The 'hump' effect in solid propellant combustion

Gawtum NAMAH

Laboratoire de Mathematiques, Université de Franche-Comté
UMR CNRS 6623
16, Route de Gray, 25030 Besancon Cedex, France

Jean-Michel ROQUEJOFFRE

UFR-MIG, Université de Toulouse III
UMR CNRS 5640
118, route de Narbonne, 31062 Toulouse Cedex, France

ABSTRACT. – An eikonal equation modelling the propagation of combustion fronts in striated media is studied *via* a level set formulation. Travelling fronts are obtained, and their speeds turn out to be monotone with respect to the angle of the striations. An effective equation for thin meniscus striations is derived from a homogenization process, thus explaining the so-called 'hump' effect. Finally, the time-asymptotic stabilization of unsteady fronts propagating in straight striations is proved.

Key words: Level set formulation, travelling fronts, homogenization, time-asymptotic stabilization.

1. Introduction

The goal of this paper is to explain a heterogeneity encountered during the combustion of solid propellants in certain rocket motors. The burning process usually occurs in heterogeneous media with periodic striations. They consist in thin binder-rich layers which are mainly due to casting processes during fabrication, cf. [7, 19]. In the context of solid propellant combustion, propagations through striations in the form of meniscii is what seems closest to realistic cases, cf. [19].

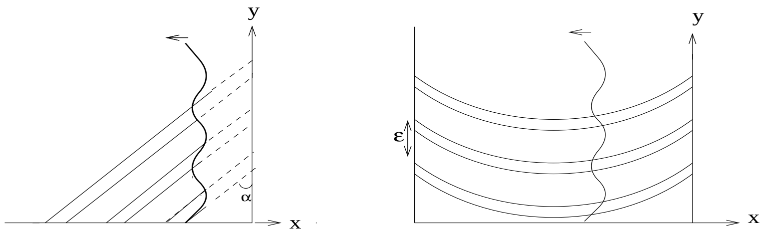

Figure 1.1: $O(1)$ oblique and ε-wide meniscus striations

As one may easily expect, no steady state is allowed to result from a propagation through striations having the form of meniscii. However, when the striations are straight - this situation may be thought of as a blow-up of the former one - one may observe the onset of periodic fronts, as shown below.

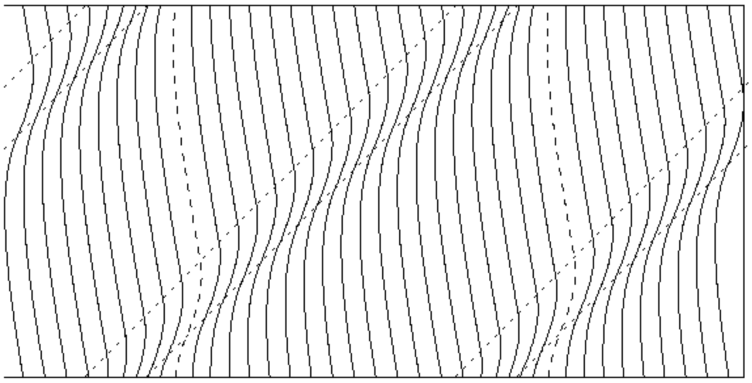

Figure 1.2: Propagation of a front in oblique striations. Space periodicity generates a time-periodic pattern.

Of course, the form of the striations play a crucial role as concerning the possible qualitative effects that this kind of homogeneity might have on the propagative process. In the meniscus case, the mean velocity has a very special profile, depicted on figure 1.3: this is the 'hump' effect; see [7].

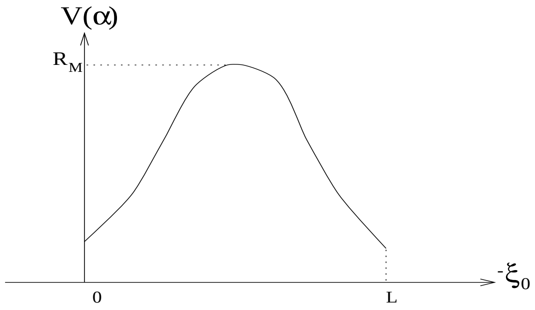

Figure 1.3. The 'hump' profile of the mean speed in the phase plane. R_M is the maximum of the local normal velocity, ξ_0 is the location of the front.

The goal of the paper is to explain the above facts. A minimal flame propagation model will be used in this paper; namely the local burning velocity will be a given function, that will be assumed to carry all the characteristics of the material under consideration. This will lead us to an eikonal equation, for which we will first write a level set formulation - Part 2. A steady state analysis will be given in Part 3, thus explaining part of the 'hump' profile of Figure 1.2. This will be completed by a homogenization of the eikonal equation in Part 4, thus providing a complete explanation of the dynamics of the effective flame front. Part 5 will be devoted to the time-asymptotic behaviour of the fronts propagating along straight striations, with any angle. Finally, numerical simulations will be presented in Part 6.

2. Model, level set formulation, main results

To allow the onset of a front, we consider our domain to be the plane $\mathbb{R}^2$ in which the striations are supposed to be curves parallel to each other. Let us denote by g the function - at least continuous -, defined up to additive constants, which describes the striations. If the flame front is also assumed to be a curve, the simplest propagation model relates its normal speed to its position, namely

$$V_N = \tilde{R}(x, y) \tag{2.1}$$

where the function $\tilde{R}$ is assumed to have as much periodicity as is allowed by the medium. In other words, if (x, y) is a given point and $(0, z)$ the intersection of the unique line passing through it with the y axis - cf. figure 2.1, then z has to have the form $z(x, y) = y - g(x)$.

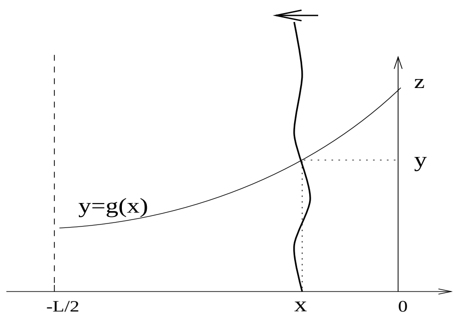

Figure 2.1: level lines of the function $\tilde{R}$.

In all cases, we will assume the following about $\tilde{R}$.

(2.2) $\quad \tilde{R}(x, y)$ is positive, smooth enough and 1-periodic in y.

We finally mention that, in the case of oblique striations of slope α, the corresponding coordinate z will simply be given by $z(x, y) = y - \alpha x$.

Let the initial front be given as the zero level line of a continuous function $u_0(x,y)$. Let $u(t,x)$ be the unique viscosity solution of

$$
\begin{aligned}
u_t + \tilde{R}(x,y)|Du| &= 0 \qquad (t,x,y) \in \mathbb{R} \times \mathbb{R}^2 \\
u(0,x,y) &= u_0(x,y)
\end{aligned}
\tag{2.3}
$$

it is known - see, for instance, Barles [1], Barles-Soner-Souganidis [3], Chen-Giga-Goto [9], Evans-Spruck [11], that
(i) at each time t, the zero level set of $u(t,.)$ depends only on the zero level set of u_0;
(ii) the zero level set of $u(t,.)$ is precisely the front that we look for. In particular, the empty interior assumption - i.e. the interior of the zero set of $u(t,.)$ is empty - is satisfied.

Hence all the studies about equation (2.1) will be carried out in the framework of equation (2.3); this will be immediately translated into information about equation (2.1). Because we wish to model flame fronts which basically propagate in the x-direction, we impose the additional condition

$$
\lim_{x \to -\infty} u(t,x,y) = -1, \quad \lim_{x \to +\infty} u(t,x,y) = 1 \quad \text{uniformly with respect to } y \tag{2.4}
$$

Three questions will be examined.

First, when the striations are straight, i.e. equation (2.3) takes the form

$$
u_t + R(y - \alpha x)|Du| = 0 \qquad (t,x,y) \in \mathbb{R} \times \mathbb{R}^2 \tag{2.5}
$$

with R smooth and 1-periodic. We look for solutions of the form

$$
v(x - V_\alpha t - \phi_\alpha(z)), \qquad z = y - \alpha x, \tag{2.6}
$$

where $u \mapsto v(u)$ is a C^1 function of the variable u, satisfying condition (2.4) and such that $v' > 0$. The function $z \mapsto \phi_\alpha(z)$ is assumed to be periodic. Let us denote by $\Gamma_\alpha(t)$ the line of the plane whose equation is given by

$$
x = \Phi_\alpha(t,z) := V_\alpha t + \phi_\alpha(z); \tag{2.7}
$$

such a line will be called a *travelling front* and V_α will be its speed. Note that, as usual, if we add a constant to Φ in equation (2.7), the resulting line is still a travelling front.

The following result states the existence of travelling fronts, but also - and this is the main information - a monotonicity result about their speeds. A partial result in that direction is obtained in Chen-Namah [8] for a model with vanishing viscosity.

Proposition 2.1 *For every $\alpha \in \mathbb{R}$ there is a unique $V_\alpha > 0$ such that travelling fronts of speed V_α exist. Moreover the function $\alpha \mapsto V_\alpha$ is an even C^1 function, which is nonincreasing on $\mathbb{R}_+$.*

As announced in the introduction, this is the first part of the explanation of the 'hump' effect.

Second, we present a homogenization analysis, when the striations are ε-wide, and of meniscus form. We model them by the problem

$$
\begin{aligned}
u_t + R(\frac{y-g(x)}{\varepsilon})|Du| &= 0 \qquad (t,x,y) \in \mathbb{R} \times \mathbb{R}^2 \\
u(0,x,y) &= u_0(x)
\end{aligned}
\tag{2.8}
$$

where the function $u_0(x)$ satisfies condition (2.4) - i.e. we start with a flat front.

Proposition 2.2 *Let u^ε be the solution of (2.8), and let $T > 0$ be given. Then u^ε converges uniformly on $[0,T] \times \mathbb{R}^2$ towards the function $u_0(x - \zeta(t))$, where ζ is the solution of the differential equation*

$$
\dot{\zeta} = V_{g'(\zeta)}, \qquad \zeta(0) = 0. \tag{2.9}
$$

This result says exactly that the homogenized front remains flat, and that its effective speed is a sole function of the common slope of the striations at its position. This, combined with the monotonicity property of Proposition 2.1, explains the 'hump' effect.

Third, we come back to straight striations and investigate the time-asymptotic stabilization of the unsteady fronts. The main result of this paper is the following.

Theorem 2.1 *Let $u(t)$ be the solution of (2.3), with an initial datum u_0 such that the support of the function $x \mapsto u_0(x,y) - (H(x) - H(-x))$ is uniformly bounded with respect to y - H is the Heaviside function.*
Then the function $u(t, x - V_\alpha t, y - \alpha V_\alpha t)$ converges uniformly to a function u^∞ as $t \to +\infty$. Moreover, for all $\mu \in]-1,1[$, the boundaries of the sets $\{u^\infty = \mu\}$ are travelling fronts.

Most probably, Theorem 2.1 is true with an initial datum satisfying only Condition (2.4). Removing the compact support assumption of the theorem would, however, imply some technicalities to recover the compactness of the solutions in $C(\mathbb{R}^2)$, that we think unnecessary.

Due to the non-strict convexity of the function $p \mapsto R|p|$, Theorem 2.1 does not fall into the recent time-asymptotic behaviour results [4], [12], [16] - although a part of this theorem could be treated *via* [4] or [16]. To illustrate that point, let us consider the extreme case when the striations are vertical, which amounts to setting $\alpha = +\infty$ or, equivalently:

$$\tilde{R}(x, y) = R(x)$$

the front is expected to become flat, and that its abscissa have a time-periodic speed. Let $\xi^\infty(t, x)$ denote the solution of the scalar equation

$$\dot{\xi}^\infty + R(\xi^\infty) = 0, \qquad \xi(0, x) = x. \tag{2.10}$$

The theorem that comes out is the following

Theorem 2.2 *Let $u(t)$ be the solution of*

$$u_t + R(x)\sqrt{u_x^2 + u_y^2} = 0, \qquad \lim_{x \to \pm\infty} u(x, y) = \pm 1 \tag{2.11}$$

with an initial datum u_0 such that the support of the function $x \mapsto u_0(x, y) - H(x)$ is uniformly bounded with respect to y.
Then there exists a continuous increasing function f and $t_0 \in \mathbb{R}$ such that $u(t, x, y) - f(\xi^\infty(t + t_0, x))$ tends to 0 as $t \to +\infty$.

A last point is the following: here we are not able to prove that the empty interior property holds for infinite times; we think, however, that the property is true. Hence, in Theorem 2.1, the level lines of u^∞ may look quite messy.

3. Travelling fronts

First we perform the change of variables $(x, y) \mapsto (x, z = y - \alpha x)$ in (2.5); this results in the equation

$$u_t + R(z)\sqrt{(u_x - \alpha u_z)^2 + u_z^2} = 0 \qquad (t, x, z) \in \mathbb{R} \times \mathbb{R}^2 \tag{3.1}$$

and we note that property (2.4) is preserved:

$$\lim_{x \to -\infty} u(t, x, z) = -1, \quad \lim_{x \to +\infty} u(t, x, z) = 1 \ \text{ uniformly with respect to } z. \tag{3.2}$$

Plugging (2.6) in (3.1) and using the strict positivity of v, we end up with an equation of the sole unknowns V_α and ϕ_α - that we denote by V and ϕ for simplicity:

$$R(z)\sqrt{(1 + \alpha\phi')^2 + \phi'^2} - V = 0, \tag{3.3}$$

this equation being assumed to hold in the viscosity sense.
To solve (3.3) we proceed as in [18]; we will not rewrite the argument but recall that, for the equation

$$u_t + H(z, u_z) = 0, \qquad u \text{ 1-periodic in } z$$

with H strictly convex with respect to p, uniformly in z. This equation is known to admit solutions of the form $Vt + \phi(z)$: either $V = V_0 := \max_z \min_p H(z,p)$, or $V > V_0$ and there exists a unique one-dimensional manifold of smooth solutions of the above type. In the former case, there is a smooth function $p(z)$ such that

$$\forall z, \quad H(z, p(z)) = \min_p H(z,p)$$

This equation is also known to have an infinity of solutions as soon as the function $z \mapsto H(z, p(z))$ assumes the value V_0 more than once - see [14].

Therefore we first look for smooth solutions, which amounts to solving

$$(1+\alpha^2)\phi'^2 + 2\alpha\phi' = \frac{V^2}{R^2} - 1.$$

This forces V to satisfy $\dfrac{V^2}{R^2} \geq \dfrac{1}{1+\alpha^2}$, which in turn implies

$$V \geq \frac{\|R\|_\infty}{\sqrt{1+\alpha^2}} = \max_z \min_p R(z)\sqrt{(1+\alpha p)^2 + p^2}. \tag{3.4}$$

For V satisfying such a condition, there are two functions $p_\pm(z)$ satisfying equation (3.3):

$$p_\pm(z) = \frac{-\alpha \pm \sqrt{(1+\alpha^2)\dfrac{V^2}{R^2(z)} - 1}}{1+\alpha^2} \tag{3.5}$$

The condition ensuring that $\phi(z) = \int_0^z p_\pm(y)\, dy$ is indeed an admissible solution writes: $\int_0^1 p_\pm(y)\, dy = 0$; this amounts to, if we assume $\alpha > 0$:

$$\alpha = \int_0^1 \sqrt{(1+\alpha^2)\frac{V^2}{R^2(z)} - 1}\; dz := g(\alpha, V). \tag{3.6}$$

Note that we do not lose any generality if we assume $\alpha > 0$. In fact, the case $\alpha \leq 0$ just consists in taking the the negative root in 3.5.

We have, for fixed $\alpha > 0$:

- $g(\alpha, \|R\|_\infty) \geq \alpha$, with strict inequality if R is not a constant,
- $g(\alpha, \dfrac{\|R\|_\infty}{\sqrt{1+\alpha^2}}) = \displaystyle\int_0^1 \sqrt{\frac{\|R\|_\infty^2}{R^2(z)} - 1}\, dz.$

Therefore we come to the following conclusion:

- if $\alpha > \alpha_0 := \displaystyle\int_0^1 \sqrt{\frac{\|R\|_\infty^2}{R^2(z)} - 1}\, dz$, there exists a unique V_α satisfying (3.4) such that $g(\alpha, V_\alpha) = \alpha$; in this case System 3.3 has a unique - up to the addition of constants - smooth solution ϕ.
- if $\alpha \leq \alpha_0$, then $V_\alpha = \dfrac{\|R\|_\infty}{\sqrt{1+\alpha^2}}$.

PROOF OF PROPOSITION 2.1. We only need now to prove the monotonicity of $\alpha \mapsto V_\alpha$. If $\alpha \leq \alpha_0$, it can be seen by inspection. If $\alpha > \alpha_0$, let us set

$$h(\alpha, V) = g(\alpha, V) - \alpha.$$

We notice that, trivially, we have $\dfrac{\partial h}{\partial V} > 0$. Let us prove that $\dfrac{\partial h}{\partial \alpha} > 0$ if R is nonconstant; to this end we compute

$$\begin{aligned} \frac{\partial g}{\partial \alpha}(\alpha, V) = &\ \frac{\alpha}{1+\alpha^2} \int_0^1 \frac{(1+\alpha^2)\dfrac{V^2}{R^2}}{\sqrt{(1+\alpha^2)\dfrac{V^2}{R^2} - 1}}\, dz \\ = &\ \frac{\alpha}{1+\alpha^2}(g(\alpha, V) + \int_0^1 \frac{dz}{\sqrt{(1+\alpha^2)\dfrac{V^2}{R^2} - 1}}) \end{aligned}$$

We apply Jensen's inequality to the second term of $\dfrac{\partial g}{\partial \alpha}$ and obtain

$$\frac{\partial g}{\partial \alpha}(\alpha, V) > \frac{\alpha}{1+\alpha^2}(g(\alpha, V)) + \frac{1}{g(\alpha, V)}) = \frac{\alpha}{1+\alpha^2} \frac{1 + g(\alpha, V)^2}{g(\alpha, V)}.$$

Remembering that $\alpha = g(\alpha, V_\alpha)$ we end up with $\dfrac{\partial g}{\partial \alpha}(\alpha, V_\alpha) > 1$. Hence $\dfrac{\partial h}{\partial \alpha}(\alpha, V_\alpha) > 0$, which in turn implies:

(i) $\alpha \mapsto V_\alpha$ is C^1 on $]\alpha_0, +\infty[$;

(ii) $\dfrac{dV_\alpha}{d\alpha}(\alpha) = -\dfrac{\partial h}{\partial \alpha}(\alpha, V_\alpha)\Big(\dfrac{\partial h}{\partial V}(\alpha, V_\alpha)\Big)^{-1} < 0.$

This ends the proof of Proposition 2.1. $\bullet$

Finally, set

$$\overline{R} = \Big(\int_0^1 \frac{dz}{R(z)}\Big)^{-1} \tag{3.7}$$

This implies the following corollary:

Corollary 3.1 *The function $\alpha \mapsto V_\alpha$ is C^1. Moreover there holds*

$$\lim_{\alpha\to+\infty} V_\alpha = \overline{R}. \tag{3.8}$$

PROOF. The only nontrivial fact is the matching of derivatives at $\alpha = \alpha_0$. We notice that we have:

$$\frac{dV_\alpha}{d\alpha} = -\frac{\alpha V_\alpha}{1+\alpha^2} + \frac{1}{\displaystyle\int_0^1 \frac{(1+\alpha^2)\frac{V_\alpha}{R^2}}{\sqrt{(1+\alpha^2)\frac{V_\alpha{}^2}{R^2}-1}}}$$

Because R is smooth, we have, at a maximum point z_{max} of R: $R(z) = \|R\|_\infty + O((z-z_{max})^2)$. Hence

$$\lim_{\alpha\to\alpha_0^+} \frac{dV_\alpha}{d\alpha} = -\frac{\alpha_0 V^{\alpha_0}}{1+\alpha_0^2},$$

which is trivially seen to coincide with the left derivative of V_α at $\alpha = \alpha_0$. $\bullet$

The proof of Proposition 2.1 is therefore over. To end this section, let us come back to the time-periodicity generated by the space periodicity of Figure 1.2 of the introduction: reverting to the old coordinates (x, y) we observe that the equation of a front is

$$x = V_\alpha t + \phi^\alpha(y - \alpha x). \tag{3.9}$$

If this equation could be inverted in x, we would indeed get an equation of the form $x = \dfrac{V_\alpha}{\alpha}t + \psi^\alpha(t, y)$; the function ψ^α being $\dfrac{1}{V_\alpha}$-periodic in t. This is not, however, possible, as shown by the following study. The function ϕ^α being - see Jensen-Souganidis [13] - piecewise C^1, a necessary and sufficient condition is $|\phi^{\alpha\prime}| < \dfrac{1}{\alpha}$.

We pick any smooth positive function R_0, and we normalize its mximum to 1. We set

$$\alpha_{00} = \alpha_0(R_0) = \int_0^1 \sqrt{\frac{1}{R_0^2(z)} - 1}\; dz; \tag{3.10}$$

and we consider the set $\mathcal{R}$ of all positive functions R such that $\|R\|_\infty = 1$ and $\alpha_0(R) = \alpha_{00}$. We now select $\alpha > 0$ and we consider all the possible functions ϕ^α obtained by solving (3.3) with a rate function $R \in \mathcal{R}$. We wish to show that all situations may occur.

Case 1. $\alpha \leq \alpha_{00}$. In this case we have $V_\alpha = \dfrac{1}{1+\alpha^2}$ and

$$\phi^{\alpha\prime} = \frac{-\alpha + \sqrt{(1+\alpha^2)\dfrac{1}{R^2(z)} - 1}}{1+\alpha^2}.$$

We notice that we always have $\dfrac{\alpha}{1+\alpha^2} < \dfrac{1}{\alpha}$; hence we have $|\phi^{\alpha\prime}| < \frac{1}{\alpha}$ as soon as R_{min} is close enough to 1. On the other hand, if R_{min} is small enough - while R stays in $\mathcal{R}$ - then $\phi^{\alpha\prime}$ has extrema whose norms may be $> \dfrac{1}{1+\alpha^2}$.

Case 2. $\alpha > \alpha_{00}$. This time we have

$$\phi^{\alpha\prime}(z) = \frac{-\int_0^1 \sqrt{(1+\alpha^2)\dfrac{V^2}{R^2(z)} - 1}\, dz + \sqrt{(1+\alpha^2)\dfrac{V^2}{R^2(z)} - 1}}{1+\alpha^2} \tag{3.11}$$

Once again it is easily seen that, if R_{min} is close to 1 - that is, R is almost constant, we have $\phi^{\alpha\prime}(z)$ almost equal to 0. On the other hand, let R^∞ be a periodic, piecewise linear function, vanishing at only one point in $[0,1]$, such that $\alpha_0(R^\infty) = \alpha_{00}$ and approximate it by a sequence of smooth functions $(R^\varepsilon)_\varepsilon$ such that $R^\varepsilon_{min} \geq \varepsilon$; then the maximum of the function $\phi^{\alpha\prime}$, with ϕ^α solving (3.3) with $R = R^\varepsilon$, tends to $+\infty$ as $\varepsilon \to 0$ - whereas the integral in (3.11) tends to a finite value.

This study raises the question of the existence of periodic fronts propagating in periodic media that are not linear in one direction. This will be the matter of a future work.

4. Homogenization

Let us now consider the case of periodic meniscus striations. Setting $z = y - g(x)$ we transform equation (2.8) into

$$\begin{aligned} u_t + R(\frac{z}{\varepsilon})\sqrt{(u_x - g'(x)u_z)^2 + u_z^2} &= 0 \qquad (t,x,z) \in \mathbb{R}\times\mathbb{R}^2 \\ u(0,x,z) &= u_0(x) \end{aligned} \tag{4.1}$$

Let $u^\varepsilon(t,x,z)$ be the unique viscosity solution of this system. Notice that x is a slow variable, and z a fast variable. From Lions, Papanicolaou, Varadhan [15], the sequence $(u^\varepsilon(t,x,z))_\varepsilon$ converges uniformly on $[0,T]\times\mathbb{R}^2$ towards the solution of a Hamilton-Jacobi equation of the form

$$\begin{aligned} u_t + H(x,z,Du) &= 0 \qquad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2 \\ u(0,x,y) &= u_0(x) \end{aligned} \tag{4.2}$$

where the effective hamiltonian $(x,z,p_x,p_z)\mapsto H(x,z,p_x,p_z)$ is the unique $\lambda>0$ such that the steady equation, with unknown $u(x,Z)$

$$\begin{aligned} R(Z)\sqrt{(p_x-(p_z+u_Z)g'(x))^2+(p_z+u_Z)^2} &= \lambda \\ \lim_{x\to\pm\infty} u(x,Z) &= \pm 1 \text{ uniformly in } Z \\ u \text{ is periodic in } & \quad z \end{aligned} \tag{4.3}$$

has solutions. A by-product of the analysis of [15] is that there is precisely a unique $\lambda>0$ such that the above system has solutions.
Notice that the only derivative involved in (4.3) is the Z-derivative, just because x is not a fast variable. As a consequence, a solution of equation 4.2 which is initially independent of z will remain independent of z, which allows us to compute the homogenized hamiltonian only at the points $(x,p_x,0)$. The key to Proposition 2.2 is therefore the

Lemma 4.1 *There holds $H(x,z,p_x,0)=V_{g'(x)}|p_x|$.*

PROOF. Obviously, the effective hamiltonian H does not depend on z. On the other hand, if u is a solution to (4.3), and if we choose any $r>0$, then ru is a solution to (4.3), with λ replaced by λr, and the vector (p_x,p_z) replaced by $r(p_x,p_z)$. Hence we have, denoting $\mathbf{p}=(p_x,p_z)$:

$$H(x,\mathbf{p}) = |\mathbf{p}|H(x,\frac{\mathbf{p}}{|\mathbf{p}|}). \tag{4.4}$$

Finally, let us set $p_z=0$ in (4.3). Using (4.4) and postulating that u does not depend on z - if we indeed find a solution satisfying this assumption, this will be true for all solutions - we end up with the equation

$$\begin{aligned} R(Z)\sqrt{(1-g'(x)u_Z)^2+u_Z^2} &= \lambda \\ u \text{ is periodic in } & \quad x \end{aligned} \tag{4.5}$$

This equation has solutions only if $\lambda=V_{g'(x)}$; hence the lemma. $\bullet$

PROOF OF PROPOSITION 2.2. As in the previous lemma, let us postulate that the solution $u(t,x,z)$ of (4.2) does not depend on z; if we indeed find a solution satisfying this requirement, we will be done. We model an initially planar front by a strictly increasing function $u_0(x)$, going to ± 1 as $x \to \pm\infty$. Because $u_0' > 0$, the function u will be nondecreasing; hence we have

$$u_t + V_{g'(x)} u_x = 0, \qquad u(0,x) = u_0(x);$$

which implies $u(t,x) = u_0(x - \zeta(t))$, where ζ is the solution of the differential equation (2.9). The Cauchy Problem for this equation is well-posed due to Corollary 3.1. •

5. Time-asymptotic behaviour for straight striations

We recall that we are now dealing with equation (3.1); we perform the transform $(x,z) \mapsto (x - V_\alpha t, z) := (x,z)$; we therefore have to solve

$$\begin{array}{rlr} u_t + R(z)\sqrt{(u_x - \alpha u_z)^2 + u_z^2} - V_\alpha u_x & = 0 & (t,x,z) \in \mathbb{R} \times \mathbb{R}^2 \\ u \text{ is 1-periodic in} & z & \\ \lim_{x \to -\infty} u(t,x,z) = -1, & \lim_{x \to +\infty} u(t,x,z) = 1 & \end{array} \tag{5.1}$$

the limits at $\pm\infty$ being uniform with respect to z. We supplement this problem with a Cauchy datum

$$u(0,x,z) = u_0(x,z) \tag{5.2}$$

which is 1-periodic in z, and for which there exists $R > 0$ such that:
- if $x \le -R$, then $u_0(x,y) = 0$,
- if $x \ge R$, then $u_0(x,y) = 1$.

Let us still denote by Γ_0 the zero level set of u_0, and Γ_t the zero level set of $u(t)$. The zero set of a travelling front will be generically denoted by $\Gamma_\alpha(t)$. The steady problem reads

$$\begin{array}{rlr} R(z)\sqrt{(u_x - \alpha u_z)^2 + u_z^2} - V_\alpha u_x & = 0 & (x,z) \in \mathbb{R} \times \mathbb{R}^2 \\ u \text{ is 1-periodic in} & z & \\ \lim_{x \to -\infty} u(t,x,z) = -1, & \lim_{x \to +\infty} u(t,x,z) = 1 & \end{array} \tag{5.3}$$

Let $C_{per,z}(\mathbb{R}^2)$ denote the space of all bounded, uniformly continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$, 1-periodic in the z variable. Equation (5.1) defines a nonlinear semigroup $\mathcal{S}(t)$ on the space

$$X = \{u \in C_{per,z}(\mathbb{R}^2) : \lim_{x \to \pm\infty} u(x,z) = \pm 1 \text{ uniformly in } z.\}. \tag{5.4}$$

Also, for every $u_0 \in X \cap W^{1,\infty}(\mathbb{R}^2)$ we may define the ω-limit set of u_0, denoted by $\omega(u_0)$, in the sense of uniform convergence. We first start with the easy

Lemma 5.1 *If $x \leq -R - 1 - \|\phi^\alpha\|_\infty$, then $\mathcal{S}(t)u_0(x,z) = -1$; if $x \geq 1 + R + \|\phi^\alpha\|_\infty$, then $\mathcal{S}(t)u_0(x,z) = 1$.*

PROOF. Let $\Gamma(u)$ be a smooth nondecreasing function, that differs from $H(u) - H(-u)$ on the interval $[0,1]$. Let us then notice that we have

$$\Gamma(x - R - \|\phi^\alpha\|_\infty - \phi^\alpha(z)) \leq u_0(x,z) \leq \Gamma(x + R - +\|\phi^\alpha\|_\infty + \phi^\alpha(z));$$

hence we have, by the maximum principle,

$$\Gamma(x - R - \|\phi^\alpha\|_\infty - \phi^\alpha(z)) \leq \mathcal{S}(t)u_0(x,z) \leq \Gamma(x + R - +\|\phi^\alpha\|_\infty + \phi^\alpha(z)).$$

This implies the lemma. •

The following proposition will always be used:

Proposition 5.1 *(i). L^∞ stability:*

$$\forall (u_0, v_0) \in X^2, \quad \|\mathcal{S}(t)u_0 - \mathcal{S}(t)v_0\|_\infty \leq \|u_0 - v_0\|_\infty \tag{5.5}$$

(ii). For every $u_0 \in X$, $\omega(u_0)$ is nonempty and compact.

Point (i) is classical [14]; Point (ii) is deduced from Lemma 5.1 and Propositions 4 and 5 of [16].

According to the position of α with respect to $\alpha_0 := \int_0^1 \sqrt{\frac{\|R\|_\infty^2}{R^2(z)} - 1}\, dz$, the periodic fronts behave in two different ways. The convergence proofs also differ.

5.1. The case $\alpha \leq \alpha_0$

This case is quite similar to the eikonal equation $u_t = |Du| - f(X)$ treated in [16], modulo the fact that we have to describe the level lines of the limiting state. In this case we have $V_\alpha = \dfrac{\|R\|_\infty}{\sqrt{1+\alpha^2}}$, and we set

$$\mathcal{Z} = \mathbb{R} \times \{z \in \mathbb{R} : \; R(z) = \|R\|_\infty\}. \tag{5.6}$$

As in [16] we have

Lemma 5.2 *The two following properties are true.*
(i). Any solution of (5.3) is nondecresing in x.
(ii). For all $(x,z) \in \mathcal{Z}$, the function $t \mapsto u(t,x,z)$ is time-nonincreasing.

PROOF. Property (i) is formally obvious, and may be proved rigorously by using the fact that equation (5.3) holds almost everywhere in (x,z), and then integrating it over a rectangle of the form $[x,x'] \times [z-\varepsilon, z+\varepsilon]$. As for Property (ii), if $R(z) = \|R\|_\infty$ and $V_\alpha = \dfrac{\|R\|_\infty}{\sqrt{1+\alpha^2}}$, it can be seen by inspection that, for all $(p_x, p_z) \in \mathbb{R}^2$, we have

$$R(z)\sqrt{(p_x - \alpha p_z)^2 + p_z^2} - V_\alpha p_x \geq 0. \tag{5.7}$$

This formal argument may be made rigorous once again by integrating (5.1) in a small box around a point (t,x,z) with $(x,z) \in \mathcal{Z}$. •

Next we have the

Lemma 5.3 *Let $\phi(x,z)$ be a Lipschitz function over $\mathcal{Z}$. Problem (5.3), with the Dirichlet datum $u = \phi$ on $\mathcal{Z}$, has at most one solution.*

PROOF. We note that, if $\psi(z)$ is a strict subsolution to the steady front problem (3.1) on every compact connected subset of $\mathbb{R}\backslash\{R(z) = \|R\|_\infty\}$, and $f(u)$ is a C^1 function such that $f'(u) > 0$ on $]-\infty, +\infty[$, then the function $f(x - \psi(z))$ is a strict subsolution to (5.3) on every compact connected subset of $\mathbb{R}^2\backslash\mathcal{Z}$. Now, any function ψ satisfying $\psi'(z) = -\dfrac{\alpha}{1+\alpha^2}$ is a strict subsolution to (5.3). This implies the desired uniqueness property: if two solutions of the Dirichlet problem u_1 and u_2 do not satisfy $u_1 \leq u_2$, then the condition at $\pm\infty$ implies the existence of a positive maximum for $u_1 - u_2$ that is reached at a point outside $\mathcal{Z}$. The usual test function for viscosity solutions

$$h^\varepsilon(X,Y) = u_1(X) - u_2(Y) - \frac{|X-Y|^2}{\varepsilon^2}$$

also has a maximum that converges, as $\varepsilon \to 0$, to the maximum of $u_1 - u_2$. And one concludes exactly as in Barles [2], Theorem 2.7. •

The convergence proof can now be concluded just as in [16], by examining the relaxed semi-limits of u as $t \to +\infty$. Let u^∞ be the limiting solution and set

$$\overline{\Gamma}(u^\infty) = \partial(\{u^\infty > 0\}), \quad \underline{\Gamma}(u^\infty) = \partial(\{u^\infty < 0\}). \tag{5.8}$$

As said in Section 2 of this paper, there is no real reason why there should hold $\overline{\Gamma}(u^\infty) = \underline{\Gamma}(u^\infty)$; however, due to Lemma 5.3 (i), we know the existence of two functions $\overline{\psi}(z)$, $\underline{\psi}(z)$, such that

$$\overline{\Gamma}(u^\infty) = \{x = \overline{\psi}(z)\}), \quad \underline{\Gamma}(u^\infty) = \{x = \underline{\psi}(z)\}. \tag{5.9}$$

To end the proof of Theorem 2.1, we first prove the

Lemma 5.4 *$\overline{\psi}$ (resp. $\underline{\psi}$) is a viscosity sub- (resp. super-) solution to (3.1).*

PROOF. Assume the existence of $z_0 \in \mathbb{R}$ and a C^1 function $\phi(z)$ such that z_0 is a maximum point of $\underline{\psi} - \phi$, with $\underline{\psi}(z_0) = \phi(z_0)$. If th denotes the hyperbolic tangent, let us set

$$u^{\infty,*}(x,z) = \limsup_{\varepsilon\to 0,\ x'\to x,\ z'\to z} \operatorname{th}\left(\frac{u^\infty(x',z')}{\varepsilon}\right). \tag{5.10}$$

We note that $u^{\infty,*}$ is upper semicontinuous. Moreover - see [3] - $u^{\infty,*}$ is a viscosity subsolution to the steady problem (5.3). Take any C^1 function $f(u)$ such that $f'(u) > 0$, $f(-\infty) = \frac{1}{2}$, $f(+\infty) = \frac{3}{2}$ and $f(0) = 1$. Set $x_0 = \underline{\psi}(z_0)$; we have:

- $u^{\infty,*}(x,z) - f(x - \phi(z)) \leq -\frac{1}{2}$ if $x < \underline{\psi}(z)$, just because we have then $u^{\infty,*}(x,z) = -1$;

- if $x \geq \underline{\psi}(z)$ we have $f(x - \phi(z)) \geq f(x - \underline{\psi}(z)) \geq 1$, hence we have $u^{\infty,*}(x,z) - f(x - \phi(z)) \leq 0$;
- $u^{\infty,*}(x_0,z_0) = 1 = f(x_0 - \phi(z_0)) = f(x_0 - \underline{\psi}(z_0))$.

Because $u^{*,\infty}$ is a subsolution to (5.3) we obtain

$$R(z_0)f'(0)\sqrt{(1+\alpha\phi'(z_0))^2 + (\phi'(z_0))^2} - V_\alpha f'(0) \leq 0,$$

which proves the desired viscosity inequality because $f' > 0$.
The case of $\overline{\psi}$ can be treated in a similar manner. •

PROOF OF THEOREM 2.1 (Case $\alpha \leq \alpha_0$). Let us prove that $\underline{\psi}$ is a super-solution to (3.1); this will end the proof. We note that what has been done for the zero set of u^∞ can be done for any value of u^∞; hence we infer that, for every $\varepsilon > 0$, the set $\partial(u^\infty > -\varepsilon)$ is a graph of the form $\{x = \overline{\psi}^\varepsilon(z)\}$, where $\overline{\psi}^\varepsilon$ is a sub-solution to (3.1). However, from the continuity of u^∞ we have

$$\underline{\psi} = \lim_{\varepsilon\to 0} \overline{\psi}^\varepsilon,$$

the limit being uniform in z. From the stability of sub-solutions under uniform convergence, we infer that $\underline{\psi}$ is also a sub-solution to (3.1). •

5.2. The case $\alpha > \alpha_0$

This time the periodic front is unique - up to the x-translations - and smooth. For every $u^\infty \in \omega(u_0)$, we will use the sets

$$
\begin{aligned}
\Omega_-(u^\infty) &= \{(x,z) \in \mathbb{R}^2 : \ x \le \sup\{x' : \ \forall x'' \le x', \ u(x'',z) < 0\}\} \\
\Omega_+(u^\infty) &= \{(x,z) \in \mathbb{R}^2 : \ x \ge \inf\{x' : \ \forall x'' \ge x', \ u(x'',z) > 0\}\}
\end{aligned}
\tag{5.11}
$$

We will also denote by $\Omega(u^\infty)$ the zero set of u^∞. Note that the boundaries of $\Omega_+(u^\infty)$ and $\Omega_-(u^\infty)$ are graphs. To make things definite, we denote by Γ^α the unique front passing through the point $(0,0)$; for all $h \in \mathbb{R}$ we will denote by τ_h the translation of vector $(h,0)$.
The convergence proof that we will follow is extremely close in spirit to the convergence proof to multidimensional travelling waves given in [17], which was itself the parabolic version of the Berestycki-Nirenberg [5] sliding method. This method will work because we almost have a strong comparison principle. Let us set, for every $u^\infty \in \omega(u_0)$

$$
\begin{aligned}
x(u^\infty) = &\ \sup\{h \in \mathbb{R} : \ \forall h' \le h, \ \tau_{h'}\Gamma^\alpha \subset \Omega_-(u^\infty)\} \\
y(u^\infty) = &\ \inf\{h \in \mathbb{R} : \ \forall h' \ge h, \ \tau_{h'}\Gamma^\alpha \subset \Omega_+(u^\infty)\}
\end{aligned}
\tag{5.12}
$$

Because $\omega(u_0)$ is compact we may define

$$
h_0 = \min_{u^\infty \in \omega(u_0)} (y(u^\infty) - x(u^\infty)).
\tag{5.13}
$$

The result that replaces the strong comparison principle of [17] is the following.

Lemma 5.5 *Choose $u^\infty \in \omega(u_0)$, and $h \in \mathbb{R}$ such that*

- $\tau_h\Gamma^\alpha \subset \Omega_-(u^\infty)$,

- $\tau_h\Gamma^\alpha \ne \partial\Omega_-(u^\infty)$.

Then there exists $\delta > 0$ and $t_0 > 0$ such that

$$
\forall t \ge t_0, \ \tau_{h+\delta}\Gamma^\alpha \subset \Omega_-(\mathcal{S}(t)u^\infty).
\tag{5.14}
$$

PROOF. **1.** The key step to the lemma is a strong comparison principle for the solutions $\zeta(t,z)$ of the equation

$$
\begin{aligned}
-\zeta_t + &\ R(z)\sqrt{(1+\alpha\zeta_z)^2 + \zeta_z^2} - V = 0, \\
&\ \zeta \text{ is 1-periodic in } z
\end{aligned}
\tag{5.15}
$$

Let ϕ^α be a solution of (3.1), and ζ_0 be a continuous Cauchy datum for (5.14). We assume the following:

$$\zeta_0(z) \geq \phi^\alpha(z), \qquad \zeta_0 \neq \phi^\alpha.$$

We claim the existence of $\delta > 0$ and $t_0 \geq 0$ such that

$$\forall t \geq t_0,\ \zeta(t) \geq \phi^\alpha + \delta.$$

To see this, we introduce the characteristic curves $X(t,z)$ of the linearized version of (5.15) around ϕ^α by setting $H(z,p) = R(z)\sqrt{(1+\alpha\zeta_z)^2 + \zeta_z^2} + V$ and

$$\dot{X} = -H_p(X, (\phi^\alpha)'(X)), \qquad X(0) = z.$$

Arguing as in [18], we infer the existence of $k_0 > 0$ such that $\zeta(t, X(t,z)) - \phi^\alpha(X(t,z)) \geq \underline{\zeta}(t,z)$ with

$$-\underline{\zeta}_t + k_0 \underline{\zeta}_z^2 = 0, \qquad \underline{\zeta}(0,z) = \zeta_0(z) - \phi^\alpha(z) \geq 0.$$

The function $\underline{\zeta}(t,z)$ is given by

$$\underline{\zeta}(t,z) = \sup_y (\zeta_0(y) - \phi^\alpha(y) - \frac{(z-y)^2}{2t}),$$

which implies, together with the 1-periodicity of $\zeta_0 - \phi^\alpha$, the uniform convergence of $\underline{\zeta}(t)$ to $\|\zeta_0 - \phi^\alpha\|_\infty > 0$. This proves our claim.

2. Assume $\tau_h\Gamma^\alpha \subset \Omega_-(u^\infty)$ and $\tau_h\Gamma^\alpha \neq \partial\Omega_-(u^\infty)$. The argument is best formulated if, following [3], we compare $(\mathcal{S}(t)u^\infty)^*$ to $\mathrm{sgn}_*(x - \phi^\alpha(z))$, where

- $\mathrm{sgn}_*(u) = 1$ if $u > 0$, $\mathrm{sgn}_*(u) = -1$ if $u \leq 0$,

- if u is a continuous function, u^* is given by the lower semi-continuous envelope of $\mathrm{sgn}_*(u)$.

Indeed, these two quantities are respectively discontinuous super- and sub- solutions to (5.1), which implies [3] that they compare.

Due to the assumptions on Γ^α and $\Omega_-(u^\infty)$, and due to the fact that they are both closed, we may construct a function $\underline{\zeta}_0(z)$ such that $u^{\infty *} \leq \mathrm{sgn}_*(x - \phi^\alpha(z) - \underline{\zeta}_0(z))$; hence we have, from Point **1**:

$$(\mathcal{S}(t)u^\infty)^* \leq \mathrm{sgn}_*(x - \phi^\alpha(z) - \underline{\zeta}(t,z)),$$

which implies the desired result. •

PROOF OF THEOREM 2.1 (Case $\alpha > \alpha_0$). Due to the compactness of $\omega(u_0)$, there exists $u^\infty \in \omega(u_0)$ satisfying $y(u^\infty) - x(u^\infty) = h_0$. First, we claim the existence of h such that $\Omega_-(u^\infty) = \tau_h\Gamma^\alpha$: if this were not so, Lemma 5.5 would apply, yielding a constant $\delta > 0$ and an instant $t_0 \geq 0$ such that, for all $t \geq t_0$, $\tau_{h+\delta}\Gamma^\alpha \subset \Omega_-(\mathcal{S}(t)u^\infty)$. On the other hand, the maximum principle implies $y(\mathcal{S}(t)u^\infty) \leq y(u^\infty)$. Therefore we have $y(\mathcal{S}(t)u^\infty) - x(\mathcal{S}(t)u^\infty) \leq h_0 - \delta$ for $t \geq t_0$, contradicting the definition of h_0. By the same argument, we also get: $\tau_h\Gamma^\alpha = \Omega_-(\mathcal{S}(t)u^\infty)$ for all $t \geq 0$. However, because $u^\infty \in \omega(u_0)$ and because of the stability property - Proposition 5.1, (ii) - we conclude that **for all** $v^\infty \in \omega(u_0)$, we have:

$$\tau_h\Gamma^\alpha = \Omega_-(v^\infty).$$

The same argument as above may be repeated for all level set of u^∞; more precisely we obtain the fact that, for all $\mu \in]-1,1[$, the boundary of the set

$$\Omega_-^\mu(u^\infty) = \{(x,z) \in \mathbb{R}^2 : \ x \leq \sup\{x' : \ \forall x'' \leq x', \ u(x'',z) < \mu\}\}$$

is a translate $\tau_{h(\mu)}$ of Γ^α. We wish to conclude that, if $u^\infty \in \omega(u_0)$, then u^∞ is a solution of (5.3); for this we need a compatibility argument. If we summarize all that has already been said, we know the existence of $h \in \mathbb{R}$ such that $\Omega_-(u^\infty) = \{x \leq \phi^\alpha(z) + h\}$. Let us set

$$\Omega = \{(x,z) : \ \phi^\alpha(z) + h < x\}; \tag{5.16}$$

this is a smooth open subset of the plane. However we have - see [14]:

$$\mathcal{S}(t)u^\infty(x,z) = \inf_{\gamma\in\mathcal{C}_\alpha, \ \gamma(t)=(x,z)} u^\infty(\gamma(0)), \tag{5.17}$$

where the set $\mathcal{C}_\alpha$ is the set of all Lipschitz curves $(\gamma_x(t), \gamma_z(t))$ such that

$$\sqrt{(\dot\gamma_x + V_\alpha)^2 + (\alpha\dot\gamma_x + \alpha V_\alpha + \dot\gamma_z)^2} \leq R(\gamma_z).$$

This equality is also valid when $(x,z) \in \partial\Omega$; in this case $\mathcal{S}(t)u^\infty = 0$. Take any $(x_0,z_0) \in \Omega$, and let $\tilde{x}_0 = \phi^\alpha(z_0) + h$. There exists $t > 0$ and a curve $(\gamma_x(s), \gamma_z(s)) \in \mathcal{C}_\alpha$ such that $(\gamma_x(t), \gamma_z(t)) \in \partial\Omega$: we may indeed take $\dot\gamma_x = -V_\alpha$, $\dot\gamma_z = 0$ and $t = \dfrac{x_0 - \tilde{x}_0}{V_\alpha}$. Because of (5.17) we have

$$\mathcal{S}(t)u^\infty(\tilde{x}_0, z_0)) \leq u^\infty(x_0,z_0),$$

implying $u^\infty(x_0,z_0) \geq 0$.

This argument may be iterated over all values of u^∞ between -1 and 1, yielding the fact that u^∞ is nondecreasing. Consequently - we only have to repeat the above argument - $\Omega_+(u^\infty)$ is also a translate of Γ^α; in fact we have

$$\Omega_+(u^\infty) = \tau_{h+h_0}\Gamma^\alpha.$$

And this may once again be repeated for all the level sets of u^∞. Hence there exists a continuous nondecreasing function $f(u)$ such that $u^\infty(x,z) = f(x-\phi^\alpha(x,z))$. If u^∞ has such a form, it is clearly a viscosity solution to (5.3): if f is smooth, one can see it by inspection. If f is not smooth and if δ^ε is an approximation of the identity, then $f * \delta^\varepsilon(x-\phi^\alpha(x,z))$ is a viscosity solution to (5.3). One only has to pass to the limit $\varepsilon \to 0$: $(f * \delta^\varepsilon(x-\phi^\alpha(x,z)))_\varepsilon$ converges uniformly to $f(x-\phi^\alpha(x,z))$, hence the limit is also a viscosity solution to (5.3). •

PROOF OF THEOREM 2.2. For all $t \in \mathbb{R}$, the function $\xi^\infty(t,.)$ given by equation (2.10), defines a diffeomorphism of $\mathbb{R}$. For convenience we drop the superscript $^\infty$ and argue in the $(\tau = t, \xi)$-coordinates. Let $X(\tau,\xi)$ be the inverse transform of ξ; the function $u(t,\xi)$ is a solution of

$$u_\tau + R(X(\tau,\xi))\sqrt{\frac{R^2(\xi)}{R(X(\tau,\xi))^2}u_\xi^2 + u_y^2} - R(\xi)u_\xi = 0. \tag{5.18}$$

Hence the function u is nonincreasing in τ, and thus converges as $\tau \to +\infty$. However the unique steady points of the above equation satisfy

$$R(\xi)(|u_\xi| - u_\xi) = 0$$

in the viscosity sense, which implies the result. •

To end this section, let us point out that, if the initial front is a graph $\{x = \psi_0(z)\}$, it will remain a graph for all later time. We may indeed solve Problem (5.1) with a datum of the form $u_0(x,z) = f(x-\psi_0(z))$ with $f' > 0$, and we will have, for all later time: $\mathcal{S}(t)u(x,z) = f(x-\psi(t,z))$ where

$$\psi_t - R(z)\sqrt{(1-\alpha\psi_z)^2 + \psi_z^2} = 0, \qquad \psi(0,z) = \psi_0(z).$$

This is once again due to the uniqueness results of [3].

6. Simulations

We now consider propagation through a heterogeneous medium which consists of a layering of two solid materials A and B. In our applications, A and B represent two propellants of different compositions, one being binder-rich and which will therefore burn at a slower rate. The results to be presented were obtained with the following standard data for the burn rate (c), and the widths of the layers (w):

	A	B
c	$c_A = 0.5$	$c_B = 0.3$
w	5	1

It is then possible to build a function $\tilde{R}(x, y)$ by imposing $\tilde{R} = c_A$ - resp. c_B - if $(x, y) \in A$ - resp. B.

6.1. Numerical scheme

In order to avoid any numerical difficulties linked to the approximation of level lines, we have decided that the fronts would be sought under the form of graphs in the (x, y) coordinates; namely: $x = \xi(t, y)$. The function ξ is then a solution of

$$\xi_t = \tilde{R}(\xi, y)\sqrt{1 + \xi_y^2}. \tag{6.1}$$

We point out - see also the concluding remark of Section 3 - that we do not know in general whether the Cauchy for (6.1) is well-posed.

To integrate (6.1) numerically, we use a standard monotone Crandall-Lions scheme - [10], with periodic boundary conditions.

6.2. The profile of the speed

In order to illustrate Proposition 2.1, we want to compute the speed of the propagation as a function of the angle of inclination. **In this subsection only**, denote by l the spatial period of the oblique striations of angle α $(0 < \alpha < \pi/2)$. Then the periods in the x and y direction will be respectively given by $X^\alpha = l/\cos\alpha$ and $Y^\alpha = l/\sin\alpha$. Once the regime is settled, the front is seen to propagate according to a periodic pattern in time of period T^α - see Figure 1.2. In fact we have

$$\xi^\alpha(y, t + T^\alpha) = \xi^\alpha(y, t) - X^\alpha, \quad y \in I\!R, \ t > 0, \tag{6.2}$$

in addition to the spatial periodicity *i.e.* $\xi^\alpha(y + Y^\alpha, t) = \xi^\alpha(y, t)$. The speed of the front is therefore $T^\alpha-$periodic in time and $Y^\alpha-$periodic in space and its mean value over one period in space and time is given by

$$V_\alpha = -\frac{1}{Y^\alpha}\frac{1}{T^\alpha}\int_0^{Y^\alpha}\int_0^{T^\alpha} \xi_t^\alpha(y, t)\, dy\, dt = \frac{X^\alpha}{T^\alpha}. \tag{6.3}$$

Figure 6.1 shows the profile of the mean propagating speed as a function of the angle and confirms that the speed is an increasing function of the angle of inclination of the striations.

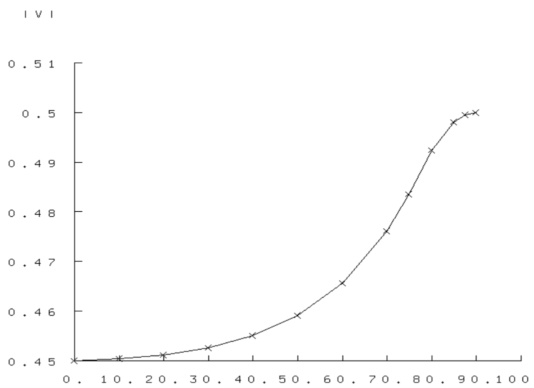

Figure 6.1: The speed is an increasing function of the angle.

6.3. The meniscus case.

In the context of the combustion of solid propellant in cylindrical grains, propagation through striations in the form of meniscii is what seems closest to realistic cases, see [19]. As previously, we consider striations disposed in a periodic fashion.

For the computations, we consider our domain to be the infinite cylinder $(-10, 0) \times I\!R$ in which the meniscii are supposed to be arcs of circles of fixed radius and whose centres lie on the axis $x = -5$. Under these assumptions, the arcs will be vertical translations of each other and thus we have a periodical set up in the y direction. Note that here, the burn rate is periodic only in the y direction although it depends on both x and y coordinates. Therefore it is clear that one does not expect the setting up of a steady state.

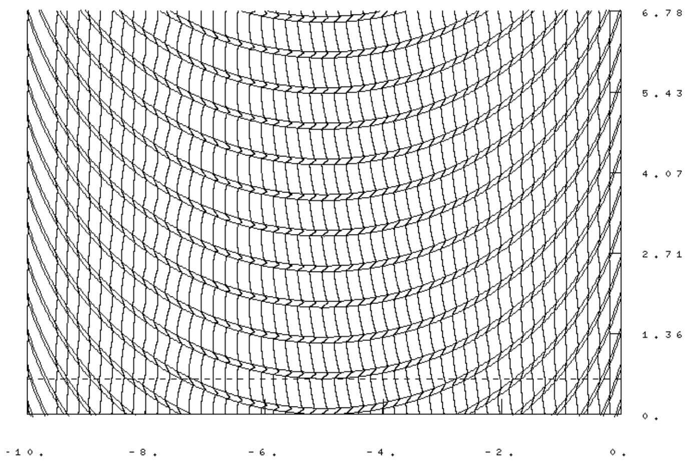

Figure 6.2: Front propagation through periodic striations of meniscus form.

By starting with a vertical initial front ($\xi_0(y) = 0$), Figure 6.2 above shows the evolution of the front in time. One can observe that the front becomes more and more corrugated as the local oblique angle increases, *i.e.* when we go from the boundaries towards the centre of the domain.

Figure 6.3 below shows the corresponding mean speed of propagation as a function of the time history. The numerical simulations reproduce the 'hump' effect mentionned in the introduction.

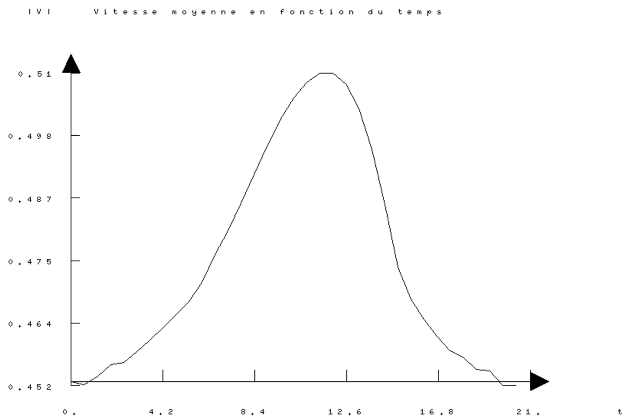

Figure 6.3: The speed as a function of time. This profile agrees with the 'hump' effect observed in industrial applications.

Therefore it is clear that the 'hump' effect, observed in industrial applications during propellant combustion, is related to the presence of striations in the propellant grain which *de facto* is not homogeneous. The overspeed observed in the midweb portion of the firing history is due to the fact that the local angle of inclination of the striations increases as we go from the walls of the domain towards the centre where it reaches its maximum value.

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