A THRESHOLD PHENOMENON IN THE PROPAGATION OF A POINT SOURCE INITIATED FLAME

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ABSTRACT. This work concerns an asymptotic model for spherical flame propagation, derived by Joulin. In Joulin's article, the flame radius is governed by a singular integro-differential equation, and its long-time behaviour is seen numerically to depend on an energy input parameter. Three regimes are identified: the flame radius may either shrink to 0, or tend to $+\infty$, or tend to a finite value.

The purpose of the present work is to give a mathematically rigorous proof of these results.

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INTRODUCTION

The purpose of this work is the analysis of a fractional differential equation describing the time-evolution of the radius $R(t)$ of a spherical flame. Let $q(t)$ be a smooth nonnegative function defined on $\mathbb{R}_+$, having a unit total mass over $\mathbb{R}_+$. Then the function $R(t)$ satisfies the following equation

$$R\partial_{1/2}R = R\mathrm{Log}R + Eq(t), \qquad R(0) = 0 \tag{1}$$

where $\partial_{1/2}$ denotes the time-derivative of order $\frac{1}{2}$:

$$\partial_{1/2}R := \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{R(s)}{\sqrt{t-s}}\,ds = \frac{1}{\sqrt{\pi}}\int_0^t \frac{\dot{R}(s)}{\sqrt{t-s}}\,ds \tag{2}$$

the last equality being valid because $R(0) = 0$. This model is derived in Joulin [7] through high activation energy asymptotics. In this context, the function q is a time-varying point source energy, and the parameter E roughly represents the intensity of the heat source.

The numerical simulations in [7] indicate that, for a given function $q(t)$, there exists a critical value $E_{cr}(q)$, such that the flame manages to propagate - i.e. its radius tends to $+\infty$ - if $E > E_{cr}(q)$, and quenches if $E < E_{cr}(q)$. If $E = E_{cr}(q)$, the flame radius tends to 1.

The following problem is also addressed in [7]: given a family of functions $(q_\lambda)_{\lambda\in\Lambda}$, does there exist a value of λ minimizing the critical energy over all possible values of λ? To investigate this question, the author computes the critical energy value for a two-parameter family of pulse-like energy inputs, and finds a minimizing value of the pulse duration. A natural question is therefore the minimization of the critical energy over all functions of unit total mass.

Our purpose in this first part is to perform a mathematically rigorous investigation of this problem. In this paper, the global behaviour of the initial value problem for (1) will be considered; sharper asymptotics and the optimisation problem will be treated in the forthcoming Part 2.

The main tool of the analysis will be a representation result starting from an idea of G. Montseny [12]. The function $R(t)$ may easily be handled as the trace at $x = 0$ of the solution of the singular parabolic equation

$$u_t - u_{xx} = 2\delta_{x=0}\big(\mathrm{Log}u + \frac{Eq(t)}{u}\big), \quad u(0,x) = 0 \tag{3}$$

posed on $\mathbb{R}$. This formulation has, according to us, two important advantages.
- Observe that (1) is **not** a Cauchy Problem; indeed, there is no semigroup property for such equations, even with time-independent data. Transferring its analysis to the study of a - possibly more complicated - parabolic equation reduces the global existence problem to sufficiently strong *a priori* estimates. Of course, (3) is a singular Cauchy problem, and the singularity at $u = 0$ will have to be handled with care.

- Problems such as (1) have no obvious monotonicity properties, whereas the maximum principle will hold for our parabolic equation. This will allow us, in particular, to use sub and super-solutions to study the generic behaviour of the flame - i.e. propagation or extinction - and this will be obtained in a straightforward way. We found out after completing the proof that part of the generic behaviour could be attacked directly through the integral equation, at the expense of much more tedious calculations. However, the study of the non-generic behaviour, i.e., the trend of the flame radius to 1, requires the use of results specific to parabolic equations - monotone dynamical systems, non increase of the zero number principle - which have no equivalent in the integral equation framework that we are aware of.

The energy inputs that we will take into account will always be smooth, nonnegative, with connected supports, bounded derivatives and unit total mass. This in particular implies that $q(t)$ tends to 0 as $t \to +\infty$. We also make the following essential additional assumption on the initial values of q.

$(\mathcal{Q})$ $q(t) \sim q_0 t^\beta$, as $t \to 0$, with $0 \le \beta < \dfrac{1}{2}$.

All these assumptions will always be understood to hold. We will see in Sections **1** and **2** why Assumption $(\mathcal{Q})$ is crucial.

A solution $R(t)$ of (1) will be in this paper a *strong* solution, i.e: $R(t)$ will be continuous at $t = 0$, positive and C^1 in the interior of its life interval. With this definition in mind, the main results of this paper are the following.

Theorem 1. *Assume q to be positive on $\mathbb{R}_+^*$. Then (1) has a unique global solution $R(t)$. Moreover there exists $E_{cr}(q) > 0$ such that*
- if $E < E_{cr}(q)$, then $\lim_{t\to+\infty} R(t) = 0$,
- if $E > E_{cr}(q)$, then $\lim_{t\to+\infty} R(t) = +\infty$,
- if $E = E_{cr}$, then $\lim_{t\to+\infty} R(t) = 1$.

This theorem tells in particular that finite-time quenching *never* occurs. When the support of q is bounded, the result is slightly different.

Theorem 2. *Assume the existence of $t_0 > 0$ such that q is positive only on $[0, t_0[$. Then (1) admits a unique $R(t)$. Moreover there exists $E_{cr}(q) > 0$ such that*
- if $E > E_{cr}(q)$ or $E = E_{cr}(q)$, the results of Theorem 1 hold.
- if $E < E_{cr}(q)$, then there exists a finite $t_{max} > t_0$ such that the life span of $R(t)$ is $[0, t_{max}[$; moreover

$$\lim_{t\to t_{max}} R(t) = 0, \tag{4}$$

The paper is organized in five sections. In Section 1, the early development of the flame is investigated, giving rise to a local existence result. Sufficient conditions for global existence are established in Section 2. In Section 3, the qualitative behaviours decribed in Theorem 1 are exhibited on a class of related smooth parabolic problems, which provides the main ideas for the study of (1). The time-asymptotic analysis is performed in Section 4 and 5, by means of a shooting method: Section 4 is devoted to the generic behaviour of the flame radius, and Section 5 to the non-generic behaviour.

1. LOCAL EXISTENCE

Let $I_{1/2}$ denote the Abel integrator of order 1/2. For any function $f(t)$ locally bounded on $\mathbb{R}_+^*$ it is defined as [4]

$$I_{1/2}f = \frac{1}{\sqrt{\pi}} \int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\, d\tau. \tag{1.1}$$

Then, for any function $f(t)$ locally bounded on $\mathbb{R}_+^*$, any constant u_0 and any function $u(t)$ continuous on $\mathbb{R}_+$ there holds

$$\left(\partial_{1/2} u = \frac{u_0}{\sqrt{\pi t}} + f\right) \text{ if and only if } \left(u = u_0 + I_{1/2} f\right).$$

The main tool of this section will be the generalized Gronwall lemma, which we state as follows.

Generalized Gronwall Lemma [5]. *Let $H(t)$ be a nonnegative, locally integrable function over $\mathbb{R}_+$. Let $u(t)$ be a locally bounded function over $\mathbb{R}_+$ and $a \in \mathbb{R}_+$ such that*

$$0 \le u(t) \le a + \int_0^t H(t-s) u(s)\, ds.$$

Then $0 \le u(t) \le a \exp \int_0^t H(s)\, ds$.

Let us also state here the following proposition, which will be useful in the subsequent sections.

Proposition 1.1. *Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function, locally lipschitz with respect to its second variable and let h belong to $C^{1/2}(\mathbb{R}_+)$. There exists $t_0 > 0$ such that the problem*

$$u = I_{1/2} f(t,u) + h(t), \quad 0 \le t \le t_0 \tag{1.2}$$

has a unique solution in $C^{1/2}([0,t_0])$. Moreover, if f is C^2 with respect to t and u and if h belongs $C^{3/2}(\mathbb{R}_+)$, then the solution u belongs to $C^{3/2}(]0,t_0])$.

PROOF. The existence and uniqueness part of this proposition is identical to the proof of the Cauchy-Lipschitz Theorem, just because of the Generalized Gronwall Lemma. The $C^{1/2}$ regularity follows that of $I_{1/2} f(.,u)$ [4]. A little more involved is the regularity, due to the unpleasant fact that the operator $I_{1/2}$ is not regularizing when applied to functions that do not vanish at the origin.

Let w be the solution of the linear weakly singular equation

$$w = k(t) + \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial_u f(s,u(s))}{\sqrt{t-s}} w(s)\, ds,$$

where

$$k(t) = h'(t) + \frac{f\big(0, u(0) + h(0)\big)}{\sqrt{\pi t}} + \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial_t f(s, u(s))}{\sqrt{t-s}} \, ds$$

to which the preceding result applies directly, and asserts that $w - k \in C^{1/2}([0, t_0]$. Then $u(t) := h(0) + \displaystyle\int_0^t w(s)\, ds$ is easily verified to be a solution of (1.2), so that $u \in C^{3/2}(]0, t_0])$. •

The following proposition is the main result of this section. Let us recall that Problem (1) has now the form

$$R = I_{1/2}\big(\mathrm{Log}R + \frac{Eq}{R}\big), \quad t \le t_0. \tag{1.3}$$

Let B denote the Euler Beta function

$$B(\alpha, \beta) = \int_0^1 \sigma^{\alpha-1}(1-\sigma)^{\beta-1}\, d\sigma. \tag{1.4}$$

Proposition 1.2. *(i). Let q be positive on $[0, t_0]$, with $q(0) = q_0$. There exists $t_1 \in]0, t_0[$ such that (1.3) has a solution on $[0, t_1]$ which satisfies:*

$$R(t) \sim R_0 t^{1/4}, \quad \text{with } R_0^2 = \frac{Eq_0}{\sqrt{\pi}} B(\frac{3}{4}, \frac{1}{2}). \tag{1.5}$$

(ii). Let q be positive on $]0, t_0]$ and be such that there exists $q_0 > 0$ and $\beta \in [0, \frac{1}{2}[$ with $q(t) \sim t^\beta(q_0 + O(t))$, as $t \to 0$. Then there exists $t_1 \in]0, t_0[$ such that (1.3) has a solution on $[0, t_1]$ which satisfies:

$$R(t) \sim R_0 t^{\frac{1+2\beta}{4}}, \quad \text{with } R_0^2 = \frac{Eq_0}{\sqrt{\pi}} B(\frac{3-2\beta}{4}, \frac{1}{2}). \tag{1.6}$$

In both cases, $R \in C^{3/2}(]0, t_1])$.

PROOF. Only Part (i) will be proved, the proof of Part (ii) being similar. The behaviour of a solution of (1.3) near 0 can be formally obtained as follows: assume that $R(t) \sim R_0 t^\alpha$ as $t \to 0$; then

$$\begin{aligned} R_0 t^\alpha &\sim \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} \big(\mathrm{Log}(R_0 s^\alpha) + \frac{Eq_0}{R_0 s^\alpha}\big)\, ds \\ &\sim \frac{Eq_0}{R_0\sqrt{\pi}} t^{1/2-\alpha} \int_0^1 \frac{ds}{s^\alpha \sqrt{1-s}} \end{aligned}$$

This yields $\alpha = 1/4$, hence (1.5). In order to make this rigorous, let us look for a solution of (1.3) under the form

$$R(t) = R_0 t^{1/4}\big(1 + \rho(t)\big), \quad \lim_{t \to 0} \rho(t) = 0.$$

The equation for ρ may be written as follows:

$$\rho(t) + \frac{Eq_0}{\sqrt{\pi}R_0^2}\int_0^1 \frac{\rho(t\sigma)}{\sigma^{1/4}\sqrt{1-\sigma}}\, d\sigma = F_1(t,\rho) + t^{1/16}F_2(t,\rho).$$

Both F_1 and F_2 are mappings of the space $C([0,T],I\!R)$ onto itself. For a given $T\in]0,1[$, let us introduce the space

$$X_T = \{\rho \in C([0,T],I\!R) : \sup_{0<t\leq T} t^{-1/8}|\rho(t)| < +\infty\},$$

endowed with its natural norm denoted by $||.||$. Let us notice that any element u of X_T has $u(0)=0$. Both F_1 and F_2 are also mappings from a ball of X_T with sufficiently small radius onto itself. Moreover they have the following properties:

- For any $r>0$ and any (ρ_1,ρ_2) in the ball of X_T with centre 0 and radius r there is a constant $K(r)>0$ such that:

$$||F_1(\rho_1) - F_1(\rho_2)|| \leq K(r)\big(||\rho_1|| + ||\rho_2||\big)\ ||\rho_1 - \rho_2||$$
$$||F_1(\rho_1)|| \leq K(r)||\rho_1||^2.$$

- The function F_2 is locally lipschitz with respect to $\rho \in X_T$.

In these conditions, the theorem will be proved if we manage to prove that the operator $I+K$ of X_T, with

$$K\rho(t) = \frac{Eq_0}{\sqrt{\pi}R_0^2}\int_0^1 \frac{\rho(t\sigma)}{\sigma^{1/4}\sqrt{1-\sigma}}\, d\sigma$$

is invertible in X_T for all T. Indeed, it will then suffice to reduce the size of T and make use of the Banach Fixed Point Theorem to the function

$$\rho \mapsto (I+K)^{-1}\big(F_1(.,\rho) + t^{1/16}F_2(.,\rho)\big).$$

Therefore the problem is reduced to proving that the norm of K in X_T is strictly less than 1. To see this, let us write

$$|K\rho(t)| \leq \frac{Eq_0}{\sqrt{\pi}R_0^2}t^{1/8}||\rho||\int_0^1 \frac{\sigma^{1/8}}{\sigma^{1/4}\sqrt{1-\sigma}}\, d\sigma;$$

hence

$$\sup_{0\leq t\leq T} t^{-1/8}|K\rho(t)| \leq \frac{B(\frac{7}{8},\frac{1}{2})}{B(\frac{3}{4},\frac{1}{2})} < 1$$

This implies that $||K||_{\mathcal{L}(X,X)} < 1$, hence the invertibility of $I+K$.

The regularity may be inferred as follows. For any $\varepsilon > 0$ let us write

$$\begin{aligned} R(t) &= \int_0^\varepsilon \big(\mathrm{Log} R + \frac{Eq(s)}{R}\big)\frac{ds}{\sqrt{t-s}} + \int_\varepsilon^t \big(\mathrm{Log} R + \frac{Eq(s)}{R}\big)\frac{ds}{\sqrt{t-s}} \\ &:= \varphi(t) + I_{1/2}R(\varepsilon + t) \end{aligned}$$

Hence Proposition 1.1 applies to the function $t \geq 0 \mapsto R(\varepsilon + t) - \varphi(\varepsilon + t)$. •

Let us end this section by the follwing problem, which will naturally come up in the forthcoming Section 2. Let $h(t)$ be smooth on $]0, +\infty[$, and let us consider

$$R = I_{1/2}\big(\mathrm{Log} R + \frac{Eq}{R}\big) + h(t), \quad t \leq t_0. \tag{1.7}$$

Proposition 1.3. *Assume that*

$$h(t) = O(\sqrt{t}) \ \text{ as } t \to 0^+.$$

(i). Let q be positive on $[0, t_0]$, with $q(0) = q_0$. There exists $t_1 \in]0, t_0[$ such that (1.3) has a solution on $[0, t_1]$ which satisfies:

$$R(t) \sim R_0 t^{1/4}, \quad \text{with } R_0^2 = \frac{Eq_0}{\sqrt{\pi}} B\big(\frac{3}{4}, \frac{1}{2}\big). \tag{1.8}$$

(ii). Let q be positive on $]0, t_0]$ and be such that there exists $q_0 > 0$ and $\beta \in [0, \frac{1}{2}[$ with $q(t) \sim t^\beta(q_0 + O(t))$, as $t \to 0$. Then there exists $t_1 \in]0, t_0[$ such that (1.3) has a solution on $[0, t_1]$ which satisfies:

$$R(t) \sim R_0 t^{\frac{1+2\beta}{4}}, \quad \text{with } R_0^2 = \frac{Eq_0}{\sqrt{\pi}} B\big(\frac{3-2\beta}{4}, \frac{1}{2}\big). \tag{1.9}$$

In both cases, $R \in C^{3/2}(]0, t_1])$.

The proof of this proposition is quite similar to the proof of Proposition 1.2, and will be omitted.

Let us end this section by performing the first step in the proof of the fundamental character of Assumption $\mathcal{Q}$. To see this, let us assume

$$\exists \beta \geq \frac{1}{2}: \qquad q(t) \sim q_0 t^\beta, \text{ as } t \to 0. \tag{$\mathcal{Q}'$}$$

Let us consider this time the integral equation

$$R = I_{1/2}\big(\mathrm{Log}|R| + \frac{Eq}{R}\big). \tag{1.10}$$

Performing a dimensional analysis just as the beginning of the section, we see that we may expect a solution $R(t)$ to (1.10) such that

$$R(t) \sim R_0 \sqrt{t}\mathrm{Log}t, \qquad R_0 = \frac{1}{2\sqrt{\pi}} B(\frac{1}{2}, \frac{1}{2}) \tag{1.11}$$

This leads us to

Proposition 1.4. *Let Assumption $(\mathcal{Q}')$ be true. Then Problem (1.10) has a local solution satisfying (1.11).*

PROOF. Let us look for a solution of the form

$$R(t) = R_0(1 + \rho(t)), \qquad \rho \text{ continuous, } \rho(0) = 0. \tag{1.12}$$

The function ρ is a solution of

$$\rho = \frac{1}{R_0\sqrt{t}\mathrm{Log}t}\Big(I_{1/2}(\mathrm{Log}(1+\rho)) + I_{1/2}(\frac{Eq}{1+\rho})\Big) + O(\frac{\mathrm{Log}|\mathrm{Log}t|}{\mathrm{Log}t})$$

For two continuous functions ρ_1 and ρ_2, defined and bounded by $\frac{1}{2}$ on an interval $[0, t_0]$ there holds

$$\begin{aligned}
&\Big\|\frac{1}{R_0\sqrt{t}\mathrm{Log}t}\Big(I_{1/2}(\mathrm{Log}(1+\rho_1)) - I_{1/2}(\mathrm{Log}(1+\rho))\Big)\Big\|_\infty \\
&\qquad = O(\frac{\mathrm{Log}|\mathrm{Log}t_0|}{\mathrm{Log}t_0})\|\rho_1 - \rho_2\|_\infty \\
&\Big\|\frac{1}{R_0\sqrt{t}\mathrm{Log}t}\Big(I_{1/2}(\frac{Eq}{1+\rho_1}) - I_{1/2}(\frac{Eq}{1+\rho_2})\Big)\Big\|_\infty \\
&\qquad = O(\frac{\mathrm{Log}|\mathrm{Log}t_0|}{\mathrm{Log}t_0})\|\rho_1 - \rho_2\|_\infty
\end{aligned}$$

Therefore, for $t_0 > 0$ sufficiently small, (1.10) has a solution. •

It will be our task, in the next section, to prove that (1.10) cannot have any other solution.

Our spherical flame has begun to develop. To see more of its expansion, we could still work on the integral formulation (1.3), with the aid of the generalized Gronwall Lemma. It is indeed possible to write it under the form of a local semiflow; see for example [11]. We will not do so because, such a formulation will not, as far as we are aware of, give us any information on the long-time behaviour of $R(t)$, which is really the problem we are interested in.

As we shall see in the next section, the diffusive formulation is much more tractable in view of the derivation of simple continuation rules, and because of its monotonicity properties.

2. DIFFUSIVE FORMULATION, COMPARISON PRINCIPLES, GLOBAL EXISTENCE

As a consequence of the diffusive representation of fractional operators [12], for any locally bounded $f(t)$ on $I\!R_+$, and any continuous function $R(t)$ over $I\!R_+$, the identity $R = I_{1/2} f$ is equivalent to

$$R(t) = u(t,0),$$

with

$$\begin{cases} u_t - u_{xx} = 2f(t)\delta_{x=0}, & x \in I\!R \\ u(0,x) = 0 \end{cases}$$

The 'only if' part can easily be inferred from the explicit formula for $u(t,x)$ as a solution of the nonhomogeneous heat equation; the 'if' part is more difficult and involves Laplace transform arguments.

We are therefore led to the study of the general Cauchy Problem

$$\begin{aligned} u_t - u_{xx} &= 2(\mathrm{Log} u + \frac{Eq(t)}{u})\delta_{x=0}, \qquad x \in I\!R \\ u(0,x) &= u_0(x) \end{aligned} \tag{2.1}$$

where the assumptions on u_0 are given below. The main result of this section is summarized in the following theorem.

Theorem 2.1. *Let u_0 be an even, square-integrable, nonnegative Lipschitz function. Assume the function $q(t)$ to satisfy Assumption $(\mathcal{Q})$. Also assume the existence of $t_0 > 0$ such that $q(t) > 0$ if $0 < t < t_0$, and $q(t) = 0$ if $t \geq t_0$.*

(i). If $t_0 = +\infty$, then (2.1) has a unique positive global solution u which is positive except at $t = 0$. Furthermore, u is C^∞ at each point (t,x) of $I\!R_+^* \times I\!R_+^* \cup I\!R_+^* \times I\!R_-^*$, and the function $t \mapsto u(t,0)$ is C^∞ on $I\!R_+^*$.

(ii). If $t_0 < +\infty$, then (2.1) has a unique positive maximal solution u, defined on an interval of the form $[0, t_{max}[$ which is positive except at $t = 0$. Furthermore, u is C^∞ at each point (t,x) of $]0, t_{max}[\times I\!R_+^* \cup]0, t_{max}[\times I\!R_-^*$, and the function $t \mapsto u(t,0)$ is C^∞ on $]0, t_{max}[$.

If $t_{max} < +\infty$ there exists a sequence $(t_n)_n$ converging to t_{max} such that

$$\lim_{n \to +\infty} u(t_n, 0) = 0. \tag{2.2}$$

Theorem 2.1 admits an obvious corollary.

Corollary 2.2. *Under the notations and assumptions of Theorem 2.1 then*

(i). if $t_0 = +\infty$, then (1) has a unique smooth global solution - which is positive on $I\!R_+^*$;

(ii) if $t_0 < +\infty$, then (1) has a unique maximal smooth solution $R(t)$, defined on an interval of the form $[0, t_{max}[$ - which is positive except at $t = 0$. Furthermore,

$R(t)$ is smooth on $]0, t_{max}[$ and there exists a sequence $(t_n)_n$ converging to t_{max} such that

$$\lim_{n\to+\infty} R(t_n) = 0.$$

The proof of Corollary 2.2 is obvious: as a consequence of Theorem 2.1 and [12], $R(t)$ is a solution of (1) if and only if $R(t) = u(t, 0)$, u being the solution of (2.1) with $u_0 \equiv 0$.

The proof of Theorem 2.1 will be carried out as follows. Recall that Proposition 1.2 gave us a local $C^{3/2}$ solution $R(t)$ to (1), thus allowing the use of parabolic regularity. It will then be possible to state a maximum principle, and use a sub-solution method, leading us to global existence.

Proposition 2.3. *Let $[0, t_1]$ be the interval on which the solution $R(t)$ of (1) was constructed in Proposition 1.3. Then (2.1) admits a unique solution $u(t, x)$ which is C^∞in $t \in]0.T]$, and C^∞ in $x \in I\!R^*_+ \cup I\!R^*_-$. Furthermore, the map $u_0 \mapsto u$ is continuous from $W^{1,\infty}(I\!R)$ to $C([0, T] \times I\!R)$.*

PROOF. Let us set

$$(2.3)\qquad \begin{aligned} f(t) &= 2\big(\mathrm{Log}R(t) + \frac{Eq(t)}{R(t)}\big), \\ \varphi(t) &= f(t)\delta_{x=0} \in L^2([0, T], H^{-1}(I\!R)) \end{aligned}$$

Problem (2.1) is equivalent to solving

$$(2.4)\qquad \begin{aligned} u_t - u_{xx} &= \varphi(t), \qquad x \in I\!R \\ u(0, x) &= u_0(x) \end{aligned}$$

From the Lions Theorem [9], (2.4) has a unique solution $u(t, x) \in L^2([0, T], H^1(I\!R))$. As a consequence, the solution $u(t, x)$ of (2.1) is an even function, the restriction of which on $I\!R_+$ is the unique solution of

$$(2.5)\qquad \begin{aligned} &u_t - u_{xx} = 0, \qquad x > 0 \\ &u_x(t, 0) = -\frac{1}{2}f(t), \qquad u(0, x) = u_0(x) \end{aligned}$$

From Proposition 1.2, $f \in C^{3/2}(]0, T])$. From parabolic regularity [8], there holds $u \in C^{5/2,3/2}(]0, T] \times I\!R_+)$, and additional smoothness is easily derived by iteration. Therefore, the continuity of $u_0 \mapsto u$ is obvious. •

The main consequence of the diffusive formulation (2.1) is the following comparison result.

Theorem 2.4. *Let $q_1(t)$ and $q_2(t)$ two smooth functions satisfying the assumptions of Theorem 2.1. Assume that $E_1 q_1 \le E_2 q_2$, and let $u_1(t)$ and $u_2(t)$ be two solutions of (2.1), with $q(t) = q_i$ and with respective initial data $u_{10}(x) \le u_{20}(x)$. Let $[0, T]$ be*

a time-interval on which both u_1 and u_2 are defined.
Then $u_1 \leq u_2$ on $[0,T] \times I\!R$.

Without the singularity at $t = 0$ and $u = 0$, this would be the standard maximum principle. But the presence of this singularity forces us to be more careful.

PROOF. Assume $q_i(0) > 0$; the general case is once again proved along the same lines.

1. Let us first assume that $u_i(t,0) = R_i(t)$ when t is small; the functions $R_i(t)$ are the solutions of (1.3) - resp. (1.7) - constructed in Part 1.

Let us set, as usual, $v = u_1 - u_2$. In the $L^2([0,T], H^{-1})$ sense there holds

$$v_t - v_{xx} = 2\Big(\mathrm{Log} u_1 - \mathrm{Log} u_2 + \frac{1}{u_1(t,0)}(E_1 q_1 - E_2 q_2) + E_2 q_2\big(\frac{1}{u_1(t,0)} - \frac{1}{u_2(t,0)}\big)\Big)\delta_{x=0};$$

using v^+ - i.e. the positive part of v - as a test function we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v^+\|_{L^2}^2 + \|v_x^+\|_{L^2}^2 &\leq 2\big(\mathrm{Log} u_1(t,0) - \mathrm{Log} u_2(t,0)\big)v^+(t,0)\\ &\leq \frac{2}{u_2(t,0)}\big(v^+(t,0)\big)^2\end{aligned}$$

On the other hand we have

$$\frac{2}{u_2(t,0)}\big(v^+(t,0)\big)^2 = -\frac{2}{u_2(t,0)}\int_0^{+\infty}\big(\partial_x(v^+)\big)^2\,dx \leq \frac{4}{u_2(t,0)^2}\|v^+\|_{L^2}^2 + \|v_x^+\|_{L^2}^2.$$

Hence

$$\frac{d}{dt}\int_{I\!R}(v^+)^2\,dx \leq \frac{4}{u_2(t,0)^2}\int_{I\!R}(v^+)^2. \tag{2.6}$$

The function $\dfrac{4}{u_2(t,0)^2}$ belongs to $L^1([0,T])$; this is a consequence of Proposition 1.2. This implies $v^+ \equiv 0$ on $[0,T] \times I\!R$.

2. The preceding step shows us that, if $u_i(t,0) = R_i(t)$ when t is small, then $u_1 \leq u_2$. The problem is that we are not sure whether $u_i(t,0)$ coincides with R_i, even for short times, and this is what we will be concerned about in the next step. Let $i \in \{1,2\}$ be given; let $\bar{u}_i$ be another potential solution with the same point source function $E_i q_i(t)$. A particular case of the preceding step is when $E_1 q_1 = E_2 q_2$, $u_2 := u_1$ and $u_1 := \bar{u}_1$. Scanning the above proof, we notice that inequality (2.6) still holds; because $\dfrac{1}{u_1(t,0)^2}$ is integrable we have $\bar{u}_1(t,0) \leq u_1(t,0)$; as a consequence we have, for $t > 0$ small enough:

$$\bar{u}_1(t,0) \leq C t^{1/4}. \tag{2.7}$$

Therefore, any positive solution $R(t)$ of Problem (1.7) satisfies $r(t) \leq C t^{1/4}$.

3. To conclude, let $R(t)$ be any positive solution of Problem (1.7). We have the following sequence of inequalities, where c denotes a generic constant:

$$\begin{aligned} R(t) &\geq I_{1/2}\left(\mathrm{Log}R + \frac{Eq}{R}\right) \\ &\geq cI_{1/2}\left(\frac{1}{R}\right) \\ &\geq c\int_0^t \frac{ds}{s^{1/4}\sqrt{t-s}} \\ &\geq ct^{1/4} \end{aligned}$$

Hence, for any positive solution u of (2.1), the function $\dfrac{1}{u(t,0)^2}$ belongs to $L^1([0,T])$. Consequently, the considerations of Step **1** apply to any solution of Problem (2.1). •

It now remains to see what would prevent the solution $u(t)$ of (2.1) from being global. This is given in the next proposition, which is an adapted version of the Cauchy-Lipschitz Theorem in its 'global' version.

Proposition 2.5. *Problem (2.1) has a unique maximal solution $u(t)$, defined on a time-interval $[0, t_{max}[$. Moreover,*

(i) either $t_{max} = +\infty$,

(ii) or there exists a sequence $(t_n)_n$ converging to t_{max} such that

$$\lim_{n\to+\infty} u(t_n, 0) = 0.$$

Because of Theorem 2.4, Problem (2.1) has a unique local solution. Hence the existence of t_{max} is obvious. Let us write an equally easy, although quite important, consequence of the above considerations: it describes how the memory effects are encoded when one wishes to write a variation-of-constant type formula.

Proposition 2.6. *Let $\mathcal{T}$ be the heat semigroup, $u(t)$ the solution of (2.1) with $u_0 = 0$, and $R(t)$ the solution of (1). Let $[0, t_{max}[$ be the life interval of u. Then for any $0 < t_0 < t < t_{max}$ we have*

$$R(t) = \mathcal{T}(t-t_0)u(t_0)(0) + \frac{1}{\sqrt{\pi}}\int_{t_0}^t \frac{1}{\sqrt{t-s}}\left(\mathrm{Log}R(s) + \frac{Eq(s)}{R(s)}\right) ds. \tag{2.8}$$

In view of the above, this formula is obvious. Let us proceed to the

PROOF OF PROPOSITION 2.5. Let us assume that t_{max} does not fall into one of the two above cases, i.e. there exists $\delta > 0$ and $t_0 \in]0, t_{max}[$ be such that

$$u(t,0) := R(t) \geq \delta \text{ in } [t_0, t_{max}[.$$

Setting $g(t) := \mathcal{T}(t-t_0)u(t_0)(0)$ and using Proposition 2.6 we see that $R(t-t_0)$ satisfies (1.7); hence there holds, for $t \geq t_0$:

$$\sqrt{\pi}R(t) \leq \frac{1}{\delta}\int_0^t \frac{Eq(s)}{\sqrt{t-s}}\, ds + \sqrt{\pi}|g(t)| + \frac{1}{\delta}\int_{t_0}^t \frac{R(s)-1}{\sqrt{t-s}}\, ds$$

This implies, with the aid of the Generalized Gronwall Lemma, that $R(t)$ is bounded in $[t_0, t_{max}[$. Since it is bounded away from 0, Proposition 2.3 applies, and yields a local solution to (2.1) in the vicinity of t_{max}, whose life span will overlap $[0, t_{max}[$. This is a contradiction. •

We may now end this section with the

PROOF OF THEOREM 2.1. The only thing to prove is that the maximal solution to (2.1) is global as soon as the function $q(t)$ does not vanish on $I\!R_+$. Let us therefore assume that $t_{max} < +\infty$; this implies the existence of a sequence $(t_n)_n$ converging to t_{max} such that

$$\lim_{n\to+\infty} u(t_n, 0) = 0.$$

Recall that Proposition 1.2 ensures the existence of a positive solution to (1.3) on a time interval $0, T_q$, the real number T_q depending only on the C^1 norm of q in the vicinity of 0. Therefore let $q_{min} > 0$ be the infimum of q on $[0, t_{max}]$ and n_0 large enough so that

$$t_{max} - t_{n_0} < \frac{T_q}{2}.$$

The following problem

$$\begin{aligned} &u_t - u_{xx} = 2(\mathrm{Log}u + \frac{Eq_{min}}{u})\delta_{x=0}, \qquad (t,x) \in]t_{n_0}, t_{n_0} + T_q] \times I\!R \\ &u(t_{n_0}, x) = 0 \end{aligned}$$

is uniquely solvable; moreover, due to Theorem 2.4, its solution $\underline{u}(t)$ satisfies $\underline{u}(t) \leq u(t)$ on $[t_{n_0}, t_{max}[$; hence $u(t,0)$ is larger than a positive constant in a small left neighbourhood of t_{max}, a contradiction. •

To finish up with Theorem 2.1, let us say that the solution - or the flame - *quenches in finite time* if Assertion (ii) of Proposition 2.5 occurs.

To conclude this section, let us see why Assumption $\mathcal{Q}$ is fundamental. The corresponding equation to (1.10) is

$$\begin{aligned} &u_t - u_{xx} = 2(\mathrm{Log}|u| + \frac{Eq(t)}{u})\delta_{x=0}, \qquad x \in I\!R \\ &u(0, x) = 0 \end{aligned} \tag{2.9}$$

Following the above discussion we may prove the

Proposition 2.7. *Let Assumption ($\mathcal{Q}'$) be true. Then Problem (2.9) has a unique local solution $u(t,x)$ satisfying $u(t,0) \sim R_0\sqrt{t}\mathrm{Log}t$ as $t \to 0$.*

PROOF. The discussion follows exactly Points **1** to **3** of Theorem 2.4. This discussion is still valid because the solution $R(t)$ of (1.10) satisfies $R(t) \sim R_0\sqrt{t}\mathrm{Log}t$

as $t \to 0$, and the function $\dfrac{1}{\sqrt{t}\mathrm{Log} t}$ is square-integrable on $[0, t_0]$ for $t_0 > 0$ small enough. •

The above discussions call two remarks.

Remark 1. Theorem 2.1 remains true if, instead of Assumption $(\mathcal{Q})$, we had assumed $q(t) \geq q_0 t^\beta$, with $\beta < \dfrac{1}{2}$. One only has to approximate q by a sequence of functions satisfying $\mathcal{Q}$, and pass to the limit by using the continuity of the solution with respect to the parameters of the equation.

Remark 2. Under Assumption $(\mathcal{Q}')$, Problem (2.1) cannot have a solution: if this were true, this solution would also be a solution of (2.9), and hence would be negative in a vicinity of 0; this is impossible. Similarly to the above remark, the same is true if we had assumed $q(t) \geq q_0 t^\beta$, with $\beta \geq \dfrac{1}{2}$.

3. A SMOOTH MODEL PARABOLIC PROBLEM

Let Ω be a smooth bounded open subset of $I\!R^N$, with outer normal ν; let g be a smooth bounded function from $I\!R$ to $I\!R$ such that $g(0) = 0$. Let $\mathcal{L}$ be the usual elliptic operator

$$\mathcal{L} = -a_{ij}(x)\partial_{x_i x_j} + b_i(x)\partial_{x_i} + c(x),$$

all the coefficients being bounded and Hölder-continuous in $\bar{\Omega}$ and $c(x) > 0$ in $\bar{\Omega}$. Finally let $q(t) \geq 0$ be a nonnegative nonzero smooth compactly supported function, whose support contains an interval of the form $[0, t_0]$. The problem under study is

$$\begin{aligned} &u_t + \mathcal{L}u = 0, \qquad (\Omega) \\ &u_\nu = g(u) + Eq(t) \qquad (\partial\Omega) \end{aligned} \tag{3.1}$$

The Cauchy Problem for (3.1) with $u(0) = 0$ admits a unique global solution $u_E(t, x)$. Problem (3.1) shares many a qualitative feature with (2.1). To reinforce its resemblance let us make the following assumption: the steady problem

$$\begin{aligned} &\mathcal{L}u = 0, \qquad (\Omega) \\ &u_\nu = g(u) \qquad (\partial\Omega) \end{aligned} \tag{3.2}$$

has exactly three hyperbolic solutions: $0 < u_- < u_+$, such that 0 and u_+ are stable, and u_- is unstable.

The function q being given, we wish to choose E so that $u_E(t)$ converges to u_- as $t \to +\infty$.

Let us at once notice that, due to the hyperbolicity and stability of the equilibrium 0, there holds $\|u_E(t)\|_\infty = O(e^{-\omega t})$ for some $\omega > 0$, as soon as E is small enough. On the other hand, for large enough E, the function $\dfrac{u(t)}{E}$ stays above the solution $v(t)$ of

$$v_t + \mathcal{L}v = 0, \qquad (\Omega)$$
$$v_\nu = \frac{q(t)}{2} \qquad (\partial\Omega)$$
$$v(0) = 0$$

which implies that $u(t_0) > u_-$ if E is large enough. Problem (3.1), with $q \equiv 0$, generates a strongly order-preserving semiflow; hence - Hirsch [6], Smith and Thieme [14] -, due to the isolatedness of the solutions of (3.2), each of its orbits is convergent. As a consequence, $u_E(t)$ converges towards u_+ as $t \to +\infty$.

The following sets:

$$X_+ = \{E > 0, \quad \lim_{t\to+\infty} u_E(t) = +\infty\}$$
$$X_- = \{E > 0, \quad \lim_{t\to+\infty} u_E(t) = 0\} \tag{3.3}$$

are both open, due to the previous considerations. Therefore the connectedness of $\mathbb{R}_+^*$ implies the existence of E_{cr} belonging to $X_0 := \mathbb{R}_+ \backslash (X_+ \cup X_-)$; for every $E \in X_0$, $u_E(t)$ converges towards u_-. From the maximum principle, X_0 is an interval; should it not be reduced to one point, there would exist a positive solution of

$$v_t + \mathcal{L}v = 0, \qquad (\Omega)$$
$$v_\nu = (g'(u_-) + o(1))v \qquad (\partial\Omega) \tag{3.4}$$
$$v(0) = 0$$

converging to 0 as $t \to +\infty$. On the other hand, for small $\varepsilon > 0$, let $-\lambda_-^\varepsilon < 0$ be the first eigenvalue to

$$\mathcal{L}u = \lambda u \quad (\Omega), \qquad\qquad u_\nu = (g'(u_-) - \varepsilon)u \quad (\partial\Omega)$$

with eigenfunction ϕ_-^ε. For small enough $\eta > 0$ and large enough $t_0 > 0$, there holds $v(t) \geq \eta\phi_-^\varepsilon e^{\lambda_-^\varepsilon(t-t_0)}$, a contradiction. This argument comes directly from Lions [10].

The preceding argument tells us the right strategy to adopt if we wish to deal with Problem (2.1). It is not, however, possible to conclude directly to the existence and uniqueness of the critical energy from the considerations of this section. There are three main obstacles.

(i) The problem is singular at $u = 0$;

(ii) Problem (2.1) is not posed on a bounded domain, and the fixed point 1 is not hyperbolic.

Point (i) is no true difficulty, as will be shown in the forthcoming Section 4. Point (ii) is a more fundamental problem; moreover the operator $-\partial_{xx}^2$ is not coercive at $\pm\infty$: an unpleasant consequence of this fact is the nonexistence of ω-limit sets orbits in any uniform topology. This explains why we shall need all the technicalities of Section 5.

4. LONG-TIME BEHAVIOUR WHEN E IS SMALL OR LARGE

The results of this section are not at all unlike the long-time behaviours observed by Aronson and Weinberger [2], [3]. Let us indeed start this section by stating, without proof, the exact equivalent of Sattinger's results [13] on sub and super-solutions of elliptic equations. If f is a locally Lipschitz real-valued function, we shall say that $\underline{u}$ (resp. $\overline{u}$) is a subsolution (resp. supersolution) to

$$-u'' = \delta_{x=0} f(u), \qquad x \in I\!R \tag{4.1}$$

if and only if there holds, in the H^{-1} sense:

$$-\underline{u}'' \le \delta_{x=0} f(\underline{u}), \qquad \text{(resp.)} \qquad -\bar{u}'' \ge \delta_{x=0} f(\bar{u}).$$

Let us consider the evolution problem

$$u_t - u_{xx} = \delta_{x=0} f(u), \qquad x \in I\!R \tag{4.2}$$

Theorem 4.1. *(i). Assume the existence of a subsolution $\underline{u}$ to problem (4.1). Then the solution $u(t)$ of Problem (4.2), with initial datum $\underline{u}$ is time-increasing on its life interval. If moreover this interval is finite and u belongs to $L^\infty(I\!R_+ \times I\!R)$, then $u(t)$ converges, locally on every compact subset of $I\!R$, towards the minimal solution u_∞ of (4.1) such that $u_\infty \ge \underline{u}$.*
(ii). Assume the existence of a supersolution $\underline{u}$ to problem (4.1). Then the solution $u(t)$ of Problem (4.2), with initial datum $\underline{u}$ is time-decreasing on its life interval. If moreover this interval is finite and u belongs to $L^\infty(I\!R_+ \times I\!R)$, then $u(t)$ converges, locally on every compact subset of $I\!R$, towards the maximal solution u_∞ of (4.1) such that $u_\infty \le \underline{u}$.

The proof of this result is a direct consequence of [13].

This leads us directly to the fact that, for small energies E, there is extinction in finite or infinite time. Let us now call $u_E(t)$ the solution of Problem (2.1), and let us set $R_E(t) := u_E(t, 0)$.

Proposition 4.2. *(i). Assume that the function $q(t)$ is compactly supported. There exists $E_0 > 0$ such that, for all $E \le E_0$, the solution quenches in finite time.*

(ii). Assume that $q(t) > 0$ for every $t > 0$. Then there exists $E_0 > 0$ such that, for all $E \le E_0$, there holds

$$\lim_{t \to +\infty} R_E(t) = 0.$$

The proof of this fact relies on the following lemma.

Lemma 4.3. *Assume that $E \le E_0 := \dfrac{\mathrm{Log} 2}{2\|q_\infty\|}$. Then $R_E(t) \le \dfrac{1}{2}$ on the life interval of R_E.*

PROOF. Let us set $\bar{u}(t) = \dfrac{1}{2}$; for $E \le E_0$ there holds obviously $\mathrm{Log}\bar{u} + \dfrac{Eq(t)}{\bar{u}} < 0$; Theorem 2.4 then applies and yields $u_E \le \bar{u}$. •

PROOF OF PROPOSITION 4.2. Assume that Case (i) holds, and suppose that the solution $u(t)$ is global. Let $]0, t_0[$ be the support of q; from (2.8) we have

$$\begin{aligned} R_E(t) &= \mathcal{T}(t-t_0)u_E(t_0)(0) + \frac{1}{\sqrt{\pi}} \int_{t_0}^t \frac{\mathrm{Log} R_E(s)}{\sqrt{t-s}}\, ds \\ &\le \mathcal{T}(t-t_0)u_E(t_0)(0) - \frac{\mathrm{Log}2}{\sqrt{\pi}}\sqrt{t-t_0} \end{aligned}$$

This implies that the solution does not live after the time $t_0 + \dfrac{\|u_E(t_0)\|_\infty^2 \pi}{(\mathrm{Log}2)^2}$, a contradiction.

Assume now that Case (ii) holds, i.e the support of q contains $\mathbb{R}_+^*$. Let us prove that $R_E(t)$ tends to 0 as $t \to= \infty$. For $\varepsilon > 0$ such that $\varepsilon \le \|q\|_\infty$, let $t_\varepsilon > 0$ be such that $q(t) \le \varepsilon$ for $t \ge t_\varepsilon$. Therefore $u_E(t,x) \le \bar{u}_\varepsilon(t,x)$ where

$$(\partial_t - \partial_{xx})\bar{u}_e = 2\delta_{x=0}(\mathrm{Log}\bar{u}_\varepsilon + \frac{\varepsilon E}{\bar{u}_\varepsilon}),\ t \ge t_\varepsilon,\ x \in \mathbb{R}; \qquad \bar{u}(t_0, x) = \frac{1}{2}.$$

The function $\bar{u}(t_0)$ being a supersolution to (4.1) with $f(u) := f_\varepsilon(u) := 2\big(\mathrm{Log}u + \dfrac{\varepsilon E}{u}\big)$, Theorem 4.1 applies. The function f_ε has a unique zero $u_{\varepsilon,\infty}$ with $\lim\limits_{\varepsilon \to 0} u_{\varepsilon,\infty} = 0$. Moreover, it is the only nonnegative solution of (4.1) with $f = f_\varepsilon$. As a consequence, $\bar{u}_\varepsilon(t)$ converges locally to $u_{\varepsilon,\infty}$, which proves our result since ε can be arbitrarily small. •

Let us turn to the propagation case. In order to do so, let us introduce a family of subsolutions to (2.1) which will have an immediate use in the proposition below, but that we shall encounter under various disguises in the sequel. To construct them, it suffices to show that the only solutions of (4.1), with $f(u) = 2\mathrm{Log}u$, are the functions $\phi_a(x)$, with

$$\phi_a(x) = -a|x| + e^a. \tag{4.3}$$

When a is positive, these functions are positive in $]-\dfrac{e^a}{a}, \dfrac{e^a}{a}[$ and vanish at the end of the interval. This implies the following

Corollary 4.4. *For every $a > 0$, the function $\underline{\phi}_a(x) = \max(0, \phi_a(x))$ is a subsolution to (4.1) with $f(u) = 2\mathrm{Log}u$.*

This leads us immediately to

Proposition 4.5. *Let $q(t)$ satisfy the assumptions of Theorem 2.1. There exists $E_1 > 0$ such that, for every $E > E_1$ we have*

$$\lim_{t \to +\infty} R_E(t) = +\infty.$$

PROOF. It can be assumed, without loss of generality, that $q(0)$ is positive; if not we may start the argument at $t=\varepsilon$ sufficiently small. Let us pick any $t_1>0$ such that $q(t)>0$ on $[0,t_1]$ and let us denote by q_{min} its infimum. Let us choose $E>0$ large enough so that

$$\forall R>0, \qquad \mathrm{Log}R+\frac{Eq_{min}}{2R}>0. \tag{4.4}$$

Let us consider the solution $\underline{R}_E(t)$ of

$$\underline{R}_E(t)=\frac{1}{2}I_{1/2}\left(\frac{Eq_{min}}{\underline{R}_E}\right), \qquad 0\le t\le t_1. \tag{4.5}$$

Looking back at Theorem 2.1 we see that $\underline{R}_E(t)$ is globally defined, and Theorem 4.1 tells us that $\underline{R}_E(t)\le R_E(t)$ for $0\le t\le t_1$. Let us set

$$\underline{\rho}(t)=\frac{\underline{R}_E(t)}{\sqrt{Eq_{min}}} \tag{4.6}$$

and let $\underline{\mu}(t)$ the solution of (4.2) with $f(u)=\dfrac{1}{u}$ satisfying $\underline{\mu}(0)=0$; recall that $\underline{\mu}(t,0)=\underline{\rho}(t)$. From the strong maximum principle, there exists a constant $\delta>0$ such that $\underline{\mu}(t_1,x)\ge\delta$ on the interval $[-e,e]$. Consequently, if $E>0$ is large enough, we have $\sqrt{Eq_{min}}\underline{\mu}(t_1,x)\ge\phi_1(x)$ for $|x|\le e$; hence from Theorem 2.4:

$$\forall x\in I\!R, \qquad \phi_1(x)\le u(t_1,x).$$

Let us consider the evolution problem

$$\begin{aligned} u_t-u_{xx}&=2\delta_{x=0}\mathrm{Log}u, \qquad t\ge t_1, x\in I\!R\\ u(t_1,x)&=\phi_1(x)\end{aligned}$$

From Theorem 2.4 its unique solution $\underline{u}_E(t)$ satisfies $\underline{u}_E(t)\le u_E(t)$ for every $t\ge t_1$. Moreover, from Theorem 4.1, $\underline{u}_E$ is time-increasing, hence globally defined in time. Let us finally notice the non existence of a solution to (4.1) with $f(u)=2\mathrm{Log}u$, greater than $\underline{\phi}_1$: the only nonnegative solutions are the ϕ_a with $a\le 0$, but they have $\phi_a(0)\le 1<\phi_1(0)$. Therefore there holds, locally in $x\in I\!R$:

$$\lim_{t\to+\infty}\underline{u}_E(t)=+\infty,$$

which implies the proposition. •

We have therefore just seen that, by conveniently tuning the energy parameter, we could cause propagation or extinction of the flame. More information on the quantitative or qualitative behaviour of the flame in these ranges of parameters will be derived in the forthcoming Part 2.

5. EXISTENCE AND UNIQUENESS OF THE THRESHOLD

The existence of the threshold will be inferred from the same shooting method as in Section 3. Let us set, as in Section 3:

$$\begin{aligned} X_+ &= \{E > 0, \ \lim_{t\to+\infty} R_E(t) = +\infty\} \\ X_- &= \{E > 0, \ \lim_{t\to+\infty} R_E(t) = 0 \text{ or } R_E(t) \text{ quenches in finite time}\} \end{aligned} \tag{5.1}$$

Let us first prove that the radius of the flame can have other behaviours than quenching or propagation.

Proposition 5.1. *Assume that $q > 0$ on $\mathbb{R}_+^*$. Then the sets X_+ and X_- are open subsets of $\mathbb{R}_+^*$.*

PROOF. Let us first suppose that $E \in X_+$. Then there exists $t_0 > 0$ such that, for every $t \geq t_0$, we have $R_E(t) \geq 2$. Now, the solution of the heat equation on $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) with prescribed value $e+2$ at the boundary $\{x = 0\}$ tends to $e+2$ uniformly on every compact subset of $\mathbb{R}_+$. This means, in particular, the existence of a time t_1 such that $u_E(t_1, x) \geq e+1$ on $[-2e, 2e]$.

Now, we claim that the solution $u_E(t)$ is continuous with respect to E, in the $C([0, 2t_1] \times \mathbb{R})$ topology; this can be seen by inspection of the proof of Proposition 2.3. Hence, if E' is close enough to E, there will hold $u_{E'}(t_1, x) \geq e$ for x in $[-2e, 2e]$. Hence $u_{E'}(t_1)$ will be greater than $\underline{\phi}_1(x)$ on $\mathbb{R}$, ensuring the divergence of the function $R_{E'}(t)$.

Let now E belong to X_-. Let $\varepsilon > 0$ such that $2\varepsilon E - \mathrm{Log}2 < 0$, and let $t_\varepsilon > 0$ be such that $q(t) \leq \varepsilon$, and $u_E(t) \leq \dfrac{1}{4}$ for $t \geq t_\varepsilon$. By continuity, there holds $u_{E'}(t_\varepsilon) \leq \dfrac{1}{2}$ for all E' close enough to E; hence Lemma 4.3 applies. This implies the convergence of R_E to 0. •

This proposition implies, by connectedness of $\mathbb{R}_*^+$, that the set

$$X_0 = \mathbb{R}_+^* \backslash (X_- \cup X_+) \tag{5.2}$$

is nonempty. It is also obvious, by Theorem 2.4, that X_0, when not reduced to one point, is a closed interval. The rest of the section is therefore devoted to proving that, for all $E \in X_0$, the corresponding flame radius goes to 1. This will be proved as follows: on the one hand, it will be proved that the zero set of $u_E - 1$ has at most two elements; on the other hand, it will be proved that $R_E(t)$ is bounded. This will allow the identification of the limiting trajectories, and, in the end, that 1 is the only possible one.

At this stage, it will be easy to infer, from a stability argument, that X_0 is reduced to one point. Let us however notice that, in view of the above proof, it is not obvious at all to prove that X_- is open when q is compactly supported.

Theorem 5.2. *Let $E > 0$ be given. For every $t > 0$ the function $x \mapsto u_E(t, x) - 1$ vanishes at most twice on $\mathbb{R}$.*

This result will be proved in two steps, aimed at taking into account whether or not the function q is compactly supported.

Lemma 5.3. *Let $t_0 > 0$ such that $q(t) > 0$ on $[0, t_0]$. For every $t \in]0, t_0]$ the function $x \mapsto u_E(t, x) - 1$ vanishes at most twice on $I\!R$.*

PROOF. If $R_E(t)$ remains below 1, the maximum principle implies that the same occurs for u_E and there is nothing to prove. Let us therefore assume the contrary, and let t_0 be the first time when $R_E(t)$ hits the value 1. Still by the maximum principle, there holds $u_E(t, x) < 1$ if $(t, x) \in [0, t_0] \times I\!R_+^*$. Moreover we have $[\partial_x u_E(t_0)] = -2Eq(t_0)$, where $[f]$ denotes the jump of f at 0. By continuity, there exists $\delta > 0$, $\tau < t_0$ and $\varepsilon < \delta$ such that

$$\forall (t,x) \in [t_0 - \tau, t_0 + \tau] \times [-\varepsilon, \varepsilon], \quad \begin{array}{l} \partial_x u_E(t,x) \leq -\delta \text{ if } x \leq 0 \\ \partial_x u_E(t,x) \geq \delta \text{ if } x \geq 0 \\ u_E(t, \pm\varepsilon) < 1 \end{array} \tag{5.3}$$

As a consequence, we have $u_E(t, x) < 1$ for $(t, x) \in]t_0, t_0 + \tau[\times \Big(I\!R \backslash] - \varepsilon, \varepsilon[\Big)$, and this implies that $u_E(t, x) - 1$ has at most two zeroes at each time between t_0 and $t_0 + \tau$: two if $R_E(t) > 1$, and none if $R_E(t) = 1$. Now, the non-increase of the zero number theorem - see Angenent [1], Theorem D for the most recent exposition - implies that, as long as $R_E(t)$ stays above 1, $u_E(t, x) - 1$ has exactly two zeroes at each time.

Assume now that $u_E(t, 0)$ falls again below 1, and let t_1 be the first instant when this happens. Because of the positivity of q there still holds $[\partial_x u_E(t_1)] < 0$, and we claim that, for every larger t than, but close to t_1, we have $u_E(t) < 1$. Indeed, because the spatial zeroes of $u_E - 1$ are simple, there exist two functions $x_i(t)$, $i \in \{1, 2\}$ and $x_1(t) = -x_2(t)$, belonging to $C^\infty([t_0, t_1[)$, such that $u_E(t, x_i(t)) = 1$. We infer, locating the zeroes of $u_E - 1$ by inspection of the sign of its x-derivative as above, that

$$\lim_{t \to t_1} \big((x_1(t) - x_2(t)\big) = 0.$$

By the maximum principle, we have $u_E(t, x) < 1$ for $x \notin [x_1(t), x_2(t)]$; hence by continuity: $u_E(t_1, x) \leq 1$ for all $x \in I\!R$. Let now $t_2 \geq t_1$ be the maximal time such that $u_E(t, 0) = 1$ for $t \in [t_1, t_2]$. If $t_1 < t_2$, then the strong maximum principle implies that $u_E(t, x) < 1$ for $t \in [t_1, t_2]$ and $x \neq 0$. As for what happens next, three cases may occur: either $R_E(t) < 1$ for $t < t_2$ close to t_2, and $u_E(t) < 1$ by the strong maximum principle. Or $R_E(t) > 1$, and the same analysis as above applies. Or, at last: there exists a decreasing seqence $(t_n)_n$ going to t_2 such that $R_E(t_{2n}) < 1$ and $R_E(t_{2n+1}) > 1$; but in this case, due to the continuity of u_E and the fact that the jump of derivatives is < 0 for large n, we may infer that $u_E(t) - 1$ has at most two zeroes for $t \in [t_2, t_{2n+1}]$ for large n, and restart the analysis from t_{2n+1}.

To conclude the proof, it remains to use a partition of $[0, t_0]$ in time intervals according to the position of $R_E(t)$ with respect to 1, just as above. •

PROOF OF THEOREM 5.2. If q is positive on $I\!R_+$, one merely applies the preceding lemma with $t_0 = +\infty$. If not, let $]0, t_0[$ be the support of q. Then $u_E(t)$ vanishes at most twice for $t \leq t_0$. For $t \geq t_0$, let us recall that u_E satisfies the formulation (2.5); hence the function $v_E(t) := u_E(t) - 1$ is a solution of the heat equation on $\mathbb{R}_+$, with a boundary condition of the form

$$\partial_x v_E(t,0) = -\frac{\mathrm{Log} R_E(t)}{R_E(t) - 1} v_E(t,0) := \varphi(t) v_E(t,0) \tag{5.4}$$

where $\varphi(t)$ is a locally smooth and bounded function, defined on the life interval of R_E. Therefore the function

$$w_E(t,x) = e^{-x\varphi(t)} \tag{5.5}$$

may be extended on the whole line, and satisfies a homogeneous parabolic equation with bounded coefficients, and unit second-order coefficient. Hence the non-increase of the zero set principle is applicable to w_E.

At $t = t_0$, either $R_E(t_0) > 1$ and $u_E(t_0) - 1$ has exactly two zeroes, or $R_E(t_0) = 1$ and $u_E(t_0) < 1$ on $\mathbb{R}^*$, or $u_E(t_0) < 1$ on $\mathbb{R}$. In the last two cases, the strong maximum principle and the Hopf Lemma imply $u_E(t) < 1$ for all $t > t_0$; in particular, due to the integrability of $u_E(t_0)$ there exists $\alpha < 1$ such that $u_E(t_0 + 1) \leq \alpha$ on $\mathbb{R}$. The case of interest is therefore when $u_E(t_0) - 1$ has exactly two zeroes; formulation (5.4) renders the non-increase of the zero number Theorem applicable, for example [1], Theorem C. A consequence of this result is the following:

- if $R_E(t) > 1$ for all times, then a mere application of this result proves our theorem.

- If there exists $t_1 > t_0$ such that $R_E(t_1) = 1$, then, because of the boundary condition in (5.4), 0 is a double zero for 1, which implies $u_E(t_1) \leq 1$. As a consequence, $u_E(t) < 1$ for $t > t_1$.

This proves Theorem 5.2. •

Theorem 5.2 has an immediate by-product:

Corollary 5.4. *(i). Let $t > 0$ be given. The function $u_E(t)$ is below 1 if and only if $R_E(t) \leq 1$.*
(ii). Let the support of q be the interval $]0, t_0[$ with $t_0 < +\infty$. Then E belongs to X_- if and only if there exists $t_1 > t_0$ such that $R_E(t_1) < 1$.

This immediately implies the

Proposition 5.5. *Assume that q is compactly supported. Then the sets X_+ and X_- are open subsets of $\mathbb{R}_+^*$.*

PROOF. Follows directly from Corollary 5.4 and the continuity of R_E with respect to E. •

The proof of Theorem 5.2 was valid for any $E \in \mathbb{R}_+^*$. From now on, **let us choose $\mathbf{E} \in \mathbf{X_0}$.** Let now introduce, for every Lipschitz square integrable u_0, the ω-limit set $\omega(u_0)$ of u_0 with respect to (2.1):

$$\begin{aligned} \omega(u_0) = \{\psi \text{ is } C^\infty \text{ over } \mathbb{R}_-^* \text{ and } \mathbb{R}_+^*,\ \psi \neq 0, \text{ and} \\ \exists (t_n)_n \to +\infty :\ \lim_{n \to +\infty} u(t_n, .) = \psi\} \end{aligned} \tag{5.6}$$

Let us pause for one moment to examine this definition, for the very existence of $\omega(u_0)$ is once again anything but obvious. Indeed, System (2.1) is not a relatively compact evolution process and $\omega(0)$ is empty if $E \in X_+ \cup X_-$.

Proposition5.6. *Assume $\omega(0)$ to be nonempty. Then there exists $\psi \in \omega(0)$ satisfing one of the three following possibilities.*

(i). $\psi \equiv 1$,

(ii). $\psi(x) > 1$ for all $x \in I\!R$,

(iii). $\psi(x) < 1$ for all $x \in I\!R$.

PROOF. Assume that none of the above happens, and choose $\psi \in \omega(0)$. From Theorem 5.2, either $\psi - 1$ has exactly two zeroes and $\psi(0) > 1$, or $\psi \le 1$ and $\psi(0) = 1$. If the latter occurs, let us introduce the semigroup $\mathcal{S}(t)$: $u(t,x) = \mathcal{S}(t)u_0$ if and only if

$$u_t - u_{xx} = 2\delta_{x=0}\mathrm{Log}u, \quad x \in I\!R, \qquad u(0,x) = u_0(x). \tag{5.7}$$

Let ψ belong to $\omega(0)$ satisfying this possibility; from Proposition 2.3, $\mathcal{S}(t)\psi$ is defined on a nontrivial time-interval; for all $t > 0$ in this time-interval there holds $\mathcal{S}(t)\psi(0) < 1$; if not there would hold, due to formulation (2.5): $\big(\partial_x \mathcal{S}(t)\psi\big)(0) = 0$, a contradiction to the Hopf Lemma.

Assume therefore the former situation to occur; let $x_i(t)$ be the zeroes of $\mathcal{S}(t)\psi$. These two functions are smooth. If $\mathcal{S}(t)\psi(0)$ is unbounded, we claim the existence of $a > 0$ and $t_0 > 0$ such that $\phi_a \le \mathcal{S}(t_0)\psi$. Indeed, let $(t_n)_n \to +\infty$ such that $\mathcal{S}(t_n)\psi \to +\infty$. We may arrange that

$$t_{n+1} \ge t_n + 3 \quad \text{and} \quad \sigma_n := \mathcal{S}(t_n)\psi = \|\mathcal{S}(t)\psi\|_{L^\infty([0,t_n]\times I\!R)}$$

We claim that $\mathcal{S}(t)\psi$ never drops below 1; otherwise Corollary 5.4 would apply, contradicting the fact that Case (ii) of the present proposition does not occur. The generalized Gronwall Lemma implies

$$\|\mathcal{S}(t)\psi\|_{L^\infty([t_n,t_n+2]\times I\!R)} \le C\sigma_n$$

and the parabolic smoothness estimates imply

$$\|\partial_t\big(\mathcal{S}(t)\psi\big)\|_{L^\infty([t_n,t_n+1]\times I\!R)} \le C\|\mathcal{S}(t)\psi\|_{L^\infty([t_n-1,t_n+2]\times I\!R)} \le C\sigma_n$$

This implies, for all $t \in [t_n, t_n+1]$:

$$\mathcal{S}(t)\psi(0) \ge \big(1 - C(t-t_n)\big)\sigma_n.$$

Let $f(t,x)$ satisfy the heat equation on $I\!R_+^*$, such that $f(0,x) = 0$ and $f(t,0) = \dfrac{1}{2}$. The maximum principle implies:

$$\mathcal{S}\big(t_n + \frac{1}{2C}, x\big) \ge \sigma_n f\big(\frac{1}{2C}, x\big).$$

Thus for $t_0 > 0$ large enough, there exists $a > 0$ such that $\phi_a \leq \mathcal{S}(t_0)\psi$, hence $\phi_a \leq u_E$ for some possibly different - but close a. Hence $\omega(0)$ is void, a contradiction.

Therefore $\mathcal{S}(t)\psi(0)$ is bounded, bounded away from 0, and $\omega(\psi)$ is nonempty; moreover $\omega(\psi) \subset \omega(0)$. Now, if the functions $x_i(t)$ are bounded away from 0, Cases (i) or (ii) occur for all $\tilde{\psi} \in \omega(\psi)$. If not, there exists there exists $\tilde{\psi} \in \omega(\psi)$ for which Cases (i) or (iii) happen. •

If $\omega(0) = \{1\}$, this obviously implies the convergence of $\mathcal{S}(t)\psi$ towards 1.

It now remains to prove that Cases (ii) or (iii) may not happen. For any $L > 0$, let us denote by $\mathcal{S}_L(t)$, defined by: $u(t,x) = \mathcal{S}_L(t)u_0$ if and only if u satisfies (5.7) on $]-L, L[$ and the Dirichlet conditions $u(t, \pm L) = 1$.

Proposition 5.7. *Choose $L > 1$. The semigroup $\mathcal{S}_L(t)$ has two fixed points: the unstable solution 1, and a stable solution $\psi_L^+ > 1$. Furthermore, there holds*

$$\lim_{L \to +\infty} \psi_L^+ = +\infty,$$

uniformly on compact subsets of $I\!R$.
Let $u_0 \in C^1([-L, L]$ be an even function satisfying $u_0(\pm L) = 1$ and $u(0) \neq 0$. If $u_0 \geq 1$ then $\mathcal{S}_L(t)u_0$ converges to ψ_+ on every compact of $]-L, L[$.

PROOF. Let us recall that $\mathcal{S}_L(t)u_0$ satisfies formulation (2.5), with $q \equiv 0$. Hence the abstract semilinear parabolic equations treatment [5] applies to it. In particular, $\mathcal{S}_L(t)$ is a compact semigroup; hence the stability of its fixed points is determined by the eigenvalues of the linearized steady operator.

Finding the fixed points of $\mathcal{S}_L(t)$ amounts to finding the functions $\psi(x)$ of the form $\dfrac{1-a}{L}|x| + a$, with the condition

$$\frac{1-a}{L} = \mathrm{Log} a, \tag{5.8}$$

which proves both the existence of ψ_L^- and its asymptotic behaviour as $L \to +\infty$.

Let us now investigate the stability of the steady states. If ψ is a fixed point of $\mathcal{S}_L(t)$, everything amounts to solving the eigenvalue problem

$$\begin{gathered} -v'' = \lambda v, \qquad x \in]0, L[\\ v'(0) = -\frac{v(0)}{\psi(0)}, \quad v(L) = 0 \end{gathered} \tag{5.9}$$

that we may complement with the initial datum $v(0) = 1$. It is a routine calculation to verify that (5.9) has only real eigenvalues; moreover $(-\lambda)$ is a negative eigenvalue if and only if $\mu := L\sqrt{\lambda}$ is a solution of

$$\mathrm{th}\mu = \frac{\mu}{L}\psi(0). \tag{5.10}$$

This equation has a solution if and only if $\frac{1}{L}\psi(0) > 1$. Now, if $\psi = 1$ and if $L > 1$, there is a unique positive solution to (5.10), hence the solution 1 is unstable. On the other hand, if $L > 1$, (5.8) implies $\frac{1}{L}\psi_L^+(0) < 1$, hence the stability of ψ_L^+.

Let now u_0 be a Cauchy datum greater than 1. The semi-group $\mathcal{S}_L(t)$ is a strongly order-preserving semigroup, with isolated fixed points. If we prove that every solution is bounded, this will imply the convergence of every trajectory to a fixed point. This boundedness property is not too difficult; indeed if u_0 belongs to $C^1([0, L]$ there exists $L_1 > L$ such that $\psi_{L_1}^+ > u_0$; this implies that the function defined by

$$\underline{u}(t) = \begin{cases} \mathcal{S}_L(t)u_0 & \text{if } x < L \\ 1 & \text{if } x \geq \text{Ł} \end{cases}$$

is a subsolution to $\mathcal{S}_{L_1}(t)$, initially less than $\psi_{L_1}^+$. Then it has to be less than $\psi_{L_1}^+$, hence the boundedness of $\mathcal{S}_L(t)u_0$.

It remains to prove that $\mathcal{S}_L(t)u_0$ cannot tend to 1, but should it be the case, the fact that $\mathcal{S}_L(t)u_0 > 1$ would imply a nontrivial component on the unstable manifold of 1, a contradiction. •

Proposition 5.7 will be quite suitable for the treatment of compactly supported functions q. For positive integrable functions q, a slight modification is required. For $\varepsilon > 0$ let us consider the following system:

$$\begin{aligned} & u_t - u_{xx} = 2\delta_{x=0}(\text{Log} u + \frac{\varepsilon}{u}) \\ & u(t, \pm L) = 1 \end{aligned} \tag{5.11}$$

The associated semigroup will be denoted by $\mathcal{S}_{L,\varepsilon}(t)$. The results of Proposition 5.7 perturb to the following

Proposition 5.8. *Choose $L > 1$. The semigroup $\mathcal{S}_{L,\varepsilon}(t)$ has three fixed points, $\psi_{L,\varepsilon}^- < \psi_{L,\varepsilon}^0 < \psi_{L,\varepsilon}^+$. The points $\psi_\varepsilon^\pm$ are stable, whereas $\psi_{L,\varepsilon}^0$ is unstable. Moreover there holds, on every compact subset of $[0, L]$:*

$$\lim_{\varepsilon \to 0} \psi_{L,\varepsilon}^- = \psi_L^-, \quad \lim_{\varepsilon \to 0} \psi_{L,\varepsilon}^0 = 1, \quad \lim_{\varepsilon \to 0} \psi_{L,\varepsilon}^+ = +\infty.$$

The first two limits are uniform on $[-L, L]$.
The function $\psi_{L,\varepsilon}$ attracts all the trajectories starting below $\psi_{L,\varepsilon}^0$, and the functions $\psi_{L,\varepsilon}^+$ attracts all the trajectories starting above $\psi_{L,\varepsilon}^+$.

The proof is left to the reader, as an easy copy of the proof of Proposition 5.7. This leads us to the main result of this section.

Theorem 5.9. *The set X_0 is reduced to one point E_{cr}. Moreover, there holds*

$$\lim_{t \to +\infty} R_{E_{cr}}(t) = 1. \tag{5.12}$$

PROOF. Let us first prove that, if $E \in X_0$, then $R_E(t)$ tends to 1.
Case 1. The function q is compactly supported. Arguing as in the proof of Proposition 5.6, we may prove that u_E is bounded and R_E bounded away from 0. Choose $\psi \in \omega(0)$; if Case (ii) holds let $L > 0$ large enough so that Proposition 5.7 holds; define again

$$\underline{u}(t) = \begin{cases} \mathcal{S}_L(t)\psi & \text{if } x < L \\ \quad 1 & \text{if } x \geq \mathrm{L} \end{cases}$$

The function $\underline{u}(t)$ goes to $+\infty$ as $t \to +\infty$, and is less than $\mathcal{S}(t)\psi$. Therefore coming back to the solution $u_E(t)$ we see that, for t large enough, it is larger than a function ϕ_a, and therefore goes to $+\infty$ as t goes to $+\infty$. Assume that Case (iii) occurs; this means the existence of $t_0 > 0$ not belonging to the support of q, such that $R_E(t_0) < 1$. Corollary 5.4 applies, leading to a contradiction.
Case 2. The function q is positive for all time. The above analysis proves that, for all $\psi \in \omega(0)$, there holds $\omega(\psi) = \{1\}$. It remains therefore to prove that $R_E(t)$ is bounded and bounded away from 0. The boundedness of R_E is proved as above; let us now prove that, for $t > 0$ large enough, $R_E(t)$ remains above 1. Let $\varepsilon > 0$ small enough so that Proposition 5.8 applies with, say, $L = 2$. On the other hand, assume the existence of a sequence $(t_n)_n$ going to $+\infty$ such that $R_E(t_n) \leq 1$; Corollary 5.4 applies and tells us that $u_E(t_n) \leq 1$. Choose n large enough so that $q(t) \leq \varepsilon$ for $t \geq t_n$; we have

$$R_E(t) \leq \mathcal{S}_{2,\varepsilon}(t - t_n)u_E(t_n);$$

hence $R_E(t) = O(\varepsilon)$ as $t \to +\infty$. The real number ε being arbitrarily small, there holds $E \in X_-$, a contradiction.

We have therefore proved (5.12) for every E belonging to X_0. It remains to see that this set is reduced to one single point. To do so, assume the existence of $E_1 < E_2$ belonging to X_0; because the functions R_{E_i} satisfy (5.12) the function

$$v(t,x) = u_{E_2}(t,x) - u_{E_1}(t,x) \tag{5.14}$$

satisfy an inequation of the form

$$v_t - v_{xx} \geq f(t)\delta_{x=0}v, \tag{5.15}$$

where $f(t)$ tends to a positive constant as $t \to +\infty$; hence for, say, $t \geq t_0$ there will hold $2f(t) \geq k > 0$. Moreover $v(t)$ is a positive function. Let $L > 0$ be large enough so that 0 is an unstable solution of the problem

$$\begin{aligned} v_t - v_{xx} &= k\delta_{x=0}v, \qquad x \in]-L, L[\\ v(t, \pm L) &= 0 \end{aligned} \tag{5.16}$$

This is possible; see Proposition 5.7. Let $\underline{v}(t)$ be the solution of (5.16), with $\underline{v}(t_0, 0) = v(t_0, 0)$. Then $\underline{v}(t)$ tends to $+\infty$, and so does $v(t)$. This is a contradiction. •

To end this section, it remains, in the case of a compactly supported function $q(t)$, to make more precise the behaviour of $R_E(t)$ near its quenching point, namely to prove (4). This will be a consequence of the following lemma. Let us assume $q(t)$ to be supported in the interval $[0, t_0]$.

Lemma 5.10. *Let t_{max} be the quenching time. There exists $t_1 \in]t_0, t_{max}[$ such that $\dot{R}_E(t) < 0$ for $t \in]t_1, t_{max}[$.*

PROOF. Due to parabolic estimates, there exists a constant $C > 0$, independent of t_{max}, such that

$$\|\partial_{tt} u_E\|_{L^\infty([t_0,t_{max})\times(\mathbb{R}\setminus]-1,1[)} \leq C.$$

This implies the existence of $t_1 \in]t_0, t_{max}[$, $x_1 > 1$ and $\delta > 0$ such that

$$\forall t \in [t_1, t_{max}[, \qquad |\partial_t u_E(t, \pm x_1)| > \delta.$$

Let us now recall that $v_E := \partial_t u_E$ satisfies the heat equation on $[t_0, t_{max}[\times\mathbb{R}_+$, with the boundary condition

$$\partial_x v_E(t, 0) = -\frac{v_E(t, 0)}{u_E(t, 0)}.$$

This makes once again the non-zero increase theorem applicable, but we must this time notice that, due to the parity of v_E, the zero number of v_E drops by a number of at least two as soon as $v_E(t, 0)$ vanishes. Due to the finiteness of the zero number of v_E on $[-x_1, x_1]$, there exists $t_2 > t_1$ such that $v_E(t, 0)$ has a constant sign for $t \in [t_2, t_{max}[$. This implies the result, because we are in the quenching assumption. •

The preceding lemma trivially implies (4), and Theorems 1 and 2 are proved.

Let us now make the following remark. The semi-groups $\mathcal{S}_L(t)$ and $\mathcal{S}_{L,\varepsilon}(t)$ are gradient systems; as such they have a Liapounov functional

$$V(t, u(t)) = \frac{1}{2}\int_{-L}^{L} u_x^2 \, dx - 2F\big(u(t, 0)\big),$$

where $F(u)$ is a primitive of $u \mapsto \mathrm{Log}u$ or $u \mapsto \mathrm{Log}u + \dfrac{\varepsilon}{u}$, whether $\mathcal{S}_L(t)$ or $\mathcal{S}_{L,\varepsilon}(t)$ is considered. As a consequence all bounded trajectories are quasiconvergent, hence convergent due to the isolatedness of the fixed points.

Such an argument does not seem to be easily carried out in the infinite domain case. One indeed has a Liapounov functional as above, where the integration over $[-L, L]$ is replaced by an integration over $\mathbb{R}$. This indeed says that, if $E \in X_0$, then $1 \in \omega(0)$. This is not, however, sufficient to conclude that $\omega(0)$ is reduced to $\{1\}$, for the compactness available in (2.1) is only compactness in the compact-open topology; hence if $(u(t_n))_n$ converges to u in this topology, with all its derivatives, it cannot be inferred that $V(t + t_n, u(t + t_n))$ converges to $V(t, u(t)$; in fact one only has

$$V(t, u(t)) \leq \lim_{n\to+\infty} V(t + t_n, u(t + t_n)).$$

Therefore, if $\psi \in \omega(0)$ and ψ is not an equilibrium state, no contradiction can be derived from the above inequality.

Let us finally notice that the study of Parts 4 and 5 is valid for any solution of (2.1), solved with a nonnegative Lipschitz even initial datum $u_0 \leq 1$, bounded away from 1 as $x \to +\infty$.

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