon)^{-1}(\frac{2}{r}\underline{p}' + Q(\underline{p} - P_{w-\delta,R}). \tag{2.13}$$

with $Q(u) = O(u^2)$. However we have $\frac{2}{r} = O(\varepsilon)$; hence equation (2.13), with unknown $\underline{p} - P_{w-\delta,R}$ has a solution which is $O(\varepsilon|\mathrm{Log}\varepsilon|)$ in C^2 norm. From (2.11) we have - in fact, much more is true but we will only need the following estimate for the time being:

$$w - \underline{p}(\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon) = O(\varepsilon|\mathrm{Log}\varepsilon|). \tag{2.14}$$

Now, for $\omega > 0$, the function $\tilde{p}(|x|)$ given by

$$\tilde{p}(|x|) = w - \delta \frac{\mathrm{sh}(\omega|x|)}{\mathrm{sh}\Big(\omega(\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon)\Big)} \frac{\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon}{|x|} \tag{2.15}$$

solves

$$-\Delta\tilde{p} = -\omega^2(\tilde{p} - w) \text{ in } (B_{R/\varepsilon + BLog\varepsilon}), \quad \tilde{p}(|x| = \frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon) = w - \delta;$$

hence it is a sub-solution to (2.8) as soon as

$$\omega \leq \sqrt{2Le}\, e^{w-\delta/2}.$$

Further we have

$$\tilde{p}_r(|x| = \frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon) = -\delta\Big(\omega\frac{\mathrm{ch}(\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon)}{\mathrm{sh}(\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon)} - \frac{1}{\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon}\Big) = -\delta\omega + O(\varepsilon);$$

hence the function defined by

$$\underline{p}_{w,R}(x) = \begin{cases} \underline{p}(|x|) & \text{if } \dfrac{R}{\varepsilon} + B\mathrm{Log}\varepsilon \leq |x| \leq \dfrac{R}{\varepsilon} \\ \tilde{p}(|x|) & \text{otherwise} \end{cases}$$

is a subsolution to (2.5) as soon as ε is small enough, provided that ω assumes the correct value.

2. Construction of $p_{w,R}$. Obviously, the function w is a super-solution to (2.5) as soon as w is $O(1)$ with respect to ε; therefore there is a solution of (2.5) between $\underline{p}_{w,R}$ and w. However, we are not completely sure that this solution meets all the requirements of the theorem, namely to take the value $\dfrac{-C+w}{2}$ in $B_{R/\varepsilon} \backslash B_{R/2\varepsilon}$. To remedy this, let us notice that $\underline{p}_{(-C+w)/2,R/2}$, extended by $A\mathrm{Log}\varepsilon$ outside $B_{R/2\varepsilon}$, is still a subsolution to (2.5); hence there is a solution of (2.5) between $\underline{p}_{(-C+w)/2,R/2}$ and w; this is the solution that we are going to call $p_{w,R}$.

Let us investigate some properties of the solutions of (2.5). First, let us notice that $p'_{w,R} \leq 0$: this comes from the fact that $A\mathrm{Log}\varepsilon$ is a subsolution, and from Gidas-Ni-Nirenberg [12].

Then let us prove that all solutions to (2.5) are above $\underline{p}_{w,R+B\varepsilon Log\varepsilon}$; assume indeed that this is not true: assume first that there is a a maximal $\overline{w} < w$ between $\dfrac{-C+w}{2}$ and w such that, for a all $w' \in [\dfrac{-C+w}{2}, \overline{w}]$, there holds $\underline{p}_{w',R/2} \leq p_{w,R}$; due to the boundary conditions there is $\overline{r} \in [0, \dfrac{R}{2}[$ such that $\underline{p}_{\overline{w},R/2}(\overline{r}) = p_{w,R}$. If $\underline{p}_{\overline{w},R/2}$ is smooth in the vicinity of $\overline{r}$, there is a contradiction with the strong maximum principle. If not, let us simply notice that there is a solution to (2.5) between $\underline{p}_{\overline{w},R/2}$ and $p_{w,R}$, that is therefore equal to $p_{w,R}$ at $r = \overline{r}$; this is again a contradiction. Hence we have proved

$$\underline{p}_{w,R/2} \leq p_{w,R}.$$

Now, to go from $\dfrac{R}{2}$ to $R+B\varepsilon\mathrm{Log}\varepsilon$, one compares exactly $\underline{p}_{w,R'}$ for all $R' \in [\dfrac{R}{2}, R+B\varepsilon\mathrm{Log}\varepsilon]$. This proves that the constructed solution $p_{w,R}$ reaches C somewhere between $r = \dfrac{R}{2\varepsilon}$ and $r = \dfrac{R}{\varepsilon}$.

3. Spectral properties. Consider $u \in H_0^1(B_{R/\varepsilon})$; we have

$$\begin{aligned}
&\int_{B_{R/\varepsilon}} (|\nabla u|^2 - g'_w(p_{w,R})u^2)\ dx \\
= &\int_{R/\varepsilon + BLog\varepsilon \le r \le R/\varepsilon} + \int_{r \le R/\varepsilon + BLog\varepsilon} \\
\ge &\int_{R/\varepsilon + BLog\varepsilon \le r \le R/\varepsilon} (|\nabla u|^2 - g'_w(P_{w,R})u^2)\ r^2 dr + O(\varepsilon \mathrm{Log}\varepsilon) \int_{R/\varepsilon + BLog\varepsilon \le r \le R/\varepsilon} u^2\ dx \\
&+ \frac{|g'_w(w)|}{2} \int_{r \le R/\varepsilon + BLog\varepsilon} u^2\ dr \\
= &\ I + II + III
\end{aligned}$$

From [14], Prop. 2.1, we have, for some $C > 0$:

$$\begin{aligned}
I \ge &\ (\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon)^2 \int_{R/\varepsilon + BLog\varepsilon}^{\frac{R}{\varepsilon}} ((u')^2 - g'_w(P_{w,R})u^2)\ dr \\
\ge &\ (\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon)^2 \frac{1}{|\mathrm{Log}\varepsilon|} \int_{R/\varepsilon + BLog\varepsilon}^{\frac{R}{\varepsilon}} u^2\ dr \\
\ge &\ \frac{C}{|\mathrm{Log}\varepsilon|} \int_{R/\varepsilon + BLog\varepsilon \le r \le R/\varepsilon} u^2\ dx
\end{aligned}$$

This all in all implies

$$\int_{B_{R/\varepsilon}} (|\nabla u|^2 - g'_w(p_{w,R})u^2)\ dx \ge \frac{C}{|\mathrm{Log}\varepsilon|} \int_{B_{R/\varepsilon}} u^2\ dx.$$

By interpolation, we see that $\mathcal{U}_{\varepsilon,R}$ is invertible from its domain into $C(B_{R/\varepsilon})$, with norm of at most of $\mathcal{U}_{\varepsilon,R}^{-1}$ of order $|\mathrm{Log}\varepsilon|$.

4. Uniqueness and smoothness with respect to the parameters. If $p_{w,R}$ and $\tilde{p}_{w,R}$ are two solutions of (2.5), we wish to prove that

$$\|p_{w,R} - \tilde{p}_{w,R}\|_{L^\infty(B_{R/\varepsilon})} = O(\varepsilon \mathrm{Log}^2\varepsilon). \tag{2.16}$$

Let $p_{w,R}$ be any solution. First, let us notice that step **2** implies the inequality

$$\underline{p}_{w,R+B\varepsilon Log\varepsilon} \le p_{w,R} \le w \tag{2.17}$$

which in turn implies, together with (2.15), the existence of $\omega > 0$ - depending on w but bounded away from 0 if w is kept in a fixed compact set - such that

$$\forall r \le \frac{R}{\varepsilon} + \sqrt{B}\mathrm{Log}\varepsilon, \qquad |p_{w,R}(r) - w| \le Ce^{-\omega(R/\varepsilon - r)}. \tag{2.18}$$

Hence, by elliptic regularity, $p'_{w,R}$ is estimated by the same quantity. This has a consequence: Because of (2.18) and (2.10) we have

$$(2.19)\quad \frac{1}{2}(p'_{w,R})^2 - 2\int_r^{R/\varepsilon+BLog\varepsilon} \frac{(p'_{w,R})^2}{r}\, dr - G_w(p_{w,R}) = O(\varepsilon^{\omega B}) = O(\varepsilon)$$

if B is large enough. Remember that $p'_{w,R} < 0$; hence equation (2.19) may be thought of as an equation on $p'_{w,R}$ in terms of $p_{w,R}$, the values of $p_{w,R}$ ranging from $A\mathrm{Log}\varepsilon$ to $w + O(\varepsilon^{\omega B})$. Let us consider $\delta > 0$ such that

$$\forall p \geq w - \delta, \qquad g'_w(p) \leq \frac{g'_w(w)}{2} < 0.$$

Consider the range of values $p_{w,R} \in [A\mathrm{Log}\varepsilon, w-\delta]$; there exists a constant $k_\delta > 0$ such that equation (2.19) and inequality (2.17) imply the following inequalities:

$$(2.20)\qquad 1 - k_\delta \varepsilon|\mathrm{Log}\varepsilon| \leq \int_{ALog\varepsilon}^{p_{w,R}} \frac{dp}{\sqrt{G_w(p)}} \leq 1 + k_\delta\varepsilon|\mathrm{Log}\varepsilon|.$$

This readily implies that $p_{w,R}(r) = \tilde{p}_{w,R}(r) + O(\varepsilon\mathrm{Log}\varepsilon)$ for all r such that $p_{w,R}(r) \in [A\mathrm{Log}\varepsilon, w-\delta]$ and for any other solution $p_{w,R}$ of (2.5) satisfying the requirements of Theorem 2.1. Because $p_{w,R}$ is decreasing in r, the complementary of this set is a ball $B_{\frac{R_\delta}{\varepsilon}}$; we have R_δ controlled from below due to the inequality (2.17). Inside $B_{\frac{R_\delta}{\varepsilon}}$, because both $p_{w,R}$ and $\tilde{p}_{w,R}$ are decreasing, we have

$$h_w(r) := \frac{g_w(p_{w,R}) - g_w(\tilde{p}_{w,R})}{p_{w,R} - \tilde{p}_{w,R}} \leq \frac{g'_w(w)}{2}.$$

The difference $p := p_{w,R} - \tilde{p}_{w,R}$ satisfies the elliptic equation

$$\begin{cases} -\Delta p - h_w(r)p & = 0 \quad (r \leq \dfrac{R_\delta}{\varepsilon}) \\ \qquad p(\dfrac{R_\delta}{\varepsilon}) & = O(\varepsilon\mathrm{Log}\varepsilon) \end{cases}$$

This implies $p_{w,R} - \tilde{p}_{w,R} = O(\varepsilon\mathrm{Log}^2\varepsilon)$ in $B_{R_\delta/\varepsilon}$; hence everywhere.

At this stage, uniqueness is not too difficult: all solutions of (2.5) that manage to reach the value $\dfrac{-C+w}{2}$ sufficiently fast are $\varepsilon\mathrm{Log}^2\varepsilon$ close to each other; on the other hand, given any such solution $p_{w,R}$, we have $\mathcal{U}_{\varepsilon,R}$ invertible with $\dfrac{1}{|\mathrm{Log}\varepsilon|}$ norm: this implies uniqueness. From the implicit function theorem, $p_{w,R}$ is smooth with respect to all its parameters; and the estimates for the derivatives are straightforward. $\square$

Corollary 2.1 *There is a function $\pi_e(R,w)$, which is C^2 in R and w, with C^2 estimates that are uniform in ε, such that*
(i) $p'_{w,R}(R/\varepsilon) = \pi_\varepsilon(R,w)$,
(ii) we have $\pi_\varepsilon(R,w) = -\sqrt{2G_w(-\infty)}+O(\varepsilon|Log^2\varepsilon|) = -\sqrt{Le}\exp\dfrac{w}{2}+O(\varepsilon|Log^2\varepsilon|)$, in C^1.

The proof follows directly from Theorem 2.1 and is omitted.

Remark 2.2 *From [12] once again, any solution to (2.3) is decreasing in r; hence $f_\varepsilon(T)$ is less than ε^A: a very small quantity. In order to keep the already heavy estimates to a minimum, we will adopt the convention that it is exactly 0. This could be made rigorous at low cost, simply by imposing the ignition temperature at, say, $\dfrac{1}{Le}+\dfrac{A}{2}\varepsilon Log\varepsilon$.*

Proof of Proposition 2.1. Consider $M>0$ large, and $R\in[\dfrac{1}{M},M]$. Let T be a solution of

$$\begin{cases} -\Delta T = \dfrac{Le}{\varepsilon^2}(\varepsilon w - T + Le^{-1})e^{\frac{(T-1/Le)}{\varepsilon}} & (|x|\le R) \\ T(|x|=R) = \dfrac{1}{Le} + A\varepsilon\mathrm{Log}\varepsilon \end{cases}$$

such that the scaled version of T:

$$p(r) = \frac{T(\varepsilon r) - \dfrac{1}{Le}}{\varepsilon} \tag{2.21}$$

reaches a constant $C\in]A\mathrm{Log}\varepsilon, w[$ within the annulus $\left\{\dfrac{R}{2\varepsilon}\le r\le\dfrac{R}{\varepsilon}\right\}$. From Theorem 2.1, we have $p(r)=p_{w,R}(r)$ and $T'(R)=\pi_\varepsilon(R,w)$. Using Remark 2.2 we have

$$T(r) = \frac{1}{Le} + A\varepsilon\mathrm{Log}\varepsilon + (R-\frac{R^2}{r})\pi_\varepsilon(R,w)$$

for $r>R$. Therefore if we want the limit of T to be εw as $|x|\to+\infty$, we must have

$$\varepsilon w = \frac{1}{Le} + A\varepsilon\mathrm{Log}\varepsilon - \sqrt{Le}\,e^{w/2}R + O(\varepsilon\mathrm{Log}^2\varepsilon);$$

hence the limiting value for R.

On the other hand, if T is a solution of (2.1)-(2.2), and if $R\in[\dfrac{1}{M},M]$ is its radius, then

$$RT'(R) = -\frac{1}{Le} + A\varepsilon\mathrm{Log}\varepsilon - \varepsilon w.$$

The function p defined by (2.21) has its derivatives bounded in $B_{R/\varepsilon}$; moreover, taking into account that it is decreasing in r, we have

$$-\frac{p'^2}{2}-\frac{1}{R}(\frac{1}{Le}+A\varepsilon\mathrm{Log}\varepsilon+\varepsilon w)=\int_{ALog\varepsilon}^{p} g_w(p')dp',$$

which implies that p reaches a nontrivial value in $B_{R/\varepsilon}\backslash B_{R/2\varepsilon}$. Uniqueness follows from Theorem 2.1. □

In order to construct the Zeldovich solution, it was natural to work with the temperature, because this is the quantity that controls the flame radius. However, when we look at the full evolution system, it will be more convenient to work in the couple of variables (mass fraction, enthalpy). In the notations of Theorem 2.1, let us set

$$q_{w,R}(\xi)=Le(w-p_{w,R}(\xi)),\qquad h_w(q)=-Lee^{w-q/Le}q. \tag{2.22}$$

Notation

If R is the radius of the flame ball, we define R_1 as the radius such that $q_{w,R}(r=\dfrac{R_1}{\varepsilon})=Le$; there exists a constant $k>1$ independent of A - A is defined by (1.6) - such that

$$\frac{1}{k}A\varepsilon|\mathrm{Log}\varepsilon|\le R-R_1\le kA\varepsilon|\mathrm{Log}\varepsilon| \tag{2.23}$$

Let us also introduce the norm $N^0_{\omega,R,\varepsilon}$ given by

$$N^0_{\omega,R,\varepsilon}(v)=\|v\|_\infty+\sup_{r\le R/\varepsilon+Blog\varepsilon}\Big(\exp\Big(\omega(\frac{R}{\varepsilon}+B\mathrm{Log}\varepsilon-r)\Big)|v(r)|\Big). \tag{2.24}$$

For all k in $\mathbb{N}$, $N^k_{\omega,R,\varepsilon}(v)$ is the sum of all $N^0_{\omega,R,\varepsilon}(v^{(i)})$; $i\le k$. Let us then introduce the operator

$$\mathcal{T}_{\varepsilon,R}:=-\Delta-h'_w(q_{w,R}) \tag{2.25}$$

with domain

$$D(\mathcal{T}_{\varepsilon,R})=\{v\in C^2(B_{R/\varepsilon}),\ v(|x|=\frac{R}{\varepsilon})=0\}.$$

The operator $\mathcal{T}_{\varepsilon,R}$ has the same properties as the operator $\mathcal{U}_{\varepsilon,R}$, and we will not repeat the proof. Instead, we will prove a more precise spectral result which will be of direct use in the derivation of spectral estimates for the full linearized problem. Start with the following

Proposition 2.2 *$\mathcal{T}_{\varepsilon,R}$ is self-adjoint in $L^2(B_{R/\varepsilon})$, and there is $b>0$ and such that*

$$\mu_1(\mathcal{T}_{\varepsilon,R})\ge\frac{b}{|Log\varepsilon|} \tag{2.26}$$

Here $\mu_1(\mathcal{T}_{\varepsilon,R})$ is the bottom of the spectrum of $\mathcal{T}_{\varepsilon,R}$. The proof of the proposition is omitted.

Consider a large $D > 10^{10}b$ and $\rho \in]\dfrac{2D}{|\mathrm{Log}\varepsilon|}, 1[$. Consider the domain of the complex plane

$$\Delta_{\varepsilon,\rho} = \{\mu \in \mathbb{C}, \quad |\mu| \le \rho, \ |\arg(\mu + \frac{b}{2\mathrm{Log}\varepsilon})| \le \frac{\pi}{4}\}. \tag{2.27}$$

Proposition 2.3 *For $\varepsilon > 0$ small enough, and for all $\mu \notin \Delta_{\varepsilon,\rho}$, the operator $\mathcal{T}_{\varepsilon,R} - \mu I$ is an isomorphism between its domain and $C(B_{R/\varepsilon})$; moreover*

$$N^0_{\omega,R,\varepsilon}(\mathcal{T}_{\varepsilon,R} - \mu I) \le \frac{C}{|\ \mu + \dfrac{b}{Log\,\varepsilon}|}.$$

This estimate is uniform for R bounded and bounded away from 0.

Proof. **1. L^∞ estimate.** Let us consider $\mu := \mu_1 + i\mu_2 \in \Delta_{\varepsilon,\rho}$, $f \in L^2(B_{R/\varepsilon})$ and $v \in H^2(B_{R/\varepsilon}) \cap H^1_0(B_{R/\varepsilon})$ such that

$$\mathcal{T}_{\varepsilon,R} v = \mu v + f; \tag{2.28}$$

taking, as usual, the L^2 scalar product and applying Proposition 2.2 we obtain

$$\frac{b}{|\mathrm{Log}\varepsilon|}\|v\|_2^2 \le \mu_1\|v\|_2^2 + \mathrm{Re}(\int_{B_{R/\varepsilon}} f\overline{v}), \qquad 0 = \mu_2\|v\|_2^2 + \mathrm{Im}(\int_{B_{R/\varepsilon}} f\overline{v});$$

this implies, due to the fact that $\mu \in \Delta_{\varepsilon,\rho}$ $\Big($if $-\mu_1 - \dfrac{b}{\mathrm{Log}\varepsilon} \le 0$ we just use the second inequality and bound from below a portion of $|\mu_2|$ by a small portion of $|\mu_1 + \dfrac{b}{\mathrm{Log}\varepsilon}|\Big)$:

$$\Big(|\mu_1 + \frac{b}{\mathrm{Log}\varepsilon}| + |\mu_2|\Big)\|v\|_2 \le \|f\|_2,$$

hence the spectral inequality for the L^2 norm. Sobolev imbeddings and the fact that we have $|\mu| \le 1$ yields

$$\|\mathcal{T}_{\varepsilon,R} - \mu I\|_{\mathcal{L}\left(C(B_{R/\varepsilon})\right)} \le \frac{C}{|\mu + \dfrac{b}{\mathrm{Log}\varepsilon}|}.$$

2. Exponential estimate. For $r \le \dfrac{R}{\varepsilon} + B\mathrm{Log}\varepsilon$, with $B > 0$ large enough, we have

$$h'_w(q_{w,R}) = -Le\,e^{w - q_{w,R}/Le}(1 - \frac{q_{w,R}}{Le}) = -Le\,e^{w}(1 + O(e^{-\omega(R/\varepsilon + BLog\varepsilon - r)}))$$

Consider therefore once again $f \in C(B_{R/\varepsilon})$ and $v \in D(\mathcal{T}_{\varepsilon,R})$ solution of (2.28); if ρ is less than $\dfrac{Le\,e^w}{2}$, then

- $\mu - Le\, e^w \notin I\!R_+^*$, hence we may define $\sqrt{Le\, e^w - \mu}$;

- if we set
$$v_1(r) = v(\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon)\frac{\mathrm{sh}(\sqrt{Le\, e^w - \mu}\, r)}{\mathrm{sh}\left(\sqrt{Le\, e^w - \mu}(\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon)\right)}\frac{\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon}{r}$$
then there exists $C > 0$ independent of ε and w bounded and R bounded and bounded away from 0 such that

$$N^0_{\omega,R,\varepsilon}(v_1) \leq C\|f\|_\infty. \tag{2.29}$$

Set
$$\tilde{v} = v - v_1;$$
we have, in $B_{R/\varepsilon + BLog\varepsilon}$:
$$-\Delta\tilde{v} + (Le\, e^w - \mu)\tilde{v} = f + O(e^{-\omega(R/\varepsilon + BLog\varepsilon - r)})v := \tilde{f};$$
because of Step 1 we readily have
$$N^0_{\omega,R,\varepsilon}(\tilde{f}) \leq CN^0_{\omega,R,\varepsilon}(f). \tag{2.30}$$
On the other hand, we have
$$\begin{aligned} r\tilde{v}(r) = & \int_0^r \frac{\mathrm{sh}(\sqrt{Le\, e^w - \mu}(r - r^*))}{\sqrt{Le\, e^w - \mu}}\tilde{f}(r^*)\; dr^* \\ & - \frac{\mathrm{sh}(\sqrt{Le\, e^w - \mu}r)}{\mathrm{sh}(\sqrt{Le\, e^w - \mu}(\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon))} \\ & \times \int_0^{R/\varepsilon + BLog\varepsilon} \frac{\mathrm{sh}(\sqrt{Le\, e^w - \mu}(\frac{R}{\varepsilon} + B\mathrm{Log}\varepsilon - r^*))}{\sqrt{Le\, e^w - \mu}}\tilde{f}(r^*)\; dr^* \end{aligned} \tag{2.31}$$
Although this expression looks rather formidable, it is only an exercise to prove that $N^0_{\omega,R,\varepsilon}(\tilde{v}) \leq CN^0_{\omega,R,\varepsilon}(\tilde{f})$. Gathering this information with (2.29) and (2.30) yields the desired estimate. □

3. Uniform spectral estimates, dispersion relation and instability of the Zeldovich flame ball

In this section the temperature of the flame ball will be assumed to go to 0 at $r = +\infty$ - hence $w = 0$. Let $\mathcal{L}^\varepsilon$ be the linearized operator about the steady flame ball:

$$\mathcal{L}^\varepsilon(u,v) = \begin{pmatrix} -\Delta u - f_\varepsilon'(T_0)Y_0 u - f_\varepsilon(T_0)v \\ \dfrac{\Delta v}{Le} + f_\varepsilon'(T_0)Y_0 u + f_\varepsilon(T_0)v \end{pmatrix} \tag{3.1}$$

The radius of this flame ball is denoted by R_0. We are interested in the eigenvalue problem

$$\mathcal{L}^\varepsilon(u,v) = \lambda(u,v), \qquad (u,v) \in L^2(I\!R^3). \tag{3.2}$$

We wish to prove the following:

Theorem 3.1 *Assume $Le < 1$: $Le = 1 - \delta$, $\delta > 0$. There exists $\delta_0 > 0$ such that, for all $\delta \in]0, \delta_0[$, the following is true: there exist three constants $m > 0$, $M > 0$ and $\theta \in [0, \frac{\pi}{2}[$, possibly depending on δ, such that the only eigenvalue of (3.2) outside the set*

$$\{\lambda \in C: \ \arg(\lambda + M) \le \theta, \quad \mathrm{Re}\lambda \ge m\}$$

is a complex number $\varepsilon^2\lambda_\varepsilon$ such that

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon = -\Big(\frac{1+\sqrt{Le}}{\delta}\Big)^2 R_0^2 Le^6 e^{2w_0}.$$

We will also see that the segment $[0, m]$ is in the essential spectrum of $\mathcal{L}^\varepsilon$.

Remark 3.1 *In the whole section, a sub or superscript ε should appear with almost every function of quantity that we will have to consider. However, we consider that the notations are already heavy enough, and that it will be clear from the context that, every time that we will allow ourselves to do without this sub or superscript, the dependence in ε will be smooth.*

3.1. Notations

The different proofs will involve various subsets of the complex plane; therefore it is worth gathering here all the notations that we are going to use.

- If $\rho > 0$ and $z \in \mathbb{C}$, then $D(z, \rho)$ is the disk of centre z and radius ρ.
- If $z \in \mathbb{C}$ and $\theta \in [0, \frac{\pi}{2}[$, then $\Gamma(z, \theta)$ is the cone

$$\Gamma(z,\theta) = \{\lambda \in \mathbb{C}, \quad |\arg(\lambda - \mathrm{z})| \le \theta\}. \tag{3.3}$$

- If $z \in \mathbb{C}$ and $\theta \in [0, \frac{\pi}{2}[$, then $\gamma(z,\theta)$ is the half-line

$$\gamma(z,\theta) = \{z + \rho e^{i\theta}, \quad \rho > 0\}. \tag{3.4}$$

- Recall that, for a large $D > 10^{10}b$ - The real number b is defined in Proposition 2.2 as the bottom of the spectrum of $\mathcal{T}_{\varepsilon,R_0}$ - and $\rho \in]\frac{2D}{|\mathrm{Log}\varepsilon|}, 1[$, $\Delta_{\varepsilon,\rho}$ is the domain of the complex plane given by

$$\Delta_{\varepsilon,\rho} = \{\mu \in \mathbb{C}, \quad |\mu| \le \rho, \ |\arg(\mu + \frac{\mathrm{b}}{2\mathrm{Log}\varepsilon})| \le \frac{\pi}{4}\}. \tag{3.5}$$

- Finally, if E is a subset of $\mathbb{C}$, we will set

$$E^* = E \backslash (E \cap \mathbb{R}^+). \tag{3.6}$$

Next, we need to define points where to match the derivatives; it appears that $\frac{R_0}{\varepsilon}$ is a good candidate, at least when the eigenvalues that we are investigating are not too big. As will be clear in the sequel, this is not the most clever choice when the eigenvalues exceed $\frac{b}{\varepsilon^2 \mathrm{Log}\varepsilon}$. Therefore we define the following quantities:

- We call R_{01} the radius defined in (2.23); recall that $q_0(\frac{R_{01}}{\varepsilon}) = Le$ and that $R_{01} - R_0 = O(1)A\varepsilon\mathrm{Log}\varepsilon$, where the $O(1)$ depends neither on A nor ε.
- Let $a > 0$ be small; for instance $a \le 10^{-100}\min(A, B, D, b)$ - these constants are defined in various places of the preceding section.
- Let us consider ρ defined as above and $\mu \notin \Delta^*_{\varepsilon,\rho}$. We define the radius $R(\mu)$ as follows: if $|\mu| \le \frac{a}{|\mathrm{Log}\varepsilon|}$, then $R(\mu) = R_0$. If not, select a small real number $\alpha > 0$, and choose $R(\mu)$ so that

$$\mu_1(\mathcal{T}_{\varepsilon,R(\mu)}) = |\mu|^\alpha. \tag{3.7}$$

  This is possible if ρ is small enough. From Proposition 2.1 and inspection of the proof of Proposition 2.1 of [14], we have

$$\frac{R_{01} - R(\mu)}{\varepsilon} = O(\frac{1}{|\mu|^\alpha}), \qquad q_0(R(\mu)) \sim k_0|\mu|^{-\alpha}, \tag{3.8}$$

  where $k_0 = Q_0'(+\infty)(1 + O(\varepsilon\mathrm{Log}\varepsilon))$.

3.2. Eigenvalue estimates

The goal of this subsection is to rule out the large eigenvalues. When we talk about large eigenvalues, we mean eigenvalues of size larger than M; where M is a large constant. In this section it will be convenient to work in rescaled coordinates: we set

$$\xi = \frac{r}{\varepsilon}, \qquad q_0(\xi) = \frac{Y_0(\xi)}{\varepsilon}, \quad w_0(\xi) = \frac{T_0(\xi) + \frac{Y_0(\xi)}{Le} - \frac{1}{Le}}{\varepsilon}$$

and, because no confusion will be possible in this section we rename the rescaled coordinate:

$$r := \xi.$$

Then we express the eigenvalue problem in terms of (v, w), the linear mass fraction and enthalpy. Namely we have to investigate the problem

$$\left\{ \begin{array}{ll} -\Delta v = & h_0'(q_0)v - Le\ q_0 e^{w_0 - q_0/Le} w + \mu v \\ -\Delta w & = \mu w + (1 - \frac{1}{Le})\mu v. \end{array} \right. \tag{3.9}$$

In the rescaling process, we have also rescaled the eigenvalues; therefore large eigenvalues are now the ones of size larger than $M\varepsilon^2$. The goal of this section is to prove the following result.

Theorem 3.2 *Consider $\delta > 0$. Then there exists $\delta_0 > 0$ such that, for all $\delta \in [0, \delta_0[$, the following holds.*

- *Assume $Le = 1 - \delta$. Then there exists $M > 0$ and $\theta \in]0, \frac{\pi}{2}[$ such that (3.9) has no eigenvalues outside the set*

$$D(0, M\varepsilon^2) \cup [0, \frac{b}{|2Log\varepsilon|}[\cup \Gamma(\frac{b}{|2Log\varepsilon|}, \theta).$$

- *Assume $Le = 1 + \delta$. There exists a smooth function $k(\delta) > 0$, going to 0 as $\delta \to 0$, and such that $(-k(\delta))$ is an eigenvalue to (3.9).*

This theorem says exactly that, when $Le > 1$, there is an unstable eigenvalue for $\mathcal{L}^\varepsilon$ of order $\frac{1}{\varepsilon^2}$. Its proof is broken into a series of lemmas, each of them being devoted to a particular section of the complex plane.

Lemma 3.1 *There exist $M_1 >$ and $\theta > 0$ such that (3.9) has no eigenvalues outside the cone $\Gamma(-M_1, \theta)$.*

PROOF. The zero order coefficients being bounded, the proof is exactly as in Henry [16], Chapter 1 - proof that the Laplacian is sectorial in L^p. $\square$

In the next lemma, we start shrinking the domain in which we have to seek for eigenvalues. This lemma plays the same role as Proposition 2.4 in [14], up to the difference that, in [14], the effect of the proposition was to make the order of a possible eigenvalue drop by one order in $\frac{1}{\varepsilon}$; this was due to the fact that we had $Le = 1 + O(\varepsilon)$.

Lemma 3.2 *Choose any $\theta_1 > \theta$. There exists $C_0 > 0$ such that, if $|Le - 1| = \delta$, then, for all $\delta \leq C_0\rho$, Problem (3.9) has no eigenvalues in*

$$\Lambda(\rho) := (\Gamma(-M_1, \theta)\backslash D(0, \rho)) \cap \{\mathrm{Re}\mu \leq |\frac{b}{|Log\varepsilon|} + \rho e^{i\pi/4}|\}. \tag{3.10}$$

PROOF. Let $\mu = \mu_1 + i\mu_2$ be a possible eigenvalue. Multiplying the equation for w by $\overline{w}$ we obtain

$$\int |\nabla w|^2 \, dx - \mu \int |w|^2 = (1 - \frac{1}{Le})\mu \int v\overline{w};$$

hence we have

$$\begin{aligned} -\mu_1 \|w\|_2 \leq & \ \frac{\delta}{1-\delta}|\mu|\|v\|_2 \\ |\mu_2|\|w\|_2 \leq & \ \frac{\delta}{1-\delta}|\mu|\|v\|_2. \end{aligned} \tag{3.11}$$

On the other hand - see the proof of Proposition 2.1 - we have, for every $v \in H^1(I\!R^3)$:

$$\int (|\nabla v|^2 - h_0'(q_0)|v|^2) \geq 0. \tag{3.12}$$

Hence we have, multiplying the equation for v by $\overline{v}$:

$$\begin{aligned} -\mu_1 \int |v|^2 \leq & \ \mathrm{Re}(\int q_0 e^{w_0 - q_0/Le} w\overline{v}) \\ |\mu_2| \int |v|^2 = & \ |\mathrm{Im}(\int q_0 e^{w_0 - q_0/Le} w\overline{v})|. \end{aligned} \tag{3.13}$$

Let us notice that, if $\mu \in \Lambda(\rho)$, then:

$$\text{if } |\mu_2| \leq \frac{\rho}{\sqrt{2}}, \text{ then } \mu_1 \leq -\frac{\rho}{\sqrt{2}}. \tag{3.14}$$

Take

$$\delta_0(\rho) = \frac{\rho}{2\|q_0 e^{w_0 - q_0/Le}\|_\infty}; \tag{3.15}$$

inequations (3.13), (3.11) and (3.14) for μ imply the lemma. $\square$

We still consider a small $\rho > 0$ such that

$$\rho \in]\frac{2D}{|\mathrm{Log}\varepsilon|}, 1[\tag{3.16}$$

and we attack the dangerous region $D(0,\rho)^*$. Because the possible rescaled eigenvalues are now small, we are able to do rigorous explicit computations. Let us therefore consider a nontrivial solution (μ, v, w) of (3.9). Our first task is to reduce the whole problem to a scalar equation posed in $B_{R(\mu)/\varepsilon}$. To this end, let us express w in terms of v: if we set

$$l(\mu) = v(r = \frac{R(\mu)}{\varepsilon}) \tag{3.17}$$

and if $\mu \notin I\!R^+$, we have:

$$\begin{aligned} w(x) &= (1 - \frac{1}{Le})\frac{\mu}{4\pi}\int_{I\!R^3} \frac{e^{-\sqrt{-\mu}|x-y|}}{|x-y|} v(y)\, dy \\ &= (1 - \frac{1}{Le})\frac{\mu}{4\pi}(\int_{B_{R(\mu)/\varepsilon}} ... + \int_{I\!R^3 \setminus B_{R(\mu)/\varepsilon}} ...) \qquad (3.18) \\ &= (1 - \frac{1}{Le})\frac{\mu}{4\pi}\int_{B_{R(\mu)/\varepsilon}} ... + \frac{\delta}{(1+\sqrt{Le})Le}\frac{\mathrm{sh}(\sqrt{-\mu}r)}{r} e^{-\sqrt{-\mu}R(\mu)/\varepsilon}\frac{l(\mu)}{\varepsilon}. \end{aligned}$$

Denote by

$$\mathcal{R}(\mu)v = \int_{B_{R(\mu)/\varepsilon}} \frac{e^{-\sqrt{-\mu}|x-y|}}{|x-y|} v(y)\, dy; \tag{3.19}$$

the following lemma is obvious, but will be the key to the rigorous estimates below.

Lemma 3.3 *There exists a constant $C > 0$, depending only on $\omega > 0$, such that, for all $\mu \in [-\rho, \rho] \setminus \Delta_{\varepsilon,\rho}$, we have*

$$\forall v \in C(B_{R(\mu)/\varepsilon}), \quad \|\mathcal{R}(\mu)v\|_\infty \le C N^0_{\omega,R(\mu),\varepsilon}(v).$$

This will indeed allow us to perform fixed point arguments in the sequel: the unknown w is indeed multiplied by $q_0 e^{w_0 - q_0/Le}$, a quantity whose norm $N^0_{\omega,R(\mu),\varepsilon}$ is controlled independently of ε. Such a situation would not have been possible if we had worked with the equation for u.

Lemma 3.4 *There exists $\rho_0 > 0$ such that, for all $\rho \in [0, \rho_0]$, the operator $\mathcal{T}_{\varepsilon,R(\mu)} - \mu I - (1 - \frac{1}{Le})\mu\mathcal{R}(\mu)$ is invertible from $D(\mathcal{T}_{\varepsilon,R(\mu)})$ to $C(B_{R(\mu)/\varepsilon})$ as soon as $\mu \in D(0,\rho)\setminus\Delta^*_{\varepsilon,\rho}$. The $N_{\omega,R(\mu),\varepsilon}$ norm of its inverse is at most*

- $|\frac{b}{Log\varepsilon}+\mu|^{-1}$ *if* $|\mu|\leq a|Log\varepsilon|^{-1}$,
- $O(|\mu|^{-\alpha})$ *otherwise.*

PROOF If $\mu\notin\Delta_{\varepsilon,\rho}$, then Proposition 2.3 implies the invertibility of $\mathcal{T}_{\varepsilon,R(\mu)}-\mu I$, with inverse $N_{\omega,R(\mu),\varepsilon}$ norm at most

- $|\frac{b}{\mathrm{Log}\varepsilon}+\mu|^{-1}$ if $|\mu|\leq a|\mathrm{Log}\varepsilon|^{-1}$,
- $O(|\mu|^{-\alpha})$ otherwise.

Then we have

$$|1-\frac{1}{Le}|N_{\omega,R(\mu),\varepsilon}(\mu\mathcal{R}(\mu)(\mathcal{T}_{\varepsilon,R(\mu)}-\mu I)^{-1})\leq\begin{cases}O(a) & \text{if } |\mu|\leq a|\mathrm{Log}\varepsilon|^{-1}\\ O(|\mu|^{1-\alpha}) & \text{otherwise}\end{cases}$$

which yields the invertibility of $\mathcal{T}_{\varepsilon,R(\mu)}-\mu I-(1-\frac{1}{Le})\mu\mathcal{R}(\mu)$ as soon as a and ρ are small enough. The norm of its inverse is then straightforward. $\square$

Remark 3.2 *We notice here that we do not need* $Le-1$ *small. The only place where it was needed was Lemma 3.2.*

We now start restricting ρ and δ.

First restriction of δ and ρ

ρ and δ will be chosen so that

- ρ is once and for all chosen so that the invertibility conditions of Lemma 3.4 hold.
- $\delta\leq\delta_0$, where δ_0 is chosen as in (3.15).

The first consequence of the preceding lemma is that $l(\mu)\neq 0$. If this were so, the function v would cancel the operator $\mathcal{T}_{\varepsilon,R(\mu)}-\mu I-(1-\frac{1}{Le})\mu\mathcal{R}(\mu)$; hence v would be zero. This would imply $v=0$ outside $B_{R(\mu)/\varepsilon}$ and $w=0$ everywhere, since $\mu\notin I\!R^+$. As a consequence, we may assume $l(\mu)=1$. Let us now consider a function $\tilde{w}(\mu,r)\in C((D(0,\rho)\setminus\Delta^*_{\varepsilon,\rho})\times B_{R(\mu)/\varepsilon})$; consider the scalar PDE

$$\begin{aligned}\Big(\mathcal{T}_{\varepsilon,R(\mu)}-\mu I-\frac{\delta}{1-\delta}\mu\mathcal{R}(\mu)\Big)v &= -q_0e^{w_0-q_0/Le}\tilde{w} \\ v(\frac{R(\mu)}{\varepsilon}) &= 1\end{aligned}\qquad (B_{R(\mu)/\varepsilon}) \tag{3.20}$$

Lemma 3.4 implies the existence of a unique solution to (3.20), and our goal is to compute an approximation of $v_r(\frac{R(\mu)}{\varepsilon})$. Let $v_1(r)$ be the unique solution of

$$(3.21)\quad \begin{aligned} \Big(\mathcal{T}_{\varepsilon,R(\mu)} - \mu I - \frac{\delta}{1-\delta}\mu\mathcal{R}(\mu)\Big)v &= -q_0 e^{w_0-q_0/Le}\tilde{w}(\mu,\frac{R(\mu)}{\varepsilon}) \qquad (B_{R(\mu)/\varepsilon}) \\ v(\frac{R(\mu)}{\varepsilon}) &= 1 \end{aligned}$$

The quantity $\frac{\partial v_1}{\partial r}(\frac{R(\mu)}{\varepsilon})$ is a good candidate for the approximation, if we think that $\tilde{w}$ varies on a $\sqrt{-\mu}$ scale. Classically, it is obtained by multiplying both sides of the PDE in (3.21) by the unique solution $e(\mu)$ of

$$(3.22)\quad \begin{aligned} (\mathcal{T}_{\varepsilon,R(\mu)} - \mu I)e &= 0 \quad (B_{R(\mu)/\varepsilon}) \\ e(\frac{R(\mu)}{\varepsilon}) &= 1 \end{aligned}$$

and to use Green's formula. We first need estimates for $e(\mu)$.

Lemma 3.5 *The following holds for e:*

- *we have*

$$N^2_{\omega,R(\mu),\varepsilon}(e(0)-q_0') \le \begin{cases} C\varepsilon|Log\varepsilon| & \text{if } |\mu| \le a|Log\varepsilon|^{-1} \\ C|\mu|^{1-\alpha} & \text{otherwise} \end{cases}$$

- *(derivative at $r = \frac{R(\mu)}{\varepsilon}$.) We have*

$$|e_r(\mu,\frac{R(\mu)}{\varepsilon})| \le \begin{cases} Ca & \text{if } |\mu| \le a|Log\varepsilon|^{-1} \\ C|\mu|^{1-\alpha} & \text{otherwise} \end{cases}$$

- *The function*

$$(3.23)\quad \psi(\mu) = e_r(\mu,\frac{R_0}{\varepsilon}) - q_0'(\frac{R_0}{\varepsilon})$$

is analytic.

Proof. **1.** We have $\mathcal{T}_{\varepsilon,R(\mu)}(e(0)-q_0') = \frac{2}{r}q_0'$ in $B_{R(\mu)/\varepsilon}$, with Dirichlet conditions, and $N^0_{\omega,R(\mu),\varepsilon}(\frac{2}{r}q_0')$ satisfies the estimates of the lemma - see Proposition 2.1. Hence the first point.
2. We write $(\mathcal{T}_{\varepsilon,R(\mu)} - \mu I)e(\mu) = \mu e(0)$ in $B_{R(\mu)/\varepsilon}$, again with Dirichlet conditions. Hence

$$N^0_{\omega,R(\mu),\varepsilon}(e(\mu)) \le |\mu| N_{\omega,R(\mu),\varepsilon}((\mathcal{T}_{\varepsilon,R(\mu)} - \mu I)^{-1}) N_{\omega,R(\mu),\varepsilon}(e_0),$$

and we apply Lemma 3.4.

3. The last point results from the invertibilty of $\mathcal{T}_{\varepsilon,R(\mu)} - \mu I$, $\mu \notin \Delta^*_{\varepsilon,\rho}$. $\square$

Using the same arguments as in this lemma, we may also prove the existence of a unique solution $e^*(\mu)$ to the problem

$$(3.24)\qquad \begin{aligned} (\mathcal{T}_{\varepsilon,R(\mu)} - \mu I - (1-\frac{1}{Le})\mu\mathcal{R}(\mu)^*)e = &\ 0 \quad (B_{R(\mu)/\varepsilon}) \\ v(\frac{R(\mu)}{\varepsilon}) = &\ 1 \end{aligned}$$

where $\mathcal{R}(\mu)^*$ is the adjoint operator of $\mathcal{R}(\mu)$. Moreover:

Lemma 3.6 *The function $e^*(\mu)$ has the same estimates as $e(\mu)$.*

Before we go any further, we need another lemma.

Lemma 3.7 *We have*

$$(3.25)\qquad \frac{\varepsilon^2}{R_0^2}\int_0^{R(\mu)/\varepsilon} q_0 e^{w_0 - q_0/Le} q_0'(r)\, r^2\ dr = e^{w_0} Le^2(1 + O(e^{-\rho^{-\alpha}})),$$

PROOF. Use (3.8) and the fact that q_0 grows at least linearly in r as r departs from $\frac{R_{01}}{\varepsilon}$; this implies the exponential decay of $e^{-q_0/Le}$ in this region. For $r \le R_{01}$ one uses the exponential decay of q_0. $\square$

Second restriction on ρ The quantity in (3.25) is > 0. This only means that ρ is shrunk independently of all other constants.

Remark 3.3 *There is a relationship between w_0 and R_0, but we do not wish to use it: first, because we do not need it. Second, because we wish to examine linear operators around functions (T, Y) that look like Zeldovich flame balls at finite distance, but that are not flame balls, namely: quasi-steady flame balls. This will be clear in the last section.*

Hence we have the

Lemma 3.8 *The function $\mu \mapsto \frac{\partial v_1}{\partial r}(\frac{R(\mu)}{\varepsilon})$ is analytic. Moreover we have*

$$\frac{\partial v_1}{\partial r}(\frac{R(\mu)}{\varepsilon}) - Le^3 e^{w_0}(1 + O(e^{-\rho^{-\alpha}})) = \tilde{w}(\mu, \frac{R(\mu)}{\varepsilon})$$

$$(3.26)\qquad = \begin{cases} O(a(1 + |\tilde{w}(\mu, \frac{R(\mu)}{\varepsilon})|) + \varepsilon|Log\varepsilon|) & \text{if } |\mu| \le \frac{a}{|Log\varepsilon|} \\ O(|\mu|^{1-\alpha}(1 + |\tilde{w}(\mu, \frac{R(\mu)}{\varepsilon})|) + \varepsilon|Log\varepsilon|) & \text{otherwise} \end{cases}$$

Proof. Multiplication by $e(\mu)$ and integration by parts yields

$$\begin{aligned}
&-\frac{\partial v_1}{\partial r}(\frac{R(\mu)}{\varepsilon}) + e_r(\mu, \frac{R(\mu)}{\varepsilon}) \\
&= Le(1-\frac{1}{Le})\frac{\varepsilon^2}{R(\mu)^2}\mu\int_0^{R(\mu)/\varepsilon} \mathcal{R}(\mu)v_1 e(\mu) r^2\, dr + O(\varepsilon \mathrm{Log}\varepsilon) \\
&= O(|\mu| \|v_1\|_\infty) + O(\varepsilon \mathrm{Log}\varepsilon)
\end{aligned}$$

Then we have

$$N^0_{\omega,R(\mu),\varepsilon}(v_1) \leq |\tilde{w}(\mu, \frac{R(\mu)}{\varepsilon})| N^0_{\omega,R(\mu),\varepsilon}(\mathcal{T}_{\varepsilon,R(\mu)} - \mu I - (1-\frac{1}{Le})\mu\mathcal{R}(\mu))^{-1})$$

and we use Lemma 3.4 to control the right hand side. □

We gather the informations of Lemmas 3.5 and 3.8 into the following proposition.

Proposition 3.1 *Assume $\tilde{w}$ to be analytic with respect to μ for all r, and to belong to $C(B_{R(\mu)/\varepsilon})$ uniformly with respect to $\mu \in (D(0,\rho)\backslash\Delta_{\varepsilon,\rho})$. The function $\mu \mapsto v_r(\frac{R(\mu)}{\varepsilon})$ is continuous, and analytic if $|\mu| \leq \frac{a}{|Log\varepsilon|}$. Moreover we have*

$$\frac{\partial v}{\partial r}(\frac{R(\mu)}{\varepsilon}) - Le^3 e^{w_0}(1+O(e^{-\rho^{-\alpha}}))\tilde{w}(\mu,\frac{R(\mu)}{\varepsilon})$$

$$= \begin{cases} O(a(1+|\tilde{w}(\mu,\frac{R(\mu)}{\varepsilon})|) + \|\tilde{w}'\sqrt{q}_0 e^{-q_0/2}\|_\infty + \varepsilon|Log\varepsilon|) & \text{if } |\mu| \leq \frac{a}{|Log\varepsilon|} \\ O(|\mu|^{1-\alpha}(1+|\tilde{w}(\mu,\frac{R(\mu)}{\varepsilon})|) + \|\tilde{w}'\sqrt{q}_0 e^{-q_0/2}\|_\infty + \varepsilon|Log\varepsilon|) & \text{otherwise} \end{cases} \tag{3.27}$$

Proof. There holds

$$\forall \mu \notin \Delta^*_{\varepsilon,\rho},\quad N^2_{\omega,R(\mu),\varepsilon}(v-v_1) \leq N^0_{\omega,R(\mu),\varepsilon}(\mathcal{T}_{\varepsilon,R(\mu)} - \mu I - (1-\frac{1}{Le})\mu\mathcal{R}(\mu))^{-1})\|w'\sqrt{q}_0 e^{-q_0/2}\|_\infty.$$

It will become clear in the proof of Theorem 3.2 why we have to append the factor $\sqrt{q}_0 e^{-q_0/2}$ to $\tilde{w}'$. On the other hand, we proceed as above: multiply by the equation for $v - v_1$ by $e(\mu)$ and integrate by parts. Then control the term

$$\frac{\varepsilon^2}{R(\mu)^2}\int_0^{R(\mu)/\varepsilon} \mathcal{R}(\mu)(v-v_1)e(\mu)r^2\, dr$$

by the above estimate. Continuity and analyticity are standard and omitted. □

So much for the inner derivative of v. As for the outer derivative, the task is somewhat easier.

Proposition 3.2 *Suppose that $\mu \notin \Delta^*_{\varepsilon,\rho}$. Then*

- *Assume $|\mu| \leq \dfrac{a}{|Log\varepsilon|}$. Then*

$$v(r) = \frac{R_0}{\varepsilon r} e^{-\sqrt{-\mu Le}(r - R_0/\varepsilon)}, \quad v_r(\frac{R_0}{\varepsilon}) = -(\sqrt{-\mu Le} + \frac{\varepsilon}{R_0}).$$

- *Assume $|\mu| \geq \dfrac{a}{|Log\varepsilon|}$. Then we have, for some $C > 0$:*

$$v(r) = \frac{R_0}{\varepsilon r} e^{-\sqrt{-\mu Le}(r - R_0/\varepsilon)} + O(e^{-C|\mu|^{-\alpha}}), \quad v_r(\frac{R_0}{\varepsilon}) = -(\sqrt{-\mu Le} + \frac{\varepsilon}{R_0}) + O(e^{-C|\mu|^{-\alpha}}).$$

Proof. The first point is straightforward, since we said that we were neglecting the $e^{10^{-10}}$ terms. As for the first point, we notice that

$$\forall r \geq R(\mu), \qquad e^{w_0 - q_0/Le} = O(e^{-2C|\mu|^{-\alpha}})$$

for some constant C; then write $\tilde{v}(r) = v(r) - \dfrac{R_0}{\varepsilon r} e^{-\sqrt{-\mu Le}(r - R_0/\varepsilon)}$, and

$$2\sqrt{-\mu} r \tilde{v}(r) = O(e^{-2C|\mu|^{-\alpha}}) \Big(\int_{R_0/\varepsilon}^{r} e^{-\sqrt{-\mu}(r - r^*)} v(r^*)\ dr^* + \int_{r}^{+\infty} e^{\sqrt{-\mu}(r - r^*)} v(r^*)\ dr^*$$
$$+ \int_{R_0/\varepsilon}^{+\infty} e^{-\sqrt{-\mu}(r + r^*)} v(r^*)\ dr^* \Big)$$

Then use the fact that $|\mu| \leq \rho$ is small to perform a contraction argument. □

We have now everything to perform the matching.

Proof of Theorem 3.2. We take

$$\tilde{w}(\mu, r) = \frac{\delta}{(1 + \sqrt{Le})Le} \frac{\mathrm{sh}(\sqrt{-\mu} r)}{\varepsilon r} e^{-\sqrt{-\mu} R(\mu)/\varepsilon}; \tag{3.28}$$

this implies

$$\|\tilde{w}'(\mu, .)\sqrt{q}_0 e^{-q_0/2}\|_\infty = O(|\delta(\sqrt{-\mu} + e^{-M})|). \tag{3.29}$$

The proof of Theorem 3.2 is now done in three steps.

1. Restriction of Le. We now withdraw the small disk $D(0, M\varepsilon^2)$ from $D(0, \rho)\backslash\Delta_{\varepsilon,\rho}$, and call the new domain $\tilde{\Delta}(\varepsilon, \rho)$. We first assume that

$$|\mu| \geq \frac{a}{|\mathrm{Log}\varepsilon|}.$$

From (3.28) and (3.25) we have

$$\frac{\partial v}{\partial r}(\frac{R(\mu)}{\varepsilon}^-) = \frac{1}{2} Le^3 e^{w_0}(1 + O(e^{-\rho^{-\alpha}})) \frac{\delta}{1 + \sqrt{Le}} + \psi(\varepsilon, Le, a, M, \mu) \tag{3.30}$$

with ψ continuous in μ; moreover we have, from Proposition 3.1 and (3.29):

$$|\psi(Le, a, M, \mu)| \leq C\Big(\delta(|\sqrt{-\mu}| + e^{-M}) + |\mu|^{1-\alpha} + \varepsilon|\mathrm{Log}\varepsilon|\Big). \tag{3.31}$$

A matching of the inner and outer derivatives of v yields

$$\begin{aligned} & -(\sqrt{-\mu Le} + O(e^{-C|\mu|^{-\alpha}}) \\ = \ & \frac{1}{2} Le^3 e^{w_0}(1 + O(e^{-\rho^{-\alpha}}))\frac{\delta}{1+\sqrt{Le}} + \psi(\varepsilon, Le, a, M, \mu). \end{aligned} \tag{3.32}$$

Recall that ρ is restricted so that Lemma 3.4 holds and so that the integral

$$\int_0^{R(\mu)/\varepsilon} q_0 e^{w_0 - q_0/Le} q_0' \, dr$$

is controlled from below.

Second restriction on δ:

We choose $\delta > 0$ small enough so that the possible solutions of equation (3.32) in $\tilde{\Delta}_{\varepsilon,\rho}$ are of order δ^2 - $\varepsilon > 0$ is much smaller than ρ and does not interfere with it. This is possible if

$$\delta << \min(\int_0^{R(\mu)/\varepsilon} q_0 e^{-q_0/Le} q_0' \, dr, \frac{1}{2}).$$

and we already saw that the integral was controlled by $Le^2(1 + O(e^{-\rho^{-\alpha}}))$.

Final choice of constants

Now, δ and ρ are fixed. All the other constants ε, M^{-1},... are chosen so that the quantities where they are involved are much smaller than, say, a large power of δ. This will always be implicit in the sequel.

2. The region $D(0,\rho)\backslash(\Delta_{\varepsilon,\rho} \cup D(0, \frac{a}{|\mathrm{Log}\varepsilon|}))$. It is only at this stage that we care about the sign of $1 - Le$.

Let us first assume that $Le > 1$. The asymptotic equation to (3.32) is

$$\sqrt{-\mu Le} = \frac{1}{2} e^{w_0} Le^3 (1 + O(e^{-\rho^{-\alpha}}))\frac{Le - 1}{1 + \sqrt{Le}}, \tag{3.33}$$

because $R(\mu)$ is $\varepsilon\mathrm{Log}\varepsilon$ close to R_0. This equation obviously has a real negative $O(\delta^2)$ solution outside $\Delta_{\varepsilon,\rho}$, that we call $\mu^\varepsilon(\delta)$. The intermediate value theorem tells us that equation (3.32) also has a solution, which is $o(\delta^2)$ close to $\mu^\varepsilon(\delta)$.

Assume now that $Le < 1$, and set $\lambda = \sqrt{-\mu}$. Equation (3.32), expressed in terms of λ, obviously has the solution

$$\lambda(\delta) = \frac{1}{4} e^{2w_0} Le^6 (1 + O(e^{-\rho^{-\alpha}}))^2 \frac{(Le-1)^2}{(1+\sqrt{Le})^2};$$

hence, if $\mu \notin \Delta^2_{\varepsilon,\rho}$ there exists $C > 0$ such that $|\lambda - \lambda(\delta)| \geq C\delta^2$; as a consequence call $\mathcal{D}^\varepsilon(\lambda, Le)$ the difference between the two sides of equation (3.32): we have - taking into account the spurious constants ε, M,... an inequality of the form

$$|\mathcal{D}^\varepsilon(\lambda, Le)| \geq C\delta^2, \tag{3.34}$$

implying that equation (3.32) has no solutions in $D(0,\rho)\backslash(\Delta_{\varepsilon,\rho} \cup D(0\frac{a}{|\text{Log}\varepsilon|}))$.

3. The region $D(0\frac{a}{|\text{Log}\varepsilon|})\backslash D(0, M\varepsilon^2)$. Examination of equation (3.32) and application of Proposition 3.1 implies that (3.34) also holds in this region. This ends the proof of Theorem 3.2. □

3.3. The dispersion relation

We leave for good the case $Le > 1$ and concentrate on the case $Le < 1$. And we still work in rescaled coordinates.

The estimates of the preceding paragraph, although quite sufficient to exclude large eigenvalues, are not precise enough to detect small eigenvalues; remember that we wish to track $O(\varepsilon^4)$ eigenvalues, due to the rescaling. On the other hand we are now operating in a range of eigenvalues of the size ε^2 at most, and this will allow us to perform specific estimates.

We work in the disc of the complex plane $D(0, M\varepsilon^2)^*$, where M is a large, but finite, real number. Therefore we have $R(\mu) = R_0$ and the operator $\mathcal{T}_{\varepsilon,R_0} - \mu I - \mu\mathcal{R}(\mu)$ is invertible; the norm of its inverse is of order $|\text{Log}\varepsilon|$.

1. Let us start with the solution v of equation (3.20). We have the

Proposition 3.3 *Recall that $\mu \in D(0, M\varepsilon^2)$. Then*

$$\begin{aligned} v_r(\mu, \frac{R_0}{\varepsilon}) = \; & e_r(0, \frac{R_0}{\varepsilon}) \\ & + \frac{\varepsilon^2 Le}{R_0^2} \int_0^{R_0/\varepsilon} q_0 e^{w_0 - q_0/Le} e(0,r)\tilde{w}(\mu, r) r^2 \, dr + O(|\mu Log\varepsilon|) \end{aligned} \tag{3.35}$$

The proof of this result is exactly along the lines of the preceding approximation results, and will therefore be omitted. We will rather concentrate of the effective computation of the terms that come into play.

2. As in the preceding paragraph, let us take

$$\tilde{w}(\mu,r) = \frac{\delta}{(1+\sqrt{Le})Le}\frac{\mathrm{sh}(\sqrt{-\mu}r)}{\varepsilon r}e^{-\sqrt{-\mu}R_0/\varepsilon}. \tag{3.36}$$

Let us also remark that we have, this time up to almost transcendental order in ε:

$$v_r(\frac{R_0}{\varepsilon}) = -(\sqrt{-\mu Le}+\frac{\varepsilon}{R_0}).$$

The dispersion relation is therefore readily obtained:

$$\begin{aligned} &-(\sqrt{-\mu Le}+\frac{\varepsilon}{R_0}) = e_r(0,\frac{R_0}{\varepsilon}) \\ &+\frac{\varepsilon}{R_0^2}\frac{\delta}{1+\sqrt{Le}}e^{-\sqrt{-\mu}R_0/\varepsilon}\int_0^{R_0/\varepsilon} q_0 e^{w_0-q_0/Le}e(0,r)\mathrm{sh}(\sqrt{-\mu}r)r\ dr+\psi(\mu); \end{aligned} \tag{3.37}$$

ψ being analytic in μ and $\psi(\mu)=O(\mu)$ for small μ - for instance, μ in the range of eigenvalues that we are investigating. We are now ready to complete the

Proof of Theorem 3.1. Let us first prove that, if μ is a solution of (3.37) in $D(0,M\varepsilon^2)\cap\{\mathrm{Re}\,\mu\le m\varepsilon^2\}$ where m is suitably chosen, then $\mu=O(\varepsilon^4|\mathrm{Log}\varepsilon|)$. Indeed we have, for $B>0$ large enough:

$$\frac{\varepsilon}{R_0^2}\frac{\delta}{1+\sqrt{Le}}e^{-\sqrt{-\mu}R_0/\varepsilon}\int_0^{R_0/\varepsilon} q_0 e^{w_0-q_0/Le}e(0,r)\mathrm{sh}(\sqrt{-\mu}r)r\ dr$$

$$\begin{aligned} &= \int_{R_0/\varepsilon+BLog\varepsilon}^{R_0/\varepsilon}\ldots+\int_{R_0/\varepsilon+BLog\varepsilon}^{R_0/\varepsilon}\ldots \\ &= O(\varepsilon^2)+\frac{\varepsilon}{R_0^2}\frac{\delta}{1+\sqrt{Le}}e^{-\sqrt{-\mu}R_0/\varepsilon}\int_{R_0/\varepsilon+BLog\varepsilon}^{R_0/\varepsilon} q_0e^{w_0-q_0/Le}e(0,r)\mathrm{sh}(\sqrt{-\mu}r)r\ dr \\ &= \frac{1}{R_0}\frac{\delta}{1+\sqrt{Le}}e^{-\sqrt{-\mu}R_0/\varepsilon}\mathrm{sh}(\frac{\sqrt{-\mu}R_0}{\varepsilon})\int_{R_0/\varepsilon+BLog\varepsilon}^{R_0/\varepsilon} q_0e^{w_0-q_0/Le}e(0,r)\ dr+O(\varepsilon|\mathrm{Log}\varepsilon|) \\ &= \frac{1}{R_0}\frac{\delta}{1+\sqrt{Le}}e^{-\sqrt{-\mu}R_0/\varepsilon}e^{w_0}Le^3\mathrm{sh}(\frac{\sqrt{-\mu}R_0}{\varepsilon})+O(\varepsilon|\mathrm{Log}\varepsilon|) \end{aligned}$$

Choose now $m>0$ such that there exists $C>0$ so that, for all $\mu\in D(0,M\varepsilon^2)\cap\{\mathrm{Re}\mu\le m\varepsilon^2\}$, we have

$$|e^{-\sqrt{-\mu}R_0/\varepsilon}\mathrm{sh}(\frac{\sqrt{-\mu}R_0}{\varepsilon})|\ge\frac{C}{\varepsilon}|\sqrt{-\mu}|. \tag{3.38}$$

This is simply done by the examination of the zeroes of the hyperbolic sine. But then we have the required estimate; and the resolution of the dispersion relation is simply done by remembering that

$$e_r(0,\frac{R_0}{\varepsilon}) = -\frac{2\varepsilon}{R_0}$$

and thus by solving the following approximate equation in $D(0, \varepsilon^4|\mathrm{Log}\varepsilon|)$:

$$\frac{1}{R_0}\frac{\delta}{1+\sqrt{Le}}e^{w_0}Le^3\mathrm{sh}(\frac{\sqrt{-\mu}R_0}{\varepsilon}) = \frac{\varepsilon}{R_0}. \tag{3.39}$$

This equation has a unique simple solution, and so has the complete dispersion relation by Rouché's Theorem. □

4. Construction of an approximate solution: Joulin's derivation revisited

The goal of this section is to compute, to sufficiently high order, an approximate solution of the full reaction-diffusion system (1.1)-(1.2). For commodity, we set $u = (T, Y)$ and write (1.1)-(1.2) in abstract form

$$\dot{u} + Au = F_\varepsilon(u), \tag{4.1}$$

with

$$Au = -\mathrm{diag}(1, \frac{1}{Le})\Delta u. \tag{4.2}$$

The main result of this section is the

Theorem 4.1 *Choose $\nu \in]0, \frac{1}{4}[$. Take any $R_0 > 0$. Consider the couple $(T_0, Y_0) \in C(I\!R^3)$ such that*

- *we have $(T_0, Y_0) = (\frac{1}{Le}, 0)$ in $B_{R_0}(0)$;*
- *(T_0, Y_0) is harmonic outside B_{R_0} and satisfies $T_0(+\infty) = 0$; $Y_0(+\infty) = 1$.*

There is a family of functions $(T_J^\varepsilon, Y_J^\varepsilon)$ and a real number $\tau_{max} > 0$ such that, for all $\tau_0 < \tau_{max}$ the following is true:

[i]. $(T_J^\varepsilon, Y_J^\varepsilon)$ is continuous on $[0, \tau_0\varepsilon^{-2}] \times I\!R^3$; moreover it is smooth on each side of $\partial B(0, \varepsilon^{-\nu})$;

[ii]. $(T_J^\varepsilon(0), Y_J^\varepsilon(0))_\varepsilon$ converges to (T_0, Y_0) as $\varepsilon \to 0$.

[iii]. Set

$$(4.3)\ \varphi_T^\varepsilon(\tau) := \frac{1}{\sqrt{4\pi\tau}}\int_{\varepsilon^{1-\nu}}^{+\infty}\left\{\rho\overline{T}_0(\rho) - \varepsilon^{1-\nu}\overline{T}_0(\varepsilon^{1-\nu})\right\}\frac{\rho-\varepsilon^{1-\nu}}{\tau}e^{-(\rho-\varepsilon^{1-\nu})^2/4\tau}\,d\rho$$

and

$$(4.4)\varphi_Y^\varepsilon(\tau) := \frac{\sqrt{Le^3}}{\sqrt{4\pi\tau}}\int_{\varepsilon^{1-\nu}}^{+\infty}\left\{\rho\overline{Y}_0(\rho) - \varepsilon^{1-\nu}\overline{Y}_0(\varepsilon^{1-\nu})\right\}\frac{\rho-\varepsilon^{1-\nu}}{\tau}e^{-Le(\rho-\varepsilon^{1-\nu})^2/4\tau}\,d\rho$$

with

$$\overline{T}_0(\xi) = \frac{1}{\varepsilon} T^\varepsilon_J\Big(0, \frac{\xi}{\varepsilon}\Big)\,; \quad \overline{Y}_0(\xi) = \frac{1}{\varepsilon}\Big(Y^\varepsilon_J\Big(0, \frac{\xi}{\varepsilon}\Big) - 1\Big) \tag{4.5}$$

The functions $T^\varepsilon_J(0)$ *and* $Y^\varepsilon_J(0)$ *are such that* $\varphi^\varepsilon_T(\tau\varepsilon^{-2})$ *and* $\varphi^\varepsilon_Y(\tau\varepsilon^{-2})$ *converge uniformly on* $[0, \tau_{max}[$ *to continuous functions* $\varphi_T(\tau)$ *and* $\varphi_Y(\tau)$*. Moreover the equation with unknown* $\overline{R}(\tau)$

$$(1 - \sqrt{Le})\partial_{1/2}\overline{R} = 2LeLog(\sqrt{Le^3}\overline{R}) + Le\varphi_T(\tau) + \varphi_Y(\tau) \tag{4.6}$$

and initial datum R_0 *has a solution on* $[0, \tau_{max}[$.

[iv]. Let $R^\varepsilon_J(t)$ *be the radius of the function* $T^\varepsilon_J(t)$*. Then the sequence* $(R^\varepsilon_J(\varepsilon^{-2}\tau))_\varepsilon$ *converges uniformly to* $\overline{R}$ *on* $[0, \tau_0]$.

[v]. The function $u^\varepsilon_J = (T^\varepsilon_J, Y^\varepsilon_J)$ *satisfies the equation*

$$\begin{aligned} \dot{u} + Au = \;& F_\varepsilon(u) \\ & + \varepsilon^{3-\nu}(f_\varepsilon(t)\mathbf{1}_{r\le R^\varepsilon_J(t)} + \varepsilon^{1-3\nu}h_\varepsilon(t)\mathbf{1}_{R_J(t)\le r\le \varepsilon^{-\nu}} + \varepsilon^{1-3\nu}k_\varepsilon(t)\delta_{r=\varepsilon^{-\nu}}) \end{aligned} \tag{4.7}$$

where f_ε*,* h_ε *and* k_ε *are respectively uniformly bounded with respect to* ε *in* $L^\infty([0, \tau_0\varepsilon^{-2}] \times I\!R^3)$ *and* $L^\infty([0, \tau_0\varepsilon^{-2}])$*. Moreover the norm* $N_{\omega, R^\varepsilon_J(t), \varepsilon}(f_\varepsilon(t))$ *is uniformly bounded with respect to* ε *in* $L^\infty([0, \tau_0\varepsilon^{-2}])$.

[vi]. Given $w_0 \in I\!R$*, if* $R_0 = \dfrac{1}{\sqrt{Le^3}} \exp\Big(-\dfrac{w_0}{2}\Big)$*, we can make this construction in such a way that the initial datum* $(T^\varepsilon_J(0), Y^\varepsilon_J(0))$ *coincides with the Zeldovich flame* $(T^\varepsilon_{w_0}, Y^\varepsilon_{w_0})$ *in the ball* $B(0, \varepsilon^{-\nu})$.

The construction is done in Joulin's spirit: the solution that we are looking for will be quasisteady in a large ball, namely $B(0, \varepsilon^{-\nu})$, and a solution of the full evolution system outside this ball.

As usual in the construction of approximate solutions, the difficult step here is the construction of the lower order term in the expansion, the subsequent terms solving linear equations. Therefore the main part of the section will be devoted to this task, which will produce a solution up to ε^2 errors. This is not quite sufficient for our purpose; as will be shown in the next section, the large-time estimates for the linear semigroup are not good enough to allow us to stop the expansion at the first significant order.

The first three subsections will be devoted to the construction of the first term of the expansion. The strategy - i.e. the idea of Joulin's paper - is explained in the first one; the second one explores what happens inside $B(0, \varepsilon^{-\nu})$. The matching - together with the definition of initial conditions - is done in subsection 3. Subsection 4 sums up the construction of the solution up to ε^2 errors. The last - and more standard - subsection will indicate what modifications are necessary in order to obtain the higher order.

4.1. Idea of the construction

The solution is, as said above, looked for as quasisteady in $B(0,\varepsilon^{-\nu})$. This is, (T,Y) is a solution to

$$(4.8) \qquad \begin{cases} -\Delta T = & Yf_\varepsilon(T) \\ -\dfrac{\Delta Y}{Le} = & -Yf_\varepsilon(T) \end{cases}$$

in $(0,+\infty)\times B(0,\varepsilon^{-\nu})$, and

$$(4.9) \qquad \begin{cases} T_t - \Delta T = & Yf_\varepsilon(T) \\ Y_t - \dfrac{\Delta Y}{Le} = & -Yf_\varepsilon(T) \end{cases}$$

in $(0,+\infty)\times(I\!R^3 \setminus B(0,\varepsilon^{-\nu}))$.

For each $t>0$, (T,Y) is a Zeldovich flame in $B(0,\varepsilon^{-\nu})$ and we let $R(t)$ be such that $T(t,R(t)) = \dfrac{1}{Le} + A\varepsilon\mathrm{Log}\varepsilon$.

On the other hand, we have for a certain (constant in space) $w(t)$, that $W := T + \dfrac{Y}{Le} - \dfrac{1}{Le} = \varepsilon w$ in $B(0,\varepsilon^{-\nu})$.

Therefore, if the functions $R(t)$ and $w(t)$ were known, (T,Y) would be determined in $B(0,\varepsilon^{-\nu})$ as the solution to (4.8) in $\{|x|<R(t)\}$ such that

$$(4.10) \qquad T(t,R(t)) = \frac{1}{Le} + A\varepsilon\mathrm{Log}\varepsilon \quad , \quad Y(t,R(t)) = Le(\varepsilon w - A\varepsilon\mathrm{Log}\varepsilon)$$

with

$$(4.11) \qquad \Delta T = \Delta Y = 0$$

in $\{(t,x)\,/\,R(t)<|x|<\varepsilon^{-\nu}\}$.

On the other hand, (T,Y) is a solution to (4.9) in $(0,+\infty)\times(I\!R^3\setminus B(0,\varepsilon^{-\nu}))$ and it would be completely determined by its initial condition and its Dirichlet condition at $|x|=\varepsilon^{-\nu}$. So, the jump of the derivatives T_r and Y_r on $|x|=\varepsilon^{-\nu}$ can be stated in terms of the functions (which are actually unknown) $R(t)$ and $w(t)$.

Thus, the idea is to choose $R(t)$ and $w(t)$ in such a way that these jumps are $O(\varepsilon^2)$.

4.2. Inside $B(0,\varepsilon^{-\nu})$

Following the ideas of Section 2 we first look at the rescaled problem (with $p(t,r) = \dfrac{1}{\varepsilon}(T(t,\varepsilon r) - \dfrac{1}{Le}))$.

Let $R(t)$ and $w(t)$ be smooth functions to be determined. Let p be the solution to

$$(4.12) \qquad \begin{aligned} -\Delta p &= g_w(p) \qquad (|x| < \frac{R(t)}{\varepsilon}) \\ p(\frac{R(t)}{\varepsilon}) &= A\mathrm{Log}\varepsilon \end{aligned}$$

with $g_w(p) = Le(w-p)e^p$, given by Theorem 2.1. Following the notation of Corollary 2.1 we let $\pi_\varepsilon(R,w)$, a C^2 function of its arguments, such that $p_r(R(t)/\varepsilon) = \pi_\varepsilon(R(t),w(t)) = -\sqrt{Le}\,\exp\big(\frac{w(t)}{2}\big) + O(\varepsilon|\mathrm{Log}^2\varepsilon|)$ in C^1.

Let $q(t,r) = Le(w(t)-p(t,r))$. Then, if $Y(t,r) = \varepsilon q(t,r/\varepsilon)$ there holds that (T,Y) is a solution to (4.8) in $\{|x|<R(t)\}$ satisfying (4.10).

We now take as T and Y harmonic functions in $\{(t,x)\,/\,R(t)<|x|<\varepsilon^{-\nu}\}$. Thus, the matching of the functions and first derivatives at $r=R(t)$ gives the following formulas for T and Y

$$\begin{aligned} T(t,r) &= \frac{1}{Le} + A\varepsilon\mathrm{Log}\varepsilon + R\pi_\varepsilon(R(t),w(t)) - \frac{R^2(t)}{r}\pi_\varepsilon(R(t),w(t)) \\ Y(t,r) &= Le\{\varepsilon w - A\varepsilon\mathrm{Log}\varepsilon - R\pi_\varepsilon(R(t),w(t)) + \frac{R^2(t)}{r}\pi_\varepsilon(R(t),w(t))\}. \end{aligned} \tag{4.13}$$

From (4.13) we get

$$\begin{aligned} T(t,\varepsilon^{-\nu}) &= \varepsilon k(t) - \varepsilon^{\nu}R^2(t)\pi_\varepsilon(R(t),w(t)) \\ Y(t,\varepsilon^{-\nu}) &= Le\{\varepsilon(w(t)-k(t)) + \varepsilon^{\nu}R^2(t)\pi_\varepsilon(R(t),w(t))\} + 1 \end{aligned} \tag{4.14}$$

where

$$k(t) := \varepsilon^{-1}\Big\{\frac{1}{Le} + A\varepsilon\mathrm{Log}\varepsilon + R(t)\pi_\varepsilon(R(t),w(t))\Big\}. \tag{4.15}$$

4.3. Outside $B(0,\varepsilon^{-\nu})$

Now, we will consider the full evolution problem with boundary data on the lateral boundary given by (4.14). But, we will write this problem in rescaled variables. This is, we let

$$\overline{T}(\tau,\xi) = \frac{1}{\varepsilon}T\big(\frac{\tau}{\varepsilon^2},\frac{\xi}{\varepsilon}\big) \tag{4.16}$$

$$\overline{Y}(\tau,\xi) = \frac{1}{\varepsilon}\Big(Y\Big(\frac{\tau}{\varepsilon^2},\frac{\xi}{\varepsilon}\Big) - 1\Big) \tag{4.17}$$

Thus, $\overline{T}$ is a solution to

$$\begin{cases} \overline{T}_\tau - \Delta\overline{T} = 0 & \text{for } |\xi|>\varepsilon^{1-\nu},\ \tau>0 \\ \overline{T}(\tau,\varepsilon^{1-\nu}) = \overline{k}(\tau) - \\ \qquad \overline{R}(\tau)^{\,2}\varepsilon^{-1+\nu}\pi_\varepsilon(\overline{R}(\tau),\overline{w}(\tau)) & \text{for } \tau>0 \\ \overline{T}(0,\xi) = \dfrac{T_0(\xi/\varepsilon)}{\varepsilon} & \text{for } |\xi|>\varepsilon^{1-\nu} \end{cases} \tag{4.18}$$

and $\overline{Y}$ is a solution to

$$\begin{cases} \overline{Y}_\tau - \dfrac{1}{Le}\Delta\overline{Y} = 0 & \text{for } |\xi|>\varepsilon^{1-\nu},\ \tau>0 \\ \overline{Y}(\tau,\varepsilon^{1-\nu}) = Le\Big(\overline{w}(\tau) - \overline{k}(\tau) \\ \qquad +\overline{R}^{\,2}\varepsilon^{-1+\nu}\pi_\varepsilon(\overline{R}(\tau),\overline{w}(\tau))\Big) & \text{for } \tau>0 \\ \overline{Y}(0,\xi) = \dfrac{(Y_0(\xi/\varepsilon)-1)}{\varepsilon} & \text{for } |\xi|>\varepsilon^{1-\nu}. \end{cases} \tag{4.19}$$

Here we have taken

$$\overline{R}(\tau) := R(\tau/\varepsilon^2), \quad \overline{w}(\tau) := w(\tau/\varepsilon^2), \quad \overline{k}(\tau) := k(\tau/\varepsilon^2).$$

Recall that we are only interested in the jump of the radial derivative $\overline{T}_\rho$ and $\overline{Y}_\rho$ at $\rho = |\xi| = \varepsilon^{1-\nu}$.

Let g be a radially symmetric solution to

$$g_\tau - \frac{1}{\lambda}\Delta g = 0 \quad \text{for } |\xi| > \varepsilon^{1-\nu}, \ \tau > 0. \tag{4.20}$$

Since the function $u(\tau,\rho) = \rho g(\tau,\rho)$ is a solution of the equation $u_\tau - \frac{1}{\lambda}u_{\rho\rho} = 0$ in $\{\rho > \varepsilon^{1-\nu}\}$ there holds that the function

$$\overline{u}(\tau,\rho) = \begin{cases} \quad u(\tau, \frac{1}{\sqrt{\lambda}}\rho + \varepsilon^{1-\nu}) - \phi(\tau) & \text{for } \rho > 0 \\ - \quad u(\tau, -\frac{1}{\sqrt{\lambda}}\rho + \varepsilon^{1-\nu}) + \phi(\tau) & \text{for } \rho < 0 \end{cases}$$

(where $\phi(\tau) = u(\tau, \varepsilon^{1-\nu})$) is a solution to

$$\begin{aligned} \overline{u}_\tau - \overline{u}_{\rho\rho} &= \begin{cases} - \quad \dot{\phi}(\tau) & \rho > 0 \\ \quad \dot{\phi}(\tau) & \rho < 0 \end{cases} \\ \overline{u}(0,\rho) &= \begin{cases} \quad u(0, \frac{1}{\sqrt{\lambda}}\rho + \varepsilon^{1-\nu}) - \phi(0) & \rho > 0 \\ - \quad u(0, \frac{1}{\sqrt{\lambda}}\rho + \varepsilon^{1-\nu}) + \phi(0) & \rho < 0. \end{cases} \end{aligned} \tag{4.21}$$

Thus,

$$\begin{aligned} \overline{u}(\tau,\rho) = \quad & \frac{1}{\sqrt{4\pi\tau}}\int_0^{+\infty} \left\{u(0, \frac{x}{\sqrt{\lambda}} + \varepsilon^{1-\nu}) - \phi(0)\right\}\left[e^{-|x-\rho|^2/4\tau} - e^{-|x+\rho|^2/4\tau}\right] dx \\ & - \int_0^\tau \int_0^{+\infty} \dot{\phi}(s) \frac{e^{-|x-\rho|^2/4(\tau-s)} - e^{-|x+\rho|^2/4(\tau-s)}}{\sqrt{4\pi(\tau-s)}} \, dx ds. \end{aligned}$$

Therefore,

$$\overline{u}_\rho(\tau,0) = \frac{1}{\sqrt{4\pi\tau}}\int_0^{+\infty} \left\{u\Big(0, \frac{x}{\sqrt{\lambda}} + \varepsilon^{1-\nu}\Big) - \phi(0)\right\}\frac{x}{\tau}e^{-x^2/4\tau} \, dx - \int_0^\tau \frac{\dot{\phi}(s)}{\sqrt{\pi(\tau-s)}} \, ds.$$

Thus,

$$\begin{aligned} u_\rho(\tau,\varepsilon^{1-\nu}) = \quad & \sqrt{\lambda}\,\overline{u}_\rho(\tau,0) \\ = \quad & \frac{\sqrt{\lambda^3}}{\sqrt{4\pi\tau}}\int_{\varepsilon^{1-\nu}}^{+\infty} \left\{u(0,x) - \phi(0)\right\}\frac{x - \varepsilon^{1-\nu}}{\tau}e^{-\lambda(x-\varepsilon^{1-\nu})^2/4\tau} \, dx. \\ & -\sqrt{\lambda}\,\partial_{1/2}\phi(\tau) \end{aligned} \tag{4.22}$$

Let us apply equation (4.22) first to the function $g = \overline{T}$ with $\lambda = 1$ and then to $g = \overline{Y}$ with $\lambda = Le$. There holds,

$$
\begin{aligned}
&\frac{\partial}{\partial \rho}(\rho \overline{T}(\tau,\rho))|_{\rho=\varepsilon^{1-\nu}+0} = \\
&= \frac{1}{\sqrt{4\pi\tau}} \int_{\varepsilon^{1-\nu}}^{+\infty} \Big\{ x\overline{T}_0(x) - \varepsilon^{1-\nu}\overline{T}_0(\varepsilon^{1-\nu}) \Big\} \frac{x-\varepsilon^{1-\nu}}{\tau} e^{-(x-\varepsilon^{1-\nu})^2/4\tau}\, dx \\
&\quad -\varepsilon^{1-\nu}\partial_{1/2}\overline{T}(\tau,\varepsilon^{1-\nu})
\end{aligned} \tag{4.23}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial \rho}(\rho \overline{Y}(\tau,\rho))|_{\rho=\varepsilon^{1-\nu}+0} = \\
&= \frac{\sqrt{Le^3}}{\sqrt{4\pi\tau}} \int_{\varepsilon^{1-\nu}}^{+\infty} \Big\{ x\overline{Y}_0(x) - \varepsilon^{1-\nu}\overline{Y}_0(\varepsilon^{1-\nu}) \Big\} \frac{x-\varepsilon^{1-\nu}}{\tau} e^{-Le(x-\varepsilon^{1-\nu})^2/4\tau}\, dx \\
&\quad -\sqrt{Le}\ \varepsilon^{1-\nu}\partial_{1/2}\overline{Y}(\tau,\varepsilon^{1-\nu}).
\end{aligned} \tag{4.24}
$$

Now we let

$$
\varphi_T(\tau) := \frac{1}{\sqrt{4\pi\tau}} \int_{\varepsilon^{1-\nu}}^{+\infty} \Big\{ \rho\overline{T}_0(\rho) - \varepsilon^{1-\nu}\overline{T}_0(\varepsilon^{1-\nu}) \Big\} \frac{\rho-\varepsilon^{1-\nu}}{\tau} e^{-(\rho-\varepsilon^{1-\nu})^2/4\tau}\, d\rho \tag{4.25}
$$

and

$$
\varphi_Y(\tau) := \frac{\sqrt{Le^3}}{\sqrt{4\pi\tau}} \int_{\varepsilon^{1-\nu}}^{+\infty} \Big\{ \rho\overline{Y}_0(\rho) - \varepsilon^{1-\nu}\overline{Y}_0(\varepsilon^{1-\nu}) \Big\} \frac{\rho-\varepsilon^{1-\nu}}{\tau} e^{-Le(\rho-\varepsilon^{1-\nu})^2/4\tau}\, d\rho \tag{4.26}
$$

Observe that

$$
\varphi_T(\tau) = \frac{\partial u_T}{\partial \rho}(\tau,\varepsilon^{1-\nu})
$$

where u_T is the solution to

$$
\begin{array}{ll}
u_\tau - u_{\rho\rho} = 0 & \rho > \varepsilon^{1-\nu}, \tau > 0 \\
u(\tau,\varepsilon^{1-\nu}) = 0 & \tau > 0 \\
u(0,\rho) = \rho\overline{T}_0(\rho) - \varepsilon^{1-\nu}\overline{T}_0(\varepsilon^{1-\nu}) & \rho > \varepsilon^{1-\nu}
\end{array} \tag{4.27}
$$

and analogously

$$
\varphi_Y(\tau) = \frac{\partial u_Y}{\partial \rho}(\tau,\varepsilon^{1-\nu})
$$

where u_Y is the solution to

$$
\begin{array}{ll}
u_\tau - \frac{1}{Le} u_{\rho\rho} = 0 & \rho > \varepsilon^{1-\nu}, \tau > 0 \\
u(\tau,\varepsilon^{1-\nu}) = 0 & \tau > 0 \\
u(0,\rho) = \rho\overline{Y}_0(\rho) - \varepsilon^{1-\nu}\overline{Y}_0(\varepsilon^{1-\nu}) & \rho > \varepsilon^{1-\nu}.
\end{array} \tag{4.28}
$$

Now, recall that

$$\frac{\partial}{\partial \rho}(\rho \overline{T}(\tau,\rho))|_{\rho=\varepsilon^{1-\nu}-0} = \overline{k}(\tau),$$

and

$$\frac{\partial}{\partial \rho}(\rho \overline{Y}(\tau,\rho))|_{\rho=\varepsilon^{1-\nu}-0} = Le\Big(\overline{w}(\tau) - \overline{k}(\tau)\Big).$$

Thus, we obtain from (4.23) and (4.24) the following equations

$$Le\left[\frac{\partial}{\partial \rho}(\rho \overline{T}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} + \left[\frac{\partial}{\partial \rho}(\rho \overline{Y}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} = \varphi_T(\tau) - \overline{k}(\tau) - \varepsilon^{1-\nu}\partial_{1/2}\overline{k} + \partial_{1/2}\Big(\overline{R}^2\pi_\varepsilon(\overline{R},\overline{w})\Big) \tag{4.29}$$

and

$$\begin{aligned}\left[\frac{\partial}{\partial \rho}(\rho \overline{Y}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} = \; & \varphi_Y(\tau) + Le\Big(\overline{k}(\tau) - \overline{w}(\tau)\Big) + \\ & + \varepsilon^{1-\nu}\sqrt{Le^3}\;\partial_{1/2}(\overline{k} - \overline{w}) - \sqrt{Le^3}\;\partial_{1/2}\Big(\overline{R}^2\pi_\varepsilon(\overline{R},\overline{w})\Big).\end{aligned} \tag{4.30}$$

In particular,

$$\begin{aligned} & Le\left[\frac{\partial}{\partial \rho}(\rho \overline{T}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} + \left[\frac{\partial}{\partial \rho}(\rho \overline{Y}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} \\ & = Le\,\varphi_T(\tau) + \varphi_Y(\tau) - \varepsilon^{1-\nu} Le(1-\sqrt{Le})\partial_{1/2}\overline{k}(\tau) - Le\,\overline{w}(\tau) \\ & -\varepsilon^{1-\nu}\sqrt{Le^3}\;\partial_{1/2}\overline{w} + Le\,(1-\sqrt{Le})\,\partial_{1/2}(\overline{R}^2\pi_\varepsilon(\overline{R},\overline{w})).\end{aligned} \tag{4.31}$$

The idea is to choose $\overline{R}$ and $\overline{w}$ in such a way that the right hand sides of (4.29) and (4.30) are small. Or, what is equivalent, so that the right hand sides of (4.29) and (4.31) are small.

Recall that $\overline{k}$ is determined as a function of $\overline{R}$ and $\overline{w}$ by the relation (4.15).
By Corollary 2.1

$$\pi_\varepsilon(R,w) = -\sqrt{Le}\;\exp\Big(\frac{w}{2}\Big) + O(\varepsilon|\mathrm{Log}^2\varepsilon|) \qquad \text{in } C^1.$$

So that, there holds that $\dfrac{\partial \pi_\varepsilon}{\partial w} \neq 0$ and (4.15) determines w as a function $w_\varepsilon(R,k)$ with $w_0(R,k) = \lim_{\varepsilon\to 0} w_\varepsilon(R,k)$ the solution to

$$\frac{1}{Le} = \sqrt{Le}\;R\exp\Big(\frac{w}{2}\Big)$$

This is,

$$w_\varepsilon(R,k) = -2\mathrm{Log}(\sqrt{Le^3}\;R) + \phi_1^\varepsilon(R,k) \tag{4.32}$$

with ϕ_1^ε a C^1 function of its arguments with $\phi_1^\varepsilon(R,k) = O(\varepsilon|\mathrm{Log}^2\varepsilon|)$ in C^1.

Once again, by (4.15)

$$R^2\pi_\varepsilon(R,w) = \varepsilon Rk - (\frac{1}{Le} + A\varepsilon\mathrm{Log}\varepsilon)R = -\frac{1}{Le}R + \phi_2^\varepsilon(R,k)$$

with $\phi_2^\varepsilon(R,k) = O(\varepsilon|\mathrm{Log}\varepsilon|)$ in C^1, a C^1 function of its arguments.

Thus,

$$\partial_{1/2}(\overline{R}^2\pi_\varepsilon(\overline{R},\overline{w})) = -\frac{1}{Le}\partial_{1/2}\overline{R} + \partial_{1/2}\phi_2^\varepsilon(\overline{R},\overline{k}).$$

All these computations give for the right hand side of (4.31)

$$\begin{aligned} r.h.s.(4.31) = \quad & Le\,\varphi_T(\tau) + \varphi_Y(\tau) - \varepsilon^{1-\nu}Le(1-\sqrt{Le})\partial_{1/2}\overline{k}(\tau) + \\ & +2Le\,\mathrm{Log}(\sqrt{Le^3}\;\overline{R}) - Le\,\overline{\phi_1^\varepsilon} + 2\varepsilon^{1-\nu}\sqrt{Le^3}\,\partial_{1/2}(\mathrm{Log}(\sqrt{Le^3}\,\overline{R})) - \\ & -\varepsilon^{1-\nu}\sqrt{Le^3}\,\partial_{1/2}\overline{\phi_1^\varepsilon} - (1-\sqrt{Le})\partial_{1/2}\overline{R} + Le\,(1-\sqrt{Le})\partial_{1/2}\overline{\phi_2^\varepsilon}. \end{aligned}$$

So let us take as $\overline{R}$ the solution to

$$(1-\sqrt{Le})\partial_{1/2}\overline{R} - 2Le\,\mathrm{Log}(\sqrt{Le^3}\;\overline{R}) = Le\,\varphi_T + \varphi_Y. \tag{4.33}$$

On the other hand,

$$r.h.s.(4.29) = \varphi_T - \overline{k} - \varepsilon^{1-\nu}\partial_{1/2}\overline{k} - \frac{1}{Le}\partial_{1/2}\overline{R} + \partial_{1/2}\overline{\phi_2^\varepsilon}$$

and we take

$$\begin{aligned} \overline{k} = \quad & \varphi_T - \frac{1}{Le}\partial_{1/2}\overline{R} = \\ & = -\frac{2}{1-\sqrt{Le}}\mathrm{Log}(\sqrt{Le^3}\;\overline{R}) - \frac{\sqrt{Le}}{1-\sqrt{Le}}\,(\,\varphi_T + \frac{1}{\sqrt{Le^3}}\varphi_Y). \end{aligned} \tag{4.34}$$

Finally, we take $w = w_\varepsilon(R,k)$ given by (4.32).

It is clear that, if $\overline{R}$ is bounded and bounded away from zero, if $\dot{\overline{R}}$ is also bounded and φ_T, φ_Y are $C^{1+\alpha/2}$ for a certain $0<\alpha<1$ with bounds uniform in τ and ε, then there holds that $\overline{k}$ and $\partial_{1/2}\overline{k}$ are bounded and also $\overline{\phi_2^\varepsilon}$ and $\partial_{1/2}\overline{\phi_2^\varepsilon}$ will be $O(\varepsilon|\mathrm{Log}\varepsilon|)$, and $\overline{\phi_1^\varepsilon}$ and $\partial_{1/2}\overline{\phi_1^\varepsilon}$ will be $O(\varepsilon|\mathrm{Log}^2\varepsilon|)$. Therefore,

$$\begin{aligned} r.h.s.(4.29) = O(\varepsilon^{1-\nu}), \\ r.h.s.(4.31) = O(\varepsilon^{1-\nu}). \end{aligned} \tag{4.35}$$

Since $\overline{T}$ and $\overline{Y}$ are continuous at $\rho = \varepsilon^{1-\nu}$ there holds that

$$\begin{aligned} \left[\frac{\partial}{\partial\rho}(\rho\overline{T}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} &= \varepsilon^{1-\nu}\left[\overline{T}_\rho(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} = \varepsilon^{-1-\nu}\left[T_r\right]_{r=\varepsilon^{-\nu}} \\ \left[\frac{\partial}{\partial\rho}(\rho\overline{Y}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} &= \varepsilon^{1-\nu}\left[\overline{Y}_\rho(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} = \varepsilon^{-1-\nu}\left[Y_r\right]_{r=\varepsilon^{-\nu}} \end{aligned}$$

and we will get from (4.35)

$$[T_r]_{r=\varepsilon^{-\nu}} = O(\varepsilon^2), \qquad [Y_r]_{r=\varepsilon^{-\nu}} = O(\varepsilon^2). \tag{4.36}$$

Therefore, it only remains to study the solution $\overline{R}$ to Joulin's equation (4.33).

Let us observe that, if $u(\tau, x)$ is a solution in $]0, \tau_0[\times I\!R_{>0}$ to: $u(\tau, 0) < R_0$ and

$$\left\{ \begin{array}{ll} u_\tau - u_{xx} = 0 & x > 0, \quad 0 < \tau < \tau_0 \\ u_x(\tau, 0) = \dfrac{2Le}{1-\sqrt{Le}} \mathrm{Log}\Big(\sqrt{Le^3}(R_0 - u(\tau, 0))\Big) + \dfrac{\phi(\tau)}{1-\sqrt{Le}} & 0 < \tau < \tau_0 \\ u(0, x) = 0 & x > 0 \end{array} \right. \tag{4.37}$$

there holds that the function $\overline{R}(\tau) := R_0 - u(\tau, 0)$ is a solution to

$$(1 - \sqrt{Le})\partial_{1/2}\overline{R}(\tau) = 2Le\,\mathrm{Log}(\sqrt{Le^3}\ \overline{R}(\tau)) + \phi(\tau)$$

since $\partial_{1/2}\overline{R}(\tau) = u_x(\tau, 0)$.

Therefore, in order to see that there is a solution of (4.33), smooth up to $\tau = 0$, we have to study problem (4.37).

Proposition 4.1 *Let $R_0 > 0$ and let $\phi \in C^{(1+\alpha)/2}([0, \overline{\tau}])$ for a certain $0 < \alpha < 1$, $\overline{\tau} > 0$. Assume that*

$$2Le\ Log(\sqrt{Le^3}\ R_0) + \phi(0) = 0 \tag{4.38}$$

There exists $0 < \tau_{max} \le \overline{\tau}$ and a unique maximal solution $u \in C^{1+\alpha/2, 2+\alpha}([0, \tau_{max}[\times I\!R_+)$ with $u(\tau, 0) < R_0$ to (4.37).

Proof. The proof is standard. We sketch it here for the sake of completeness. For every $n \in I\!N$, $\dfrac{1}{n} < R_0 < n$ we let

$$f_n(\tau, u) = \left\{ \begin{array}{ll} 2Le\,\mathrm{Log}(\sqrt{Le^3}(R_0 - u)) + \phi(\tau) & \text{if } \dfrac{1}{n} < R_0 - u < n \\ 2Le\,\mathrm{Log}(\sqrt{Le^3}\ \dfrac{1}{n}) + \phi(\tau) & \text{if } R_0 - u \le \dfrac{1}{n} \\ 2Le\,\mathrm{Log}(\sqrt{Le^3}\ n) + \phi(\tau) & \text{if } R_0 - u \ge n \end{array} \right.$$

Then, f_n is uniformly Lipschitz continuous in u and $C^{(1+\alpha)/2}$ in τ. Standard results give the existence of a unique solution $u_n \in C^{1+\alpha/2, 2+\alpha}([0, \overline{\tau}] \times I\!R_+)$ to

$$\left\{ \begin{array}{ll} u_\tau - u_{xx} = 0 & x > 0, \quad 0 < \tau < \overline{\tau} \\ u_x(\tau, 0) = \dfrac{1}{1-\sqrt{Le}} f_n(\tau, u(\tau, 0)) & 0 < \tau < \overline{\tau} \\ u(0, x) = 0 & x > 0 \end{array} \right.$$

Let $0 < t_n \le \overline{\tau}$ be such that $R_0 - n \le u_n(\tau, 0) \le R_0 - \dfrac{1}{n}$ for $0 \le \tau \le t_n$. Then, u_n is a solution to (4.37) for $0 \le \tau \le t_n$. Observe that (4.37) has a unique solution such that $R_0 - \gamma^{-1} < u(\tau, 0) < R_0 - \gamma$ for a positive constant γ. Thus, in this way, we see that there is a unique maximal solution to (4.37) and this solution satisfies that $u \in C^{1+\alpha/2, 2+\alpha}([0, \tau_{max}[\times \mathbb{R}_{\ge 0})$, $u(\tau, 0) < R_0$ for $0 \le \tau < \tau_{max}$. $\square$

Now, let $\overline{R}(\tau) := R_0 - u(\tau, 0)$. There holds that $\overline{R} \in C^{1+\alpha/2}([0, \tau_{max}[)$ is the unique maximal solution to (4.33) with $\overline{R}(0) = R_0$.

Recall that in our problem the function $\phi(\tau) = Le\, \varphi_T(\tau) + \varphi_Y(\tau)$. Thus, we have the following remark,

Remark 4.1 *If $\varphi_T,\ \varphi_Y \in C^{1+\alpha/2}(\mathbb{R}_+)$ the function $\overline{k}$ given by (4.34) also belongs to $C^{1+\alpha/2}([0, \tau_{max}[)$.*

Since

$$\varphi_T(\tau) = \frac{\partial u_T}{\partial \rho}(\tau, \varepsilon^{1-\nu})$$

where u_T is the solution to (4.27) and

$$\varphi_Y(\tau) = \frac{\partial u_Y}{\partial \rho}(\tau, \varepsilon^{1-\nu})$$

where u_Y is the solution to (4.28), in order to have $\varphi_T,\ \varphi_Y \in C^{(1+\alpha)/2}([0, \infty[)$ with norm independent of ε, we must ask that

$$\begin{aligned} &|\rho \overline{T}_0(\rho)|,\ |\rho \overline{Y}_0(\rho)| \le C \quad \text{for } \rho > \varepsilon^{1-\nu} \quad \text{with } C \text{ independent of } \varepsilon \\ &\rho \overline{T}_0(\rho),\ \rho \overline{Y}_0(\rho) \in C^{2+\alpha} \quad \text{near } \rho = \varepsilon^{1-\nu} \\ &\frac{\partial^2}{\partial \rho^2}\big(\rho \overline{T}_0(\rho)\big)|_{\rho=\varepsilon^{1-\nu}} = \frac{\partial^2}{\partial \rho^2}\big(\rho \overline{Y}_0(\rho)\big)|_{\rho=\varepsilon^{1-\nu}} = 0. \end{aligned} \tag{4.39}$$

If we want $\varphi_T,\ \varphi_Y \in C^{1+\alpha/2}$ we have to ask, in addition, that

$$\rho \overline{T}_0(\rho),\ \rho \overline{Y}_0(\rho) \in C^{3+\alpha} \quad \text{near } \rho = \varepsilon^{1-\nu}. \tag{4.40}$$

4.4. Summary

We have now all the elements for the proof of the main result of this section, namely the solution to (1.1)-(1.2) up to ε^2 errors.

Theorem 4.2 *Let $\varepsilon > 0$, $0 < \nu < 1$ and $R_0^\varepsilon \in \mathbb{R}$ with $R_0^\varepsilon \to R_0 > 0$. Let $w_0^\varepsilon \in \mathbb{R}$ be such that*

$$\frac{1}{Le} + A\varepsilon\, Log\, \varepsilon + R_0^\varepsilon \pi_\varepsilon(R_0^\varepsilon, w_0^\varepsilon) = O(\varepsilon) \tag{4.41}$$

with π_ε *as in Corollary 2.1.*

Let

$$k_0^\varepsilon = \varepsilon^{-1}\Big(Le^{-1} + A\varepsilon\, Log\,\varepsilon + R_0^\varepsilon \pi_\varepsilon(R_0^\varepsilon, w_0^\varepsilon)\Big)$$

and $a_0^\varepsilon,\ b_0^\varepsilon$ *the solution to*

$$(4.42)\qquad \begin{array}{ll} (a) & a_0^\varepsilon + Le^{-1}\, b_0^\varepsilon = -2Log(\sqrt{Le^3}\, R_0^\varepsilon) \\ (b) & \sqrt{Le}\, a_0^\varepsilon + Le^{-1}\, b_0^\varepsilon = -(1-\sqrt{Le})k_0^\varepsilon - 2Log(\sqrt{Le^3}\, R_0^\varepsilon). \end{array}$$

Let $(T_0^\varepsilon, Y_0^\varepsilon)$ *be defined in* $\{|x| \geq \varepsilon^{-\nu}\}$*, radially symmetric and continuous such that, with* $\overline{T}_0(x) = \dfrac{1}{\varepsilon} T_0^\varepsilon(\dfrac{x}{\varepsilon})$, $\overline{Y}_0(x) = \dfrac{1}{\varepsilon}\Big(Y_0^\varepsilon(\dfrac{x}{\varepsilon}) - 1\Big)$, $\rho = |x|$,

$$\begin{array}{rl}
[i]. & \|\,|x|\overline{T}_0(x)\|_{L^\infty(|x|>\varepsilon^{1-\nu})},\ \|\,|x|\overline{Y}_0(x)\|_{L^\infty(|x|>\varepsilon^{1-\nu})} \leq C \quad \textit{independent of } \varepsilon \\
[ii]. & \|\,|x|\overline{T}_0(x)\|_{C^{3+\alpha}(\varepsilon^{1-\nu}\leq|x|\leq\varepsilon^{1-\nu}+r_0)},\ \|\,|x|\overline{Y}_0(x)\|_{C^{3+\alpha}(\varepsilon^{1-\nu}\leq|x|\leq\varepsilon^{1-\nu}+r_0)} \leq C \\
& \textit{with } C,\ r_0>0,\ 0<\alpha<1 \textit{ independent of } \varepsilon \\
[iii]. & \dfrac{\partial^2}{\partial\rho^2}(|x|\overline{T}_0(x))\Big|_{|x|=\varepsilon^{1-\nu}+0} = \dfrac{\partial^2}{\partial\rho^2}(|x|\overline{Y}_0(x))\Big|_{|x|=\varepsilon^{1-\nu}+0} = 0 \\
[iv]. & (\rho+\varepsilon^{1-\nu})\overline{T}_0(\rho+\varepsilon^{1-\nu}),\ (\rho+\varepsilon^{1-\nu})\overline{Y}_0(\rho+\varepsilon^{1-\nu}) \quad \textit{converge uniformly} \\
& \textit{on compact subsets of } \{\rho\geq 0\} \textit{ as } \varepsilon\to 0 \\
[v]. & \varepsilon^{1-\nu}\overline{T}_0(\varepsilon^{1-\nu}) = \varepsilon^{1-\nu}k_0^\varepsilon - (R_0^\varepsilon)^2\pi_\varepsilon(R_0^\varepsilon, w_0^\varepsilon) \\
[vi]. & \varepsilon^{1-\nu}\overline{Y}_0(\varepsilon^{1-\nu}) = Le\Big(\varepsilon^{1-\nu}(w_0^\varepsilon - k_0^\varepsilon) + (R_0^\varepsilon)^2\pi_\varepsilon(R_0^\varepsilon, w_0^\varepsilon)\Big) \\
[vii]. & \dfrac{\partial}{\partial\rho}(\rho\overline{T}_0(\rho))\Big|_{\rho=\varepsilon^{1-\nu}} = a_0^\varepsilon \\
[viii]. & \dfrac{\partial}{\partial\rho}(\rho\overline{Y}_0(\rho))\Big|_{\rho=\varepsilon^{1-\nu}} = b_0^\varepsilon.
\end{array}$$

(4.43)

Finally, let $(T_0^\varepsilon, Y_0^\varepsilon)$ *be defined for* $|x| < \varepsilon^{-\nu}$ *as the solution to (4.8) in* $\{|x| < R_0^\varepsilon\}$ *with*

$$T_0^\varepsilon(R_0^\varepsilon) = \frac{1}{Le} + A\varepsilon\, Log\,\varepsilon \quad , \quad Y_0^\varepsilon(R_0^\varepsilon) = Le(\varepsilon w_0^\varepsilon - A\varepsilon\, Log\,\varepsilon)$$

with T_0^ε *and* Y_0^ε *harmonic in* $\{R_0^\varepsilon < |x| < \varepsilon^{-\nu}\}$ *and* $C^1(B(0,\varepsilon^{-\nu}))$.

Then, there exists $\tau_{max} > 0$*, such that for every* $0 < \tau_0 < \tau_{max}$ *there exists* $\varepsilon_0 > 0$ *such that, if* $0 < \varepsilon < \varepsilon_0$ *there exists an approximate solution* $(T_J^\varepsilon, Y_J^\varepsilon)$ *to (4.8), (4.9) that is defined and continuous in* $[0, \tau_0/\varepsilon^2] \times I\!R^3$, $C_x^1([0,\tau_0/\varepsilon^2] \times \{|x| \leq \varepsilon^{-\nu}\})$, $C^{1,2}(]0,\tau_0/\varepsilon^2] \times \{|x| \geq \varepsilon^{-\nu}\})$*, that satisfies the jump condition (4.36) on the derivative and such that* $T_J^\varepsilon(0,x) = T_0^\varepsilon(x)$, $Y_J^\varepsilon(0,x) = Y_0^\varepsilon(x)$.

Moreover $(T_J^\varepsilon, Y_J^\varepsilon)$ *satisfies*

$$(4.44)\qquad \begin{array}{l} T_t - \Delta T = Y f_\varepsilon(T) + F_1^\varepsilon(x,t) + \mu_1^\varepsilon \\ Y_t - \dfrac{\Delta Y}{Le} = -Y f_\varepsilon(T) + F_2^\varepsilon(x,t) + \mu_2^\varepsilon \end{array}$$

where the functions F_1^ε, F_2^ε are supported on $[0,\tau_0/\varepsilon^2]\times B(0,\varepsilon^{-\nu})$, $\mu_i^\varepsilon = h_i^\varepsilon\, dS_\varepsilon$ where h_i^ε is a continuous function defined on $[0,\tau_0/\varepsilon^2]\times \partial B(0,\varepsilon^{-\nu})$, dS_ε is the surface measure of $[0,\tau_0/\varepsilon^2]\times\partial B(0,\varepsilon^{-\nu})$ and there exists $C_{\tau_0}>0$ such that $|F_i^\varepsilon(x,t)|\le C_{\tau_0}\varepsilon^2$ for $0\le t\le \tau_0/\varepsilon^2$, $x\in B(0,\varepsilon^{-\nu})$ and $|h_i^\varepsilon(x,t)|\le C_{\tau_0}\varepsilon^2$ for $0\le t\le\tau_0/\varepsilon^2$, $x\in\partial B(0,\varepsilon^{-\nu})$, for $i=1,2$.

Moreover, let $R_\varepsilon(t)$ be the largest radius such that $T_J^\varepsilon(t,r)=\dfrac{1}{Le}+A\varepsilon Log\,\varepsilon$ and $\overline{R}_\varepsilon(\tau):=R_\varepsilon(\tau/\varepsilon^2)$ then,

$$(1-\sqrt{Le})\partial_{1/2}\overline{R}_\varepsilon = 2Le\, Log(\sqrt{Le^3}\,\overline{R}_\varepsilon)+\phi_\varepsilon(\tau) \tag{4.45}$$

with $\overline{R}_\varepsilon(0)=R_0^\varepsilon$, where the function $\phi_\varepsilon\in C^{1+\alpha/2}(I\!R_+)$ is determined by the values of T_0^ε and Y_0^ε for $|x|\ge\varepsilon^{-\nu}$ and converges to a function ϕ in $C^{1+\alpha'/2}(I\!R_+)$ for $0<\alpha'<\alpha$, as $\varepsilon\to 0$. Thus, there exists

$$\lim_{\varepsilon\to 0}\overline{R}_\varepsilon=\overline{R}\quad in\ C^1([0,\tau_0])$$

and $\overline{R}$ is the solution to

$$(1-\sqrt{Le})\partial_{1/2}\overline{R} = 2Le\, Log(\sqrt{Le^3}\,\overline{R})+\phi(\tau) \tag{4.46}$$

with $\overline{R}(0)=R_0$.

Proof. Let
$$\phi_\varepsilon(\tau)=Le\,\varphi_T^\varepsilon(\tau)+\varphi_Y^\varepsilon(\tau).$$
with $\varphi_T^\varepsilon(\tau),\varphi_Y^\varepsilon(\tau)$ given by (4.25), (4.26).

By (4.43) $\phi_\varepsilon(\tau)$ converges in $C^{1+\alpha'/2}(I\!R_+)$ to a function $\phi(\tau)$ for any $0<\alpha'<\alpha$.

Let $\tau_{max}>0$ be the lifetime of the solution $\overline{R}$ to (4.46) with initial value R_0 and let $0<\tau_0<\tau_{max}$.

Now, let $\varepsilon_1>0$ be such that for $0<\varepsilon<\varepsilon_1$ the maximal solution $\overline{R}_\varepsilon(\tau)$ to (4.45) with initial value R_0^ε is defined in $[0,\tau_0]$, and let $R_\varepsilon(t)=\overline{R}_\varepsilon(\varepsilon^2 t)$. Observe that, due to (4.42) and (4.43) – in particular, relations [vii] and [viii] – we are under the hypotheses of Proposition 4.1.

Let (T,Y) be the solution to (4.8) in $B(0,R_\varepsilon(t))$ with

$$T(t,R_\varepsilon(t))=\frac{1}{Le}+A\varepsilon\mathrm{Log}\,\varepsilon,\quad Y(t,R_\varepsilon(t))=Le(\varepsilon w_\varepsilon(t)-A\varepsilon\mathrm{Log}\,\varepsilon),$$

where $w_\varepsilon(t)=\overline{w}_\varepsilon(\varepsilon^2 t)$ and $\overline{w}_\varepsilon(\tau)=w_\varepsilon(\overline{R}_\varepsilon(\tau),\overline{k}_\varepsilon(\tau))$ given by (4.32) with $\overline{k}_\varepsilon$ given by (4.34); and let $k_\varepsilon(t)=\overline{k}_\varepsilon(\varepsilon^2 t)$. Note at this point that, due to (4.43) [vii] and [viii] and (4.42), $\overline{k}_\varepsilon(0)=k_0^\varepsilon$.

Define $T,\ Y$ in $R_\varepsilon(t)<|x|<\varepsilon^{-\nu}$ to be harmonic functions with

$$T,\ Y\in C_x^1(]0,\tau_0/\varepsilon^2[\times B(0,\varepsilon^{-\nu})).$$

Finally, let (T, Y) be the solution to (4.9) in $]0, \tau_0/\varepsilon^2[\times(I\!R^3 \setminus B(0, \varepsilon^{-\nu}))$ with $T(0, x) = T_0^\varepsilon(x)$, $Y(0, x) = Y_0^\varepsilon(x)$ for $|x| > \varepsilon^{-\nu}$, with $T,\ Y \in C(]0, \tau_0/\varepsilon^2[\times I\!R^3)$. Then, we will see that T and Y are continuous up to $t = 0$ and, as a consequence of the derivations in this section, the jump of the gradient of T and Y on $|x| = \varepsilon^{-\nu}$ is of order ε^2.

In fact, let us see that T, Y are continuous in $[0, \tau_0/\varepsilon^2] \times \{|x| \leq \varepsilon^{-\nu}\}$ and T_t and Y_t are bounded by $\overline{C}_{\overline{t}}\, \varepsilon^2$ in $\{|x| < \varepsilon^{-\nu}\}$.

In fact, by Theorem 2.1, there holds, in $|x| \leq R_\varepsilon(t)$,

$$\begin{aligned} T(t, x) &= \mathcal{T}^\varepsilon(x, R_\varepsilon(t), w_\varepsilon(t)) \\ Y(t, x) &= \mathcal{Y}^\varepsilon(x, R_\varepsilon(t), w_\varepsilon(t)) \end{aligned}$$

with $\mathcal{T}^\varepsilon$, $\mathcal{Y}^\varepsilon$, C^2 functions of their arguments with derivatives bounded independently of ε. In particular, T, Y are C^1 functions of t up to $t = 0$ and

$$\begin{aligned} |T_t(t, x)| &= |\mathcal{T}_R^\varepsilon\, \dot{R}_\varepsilon + \mathcal{T}_w^\varepsilon\, \dot{w}_\varepsilon| \leq C(|\dot{R}_\varepsilon| + |\dot{w}_\varepsilon|) \\ |Y_t(t, x)| &= |\mathcal{Y}_R^\varepsilon\, \dot{R}_\varepsilon + \mathcal{Y}_w^\varepsilon\, \dot{w}_\varepsilon| \leq C(|\dot{R}_\varepsilon| + |\dot{w}_\varepsilon|). \end{aligned}$$

Since, by construction $w_\varepsilon = w_\varepsilon(R, k)$ is a C^1 function of its arguments, and

$$\begin{aligned} |\dot{R}_\varepsilon(t)| &= \varepsilon^2 |\dot{\overline{R}}_\varepsilon(\varepsilon^2 t)| \leq \overline{C}_{\tau_0}\, \varepsilon^2 \quad \text{if } 0 \leq t \leq \tau_0/\varepsilon^2 \\ |\dot{k}_\varepsilon(t)| &= \varepsilon^2 |\dot{\overline{k}}_\varepsilon(\varepsilon^2 t)| \leq \overline{C}_{\tau_0}\, \varepsilon^2 \quad \text{if } 0 \leq t \leq \tau_0/\varepsilon^2, \end{aligned}$$

there holds that

$$\begin{aligned} |T_t| &\leq \overline{C}_{\tau_0}\, \varepsilon^2 \\ |Y_t| &\leq \overline{C}_{\tau_0}\, \varepsilon^2 \end{aligned}$$

for $|x| \leq R_\varepsilon(t)$, $0 \leq t \leq \tau_0/\varepsilon^2$.

Moreover,

$$\begin{aligned} \lim_{t\to 0} T(t, x) &= \mathcal{T}^\varepsilon(x, R_0^\varepsilon, w_0^\varepsilon) = T_0^\varepsilon(x) \\ \lim_{t\to 0} Y(t, x) &= \mathcal{Y}^\varepsilon(x, R_0^\varepsilon, w_0^\varepsilon) = Y_0^\varepsilon(x). \end{aligned}$$

On the other hand, in $R_\varepsilon(t) \leq |x| \leq \varepsilon^{-\nu}$,

$$\begin{aligned} T(t, x) &= \varepsilon k_\varepsilon(t) - \frac{R_\varepsilon^2(t)}{|x|} \pi_\varepsilon(R_\varepsilon(t), w_\varepsilon(t)) \\ Y(t, x) &= \varepsilon Le(w_\varepsilon(t) - k_\varepsilon(t)) + Le \frac{R_\varepsilon^2(t)}{|x|} \pi_\varepsilon(R_\varepsilon(t), w_\varepsilon(t)). \end{aligned}$$

Since R_ε, k_ε, and w_ε belong to $C^1([0, \frac{\tau_0}{\varepsilon^2}])$ and π_ε is a C^1 function of its arguments with derivatives bounded uniformly in ε for $\varepsilon \leq \varepsilon_2$ and $\frac{1}{M} \leq R \leq M$ with $M > 0$, there holds that T, $Y \in C^{1,2}(\{R_\varepsilon(t) \leq |x| \leq \varepsilon^{-\nu}\})$, continuous up to $t = 0$ with $T(0, x) = T_0^\varepsilon(x)$, $Y(0, x) = Y_0^\varepsilon(x)$ in $\{R_0 < |x| < \varepsilon^{-\nu}\}$ and

$$|T_t|,\ |Y_t| \leq C\Big(|\dot{R}_\varepsilon(t)| + |\dot{k}_\varepsilon(t)| + |\dot{w}_\varepsilon(t)|\Big) \leq \overline{C}_{\tau_0}\, \varepsilon^2$$

if $0 \le t \le \dfrac{\tau_0}{\varepsilon^2}$.

Since T is decreasing there holds that $F_\varepsilon(T) = 0$ for $r > R_\varepsilon$. Thus, (T, Y) is a solution to (4.44) with F_i^ε, μ_i^ε as stated.

Moreover, $\lim_{t\to 0} T(t,x) = T_0^\varepsilon(x)$, $\lim_{t\to 0} Y(t,x) = Y_0^\varepsilon(x)$ for $x \in \mathbb{R}^3$.

□

Remark 4.2 *Let $w_0 \in \mathbb{R}$. If in Theorem 4.1 we take as R_0^ε the radius R_{ε,w_0} of a Zeldovich flame, there holds that*

$$\frac{1}{Le} + A\varepsilon \, Log\,\varepsilon + R_0^\varepsilon \pi_\varepsilon(R_0^\varepsilon, w_0) = \varepsilon w_0 = O(\varepsilon).$$

Thus, we may take $w_0^\varepsilon = w_0$. With this choice we have $R_0 = \dfrac{1}{\sqrt{Le^3}} \exp\Big(-\dfrac{w_0}{2}\Big)$.

4.5. Proof of Theorem 4.1

We have just seen that the whole construction was actually governed by two functions of t: the radius - we drop the superscript $^\varepsilon$ - R_J and the function k, given by (4.15), and denoted by k_J in the sequel. We look for R_J and k_J under the form

$$(4.47) \qquad \begin{aligned} R_J(t) &= \sum_{i=1}^{3} \varepsilon^{i-1} R_{i,J}(t) \\ k_J(t) &= \sum_{i=1}^{3} \varepsilon^{i-1} k_{i,J}^i(t) \end{aligned}$$

where $R_{1,J}$ and $k_{1,J}$ are essentially the solutions produced by Theorem 4.2. We indeed saw in the previous subsections that an error of order ε in R_J and k_J implied the equation to be verified up to ε^2 errors.

4.5.1. Inside $B_{R_J(t)}$

Provided that the Lipschitz constants of $R_J(t/\varepsilon^2)$ and $k_J(t/\varepsilon^2)$ are uniformly bounded with respect to ε, the procedure of Subsection 4.2 yields a solution (T_J, Y_J) satisfying the reaction-diffusion system up to $\varepsilon^3|\mathrm{Log}^2\varepsilon|$ errors. Moreover the $N_{\omega,R_J(t),\varepsilon}$ error in the equation for Y is also of order $\varepsilon^3|\mathrm{Log}^2\varepsilon|$.

4.5.2. Inside $B(0,\varepsilon^{-\nu})\backslash B_{R_J(t)}$

Assuming R_J and k_J to be given, let us set

$$(4.48) \qquad \begin{aligned} T_J^1(t,r) &= \varepsilon k_J(t) - \frac{R_J^2(t)}{r} \pi_\varepsilon(R_J(t), w_J(t)) \\ Y_J^1(t,r) &= 1 + Le\{\varepsilon(w_J - k_J) + \frac{R^2(t)}{r}\pi_\varepsilon(R(t), w(t))\} \end{aligned}$$

Let $T_J^2(t,r)$ and $Y_J^2(t,r)$ be given by the system

$$\begin{aligned} \Delta T_J^2 = &\ \partial_t T_J^1, \qquad (R_J(t) < r < \varepsilon^{-\nu}) \\ T_J^2(t, R_J(t)) = &\ \partial_r T_J^2(t, R_J(t)) = 0 \end{aligned} \tag{4.49}$$

$$\begin{aligned} \frac{\Delta Y_J^2}{Le} = &\ \partial_t Y_J^1 \qquad (R_J(t) < r < \varepsilon^{-\nu}) \\ Y_J^2(t, R_J(t)) = &\ \partial_r Y_J^2(t, R_J(t)) = 0 \end{aligned} \tag{4.50}$$

An easy computation - see subsection 4.3; compute in fact rT_J^2 and rY_J^2 - shows that

$$T_J^2(t,r) = O(\varepsilon^{2-\nu}), \qquad Y_J^2(t,r) = O(\varepsilon^{2-\nu}). \tag{4.51}$$

Still provided that, this time, the Lipschitz constants of $\dot{\overline{R}}_J$ and $\dot{\overline{k}}_J$ are uniformly bounded w.r.t ε, the functions

$$T_J(t,r) = T_J^1(t,r) + T_J^2(t,r), \qquad Y_J(t,r) = Y_J^1(t,r) + Y_J^2(t,r) \tag{4.52}$$

satisfy system (4.1) up to $O(\varepsilon^{4-4\nu})$ errors.

Observe that if $\dot{\overline{R}}_J(0) = \dot{\overline{k}}_J(0) = 0$ there holds that $T_J^2(0,r) = Y_J^2(0,r) = 0$ so that this modification does not affect the initial datum.

4.5.3. Outside $B(0, \varepsilon^{-\nu})$

This is where the definition of the functions $R_{i,J}$ and $k_{i,J}$ comes out. First, we take $R_0^\varepsilon \to R_0$ and w_0^ε such that the condition on k_0^ε in Theorem 4.2 is satisfied (and define k_0^ε accordingly).

Writing down the formulae for

$$Le\left[\frac{\partial}{\partial\rho}(\rho\overline{T}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} + \left[\frac{\partial}{\partial\rho}(\rho\overline{Y}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} \tag{4.53}$$

and

$$\left[\frac{\partial}{\partial\rho}(\rho\overline{T}(\tau,\rho))\right]_{\rho=\varepsilon^{1-\nu}} \tag{4.54}$$

and remembering the expansion of R_J and k_J it is easy to see that (4.53) and (4.54) become of order $e^{3-3\nu}$ if we define recursively $R_{i,J}$ and $k_{i,J}$ in the following way:

First $R_{1,J}(t) = \overline{R}_1(\varepsilon^2 t)$ where $\overline{R}_1$ is the solution of Joulin's equation with initial datum R_0^ε. Then, $k_{1,J}(t) = \overline{k}_1(\varepsilon^2 t)$ where

$$\overline{k}_1 = -\frac{2}{1-\sqrt{Le}}\mathrm{Log}(\sqrt{Le^3}\,\overline{R}_1) - \frac{\sqrt{Le}}{1-\sqrt{Le}}\varphi_T - \frac{1}{Le(1-\sqrt{Le})}\varphi_Y.$$

Then, $R_{2,J}(t) = \overline{R}_2(\varepsilon^2 t)$ where $\overline{R}_2$ is the solution to the linearized Joulin equation with forcing term equal to the $O(\varepsilon^{1-\nu})$ terms in the expression of (4.53). This is,

$$
\begin{aligned}
(1-\sqrt{Le})\,\partial_{1/2}\overline{R}_2 \quad -2Le\frac{\overline{R}_2}{\overline{R}_1} &= \varepsilon^{-1} Le\Big(-\varepsilon^{1-\nu}(1-\sqrt{Le})\,\partial_{1/2}\overline{k}_1 - \phi(\overline{R}_1,\overline{k}_1) \\
&\quad +2\varepsilon^{1-\nu}\sqrt{Le}\,\partial_{1/2}\Big(\mathrm{Log}(\sqrt{Le^3}\,\overline{R}_1)\Big) - \varepsilon^{1-\nu}\sqrt{Le}\,\partial_{1/2}\phi_1(\overline{R}_1,\overline{k}_1) \\
&\quad +(1-\sqrt{Le})\,\partial_{1/2}\phi_2(\overline{R}_1,\overline{k}_1)\Big)
\end{aligned}
$$

with initial datum

$$
\overline{R}_2(0) = \frac{\varepsilon^{-1}}{2} R_0^\varepsilon\,\phi_1(R_0^\varepsilon, k_0^\varepsilon).
$$

In this way the initial datum $\overline{R}_2(0)$ is compatible with the (linear) equation and the solution $\overline{R}_2(\tau)$ is smooth in $[0,\tau_0]$.

Here we have used that $\overline{R}_1$ and $\overline{k}_1$ have bounded first derivatives in $[0,\tau_0]$ so that all the 1/2 derivatives appearing in the right hand side of the equation vanish at $\tau = 0$.

In order for $\dot{\overline{R}}_1$ and $\dot{\overline{k}}_1$ to be bounded we ask that the compatibility conditions (4.43) vii) and viii) hold, as well as the regularity conditions (4.43) i), ii) and iii).

Now we define $k_{2,J}(t) = \overline{k}_2(\varepsilon^2 t)$ where

$$
\overline{k}_2 = \frac{1}{Le}\partial_{1/2}\overline{R}_2 + \varepsilon^{-1}\Big(\varepsilon^{1-\nu}\partial_{1/2}\overline{k}_1 + \partial_{1/2}\phi_2(\overline{R}_1,\overline{k}_1)\Big).
$$

Observe that $\overline{k}_2(0) = 0$.

Now, $R_{3,J}(t) = \overline{R}_3(\varepsilon^2 t)$ where $\overline{R}_3$ is the solution to

$$
\begin{aligned}
&(1-\sqrt{Le})\partial_{1/2}\overline{R}_3 - 2Le\frac{\overline{R}_3}{\overline{R}_1} = \varepsilon^{-2} Le\,\left(\varepsilon^2\Big(\frac{\overline{R}_2}{\overline{R}_1}\Big)^2 + \frac{2}{3}\varepsilon^3\Big(\frac{\overline{R}_2}{\overline{R}_1}\Big)^3\right. \\
&-\varepsilon\overline{R}_2\int_0^1 \partial_R\phi_1(\overline{R}_1 + s\varepsilon\overline{R}_2, \overline{k}_1 + s\varepsilon\overline{k}_2)\,ds - \varepsilon\overline{k}_2\int_0^1 \partial_k\phi_1(\overline{R}_1 + s\varepsilon\overline{R}_2, \overline{k}_1 + s\varepsilon\overline{k}_2)\,ds \\
&\quad +2\varepsilon^{2-\nu}\sqrt{Le}\,\partial_{1/2}\Big(\frac{\overline{R}_2}{\overline{R}_1}\Big) + (1-\sqrt{le})\,\partial_{1/2}\Big[\varepsilon\overline{R}_2\int_0^1 \partial_R\phi_2(\overline{R}_1 + s\varepsilon\overline{R}_2, \overline{k}_1 + s\varepsilon\overline{k}_2)\,ds\Big] \\
&\quad\quad \left. +(1-\sqrt{le})\,\partial_{1/2}\Big[\varepsilon\overline{R}_2\int_0^1 \partial_k\phi_2(\overline{R}_1 + s\varepsilon\overline{R}_2, \overline{k}_1 + s\varepsilon\overline{k}_2)\,ds\Big]\right)
\end{aligned}
$$

with initial datum

$$
\begin{aligned}
\overline{R}_3(0) \quad &= \frac{\varepsilon^{-2}}{2} R_0^\varepsilon\Big(\varepsilon^2\Big(\frac{\overline{R}_2(0)}{R_0^\varepsilon}\Big)^2 + \frac{2}{3}\varepsilon^3\Big(\frac{\overline{R}_2(0)}{R_0^\varepsilon}\Big)^3 \\
&\quad -\varepsilon\overline{R}_2(0)\int_0^1 \partial_R\phi_1(R_0^\varepsilon + s\varepsilon\overline{R}_2(0), k_0^\varepsilon)\,ds\Big).
\end{aligned}
$$

So that the initial datum in compatible with the equation and $\dot{\overline{R}}_3$ is bounded in $[0, \tau_0]$.

Finally, we take $k_{3,J}(t) = \overline{k}_3(\varepsilon^2 t)$ where

$$\overline{k}_3 = \frac{1}{Le}\partial_{1/2}\overline{R}_3 + \varepsilon^{-2}\Big(\varepsilon^{2-\nu}\partial_{1/2}\overline{k}_2 + \partial_{1/2}\Big[\varepsilon\overline{R}_2\int_0^1 \partial_R\phi_2(\overline{R}_1 + s\varepsilon\overline{R}_2, \overline{k}_1 + s\varepsilon\overline{k}_2)\,ds\Big]$$

$$+\partial_{1/2}\Big[\varepsilon\overline{k}_2\int_0^1 \partial_k\phi_2(\overline{R}_1 + s\varepsilon\overline{R}_2, \overline{k}_1 + s\varepsilon\overline{k}_2)\,ds\Big]\Big).$$

Note that $\overline{k}_3(0) = 0$.

In this way (4.53) and (4.54) become of order $\varepsilon^{3-3\nu}$.

It is important to notice that condition (4.42) (b) together with $\overline{k}_2(0) = \overline{k}_3(0) = 0$ imply that $k_J(0) = k_0^\varepsilon$.

In order for everything to close up we need to change conditions (4.43) v) and vi) by replacing R_0^ε by $R_{0,\varepsilon} := R_0^\varepsilon + \varepsilon R_{2,J}(0) + \varepsilon^2 R_{3,J}(0)$ and w_0^ε by $w_{0,\varepsilon} := w_\varepsilon(R_{0,\varepsilon}, k_0^e)$.

Also observe that in order to have R_J, $k_J \in C^{1,1}([0,\tau_0])$ (this is needed in subsection 4.5.2) we have to replace in (4.43) ii) $C^{3+\alpha}$ by $C^{5+\alpha}$ and we have to add in (4.43) iii) that third derivatives should also vanish so that $\dot{R}_J(0) = \dot{k}_J(0) = 0$.

All these are conditions on the initial data T_0^ε, Y_0^ε that can be set up a priori. □

Remark 4.3 *In view of Remark 4.2 and the fact that $k_{2,J}(0) = k_{3,J}(0) = 0$ it is easy to check that, given $w_0 \in I\!R$ we may take $R_0^\varepsilon = R_{w_0}^\varepsilon$, the radius of the Zeldovich flame $(T_{w_0}^\varepsilon, Y_{w_0}^\varepsilon)$, and $w_0^\varepsilon = w_0$ so that $k_0^\varepsilon = w_0$ and the initial datum of Joulin's solution $(T_J^\varepsilon(0,r), Y_J^\varepsilon(0,r))$ coincides with Zeldovich's flame in the ball $B(0,\varepsilon^\nu)$.*

5. Linear evolution around the Zeldovich flame ball

The goal of this section is to obtain reasonably accurate pointwise estimates of the semigroup $e^{-t\mathcal{L}^\varepsilon}$ for time intervals of order ε^{-2}. The linear evolution equation is written in terms of the linear mass fraction and enthalpy (v, w):

$$(5.1)\qquad \begin{cases} v_t - \dfrac{\Delta v}{Le} = \dfrac{1}{\varepsilon^2}h_0'(q_0)v - \dfrac{1}{\varepsilon^2}e^{w_0 - q_0/Le}q_0 w \\ w_t - \Delta w = (\dfrac{1}{Le} - 1)v_t \end{cases} \qquad (I\!R^3)$$
$$v(0,r) = v_0(r), \qquad w(0,r) = w_0(r)$$

5.1. Outline; presentation of the results

5.1.1. Initial data

A nontrivial problem is the functional space in which we are going to look for the estimates. As should be obvious from Section 3, the whole real positive line is in the essential spectrum of $\mathcal{L}^\varepsilon$; hence we have no chance to have $\|e^{-t\mathcal{L}^\varepsilon}\|$ integrable in time with respect to any norm. On the other hand, the nonlinearity that we are studying has a very special structure which is made up of four types of data.

- **Type I.** (for the free evolution) We denote by $UC(\mathbb{R}^3, \mathbb{R}^2)$ the space of all bounded, uniformly continuous functions from $\mathbb{R}^3$ to $\mathbb{R}^2$. Set

$$X = \{(v,w) \in UC(\mathbb{R}^3,\mathbb{R}^2),\ (1+r)(v,w) \in UC(\mathbb{R}^3,\mathbb{R}^2)\} \tag{5.2}$$

We endow it with the norm:

$$\|(v,w)\|_{X,R_0,\varepsilon} = N_{\omega,R_0,\varepsilon}(v) + \|(1+r)(v,w)\|_\infty. \tag{5.3}$$

As usual, for all $\alpha \in]0,1[$ we define the set X^α as the domain of $D((I_X-\Delta)^\alpha$; see Henry [16]; Chap. 1.Its norm is the ntural norm

$$\|(v,w)\|_{X^\alpha,R_0,\varepsilon} = \|(I-\Delta)^\alpha(v,w)\|_{X,R_0,\varepsilon}. \tag{5.4}$$

- **Type II.** Functions $(v(r), w(r))$ of the form $e^{(T_J-1/Le)/\varepsilon}(v_0,w_0)$ with (v_0,w_0) of type **I**, supported in $B_{R_0}(0)$, satisfying moreover the following algebraic relation:

$$w + (1 - \frac{1}{Le}) = 0. \tag{5.5}$$

For such functions, the X norm is basically the $N_{\omega,R_0,\varepsilon}$ norm. However, we will need to define a special norm for them, that will take into account the fact that they may have a linear growth in the vicinity of $\{r = R_0\}$, and hence may be of order Logε in the vicinity of $\{T_\varepsilon = 1-\varepsilon\}$. For every $q > 0$ such that $2w_0 \geq -q$, we let $R_{0,q}$ be the unique radius such that

$$T^\varepsilon_{w_0}(R_{0,q}) = \frac{1}{Le} - q. \tag{5.6}$$

Then

$$\tilde N_{\omega,R_0,\varepsilon}(v,w) = N_{\omega,R_0,\varepsilon}\Big((1+\frac{R_0-r}{\varepsilon}\mathbf{1}_{\{R_0\varepsilon^{-1}\geq r\geq R_q\varepsilon^{-1}\}})(v,w)\Big). \tag{5.7}$$

We may then define The X^α norm of such an initial datum is then defined with the aid of $\tilde N_{\omega,R_0,\varepsilon}$ and is denoted by

$$\|(v,w)\|_{\tilde X^\alpha,R_0,\varepsilon} = \tilde N_{\omega,R_0,\varepsilon}(I-\Delta)^\alpha(v,w). \tag{5.8}$$

- **Type III.** Functions that are bounded and supported in $\{R_0 \le r \le \varepsilon^{-\nu}\}$.
- **Type IV.** Surface measures on $\{r = \varepsilon^{-\nu}\}$; $\nu > 0$ small.

Remark 5.1 *One word is in order about the algebraic relation (5.5). A major term that we shall have to compute is the effect of the right handside of System (1.1), once linearized around a steady solution - or, similarly, a quasi-steady solution. Namely, we wish to write down estimates for a problem of the following form:*

$$\text{(5.9)} \qquad \begin{cases} u_t - \Delta u - \dfrac{1}{\varepsilon^2}(e^{p_0}u + q_0 e^{p_0}v) = \dfrac{1}{\varepsilon^3}F(u,v) \\ v_t - \dfrac{\Delta v}{Le} + \dfrac{1}{\varepsilon^2}(e^{p_0}u + q_0 e^{p_0}v) = -\dfrac{1}{\varepsilon^3}F(u,v) \end{cases} \qquad (I\!R^3)$$
$$u(0,r) = v(0,r) = 0$$

where F is quadratic with respect to (u,v) - and has $O(1)$ estimates wrt to ε. Hence we have:

$$\text{(5.10)} \qquad (u(t), v(t)) = \frac{1}{\varepsilon^3}\int_0^t e^{-(t-s)\mathcal{L}^\varepsilon}(F(u,v), -F(u,v))\, ds;$$

the integral being evaluated through the estimate of $e^{-t\mathcal{L}^\varepsilon}$ applied to an initial datum whose components have a zero sum: this is exactly (5.5).

Remark 5.2 *Type* **II** *initial data may be supported, up to exponentially small terms, be assumed to supported in $B_{R_0}(0)$. This will be done when it is convenient to us.*

5.1.2. Statement of the results

The cornerstone of our study is the estimate for $e^{-t\mathcal{L}^\varepsilon}$ in the norm $L^\infty(B_{R_0})$, according to the type of initial datum. Estimates everywhere will follow as an easy corollary.

Theorem 5.1 *[i]. (L^∞ estimates) For all $t \le \dfrac{\tau_0}{\varepsilon}^{-2}$, the following table is valid.*

(5.11)

Initial datum	*Estimate for* $\Vert e^{-t\mathcal{L}^\varepsilon}(v_0, w_0)\Vert_{L^\infty(B_{R_0})}$
I	$\dfrac{1}{\sqrt{t}}\Vert(v_0,w_0)\Vert_{X,R_0,\varepsilon}$
II	$\min(1, \dfrac{\varepsilon^2}{\sqrt{t}})\tilde{N}_{\omega,R_0,\varepsilon}(v_0,w_0)$
III	$\min(1, \dfrac{\varepsilon^{1-4\nu}}{\sqrt{t}}) + \dfrac{\varepsilon^{-4\nu}}{1+t}\Vert(v_0,w_0)\Vert_{X,R_0,\varepsilon}$
IV	*Identical to Type* **III**

where $< \mu >$ denotes the mass of the measure μ.

[ii]. (Gradient estimates) Let $\|(v_0, w_0)\|$ denote the norm controlling the estimate of $\|e^{-t\mathcal{L}^\varepsilon}(v_0, w_0)\|_{L^\infty(B_{R_0})}$ in the table (5.11). Then set

$$N_\varepsilon(t) := \frac{\|e^{-t\mathcal{L}^\varepsilon}(v_0, w_0)\|_{L^\infty(B_{R_0})}}{\|(v_0, w_0)\|}, \tag{5.12}$$

$\|.\|$ corresponding, as said above, to any of the above four types. Then

$$\|\nabla(e^{-t\mathcal{L}^\varepsilon}(v_0, w_0))\|_{L^\infty(B_{R_0})} = \frac{1}{\varepsilon}O(1 + \frac{1}{\sqrt{t}})N_\varepsilon(t)\|(v_0, w_0)\|, \tag{5.13}$$

For the proof of these results, we need to know the spectrum of $\mathcal{L}^\varepsilon$ - this was the goal of Section 3 -, but also resolvent estimates. Then we conclude with a Laplace transform.

Theorem 5.2 *(Estimates in the whole space) For (v_0, w_0) of one of the types* **I** *to* **IV**, *let $N_\varepsilon(t)$ be defined as in Theorem 5.1, [ii]. Then the following pointwise estimate holds:*

$$|e^{-t\mathcal{L}^\varepsilon}(v_0, w_0)| \leq C\Big(|e^{t\Delta_{R_0}}(v_0, w_0 - \frac{v_0}{Le})| + \frac{N_\varepsilon(t)}{1+r}\|(v_0, w_0)\|\Big). \tag{5.14}$$

In this theorem we do not need gradient estimates.

Theorem 5.3 *(Time derivatives) There is a constant $C > 0$ such that, for all $t \leq \tau_0\varepsilon^{-2}$ we have*

$$\begin{aligned} \|\partial_t(e^{-t\mathcal{L}^\varepsilon}(v_0, w_0))\|_{X,R_0\varepsilon} \leq \quad & C(\varepsilon^2 + \frac{1}{t})N_\varepsilon(t)\|(v_0, w_0)\| \\ \|\partial_t(e^{-t\mathcal{L}^\varepsilon}(v_0, w_0))\|_{X,R_0\varepsilon} \leq \quad & C(\varepsilon^{2-\alpha} + \frac{1}{\varepsilon^\alpha t^{1-\alpha}})N_\varepsilon(t)\|(v_0, w_0)\| \end{aligned} \tag{5.15}$$

5.1.3. Notations, presentation of the computations

Estimating $e^{-t\mathcal{L}^\varepsilon}$ for various kinds of initial data will require the computation, then the estimation, of a very large number of Laplace transforms and integrals of various kinds. Proving them all in detail would make this - already large - paper swell beyond any reasonable limit.

We have therefore chosen, in most cases, to only sketch most of the proofs of the integral estimates - given the fact that they all reduce to standard arguments. Only a few of them will be displayed, either because they are more delicate than others, or because they are typical of the results to expect.

Notation

The Laplace transform will be denoted by $I\!L$. If t is the time variable, its dual variable is denoted by λ. For a function f bounded from $I\!R_+$ to $I\!R$, we define $I\!Lf$ in the left half complex plane by

$$I\!Lf(\lambda) = \int_0^{+\infty} e^{\lambda t} f(t)\, dt. \tag{5.16}$$

5.2. Evaluation of $e^{-t\mathcal{L}^\varepsilon}(v_0, w_0)$ for type II initial data

This is the crucial part of the program, the estimates derived here will be reused in the next subsections. Set

$$(v, w) = e^{-t\mathcal{L}^\varepsilon}(v_0, w_0), \quad l(t) = v(t, R_0).$$

If λ is the dual variable of t we set

$$(\hat{v}(\lambda), \hat{w}(\lambda)) = I\!L(v, w)(\lambda); \quad \hat{l}(\lambda) = (I\!Ll)(\lambda). \tag{5.17}$$

We use the scaled variables

$$\xi = \frac{r}{\varepsilon} \quad \mu = \varepsilon^2\lambda. \tag{5.18}$$

We have

$$\left\{ \begin{array}{lll} -\dfrac{\Delta\hat{v}}{Le} = & h_w(q_0, w_0)\hat{v} - h_w(q_0.w_0)\hat{w} + \mu\hat{v} + \varepsilon^2\hat{f} \\ -\Delta\hat{w} & = \mu\hat{w} + (1 - \dfrac{1}{Le})\mu\hat{v} \end{array} \right. \tag{5.19}$$

As soon as (5.19) has a unique solution $(\hat{v}, \hat{w})$, we set

$$\hat{l}(\lambda) = \mathcal{A}^\varepsilon(\lambda) v_0|_{r=R_0}; \tag{5.20}$$

For all $\lambda \in \rho(\mathcal{L}^\varepsilon)$, the expression $\mathcal{A}^\varepsilon(\lambda)$ is therefore a linear functional from, say, the space of all continuous functions from $B_{R_0/\varepsilon}$ to $\mathbb{C}$. The function $\lambda \mapsto \mathcal{A}^\varepsilon(\lambda)$ is holomorphic at least from $\rho(\mathcal{L}^\varepsilon)$ to the set of the linear functionals of $C(B_{R_0/\varepsilon}, \mathbb{C})$. We have

$$I\!L^{-1}\mathcal{A}^\varepsilon(t) v_0 = \frac{1}{2i\pi}\int_\gamma e^{-\lambda t}\mathcal{A}^\varepsilon(\lambda) v_0\, d\lambda, \tag{5.21}$$

where γ is the usual integration contour

$$\gamma = \{\lambda : \quad |\arg(\lambda - z_0)| = \theta\},$$

where $z_0 \in I\!R$ is suitably negative, and θ close enough to $\dfrac{\pi}{2}$. Let us deform γ: From Theorems 3.1 and 3.2, Problem (5.19), with $f = 0$, has only the trivial solution if

- $\mu \neq \varepsilon^4 \lambda_\varepsilon$ - see the definition of λ_ε in Theorem 3.1 -;
- and $\mu \notin [0, \frac{b}{|2\text{Log}\varepsilon|}[\cup \Gamma(\frac{b}{|2\text{Log}\varepsilon|}, \theta)$.

Hence, to assess whereas $\lambda = \frac{\mu}{\varepsilon^2}$ belogs to $\rho(\mathcal{L}^\varepsilon)$, it suffices - [16] - to check the coercivity at $r = +\infty$, namely to check that the problem

$$(5.22) \qquad \begin{cases} -\frac{\Delta v}{Le} = & h_0'(0)v + \mu v + \varepsilon^2 v_0 \\ -\Delta w & = \mu w + (1 - \frac{1}{Le})\mu v + \varepsilon^2(u_0 + v_0) \end{cases}$$

is well-posed. The resolvent set of the Laplacian in $I\!R^3$ being $\mathbb{C} \backslash \mathrm{R}_+$, the result is true. Hence the contour γ that we may choose is made up of three parts:

- a circle centered around $\lambda_\varepsilon \varepsilon^2$, of very small radius - say, of order ε^{100}, and called γ_1,
- a part γ_2 very close to the segment $[0, \frac{b}{|2\varepsilon^2\text{Log}\varepsilon|}[$; we may choose for definiteness:

  $$(5.23)\ \gamma_2 = \{\lambda : \quad -\varepsilon^{100} \leq \text{Re}\lambda \leq \frac{b}{|2\varepsilon^2\text{Log}\varepsilon|};\ |\arg(\lambda + \varepsilon^{100})| = \varepsilon^{100}\},$$

- the remaining part γ_3 which is basically

  $$(5.24) \qquad \gamma = \{\lambda : \quad |\arg(\lambda - \frac{b}{|2\varepsilon^2\text{Log}\varepsilon|})| = \theta\},$$

  from which two small parts of size $O(\varepsilon^{100})$ have been removed.

We now break the integral in (5.21) in three parts: $\gamma = \bigcup_{i=1}^{3} \gamma_i$, and perform a different estimate on each γ_i. We set

$$(5.25) \qquad (v_i(t,r), w_i(t,r)) = \frac{1}{2i\pi} \int_{\gamma_i} e^{-\lambda t} \mathcal{A}^\varepsilon(\lambda) v_0 \, d\lambda.$$

Remark 5.3 *Due to the locations of the singularities of $\mathcal{A}^\varepsilon(\lambda)$ - i.e the real line - its inverse Laplace transform can be, as is classical, estimated through its absolute value. This fact will be used without further mention.*

5.2.1. Integral over γ_3.

This part will not matter much in the final computation. From Lemmas 3.2 and 3.4 we have

$$\forall \lambda \in \gamma_3, \quad \|(\hat{v}, \hat{w})\|_\infty \le \frac{C}{|\lambda|}\|v_0\|_\infty. \tag{5.26}$$

We must also take care of the $N_{\omega,R_0(\mu),\varepsilon}(\hat{v})$. For this we need a lemma.

Lemma 5.1 *There exists $\omega > 0$ small enough, independent of ε such that, if*

$$\mu \in \Gamma(-M_1, \theta) \cup \Lambda(\rho),$$

then $N_{\omega,R_0(\mu),\varepsilon}(\hat{v})$ has the same estimate as $\|\hat{v}\|_\infty$.

PROOF. If $\mu \notin \Lambda(\rho)$, we first write an ODE for $re^{\omega(R(\mu)/\varepsilon - r)}(\hat{v}, \hat{w})$, then apply Henry's estimate - [16], Chapter 1: it is still valid if ω is small enough. If $\mu \in \Lambda(\rho)$, one may use the same argument but this time refer to Lemma 3.2. □

This implies the

Proposition 5.1 *There exists $d > 0$, independent of ε, such that there holds*

$$\begin{aligned} \|v_3(t,.), w_3(t,.)\|_\infty + \quad & N_{\omega,R_0(\mu),\varepsilon}(v_3(t,.)) \\ & \le C \min\left(|Log\, t|, \frac{\varepsilon^2}{t}|Log\varepsilon| e^{-dt/\varepsilon^2|Log\varepsilon|}\right) N_{\omega,R_0(\mu),\varepsilon}(v_0) \end{aligned} \tag{5.27}$$

Once Lemma 5.1 is known, the proof is a standard computation and is omitted.

5.2.2. Integral over γ_2.

The algorithm is the following: we first estimate $\hat{l}(\lambda)$, then $\hat{v}(\lambda)$ inside B_{R_0}. This will yield $v_2(t)$ inside B_{R_0}, then v_2 everywhere thanks to the maximum principle. Finally we estimate w_2 through $\hat{w}$.

We break the line γ_2 into two subsets:

$$\gamma_{21} = \{\lambda \in \gamma_2 : \quad \mathrm{Re}\lambda \le M\}, \quad \gamma_{22} = \gamma_2 \backslash \gamma_{21},$$

where the constant M is defined in Theorem 3.2. And we set, for $i = 1, 2$:

$$(v_{2i}(t,r), w_{2i}(t,r)) = \frac{1}{2i\pi}\int_{\gamma_{2i}} e^{-\lambda t}\mathcal{A}^\varepsilon(\lambda)\, v_0\, d\lambda. \tag{5.28}$$

Integral over γ_{22}. First, we restate without proof - the argument is completely similar - the analogue of Proposition 3.1.

Proposition 5.2 *With the notations of Proposition 3.1 - in particular, $\tilde{w}$ is given by (3.28), and $e(\mu)$ given by (3.22) - we have*

$$\frac{1}{l(\lambda)}\frac{\partial \hat{v}}{\partial r}(\frac{R(\mu)}{\varepsilon}) - Le^3 e^{w_0}(1+O(e^{-\rho^{-\alpha}}))\tilde{w}(\mu,\frac{R(\mu)}{\varepsilon})$$

$$(5.29)\quad = \begin{cases} O(a(1+|\tilde{w}(\mu,\frac{R(\mu)}{\varepsilon})|) + \|\tilde{w}'\sqrt{q}_0 e^{-q_0/2}\|_\infty \\ +\varepsilon^2 N_{\omega,R_0(\mu),\varepsilon}(v_0) + \varepsilon|Log\varepsilon|) & \textit{if } |\mu| \le \frac{a}{|Log\varepsilon|} \\ \\ O(|\mu|^{1-\alpha}(1+|\tilde{w}(\mu,\frac{R(\mu)}{\varepsilon})|) + \|\tilde{w}'\sqrt{q}_0 e^{-q_0/2}\|_\infty \\ +\varepsilon^2 N_{\omega,R_0(\mu),\varepsilon}(v_0) + \varepsilon|Log\varepsilon|) & \textit{otherwise} \end{cases}$$

The computation of $\frac{\partial v}{\partial r}(\frac{R(\mu)}{\varepsilon}^-)$ and the notations are exactly as in the proof of Theorem 3.2; therefore we do not give the details. Setting

$$(5.30)\quad \begin{aligned} \mathcal{D}^\varepsilon(\lambda, Le) = \; & -(\sqrt{-\mu Le} + O(e^{-C|\mu|^{-\alpha}})) \\ & -\frac{1}{2}Le^3 e^{w_0}(1+O(e^{-\rho^{-\alpha}}))\frac{1-Le}{1+\sqrt{Le}} + \psi(\varepsilon, Le, a, M, \mu) \end{aligned}$$

we have

$$(5.31)\quad |l(\lambda)| \le \frac{C\varepsilon^2}{|\mathcal{D}^\varepsilon(\lambda)|}N_{\omega,R_0(\mu),\varepsilon}(v_0) \le C\varepsilon^2 N_{\omega,R_0(\mu),\varepsilon}(v_0)$$

the last inequality being valid by (3.34). Once we have the value of $l(\lambda)$, we wish to estimate v_{22} inside B_{R_0}. For that, we just have to write down an equation for $\hat{v}_1 = \hat{v} - l(\lambda)e(\mu)$; application of Proposition 2.3 and Lemma 3.4 yields:

$$\forall\lambda \in \gamma_{22}, \qquad N_{\omega,R_0(\mu),\varepsilon}(\hat{v}) \le C\varepsilon^2|\mathrm{Log}\varepsilon|N_{\omega,R_0(\mu),\varepsilon}(v_0).$$

For $r \in [\frac{R(\mu)}{\varepsilon}, \frac{R_0}{\varepsilon}]$ the function $\hat{v}$ has basically a zero Laplacian. The boundary condition for $\hat{v}$ at $\frac{R(\mu)}{\varepsilon}$, implies:

$$(5.32)\quad \forall\lambda \in \gamma_{22}, \qquad N_{\omega,R_0,\varepsilon}(\hat{v}) \le C\varepsilon^2|\mathrm{Log}\varepsilon|N_{\omega,R_0,\varepsilon}(v_0).$$

Hence we obtain

$$(5.33)\quad \begin{aligned} N_{\omega,R_0(\mu),\varepsilon}(v_{22}(t)) \le \; & C\varepsilon^2|\mathrm{Log}\varepsilon|\Big(\int_M^{b/\varepsilon^2|Log\varepsilon|} e^{-\lambda t}\, d\lambda\Big)N_{\omega,R_0(\mu),\varepsilon}(v_0) \\ \le \; & C\frac{\varepsilon^2}{t}|\mathrm{Log}\varepsilon|(e^{-Mt} - e^{-bt/\varepsilon^2|Log\varepsilon|})N_{\omega,R_0(\mu),\varepsilon}(v_0) \end{aligned}$$

Turning to w_{22} we recall that, for all $\lambda \in \gamma_1 \cup \gamma_2$, we have to apply estimates

$$(5.34)\quad \hat{w}(\lambda, r) = (1-\frac{1}{Le})\frac{\mu}{4\pi}\mathcal{R}(\mu)\hat{v}(\lambda) + l(\lambda)\tilde{w}.$$

From (3.28), there exists $C > 0$ such that

$$\forall r \in [0, R_0], \quad |\tilde{w}(\lambda, r)| \leq C.$$

Consequently, we have

$$\|w_{22}(t)\|_{L^\infty(B_{R_0})} \leq C\frac{\varepsilon^2}{t}|\mathrm{Log}\varepsilon|(e^{-Mt} - e^{-bt/\varepsilon^2|Log\varepsilon|})N_{\omega,R_0(\mu),\varepsilon}(v_0). \tag{5.35}$$

Integral over γ_{21}. Inequality (5.31) for $l(\lambda)$ is still valid; what changes is the value of $\mathcal{D}^\varepsilon(\lambda)$. Therefore the whole above proof works, *mutatis mutandis.* Here we have, from (3.37) and (3.39):

$$|\mathcal{D}^\varepsilon(\lambda)| \geq C\Big|\frac{1}{R_0}\frac{1-Le}{1+\sqrt{Le}}e^{w_0}Le^3\mathrm{sh}(\frac{\sqrt{-\mu}R_0}{\varepsilon}) - \frac{\varepsilon}{R_0}\Big|. \tag{5.36}$$

In the range of λ's that we are investigating, this translates into

$$\frac{1}{|\mathcal{D}^\varepsilon(\lambda)|} \leq \frac{C}{\sqrt{|\lambda|}}. \tag{5.37}$$

Therefore we have

$$|l(\lambda)| \leq \frac{C\varepsilon^2}{|\mathcal{D}^\varepsilon(\lambda)|}N_{\omega,R_0,\varepsilon}(v_0) \leq C\varepsilon^2 N_{\omega,R_0(\mu),\varepsilon}(v_0) \tag{5.38}$$

In order to estimate v_{21} inside B_{R_0} we could apply Proposition 2.3 and Lemma 3.4 once again; however it would yield:

$$\forall\lambda \in \gamma_{22}, \quad N_{\omega,R_0,\varepsilon}(\hat{v}) \leq C\varepsilon^2|\mathrm{Log}\varepsilon|N_{\omega,R_0,\varepsilon}(v_0).$$

There is a spurious $\mathrm{Log}\varepsilon$, and this is where the $\tilde{N}_{\omega,R_0,\varepsilon}$ norm comes into play. Let us use the following trick: consider the operator $\mathcal{T}_{\varepsilon,R_0}$ with Dirichlet conditions not on the sphere of radius $\dfrac{R_0}{\varepsilon}$, but of radius $\dfrac{R_{0,q}}{\varepsilon}$.

Proposition 5.3 *There is $C > 0$ such that, for all $f \in C(B_{R_{0,q}/\varepsilon})$ we have*

$$N_{\omega,R_{0,q},\varepsilon}(\mathcal{T}_{\varepsilon,R_0}^{-1}f) \leq C\tilde{N}_{\omega,R_{0,q},\varepsilon}(f). \tag{5.39}$$

PROOF. If $p(\eta)$ solves

$$-p''_{w_0} = g_{w_0}(p_{w_0}),\ p_{w_0}(0) = -q,\ p_{w_0}(+\infty) = 0,$$

then the following boundary value problem, posed on $\mathbb{R}_+$:

$$-u'' + g'_{w_0}(p_0)u = f, \quad u(0) = u(+\infty) = 0$$

has a unique solution $u^0(\eta)$ whose explicit expression is

$$u^0(\eta) = p'_{w_0}(\eta) \int_0^\eta \frac{1}{p'_{w_0}(\zeta)^2} \int_\zeta^{+\infty} p_{w_0}(\zeta') f(\zeta') \ d\zeta' \ d\zeta d\zeta'$$

and we have, for any $\omega \in]0, \frac{g'_{w_0}(w_0)}{2}]$:

$$\|e^{\omega\eta} u_0\|_\infty \leq C\|(1 + \eta \mathbf{1}_{\{ALog\varepsilon \leq p_{w_0} \leq -A|Log\varepsilon|^{-1}\}}) e^{\omega\eta} f\|_\infty. \tag{5.40}$$

If χ is a cut-off function in $\{-q \leq p_{w_0} \leq -A|Log\varepsilon|^{-1}\}$ - i.e. is nonnegative; equal to 1 except in a vicinity of size 1 of the boundaries of the interval, let us denote by $\xi = \frac{x}{\varepsilon}$ the 3D rescaled variable; and set

$$\tilde{u}^0(\xi) = \chi(\frac{R_{0,q}}{\varepsilon} + |\xi|) u^0(\frac{R_{0,q}}{\varepsilon} + |\xi|).$$

If $u = \mathcal{T}^{-1}_{\varepsilon,R_0} f$, we have

$$N_{\omega,R_0,\varepsilon}\Big(\mathcal{T}_{\varepsilon,R_0}(u - \tilde{u}^0)\Big) = O(\varepsilon \text{Log}\varepsilon);$$

hence the result by application of Proposition 2.3. □

Armed with this result we may repeat the procedure for the computation of $\hat{v}_{21}(\lambda, .)$ but this time we set

$$v_{21}(\lambda, \frac{R_{0,q}}{\varepsilon}) = l_q(\lambda).$$

The quantity $\partial_r v_{21}(\lambda, \frac{R_{0,q}}{\varepsilon})$ can be estimated as in the preceding paragraph. Then Gronwall's lemma yields an estimate for $\partial_r(v_{21}(\lambda, \frac{R_0}{\varepsilon}) - l_q(\mu)e(\mu))$, and we in all we get

$$\begin{aligned} N_{\omega,R_0,\varepsilon}(v_{21}(t)) \leq \ & C\varepsilon^2 \int_0^M \frac{e^{-\lambda t}}{\sqrt{\lambda}} \ d\lambda \ \tilde{N}_{\omega,R_0,\varepsilon}(v_0) \\ \leq \ & C\varepsilon^2 \min(1, \frac{1}{\sqrt{t}}) \tilde{N}_{\omega,R_0,\varepsilon}(v_0) \end{aligned} \tag{5.41}$$

and the function $\|w_{21}(t)\|_{L^\infty(B_{R_0})}$ has the same estimates.

5.2.3. Integral over γ_1.

From Theorem 3.1 and its proof, we have, as $\lambda \sim \varepsilon^2 \lambda_\varepsilon$:

$$\frac{1}{\mathcal{D}^\varepsilon(\lambda)} = \frac{d_\varepsilon(\lambda)}{\sqrt{-\lambda} - \sqrt{\lambda_\varepsilon}} \tag{5.42}$$

where d_ε is analytic in λ, with modulus controlled from above and below independently of ε. Therefore, Proposition 2.3 and the residue theorem implies

$$N_{\omega,R_0,\varepsilon}(v_1(t)) + \|w_1(t)\|_{L^\infty(B_{R_0})} \leq C\varepsilon^3 e^{\varepsilon^2 \text{Re}\lambda_\varepsilon t} \tilde{N}_{\omega,R_0,\varepsilon}(v_0). \tag{5.43}$$

5.3. Estimates for Types III and IV initial data

Recall that, for $R > 0$, we denote by Δ_R the Dirichlet Laplacian outside B_R. For $r \geq R_0$ we have

$$\hat{v}(\lambda) = (-\Delta_{R_0} - \lambda I)^{-1} v_0 + l(\lambda) \frac{e^{-\sqrt{-\lambda}(r-R_0)}}{r} R_0.$$

Hence, with the notation of (3.28) we have - we mix the notations μ and λ:

$$(5.44) \quad -\frac{Le}{\delta}\hat{w} = \frac{\mu}{4\pi}(\tilde{w} + \mu\mathcal{R}(\mu)v) + \lambda(-\Delta - \lambda I)^{-1}(-\Delta_{R_0} - \lambda I)^{-1} v_0 + \hat{w}_0$$

with

$$(5.45) \qquad \hat{w}_0 = (-\Delta - \lambda I)^{-1}(w_0 + (1 - \frac{1}{Le})v_0).$$

We have

$$(5.46) \quad |l(\lambda)| \leq \frac{C}{|\mathcal{D}^\varepsilon(\lambda)|}(\|\hat{w}\|_\infty + \|\hat{w}_0\|_\infty + \varepsilon|(\partial_r(-\Delta_{R_0} - \lambda I)^{-1} v_0|_{r=R_0}|).$$

As said in the introduction of Section 5, we will only give the estimates for the most significant integrals; the other ones being controlled as explained above.

In particular we have, for $\lambda \in \gamma_1 \cup \gamma_{21}$, and because v_0 is supported in $B_{\varepsilon^{-\nu}}$ or its boundary:

$$(5.47) \qquad \begin{aligned} \|\lambda(-\Delta - \lambda I)^{-1}(-\Delta_{R_0} - \lambda I)^{-1} v_0\|_{L^\infty(B_{R_0})} &\leq C\sqrt{|\lambda|}\, \|v_0\| \\ \varepsilon|(\partial_r(-\Delta_{R_0} - \lambda I)^{-1} v_0|_{r=R_0}| \leq C\varepsilon N_\varepsilon(v_0) & \end{aligned}$$

where $\|v_0\|$ is

- $\varepsilon^{-3\nu}\|v_0\|_\infty$ if v_0 is of type **III**,
- $| < v_0 > |$ if v_0 is of type **IV**.

Also we have, more generally, for all u_0 supported in $B(0, \varepsilon^{-\nu})$:

$$(5.48) \qquad \|(-\Delta - \lambda I)^{-1} u_0\|_\infty \leq C\|u_0\|$$

with the same notation as above.

5.3.1. $\lambda \in \gamma_1$

Then $\lambda \sim \varepsilon^2$, and we have, because of (5.48) which is the dominant estimate:

$$(5.49) N_{\omega,R_0,\varepsilon}(v_1(t)) + \|w_1(t)\|_{L^\infty(B_{R_0})} \leq C\varepsilon e^{\varepsilon^2 \mathrm{Re}\lambda_\varepsilon t}\|(v_0, w_0)\| \leq C\varepsilon\|(v_0, w_0)\|$$

the estimate being valid for $t \in [0, \frac{\tau_0}{\varepsilon^2}]$. Here the Log$\varepsilon$ could be removed because v_0 and w_0 come in only through w, which is itself multiplied by $e^{p w_0}$, whose $\tilde{N}_{\omega,R_0,\varepsilon}$ norm is controlled by a constant.

5.3.2. $\lambda \in \gamma_{21}$

Reproducing the analysis of Paragraph 5.1.2 and keeping in mind that the dominant term is - as we saw previously - the term in $\hat{w}_0$ we obtain:

$$N_{\omega,R_0,\varepsilon}(v_{21}(t)) + \|w_{21}(t)\|_{L^\infty(B_{R_0})} = O(\varepsilon^2 + \frac{\varepsilon}{\sqrt{t}} + \frac{1}{1+t})\|(v_0, w_0)\|. \tag{5.50}$$

5.4. Proofs of the theorems

To finish the proof of Theorem 5.1, part [i] we simply decompose v_0 into

$$v_0 = (\mathbf{1}_{r\leq R_0} + \mathbf{1}_{r\geq R_0})v_0;$$

the part $\mathbf{1}_{r\leq R_0}$ is treated as a type **II** initial data. As for the other part, we only need to notice that

$$\begin{aligned} \|\lambda(-\Delta - \lambda I)^{-1}(-\Delta_{R_0} - \lambda I)^{-1} v_0\|_{L^\infty(B_{R_0})} &\leq \frac{C}{\sqrt{|\lambda|}}\|v_0\| \\ \varepsilon|(\partial_r(-\Delta_{R_0} - \lambda I)^{-1}v_0|_{r=R_0}| \leq C\|v_0\| \end{aligned} \tag{5.51}$$

whose inverse Laplace transforms can be computed explicitely. The estimate follows.

Proof of Theorem 5.1, [ii]. For that we only need to come back to system (5.19). If ρ is the stretched coordinate, the estimate for (v, w), the estimate for the gradient is

$$\|\nabla(v, w)\|_{L^\infty(B_{R_0})} \leq C(1 + \sqrt{|\mu|})\|(v, w)\|_{L^\infty(B_{R_0/\varepsilon+1})}.$$

The $\|(v, w)\|_{L^\infty(B_{R_0/\varepsilon+1})}$ is ε-close to $\|(v, w)\|_{L^\infty(B_{R_0/\varepsilon})}$. □

Proof of Theorem 5.2. We have

$$v(t) = e^{t\Delta_{R_0}}v_0 + v_1(t),$$

where $v_1(t)$ satisfies the heat equation outside B_{R_0} and the Dirichlet condition

$$v_1(t, R_0) = l(t).$$

The quantity v_1 is computed just as in Section **4.**

Finally, we have to deal with w; however we know that

$$u(t) = w(t) - \frac{v(t)}{Le}$$

and we already have estimates of $w(t, R_0)$, hence of $u(t, R_0)$. Now, we just use the fact that u is caloric outside B_{R_0}.

Proof of Theorem 5.3. The main estimate is the $N_{\omega,R_0,\varepsilon}$ one, which can be inferred by estimating $(\lambda^\alpha v, \lambda^\alpha w)$ from the Laplace integrals in Section 5.1. The details of the computation are classical and left to the reader. The L^∞ estimate comes from the maximum principle. □

6. Convergence to the Joulin solution

Everything is now ready for the proof of Theorem 1.1. Recall that Theorem 4.1 in Section 4 yields - with notations different to those of Section 5 - a solution u_J^ε of

$$\begin{aligned}\dot{u} + Au = \ & F_\varepsilon(u) \\ & + \varepsilon^{3-\nu}(f_\varepsilon(t)\mathbf{1}_{r\leq R_J^\varepsilon(t)} + \varepsilon^{1-2\nu}(\mathbf{1}_{R_J^\varepsilon(t)\leq r\leq \varepsilon^{-\nu}} h_\varepsilon(t) + k_\varepsilon(t)\delta_{r=\varepsilon^{-\nu}}))\end{aligned} \tag{6.1}$$

Let $u = (T, Y)$ be a solution of (1.1). Equation (4.1), together with (4.7), therefore yield for $v := u - u_J^\varepsilon$:

$$\begin{aligned}\dot{v} + \mathcal{L}_J^\varepsilon(t)v = \ & Q_\varepsilon(t,v).v^{(2)} \\ & + \varepsilon^{3-\nu}\Big(f_\varepsilon(t)\mathbf{1}_{r\leq R_J^\varepsilon(t)} + \varepsilon^{1-2\nu}(h_\varepsilon(t)\mathbf{1}_{r\leq\varepsilon^{-\nu}} + k_\varepsilon(t)\delta_{r=\varepsilon^{-\nu}})\Big) \\ := \ & G(\varepsilon, t, r, v, v_r)\end{aligned} \tag{6.2}$$

where $Q_\varepsilon(t,v)$ is basically $F''(u_J)$ and where we have set

$$\mathcal{L}_J^\varepsilon(t) = A - DF_\varepsilon(T_J^\varepsilon(t), Y_J^\varepsilon(t)). \tag{6.3}$$

We have

$$v(t) = \mathcal{R}^\varepsilon(0,t)v_0 + \int_0^t \mathcal{R}^\varepsilon(s,t)G(\varepsilon, s, r, v, v_r)\, ds, \tag{6.4}$$

and the proof will be concluded by a stability argument.

We have already gathered some information about the linear operator around a steady solution; however what we really need to compute is not only $e^{-t\mathcal{L}^\varepsilon}$, but the quantity $\mathcal{R}(s,t)$, such that $\mathcal{R}(s,t)v_0$ is the unique solution of

$$v_t + \mathcal{L}_J^\varepsilon(t)v = 0, \qquad v(0) = v_0 \tag{6.5}$$

Computing $\mathcal{R}(s,t)$ requires the knowledge of $e^{-t\mathcal{L}_J^\varepsilon(s_0)}$ for all s_0, but also differences of the form $e^{-t\mathcal{L}_J^\varepsilon(t_0)} - e^{-t\mathcal{L}_J^\varepsilon(s_0)}$ for all pairs (t_0, s_0). Before we actually go into it, we need a change of variables: indeed, an $O(\varepsilon^{-1})$ error appears, in the estimate of the above quantities, due to the fact that the front moves in an area of $O(1)$ gradients.

6.1. Preliminaries

6.1.1. Change of variable

Let $\Gamma_1(r)$ and $\Gamma_2(r)$ two C^∞ nonnegative functions such that

- Γ_1 is supported in $B_{2\varepsilon^{-\nu}+1}$ and equal to 1 in $B_{2\varepsilon^{-\nu}}$;
- Γ_2 is supported outside $B_{2\varepsilon^{-\nu}-1}$ and constant outside $B_{2\varepsilon^{-\nu}+2}$;

- we have for all $t \le \tau_0 \varepsilon^{-2}$: $R^\varepsilon_J(t)\Gamma'_1(r) + \Gamma'_2(r) \ge 0$; $R_J(t)^\varepsilon \Gamma_1(r) + \Gamma_2(r) > 0$.

The construction of Γ_1 and Γ_2 is an easy exercise.

For every $q > 0$ such that $2w^\varepsilon_J(t) \ge -q$, we let $R^\varepsilon_{J,q}(t)$ be the unique radius such that

$$T^\varepsilon_J(t, R^\varepsilon_{J,q}(t)) = \frac{1}{Le} - q. \tag{6.6}$$

Its existence, uniqueness and smoothness is guaranteed by the fact that T^ε_J is nondegenerate about its free boundary.

Let us define implicitely the new coordinates (t', r') as follows:

$$\begin{cases} r = & \Gamma_1(r')R^\varepsilon_{J,q}(t)r' + \Gamma_2(r')r' \\ t' = & t \end{cases} \tag{6.7}$$

This definition makes sense as long as $R^\varepsilon_{J,q}$ does not reach 0; which is rather logical. The condition $R^\varepsilon_{J,q}\Gamma_1(r) + \Gamma_2(r) > 0$ implies that, for every t:

$$\frac{\partial r}{\partial r'} > 0.$$

This implies the existence of τ'_0, bounded and bounded away from below independently of ε, such that the mappping $(t, x) \mapsto (t', x')$ is a C^1 diffeomorphism from $[0, \tau'_0 \varepsilon^{-2}] \times I\!R$ to itself.

6.1.2. Problem under study and strategy of the proof

We set, for commodity, $t := t'$ and $x := x'$. Let us set

$$\gamma_1(t, r) = (\Gamma_1(r)R^\varepsilon_{J,q}(t) + \Gamma_2(r))^2 \tag{6.8}$$

In the new coordinates, the solution v of (6.2) satisfies the equation

$$\gamma_1(t, x)\dot{v} + \mathcal{L}^\varepsilon_J(t)v = \gamma_1(t, r)G(\varepsilon, t, r, v, v_r) \tag{6.9}$$

where G has the same expression as in (6.2), but where the expression of $\mathcal{L}^\varepsilon_J(t)$ has been modified. Let us list the features of the modified operator $\mathcal{L}^\varepsilon_J$.

- the function $\gamma_1(t, .)$ is C^∞, and constant in r outside the annulus

  $$S_{\nu,\varepsilon} = \{2\varepsilon^{-\nu} - 1 \le r \le 2\varepsilon^{-\nu} + 2\}; \tag{6.10}$$

  moreover it is uniformly controlled from above and below.

- The expression of $\mathcal{L}^\varepsilon_J$ is

  $$\mathcal{L}^\varepsilon_J(t) = A + (\varepsilon^2\gamma_2(t, x)x + \gamma_3(t, x)).\nabla - \gamma_1 DF_\varepsilon(T_J(t), Y_J(t)) \tag{6.11}$$

  where the function γ_2 is C^∞, and has $\partial_t\gamma_i = O(\varepsilon^2)$. Moreover $\gamma_2(t, .)$ is supported in $B_{2\varepsilon^{-\nu}+2}$;

- the function $\gamma_3(t,.)$ has its values in $I\!R^3$ and is supported in $S_{\nu,\varepsilon}$.

Note the following simple facts.

- Let $t_0 > 0$ be given. Solving the equation

$$\gamma_1(t_0, x)\dot{v} + \mathcal{L}_J^\varepsilon(t_0)v = 0, \qquad v(0) = v_0$$

exactly amounts to computing $e^{-t\mathcal{L}_J^\varepsilon(t_0)}v_0$ in the old space-time coordinates, then going to the new ones.

- For $|x| \leq \dfrac{\varepsilon^{-\nu}}{2}$ we have, at that point:

$$\Big(\Gamma_1(r)R_{J,q}^\varepsilon(t) + \Gamma_2(r)\Big)^2 DF_\varepsilon(T_J^\varepsilon(t), Y_J^\varepsilon(t)) = {R_{J,q}^\varepsilon}^2 DF_\varepsilon(T_J^\varepsilon(t), Y_J^\varepsilon(t))$$

Our goal is to conclude that, if $v_0 = O(\varepsilon^4)$ in a big ball and 0 outside, $v(t)$ keeps the same estimates on the ε^{-2} time scale. What will be playing for us is the fact that $\mathcal{L}_J^\varepsilon(t)$ varies slowly, i.e. like ε^2. Therefore averaging arguments will apply, and we will basically have $\mathcal{R}^\varepsilon(s,t) \sim e^{\int_s^t \mathcal{L}_J^\varepsilon(\sigma)\, d\sigma}$. This is precisely the quantity that is estimated in Section 5; and the conclusion is reached basically because u_J is quasisteady, hence very close to a Zeldovich-like solution.

Final notational conventions

- The superscript $^\varepsilon$ will be removed from all quantities: T_J, Y_J, R_J, $R_{J,q}$...
- The norm $\|.\|_{X,1,\varepsilon}$, defined by (5.3), will simply be denoted by $\|.\|_X$.
- Similarly, if $\|.\|$ is any of the norms of Theorem 5.1 - basically, Types **I** and **II** - then $\|v_0\|_\alpha$ will denote $\|(I-\Delta)^\alpha v_0\|$.

6.2. Linear evolution around a frozen Joulin flame ball

The operator $\mathcal{L}_J(t_0)$ very much looks like the linearized operator around the Zeldovich flame ball $\mathcal{L}^\varepsilon$, studied in Section 5 - up to the $\varepsilon^2 x.\nabla$ modification. However we will see that it will not affect the dispersion relation - and hence the estimates of Section 5 - too much.

Theorem 6.1 *Consider $t_0 \in]0, \tau_0\varepsilon^{-2}]$. Then $e^{-t\mathcal{L}_J(t_0)}$ satisfies the estimates of Theorems 5.1 to 5.3, with a constant uniform in every compact in t_0.*

PROOF. Our task is to check that the dispersion relation for the linear problem

$$\mathcal{L}_J(t_0)v = \gamma_1(t_0, x)\lambda v \tag{6.12}$$

is not drastically modified by the change of variables (6.7) and has the same characteristics as the original linear operator around the Zeldovich flame ball. The modifications that are to be made are to change the functions (T_0, Y_0, W_0) into (T_J, Y_J, W_J), and the corresponding rescaled quantities (p_0, q_0) into

$$(p_J, q_J)(t, \xi) = \Big(\frac{T_J(t, \varepsilon\xi) - 1}{\varepsilon}, \frac{Y_J(t, \varepsilon\xi)}{\varepsilon}\Big), \quad w_J = p_J + q_J \tag{6.13}$$

and the exponential e^{w_0} into $(R_J)^2 e^{w_J}$. Then we have to investigate the derivatives $\partial_r v_1$, where v_1 is this time the first component of v - v is the solution of (6.12). Set

$$\tilde{\mathcal{L}}_J(t) = \mathcal{L}_J(t) - \varepsilon^2\gamma_2(t, x)x.\nabla \tag{6.14}$$

and let $\tilde{v}$ solve

$$\tilde{\mathcal{L}}_J(t_0)\tilde{v} = \lambda\gamma_1(t_0, x)\tilde{v}. \tag{6.15}$$

1. Inside $B_{R_J}(t_0)$. If $\mathcal{L}_J(t_0)$ was replaced by $\tilde{\mathcal{L}}_J(t_0)$, then, because $\gamma_1 \equiv {R_{J,q}}^2$ in a vicinity of B_{R_J}, $\partial_r v_1(R_J^-)$ has the same expression as in Section **3** but for a factor ${R_J}^2$. If we put the perturbation $\varepsilon^2\gamma_2(t_0, x)x.\nabla$ back in, then it is an $O(\varepsilon^3)$ perturbation of $\tilde{\mathcal{L}}_J(t_0)$ once rescaling is done. Application of Lemma 3.8 yields, together with a standard perturbation argument:

$$\frac{\partial v_1}{\partial r}(R_J^-) = \frac{\partial \tilde{v}_1}{\partial r}(R_J^-) + O(\varepsilon^2|\text{Log}\varepsilon|). \tag{6.16}$$

2. Outside $B_{R_J}(t_0)$. The same as above works for $\tilde{\mathcal{L}}_J(t_0)$. Moreover, what is playing for us is that $\mathcal{L}_J(t_0)$ is an ε^2 perturbation of $\tilde{\mathcal{L}}_J(t_0)$, which - this is important - has compact support due to the coefficients. This makes a usual fixed point work, in order to give

$$\frac{\partial v_1}{\partial r}(R_J^+) = \frac{\partial \tilde{v}_1}{\partial r}(R_J^+) + O(\varepsilon^2). \tag{6.17}$$

Hence the new eigenvalues differ from the ones of $\mathcal{L}_J(t_0)$ by an ε^4 error.
3. The outer and inner derivative of $\tilde{v}$ in (6.15) only differ from a factor $R_{J,q}$ - one only has to take into account the change of variables, which in this case reduces to a dilation. Hence the dispersion relation for(6.15) is the same as (3.37); as a consequence the dispersion relation for (6.12) differs from (3.37) by an $O(\varepsilon^2)$. The proof of Theorem 5.1 subsequently applies. $\square$

The next result is concerned about how much $e^{-t\mathcal{L}_J(t_0)}$ deviates from $e^{-t\mathcal{L}_J(s_0)}$, for different time parameters t_0 and s_0.

Theorem 6.2 *Pick* $\alpha \in]0,1[$ *- its vocation will be to be small. For* $(s_0, t_0, t) \in [0, \frac{\tau_0}{\varepsilon^2}]^2 \times [1, \frac{\tau_0}{\varepsilon^2}]$ *with* $|t_0 - s_0| = 1$*, there is a constant* $C_\alpha > 0$ *independent of* (s_0, t_0, t) *but depending on* τ_0 *such that, if* v_0 *has any of the types* **I** *to* **IV***, we have*

$$\|e^{-t\mathcal{L}_J(t_0)}v_0 - e^{-t\mathcal{L}_J(s_0)}v_0\|_X \leq C\varepsilon^2 \int_0^t N_\varepsilon(s) \, \frac{ds}{\sqrt{t-s}}\|v_0\|_\alpha. \tag{6.18}$$

The notations are those of Theorems 5.1 and 5.2.

Proof. The quantity to be estimated is

$$\tilde{v}(t) = \frac{\partial}{\partial t_0}(e^{-t\mathcal{L}_J(t_0)}v_0).$$

The function $\tilde{v}$ satisfies the evolution equation

$$\gamma_1(t_0, x)\dot{\tilde{v}} + \mathcal{L}(t_0)\tilde{v} = \sum_{i=1}^{3} f_i(t_0, t); \tag{6.19}$$

with

$$\begin{aligned} f_1(t_0,t) = \;& \gamma_1(t_0,x)\Big(\dot{R_{J,q}}(t_0)(\frac{d}{dR_{J,q}}(DF_\varepsilon)(t_0)) \\ & +\dot{w}_J(t_0)(\frac{d}{dw_J}(DF_\varepsilon)(t_0))\Big).(e^{-t\mathcal{L}_J(t_0)}v_0) \\ f_2(t_0,t) = \;& \partial_{t_0}\gamma_1(t_0,x)\partial_t(e^{-t\mathcal{L}_J(t_0)}v_0) \\ f_3(t_0,t) = \;& \varepsilon^2\partial_{t_0}\gamma_2(t_0,x)x.\nabla(e^{-t\mathcal{L}_J(t_0)}v_0) + \partial_{t_0}\gamma_3(t_0,x).\nabla(e^{-t\mathcal{L}_J(t_0)}v_0) \\ & -\partial_{t_0}\gamma_1(t_0,x)(DF_\varepsilon(t_0))(e^{-t\mathcal{L}_J(t_0)}v_0) \end{aligned} \tag{6.20}$$

Let us study what happens when each of the f_i is at the right handside of (6.19), the unknown being called v_i.

1. Estimate for v_1. The following is true:

$$\dot{R_{J,q}}(t_0) = O(\varepsilon^2) \quad , \quad \dot{w_J}(t_0) = O(\varepsilon^2). \tag{6.21}$$

Indeed, R_J and T_J satisfy (6.21); the conclusion follows then from the nondegeneracy of T_J. On the other hand the quantities $\partial_{R_J}(DF_\varepsilon)(t_0)$ and $\partial_{w_j}(DF_\varepsilon)(t_0)$ fall into the type **II** of Theorem 5.1. Let us prove, for instance, that $\partial_{R_J}(DF_\varepsilon)(t_0)$ is $O(\varepsilon^{-2})$. For this we only need to prove that the quantities

$$(\partial_{R_J}T_J)f'_\varepsilon(T_J), \quad (\partial_{R_J}T_J)Y_J f''_\varepsilon(T_J)$$

are $O(\varepsilon^{-2})$. Examine, for instance, the first quantity; rescale the coordinate: $\xi = \dfrac{x}{\varepsilon R_{J,q}}$, and define p_J and q_J as in (6.13). Then we have, with the notations of Section 2:

$$\begin{aligned} \mathcal{U}_{\varepsilon,1}\partial_{R_{J,q}}p_J = \;& 2R_J g_{w_J}(p_J) \quad , \quad |\xi| < \varepsilon^{-1} \\ \partial_{R_J}p_J(\varepsilon^{-1}) = \;& 0 \end{aligned}$$

Hence we have, from Proposition 5.3 and Theorem 2.1:

$$N_{\omega,R_{J,q}(t_0),\varepsilon}((\partial_{R_J} p_J) \leq C\tilde{N}_{\omega,R_{J,q}(t_0),\varepsilon}(p_J e^{p_J});$$

Gronwall lemma inplies:

$$N_{\omega,R_J(t_0),\varepsilon}(\partial_{R_J} p_J) \leq C\tilde{N}_{\omega,R_J(t_0),\varepsilon}(p_J e^{p_J}).$$

and the right handside quantity is finite. Then we apply Theorem 5.1 to get

$$\begin{aligned} N_{\omega,R_J(t_0),\varepsilon}(v_1(t)) \leq & \; C\varepsilon^2 \int_0^t \frac{\tilde{N}_{\omega,R_J(t_0),\varepsilon}(e^{p_J} p_J(v(s), w(s)))}{\sqrt{t-s}} \, ds \\ \leq & \; C\varepsilon^2 \int_0^t \frac{N_\varepsilon(s)}{\sqrt{t-s}} \, ds N_{\omega,R_J(t_0),\varepsilon}(v_0, w_0) \end{aligned}$$

Theorem 5.2 applies subsequently.

2. Estimate for v_2. We infer from (6.19) and Theorem 5.3 that:

$$N_{\omega,R_J,\varepsilon}(\tilde{v}_2(t)) \leq C\varepsilon^3 \int_0^t \frac{N_\varepsilon(s)}{s^{1-\alpha}\sqrt{t-s}} \, ds \|v_0\|_\alpha,$$

which is negligible in comparison to v_1 - provided that it is estimated in terms of $\|v_0\|_\alpha$, which is of course always the case. As said in the introduction, this is one of the places where the smallness of the time derivatives are of great help to us.

3. Estimate for v_3 This is a consequence of Theorem 5.2 for Types **II** and **III** initial data, together with the fact that $\partial_{t_0}\gamma_i = O(\varepsilon^2)$.

□

6.3. Putting everything together

We are now ready for the convergence proof. The idea is now very simple: break the interval $[0, \frac{\tau_0}{\varepsilon^2}]$ into $N = \left[\frac{\tau_0}{\varepsilon^2}\right]$ sub-intervals of length 1. Two intermediate results are needed; they may be regarded as quantitative versions of *a priori* estimates for evolution equations with time dependent operator - see, for instance, Pazy [22].

Lemma 6.1 *Recall that the notation $N_\varepsilon(t)$ is defined by (5.12). For all $n \leq N$ there is a constant $C > 0$, depending only on τ_0, such that, for all v_0 of type* **II** *to* **IV** *we have:*

$$\|\prod_{i=0}^{n} e^{-\mathcal{L}_J(i)} v_0\|_{\mathcal{L}(X)} \leq C\, N_\varepsilon(n) \|v_0\|. \tag{6.22}$$

PROOF. Set

$$\pi_{k,j} = \prod_{i=k}^{j} e^{-\mathcal{L}_J(i)}; \qquad p_{k,j} = \| \prod_{i=k}^{j} e^{-\mathcal{L}_J(i)} \|_{\mathcal{L}(X)}$$

by induction and Theorems 5.1 [ii], 5.2 and 6.2 we obtain:

$$\begin{aligned} p_{0,n+1} \leq\ & \|e^{-(n+1)\mathcal{L}_J(n)}\|_{\mathcal{L}(X)} \\ & + \sum_{k=1}^{n} \|(e^{-(n-k)\mathcal{L}_J(n-k)} - e^{-k\mathcal{L}_J(n-k-1)})\pi_{0,k}\|_{\mathcal{L}(X)} \\ \leq\ & N_\varepsilon(n+1) + C\varepsilon^2 \sum_{k=1}^{n} p_{0,k} \int_0^k \frac{N_\varepsilon(s)}{\sqrt{k-s}}\ ds. \end{aligned} \tag{6.23}$$

Note that, in the case of $e^{-k\mathcal{L}_J(n-k)} - e^{-k\mathcal{L}_J(n-k-1)}$, a L^∞ estimate translates into an estimate in X, due to the fact that this quantity is zero at $t=0$ and is caloric outside a ball.

We are going to prove by induction that, for a constant $C > 0$ depending only on τ_0:

$$\forall k \leq n, \quad p_{0,k} \leq CN(k). \tag{6.24}$$

We are not going to carry it out for all types of initial data; let us see how it works for type **II** initial data: it works in the same fashion for the other two types. Assume that

$$p_{0,k} \leq \overline{C}\frac{(1+C\varepsilon^2)^k}{\sqrt{k}}\varepsilon^2; \tag{6.25}$$

the constnt $\overline{C}$ to be determined. From (6.23) and Theorem 5.1 - Type **I** initial data - we have

$$p_{0,n+1} \leq C\frac{\varepsilon^2}{\sqrt{n+1}} + C\overline{C}\varepsilon^4 \sum_{k=1}^{n} \frac{(1+C\varepsilon^2)^k}{\sqrt{k}},$$

An Abel discrete integration by parts yields

$$\begin{aligned} p_{0,n+1} \leq\ & C\frac{\varepsilon^2}{\sqrt{n+1}} + \overline{C}\varepsilon^2\frac{(1+C\varepsilon^2)^{n+1}-1}{\sqrt{n}} \\ & + \overline{C}\varepsilon^4 \sum_{k=0}^{n+1} \frac{\sum_{l=0}^{k}(1+C\varepsilon^2)^l}{\sqrt{k(k+1)}(\sqrt{k+1}+\sqrt{k})} \\ \leq\ & C\frac{\varepsilon^2}{\sqrt{n+1}} + \overline{C}\varepsilon^2\frac{(1+C\varepsilon^2)^{n+1}-1}{\sqrt{n}} + \frac{\overline{C}}{2}\varepsilon^3 \\ \leq\ & \overline{C}\frac{(1+C\varepsilon^2)^{n+1}}{\sqrt{n+1}}\varepsilon^2 \end{aligned}$$

as soon as we have $n = O(\varepsilon^2)$, $\overline{C} > 3C$ and $\varepsilon > 0$ small enough. □

Lemma 6.2 *Choose any small $\alpha > 0$. Let us set, for all $s < t$:*

$$\mathcal{R}_0(s,t) = e^{-\{t-s\}\mathcal{L}_J([t-s])} \prod_{i=0}^{[t-s]} e^{-\mathcal{L}_J(s+i)} \tag{6.26}$$

where $\{\tau\}$ denotes the fractional part of the real number τ.

For every type **II** *to IV initial datum v_0, we have:*

$$\|\partial_t \mathcal{R}_0(s,t)v_0\|_X \le C(\varepsilon^2 + \frac{1}{\varepsilon^\alpha t^{1-\alpha}}) N_\varepsilon(t-s)\|v_0\|_\alpha. \tag{6.27}$$

Proof. Use Theorem 5.3 and argue as in the preceding lemma. □

We have now the elements all the elements to estimate $\mathcal{R}(s,t)$.

Theorem 6.3 *Choose any small $\alpha > 0$. For Type* **II** *to IV initial data, we have*

$$\forall v_0 \in X^\alpha, \qquad \|\mathcal{R}(s,t)v_0\|_\alpha \le CN_\varepsilon(t-s)\|v_0\|_\alpha. \tag{6.28}$$

Proof. It has a lot to do with the proof of Theorem 6.2; we only sketch it.

If $v(t) = \mathcal{R}(s,t)v_0$ we may write

$$v(t) = \mathcal{R}_0(s,t) + \int_s^t \mathcal{R}(\sigma,t)L(\sigma)[v]\ d\sigma, \tag{6.29}$$

where - obvious modification if $\sigma \ge [t-s]$:

$$L(\sigma)[v] = \int_0^1 \sum_{i=1}^3 f_i([\sigma - s] + \zeta, \sigma)\ d\zeta := \sum_{i=1}^3 L_i(\sigma)[u].$$

The f_i's have the same expression as in (6.20), but $e^{-t\mathcal{L}_J(t_0)}v_0$ is now replced by v. The dangerous term is now $L_2(\sigma)v$; by integration by parts this amounts to studying

$$I(t) := \int_s^t \partial_t\Big(\mathcal{R}(\sigma,t)\Big)\Big(\int_0^1 \partial_\sigma\gamma_1(\sigma,.)v(\sigma)\Big)\ d\sigma.$$

Just as in the proof of Theorem 6.2, item **2**, $I(t)$ may be neglected in view of the other terms, provided that this time they are estimated in terms of $\|v(t)\|_\alpha$.

All in all, we arrive at the inequality

$$\|v(t)\| \le CN_\varepsilon(t)\|v_0\|_\alpha + C\varepsilon^2 \int_0^t \frac{N_\varepsilon(t-s)}{\sqrt{t-s}}\ ds,$$

which yields

$$\|\mathcal{R}(s,t)v_0\| \le CN_\varepsilon(t-s)\|v_0\|_\alpha.$$

The estimate for $\|\mathcal{R}(s,t)v_0\|_\alpha$ then comes from the classical parabolic estimates. □

PROOF OF THEOREM 1.1. It is now only the matter of a simple stability argument. Assume $\|(v_0, w_0)\|_{X^\alpha}$ to be of the order ε^4. Define $\frac{\tau_1}{\varepsilon^2} < \frac{\tau_0}{\varepsilon^2}$ as the first time t when the quantity $\|u(t)\|_{X^\alpha}$ of (6.4) exceeds $\varepsilon^{3-\kappa}$, for some small number $\kappa > 0$. Then we have, applying Theorem 6.3 for all $t < \frac{\tau_1}{\varepsilon^2}$:

$$\|Q_\varepsilon(t,u).u^{(2)}\|_{X^\alpha} = O(\varepsilon^{3-6\kappa}).$$

Moreover, recall that $Q_\varepsilon(t,v).v^{(2)}$ is of Type **II**. The other terms in the integral are of types **III** and **IV**. Application of Theorem 6.3 yields:

$$\forall t \leq \frac{\tau_1}{\varepsilon^2}, \quad \|u(t)\|_{X^\alpha} \leq \varepsilon^{4-6\kappa}.$$

This contradicts the definition of τ_1.

It only remains to prove the converge of the radii; however we have $\|T^\varepsilon(t) - T_J(t)\|_\infty = O(\varepsilon^3)$ - at most - on one hand; on the other hand, once again from Theorem 4.1 we have

$$\partial_r T_J(t,r) = -\sqrt{2Le}e^{w_J/2} + O(\varepsilon|\text{Log}\varepsilon|) \text{ in an } \varepsilon|\text{Log}\varepsilon|\text{-neighbourhood of } R_J.$$

Hence, we have

$$R^\varepsilon(t) - R_J(t) = O(\varepsilon^2)$$

on $[0, \tau_0\varepsilon^{-2}]$.

$\square$

Remark 6.1 *Our result is valid as soon as the real nmber α is < 1. It should probably hold for α close to $\frac{1}{2}$, implying therefore the convergence of (T,Y) to (T_J, Y_J) at a rate of at least ε^2. This would imply the nondegeneracy of T, hence the convergence of the radius R of the solution to R_J.*

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